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An Introduction to Hydrodynamics and Water Waves
Volume II: Water Wave Theories

BERNARD LE MÉHAUTÉ

JULY 1969

Miami, Florida

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An Introduction to Hydrodynamics and Water Waves
Volume II: Water Wave Theories

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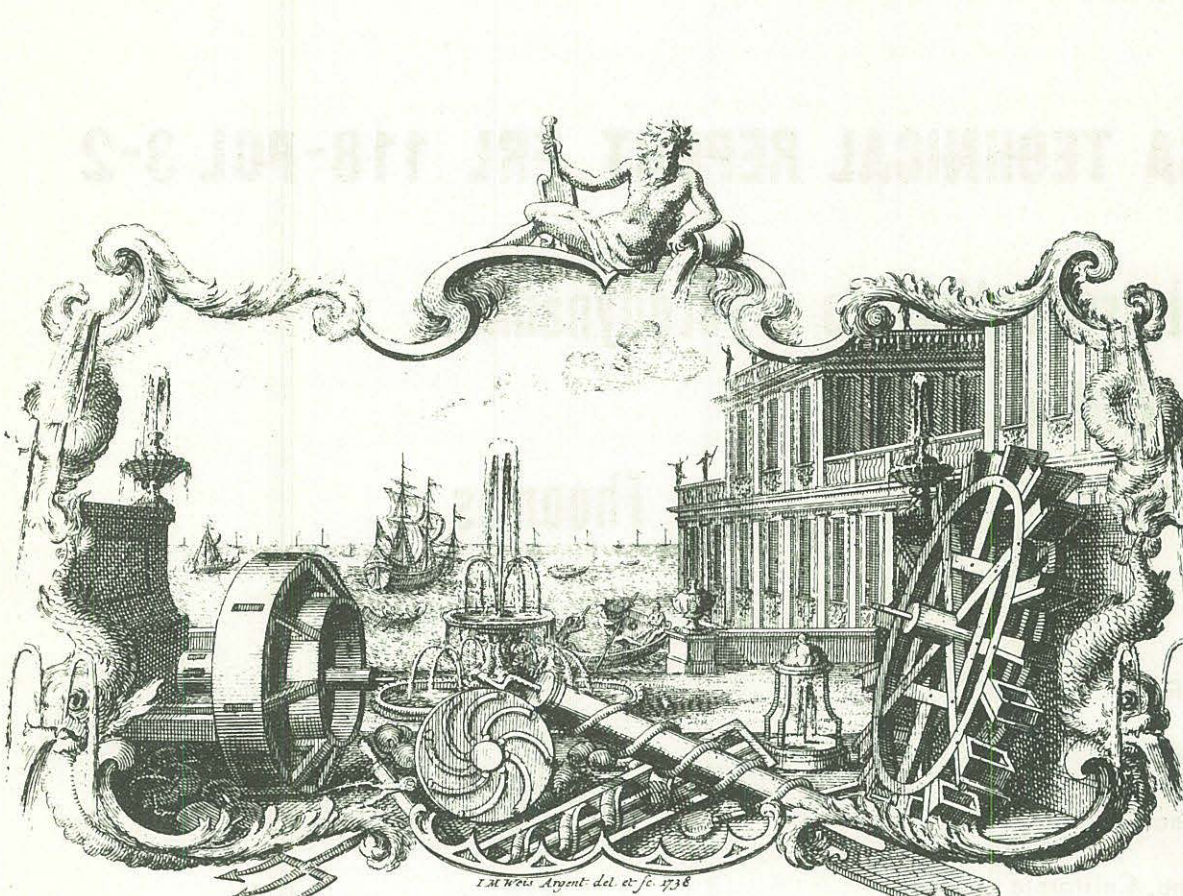
HYDRODYNAMICA,

SIVE

DE VIRIBUS ET MOTIBUS FLUIDORUM COMMENTARII.



"Remember, when discoursing about water,
to induce first experience, then reason"

Leonardo da Vinci

FOREWORD

Understanding and interpreting oceanographic observations depend on a knowledge of the basic physics governing water motion. Water waves, from the shortest ripples that roughen the sea surface, increasing wind drag, to the tides of global dimensions, with their associated currents affecting the entire ocean volume, influence the oceanic and nearshore environment. ESSA has a wide variety of interests in fluid dynamics and especially in water waves. Its interest in hydrodynamics extends from the most basic scientific aspects, which may be of academic interest only, to engineering applications, which put knowledge into use for the good of mankind. Dr. Le Méhauté does much to bridge the gap between rigorous but abstract theoretical works, which are often difficult to translate into useable applications, and pure engineering approaches to hydrodynamics, which do not go much beyond a presentation of results and so contribute little to one's basic understanding.

Gaylord R. Miller
Director
Joint Tsunami Research Effort
Pacific Oceanographic Laboratories

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FREE SURFACE FLOW

AND WATER WAVES

CHAPTER XV

AN INTRODUCTION TO WATER WAVES

XV-1 A PHYSICAL CLASSIFICATION OF WATER WAVES AND DEFINITIONS

XV-1.1 ON THE COMPLEXITY OF WATER WAVES

XV-1.1.1

The aim of this chapter is to present the theories for unsteady free surface flow subjected to gravitational forces. Such motions are called water waves, although pressure waves (such as acoustic waves) in water are also water waves. They are also called gravity waves, although atmospheric motions are also waves subjected to gravity.

From the physical viewpoint, there exist a great variety of water waves. Water wave motions range from storm waves generated by wind in the oceans to flood waves in rivers, from seiche or long period oscillations in harbor basins to tidal bores or moving hydraulic jumps in estuaries, from waves generated by a moving ship in a channel to tsunami waves generated by earthquakes or to waves generated by underwater nuclear explosions.

From the mathematical viewpoint, it is evident that a general solution does not exist.

Even in the simpler cases, approximations must be made.

One of the important aspects of water wave theories is the establishment of the limits of validity of the various solutions due to the simplifying assumptions. The mathematical approaches for the study of wave motion are as various as their physical aspects. As a matter of fact, the mathematical treatments of the water wave motions embrace all the resources of mathematical physics dealing with linear and non-linear problems as well. The main difficulty in the study of water wave motion is that one of the boundaries, namely the free surface, is one of the unknowns.

Water wave motions are so various and complex that any attempt at classification may be misleading. Any definition corresponds to idealized situations which never occur rigorously but are only approximated. For example, a pure two-dimensional motion never exists. It is a convenient mathematical concept which is physically best approached in a tank with parallel walls. Boundary layer effects and transverse components still exist although they are difficult to detect.

XV-1.1.2

It has to be expected that due to this inherent complexity, a simple introduction to the problem of water waves is a difficult, if not impossible, task. The subject is not simple, so a simple introduction would be misleading. Hence this chapter should rather be regarded as a guide for the following chapters and for continuing further study beyond the scope of this book.

It is the purpose of this chapter to cover as many theories as possible and to relate them with respect to each other. It will be better understood once the following chapters have been studied. The number of water wave theories is such that the subject is confusing to a beginner.

It has to be expected that even the present superficial exposure will be of great help for the understanding of these various approaches. The full assimilation of the subject leading to a clear cut understanding of this chapter can only come after a comprehensive study of each existing theory within or beyond the scope of the present book.

This being borne in mind, the following classification is proposed. A physical classification is given first, then the different mathematical approaches and their limits of validity are introduced. Finally, the traditional two great families of water waves are presented.

XV-1.2 OSCILLATORY WAVES

From the physical viewpoint, there exist essentially two kinds of water waves. They are the oscillatory waves and the translatory waves. In an oscillatory wave, the average transportation of fluid, i.e., the discharge or mass transportation, is nil. The wave motion is then analogous to the transverse oscillation of a rope (see Figure XV-1). A translatory wave involves by definition a transport of fluid in the direction in which the wave travels. For example, a moving hydraulic jump, so called tidal bore or simply bore, is a translatory wave.

XV-1.2.1 Progressive Waves

An oscillatory wave can be progressive or standing. Consider

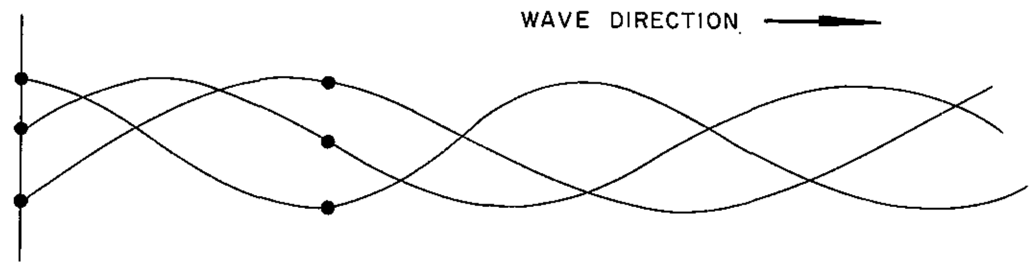


FIGURE XV-1
OSCILLATORY WAVE

a disturbance $\eta(x, t)$ such as a free surface elevation traveling along the OX axis at a velocity C. The characteristics of a progressive wave remain identical for an observer traveling at the same speed and in the same direction as the wave (Figure XV-2). In the case where η can be expressed as a function of $(x - Ct)$ instead of (x, t), a "steady state" profile is obtained. $\eta(x - Ct)$ is the general expression for a progressive wave of steady state profile traveling in the positive OX direction at a constant velocity C. In the case where the progressive wave is moving in the opposite direction, its mathematical form is expressed as a function of $(x + Ct)$. It is pointed out that the definition of wave velocity C for a non-steady state profile makes no sense, since each "wave element" travels at its own speed, so causing the wave deformation.

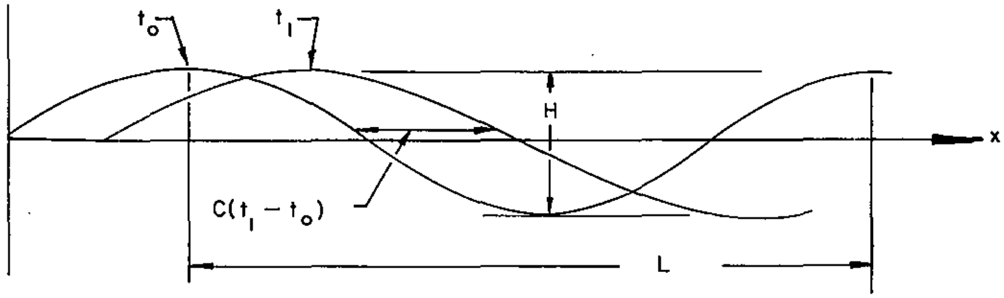


FIGURE XV-2
PROGRESSIVE WAVE

XV-1.2.2 Harmonic Waves

The simplest case of a progressive wave is the wave which is defined by a sine or cosine curve such as

$$\eta = \frac{H}{2} \begin{matrix} \sin \\ \cos \end{matrix} m(x - Ct)$$

Such a wave is called a harmonic wave where $H/2$ is the amplitude and H the wave height.

The distance between the wave crests is the wave length L, and $L = CT$ where T is the wave period. The wave number $m = 2\pi/L$ is the number of wave lengths per cycle. The frequency is $k = 2\pi/T$. Hence the previous equation can be written

$$\eta = \frac{H}{2} \begin{matrix} \sin \\ \cos \end{matrix} 2\pi \left(\frac{x}{L} - \frac{t}{T}\right).$$

XV-1.2.3 Standing Waves, Clapotis and Seiche

A standing or stationary wave is characterized by the fact that it can be mathematically described by a product of two independent functions of time and distance, such as

$$\phi = H \sin \frac{2\pi x}{L} \sin \frac{2\pi t}{T}$$

or more generally (see Figure XV-3):

$$\phi = \phi_1 (x) \cdot \phi_2 (t)$$

A standing wave can be considered as the superposition of two waves of the same amplitude and same period traveling in opposite directions. In the case where the convective inertia terms are negligible, the standing wave motion is defined by a mere linear addition to the equations for the two progressive waves. The following identity is

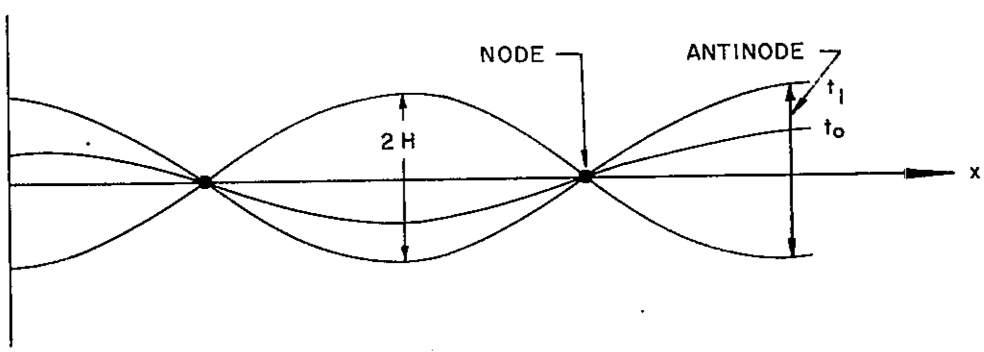


FIGURE XV-3
STANDING WAVE

easily verified:

$$\frac{H}{2} \sin \frac{2\pi}{L} (x - Ct) + \frac{H}{2} \sin \frac{2\pi}{L} (x + Ct) \quad = \quad H \sin \frac{2\pi}{L} x \cos \frac{2\pi}{T} t$$

A standing wave generated by an incident wind wave is called clapotis. In relatively shallow water ($\frac{d}{L} < 0.05$) it is called a seiche. A seiche is a standing oscillation of long period encountered in lakes and harbor basins. The amplitude at the node is H and at the antinode it is zero.

XV-1.2.4 Partial Clapotis

Two waves of same period but different amplitudes traveling in opposite directions form a "partial clapotis" and can be defined linearly by the sum of $A \sin(x - Ct) + B \sin(x + Ct)$. A partial clapotis can also be considered as the superposition of a progressive wave with a standing wave. A partial clapotis is encountered in front of an obstacle which causes a partial reflection. The amplitude at the node is $N = A + B$ and at the antinode $D = A - B$. The direct measurement of N and D yields: $A = \frac{N+D}{2}$ and $B = \frac{N-D}{2}$ and the reflection coefficient $R = \frac{N-D}{N+D}$.

XV-1.2.5 Wave Refraction, Wave Diffraction, and Wave Breaking

It will be seen that the wave velocity is in general a function of the water depth (see Section XVI-3.3). The phenomenon of refraction is encountered when a wave travels from one water depth to another water depth (Figure XV-4). The phenomenon of diffraction is encountered at the end of an obstacle (Figure XV-5). It can be considered as a process of transmission of energy in a direction parallel to the wave crest

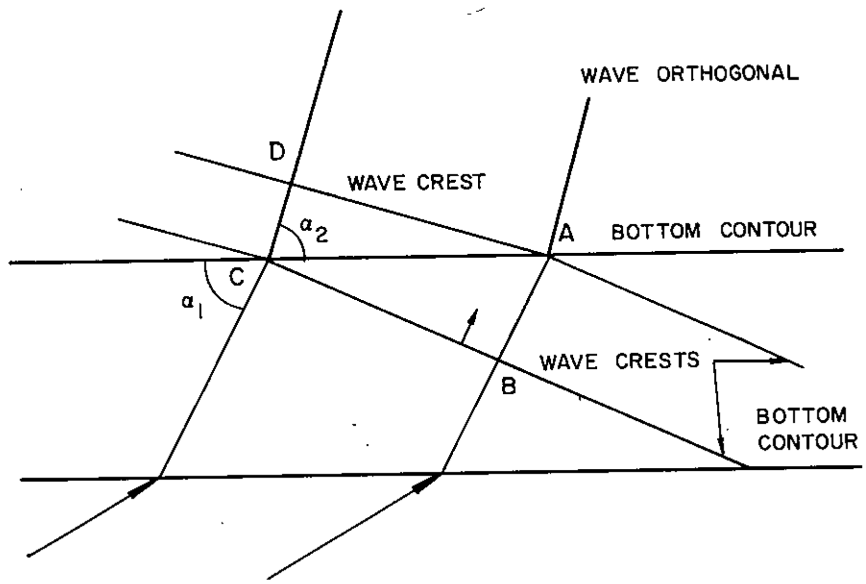


FIGURE XV-4
WAVE REFRACTION

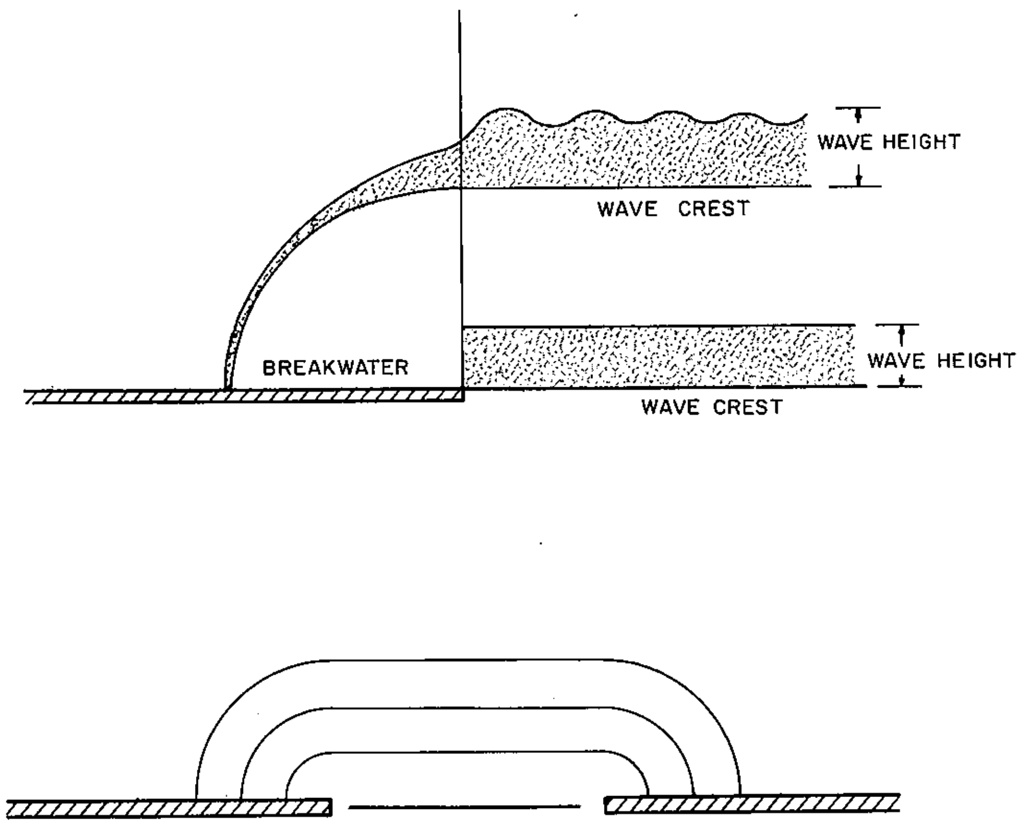


FIGURE XV-5
WAVE DIFFRACTION

although this definition oversimplifies a more complex phenomenon.

The breaking phenomenon is encountered at sea under wind action (white caps) or on beaches (breakers) and in tidal estuaries (tidal bores), see Figure XV-6. It is a shock wave phenomenon which is also encountered in gas dynamics. The breaking phenomenon is characterized by a high rate of free turbulence associated with a high rate of energy dissipation. Bores generated by wind waves breaking on beaches or by tides of high amplitude in estuaries should be regarded as translatory waves.

From the hydrodynamic viewpoint the breaking criterion is defined when the particle velocity at the crest tends to become larger than the wave velocity, or when the pressure condition (p = constant) at the free surface can no longer be satisfied, or when the particle acceleration at the crest becomes larger than the gravity acceleration, or again when the free surface tends to become a vertical wall of water. For irrotational progressive gravity waves it is found that the breaking criterion is related to a maximum wave steepness. The following formulas

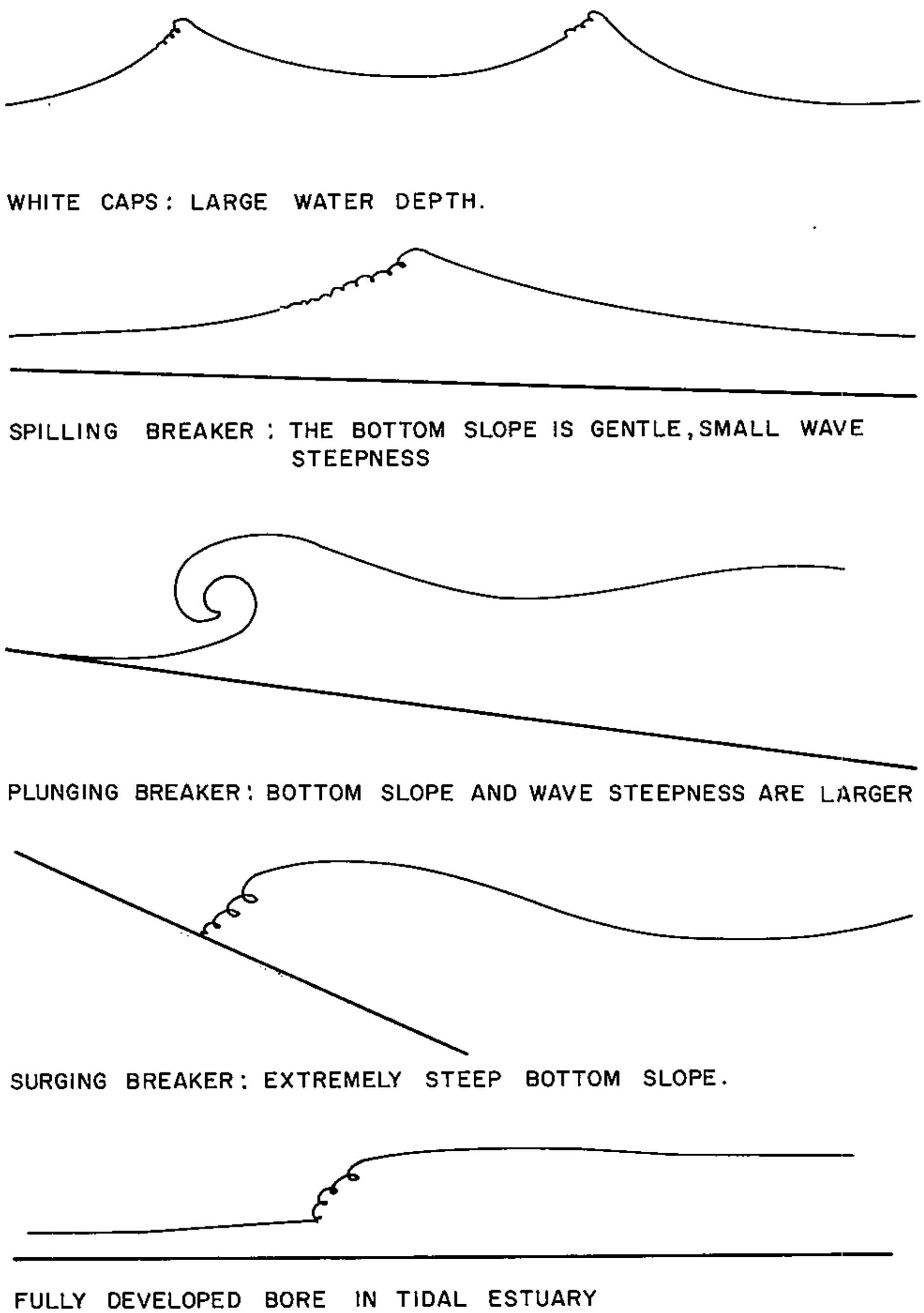


FIGURE XV-6
DIFFERENT KINDS OF WAVE BREAKERS

are generally presented

$$\frac{H}{L} < 0.142 \quad \text{in deep water (Michell limit)}$$

$$\frac{H}{L} < 0.14 \tanh \frac{2\pi d}{L} \quad \text{in intermediate water depth (Miche formula)}$$

$$\frac{H}{d} < 0.78 \quad \text{for solitary waves in shallow water.}$$

XV-1.3 TRANSLATORY WAVES

In a translatory wave, there is a transport of water in the direction of the wave travel. A number of examples are given:

Tidal bore or moving hydraulic jump

Waves generated by the breaking of a dam

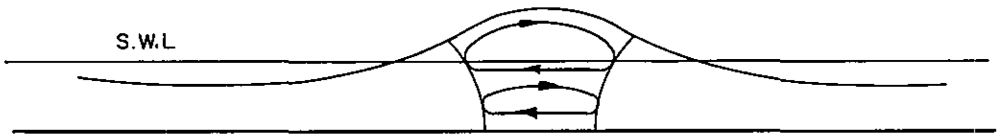
Surge on a dry bed

Undulated moving hydraulic jump

Solitary waves

Flood waves in rivers

In order to illustrate the basic difference between oscillatory waves and translatory waves, examples of a cnoidal type wave and a solitary wave are presented (see Figure XV-7). The motion of these waves is very similar:



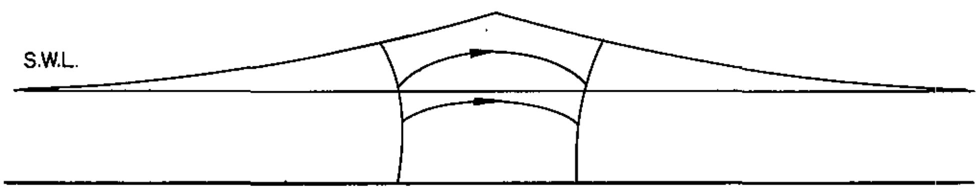


FIGURE XV-7
DIFFERENCE BETWEEN AN OSCILLATORY WAVE AND A TRANSLATORY WAVE

it consists of a fast jump ahead of the water particle under the wave crest. However, in the case of a cnoidal wave, there is a gentle slow return under a long flat trough (with a relatively small mass transport). A solitary wave motion always involves an important net mass transport. However, from the mathematical viewpoint, these two kinds of motion are of the same family, i.e., they are subjected to the same simplifying assumption and they obey the same basic equations.

XV-2 CRITERIA FOR MATHEMATICAL METHODS OF SOLUTION

XV-2.1 THE SIGNIFICANT WAVE PARAMETERS

XV-2.1.1

In an Eulerian system of coordinates a surface wave problem generally involves three unknowns: the free surface elevation (or total water depth), the pressure (generally known at the free surface), and the particle velocity.

Since a general method of solution is impossible, a number of simplifying assumptions have been made which apply to a succession of particular cases with varying accuracy.

In general, the method of solution which is used depends upon the relative importance of the convective inertia terms with respect to the local inertia.

XV-2.1.2

However, instead of dealing with these inertial terms directly, it is more convenient to relate this ratio to more accessible parameters. Three characteristic parameters are used. They are:

1) A typical value of the free surface elevation such as the wave height H.

2) A typical horizontal length such as the wave length L.

3) The water depth d.

Although the relationships between the inertial terms and these three parameters are not simple, their relative values are of considerable help in classifying the water wave theories from a mathematical viewpoint.

For example, it is easily conceived that when the free surface elevation decreases the particle velocity decreases also. Consequently, when the wave height H tends to zero, the convective inertia term, which is related to the square of the particle velocity, is an infinitesimal of higher order than the local inertia term, which is related linearly to the velocity. Consequently, the convective inertia can be neglected and the theory can be linearized.

In this way, the three possible parameters to be considered are:

$$\frac{H}{L}, \quad \frac{H}{d}, \quad \text{and} \quad \frac{L}{d}.$$

The relative importance of the convective inertia term increases as the value of these three parameters increases.

In deep water (small H/d, and small L/d), the most significant parameter is H/L which is called the wave steepness. In shallow water the most significant parameter is H/d which is called the relative height. In intermediate water depth, it will be seen that a significant parameter which also covers the three cases is $\frac{H}{L}\left(\frac{L}{d}\right)^3$.

XV-2.2 THE METHODS OF SOLUTION

Depending upon the problem under consideration and the range of values of the parameters H/L, H/d and L/d, four mathematical approaches are used. They are:

1) linearization
2) power series
3) numerical methods
4) random functions

XV-2.2.1 Linearization

The simplest cases of water wave theories are, of course, the linear wave theories, in which case the convective inertia terms are neglected completely. These theories are valid when H/L, H/d and L/d are small, i.e., for waves of small amplitude and small wave length in deep water. For the first reason they are called the "small amplitude wave theory". It is the infinitesimal wave approximation.

The linearization of the basic equation is so suitable to mathematical manipulation that the linear wave theories cover an extreme variety of water wave motions. For example, some phenomena which can be subjected to linear mathematical treatment include the phenomena of wave diffraction, the waves generated by a moving ship, waves generated by explosions, etc.

XV-2.2.2 The Power Series and the Steady State Profiles

The solution can be found as a power series in terms of a small quantity by comparison with the other dimensions. This small quantity is H/L for small L/d since in deep water the most significant parameter is H/L. It is H/d for large L/d since in shallow water the most significant parameter is H/d.

In the first case (development in terms of H/L), the first term of the power series is given by application of the linear theory. In the second case the first term of the series is already a solution of non-linear equations.

The calculation of the successive terms of the series is so cumbersome that these methods are used in a very small number of cases. The most typical case is the progressive periodic wave. In this case, the solution is assumed to be a priori that of a steady state profile, i.e., a function such as $F = f(x - Ct)$ where C is a constant equal to the wave velocity or phase velocity.

The simplification introduced by such an assumption is due to the fact that

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial (x - Ct)}$$

and

$$\frac{\partial F}{\partial t} = C \frac{\partial F}{\partial (x - Ct)}$$

such that

$$\frac{\partial F}{\partial t} = - C \frac{\partial F}{\partial x}$$

In such a way the time derivatives can be eliminated easily and replaced by a space derivative.

Typical examples of such treatments are:

1) Power series of H/L or the Stokes waves, valid in deep water. The first term of the series is obtained from linear equations and corresponds to the infinitesimal wave approximation.

2) Power series of H/d: the cnoidal wave or the solitary wave, valid for shallow water. The first terms of the series are obtained as a steady state solution of already non-linear equations, but correspond to shallow water approximation which will be developed in Section XV-4.1.

XV-2.2.3 <u>The Numerical Methods</u>

However, it may happen that a steady state profile does not exist as a solution, in which case the method to be used is often a numerical method of calculation where the differentials are replaced by finite difference. This occurs for large values of H/d and L/d which corresponds to the fact that the nonlinear terms such as $\rho u \frac{\partial u}{\partial x}$ are relatively large by comparison with the local inertia such as $\rho \frac{\partial u}{\partial t}$. This is the case of long waves in very shallow water.

XV-2.2.4

Of course, a numerical method of calculation can be used for solving a linearized system of equations. For example, the relaxation method is used for studying small wave agitation in a basin. Also, an analytical solution of a nonlinear system of equations can be found in some particular cases. Hence it must be borne in mind that these three methods and the range of application which has been given indicate more of a trend than a general rule.

XV-2.2.5 The Random Functions

Aside of the three previous methods which aim at a fully deterministic solution of the water wave problem, the description of sea state generally involves the use of random functions. The mathematical operations which follow such treatment (such as harmonic analysis) generally imply that the water waves obey linear laws, which are the necessary requirements for assuming that the principle of superposition is valid. Consequently, such method loses its validity for describing the sea state in very shallow water (large values of H/d and L/d).

XV-2.3 AN INTRODUCTION TO THE URSELL PARAMETER

XV-2.3.1

An example will illustrate the previous considerations. The potential function for a Stokes wave or irrotational periodic gravity wave traveling over a constant finite depth at a second order of approximation is found to be:

$$\phi = -\frac{H}{2}\frac{k}{m}\frac{\cosh m(d+z)}{\sinh md}\cos(mx - kt)$$

$$+ \frac{3}{8}\left(\frac{H}{2}\right)^2 k \frac{\cosh 2m(d+z)}{\sinh^4 md}\cos 2(mx - kt)$$

The series being convergent, and since the term in H is the solution obtained by taking into account the local inertia only, while the term in H^2 is the first correction due to convective inertia, i.e., the most significant one, the relative importance of the convective inertia term can be described by the ratio of the amplitudes of these two terms. In particular, in very shallow water, since $\cosh A \rightarrow 1$ and $\sinh A \longrightarrow A$, it is seen after some simple calculation that the ratio of the amplitude of the second order term to the amplitude of the first order term is

$$\frac{3}{16}\frac{1}{(2\pi)^2}\cdot\frac{H}{L}\left(\frac{L}{d}\right)^3$$

When $U_R = \frac{H}{L}\left(\frac{L}{d}\right)^3$ is very small, the small amplitude wave theory is valid.

If, instead of H, one uses the maximum elevation η_o above the still water level (η_o is equal to $\frac{H}{2}$ in the linear theory), the so-called Ursell parameter initially introduced by Korteweg and de Vries is obtained.

$$U_R = \frac{\eta_o}{L}\left(\frac{L}{d}\right)^3$$

When $\frac{\eta_o}{L} \left(\frac{L}{d}\right)^3 << 1$, the linear small amplitude wave theory applies. In principle more and more terms of the power series would be required in order to keep the same relative accuracy as the Ursell parameter increases.

Also, in the case of very long waves in shallow water such as flood waves, bore, and nearshore tsunami waves, the value of the Ursell parameter which is supposed to be $>> 1$ depends upon the interpretation given to L. The relative amplitude $\frac{H}{d}$ is then a more significant parameter for interpreting the importance of the non-linear terms. In this case the vertical component of inertia force is negligible and the only term for convective inertia is $\rho u \frac{\partial u}{\partial x}$. Then it is possible to calculate the ratio of amplitude of convective inertia to the amplitude of local inertia $\left(\rho u \frac{\partial u}{\partial x} \Big/ \rho \frac{\partial u}{\partial t}\right)$ directly. Since in very shallow water $\frac{d}{L}$ is very small and $\cosh A \rightarrow 1$ and $\sinh A \rightarrow A$, one has simply:

$$u = \frac{\partial \phi}{\partial x} = \frac{H}{2} \frac{k}{md} \sin(mx - kt)$$

and it is found that

$$\frac{\rho\, u \left.\frac{\partial u}{\partial x}\right|_{max}}{\rho \left.\frac{\partial u}{\partial t}\right|_{max}} = \frac{H}{2d}$$

which demonstrates the relative importance of the ratio $\frac{H}{d}$. Despite these difficulties of interpretation, the Ursell parameter is a useful simple guide, but is not necessarily sufficient for judging the relative importance of the non-linear effects.

XV-2.3.2

The following graph (Figure XV-8) indicates approximately the range of validity of the various theories. This graph has been established for two-dimensional periodic waves such as illustrated on Figure XV-9, but it gives an indication for any kind of water waves. Three corresponding values of the Ursell parameter have been indicated. The graph is limited by a breaking criteria which indicates that there is a maximum value for the wave steepness which is a function of the relative depth (see Section XV-1.2.5).

A deep quantitative investigation on the error which is made by using various theories over various areas of application and at the limits of separation has not been done so far; so such a kind of graph is somewhat arbitrary and qualitative.

XV-2.4 THE TWO GREAT FAMILIES OF WATER WAVES

In hydrodynamics the water wave theories are generally

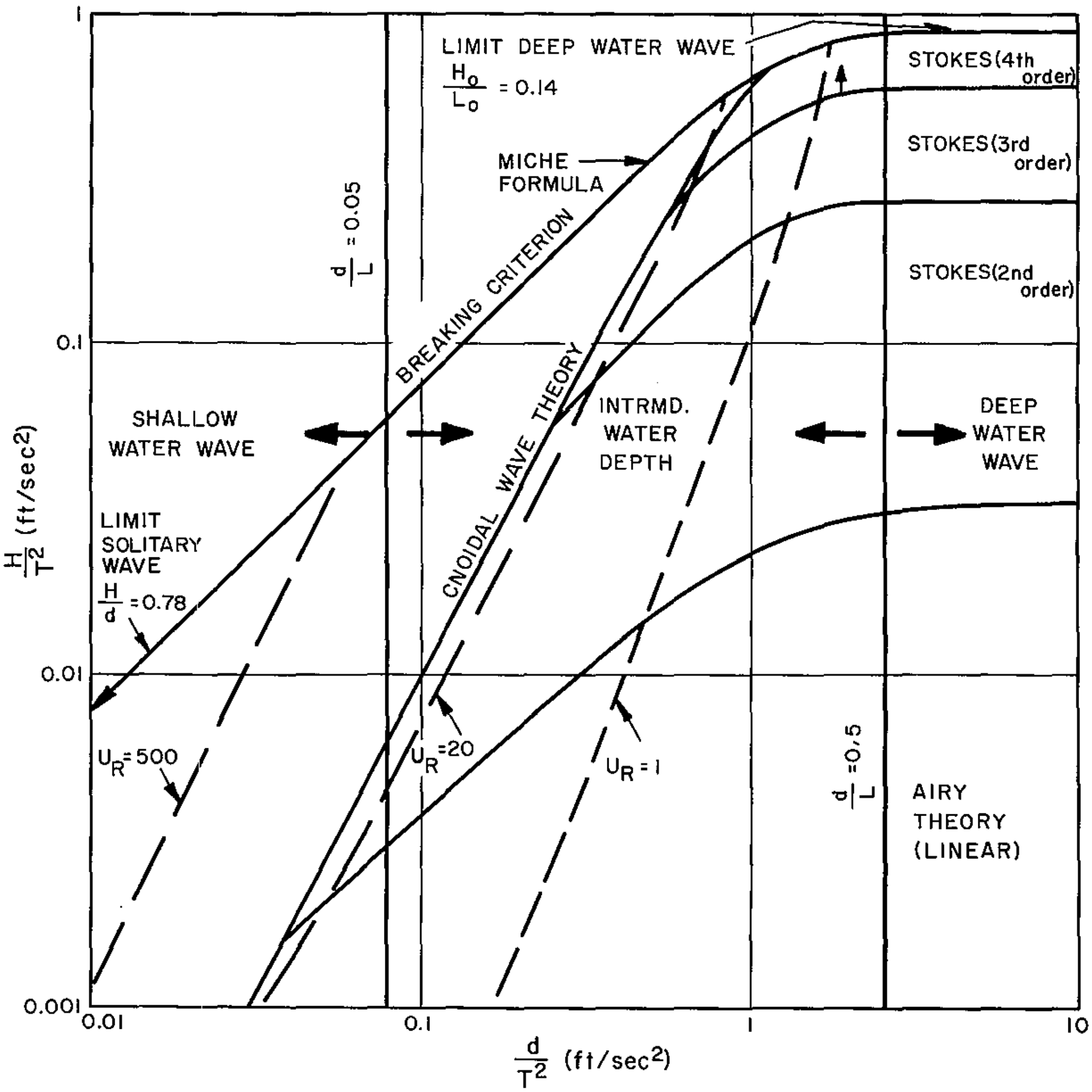


FIGURE XV-8

LIMITS OF VALIDITY FOR VARIOUS WAVE THEORIES

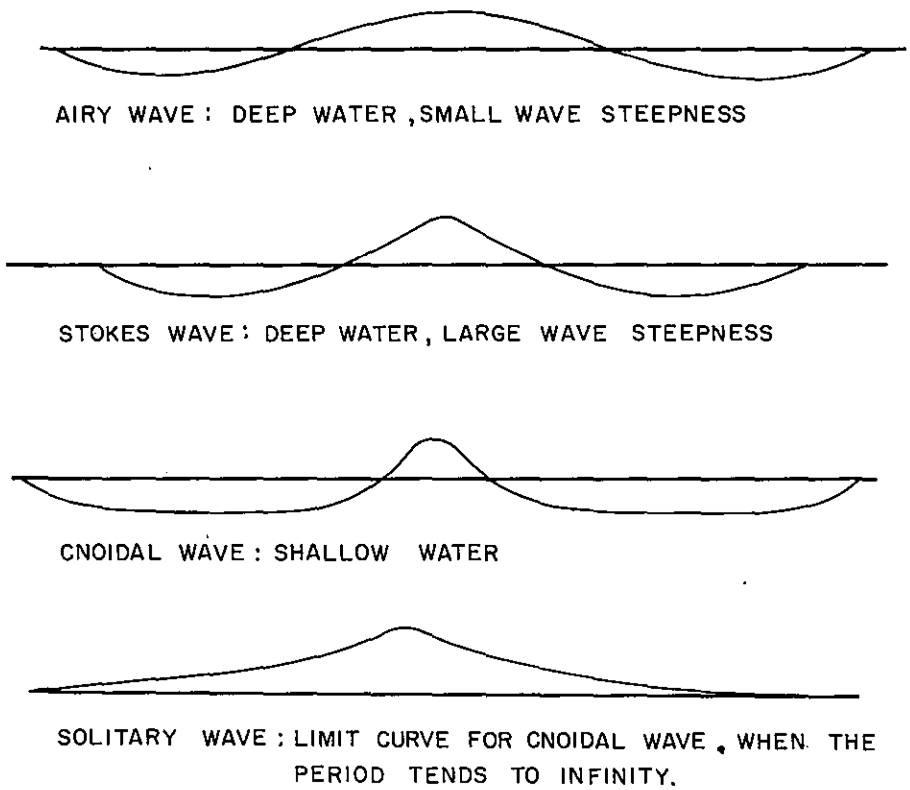


FIGURE XV-9
A PHYSICAL ILLUSTRATION OF VARIOUS WAVE PROFILES

classified in two great families. They are the "small amplitude wave theories" and the "long wave theories".

The small amplitude wave theories embrace the linearized theories and the first categories of power series, i.e., the power series in terms of H/L.

The long wave theories embrace the numerical method of solution mostly used for the non-linear long wave equations.

These two great families include a number of variations and some intermediate cases presenting some of the characteristics of both families. For example, the cnoidal wave, the solitary wave, and the monoclinal wave are considered as being particular cases (steady state profile) of the long wave theories, because they are non-linear shallow water waves.

It can be considered that there exists some arbitrariness in such classification. This arbitrariness is the heritage of the tradition, since the wave theories, as any theory, have been developed in a haphazard manner. But it is most important to understand the relative position of these theories with respect to each other, and their limits of validity. The small amplitude wave theories and the long wave theories are now considered separately.

XV-3 THE SMALL AMPLITUDE WAVE THEORY

XV-3.1 THE BASIC ASSUMPTION OF THE SMALL AMPLITUDE WAVE THEORY

It has been mentioned in the previous section that the small amplitude wave theory is essentially a linear theory, i.e., the non-linear convective inertia terms are considered as small. It is called the small amplitude wave theory because the theory is theoretically exact when the motion tends to zero even if the convective inertia terms are taken into account. Indeed, in that case the non-linear terms are infinitesimals of higher order than the linear terms.

This assumption is extremely convenient because the free surface elevation can a priori be considered as zero, i.e., the motion takes place within known boundaries. This assumption is used in order to determine the zero wave motion and such solution is assumed to be valid even if the wave motion is different from zero.

Aside of this assumption, the motion is also most often considered as irrotational. This assumption is compatible with the neglect of the quadratic convective term $\rho \vec{V} \times \overrightarrow{\text{curl}}\, \vec{V}$. Then the solution of the problem consists of determining the velocity potential function $\phi(x, y, z, t)$ satisfying the boundary conditions at the free surface and at the limit of the container.

This approach has been proven to be extremely successful even for wave motion of significant magnitude. Moreover, the assumption of linearity permits the determination of a complex motion by superposition of elementary wave motions.

XV-3.2 THE VARIOUS KINDS OF LINEAR SMALL AMPLITUDE WAVES

XV-3.2.1 Periodic Small Amplitude Wave Theory

Progressive periodic two-dimensional linear wave motion is the basic motion which leads to the understanding of many other more complex motions. Such a solution is found by assuming that the motion is of the form $A \sin \frac{2\pi}{L}(x - Ct)$ where C is a constant. This solution can be obtained in very deep water or in shallow water. In practice, when the relative depth d/L, i.e., the ratio of the water depth d to the wave length L, is larger than $1/2$, the deep water wave theory will apply. In intermediate water depth and in shallow water $(\frac{d}{L} < \frac{1}{2})$, the water depth has a significant influence on the wave motion. In shallow water $(\frac{d}{L} < \frac{1}{20})$, the linear small amplitude theory simplifies considerably: the vertical acceleration becomes negligible and the pressure is assumed to be hydrostatic, i.e., simply proportional to the distance from the free surface. It will be seen that the small amplitude wave theory becomes then a particular case (a limit case) of the long wave theory where the convective inertia is neglected.

Two periodic progressive waves of slightly different period traveling in the same direction form a succession of wave trains giving rise to a beat phenomenon (Figure XV-10). It has been seen that two progressive periodic waves of the same period and amplitude traveling in opposite directions form a standing wave or clapotis, and in the case where $\frac{d}{L} < \frac{1}{20}$, a seiche (Figure XV-11). A periodic wave reflected by a vertical wall at an angle forms a system of "short crested" waves which appear as a grid of peaks of water moving parallel to the wall.

A great number of three-dimensional periodic motions within complex boundaries can be determined by the small amplitude wave theory. They are the three-dimensional wave motions within tanks of various

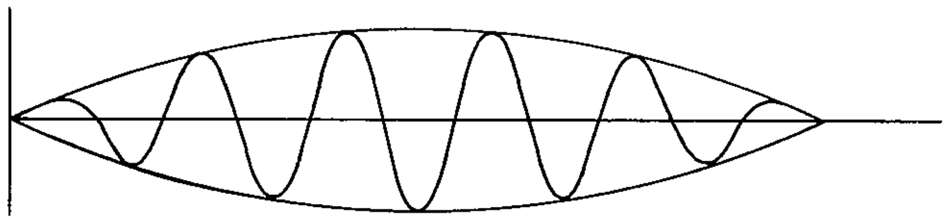


FIGURE XV-10

WAVE BEATING

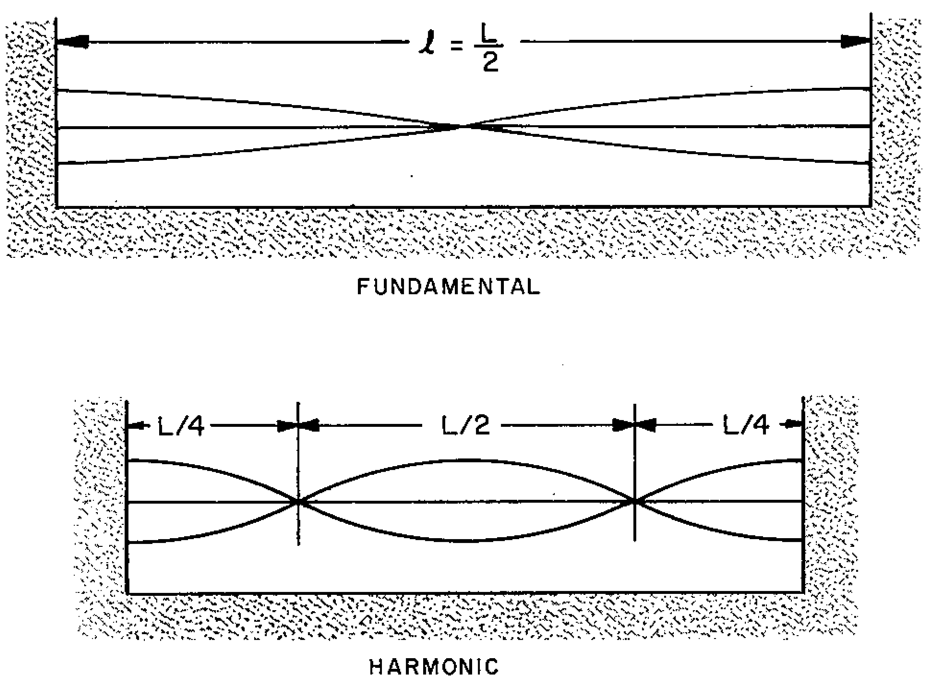


FIGURE XV-11

SEICHE MOTION IN A TWO-DIMENSIONAL BASIN shapes (rectangular, circular, etc.) with constant or varying depth. The process of wave diffraction by a vertical wall or through a breach is also thoroughly analyzed by the small amplitude wave theory.

Finally, this theory for progressive waves is essentially destined to be the foundation of the study of wind waves, although this phenomenon is random and not periodic. Hence the study of wind waves will require further analysis as described in Chapter XVIII on wave spectra (see also Section XV-5).

XV-3.2.2 Waves Created by a Local Disturbance

The small amplitude wave theory is also particularly successful in determining the wave motion created by a sudden disturbance or impulse at the free surface, or at the bottom (see Figure XV-12).

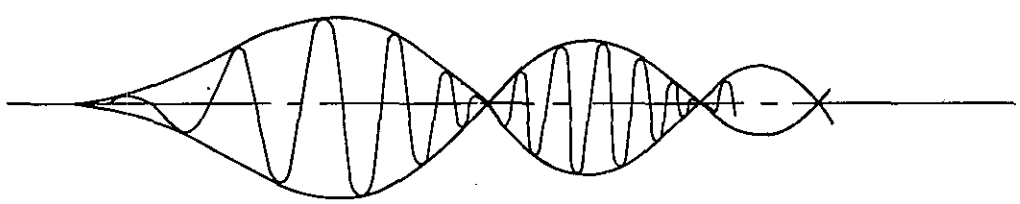


FIGURE XV-12
WAVE GENERATED BY A FREE SURFACE LOCAL DISTURBANCE

For example, tsunami waves generated by earthquakes can be treated in deep water by mathematical methods of the small amplitude wave theory. Likewise, waves generated by an underwater explosion or by the drop of a stone on the free surface of a body of water receive a similar theoretical treatment. In general, these waves have a cylindrical symmetry, but they are non-periodic. One may consider that they have a pseudo wave period defined by the time which elapses between two wave crests. This period decreases with time at a given location. Also, in general, this pseudo wave period tends to increase with the distance from the disturbance. Waves generated by disturbances often appear as a succession of wave trains, the number of waves within each wave train increasing as the distance from the disturbance increases.

The average wave height also tends to decrease with distance due to the double effect of increase of wave length with distance and radial dispersion.

XV-3.2.3 Ship Waves

Finally, the wave motion created by a moving disturbance (ship or atmospheric disturbances) can also be analyzed by application of the small amplitude wave theory. The theoretical wave patterns created by a moving ship are presented in Figure XV-13.

XV-3.3 THE NON-LINEAR SMALL AMPLITUDE WAVE THEORIES

XV-3.3.1 The Physical Aspects of Progressive Periodic Waves

The solution for a progressive harmonic linear wave over a

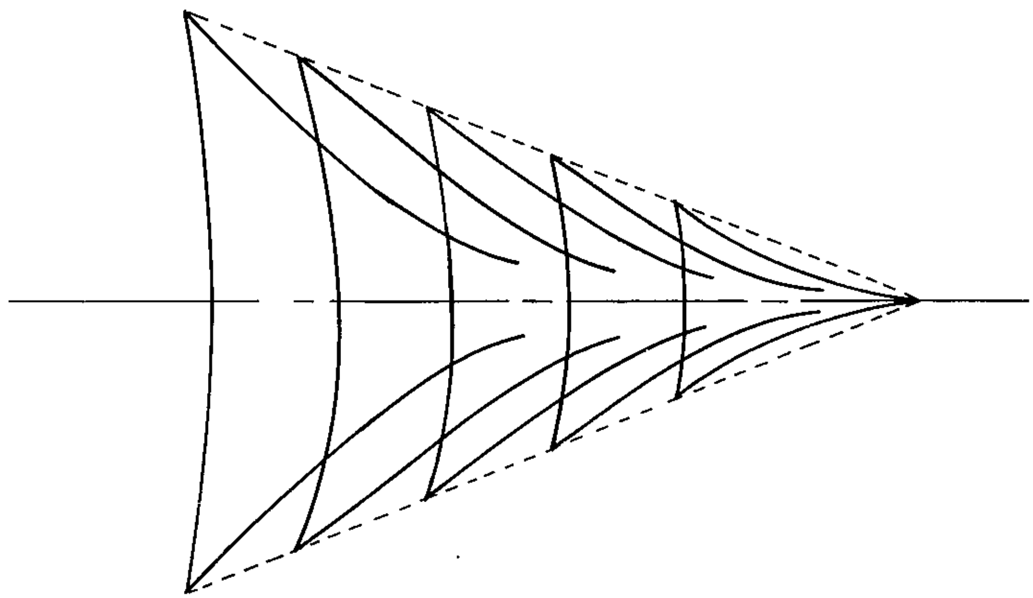


FIGURE XV-13
WAVE GENERATED BY A MOVING SHIP

horizontal bottom is a sine function of $(x - Ct)$, so the free surface is perfectly defined by a sine curve. In shallow water, the crest has a tendency to become steeper and the trough flatter, as shown in Figure XV-9b. Then the linear small amplitude wave theory is no longer valid.

XV-3.3.2 Waves Defined by a Power Series

In the simple case of periodic waves, progressive or standing, the small amplitude wave theory can be refined by taking into account the convective inertia forces to some extent. It has been indicated in a previous section (XV-2.2.2) that this is done by assuming the solution for the motion to be given by a power series in terms of a small quantity by comparison with the other dimensions. For example, in the simple case of a periodic progressive or standing two-dimensional wave, it is assumed that the solution for the motion is given as a power series in terms of the wave height H (or of the wave steepness H/L, defined as the ratio of the wave height to the wave length L). For example, the potential function $\phi(x, z, t)$ will be written:

$$\phi = H\phi_1 + H^2\phi_2 + H^3\phi_3 + H^4\phi_4$$

The first order term $H\phi_1$ is given by the linear small amplitude theory exactly, i.e., by neglecting the non-linear terms completely. The other terms are correction terms due to the non-linear convective inertia. These terms of the series are obtained successively by recurrence formulae. A third order wave theory, or theory at a third order of approximation, is a theory in which the calculation has been performed up to the corresponding power term, i.e., it includes $H\phi_1$, $H^2\phi_2$, $H^3\phi_3$. In the case of a harmonic wave, ϕ_2 and ϕ_3 are sinusoidal functions of $n(x - Ct)$ where n is an integer equal to the order of the considered term and the ϕ_n are functions of the relative depth d/L. In practice the complexity of the terms ϕ_2, ϕ_3, ... increases so much as the order of approximation increases that calculation can rarely be performed at an order of approximation higher than the fifth. The formulae for the fifth order of approximation are so complicated that for their application a set of tables obtained from a high speed computer is required.

In engineering practice, the first order wave theory is most often sufficient. However, higher order wave theory indicates some interesting trends for waves of large steepness (large H/L) in deep water. In very shallow water the convective inertia terms are relatively high and the convergence of the series becomes very slow. The series are not even necessarily uniformly valid and the function of relative depth d/L loses its meaning.

It has been mentioned that in shallow water the important parameter becomes H/d instead of H/L for deep water. A power series in terms of H/d is most convenient and would require fewer terms for a better accuracy. Such power series appear in the cnoidal and solitary wave theories which will be discussed in the following paragraphs (XV-4.4.1).

XV-3.3.3

Once all the equations of motion and all the boundary conditions have been specified, an infinite number of solutions may be found, or in a word, these equations are not sufficient for determining the wave motion.

Two other conditions are required. One is on rotationality and is considered in the following section. One should also specify whether the wave motion should be a progressive wave or a standing wave or a wave train. For example, in the first case a solution for steady state profile has to be found such that the solution appears as a function of $(x - Ct)$ where C is the constant wave velocity. It has to be pointed out that the solution for a standing wave is obtained by mere addition of two periodic gravity waves traveling in opposite direction in the linear case only. This addition will also be valid for the first term (linear term) of the power series. However, high order terms must be found independently by recurrence formulae, established for the specific type of motion (progressive or standing).

XV-3.4 ON THE EFFECT OF ROTATIONALITY ON WATER WAVES

XV-3.4.1

The other assumption for finding a unique solution to the equations of motion can be on the distribution of rotationality over a vertical. Of course, the simpler solution consists of assuming the motion to be irrotational. Most of the small amplitude wave theory is based on this assumption. The solution of the problem then consists of finding the potential function for the wave motion. In the case of a Stokes wave, i.e., the case of a periodic irrotational two-dimensional wave motion, if the calculations are carried out to a higher order of approximation than the linear theory, a mass transportation in the direction of wave travel is found, i.e., the particle path instead of being a closed orbit has a forward motion as shown in Figure XV-14. The mass transportation is a function of the relative depth and wave steepness.

XV-3.4.2

One may wonder whether this fact is a mathematical occurrence of no physical significance. And this leads us to discuss the limit of validity of the principle of irrotationality for water waves. A deep water swell, i.e., wave generated by wind traveling out of the generating area, is probably the motion which most closely approaches the condition of irrotationality. But under wind action the free surface shearing stress induces rotationality (and turbulence) in the direction of the wave travel

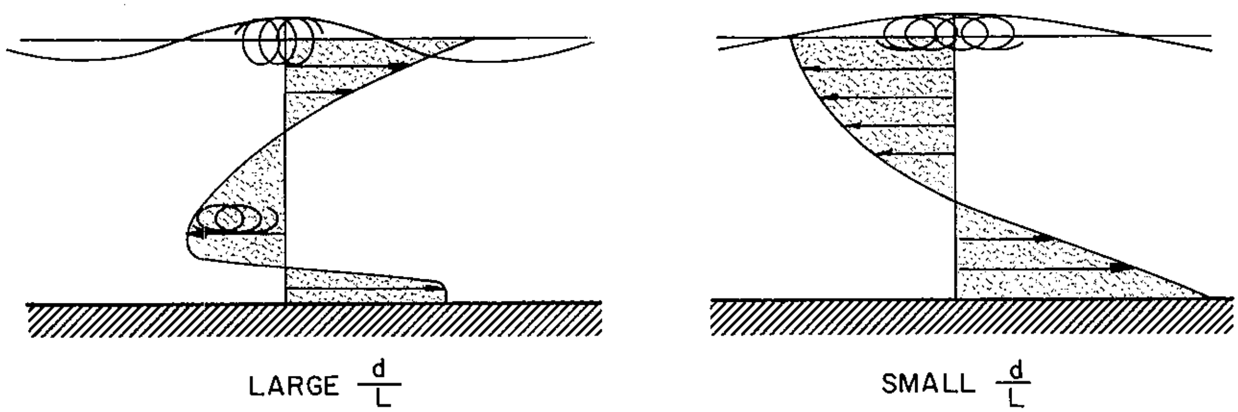


FIGURE XV-14
PERIODIC PROGRESSIVE WAVE WITH MASS TRANSPORTATION EFFECT

and increases the mass transportation in the wave travel direction near the free surface. Also, in shallow water, the bottom friction induces rotationality. Moreover, a mass transportation is not always compatible with the principle of continuity near a shore. The water piling up should have a return flow uniformly distributed or located by instability at some specific location (they are the rip currents).

For this reason, one may superpose a current in such a way that the integral of mass transportation over a vertical is nil (see Figure XV-15). In this case a rotational wave theory may be found, the distribution of rotationality being a function of the water depth.

If one assumes that not only the integral of mass transportation is zero, but the mass transportation is nil locally, a "closed orbit" theory is found. Such families of theory are also rotational. Unfortunately the

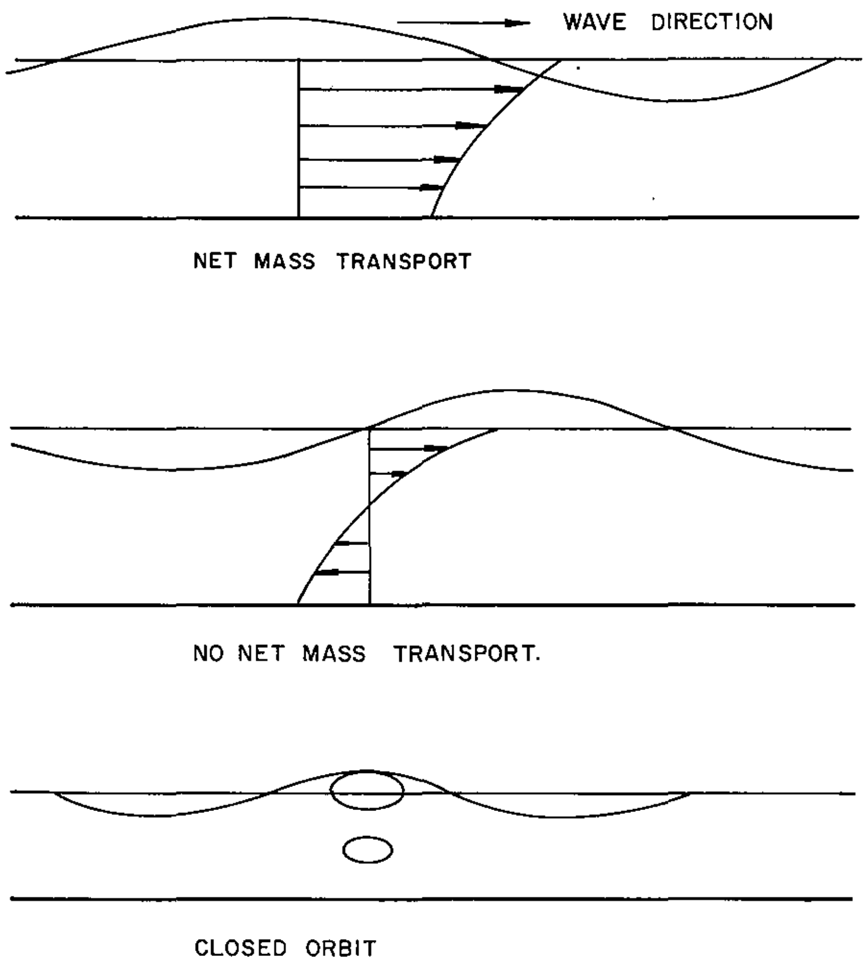


FIGURE XV-15
DIFFERENT KINDS OF ASSUMPTIONS ON MASS
TRANSPORTATION IN PERIODIC PROGRESSIVE WAVES

rotationality is in the opposite direction to what will be expected under the influence of a free surface shearing stress due to a generating wind.

The most famous closed orbit theory is the Gerstner theory, valid for deep water only, which also happens to be an exact theory, i.e., all the convective inertia terms are taken into account exactly. In shallow water, the closed orbit wave theory can only be expressed approximately as a power series in terms of the wave height H.

XV-3.4.2

Although it is not the purpose of this book to deeply analyze the phenomenon of wave breaking, the relationship between the rate of rotationality and the limit wave steepness is worthwhile mentioning.

It has been mentioned (Section XV-1.2.5) that wave breaking inception will occur when the wave profile reaches a limit wave steepness $\left.\frac{H}{L}\right|_{max}$. This limit steepness is theoretically 0.142 for a deep water irrotational periodic wave. Rotationality at the crest in the direction of the wave travel such as that due to a generating wind will reduce the limit wave steepness to a smaller value (see Figure XV-16). A deep sea wave steepness larger than 0.10 is rarely encountered.

Rotationality in the opposite direction will theoretically increase the limit wave steepness. Such a case can be observed near the coasts when the wave travels in the opposite direction to a wind blowing offshore. At the limit, according to the closed orbit Gertsner theory, the maximum limit steepness is 0.31, but the rotationality at the crest is then infinity and in the opposite direction to the wave travel. It is evident that this result of the Gertsner theory has no physical significance.

It is seen how important it would be to establish a general rotational wave theory and to relate the rotationality and mass transportation to the wind action and bottom friction. The effect of viscous friction at the bottom has already been subjected to investigation to some

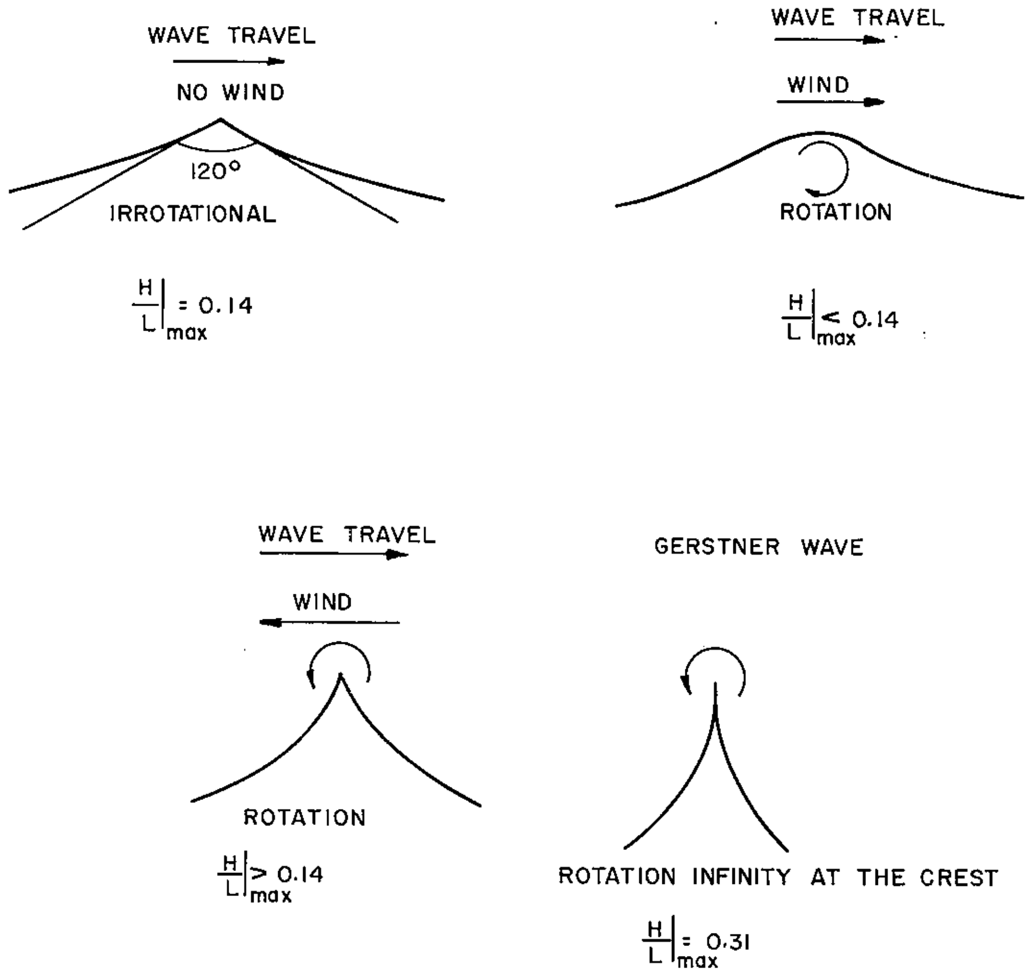


FIGURE XV-16
ROTATIONALITY AND WAVE LIMIT STEEPNESS

extent. However, a general theory of periodic waves with an arbitrary rotationality or mass transportation, since they are related, valid for any wave height, any wave period, and any water depth, remains to be done. One has to be content with existing theory valid over a limited range of the Ursell parameter.

XV-4 THE LONG WAVE THEORY

XV-4.1 THE BASIC ASSUMPTIONS OF THE LONG WAVE THEORY

The long wave theory applies when the relative depth is very small. In this case the vertical acceleration is then neglected and the path curvature is small. Consequently, the vertical component of the motion does not influence the pressure distribution, which is assumed to be hydrostatic, i.e., the pressure at a given point is assumed to be equal to the product of the specific weight of the water ρg by the distance from the free surface. However, contrary to the small amplitude wave theory, the free surface is now unknown even during the first step of the calculations. Also, the velocity distribution along a vertical is assumed to be uniform. In fact, the particle velocity should rather be considered as the average value over a vertical. As in the case of the generalized Bernoulli equation, a correction coefficient close to unity should be included where quadratic terms appear. This refinement is most often neglected.

So the only unknowns to be determined remain the free surface elevation η and the horizontal particle velocity $\vec{V}(u, v)$. As such, the small amplitude wave theory in shallow water can be considered as the limit case of the long wave theory, which also leads to these results as will be seen in detail in the following chapters (see Section XVIII-4). However, the convective terms due to the horizontal components of motion such as $\rho u \frac{\partial u}{\partial x}$ are taken into account in the non-linear long wave theory. Because the equation becomes non-linear, the number of analytical solutions is limited to a very small number of particular cases.

XV-4.2 THE PRINCIPLE OF NUMERICAL METHODS OF CALCULATION

While the small amplitude wave theory consists of finding a potential function by analytical means, the long wave theory has most often to be treated by numerical methods, graphical methods and by making use of a high speed computer.

For this purpose the first task generally consists of transforming the set of differential equations (continuity and momentum) into a finite difference scheme. Then the calculation consists of proceeding step by step, i.e., the calculation of $\eta(x, t)$ and $u(x, t)$ at a given time t_1 and at a given location x_1 is calculated from the knowledge of their values at a small interval away.

For example, consider the simple linearized long wave equation which is demonstrated in Section XVIII-4:

$$\frac{\partial u}{\partial t} = -g \frac{\partial \eta}{\partial x}$$

Consider the points 1, 2, 3, 4 in a t-x diagram separated by intervals Δx and Δt respectively (see Figure XV-17). Knowing the value of η at points 1, 2, 3, the value of u at point 4 is obtained from the equation:

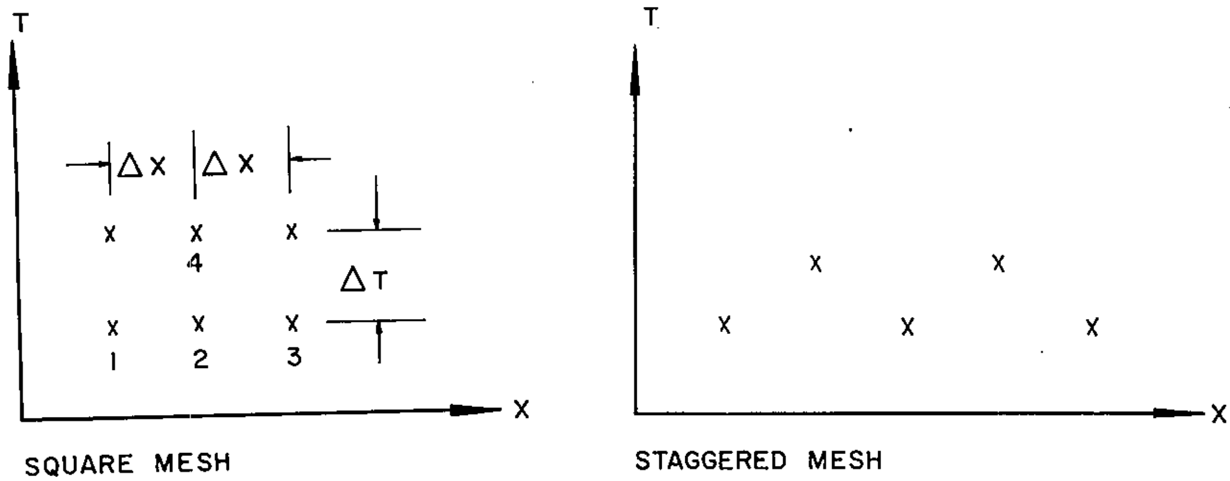


FIGURE XV-17
TWO KINDS OF FINITE DIFFERENCE SCHEMES

$$\frac{\Delta u}{\Delta t} = - g \frac{\Delta \eta}{\Delta x}$$

expressed as

$$\frac{u_4 - u_2}{\Delta t} = - g \frac{\eta_3 - \eta_1}{2 \Delta x}$$

Then one proceeds step by step calculating successively u and η for the entire diagram. The history of the wave profile is so obtained as a function of time and distance.

XV-4.3 THE VERSATILITY AND LIMITS OF VALIDITY OF NUMERICAL METHODS OF CALCULATION

XV-4.3.1

It is easily seen that the great advantage of the long wave theory treated by numerical analysis is its versatility. Numerical methods are particularly convenient over complex boundary conditions, while the search for analytical solutions is beyond the scope of the best analysts. For example, the long wave theory may be applied in a river with variable cross section. Also, terms for bottom friction, wind stress on the free surface, gradients due to complex pressure distribution can easily be taken into account. The term for bottom friction is particularly important for flood waves and tidal waves. Wind stress and free surface pressure gradients must be taken into account in the study of "storm surge". (A storm surge is the rise of water level at the shoreline by atmospheric action pushing the water toward the shoreline. Such a phenomenon also obeys the long wave theory although the inertial forces may often be neglected. It is then treated as a "quasi-static" problem.)

XV-4.3.2 Error and Limit of Validity of Numerical Procedures

The transformation of a differential equation into finite difference involves a systematic error. Indeed, it is known that by developing a differential term into finite difference by a Taylor expansion yields:

$$\frac{\partial F}{\partial x} = \frac{\Delta F}{\Delta x} - \frac{1}{2} f''(x) \ldots . - \frac{\Delta x^{n-1}}{n!} f^{(n)}(x) \ldots .$$

The first task of any numerical computation is to insure that by taking simply $\frac{\partial F}{\partial x} = \frac{\Delta F}{\Delta x}$ the cumulative error due to the neglect of high order terms does not exceed the desired accuracy.

Such a study involves the search for "stability criteria", which results in a relationship between intervals (generally space Δx and time Δt). In the case of high order derivative terms, stability criteria may not exist. The high order derivative terms have then to be replaced by a first order derivative term of another variable, so the number of unknowns increases. For example, $\frac{\partial^2 \eta}{\partial t^2}$ may be replaced by $\frac{\partial a}{\partial t}$, a being equal to $\frac{\partial \eta}{\partial t}$. Also, the choice of the interval is conditioned by the cumulative error, cost of the computing time, and the "round-off error". The round-off error is due to the fact that any numerical calculus is necessarily done with a limited number of figures or "digits". For example, most calculation done on computers is done with eight digits, sometimes 16 digits if one used "double precision", or even more. But the increasing cost of computing time offsets this advantage.

In brief, the application of the long wave theory always involves an inherent error, which increases with time and/or distance.

The study of the propagation of a bore, or an undulated wave, or the wave created by the breaking of a dam over a long distance is unreliable even if the stability criteria is satisfied because of the cumulative effect of the error. However, the study of a tidal wave in an estuary, or even of a flood wave with gentle variation of depth, is possible. Similarly, the propagation of a breaking wave (bore) over a steep beach, i.e., over a short distance because of the steep slope, may give reliable results.

XV-4.3.3 The Long Wave Paradox

Another error, inherent in the simplifying assumptions, is also encountered systematically in the treatment of the long wave theory. Since the velocity of the "wave element" is an increasing function of the water depth such as $\sqrt{gh}$, the wave elements carrying the most energy have a tendency to catch up with the first wave elements ahead of the wave (see Figure XV-18). A vertical wall of water soon results, forming a tidal bore. This phenomenon actually may occur physically but even if it occurs, it will happen much later than predicted by the long wave theory. In particular, in the case of a wave which contains high space derivatives for $\frac{\partial \eta}{\partial x}$ and $\frac{\partial u}{\partial x}$, the long wave theory may no longer be valid. Similarly, the breaking of a long wave on a beach will be predicted sooner than if it were due to the change of bottom depth only.

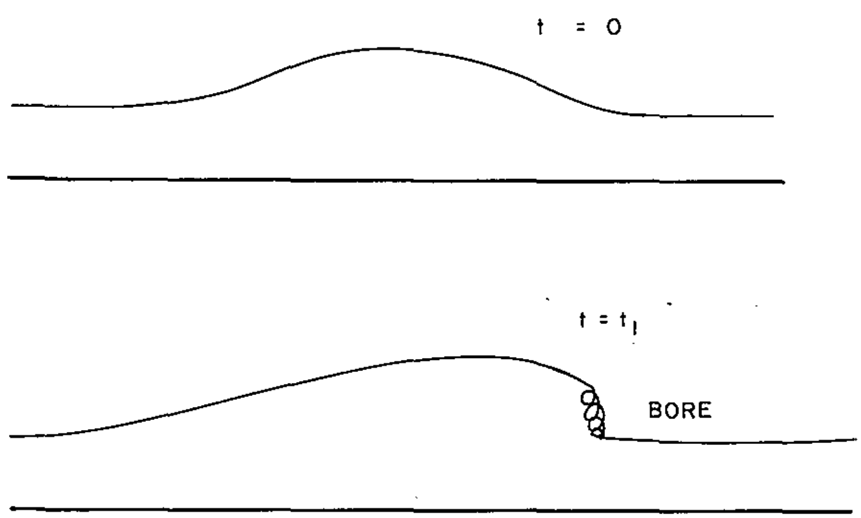


FIGURE XV-18
A PHYSICAL ILLUSTRATION OF THE LONG WAVE PARADOX

Finally, it is realized that the long wave theory and steady state profile are two concepts theoretically incompatible although steady state profiles have been observed. This inherent deficiency in the long wave theory is the long wave paradox and is also encountered in gas dynamics and non-linear acoustics.

The two stabilizing factors which explain the existence of steady state profile are the vertical acceleration and the bottom friction. They are now considered successively.

XV-4.4 STEADY STATE PROFILE

XV-4.4.1 The Solitary Wave and Cnoidal Wave Theory

If one takes vertical acceleration into account, the pressure distribution is no longer hydrostatic. In particular, due to the centrifugal force of water particles under a wave crest, the pressure at the bottom decreases by a significant order of magnitude.

Although non-negligible, the vertical acceleration can be linearized such as $\frac{dw}{dt} \cong \frac{\partial w}{\partial t}$ because the vertical component of the motion w being small, the convective terms $w \frac{\partial w}{\partial x}$ and $u \frac{\partial w}{\partial x}$ remain small. If this correction effect is taken into account in the long wave theory, the motion becomes non-linear horizontally and linear vertically. Even if the non-linear vertical components are taken into account in a solution obtained as a power series in terms of H/d, they have to be introduced in high order terms only for the sake of consistency in the approximations. If one assumes that the solution of the equation of long waves with a correction term for flow curvature is that of a steady state, this solution has to be defined as a function of $(x - Ct)$. Such kinds of solutions do exist. They are the solutions for solitary waves and cnoidal waves (although in the latter case the pressure is found to be hydrostatic at a first order of approximation).

It has been seen that in the case of very long waves the Ursell parameter loses its clear-cut significance to some extent. It is recalled that the concept of wave length loses its meaning since the "wave length" of a solitary wave is infinity. But the flow curvature under the crest is that of a cnoidal wave for which a finite wave length can be defined.

It is seen now why a significant wave length where the flow curvature action is significant should rather be considered. In this case it has been mentioned that the value of the Ursell parameter characterising cnoidal and solitary waves is around unity. (See Section XV-2.3.1.)

In the case of very long waves where the vertical acceleration and path curvature are effectively negligible, the choice of the significant length is very arbitrary. It is sufficient to notice that this kind of motion corresponds to a value of the Ursell parameter much larger than unity.

XV-4.4.2 Monoclinal Flood Wave

A quadratic, or more generally a non-linear, bottom friction has also a stabilizing effect which may balance the horizontal components of the convective inertia. Indeed, a quadratic shearing stress will tend to slow down most of the wave elements having the highest particle velocity, i.e., carrying the most energy.

Under certain conditions inherent to the characteristics of the bottom slope, the friction factor, and the water depths ahead and after the transient wave, it may happen that a steady state translatory wave also exists. It is the monoclinal flood wave which is an exact solution of the long wave equation with bottom friction.

XV-5 WAVE MOTION AS A RANDOM PROCESS

XV-5.1 A NEW APPROACH TO WATER WAVES

A brief introduction to the hydrodynamics and mathematics of periodic and other analytically defined waves in a heavy fluid bounded by a free surface has been presented in earlier sections. The developments along these lines have been continued for well over a century.

Any observation of a real sea reveals a continuously changing random pattern of bumps and hollows. Waves of different lengths travel at different speeds combining and recombining in constantly changing patterns even if they are all unidirectional. When multidirectional waves are present the patterns created are even more complex. It was thought for a long time, even by such men as Rayleigh and Stokes, that this apparently chaotic process was beyond adequate mathematical description. The best that could be done appeared to be the choice of a mean wave height and a mean wave length and then the classical wave theories were applied. To the mathematician the classical wave theories themselves were fascinating and the apparent difficulties of dealing with a real sea did not concern him.

In comparatively recent times an approach to the understanding of real seas has been developed. This approach is based on the combination of statistics, Fourier analysis and hydrodynamics. The theories of statistics have to be used to determine stable parameters for describing a random sea state; then Fourier analysis is used to break down the random process into harmonic components whose behavior can be analyzed by using classical hydrodynamic theories of wave motions.

Recent progress owes a great deal to the development of research on the statistical analysis of random noise by communications engineers. The study of sea waves has developed as a combination of time series analysis and statistical geometry governed by the laws of hydrodynamics. This section presents an introduction to this non-deterministic aspect of waves, which will be further developed in Chapter XVIII. Probability concepts and spectrum theory will be discussed.

XV-5.2 SOME DEFINITIONS AND STABILITY PARAMETERS

A part of a typical wave record is shown in Figure XV-19. The concepts of surface ordinates, "wave height", "wave periods", maxima and minima, and envelopes are illustrated. Some of the statistics of the various parameters which can be obtained from such a wave record as Figure XV-20 will be discussed. Clearly, if the wave record is very long it is impractical to keep it in its original form. Methods of condensing the gross details of the wave record are required whereas inevitably, after such a process, much detail will be lost.

The condensations of the real sea state need to have the property of stability. The term stability is used to describe a characteristic which does not change too much if the observation is repeated. For example,

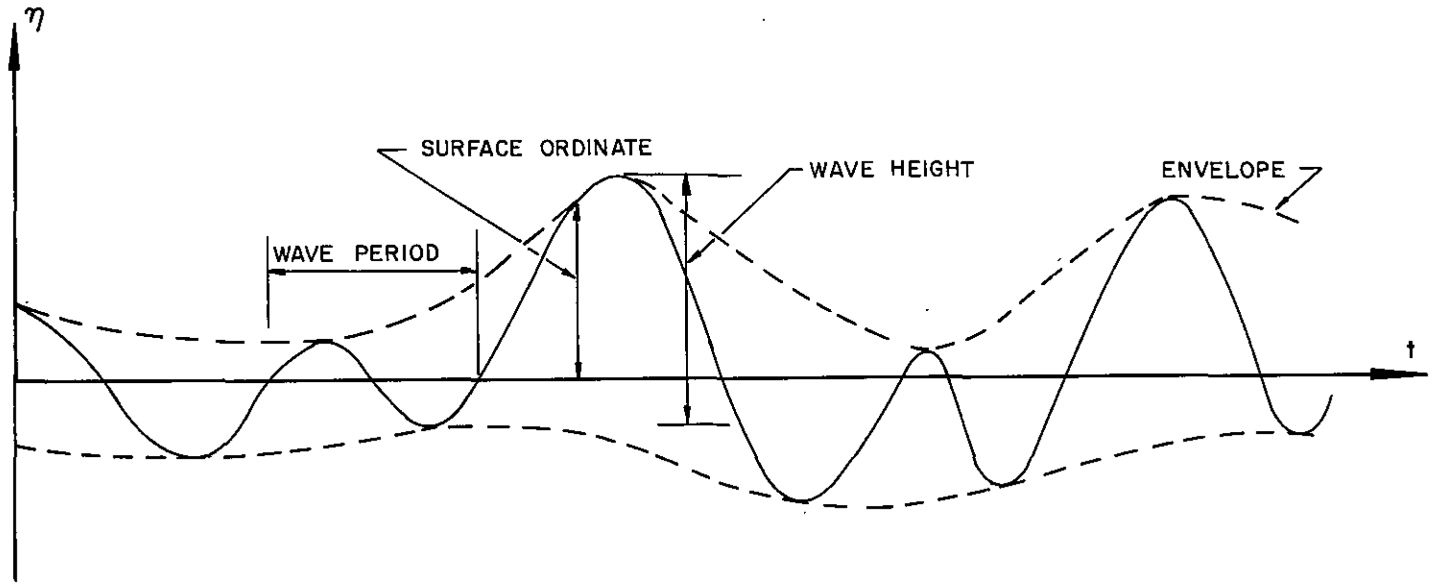


FIGURE XV-19
A TYPICAL WAVE RECORD AT SEA

suppose two wave recorders were placed in the open sea at a distance of, say, 200 feet apart. The water surface-time history of the two records taken at the same time would be completely different. The sea surface records themselves are unstable. On the other hand, such things as the average wave height, mean square water surface fluctuation, etc. of the two records would be very close to being equal (so long as the records were of reasonable length). Such statistical properties are said to be stable.

Probability densities and distributions of sea surface parameters, together with the spectral (or variance-frequency) distribution of the sea surface, have been found to be concise and useful properties of this process. The spectrum is a form of probability distribution and has very desirable stability characteristics. The spectrum retains much information on wave amplitudes and "periods" but loses all information on phase position. Probability distributions, on the other hand, lose all information on wave periods if "wave height" probabilities are computed or vice versa.

The description of sea states in terms of spectra has already been used extensively by naval architects to predict ship performance. Spectral and other probability concepts are used by mechanical and aeronautical engineers to handle vibration problems. Many branches of geophysics, such as turbulence, seismology, tide analysis, etc., use the basic communications theory to describe the processes and analyze data.

A summary of the most important probability distributions and an exposition of spectral representation of waves as random processes will be analyzed in Chapter XVIII.

XV-6 A SYNTHESIS OF WATER WAVE THEORIES

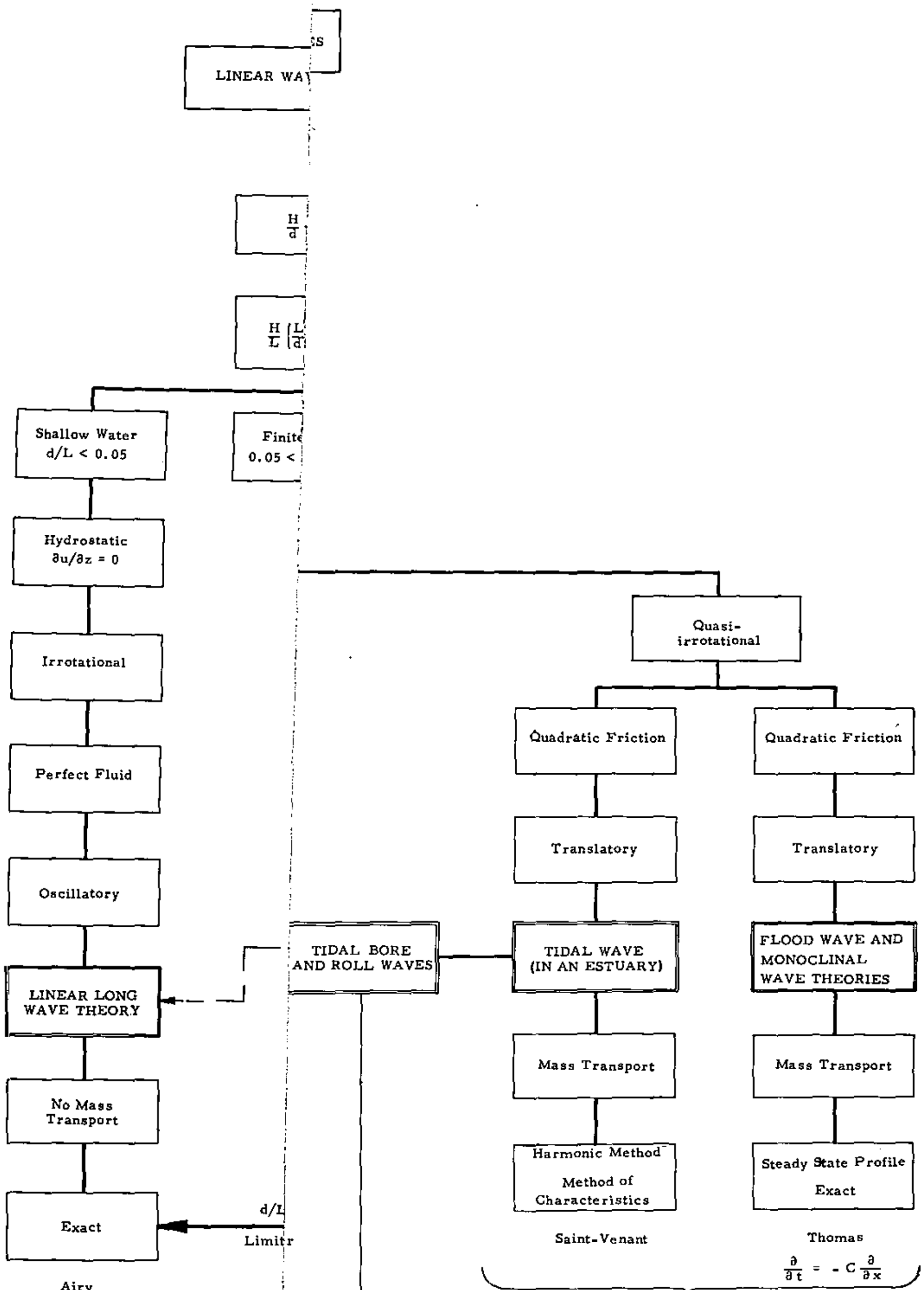
XV-6.1 A FLOW CHART FOR THE GREAT FAMILY OF WATER WAVES

The following flow chart summarizes the previous considerations. It describes the main characteristics of the great family of water waves. The two categories of motion are the linear and non-linear motion, depending upon whether the convective inertia is taken into account or not. Each of these motions may also be subdivided into motion where the pressure is assumed to be hydrostatic or motion with non-negligible flow curvature. Finally, the motion may or may not be irrotational and the bottom friction may or may not be taken into account.

Due to the limitation of mathematical methods, the most complex cases cannot be analyzed. For example, a theory for non-linear rotational waves in shallow water with non-negligible vertical acceleration and bottom friction does not exist.

Although the equations for these motions have still not been studied, they have been presented in this flow chart. These equations have been included with the idea that this flow chart can be used as a guide throughout the following chapters.

All the theories presented in the flow chart will not be presented in this book. It has been judged useful to quote the Gertsner wave theory and the cnoidal wave theory although they will not be further studied.



$$\frac{\partial}{\partial t} = -C\frac{\partial}{\partial x}$$

Airy

$$p = \rho g(\eta + z)$$

$$\frac{\partial \eta}{\partial x} = -\frac{1}{g}\frac{\partial u}{\partial t}$$

$$gh\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$$

$$\frac{\partial h}{\partial t} + \frac{\partial (hV)}{\partial x} + \frac{hV}{\ell}\frac{\partial \ell}{\partial x} = 0$$

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial h}{\partial x} = -gS - \frac{g}{C_h^2}\frac{V|V|}{h}$$

$$_2 h_2 = W(h_2 - h_1) + V_1 h_1$$

$$\frac{g}{}\left(h_2^2 - h_1^2\right) = \rho h_1 (V_2 - V_1)(W - V_1)$$

XV-6.2 A PLAN FOR THE STUDY OF WATER WAVES

The plan in this book for the study of water waves follows tradition, i.e., the two great families of water waves are separated under the titles "small amplitude wave theory" and "long wave theory".

The following chapter (XVI) is entirely devoted to the small amplitude wave theory. The case of irrotational frictionless harmonic motion only is considered. Two-dimensional and three-dimensional motions are studied.

Chapter XVII is devoted to the long wave theory and river flow. The calculation of backwater curves has been included in this chapter as a particular case of free surface flow. The theories of solitary waves and monoclinal waves are also presented. Consequently, this chapter is essentially devoted to the problems encountered in rivers and channels.

Chapter XVIII is devoted to the concept of wave spectrum as an application of linear theory to phenomena of random characteristics.

XV-1 Consider a straight shoreline with parallel bottom contours and a periodic wave arriving at an angle with the deep water bottom contours. Demonstrate that at any contour one has

$$\frac{\sin \alpha}{C} = \frac{\sin \alpha_o}{C_o} = \text{constant}$$

by application of the law of wave refraction. C is the water wave velocity, α is the angle of wave crest with the parallel bottom contours, and subscript o refers to infinite depth.

CHAPTER XVI

PERIODIC SMALL AMPLITUDE WAVE THEORIES

XVI-1 BASIC EQUATIONS AND FORMULATION OF A SURFACE WAVE PROBLEM

XVI-1.1 NOTATION AND CONTINUITY

The motion is defined with respect to a three axes system of Cartesian coordinates OX, OY, OZ. The axis OZ is vertical and positive upwards. The depth is assumed to be a constant and is defined by $z = -d$ (Figure XVI-1). Any point is defined by its three coordinates x, y and z.

The viscosity forces are neglected and the motion is assumed to be irrotational and the fluid is incompressible.

$$\overrightarrow{\text{curl}}\ \vec{V} = 0 \quad \text{or} \quad \zeta = \eta = \xi = 0$$

$$\text{div}\ \vec{V} = 0 \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

These assumptions result in a number of simplifications.

$\overrightarrow{\text{curl}}\ \vec{V} = 0$ insures the existence of a single valued velocity potential function $\phi(x, y, z, t)$ from which the velocity field

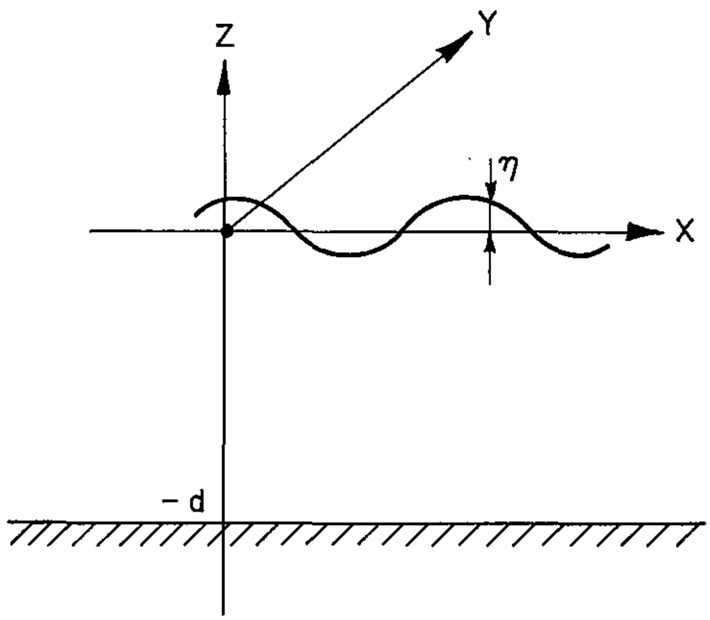


FIGURE XVI-1
NOTATION

can be derived. It has been seen (see Section II-5.3) that the potential function can arbitrarily be defined as $\vec{V} = \overrightarrow{\text{grad}}\, \phi$ or $\vec{V} = -\overrightarrow{\text{grad}}\, \phi$. The latter definition is used in the present chapter, i.e., $u = -\frac{\partial \phi}{\partial x}$, $v = -\frac{\partial \phi}{\partial y}$, $w = -\frac{\partial \phi}{\partial z}$ (see XI-3). The velocity potential function has to be found from the continuity equation, the momentum equation and the boundary conditions.

The continuity equation $\text{div}\, \vec{V} = 0$ is expressed in terms of ϕ by the well-known Laplace equation $\nabla^2 \phi = 0$, that is in Cartesian coordinates (see III-3.4)

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

XVI-1.2 THE MOMENTUM EQUATION

The momentum equation for an irrotational flow is given by the following form of the Bernoulli equation (see X-2.4) which corresponds to the new definition of ϕ

$$-\underbrace{\frac{\partial \phi}{\partial t}}_{\substack{\text{local}\\\text{inertia}\\\text{term}}} + \underbrace{\frac{1}{2} V^2}_{\substack{\text{convective}\\\text{inertia}\\\text{term}}} + \underbrace{\frac{p}{\rho}}_{\substack{\text{pres-}\\\text{sure}\\\text{term}}} + \underbrace{gz}_{\substack{\text{gravity}\\\text{term}}} = f(t)$$

in which $f(t)$ may depend on t but not on the space variables. The fact that the flow is assumed to be irrotational means that the Bernoulli law is valid for any point of the fluid and not only along streamlines.

This equation is non-linear because of the convective inertia term which may be expressed as a function of the potential function ϕ as

$$\frac{1}{2} V^2 = \frac{1}{2} \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 \right]$$

Hence, wave motion should be defined by non-linear laws. However, in the case of very slow motion, this term is neglected and the Bernoulli equation is written

$$-\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + gz = f(t)$$

Periodic gravity wave theories often satisfy the condition for slow motion with a fairly good degree of accuracy. The corresponding solutions are mathematically exact when the motion tends to be infinitely small.

XVI-1.3 FIXED BOUNDARY CONDITION

At a fixed boundary, the fluid velocity is tangential to the boundary, that is, the normal component V_n is zero. In terms of velocity potential ϕ, this condition is written $\frac{\partial \phi}{\partial n} = 0$. In particular on a horizontal bottom

$$w\Big|_{z=-d} = -\frac{\partial \phi}{\partial z}\Big|_{z=-d} = 0.$$

XVI-1.4 FREE SURFACE EQUATIONS

One of the difficulties encountered in determining the nature of wave motion is due to the fact that one of the boundaries - the free surface - is unknown, except in the case of infintely small motion in which the free surface is, at the beginning, assumed to be a horizontal line. However, in the general case, the form of the free surface is unknown as well as the potential function ϕ. Hence, another unknown $z = \eta$ appears in wave problems. If one assumes that the free surface in the most general case of a three dimensional motion is given by the equation

$$z = \eta\ (x, y, t)$$

the variation of z with respect to time t is

$$\frac{dz}{dt} = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x}\frac{dx}{dt} + \frac{\partial \eta}{\partial y}\frac{dy}{dt}$$

Introducing the values

$$\frac{dx}{dt} = u = -\frac{\partial \phi}{\partial x}, \quad \frac{dy}{dt} = v = -\frac{\partial \phi}{\partial y},$$

$$\frac{dz}{dt} = w\Big|_{z=\eta} = -\frac{\partial \phi}{\partial z}\Big|_{z=\eta}$$

the free surface equation becomes

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=\eta} = -\frac{\partial \eta}{\partial t} + \left.\frac{\partial \phi}{\partial x}\right|_{z=\eta}\frac{\partial \eta}{\partial x} + \left.\frac{\partial \phi}{\partial y}\right|_{z=\eta}\frac{\partial \eta}{\partial y}$$

This equation is non-linear and is called the kinematic equation.

Another equation - the dynamic equation - is given by the Bernoulli equation in which the pressure p is considered as constant (and equal to atmospheric pressure). Hence the free surface dynamic condition becomes

$$-\frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right]$$
$$+ g\eta = f(t)$$

Thus, generally ϕ and η appear to be given b the solution of $\nabla^2 \phi = 0$ with two simultaneous non-linear boundary conditions at the free surface and a linear boundary condition at the bottom

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=-d} = 0$$

XVI-1.5 THE FREE SURFACE CONDITION IN THE CASE OF VERY SLOW MOTION

In the case of slow motion the Bernoulli equation

$$-\frac{\partial \phi}{\partial t} + \frac{p}{\rho} + gz = f(t)$$

may become at the free surface $\left.-\frac{\partial \phi}{\partial t}\right|_{z=\eta} + g\eta = 0$

that is

$$\eta = \frac{1}{g} \left. \frac{\partial \phi}{\partial t} \right|_{z = \eta}$$

provided the function f(t) and any additive constant can be included in the value of $\frac{\partial \phi}{\partial t}$.

Since the motion is assumed to be infinitely small, η may be written $\frac{1}{g} \left. \frac{\partial \phi}{\partial t} \right|_{z = 0}$ where $\eta \cong 0$. This approximation leads to an error of the order already neglected in the convective inertia term.

Now consider the kinematic condition. $\frac{\partial \eta}{\partial x}$ and $\frac{\partial \eta}{\partial y}$ are the components of the slope of the free surface and they are small in the case of slow motion (see Figure XVI-2).

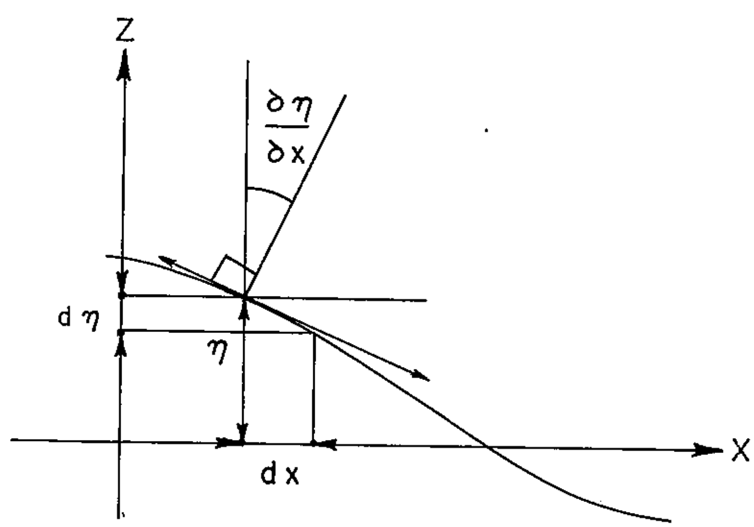


FIGURE XVI-2
NOTATION

Hence the non-linear terms $\frac{\partial \phi}{\partial x} \frac{\partial \eta}{\partial x}$ and $\frac{\partial \phi}{\partial y} \frac{\partial \eta}{\partial y}$ may be neglected. Now writing that the normal component of the fluid velocity at the free surface is equal to the normal velocity of the surface itself, gives with sufficient approximation

$$\frac{\partial \eta}{\partial t} = - \frac{\partial \phi}{\partial z} \Big|_{z = 0}$$

η may now be easily eliminated from the dynamic and kinematic conditions. The derivative of η with respect to t in the kinematic condition gives

$$\frac{\partial \eta}{\partial t} = \frac{1}{g} \frac{\partial^2 \phi}{\partial t^2} \Bigg|_{z = 0}$$

and eliminating $\frac{\partial \eta}{\partial t}$ by inserting the previous equation yields:

$$\left[\frac{\partial \phi}{\partial z} + \frac{1}{g} \frac{\partial^2 \phi}{\partial t^2}\right]_{z = 0} = 0$$

which is the so-called Cauchy-Poisson condition at the free surface.

XVI-1.6 FORMULATION OF A SURFACE WAVE PROBLEM

XVI-1.6.1 Thus ϕ and η appear to be solutions of the system

a) Continuity $\nabla^2 \phi = 0 \quad \begin{cases} -d \leq z \leq \eta(x, y, t) \\ -\infty < {x \atop y} < \infty \end{cases}$

b) Fixed boundary $\frac{\partial \phi}{\partial n} = 0$

in particular at the bottom $\frac{\partial \phi}{\partial z} \Big|_{z = -d} = 0$

c) Free surface $z = \eta\,(x, y, t)$

1) Kinematic condition

$$\frac{\partial \phi}{\partial z} = -\frac{\partial \eta}{\partial t} + \left.\frac{\partial \phi}{\partial x}\right|_{z=\eta} \cdot \frac{\partial \eta}{\partial x} + \left.\frac{\partial \phi}{\partial y}\right|_{z=\eta} \cdot \frac{\partial \eta}{\partial y}$$

2) Dynamic condition

$$-\frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] + g\eta = 0$$

where $f(t)$ is now included in $\frac{\partial \phi}{\partial t}$. However, even in this case, this last condition may be different from zero, and equal to a given function $f(x, y, t)$ in the case of a disturbance created at the free surface.

XVI-1.6.2 In the case of slow motion, η may be eliminated from the two free surface conditions resulting in the simple Cauchy-Poisson condition $\left[\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z}\right]_{z=0} = 0$. This leaves only one unknown ϕ to be determined from

$$\nabla^2 \phi = 0 \quad \begin{cases} -d \leq z \leq \eta = 0 \\ -\infty < \begin{matrix} x \\ y \end{matrix} < \infty \end{cases}$$

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=-d} = 0 \quad \text{and} \quad \left[\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z}\right]_{z=0} = 0$$

XVI-1.6.3 So formulated, the solution of the system of equations presented in Section XVI-1.6.1 is still difficult to determine. Firstly, the equations are non-linear and secondly the free surface is unknown and changes with time.

Hence, it is necessary to simplify the theory by making assumptions based on physical facts. It has already been seen that the first assumption consists of neglecting the quadratic terms in order to obtain a system of linear equations as in the previous section. The second assumption for non-linear theory is to assume that the solution may be given in the form of a power series of convergent terms.

XVI-1.7 PRINCIPLE OF SUPERPOSITION

When all the equations are homogeneous and linear, that is to say ϕ and its differential coefficients never occur in any form other than that of the first degree, the principle of superposition states that any number of individual solutions may be superimposed to form new functions which constitute solutions themselves. (See Section II-5.) For example, if ϕ_1 and ϕ_2 are two separate solutions, $a\,\phi_1 + b\,\phi_2$ is also a solution, a and b being two arbitrary constants.

This basic principle is very important and is frequently used in the following sections.

XVI-1.8 HARMONIC MOTION

Most of the solutions with which we are concerned in this chapter are harmonic. This stems from the fact that harmonic functions are quite natural solutions of the basic equations. It is also due to the fact that, by means of Fourier analysis, a great number of functions, particularly those characterizing periodic motions, may be considered as a superimposition of harmonic components (see Chapter XIX). Hence, from the principle of superposition any wave may be regarded as the resultant of a set of harmonic waves.

The solution of $\phi(x, y, z, t)$ is usually of the form

$\phi = f(x, y, z) \cos (kt + \epsilon)$ where $k = \frac{2\pi}{T}$ and T is the wave period.

or $\phi = \mathrm{Re.}\ f(x, y, z)\ e^{i(kt + \epsilon)}$

(since $e^{i(kt+\epsilon)} = \cos(kt + \epsilon) + i \sin(kt + \epsilon)$)

Re. means: real part of and ϵ is the phase of ϕ with respect to the origin of time, $t = 0$.

In the following Re. will be omitted and it is to be understood that only the real parts of the mathematical expressions are considered.

Introducing this expression of ϕ in the free surface

condition $\frac{1}{g} \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial \phi}{\partial z} = 0$

gives $\left(\frac{k^2}{g} - \frac{\partial}{\partial z}\right) \phi = 0$

XVI-1.9 METHOD OF SEPARATION OF VARIABLES

The basic equation $\nabla^2 \phi = 0$ may be solved by using the method of separation of variables based on the assumption that the solution ϕ is given by a product of functions of each variable alone.

From physical considerations, it may be expected that the solution ϕ will be given by the product of the functions of the horizontal components $U(x, y)$, of the vertical component $P(z)$

and of the time $f(t)$.

Hence $\phi = U(x, y) \cdot P(z) \cdot f(t)$

Introducing this value of $\phi(x, y, z, t)$ into the continuity equations: $\nabla^2 \phi = 0$ and after division by ϕ gives

$$- \frac{\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2}}{U(x, y)} = \frac{\dfrac{d^2 P}{d z^2}}{P(z)}$$

which may be written

$$- \frac{\nabla^2 U}{U} = \frac{P''}{P}$$

It must be said that it was not certain at the beginning that it would have been possible to separate the variables as it has been done. However, it will be seen later that this process may be performed. Only then may we be certain that solutions of the given differential equation $\nabla^2 \phi = 0$ will exist of the postulated form above.

The righthand side of the above equation is a function of z. The lefthand side is a function of x and y. Since x and y can vary independently of z and vice versa, the only way in which the function of x and y and the function of z can always be equal (as stated by the above equation) is if the lefthand side and the righthand side are both equal to the same constant m^2 where m may be real or imaginary.

It will be easily seen that if m is imaginary there is no physical meaning to the solutions in the case of wave motion: m is chosen to be real so that m^2 is always positive.

Then the equations

$$\frac{P''}{P} = - \frac{\nabla^2 U}{U} = m^2$$

are now reduced to $\quad \dfrac{d^2 P}{dz^2} - m^2 \; P(z) = 0$

and $\quad \dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} + m^2 \; U(x, y) = 0$

These equations will often be written in a shorter form

$$\left\{ \begin{array}{l} \left(\dfrac{d^2}{dz^2} - m^2 \right) P = 0 \\ (\nabla^2 + m^2) \; U = 0 \end{array} \right.$$

The last equation is the well-known Helmholtz equation of mathematical physics.

XVI-2 WAVE MOTION ALONG A VERTICAL

XVI-2.1 INTEGRATION ALONG A VERTICAL

The equation $\left(\dfrac{d^2}{dz^2} - m^2 \right) P = 0$ may easily be integrated.

The characteristic equation is $r^2 - m^2 = 0$, that is $r = \mp m$, which gives the general solution

$$P = Ae^{mz} + Be^{-mz}$$

A and B being two constant factors.

The boundary condition at the bottom

$$\left.\frac{\partial \phi}{\partial z}\right|_{z = -d} = 0$$

gives for any fixed value of x, y and t: $\left.\frac{\partial P}{\partial z}\right|_{z = -d} = 0$

since ϕ is proportional to P.

Introducing the value of P given above leads to

$$m A e^{-md} - m B e^{+md} = 0$$

Hence

$$A e^{-md} = B e^{+md}$$

Let

$$A e^{-md} = B e^{+md} = \frac{1}{2} D$$

Then

$$P = \frac{D}{2}\left(e^{m(z+d)} + e^{-m(z+d)}\right)$$

That is $P = D \cosh m(z+d)$

Now, substituting P in the expression of ϕ gives

$$\phi = D \cosh m(d+z) \;.\; U(x, y) \;.\; f(t)$$

XVI-2.2 INTRODUCTION OF THE FREE SURFACE CONDITION

The solution for f(t) is given by the Cauchy-Poisson condition at the free surface

$$\left[\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z}\right]_{z = 0} = 0$$

Introducing the above value for ϕ for $z = 0$ and dividing by ϕ gives successively

$$f''/f = -g\ (P'/P)_{z=0} = -gm \tanh md.$$

If we let $k^2 = gm \tanh md$, the solution for f is given by the equation $f'' + k^2 f = 0$.

The characteristic equation $r^2 + k^2 = 0$ gives $r = \pm ik$

Hence

$$f = \alpha e^{ikt} + \beta e^{-ikt}$$

α, β being constant coefficients depending upon the boundary conditions. The physical meaning of k is found to be a frequency $2\pi/T$, T being the wave period.

Therefore, with $\beta = 0$ and the coefficient α merged with the coefficient D, it is found

$$\phi = D \cosh m(d+z)\ U\ (x,y)\ e^{i(kt+\epsilon)}$$

Since there exist an infinite (but discrete) number of sets of values for k_n and m_n for which the equation $k_n^2 = m_n g \tanh m_n d$ is satisfied, a general solution of o is

$$\phi = \sum_{n=0}^{\infty} D_n \cosh m_n (d+z)\ U_n\ (x,y)\ e^{i(k_n t + \epsilon_n)}$$

Now considering the case of a monochromatic wave only, it is convenient to express D as a function of the wave height 2a.

From the free surface dynamic equation

$$\eta = \frac{1}{g} \left.\frac{\partial \phi}{\partial t}\right|_{z=0}$$

one obtains

$$\eta = \frac{ikD}{g} \cosh md\ U\ (x,y)\ e^{i(kt-\varepsilon)}$$

or

$$\eta = -\frac{kD}{g} \cosh md \; U(x, y) \; \sin(kt-\epsilon).$$

The expression for ϕ and η becomes more convenient if we write (aU) for the amplitude of η

Then

$$a = -\frac{kD}{g} \cosh md$$

Hence

$$D = -\frac{iag}{k} \frac{1}{\cosh md}$$

and

$$\phi = -\frac{iag}{k} \frac{\cosh m(d+z)}{\cosh md} U(x, y) \; e^{i(kt+\epsilon)}$$

Introducing the relationship $k^2 = mg \tanh md$ leads to

$$\phi = -a\frac{k}{m} \frac{\cosh m(d+z)}{\sinh md} \quad U(x, y) \; e^{i(kt+\epsilon)}$$

The function for P which is independent of x, y and t is

$$P(z) = -\frac{ak}{m} \frac{\cosh m(d+z)}{\sinh md} = -\frac{ag}{k} \frac{\cosh m(d+z)}{\cosh md}$$

With these conditions the wave height at any point is 2a U (x, y). U (x, y) is the relative value of the wave height with respect to a plane or a point where it is simply 2a.

XVI-3 TWO-DIMENSIONAL WAVE MOTION. LINEAR SOLUTION

XVI-3.1 WAVE EQUATION

Now the differential wave equation $(\nabla^2 + m^2) \; U = 0$ has to be investigated. A general solution of this equation does not exist, but a number of solutions may be found, corresponding to particular boundary conditions. Some fundamental solutions of great importance will be studied. They correspond to particular simple cases.

It is interesting to notice that the same equation governs sound waves:

$(\nabla^2 + m^2) p = 0$ where p is the pressure, and electromagnetic waves: $(\nabla^2 + m^2) \overline{E} = 0$ where $\overline{E}$ is an electric vector. This last equation is vectorial and identical to the three following ones:

$$(\nabla^2 + m^2) E_x = 0, \quad (\nabla^2 + m^2) E_y = 0,$$

$$(\nabla^2 + m^2) E_z = 0$$

where E_x, E_y, E_z are the three components of vector $\vec{E}$. Hence it is easy to realize that a great number of mathematical solutions may be found in other fields of physics and adapted to hydrodynamics.

XVI-3.2 INTEGRATION OF THE WAVE EQUATION

In the case of a two-dimensional wave such as motion encountered in a wave flume:

$\frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial U}{\partial y} = 0$ and the wave equation is reduced to:

$$\left(\frac{\partial^2}{\partial x^2} + m^2\right) \dot{U} = 0$$

The characteristic equation is:

$$r^2 + m^2 = 0 \qquad r = \mp im$$

Hence the solutions for U are given by any linear combination of e^{-imx} and e^{imx}

$$U = A'e^{imx} + B'e^{-imx}$$

In particular if $U = e^{-imx}$, then

$$\phi = -a \frac{k}{m} \frac{\cosh m(d+z)}{\sinh md} e^{i(kt-mx+\epsilon)}$$

or

$$\phi = -a \frac{k}{m} \frac{\cosh m(d+z)}{\sinh md} \cos(kt - mx + \epsilon)$$

This is the velocity potential function of a progressive wave traveling in the OX direction.

If $U = e^{imx}$, the velocity potential function of a wave traveling in the opposite direction is obtained.

If the solution for U is:

$$U = \frac{1}{2}\left(e^{imx} + e^{-imx}\right) = \cos mx$$

or

$$U = \frac{1}{2i}\left(e^{imx} - e^{-imx}\right) = \sin mx$$

then

$$\phi = -a \frac{k}{m} \frac{\cosh m(d+z)}{\sinh md} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} mx \cos(kt + \epsilon)$$

This is the velocity potential function of a standing wave or clapotis.

If A' is different from B' : a partial clapotis is obtained.

In practice, the values for A' and B' are given by boundary conditions (wave reflection...etc).

In the most general case of a two-dimensional irregular wave, as may be observed at sea, the velocity potential function ϕ is:

$$\phi = \sum_{n=0}^{\infty} -a_n \frac{k_n}{m_n} \frac{\cosh m_n(d+z)}{\sinh m_n d} e^{-im_n x} e^{i(k_n t + \epsilon_n)}$$

When there are two waves only, traveling in the same direction, the velocity potential function describing the "beating" phenomena may be obtained easily.

XVI-3.3 PHYSICAL MEANING - WAVE LENGTH

Now it is easy to see the physical meaning of the coefficient m: since ϕ and consequently η is periodic with respect to space, $m = 2\pi/L$ and L is the wave length for the free surface condition above.

Hence the wave length is given by:

$$k^2 = mg \tanh md$$

$$\left(\frac{2\pi}{T}\right)^2 = \frac{2\pi}{L} g \tanh \frac{2\pi}{L} d$$

That is:

$$L = \frac{gT^2}{2\pi} \tanh \frac{2\pi d}{L}$$

and the wave celerity:

$$C = \frac{L}{T} = \frac{gT}{2\pi} \tanh \frac{2\pi d}{L}$$

In particular when d/L is small(shallow water)

$$\tanh \frac{2\pi d}{L} \doteqdot \frac{2\pi d}{L}, \quad L = T\sqrt{gd} \text{ and } C = \sqrt{gd}$$

When d/L is large (deep water), $\tanh \frac{2\pi d}{L} = 1$, $L = \frac{gT^2}{2\pi}$, $C = \frac{gT}{2\pi}$.
The values of L and C are given as functions of the depth d and the wave period T on the two following nomographs (Figures XVI-3 and XVI-4).

XVI-3.4 FLOW PATTERNS

The velocity components are $u = -\frac{\partial \phi}{\partial x}$, $w = -\frac{\partial \phi}{\partial z}$ and the particle orbits are:

$$x = -\int_0^t \frac{\partial \phi}{\partial x} dt \qquad z = -\int_0^t \frac{\partial \phi}{\partial z} dt$$

In the case of a progressive wave:

$$u = ka \frac{\cosh m(d+z)}{\sinh md} \sin(kt - mx)$$

$$w = ka \frac{\sinh m(d+z)}{\sinh md} \cos(kt - mx)$$

The particle orbits are determined by assuming that the motion around a fixed point x_o, z_o is small, so that one can consider x and z constant in the integration.

$$x = x_o - a \frac{\cosh m(d+z_o)}{\sinh md} \cos(kt - mx_o)$$

and

$$z = z_o + a \frac{\sinh m(d+z_o)}{\sinh md} \sin(kt - mx_o)$$

Squaring and adding these two last equations to eliminate t, the equation of an ellipse is obtained:

$$\frac{(x - x_o)^2}{A^2} + \frac{(z - z_o)^2}{B^2} = 1$$

It is now seen that x_o and z_o are at the center of the ellipse, i.e., can be considered as the position of the particle at rest with the horizontal semi-major axis

$$A = a \frac{\cosh m(d+z_o)}{\sinh md}$$

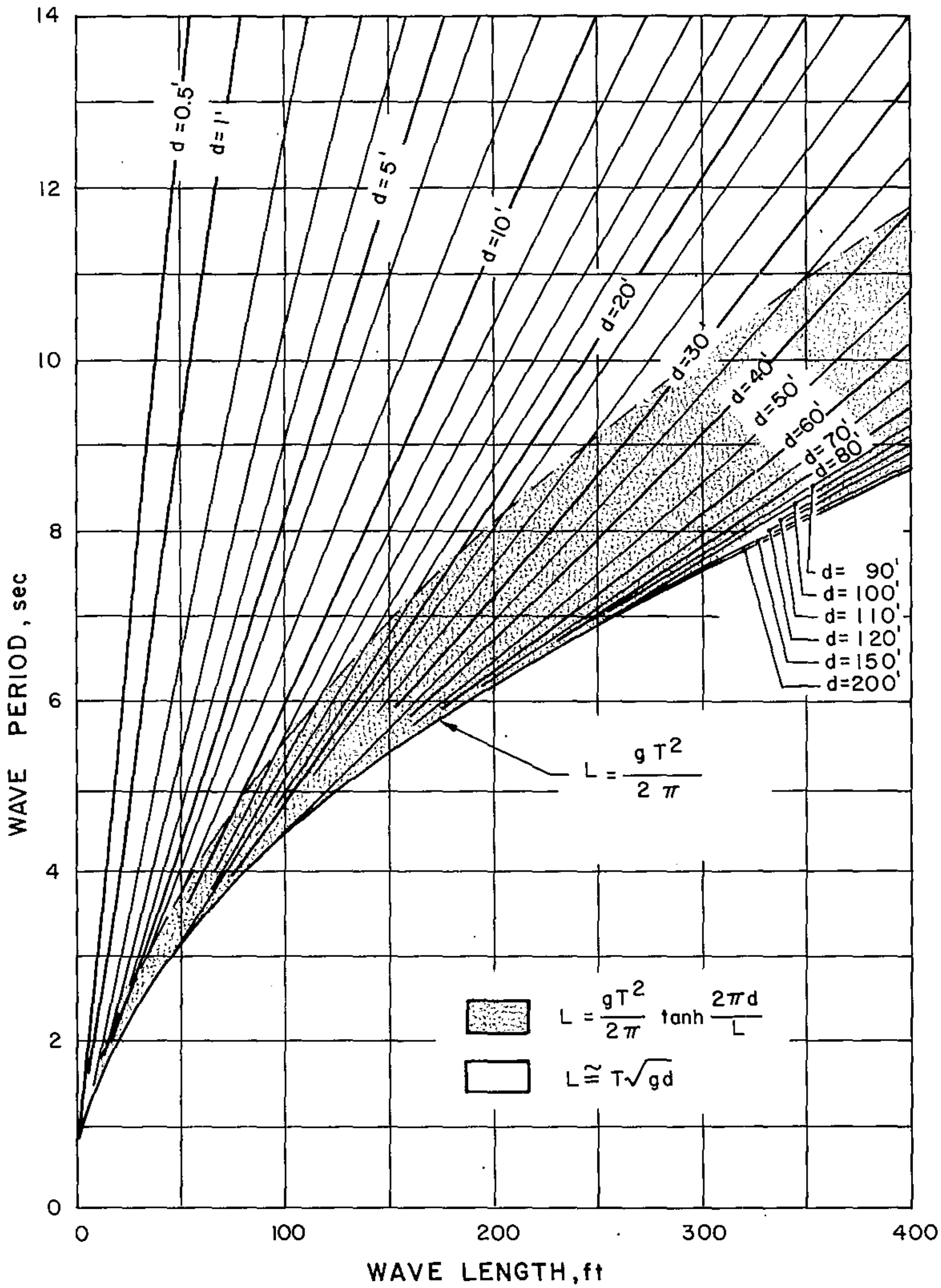


FIGURE XVI-3

WAVE LENGTH VERSUS WAVE PERIOD

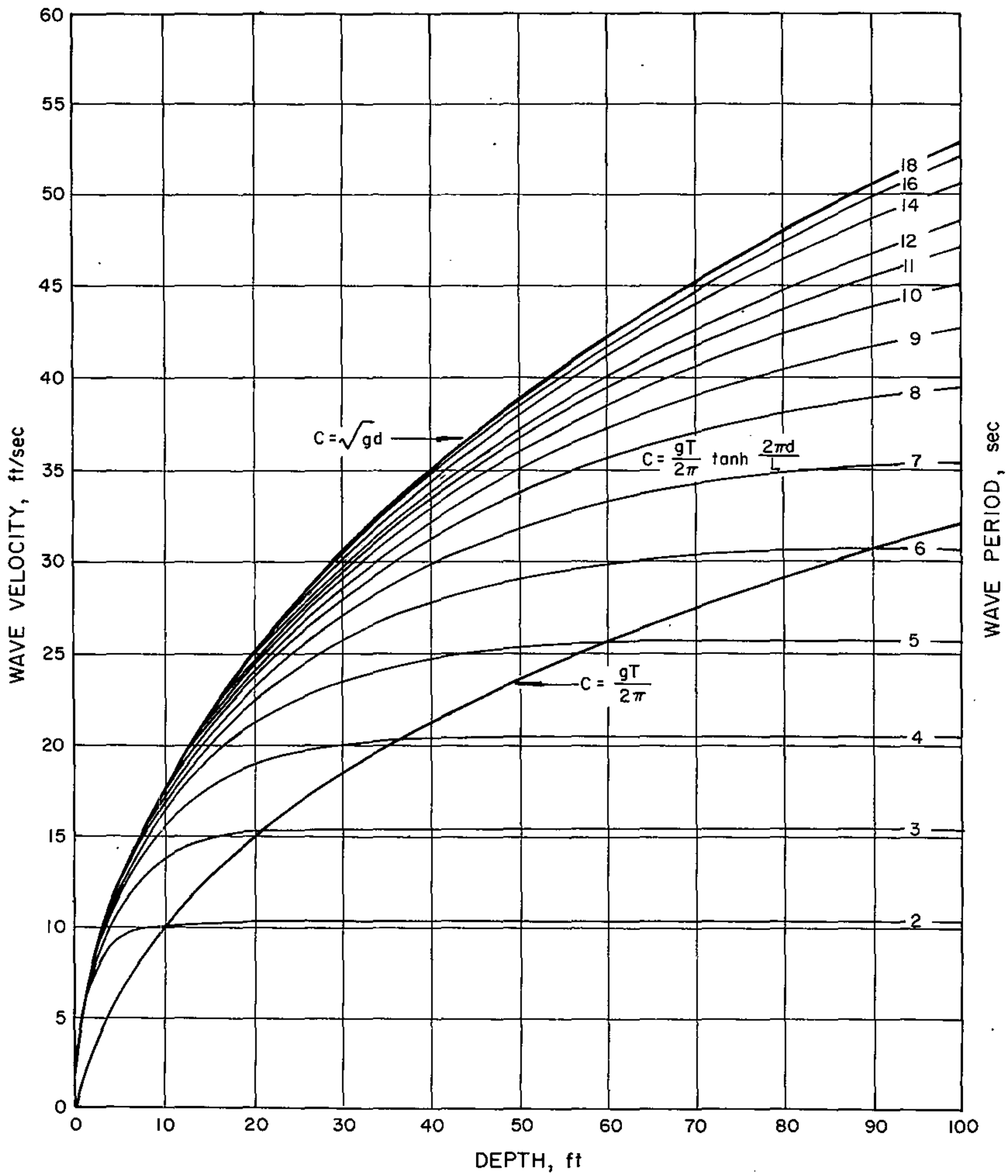


FIGURE XVI-4
WAVE VELOCITY VERSUS DEPTH

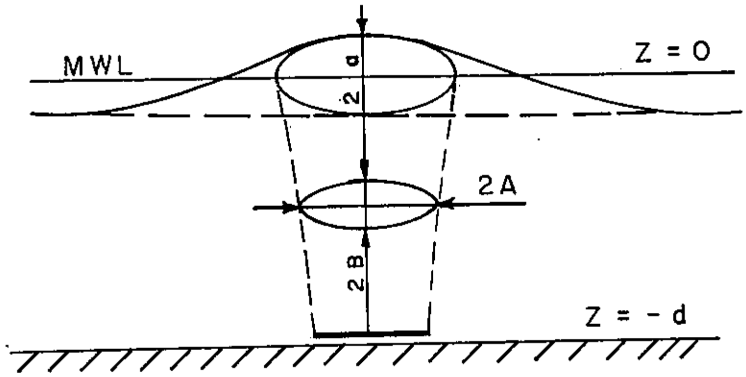


FIGURE XVI-5
TWO-DIMENSIONAL PROGRESSIVE WAVE MOTION

and the vertical semi-minor axis: $B = a \dfrac{\sinh m\,(d + z_o)}{\sinh md}$

$B = a$ at the free surface, and $B = 0$ at the bottom (Figure XVI-5).

The free surface equation is:

$$\eta = a \sin (kt - mx)$$

In the case of a clapotis, it would be easily found that the paths of particles are straight lines given by: (see Figure XVI-6)

$$\frac{z - z_o}{x - x_o} = - \tanh m(d + z_o) \cot mx_o$$

or

$$+ \tanh m(d + z_o) \tan mx_o$$

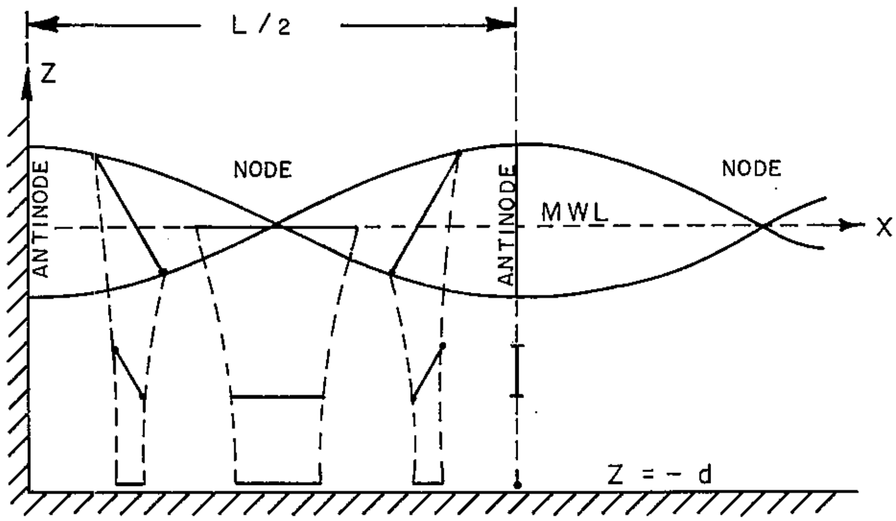


FIGURE XVI-6
CLAPOTIS

XVI-3.5 THE USE OF COMPLEX NUMBER NOTATION

In order to show the great simplicity introduced by the operation $W = \phi + i\psi$, W is calculated to a first order of approximation in the case of a two-dimensional monochromatic progressive wave motion.

First of all, it is convenient to make a change in the origin of the vertical axis, and to take the horizontal axis on the bottom, such as: $z' = d + z$.

With this new condition the velocity potential function becomes:

$$\phi = -a \frac{k}{m} \frac{\cosh mz'}{\sinh md} \cos (mx - kt)$$

or with

$$A = -a \frac{k}{m} \frac{1}{\sinh md}$$

$$\phi = A \cosh mz' \cos (mx - kt)$$

The stream function ψ is given by one of the following operations:

$$u = -\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial z'} \quad \text{or} \quad W = -\frac{\partial \phi}{\partial z'} = -\frac{\partial \psi}{\partial x}$$

which gives easily:

$$\psi = A \sinh mz' \sin (mx - kt)$$

Now:

$$W = \phi + i\psi = A \left[\cosh mz' \cos (mx - kt) + i \sinh mz' \sin (mx - kt) \right]$$

Introducing the following relationships:

$$\cosh mz' = \cos imz'$$

$$\sinh mz' = - i \sin imz'$$

One obtains:

$$W = A \left[\cos imz' \cos (mx - kt) + \sin imz' \sin (mx - kt) \right]$$

That is:

$$W = A \cos (mx - imz' - kt)$$

Introducing the complex number: $Z = x - iz'$, one obtaines the very simple relationship: $W = \phi + i\psi = A \cos (mZ - kt)$

Vice versa, the velocity potential function ϕ is given by the real part of W, while the stream function ψ is given by the imaginary part.

Similarly, the value of W for a clapotis is:

$$W = 2A \sin mZ \cos kt \quad \text{or} \quad 2A \cos mZ \cos kt$$

XVI-4 THREE-DIMENSIONAL WAVE MOTION

XVI-4.1 THREE-DIMENSIONAL WAVE MOTION IN A RECTANGULAR TANK

Using the same process of separation of variables as used before in solving the equation which governs three-dimensional motion $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + m^2\right) U = 0$, it is found that

$$\frac{\partial^2 U}{\partial x^2} + p^2 U = 0$$

$$\frac{\partial^2 U}{\partial y^2} + q^2 U = 0$$

in which p and q are constants, real or complex, such that $m^2 = p^2 + q^2$.

As an application of three-dimensional motion, the case of standing wave in a rectangular tank is analyzed.

The boundaries are shown in Figure XVI-7.

$$\frac{\partial \phi}{\partial n} = 0 \quad \text{or} \quad \frac{\partial U}{\partial n} = 0 \quad \text{for} \begin{cases} x = 0, a \\ y = 0, b \end{cases}$$

It may be easily verified that the solution of $(\nabla^2 + m^2)\ U = 0$ is $U = \cos \frac{r \pi x}{a} \cos \frac{s\pi y}{b}$ with $p = \frac{r\pi}{a}$, $q = \frac{s\pi}{b}$, where r and s are integers, and the solution for $\nabla^2 \phi = 0$ is

$$\phi = -a \frac{k}{m} \frac{\cosh m(d+z)}{\sinh md} \cos \frac{r \pi x}{a} \cos \frac{s \pi y}{b}$$

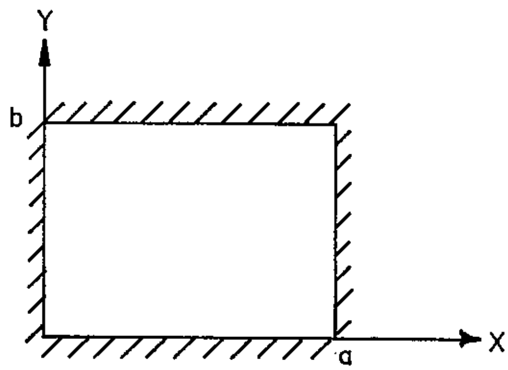


FIGURE XVI-7
RECTANGULAR BASIN - NOTATION

The general solution is

$$\phi = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} -a_n \quad \frac{k_n}{m_n} \frac{\cosh m_n(d+z)}{\sinh m_n d} \quad \cos \frac{r_i \pi x}{a} \quad \cos \frac{s_j \pi y}{b}$$

where:

$$k_n^2 = m_n g \tanh m_n d \quad \text{and} \quad p_i^2 + q_j^2 = m_n^2$$

At this point a remark concerning free surface curvature is worth-while.

The equation of a two-dimensional motion

$$\left(\frac{\partial^2}{\partial x^2} + m^2\right) U = 0$$

shows that the curvature of the free surface is proportional to the free surface elevation η since $\eta = aU$. The shape of the free surface is that of a sine curve.

In the case of three-dimensional motion, one has

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + m^2\right) U = 0$$

This is the sum of the curvatures in two directions (OX and OY) which is proportional to the free surface elevation. The name "short-crested wave" is often given to some kinds of three-dimensional wave motion.

XVI-4.2 CYLINDRICAL WAVE MOTION

The continuity equation expressed in terms of cylindrical coordinates is

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

In the case of a motion with an axis of symmetry so that $\frac{\partial^2 \phi}{\partial \theta^2} = 0$, the solution is

$$\phi = U(r) \; P(z) \; e^{i(kt + \epsilon)}$$

The resulting wave equation $(\nabla^2 + m^2)\, U = 0$ for cylindrical waves becomes

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + m^2\right) U = 0$$

which is a Bessel equation of order zero. The solutions for U are given by any linear combination of $H_o^{(1)}(mr)$ and $H_o^{(2)}(mr)$ or any linear combination of $J_o(mr)$ and $Y_o(mr)$. The Figure XVI-8 and the following table gives the values for J_o and Y_o and the relationships between these four functions. Their physical meaning is also given by mathematical analogy with the sinusoidal functions. When mr becomes very large, it is found that $H_o^{(1)}(mr) \approx A\, e^{iB}$, $H_o^{(2)}(mr) \approx A\, e^{-iB}$, $J_o(mr) \approx A \cos B$, $Y_o(mr) \approx A \sin B$ in which $A = \sqrt{\frac{2}{\pi\, mr}}$, $B = \left(mr - \frac{\pi}{4}\right)$

In a two-dimensional motion, A will be replaced by a and B will be replaced by $(mx + \epsilon)$.

It is seen that the wave height which is given by the amplitude of U, is proportional to A. It decreases as $r^{\frac{1}{2}}$. This may also be demonstrated by conservation of the transmitted wave energy (see Section XVI-6).

TWO-DIMENSIONAL WAVES		CIRCULAR WAVES	
$\left(\frac{\partial^2}{\partial x^2} + m^2\right) U = 0$		$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + m^2\right) U = 0$	
$e^{-imx} = \cos mx - i \sin mx$	Progressive wave in the positive OX direction (Wave height = cst)	$H_o^{(2)}(mr) = J_o - iY_o$	Converging wave (sink) (Wave height increases when r decreases.)
$e^{+imx} = \cos mx + i \sin mx$	Progressive wave in the negative OX direction (Wave height=cst)	$H_o^{(1)}(mr) = J_o + iY_o$	Diverging wave (source) (Wave height decreases when r increases)
$\cos mx = \frac{e^{imx} + e^{-imx}}{2}$	Standing wave (clapotis) (Horizontal velocity = 0 when x = 0)	$J_o(mr) = \frac{H_o^{(1)} + H_o^{(2)}}{2}$	Standing circular wave (Horizontal velocity = 0 when r = 0)
$\sin mx = \frac{e^{imx} - e^{-imx}}{2i}$	Standing wave (clapotis) (Horizontal velocity maximum for x=0)	$Y_o(mr) = \frac{H_o^{(1)} - H_o^{(2)}}{2i}$	Standing circular wave (Horizontal velocity infinite for r = 0)

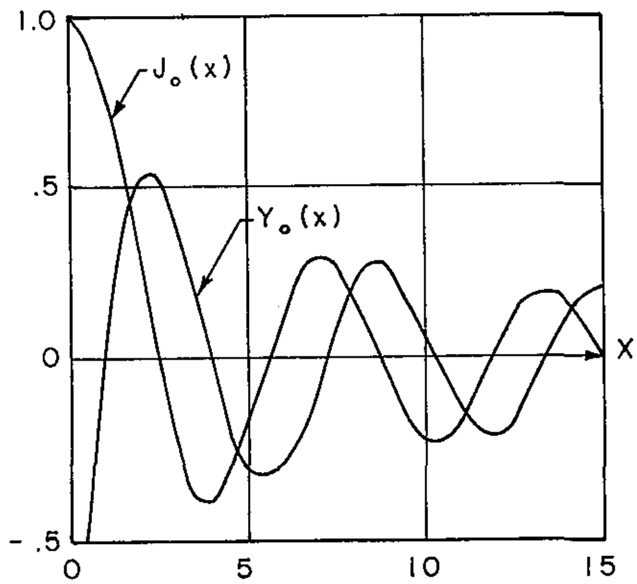


FIGURE XVI-8
BESSEL FUNCTIONS OF FIRST
ORDER

XVI-4.3 WAVE AGITATION IN A CIRCULAR TANK

The boundary condition for a circular tank is $\left.\frac{\partial \phi}{\partial r}\right|_{r=R} = 0$ where R is the radius of the tank. That is $\left.\frac{\partial U}{\partial r}\right|_{r=R} = 0$

It is found that the solution for U is: $U = J_n(mr) \cos n\theta$

where J_n is the Bessel function of order n.

When $n = 0$, $U = J_o(mr)$ and the motion is a stationary circular wave.

XVI-5 NON-LINEAR THEORY OF WAVES

XVI-5.1 GENERAL PROCESS OF CALCULATION

It is assumed that the two unknowns, the velocity potential ϕ and the free surface elevation η, may be transformed into the following power series expressions with respect to a parameter a. a is chosen arbitrarily but has the dimension of a length. This length is usually taken to be a half wave height.

$$\phi = a\phi_1 + a^2\phi_2 + a^3\phi_3 + \ldots + a^n\phi_n + \ldots$$

$$\eta = a\eta_1 + a^2\eta_2 + a^3\eta_3 + \ldots + a^n\eta_n + \ldots$$

Introducing this expression for ϕ in the continuity relationship $\nabla^2\phi = 0$, it is seen that each of these terms ϕ_n is an independent solution of the

Laplace equation $\nabla^2 \phi_n = 0$. ϕ_n also satisfies the fixed boundary condition $\frac{\partial \phi_n}{\partial n} = 0$ and the free surface condition.

XVI-5.2 FREE SURFACE CONDITIONS

The free surface conditions are expressed in terms of their values at the still water level $(z = 0)$.

Developing $\phi\,(x, y, 0 + \eta, t)$ in terms of power series in η gives:

$$\phi\,(x, y, \eta, t) = \phi(x, y, 0, t) + \eta \left[\frac{\partial \phi(x, y, 0, t)}{\partial z}\right] + \dots$$

or

$$\Big[\phi\Big]_{z=\eta} = \Big[\phi\Big]_{z=0} + \eta \left[\frac{\partial \phi}{\partial z}\right]_{z=0} + \frac{\eta^2}{2!} \left[\frac{\partial^2 \phi}{\partial z^2}\right]_{z=0} + \dots$$

Introducing this expression for ϕ into the free surface conditions given in Section XVI-1.4, one obtains for the Kinematic Condition

$$\frac{\partial}{\partial z}\left[\phi + \eta \frac{\partial \phi}{\partial z} + \dots\right] = -\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} \frac{\partial}{\partial x} \left[\phi + \eta \frac{\partial \phi}{\partial z} + \dots\right] + \frac{\partial \eta}{\partial y} \cdot \frac{\partial}{\partial y}\left[\phi + \eta \frac{\partial}{\partial z} + \dots\right]$$

and for the dynamic condition

$$-\frac{\partial}{\partial t}\left[\phi + \eta \frac{\partial \phi}{\partial z} + \dots\right] + \frac{1}{2}\left[\left\{\frac{\partial}{\partial x}\left(\phi + \eta \frac{\partial \phi}{\partial z} + \dots\right)\right\}^2 + \left\{\frac{\partial}{\partial y}\left(\phi + \eta \frac{\partial \phi}{\partial z} + \dots\right)\right\}^2\right] + g\eta = 0$$

Now the problem is completely formulated: introducing the expression of ϕ and η of XVI-5.1 in the above free surface conditions permits us to carry out the calculations.

XVI-5.3 METHOD OF SOLUTIONS

These relationships must be verified for any value of a since a is arbitrarily taken. Grouping together the terms of common powers of a, it is found respectively

$$a\left(\frac{\partial \phi_1}{\partial z} + \frac{\partial \eta_1}{\partial t}\right) + a^2\left(\frac{\partial \phi_2}{\partial z} + \frac{\partial \eta_2}{\partial t} + \eta_1 \frac{\partial^2 \phi_1}{\partial z^2} - \frac{\partial \phi_1}{\partial x}\frac{\partial \eta_1}{\partial x} - \frac{\partial \phi_1}{\partial y}\frac{\partial \eta_1}{\partial y}\right) + a^3 (\ldots) + \ldots = 0$$

and

$$a\left(-\frac{\partial \phi_1}{\partial t} + g\eta_1\right) + a^2\left(g\eta_2 - \frac{\partial \phi_2}{\partial t} - \eta_1 \frac{\partial^2 \phi_1}{\partial t\, \partial z} + \frac{1}{2}\left\{\left(\frac{\partial \phi_1}{\partial x}\right)^2 + \left(\frac{\partial \phi_1}{\partial y}\right)^2 + \left(\frac{\partial \phi_1}{\partial z}\right)^2\right\}\right) \ldots$$

$$+ a^3 (\ldots) + \ldots = 0$$

This leads to the following equations which are independent of the value of a

$$\frac{\partial \phi_1}{\partial z} + \frac{\partial \eta_1}{\partial t} = 0$$

$$\frac{\partial \phi_2}{\partial z} + \frac{\partial \eta_2}{\partial t} + \eta_1 \frac{\partial^2 \phi_1}{\partial z^2} - \frac{\partial \phi_1}{\partial x} \cdot \frac{\partial \eta_1}{\partial x} - \frac{\partial \phi_1}{\partial y}\frac{\partial \eta_1}{\partial y} = 0$$

$$\frac{\partial \phi_n}{\partial z} + \frac{\partial \eta_n}{\partial t} = f\,(\phi_{n-1}, \eta_{n-1})$$

and

$$-\frac{\partial \phi_1}{\partial t} + g\,\eta_1 = 0$$

$$-\frac{\partial \phi_2}{\partial t} + g\eta_2 - \eta_1 \frac{\partial^2 \phi_1}{\partial t\, \partial z} + \frac{1}{2}\left[\left(\frac{\partial \phi_1}{\partial x}\right)^2 + \ldots\right] = 0$$

$$-\frac{\partial \phi_n}{\partial t} + g\eta_n = f'(\phi_{n-1}, \eta_{n-1})$$

Taken in pairs, the above equations may be solved for $\phi_n \Big|_{z=0}$ and η_n when $\phi_{n-1}\Big|_{z=0}$ and η_{n-1} are known, using $\nabla^2 \phi_n = 0$ and $\frac{\partial \phi_n}{\partial z}\Big|_{z=-d} = 0$. In particular, eliminating between the two first ones, which are linear in a

$$a\left(\frac{\partial \phi_1}{\partial z} + \frac{\partial \eta_1}{\partial t}\right) = 0 \text{ and } a\left(-\frac{\partial \phi_1}{\partial t} + g\eta_1\right) = 0$$

leads to

$$\left[\frac{\partial^2 \phi_1}{\partial t^2} + g\frac{\partial \phi_1}{\partial z}\right]_{z=0} = 0$$

This may be compared with the Cauchy-Poisson condition, which was previously developed. Then the linear motion will be defined by $\phi = a\phi_1$, and $\eta = a\eta_1$.

In practice, the study of non-linear problems requires very long and tedious calculations. Sometimes, this may be reduced in the case of two-dimensional motion by using the function

$$W = \phi + i\psi$$

where ψ is the stream function and $i = \sqrt{-1}$.

XVI-5.4 THE BERNOULLI EQUATION AND THE RAYLEIGH PRINCIPLE

In the case of periodic progressive wave moving in the OX direction at a celerity C, the general solution for ϕ is

$$\phi = P(z)\ F(x - Ct)$$

and

$$\frac{\partial \phi}{\partial t} = -CP(z)\ F'(x - Ct)$$

also

$$\frac{\partial \phi}{\partial x} = P(z)\ F'(x - Ct)$$

Hence it is seen that

$$\frac{\partial \phi}{\partial t} = -C\frac{\partial \phi}{\partial x} = +Cu.$$

Introducing these relationships, it is seen that the Bernoulli equation in the case of progressive waves takes the form

$$-Cu + \frac{1}{2}(u^2 + w^2) + \frac{p}{\rho} + gz = 0$$

which is often written after some elementary transformations

$$C^2 - 2Cu + u^2 + w^2 + \frac{2p}{\rho} + 2gz = C^2$$

C^2 being a constant, and p being also a constant at the free surface, the dynamic condition at the free surface becomes after division by C^2

$$\left(\frac{u - C}{C}\right)^2 + \frac{w^2}{C^2} + \frac{2g\eta}{C^2} = \text{const.}$$

The constant can be taken equal to unity. It is seen that the motion could be considered as a steady motion in a new system of relative coordinates. In this system of coordinates, the origin of the OX axis moves at the wave celerity C. It is the Rayleigh principle.

XVI-6 ENERGY FLUX AND GROUP VELOCITY

XVI-6.1 ENERGY FLUX

The average energy flux per unit of wave crest through a fixed vertical plane parallel to the wave crest is:

$$F_{av} = \frac{1}{T}\int_t^{t+T}\int_{-d}^{\eta}\left(\rho\,\frac{V^2}{2} + p + \rho g z\right) u\, dz\, dt$$

i.e., by application of the Bernoulli equation where $f(t)$ is assumed to be taken into account in $\frac{\partial \phi}{\partial t}$:

$$F_{av} = -\rho\,\frac{1}{T}\int_t^{t+T}\int_{-d}^{\eta}\frac{\partial \phi}{\partial t}\,\frac{\partial \phi}{\partial x}\, dz\, dt$$

This formula is general and can be applied by inserting the value of the potential function ϕ for any kind of irrotational wave, linear or non-linear. In the case of a linear periodic progressive wave (see Section XVI-3.2)

$$\phi = -a\,\frac{k}{m}\,\frac{\cosh m(d+z)}{\sinh md}\cos(kt - mx)$$

Inserting this value in the above expression leads to:

$$F_{av} = \rho\,\frac{1}{T}\int_t^{t+T}\int_{-d}^{\eta} a^2\,\frac{k^3}{m}\,\frac{\cosh^2 m(d+z)}{\sinh^2 md}\,\sin^2(kt - mx)\, dz\, dt$$

which gives, after integration and the neglection of some high order terms:

$$F_{av} = \frac{1}{4}\,\rho g a^2 C\left(1 + \frac{2md}{\sinh 2md}\right)$$

which tends toward

$$F_{av}\Big|_{d\to\infty} = \frac{1}{4}\rho g a^2 \frac{gT}{2\pi}$$

in deep water and

$$F_{av}\Big|_{d\to o} = \frac{1}{2}\rho g a^2 \sqrt{gh}$$

in shallow water.

XVI-6.2 ENERGY PER WAVE LENGTH AND RATE OF ENERGY PROPAGATION

The stored energy per wave length per unit length of crest is the sum of the kinetic energy and the potential energy:

$$E = \rho\int_x^{x+L}\int_{-d}^{\eta}\left[\frac{1}{2}V^2 + gz\right] dz\, dx$$

Inserting the value of $V^2 = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2$ and neglecting some high order terms, it is found that in the case of a simple linear periodic progressive wave that the potential energy E_p equals the kinetic energy E_K and the total energy per wave length per unit length of crest is

$$E = E_p + E_K = \frac{1}{2}\rho g a^2 L$$

and per unit length is

$$E_{av} = \frac{1}{2}\rho g a^2$$

Dividing the energy flux by the stored energy gives the rate of propagation of energy:

$$U_E = \frac{F_{av}}{E_{av}} = \frac{1}{2} C \left[1 + \frac{2\,m\,d}{\sinh 2\,m\,d} \right]$$

i.e., $U_E = \frac{1}{2}\frac{gT}{2\pi}$ in deep water and $U_E = \sqrt{g\,d}$ in shallow water, which means that the rate of transmitted energy is half of the wave velocity in deep water and equal to the wave velocity in shallow water.

XVI-6.3 GROUP VELOCITY

Consider the linear superimposition of two progressive waves of same amplitude and of slightly different periods such as

$$\eta = a \sin [mx - kt] + a \sin [(m + \delta m)x - (k + \delta k)t]$$

where δm and δk are supposed to be small quantities. This expression can still be written:

$$\eta = 2a \cos \frac{1}{2}(\delta m x - \delta k t) \sin \left[\left(m + \frac{\delta m}{2}\right)x - \left(k + \frac{\delta k}{2}\right)t\right]$$

and since δm and δk are small:

$$\eta \cong 2a \cos \frac{1}{2}(\delta m x - \delta k t) \sin (mx - kt)$$

It is seen that the wave $2a \sin (mx - kt)$ is modulated by the term:

$$\cos \frac{1}{2}(\delta m x - \delta k t) = \cos \frac{1}{2} \delta m \left(x - \frac{\delta k}{\delta m} t\right)$$

which travels at a speed $U = \frac{\delta k}{\delta m}$. Since the wave velocity $C = \frac{k}{m}$, $\delta k = \delta(mC)$ and $U = \frac{\delta (mC)}{\delta m}$, i.e., at the limit when δm tends to be the infinitesimal dm, and replacing m by $\frac{2\pi}{L}$

$$U = \frac{d(mC)}{dm} = C - L\frac{dC}{dL}.$$

Inserting the value of $C = \frac{gT}{2\pi}\tanh\frac{2\pi d}{L}$ and $L = CT$, the value of

$$U = \frac{C}{2}\left[1 + \frac{2md}{\sinh 2md}\right]$$

is found. It is interesting to note that this expression is the same as the expression for the rate of energy propagation U_E. This stems from the fact that in the case of a wave train, there is no energy passing through the node of the wave train where the wave amplitude is zero. So, the velocity of energy propagation is that of a wave train or of a group of waves. For this reason, the group velocity U is equal to the rate of energy propagation. However, it is important to note that this statement holds true only in the case of a linear wave. The values of U/C_o and C/C_o where C_o is the deep water wave celerity ($C_o = gT/2\pi$) is shown on Figure XVI-9.

XVI-6.4 WAVE SHOALING

In the case of a wave traveling over a very gentle slope, it is assumed that the wave motion is the same as if the bottom were horizontal. In a word, the flow pattern deformation due to the bottom slope is neglected. Then it is assumed that the flux of transmitted energy is a constant, in which case the wave height 2a at a given depth d is known as a function of the deep water wave height $2a_o$ from the formula

$$F_{av}\Big|_d = F_{av}\Big|_{d\to\infty}$$

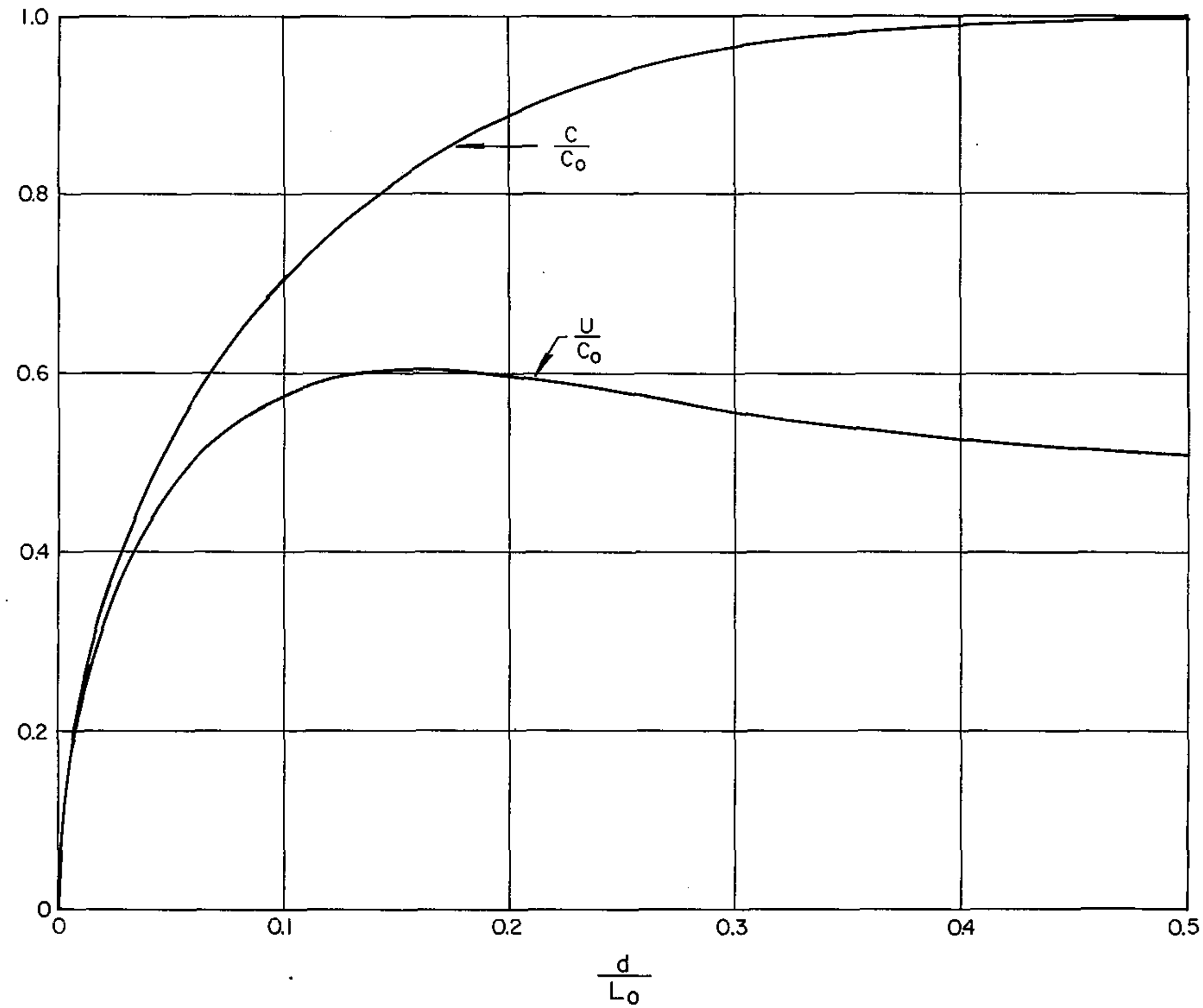


FIGURE XVI-9

RELATIVE WAVE VELOCITY AND GROUP VELOCITY VS. DEPTH i.e.,

$$\frac{1}{4}\rho g a^2 \frac{g}{k} \tanh md \left[1 + \frac{2md}{\sinh 2md}\right] = \frac{1}{4}\rho g a_o^2 \frac{g}{k}$$

i.e.,

$$\frac{2a}{2a_o} = \frac{H}{H_o} = \frac{1}{\left\{\tanh md \left[1 + \frac{2md}{\sinh 2md}\right]\right\}^{\frac{1}{2}}}$$

Inserting the expression for $\frac{L}{L_o} = \tanh \frac{2\pi d}{L}$ where L_o is the deep water wave length $L_o = \frac{gT^2}{2\pi}$, it is then possible to calculate the value of $\frac{H}{H_o}$ as a function of $\frac{d}{L_o}$ only. The results of such calculation are shown in Figure XVI-10.

XVI-7 A FORMULARY FOR THE SMALL AMPLITUDE WAVE THEORY

Figure XVI-11 gives the values

$$\sinh \frac{2\pi d}{L}, \quad \cosh \frac{2\pi d}{L}, \quad \text{and} \quad \tanh \frac{2\pi d}{L}$$

as functions of $\frac{d}{L}$. The two following tables (Tables XVI-I and II) summarize a number of formulas for linear periodic waves in deep water, in an intermediate water depth, and in shallow water. The first table presents a set of formulas for progressive waves, and the second table is for standing waves.

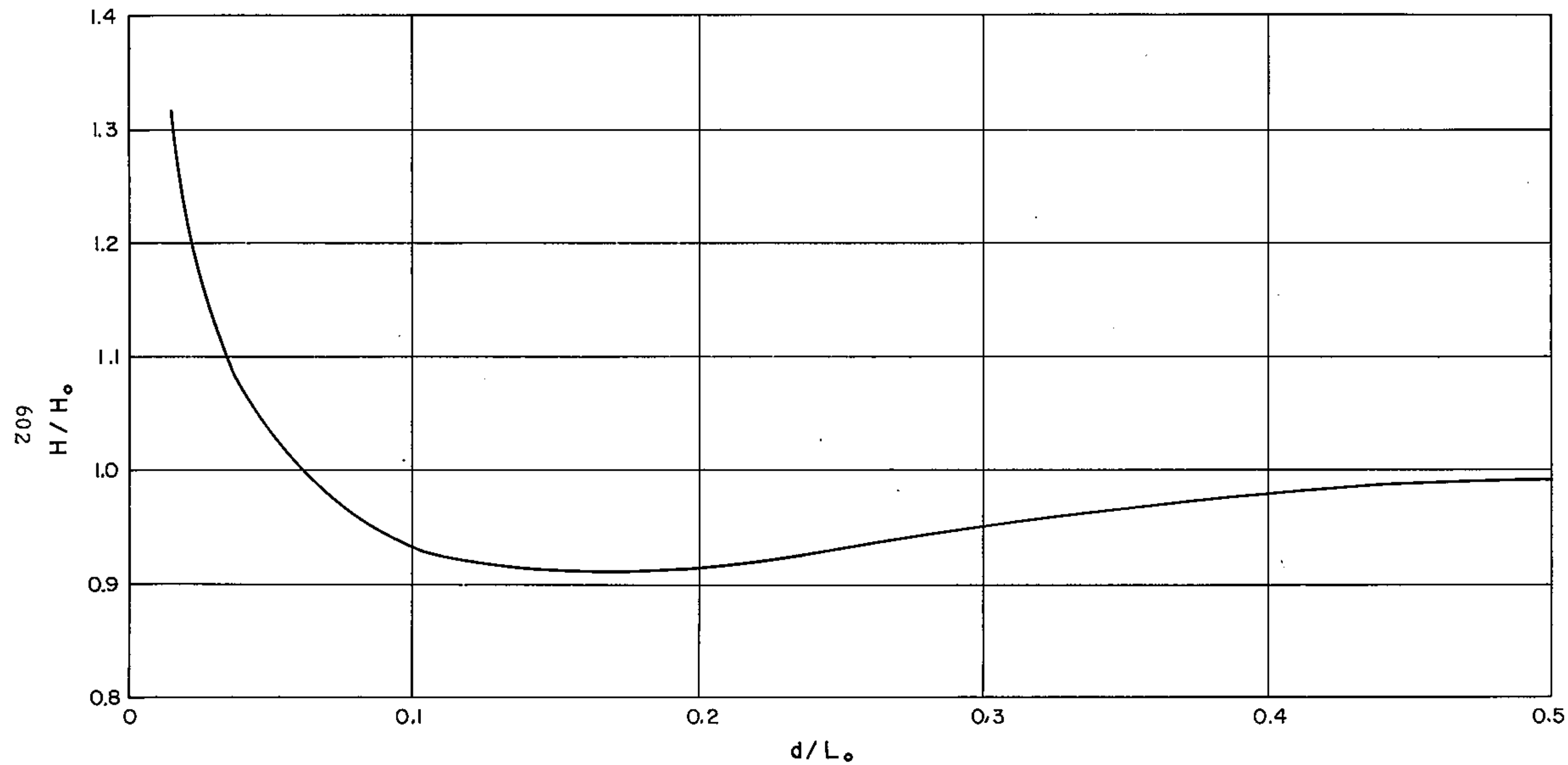


FIGURE XVI-10

RELATIVE WAVE HEIGHT VARIATION WITH DEPTH

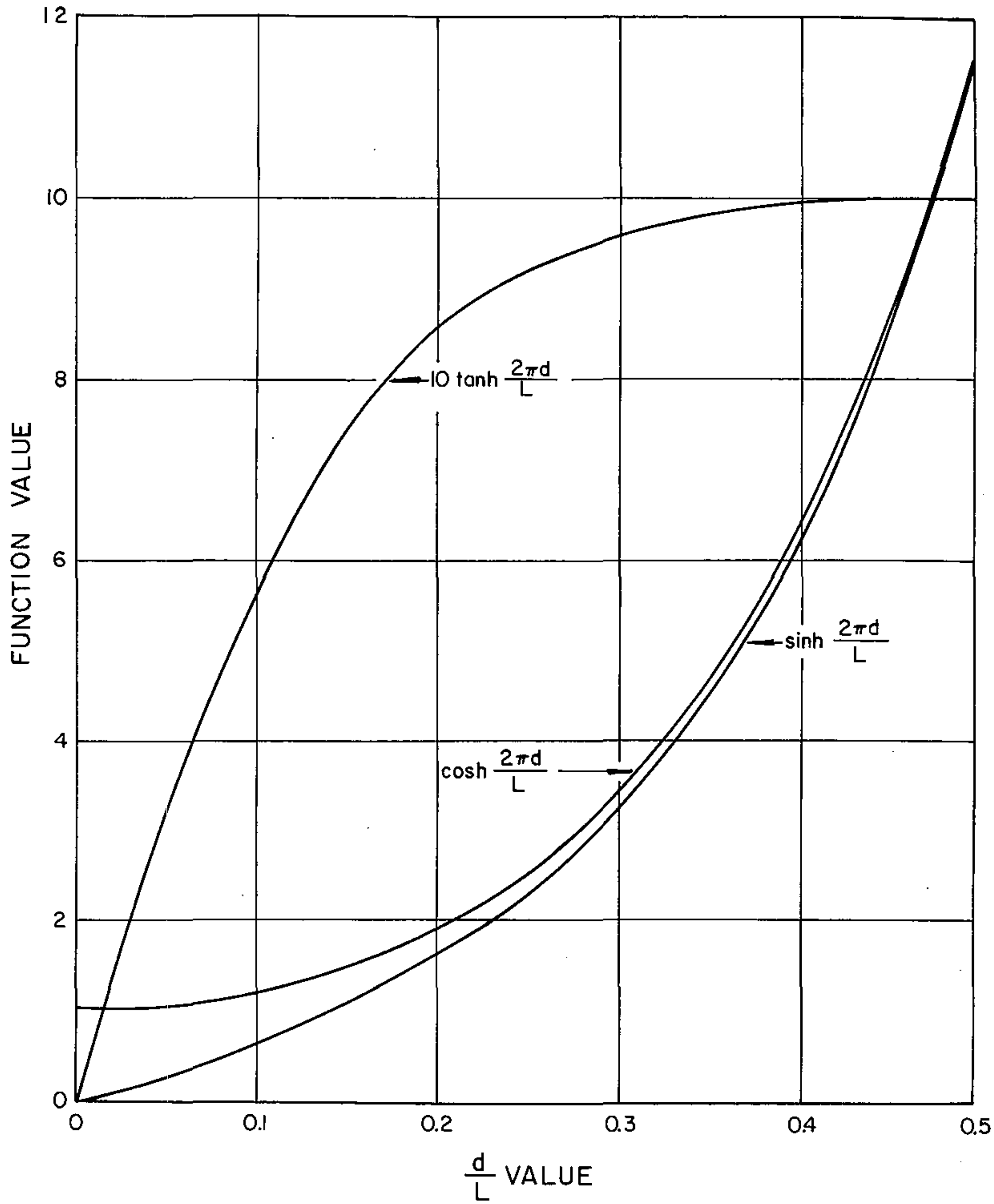


FIGURE XVI-11

HYPERBOLIC FUNCTIONS VS. RELATIVE DEPTH

		SHALLOW WATER $d/L < 0.05$
Wave velocity	$C = [$	$C = \sqrt{gd}$
Wave length	$L = g$	$L = T\sqrt{gd}$
	$k^2 = g$	$k^2 = m^2\, gd$
Group velocity	$U = \frac{C}{2}$	$U = C = \sqrt{gd}$
Potential function	$\varphi = -$	$\varphi = -\frac{H}{2}\frac{gT}{2\pi}\cos(kt - mx)$
	$\varphi = -$	$\varphi = -\frac{H}{2}\frac{g}{k}\cos(kt - mx)$
Velocity components	$u =$	$u = \frac{H}{2}\sqrt{\frac{g}{d}}\sin(kt - mx)$
	$w =$	$w = 0$
Orbits	$x - x_o = -$	$x - x_o = -\frac{H}{2}\frac{T}{2\pi}\sqrt{\frac{g}{d}}\cos(kt - mx_o)$
	$z - z_o =$	$z - z_o = 0$
	Circles of Radius:	Straight lines horizontal amplitude: $\frac{H}{2}\frac{T}{2\pi}\sqrt{\frac{g}{d}}$
Free surface	$\eta =$	$\eta = \frac{H}{2}\sin(kt - mx)$
Pressure	$p =$	$p = -\rho g z + \rho g\frac{H}{2}\sin(kt - mx)$
	$x_o)$	
Energy per wave length per unit of length of crest	$E =$	$E = \frac{1}{2}\rho g L\left(\frac{H}{2}\right)^2,\ E_K = E_P = \frac{1}{2}E$
Average energy flux per unit of length of crest	$\frac{1}{4}\rho g$	$\frac{1}{2}\rho g\left(\frac{H}{2}\right)^2\sqrt{gd}$

		SHALLOW WATER $d/L < 0.05$
Potential function	$\varphi =$	$\varphi = -\frac{H}{2}\frac{g}{k}\cos mx \sin kt$
	$\varphi =$	
Velocity components	$u = -$	$u = -\frac{H}{2}\sqrt{\frac{g}{d}}\ \sin mx \sin kt$
	$w =$	$w = 0$
Orbits	$x - x_o =$	$x - x_o = \frac{H}{2}\frac{T}{2\pi}\sqrt{\frac{g}{d}}\ \sin mx_o \cos kt$
	$z - z_o = -$	$z - z_o = 0$
	Straight lines:	Horizontal lines of amplitude: $\frac{H}{2}\frac{T}{2\pi}\sqrt{\frac{g}{d}}\ \sin mx_o$
Free surface	$\eta = -$	$\eta = -\frac{H}{2}\cos mx \cos kt$
Pressure	$p =$	$p = -\rho g z - \rho g \frac{H}{2}\cos mx \cos kt$
Energy per wave length	$E_K =$	$E_K = \frac{1}{4}\rho g \left(\frac{H}{2}\right)^2 L \sin^2 kt$
	$E_p =$	$E_p = \frac{1}{4}\rho g \left(\frac{H}{2}\right)^2 L \cos^2 kt$
	$E =$	$E = \frac{1}{4}\rho g L \left(\frac{H}{2}\right)^2$

XVI-8 DIFFERENCES BETWEEN WATER WAVES AND UNSTEADY FLOW THROUGH POROUS MEDIA

XVI-8.1 A REVIEW OF THE BASIC ASSUMPTIONS

Before finishing up the study of the small amplitude wave theory, it is particularly interesting to establish a parallel between unsteady flow through porous media and irrotational water waves, and to point out the essential differences.

It is recalled that the study of the average flow through a porous medium generally permits the neglection of all inertial forces, local and convective (see Section IX-2.3). Pressure and body forces always balance friction forces. In the case of water waves, pressure and body forces balance inertial forces, while the friction forces are neglected. As a consequence of this first equality between pressure, body force, and friction forces, a "water wave effect" for a flow through a porous medium is impossible. For example, consider a hump of the free surface elevation $\eta(x, t)$ and $h(x, t)$ with a zero velocity at time $t = 0$ such as shown on Figure XVI-12.

In the case of the water waves, the potential energy is replaced by kinetic energy as $\eta(x, t)$ tends to zero and the kinetic energy is in turn changed into potential energy. The free surface elevation oscillates around the still water level (SWL) and the initial perturbation generates a water wave.

In the case of flow through porous medium, the potential energy is at any time dissipated by the friction forces. The free surface elevation tends slowly toward the still water level. A case of practical interest

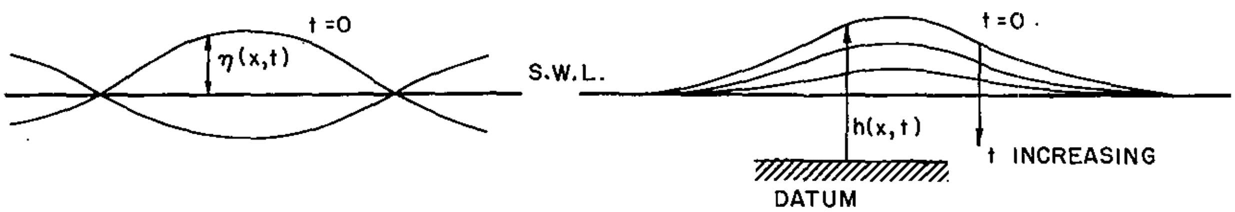


FIGURE XIV-12
FLOW THROUGH POROUS MEDIUM h(x, t) → S. WL.
WATER WAVES: η (x, t) OSCILLATES

where the two phenomena can be observed jointly is when a wave (tidal wave or wind wave) oscillates along a pervious ground or a pervious quay. The water table motion induced by the waves at sea is damped very rapidly with distance.

These facts can, of course, be demonstrated and calculated exactly. A short parallel between the most typical equations for these two kinds of motion is now presented.

XVI-8.2 DYNAMIC CONDITIONS

Both problems consist of defining a potential function ϕ satisfying the Laplace equation $\nabla^2\phi = 0$.

It is recalled (see Section IX-2.6) that the potential function for a flow through porous medium is defined in the case of two-dimensional motion by:

$$u = -K \frac{\partial \phi}{\partial x}$$

$$w = -K \frac{\partial \phi}{\partial z}$$

where the average double bar $\bar{\bar{u}}$, $\bar{\bar{w}}$ is eleminated for the sake of simplicity. It is recalled that under such a condition,

$$\phi = \frac{p}{\rho g} + z$$

In particular, $\phi = h$ at the free surface where $z = h(x,t)$ by definition. Thus the free surface equation is

$$h(x,t) = \phi(x, z, t)\Big|_{z = h}$$

or again

$$h(x,t) = \phi[x, h(x,t), t]$$

Although the word "dynamic" would now be misused, it has to be noted that this equation corresponds to the free surface dynamic condition for water waves:

$$-\frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] + g\eta = f(t)$$

It is seen that if one linearized, the basic difference is that in one case h is proportional to ϕ, and in the other case η is proportional to $\frac{\partial \phi}{\partial t}$

due to local inertia.

XVI-8.3 KINEMATIC CONDITION

It is recalled (see Section XVI-1.4) that the kinematic condition for water waves is

$$\left[\frac{\partial \phi}{\partial z} = -\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}\right]_{z=\eta}$$

A similar equation does exist for flow through porous medium. However, due to the fact that the water fills only the voids, and due to the change of definition of the potential function, this equation has to be slightly modified as a function of the void coefficient ϵ and the coefficient of permeability K as follows.

Let us consider an element $d\sigma$ of the free surface at time t (see Figure XVI-13) and at time $t + dt$. The volume of fluid within ABCD is equal to the discharge through AB times the interval of time dt. On one hand the volume of water in ABCD is $\epsilon\, d\sigma\, dn$, where ϵ is the void coefficient and $dn = \frac{\partial h}{\partial t} dt \cos \alpha$. On the other hand, the fluid discharge through AB during the time dt is

$$[u \sin \alpha\, d\sigma + w \cos \alpha\, d\sigma]\, dt$$

Inserting $\tan \alpha = -\frac{\partial h}{\partial x}$ and replacing u and w by $-K\frac{\partial \phi}{\partial x}$ and $-K\frac{\partial \phi}{\partial z}$ respectively and dividing by $\cos \alpha\, dt$ leads to:

$$\frac{\partial \phi}{\partial z} = -\frac{\epsilon}{K}\frac{\partial h}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial h}{\partial x}$$

which is the kinematic condition for flow through porous media.

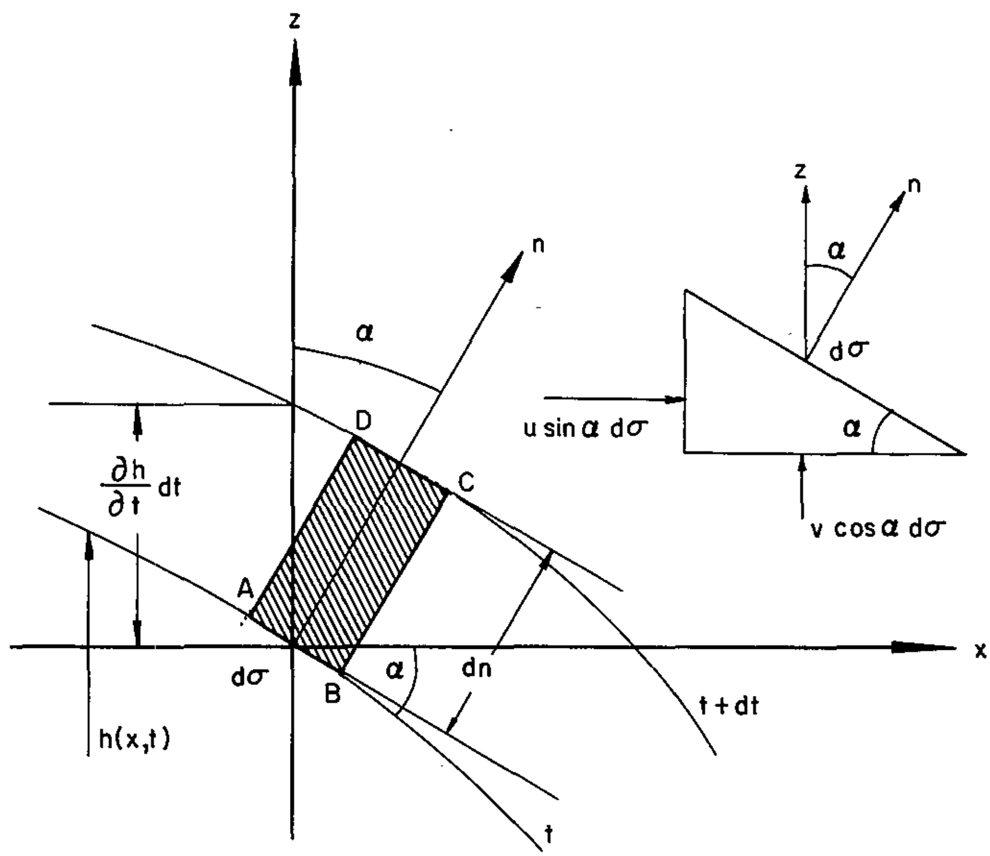


FIGURE XVI-13
NOTATION FOR THE FREE SURFACE EQUATION

Since at the free surface $z = h(x,t) = \phi\,[x,\ h(x,t),\ t]$ one has, in differentiating with respect to t and x successively (* means t or x),

$$\frac{\partial h}{\partial *} = \frac{\partial \phi}{\partial *} + \frac{\partial \phi}{\partial h}\,\frac{\partial h}{\partial *}$$

which can still be written:

$$\frac{\partial \phi}{\partial x} = (1 - \frac{\partial \phi}{\partial z}) \frac{\partial h}{\partial x} = 0$$

and

$$\frac{\partial \phi}{\partial t} = (1 - \frac{\partial \phi}{\partial z}) \frac{\partial h}{\partial t} = 0$$

Inserting these two relationships into the above equation of the previous section permits us to eliminate h which gives

$$\frac{\epsilon}{K} \frac{\partial \phi}{\partial t} = \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2 - \frac{\partial \phi}{\partial z}$$

which can still be written

$$\frac{\epsilon}{K} \frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x} [\phi \frac{\partial \phi}{\partial x}] + \frac{\partial}{\partial z} [\phi \frac{\partial \phi}{\partial z}] - \frac{\partial \phi}{\partial z}$$

since $\nabla^2 \phi = 0$. In the case where the vertical component of motion $\frac{\partial \phi}{\partial z}$ is negligible, and since $\phi = h$

$$\frac{\epsilon}{K} \frac{\partial h}{\partial t} = \frac{\partial}{\partial x} [h \frac{\partial h}{\partial x}]$$

which is the Dupuit approximation. Now, assuming that the variations of h are small with respect to h $(h = d + \eta)$

$$\frac{\epsilon}{Kd} \frac{\partial \eta}{\partial t} = \frac{\partial^2 \eta}{\partial x^2}$$

which is the heat equation.

The following table summarizes the main equations for the formulation of a two-dimensional wave and flow through porous medium

	Irrotational Water Waves	Flow Through Porous Medium
Definition of the Potential Function	$u = -\frac{\partial \phi}{\partial x}$ $w = -\frac{\partial \phi}{\partial z}$	$u = -K \frac{\partial \phi}{\partial x}$ $w = -K \frac{\partial \phi}{\partial z}$
Expression of the Potential Function	$\phi = f(x, z, t)$	$\phi = \frac{p}{\rho g} + z$
Kinematic Condition at the Free Surface	$\frac{\partial \phi}{\partial z} = -\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}$	$\frac{\partial \phi}{\partial z} = -\frac{\epsilon}{K}\frac{\partial h}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial h}{\partial x}$
Dynamic Condition at the Free Surface	$-\frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] + g\eta = f(t)$	$\phi = h$
Kinematic and Dynamic Condition gives	$\frac{\partial^2 \phi}{\partial t^2} + g\frac{\partial \phi}{\partial z} = 0$ (linearized)	$\frac{\epsilon}{K}\frac{\partial \phi}{\partial t} = \left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] - \frac{\partial \phi}{\partial z}$ or $\frac{\epsilon}{K}\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left[h\frac{\partial h}{\partial x}\right]$ (Dupuit approximation)

XVI-8.4 FORM OF SOLUTIONS

It is now possible to substantiate mathematically the fact that a water wave effect is impossible in the case of flow through a porous medium.

In the case of water waves, η is proportional to $\frac{\partial \phi}{\partial t}$ or $\frac{\partial \phi}{\partial z}$ is proportional to $\frac{\partial^2 \phi}{\partial t^2}$, i.e., with

$$\phi = f(x, z)\ e^{i\sigma t}$$

it is seen that $\frac{\partial \phi}{\partial z}$ is proportional to $\sigma^2 \phi$ which is real. The periodic water wave solution does exist.

In the case of flow through a porous medium, a wave effect is impossible since h is proportional to ϕ and $\frac{\partial \phi}{\partial z}$ is proportional to $\frac{\partial \phi}{\partial t}$. It is seen that a solution such as $\phi = f(x, z)\ e^{i\sigma t}$ for wave motion would lead to an imaginary relationship between h and ϕ unless σ is imaginary, i.e., $i\sigma$ is then real, which means that the motion is exponential in time instead of oscillatory.

A previous figure (XVI-12) illustrates these considerations physically.

XVI-1 Determine the streamlines, the equipotential lines, paths, and lines of equipressure (isobars) in a linear periodic progressive wave.

Answer:

Streamlines:

$$\frac{k}{m} \frac{\sinh m(d+z)}{\sinh md} \cos mx = \text{constant}$$

Isobars:

$$z = -a \frac{\cosh m(d+z)}{\cosh md} \cos(kt - mx)$$

XVI-2 Determine the streamlines, the equipotential lines, paths, and lines of equipressure in a linear periodic standing wave.

Answer:

Streamlines:

$$\frac{k}{m} \frac{\sinh m(d+z)}{\sinh md} \sin mx = \text{constant}$$

Isobars:

$$z = -2a \frac{\cosh m(d+z)}{\cosh md} \cos mx \sin kt$$

XVI-3 Demonstrate by applying the linear long wave theory that the free surface of two-dimensional fundamental motion of a seiche in a basin in the form of a parabola $\left[d(x) = d_o \left(1 - \frac{x^2}{a^2}\right)\right]$ is a straight line. Demonstrate that the free surface of the first harmonic motion is a parabola.

Answer:

Fundamental free surface elevation:

$$\eta = \frac{2x}{a^2} \sin \frac{2\pi}{T_1} t$$

Harmonic free surface elevation:

$$\eta = \frac{3x^2 - a^2}{a^3} \sin \frac{2\pi}{T^2} t \qquad \left(\frac{T_1}{T_2} = \sqrt{3}\right)$$

XVI-4 Calculate the value of the wave length L as a function of the water depth d in the case of a wave period T = 8 seconds, 10 seconds, and 12 seconds. Determine the minimum value of $\frac{d}{L}$ when L can be considered as the deep water wave length, and the maximum value of $\frac{d}{L}$ permitting the shallow water approximation within 5% of accuracy.

XVI-5 Demonstrate that the velocity potential function for a linear periodic pregressive wave in deep water is

$$\phi = -a\frac{k}{m} e^{-kz} \cos(kt - mx)$$

and give the expression for the pressure $p(x, z, t)$ and the free surface. Demonstrate why the criterion $\frac{d}{L} \leq 0.5$ is generally considered as the limit of validity of the above expression for ϕ. Explain why the wave refraction diagram must start at $\frac{d}{L} > 0.3$.

XVI-6 Establish the expression for $p(x, z, t)$ for a two-dimensional periodic linear progressive wave in intermediate water depth.

Answer:

$$\frac{p}{\rho g} = -z + \frac{H}{2}\frac{\cosh m(d+z)}{\cosh md} \cos(kt - mx) + \frac{pa}{\rho g}$$

XVI-7 Consider the relationship:

$$k^2 = mg \tanh md$$

and demonstrate the expression:

$$C = \sqrt{\frac{gL}{2\pi} \tanh.\frac{2\pi d}{L}}$$

XVI-8 Demonstrate that the particle path in a two-dimensional periodic linear standing wave is a straight line and sketch the corresponding flow pattern.

Answer:

$$\frac{y - y_o}{x - x_o} = -\tanh m\,(d + z_o)\ \cot m x_o$$

XVI-9 Determine the periods of free oscillation (fundamental and first ten harmonics) of a two-dimensional basin ten feet long and two feet deep. Determine the three longest periods of free oscillation of a rectangular basin (10' x 8') and two feet deep.

XVI-10 Determine the error on the value of C corresponding to the deep water approximation $(C = \frac{gT}{2\pi})$ when $\frac{d}{L} = \frac{1}{3}$. Determine the minimum value of $\frac{d}{L}$ which gives the same error for the shallow water approximation: $C = \sqrt{gd}$. Determine the common value of $\frac{d}{L}$ such that the error for the shallow water approximation equals the error for the deep water approximation.

XVI-11 The pressure $p(t)$ due to a periodic wave traveling in intermediate water depth is recorded at a fixed location on the sea bottom. Determine the operation which permits one to determine the wave height from the knowledge of this pressure fluctuation.

Answer:

$$H = \frac{2\,(p_{max} - \rho\, g\, d)}{\rho\, g \cosh m d}, \qquad \left(\frac{2\pi}{T}\right)^2 = m\, g \tanh m d$$

XVI-12 Establish the value of the pressure on a vertical wall (first approximation) and determine a corresponding method of calculation for determining the stability of a vertical breakwater.

Answer:

The maximum pressure on the vertical wall is approximated by a linear distribution between the elevation $(d+H)$ where the pressure is zero, and the bottom where the pressure is $\rho\, g\, [d + H\,(\cosh \frac{2\pi d}{L})^{-1}]$. The underpressure acting on the vertical breakwater is assumed to be distributed between this latter expression and $\rho\, g\, d$ on the harbor side of the breakwater. The pressure on the harbor side is hydrostatic. The overturning momentum and bottom stress are then determined as in a gravity dam.

XVI-13 It will be assumed that the principle of conservation of transmitted energy between wave orthogonals is valid. Moreover, it can be demonstrated that the wave breaks over a horizontal bottom when the wave steepness

$$\frac{H}{L} \longrightarrow 0.14 \tanh \frac{2\pi d}{L}$$

It will be assumed that this breaking criterion is still valid in the case of a wave breaking at an angle on a gently sloping beach. Now, consider a periodic wave arriving from deep water at an angle α_o with the bottom contours, these bottom contours being straight and parallel and defining a very gently sloping beach. Establish a method for calculating the angle of breaking wave crest α_b with the shoreline, the depth of breaking d_b and the wave height of breaking H_b as a function of α_o, the deep water wave height H_o, and the wave period T.

<u>Answer:</u>

Distance between orthogonals:

$$\frac{b_o}{b_b} = \frac{\cos \alpha_o}{\cos \alpha_b}$$

Energy flux:

$$H_o^2 \, b_o \, V_o = H_b^2 \, b_b \, V_b$$

$$\frac{L_b}{L_o} = \frac{\sin \alpha_b}{\sin \alpha_o} = \tanh \frac{2\pi \, d_b}{L_b}$$

$$\frac{L_b}{L_o} = \frac{L_b}{H_b} \times \frac{H_b}{H_o} \times \frac{H_o}{L_o}$$

Let

$$\alpha = \frac{2\pi\, d_b}{L_b}, \qquad t = \tanh \alpha, \qquad s = \sinh 2\alpha$$

Then

$$0.14\, t^{5/2} \left(1 + \frac{2\alpha}{s}\right)^{1/2} = \frac{H_o}{L_o} \left(\frac{1 - \sin^2 \alpha_o}{1 - t^2 \sin^2 \alpha_o}\right)^{1/4}$$

which gives $\frac{d_b}{L_b}$ as a function of H_o, L_o, and α_o.

So L_b is obtained from $\frac{L_b}{L_o} = t$ and d_b and finally α_b.

CHAPTER XVII

OPEN CHANNEL HYDRAULICS AND LONG WAVE THEORIES

XVII-1 STEADY FLOW IN AN OPEN CHANNEL

XVII-1.1 UNIFORM FLOW, NORMAL DEPTH, AND CRITICAL DEPTH

XVII-1.1.1 Hydraulic Radius and the Chezy Formula

XVII-1.1.1.1

Most of the basic equations of this chapter can be obtained 1) from the Eulerian equations in which a number of terms are taken to be equal to zero and an empirical friction term is included, 2) from the generalized Bernoulli equation in which the condition at the free surface $p = p_a$ (atmospheric pressure) is inserted, or 3) by a direct application of the momentum equation in which the significant terms only are included.

These three kinds of approach are used, but it has to be borne in mind that these mathematical forms all originate from the Newtonian equation, and that a criterion for choosing one rather thar another does not really exist.

XVII-1.1.1.2

Consider a uniform flow parallel to the axis OX (see Figure XVII-1). Since the motion is steady, $\frac{\partial(u, v, w)}{\partial t} = 0$, and

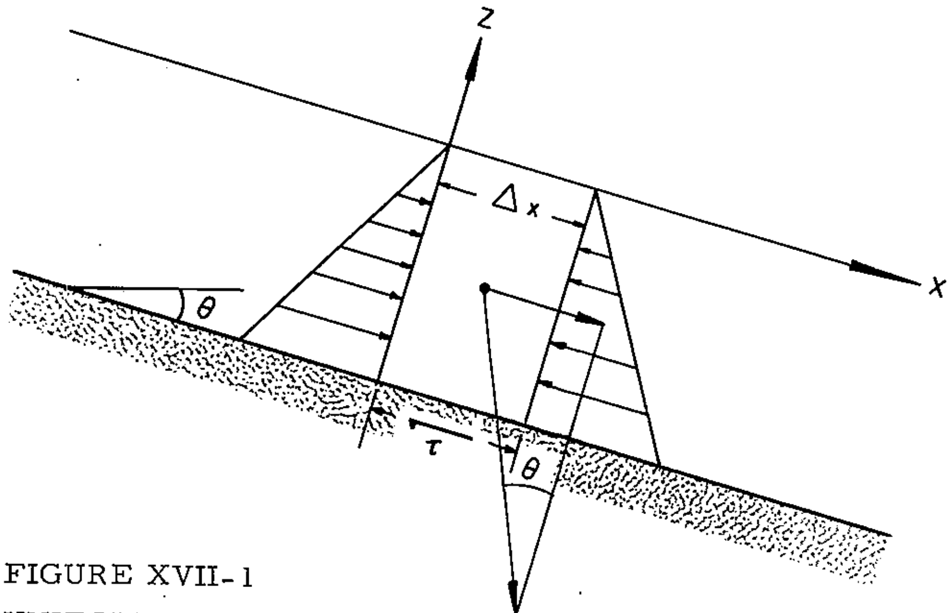


FIGURE XVII-1
UNIFORM FLOW

since the motion is uniform, v and w = 0, and $\frac{\partial u}{\partial x} = 0$. Hence it is easily verified that all the inertial terms are nil. Moreover, the pressure forces acting on each side of cross section A of an element of fluid $A \Delta x$ balance. The OZ components of pressure force and gravity also balance independently of the flow velocity. Thus the only significant forces which remain are the gravity component in the OX direction and the shearing forces:

$$\rho g A \Delta x \sin \theta = \Delta x \int_0^P \tau dP$$

where P is the "wetted perimeter", i.e., the length of the perimeter of the cross section A which is underwater, and τ is the shearing stress per unit area.

XVII-1.1.1.3

In general, τ varies from one place to another, except in circular pipe due to symmetry. However, due to secondary currents, the variation of τ can often be considered as negligible (see Section VIII-2.4).

The previous equation is then written:

$$\rho g \frac{A}{P} \sin \theta = \tau$$

$R_H = \frac{A}{P}$, which has the dimension of a length, is the hydraulic radius. It is easily verified that in the case of a rectangular section of depth h and width ℓ:

$$R_H = \frac{\ell h}{\ell + 2h}$$

and R_H tends to h when $\ell \to \infty$; i.e., in practice in the case of a large river. In the case of a circular pipe of radius R:

$$R_H = \frac{\pi R^2}{2\pi R} = \frac{R}{2}$$

XVII-1.1.1.4

In the case of a river or a channel, the Reynolds number is generally large, such that the flow is fully turbulent. The shearing stress can then be assumed to be related to the average velocity, $V = \frac{1}{A} \iint_A u \, dA$, by a quadratic function such as

$$\tau = \rho f V^2$$

where f is a dimensionless friction factor. Then one obtains

$$V = \left[\frac{g}{f}\right]^{1/2} \left[R_H \sin\theta\right]^{1/2}$$

$\left[\frac{g}{f}\right]^{1/2} = C_h$ is the Chezy coefficient of dimension $\left[LT^{-2}\right]^{1/2}$ and $f = \frac{g}{C_h^2}$. The slope S being generally small,

$$S = \tan\theta \cong \sin\theta$$

so that finally:

$$V = C_h \sqrt{R_H S}$$

This is the Chezy formula.

The discharge $Q_n = VA$ is then

$$Q_n = A C_h \sqrt{R_H S}$$

$K = A C_h \sqrt{R_H}$ is the "conveyance" of the channel and depends upon the geometry of the cross section of the channel and the water depth only. Q_n is the "normal discharge" which is defined as a function of water depth for a given channel.

XVII-1.1.1.5

The Chezy coefficient and f can only be determined by experiment. It is found that

$$C_h = \frac{1.486}{n} R_H^{1/6}$$

where R_H is in feet and n is the Manning coefficient; n is given as a function of relative roughness and in practice varies between 0.01 and 0.03. Inserting this expression in the Chezy formula gives the Manning formula

$$V = \frac{1.486}{n} R_H^{2/3} S^{1/2}$$

and the conveyance

$$K = \frac{1.486}{n} A R_H^{2/3}$$

XVII-1.1.2 Normal Depth and Transitional Depth

XVII-1.1.2.1

The normal depth h_n is defined as the distance between the lowest part of the channel and the free surface of a uniform flow. It is determined by the equality

$$\frac{Q}{\sqrt{S}} = K(h_n)$$

where $K(h_n)$ is the function characterizing the conveyance of the channel. In the case of a wide rectangular channel, since $R_H = h_n$, the normal depth h_n is:

$$h_n = \left[\frac{q}{C_h \sqrt{S}}\right]^{2/3}$$

where q is the discharge per unit width. It is seen that (for a given channel) to a given depth there corresponds a unique discharge, the normal discharge, and to a given discharge there corresponds a unique depth, the normal depth.

XVII-1.1.2.2

In the case of a nonuniform flow, the term $\frac{d}{dx}(\frac{V^2}{2})$ is no longer zero; hence gravity force and friction force do not balance exactly. The water depth h is different from the normal depth. h is then a "transitional depth" which varies with distance and can be larger or smaller than h_n. For example, the transitional depth upstream of a dam is larger than the normal depth.

It is assumed that the values or formulae for the friction coefficients f, C_h, and n obtained in the case of a steady uniform flow with normal depth are still valid in the case of a nonuniform flow with a transitional water depth. In general, the lack of accurate information on the value of the friction coefficient makes this approximation compatible with the inherent error due to this approximation. Consequently, the conveyance K is a general function of h which can also be used for nonuniform flow.

XVII-1.2 SPECIFIC ENERGY, SPECIFIC FORCE, AND CRITICAL DEPTH

XVII-1.2.1 Definition of the Specific Energy and Specific Force

The quantity

$$E = \frac{(1 + \alpha)V^2}{2g} + h = \frac{(1 + \alpha)Q^2}{2g\,A^2} + h$$

where V is the average velocity, h the maximum water depth, Q the discharge, and A the cross section, is called the specific energy. α is a coefficient > 0 which, for the sake of simplicity, will be neglected in the following (see Section XII-2.2). It is seen that the specific energy is the sum of the kinetic energy ($\frac{V^2}{2g}$) and the potential energy per unit of weight with respect to the bottom of the channel h (but not with respect to a horizontal datum)(see Figure XVII-2).

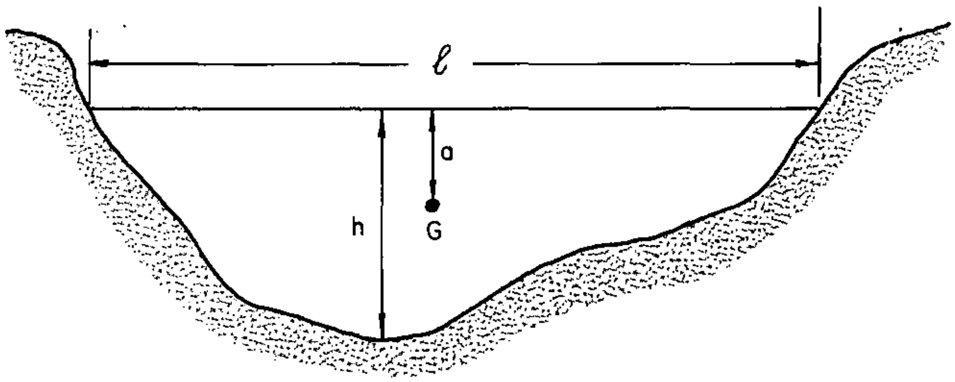


FIGURE XVII-2
NOTATION FOR AN OPEN CHANNEL

The specific force is

$$I = \iint_A \left[\frac{V^2}{g} + \frac{p}{\rho g}\right] dA = \frac{Q^2}{gA} + aA$$

where a is the distance between the center of gravity of the cross section A and the free surface. It is recalled that the specific force is the sum of the momentum per unit time and per unit weight and the integral of pressure force per unit weight of water.

In the case of a two-dimensional channel (rectangular cross section), $A = h\ell$ and $a = h/2$ and $q = \dot{Q}/\ell$ so that E and I per unit width become

$$E = \frac{q^2}{2gh^2} + h$$

$$I = \frac{q^2}{gh} + \frac{h^2}{2}$$

respectively. Writing

$$h^* = \frac{h}{E}, \quad q^* = \frac{q}{E\sqrt{2gE}}$$

one obtains the equation

$$q^* = h^*\sqrt{1 - h^*}$$

which is universal.

XVII-1.2.2 Definition of the Critical Depth

XVII-1.2.2.1 The Case of A Rectangular Channel

Let us consider the two functions $E(q, h)$ and $I(q, h)$ in the case of a rectangular channel. It is seen that for a given value q, $E(h)$, and $I(h)$ vary as shown on Figure XVII-3a and b.

In particular, the minimum values for E and I given by $\frac{\partial E}{\partial h} = 0$ and $\frac{\partial I}{\partial h} = 0$ are obtained for the same value of $h = h_c$, when

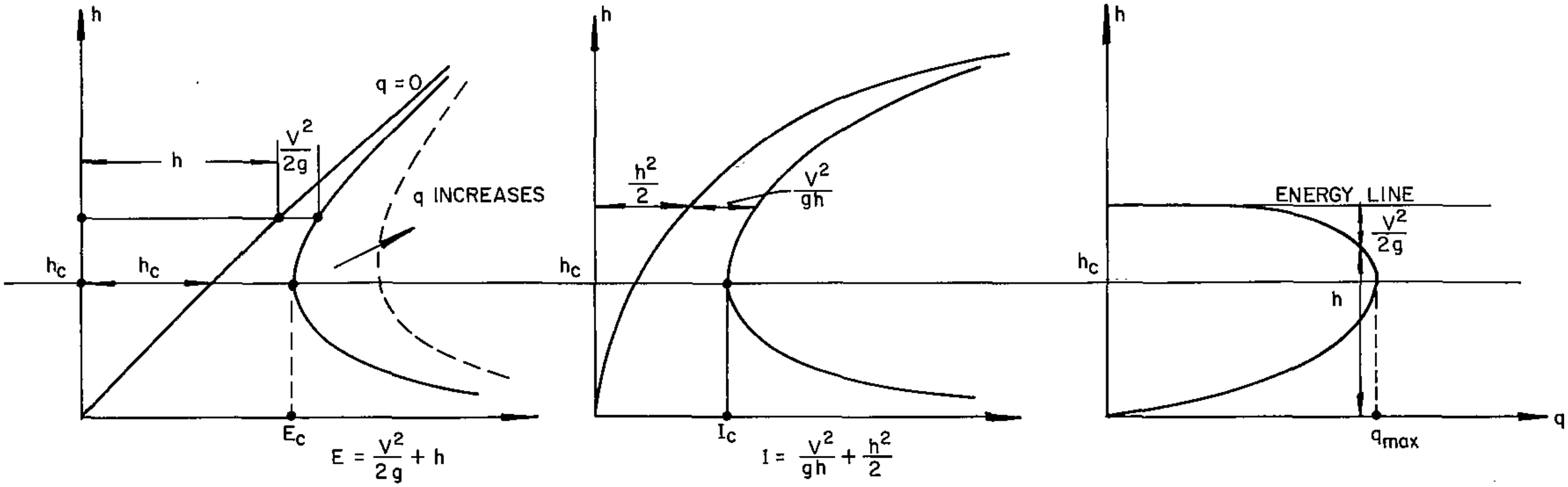


FIGURE XVII-3
SPECIFIC ENERGY AND SPECIFIC FORCE

$$\frac{q^2}{gh_c^3} = 1$$

i. e.,

$$h_c = \left[\frac{q^2}{g}\right]^{1/3} = \frac{V^2}{g} \text{ , or } \quad V = \sqrt{gh_c}$$

in which case

$$E = E_c = \frac{3}{2} h_c$$

$$I = I_c = \frac{3}{2} h_c^2$$

XVII-1.2.2.3

Similarly, the function q(h) for a given value E = constant varies as it is shown on Figure XVII-3c. It is easily verified that the minimum value for q given by $\frac{\partial q}{\partial h} = 0$, is also found to be for $h = h_c$.

XVII-1.2.2.4

In the case of a complex cross section A(h) the critical depth is defined by

$$\frac{\partial E}{\partial h} = 1 - \frac{2}{A^3} \frac{Q^2}{2g} \frac{\partial A}{\partial h} = 0$$

and since $\frac{\partial A}{\partial h} = \ell$, $\ell(h)$ being the width of the river at the free surface, one has

$$\frac{Q^2}{g} = \frac{A^3}{\ell}$$

A and ℓ being functions of h only, it is then possible to establish the curve h_c as a function of the critical discharge Q_c for any kind of cross section (see Figure XVII-4). It is easily verified that in the case of a rectangular cross section, the previous value h_c is found.

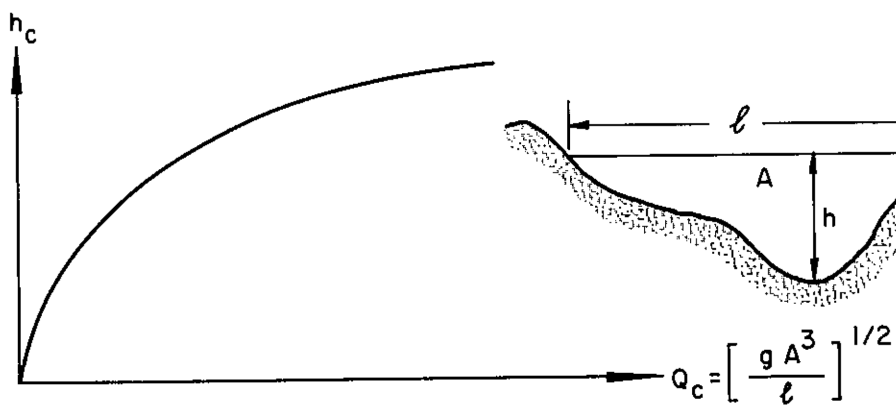


FIGURE XVII-4
VARIATION OF THE CRITICAL DEPTH WITH DISCHARGE

XVII-1.3 TRANQUIL FLOW AND RAPID FLOW

XVII-1..3.1 Conjugate Depths

There exist two possible values, h_1 and h_2, for the same values of E and q, defined by

$$E = h_1 + \frac{V_1^2}{2g} = h_2 + \frac{V_2^2}{2g}$$

and

$$q = h_1 V_1 = h_2 V_2$$

such as

$$\frac{q^2}{2g} \cdot \frac{h_1^2 - h_2^2}{h_1^2 h_2^2} = h_1 - h_2$$

and inserting the value for $h_c = \left[\frac{q^2}{g}\right]^{1/3}$ one obtains

$$\frac{2h_1^2 h_2^2}{h_2 + h_1} = h_c^3$$

h_1 and h_2 are called "conjugate" depths. The larger one, say h, corresponds to a "subcritical" or "tranquil flow" in which case $V < \sqrt{gh}$. The smaller one, say h_2, corresponds to a "supercritical" or "rapid flow" in which case $V > \sqrt{gh}$.

XVII-1.3.2 Critical Slope

It has been seen that the normal discharge is a function of the conveyance and the bottom slope: $Q_n = K\sqrt{S}$, while the critical discharge is a function of the cross section and the width of the channel only: $Q_c = \left[\frac{g A^3}{\ell}\right]^{1/3}$.

There exists a slope, the "critical slope", for which the normal discharge, i.e., the discharge of a uniform flow, is equal to the critical discharge, and consequently the normal depth h_n is equal to the critical depth h_c.

Consequently, the critical slope is defined by

$$AC_h \sqrt{R_H S_c} = \left[\frac{gA^3}{\ell}\right]^{1/3}$$

i.e.,

$$S_c = \frac{g}{C_h^2} \frac{A}{R_H \ell}$$

and in the case of a large rectangular channel

$$S_c = \frac{g}{C_h^2} = f$$

When $S > S_c$, one has $h_n < h_c$ corresponding to a value of the Froude number $\frac{V}{\sqrt{gh}} < 1$: the flow is subcritical or tranquil. When $S < S_c$, one has $h_n > h_c$ corresponding to a value of the Froude number $\frac{V}{\sqrt{gh}} > 1$, and the flow is supercritical or rapid.

XVII-1.3.3 Free Surface Disturbance and Flow Control

Since any disturbance travels at a speed $\sqrt{gh}$, at the greatest, the occurrence of $V > \sqrt{gh}$ means that any kind of perturbation cannot travel in an upstream direction. Also when the flow characteristics change from a tranquil upstream flow to a rapid downstream flow like at the top section of a weir, the discharge is controlled at this critical cross section.

XVII-2 GRADUALLY VARIED FLOW

XVII-2.1 BASIC EQUATIONS FOR GRADUALLY VARIED FLOW

XVII-2.1.1

One can apply the generalized Bernoulli equation with friction to a gently sloped free surface flow as follows. It is recalled that the generalized Bernoulli equation can be written

$$\frac{d}{dx}\left[\frac{V^2}{2g} + \frac{p}{\rho g} + z\right] = -\frac{\tau}{\rho g R_H}$$

i.e.,

$$\frac{d}{dx}\left[\frac{Q^2}{2gA^2} + \frac{p}{\rho g} + z\right] = -\frac{Q^2}{C_h^2 R_H A^2}$$

where

$$\frac{Q^2}{C_h^2 R_H A^2} = \frac{Q^2}{K^2} = S_f$$

S_f is the slope of the energy line or head loss. Then, inserting $p=p_a$ (atmospheric pressure) as a constant and calling z the elevation of the free surface with respect to a horizontal datum, since (see Figure XVII-5)

$$\frac{dz}{dx} = -S + \frac{dh}{dx}$$

yields:

$$\frac{d}{dx}\left[\frac{Q^2}{2g A^2} + h\right] = S - \frac{Q^2}{K^2}$$

The quantity between brackets is recognized to be the specific energy.

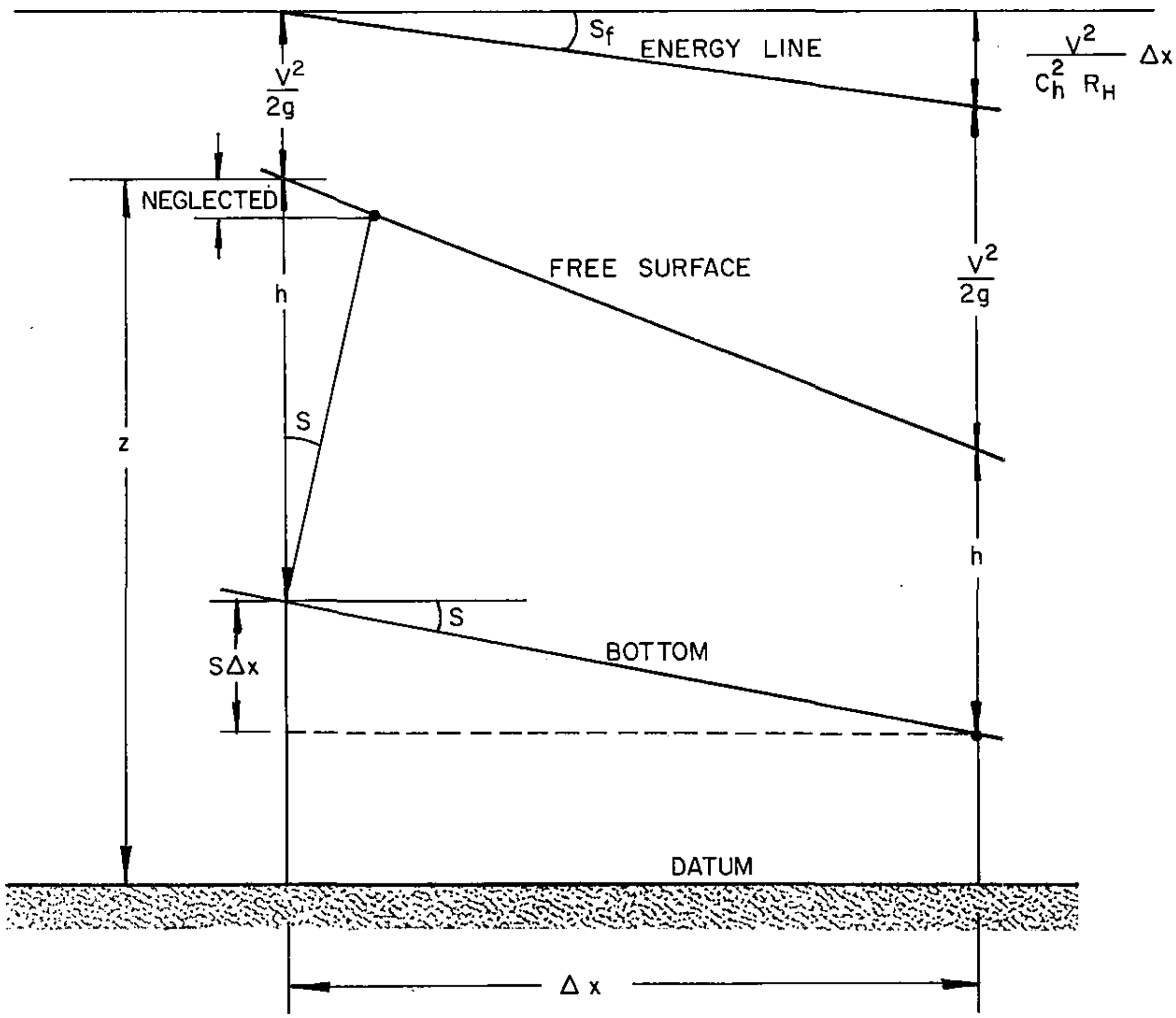


FIGURE XVII-5

NOTATION FOR GRADUALLY VARIED FLOW

XVII-2.1.2

Since

$$\frac{d}{dx}\left[\frac{Q^2}{2g\,A^2}\right] = -\frac{Q^2}{g}\,\frac{L}{A^3}\,\frac{dh}{dx}$$

one has finally,

$$\frac{dh}{dx} = S\,\frac{1 - \dfrac{Q^2}{K^2 S}}{1 - \dfrac{Q^2 L}{g\,A^3}}$$

which is the fundamental equation for gradually varied flow. Considering that $\frac{Q^2}{K^2 S} = \frac{S_f}{S}$ and $\frac{Q^2 L}{g\,A^3} = \frac{C_h^2\,R_H S_f L}{gA} = \frac{S_f}{S_c}$ this equation can be written

$$\frac{dh}{dx} = S\,\frac{1 - \dfrac{S_f}{S}}{1 - \dfrac{S_f}{S_c}}$$

or again in the case of a large rectangular channel ($R_H = h$) one obtains after some elementary operations,

$$\frac{dh}{dx} = S\,\frac{h_n^3 - h^3}{h_c^3 - h^3}$$

XVII-2.2 BACKWATER CURVES

XVII-2.2.1

In general, the free surface variation depends upon: 1) the sign of S; 2) the sign of the numerator, depending upon the water depth h by comparison with the normal depth h_n; 3) the sign of the denominator, depending upon the water depth h by comparison with the critical depth h_c.

It is to be noticed that $h > h_n$ or $h < h_n$ may involve $h > h_c$ or $h < h_c$ depending whether $h_n > h_c$ or $h_n < h_c$. In practice, the first calculation will consist of comparing the value of h_n and h_c.

For example, let us consider the case of a flow in a channel with a positive slope: $S > 0$, in which one finds $h_n > h_c$. It is seen that if $h_c < h_n$, the numerator and denominator are both negative, so $\frac{dh}{dx}$ is positive. The water depth continuously increases with distance. It is the case of a backwater curve due to a dam on a river with a gentle slope (see Figure XVII-6, case M_1).

The following table and Figure XVII-6 illustrate all the cases which may be encountered.

Area A corresponds to the case where the transitional water depth is larger than both the normal depth h_n and the critical depth h_c. Area B corresponds to the case where the transitional water depth is between h_n and h_c. Area C corresponds to the case where the transitional water depth is smaller than both h_n and h_c. B (backwater) means that the water depth tends to increase in the direction of the flow. D (drawdown) means that the water depth tends to decrease in the direction of the flow. U (uniform) means that the water depth is constant.

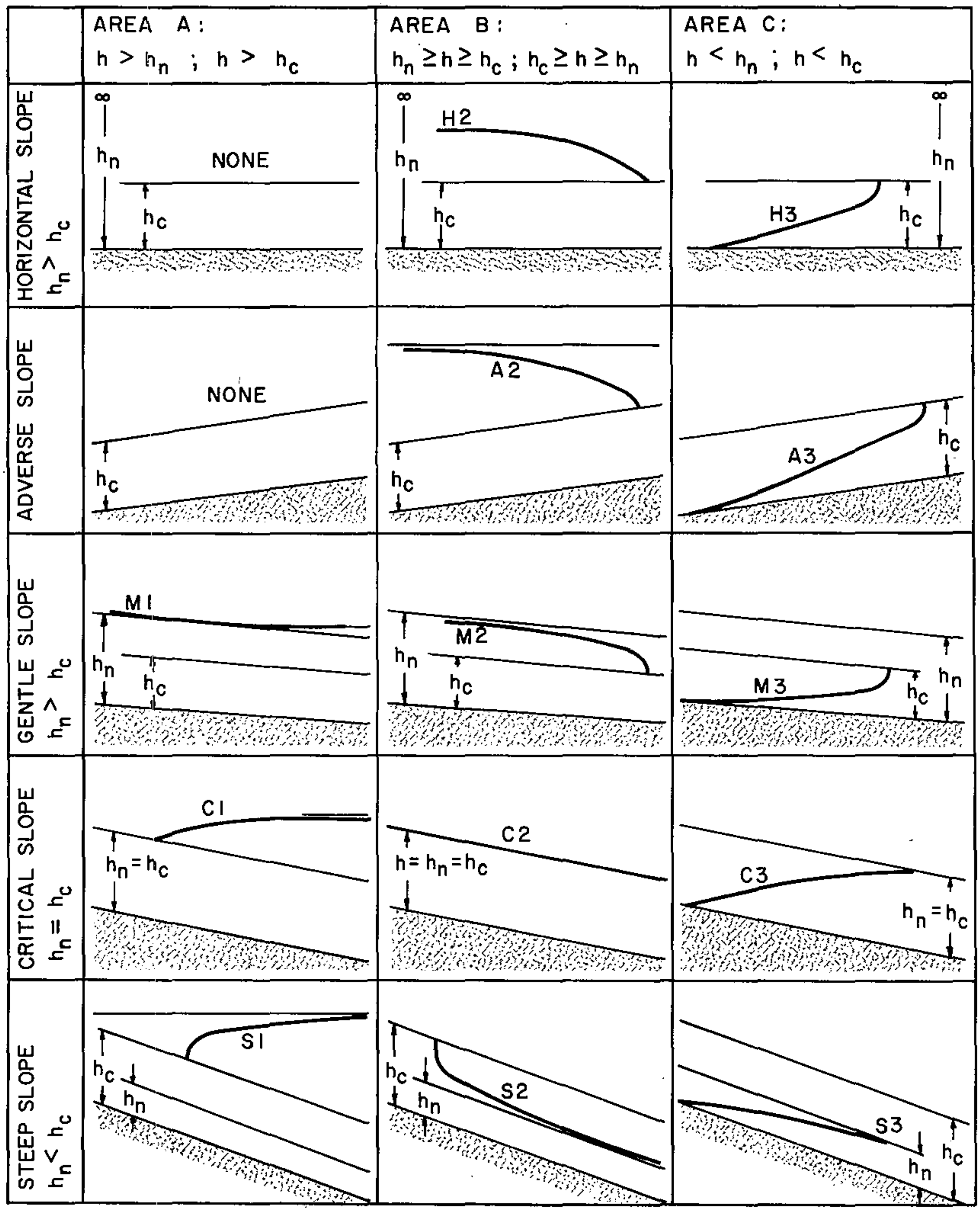


FIGURE XVII-6

DIFFERENT KINDS OF BACKWATER CURVES

Bottom Slopes	Areas			Water Depths	Water Depth Variations
	A	B	C		
Adverse $S < 0$	--			$h > \lvert h_n \rvert > h_c$	--
		A_2		$\lvert h_n \rvert > h > h_c$	D
			A_3	$\lvert h_n \rvert > h_c > h$	B
Horizontal $S = 0$	--			$h > h_n > h_c$	--
		H_2		$h_n > h > h_c$	D
			H_3	$h_n > h_c > h$	B
Gentle $0 < S < S_c$	M_1			$h > h_n > h_c$	B
		M_2		$h_n > h > h_c$	D
			M_3	$h_n > h_c > h$	B
Critical $S = S_c$	C_1			$h > h_c = h_n$	B
		C_2		$h_c = h = h_n$	U
			C_3	$h_c = h_n > h$	B
Steep $S > S_c > 0$	S_1			$h > h_c > h_n$	B
		S_2		$h_c > h > h_n$	D
			S_3	$h_c > h_n > h$	B

XVII-2.2.2

Further investigations on backwater curves will not be treated in this book. All the other cases can be found systematically by a simple analysis of the basic equation. Similarly, the backwater curves on channels of variable slope, continuous or discontinuous, can be analyzed and calculated.

XVII-2.2.3

In engineering practice, the calculation of backwater curves can be done either by an exact integration in some simple cases like in the case of a rectangular channel (it is the Bresse formula), or most often numerically by application of a finite difference method over a succession of intervals Δx. Also, a number of approximate methods and graphical methods do exist. A number of tables and graphs can be found in the technical literature for a dimensionless "channel unity". As usual, the most direct approach based on a finite difference method is the most practical and can be easily programmed for high speed computers.

XVII-2.3 RAPIDLY VARIED FLOW - EFFECT OF CURVATURE

In the case of rapidly varied flow, the pressure distribution is no longer hydrostatic. The path curvature has a nonnegligible influence on the flow behavior. Also a uniform flow near critical depth is very unstable. The flow motion may then enter the category of flow motion with path curvature.

Let us call R the radius of curvature of the free surface, such that

$$\frac{1}{R} = \frac{d^2h/dx^2}{\left[1 + (dh/dx)^2\right]^{3/2}} \cong \frac{d^2h}{dx^2}$$

Let us assume that the curvature is linearly distributed from the bottom to the free surface, such that

$$\frac{1}{R(z)} = \frac{h + z}{h} \frac{d^2h}{dx^2}$$

The centrifugal acceleration is V^2/ρ where V is the average velocity and the dynamic equation along a vertical becomes:

$$\frac{\partial p}{\partial z} = -\rho g + \rho V^2 \left[\frac{h + z}{h} \frac{d^2h}{dx^2}\right]$$

which integrated is:

$$p(z) = -\rho g z + \rho \frac{V^2}{h} \frac{d^2h}{dx^2} \left[hz + \frac{z^2}{2}\right]$$

Boussinesq has made an extensive use of this equation for determining the condition when a flow motion is stable or may become undulated, or when it results in an undulated hydraulic jump.

Also, the study of flow over a rapidly variable slope such as over a weir has been investigated by taking flow curvature into account. This subject will not be further developed in this book.

XVII-3 THE LONG WAVE THEORY; ASSUMPTIONS AND BASIC EQUATIONS

XVII-3.1 MOMENTUM EQUATIONS

XVII-3.1.1

The momentum equations of the long wave theory can also be established directly, or by application of the generalized Bernoulli equation, or again from the Eulerian equations. This last approach will be used.

Let us consider the Eulerian equations in a two-dimensional (X, Z) system of coordinates. The motion is now unsteady, but it will be assumed that the vertical component of the motion is small and that consequently the convective acceleration terms $u\frac{\partial w}{\partial z}$, $u\frac{\partial w}{\partial x}$, $w\frac{\partial w}{\partial z}$ are negligible. In the case where w is very small, $\frac{\partial w}{\partial t}$ will also be neglected in such a way that the Eulerian equation is reduced to:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{\partial p/\rho}{\partial x}$$

$$0 = \frac{\partial p/\rho}{\partial z} + g$$

It is to be noticed that the term $u\frac{\partial u}{\partial x}$ is now not neglected as in the linear wave theory.

Integrating the second equation yields (see Figure XVI-7):

$$p = \rho g\,(-z + \eta)$$

and at the bottom ($z = -d$)

$$p = \rho g\,(d + \eta)$$

The pressure distribution is hydrostatic.

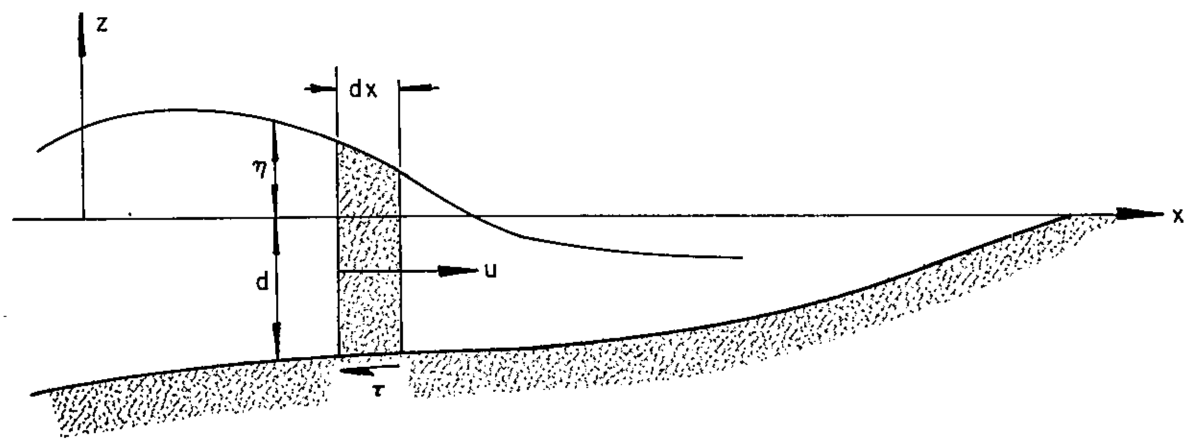


FIGURE XVII-7

NOTATION FOR LONG WAVE THEORY

XVII-3.1.2

u(z) is assumed to be constant along a vertical or more specifically is the average velocity over a vertical

$$\overline{u} = \frac{1}{\eta + d}\int_{-d}^{\eta} u(z)\,dz$$

In the following, the average sign is deleted. The average value $u\frac{\partial u}{\partial x}$ expressed in terms of the average velocity u should theoretically be corrected by a coefficient $(1 + \alpha) > 1$, taking into account the nonuniform velocity distribution. This coefficient is similar to the coefficient α obtained in the generalization of the Bernoulli equation (see Section XII-2.2). This correction factor will also be neglected for the sake of simplicity.

XVII-3.1.3

Differentiating the value of $p = \rho g(-z + \eta)$ with respect to x gives

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial \eta}{\partial x}$$

in such a way that the momentum equation along an OX axis becomes simply:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial \eta}{\partial x}$$

XVII-3.1.4

Since the wave motion at the bottom is far from being negligible, the shearing force due to friction takes in practice a relatively great importance in the long wave theory, so that the long wave equation should rather be written:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial \eta}{\partial x} - \frac{\tau}{\rho\,(d + \eta)}$$

In general, τ will be assumed to be quadratic, i.e., $\tau = \rho f u^2$. Also, the coefficient f is assumed to be independent of the variation of u with respect to time. The coefficient f is assumed to be the same as if the motion were steady. Consequently, f can be expressed in terms of the Chezy coefficient C_h: $f = \frac{g}{C_h^2}$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = - g \frac{\partial \eta}{\partial x} - \frac{g}{C_h^2} \quad \frac{u|u|}{(d + \eta)}$$

It has been seen that the friction coefficients such as f, C_h, or n are used for gradually varied flow. Most often, they are also considered valid for unsteady motion, such as due to flood waves or tidal motion in an estuary.

However, one must realize that the practical use of such empirical formulation is only due to the lack of further information on the problem of turbulent inception and friction factor for unsteady motion. This field is still in a phase of research.

One only knows that the application of the Chezy coefficient to unsteady motion is valid provided the velocity distribution is influenced by the friction up to the free surface as in the case of a steady flow. This fact occurs when the expression for the boundary layer thickness which increases with the wave period becomes much larger than the factual depth h. Still, density effects in a marine estuary have a non-negligible influence on the distribution of friction forces. They can only be known by analyzing field records.

XVII-3.2 EQUATION OF BARRÉ DE ST. VENANT

Adding to the above wave equation, the zero quantity

$$\left(g \frac{d(d)}{dx} \pm gS\right)$$

where S is the bottom slope, which can be positive or negative depending upon the motion direction with respect to the x axis direction, and replacing $(d + \eta)$ by the total depth h and u by V gives:

$$\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} + g \frac{\partial h}{\partial x} = \mp gS - \frac{g}{C_h^2} \frac{V|V|}{h}$$

Such an equation is known as the equation of Barré de St. Venant.

Due to its importance, for the study of tidal estuary, it is useful to present a special demonstration of this formula aside of introducing it as a special case of the Eulerian equation or a special form of the generalized Bernoulli equation.

Consider a layer of fluid of thickness dx of cross section h and mass $\rho h dx$. The average velocity being $V(x, t)$, the inertial forces are

$$\rho h \, dx \frac{dV(x,t)}{dt} = \rho h \, dx \left(\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x}\right)$$

These inertia forces are balanced by the forces of pressure, gravity and friction. The pressure is assumed to be hydrostatic, hence the resultant of pressure force is

$$\rho g \frac{h^2}{2} - \rho \frac{g}{2} \left[h + \frac{\partial h}{\partial x} dx\right]^2 = -\rho g h \frac{\partial h}{\partial x} dx$$

where

$$\frac{dh}{dx} = \frac{d(d)}{dx} + \frac{\partial \eta}{\partial x} = -S + \frac{\partial \eta}{\partial x}$$

The gravity component in the direction of the motion is: $-gh\,dx\,S$ and the friction force is $\rho gh \dfrac{V|V|}{C_h^2 R_H} dx$ where the hydraulic radius $R_H = h$. Adding these expressions and dividing by $\rho h\,dx$, the equation of Barré de St. Venant is found.

XVII-3.3 THE INSERTION OF PATH CURVATURES EFFECT - BOUSSINESQ EQUATION

Let us again consider the Eulerian equation along a vertical axis

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

in which one assumes that $\frac{dw}{dt} \cong \frac{\partial w}{\partial t}$, i.e., where the nonlinear terms $u\frac{\partial w}{\partial z}$ and $w\frac{\partial w}{\partial z}$ are neglected, but that the linear term $\frac{\partial w}{\partial t}$ is nonnegligible. Then the pressure distribution is no longer hydrostatic. The vertical acceleration, $\frac{\partial w}{\partial t}$ causing path curvatures, is going to modify this simple approximation as follows. On a horizontal bottom, the vertical component of velocity at the bottom w_b is nil. At the free surface w is equal to the velocity of the free surface itself $\frac{d\eta}{dt} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x}$ (see Section XVI-4.1). The nonlinear term $u\frac{\partial \eta}{\partial x}$ will also be neglected in such a way that $w_s \cong \frac{\partial \eta}{\partial t}$. Consequently w(z) increases from $w_b = 0$ to $w_s = \frac{\partial \eta}{\partial t}$. In the most general case, one can always say that w(z) is given by a power series such as:

$$w(z) = \frac{\partial \eta}{\partial t}\sum^{n} A_n\left[\frac{z}{d+\eta}\right]^n$$

where $A_1 = 1$ (z = 0 at the bottom). Considering the first term of this series only (n = 1),

$$w(z) = \frac{z}{d+\eta}\,\frac{\partial \eta}{\partial t}$$

which can be read as the assumption that w is linearly distributed from the bottom to the free surface. Moreover, assuming η to be small in comparison with d ,

$$\frac{\partial w}{\partial t} = \frac{z}{d + \eta} \quad \frac{\partial^2 \eta}{\partial t^2}$$

Inserting this value in the Eulerian equation where the nonlinear terms are neglected gives

$$\frac{z}{d + \eta} \quad \frac{\partial^2 \eta}{\partial t^2} = - \frac{\partial}{\partial z} \left(\frac{p}{\rho} + gz\right)$$

and integrating with respect to the vertical direction between a point z and the free surface at a distance $(d + \eta)$ from the origin gives

$$\frac{p(z)}{\rho} = g \left[d + \eta - z\right] + \int_z^{d+\eta} \frac{z}{d + \eta} \quad \frac{\partial^2 \eta}{\partial t^2} \, dz$$

i. e.,

$$\frac{p(z)}{\rho} = g \left[d + \eta - z\right] + \frac{\partial^2 \eta}{\partial t^2} \quad \frac{(d + \eta)^2 - z^2}{2 (d + \eta)}$$

The first term is the hydrostatic pressure, the second term is the correction due to vertical acceleration. Differentiating with respect to x and inserting $\frac{\partial p}{\partial x}$ in the Eulerian equation along a horizontal axis gives, by neglecting some of the small terms,

$$\frac{du}{dt} = -g\frac{\partial\eta}{\partial x} - \frac{\partial^3\eta}{\partial t^2\partial x}\left[\frac{(d+\eta)^2 - z^2}{2(d+\eta)}\right]$$

Now, averaging with respect to the vertical and neglecting some terms due to the fact that η is small by comparison with d gives,

$$\frac{d\bar{u}}{dt} = \frac{1}{d+\eta}\int_{d+\eta}^{0}\left[g\frac{\partial\eta}{\partial x} + \frac{\partial^3\eta}{\partial t^2\partial x}\frac{(d+\eta)^2 - z^2}{2(d+\eta)}\right] dz$$

i.e., integrating and developing $\frac{d\bar{u}}{dt}$ leads finally to:

$$\frac{\partial\bar{u}}{\partial t} + \bar{u}\frac{\partial\bar{u}}{\partial x} + g\frac{\partial\eta}{\partial x} + \frac{d+\eta}{3}\frac{\partial^3\eta}{\partial t^2\partial x} = 0$$

The first three terms of the long wave equation previously presented are recognized. The last term is an approximate correction due to flow curvature. Such an equation is known as the Boussinesq equation.

XVII-3.4 THE CONTINUITY EQUATION

XVII-3.4.1

It has been demonstrated (see Section III-3.1) that the continuity equation in the case of a two-dimensional long wave is:

$$\frac{\partial\eta}{\partial t} + \frac{\partial[u(d+\eta)]}{\partial x} = 0$$

Similarly, in the case of a three-dimensional long wave, the continuity equation becomes:

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left[u\,(d+\eta)\right] + \frac{\partial}{\partial y}\left[v\,(d+\eta)\right] = 0$$

XVII-3.4.2

In a river or an estuary with gentle variation of width ℓ, the continuity equation is

$$\frac{\partial A}{\partial t} + \frac{\partial (AV)}{\partial x} = 0$$

where A is the cross section: $h\,\ell$. Developing this equation and neglecting $\frac{\partial \ell}{\partial t}$ i.e., $\frac{\partial \ell}{\partial x} \cong \frac{d\ell}{dx}$, gives

$$\frac{\partial \eta}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{hu}{\ell}\,\frac{d\ell}{dx} = 0$$

The first two terms are easily recognized, and the last term is the correction due to a slight change of width $\frac{d\ell}{dx}$.

XVII-4 THE LINEAR LONG WAVE THEORY

XVII-4.1 BASIC ASSUMPTIONS

If one neglects the convective inertia and friction terms in the momentum equation, one obtains

$$\frac{\partial u}{\partial t} = - g \frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} = - g \frac{\partial \eta}{\partial y}$$

Similarly, if the nonlinear terms $\frac{\partial u\eta}{\partial x}$ and $\frac{\partial v\eta}{\partial y}$ are neglected in the continuity equation, one has:

$$\frac{\partial \eta}{\partial t} + \frac{\partial (ud)}{\partial x} + \frac{\partial (vd)}{\partial y} = 0$$

This set of equations characterizes the linearized long wave theory which is valid provided $\eta \ll d$ and $\frac{\eta}{L} (\frac{L}{d})^3 \ll 1$.

In the case of quasi-two-dimensional motion v is negligible and these equations can be written

$$\frac{\partial u}{\partial t} = - g \frac{\partial \eta}{\partial x}$$

$$\frac{\partial}{\partial x} (Au) + \ell \frac{\partial \eta}{\partial t} = 0$$

where A is the cross section perpendicular to the velocity vector u and ℓ the width of the container at the free surface. This system of equation has a great number of exact solutions for various geometrical bottom topography.

XVII-4.2 THE LINEAR LONG WAVE EQUATIONS

For example, in the two-dimensional case, taking the water depth d as a constant, $d = \frac{A}{\ell}$, differentiating the momentum equation with respect to x and the continuity equation with respect to t, and eliminating u gives:

$$\frac{\partial^2 \eta}{\partial t^2} + gd \frac{\partial^2 \eta}{\partial x^2} = 0$$

Similarly differentiating the momentum equation with respect to t and the continuity equation with respect to x and eliminating η yields:

$$\frac{\partial^2 u}{\partial t^2} + gd \frac{\partial^2 u}{\partial x^2} = 0$$

XVII-4.3 HARMONIC SOLUTION - SEICHE

It can easily be verified that the solution can be that of a progressive wave such as:

$$\eta = H \cos (mx - kt)$$

$$u = H\left[\frac{g}{d}\right]^{1/2} \cos (mx - kt)$$

with $m = \frac{2\pi}{L}$, $k = \frac{2\pi}{T}$, and $L = T\sqrt{gd}$, or that of a standing wave (seiche)

$$\eta = H \cos mx \cos kt$$

$$u = H(g/d)^{1/2} \sin mx \sin kt$$

It can also easily be verified that the linear long wave theory is the limit case of the small amplitude wave theory when $d/L \to 0$ (see Section XVI-3).

XVII-5 THE NUMERICAL METHODS OF SOLUTION

XVII-5.1 THE METHOD OF CHARACTERISTICS

XVII-5.1.1 Establishment of the Characteristic Equations

Let us now consider the long wave equation under the form:

Momentum

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -g\frac{\partial \eta}{\partial x}$$

Continuity

$$\frac{\partial \eta}{\partial t} + \frac{\partial[u(d+\eta)]}{\partial x} = 0$$

and let us define $c = \sqrt{g(d+\eta)}$ which has the dimension of a velocity.

Adding to the momentum equation the zero quantity

$$-\left[\frac{d(d)}{dx} + S\right] = 0$$

and since

$$\frac{\partial(d)}{\partial t} = 0$$

respectively give (g being constant):

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial[g\,(d + \eta)]}{\partial x} = -gS$$

$$\frac{\partial[g\,(d + \eta)]}{\partial t} + \frac{\partial[u\,g\,(d + \eta)]}{\partial x} = 0$$

Inserting $c^2 = g(d + \eta)$, differentiating $\frac{\partial c^2}{\partial *} = c\frac{\partial(2c)}{\partial *}$, and dividing the second equation by c yields,

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + c\frac{\partial 2c}{\partial x} = -\,gS$$

$$\frac{\partial 2c}{\partial t} + c\frac{\partial u}{\partial x} + u\frac{\partial 2c}{\partial x} = 0$$

Adding and subtracting these two equations gives:

$$\frac{\partial}{\partial t}(u \mp 2c) + (u \mp c)\frac{\partial}{\partial x}(u \mp 2c) = -\,gS$$

It is now recalled that the total derivative of an expression A(x, t) with respect to time is:

$$\frac{dA(x,t)}{dt} = \frac{\partial A}{\partial t} + \frac{\partial A}{\partial x}\;\frac{dx}{dt}$$

Similarly, the left hand side of the above equation is the total derivative

$$\frac{d}{dt}(u \mp 2c)$$

provided

$$\frac{dx}{dt} = u \mp c$$

This means that along a line of slope $\frac{dx}{dt} = u \mp c$, the relationship

$$\frac{d}{dt}(u \mp 2c) = -gS$$

will apply (see Figure XVII-8).

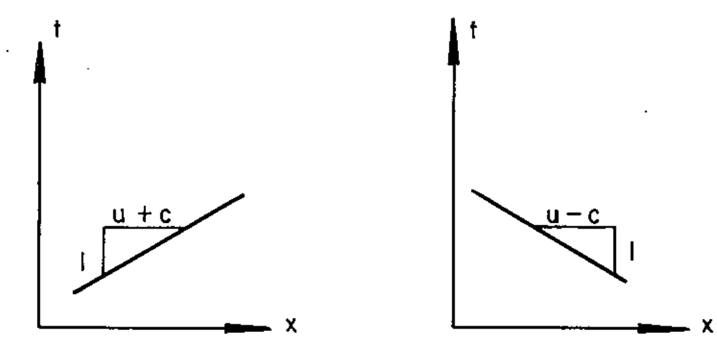


FIGURE XVII-8

ADVANCING AND RECEDING CHARACTERISTICS

The lines of slope $\frac{dx}{dt}$ = u + c are called the advancing or positive characteristics. The lines of slope $\frac{dx}{dt}$ = u - c are called the receding or negative characteristics.

It is seen that from a given point (x, t) in a T-X diagram, two lines of slopes (u + c) and (u - c) can be drawn.

XVII-5.1.2 An Application Case: The Dam Break Problem

In the case of a horizontal bottom S = 0 and $\frac{d}{dt}(u \mp 2c) = 0$ i.e., $u \mp 2c$ = constant values K_1 and K_2 along lines of slope $\frac{dx}{dt} = u \mp c$.

Let us consider the case of a vertical wall of water which is suddenly released as in the case of a dam breaking (see Figure XVII-9). It is seen that at time t = 0, u = 0, and $c = \sqrt{gd}$ where d is the water depth so that the constants K_1 and K_2 are

$$u + 2c = K_1 = 2\sqrt{gd}$$

$$u - 2c = K_2 = -2\sqrt{gd}$$

At time $t = t_o$, the water wall collapses. At the downstream water tip: $\eta = -d$ and c = 0. Hence the speed of the water tip is $\frac{dx}{dt} = u(+0)$ and

$$(u + 2c)\Big|_{t = 0} = (u + 2c)\Big|_{t = t}$$

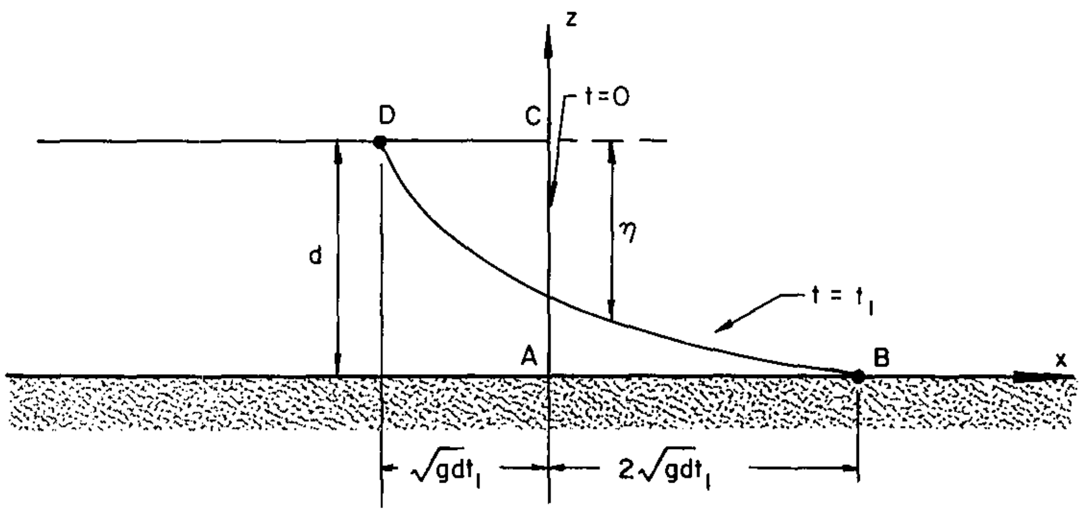


FIGURE XVII-9

NOTATION FOR THE DAM BREAK PROBLEM

AB = 2CD

becomes

$$0 + 2\sqrt{gd} = u$$

such that the water tip travels at a speed $2\sqrt{gd}$. The speed of the rarefaction wave traveling upstream where $\eta = 0$ and $u = 0$ is $\frac{dx}{dt} = 0 + c$, i.e., $(0 + 2c)\Big|_{t=0} = (0 + 2c)\Big|_{t=t'}$, i.e.,

$$c = \frac{dx}{dt} = \sqrt{gd}$$

XVII-5.1.3 Practical Method of Solution

In general, the variation of u and η (or c) are gentle enough in order that a finite interval method can be applied. The basic characteristic equations are then written for finite intervals Δx, Δt:

$$\Delta(u \mp 2c) = -gS\,\Delta t \quad \text{along} \quad \frac{\Delta x}{\Delta t} = u \mp c$$

The time history of the wave evolution can then be determined step by step as follows. Given $u(x, t_1)$ and $\eta(x, t_1)$ or $c(x, t_1)$ for a given wave at a time $t = t_1$ along an axis OX (see Figure XVII-10).

The value of u and c at time $t = t_1$ are calculated at regular intervals Δx. Then the characteristic lines of slope $\frac{\Delta x}{\Delta t} = u_1 + c_1$ is drawn from the point (1) and the characteristic line of slope $\frac{\Delta x}{\Delta t} = u_2 - c_2$ is drawn from the point (2).

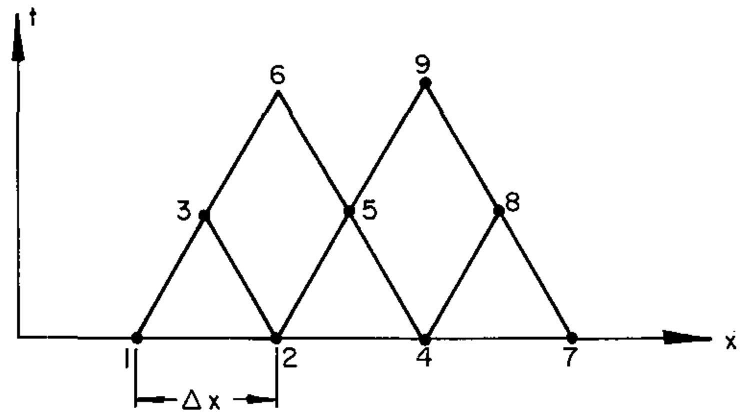


FIGURE XVII-10

APPLICATION OF THE METHOD OF CHARACTERISTICS

Their intersection at point (3) defines x_3, t_3 graphically. Then, by applying the characteristic equation along these lines, the values of u_3 and c_3 (and consequently η_3) are found from the two equations

$$u_3 + 2c_3 = u_1 + 2c_1 - gS_1 (t_3 - t_1)$$

$$u_3 - 2c_3 = u_2 - 2c_2 - gS_2 (t_3 - t_2)$$

In the particular case of the Figure XVII-10, $t_2 = t_1$. Similarly, u_5 and c_5 are found from the points (2) and (4), u_6 and c_6 from the points (3) and (5), and so on.

XVII-5.1.4 Domain of Influence and Domain of Dependence

By this process of calculation, it is seen that the state of a wave at a given point (1) (x_1, t_1)(see Figure XVII-11) has an influence upon the state of the wave at any other point between the characteristic lines issued from that point. Such characteristic lines define the domain of influence of this point. Similarly, the state of the wave at a given point (3) depends solely upon the state of the wave under the two characteristic lines crossing at point (3). Such a domain is the domain of dependence of point (3).

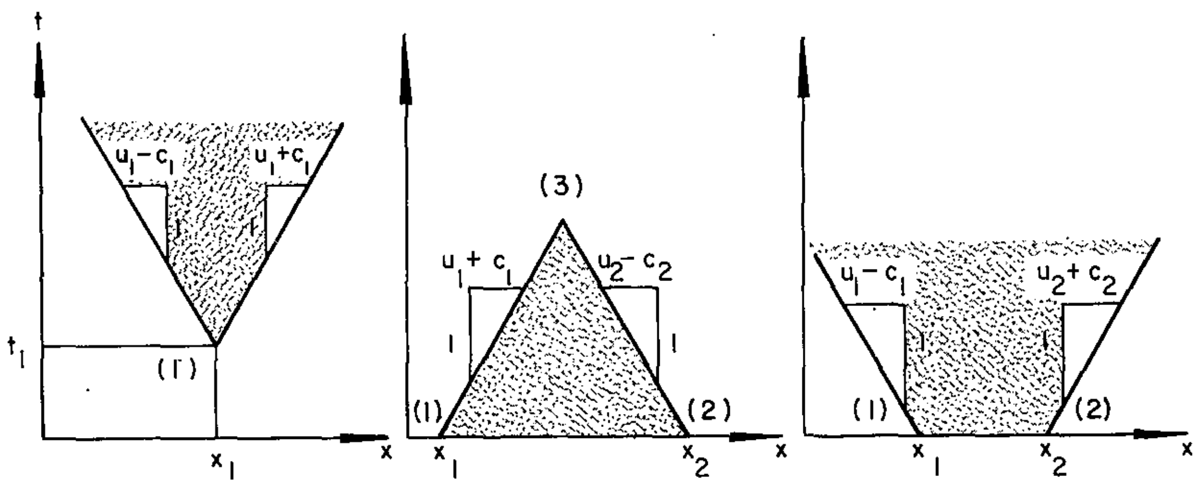


FIGURE XVII-11

DOMAIN OF INFLUENCE AND DOMAIN OF DEPENDENCE

Any disturbance arising over a finite distance $(x_1 - x_2)$ will have an influence on the water behavior within a domain defined by a negative characteristic issued from x_1 and a positive characteristic x_2.

XVII-5.1.5 The Method of Characteristics for Tidal Estuary

In the case of tidal estuary, the bottom friction and variation of cross section must be taken into account. The momentum equation can easily be modified for taking into account the bottom friction as in Section XVII-3.1.4. The continuity equation can also be modified for the variation of cross section as in Section XVII-3.4.2. Then the same calculation as previously shown leads to:

$$\Delta (u \mp 2c) = -\left[gS + \frac{g}{C_h^2}\left(\frac{u}{c}\right)^2 \mp \frac{uh}{\ell} \frac{\Delta \ell}{\Delta x} \right] \Delta t$$

in which case the method of characteristics will apply similarly.

XVII-5.2 TIDAL BORE

XVII-5.2.1 Bore Inception

When two positive characteristics $\frac{dx}{dt} = u + c$ cross each other (see Figure XVII-12), one set of values can be found from points (a) and (b): u_2 and c_2; and another set of values can be found from points (a) and (c): u_1 and c_1.

The two values for u and the two values for c at the same location indicate a discontinuity or a vertical wall of water. There is bore inception.

XVII-5.2.2 Bore Travel

The line $\frac{dx}{dt} = W$, where W is the speed of the bore, is a line of discontinuity for the mesh formed by the positive and negative

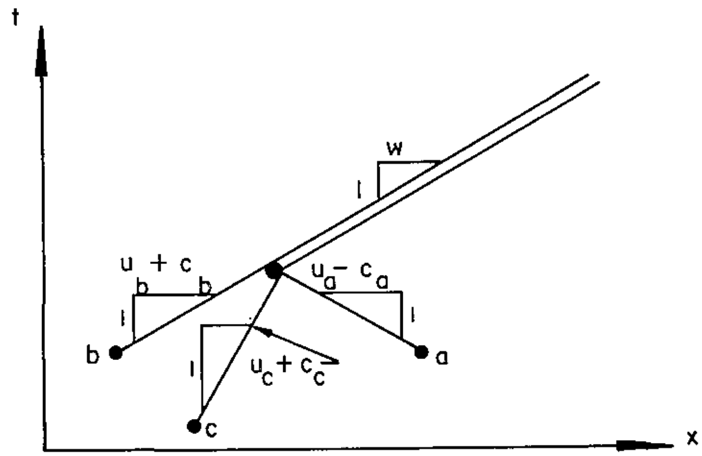


FIGURE XVII-12
BORE INCEPTION

characteristics. At the locus of the bore there are five unknowns to be determined, namely: u_1, η_1 on the low side of the bore, u_2, η_2 on the high side of the bore, and the bore velocity itself, W. Since $W > u_1 + c_1$, u_1 and η_1 are determined by application of the method of characteristics directly (see Figure XVII-13). u_2, η_2 and W are determined from the three following equations:

1) The momentum equation for a moving hydraulic jump:

$$\frac{\rho g}{2} \left[h_2^2 - h_1^2 \right] = \rho h_1 (u_2 - u_1)(W - u_1)$$

2) The continuity equation for a moving hydraulic jump:

$$u_2 h_2 = W(h_2 - h_1) + u_1 h_1$$

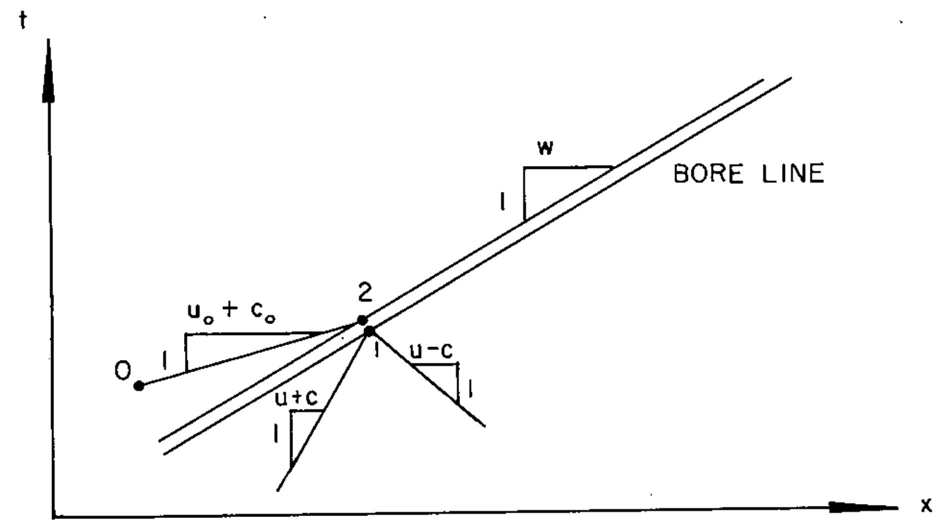


FIGURE XVII-13
BORE PROPAGATION

3) And since $u_2 + c_2$ always exceeds W, the positive characteristics equation which merges with the bore line at the considered point (x, t):

$$u_2 + 2c_2 = u_o + 2c_o - gS(t_2 - t_o)$$

The wave profile on the high side of the bore is determined afterward by making use of u_2 and c_2 by applying the method of characteristics in a straightforward manner.

The problem of a limit solitary wave (see Section XVII-6.2) travelling on an horizontal bottom and reaching a 1/10 bottom slope is treated on Figure XVII-14 as an example of applicability of the method of characteristics. The successive wave profiles are presented on Figure XVII-15 and are obtained by interpolating between the values of η at the crossing of characteristic line.

XVII-5.3 THE DIRECT APPROACH TO NUMERICAL SOLUTIONS

The application of the method of characteristics requires the solution of four unknowns at each point of crossing lines, namely: x, t, u

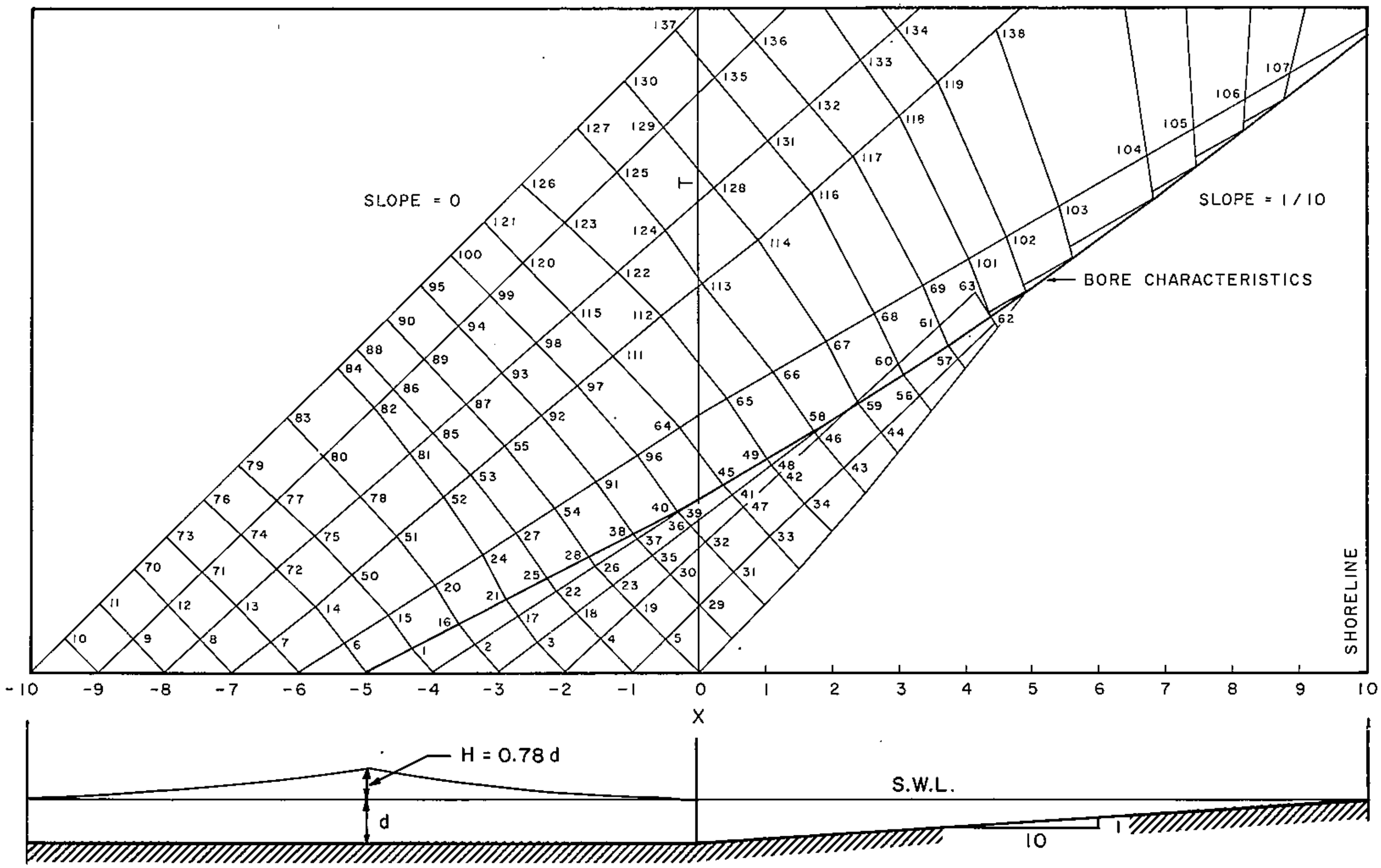


FIGURE XVII-14

APPLICATION OF THE METHOD OF CHARACTERISTICS TO A WAVE BREAKING ON A SLOPE

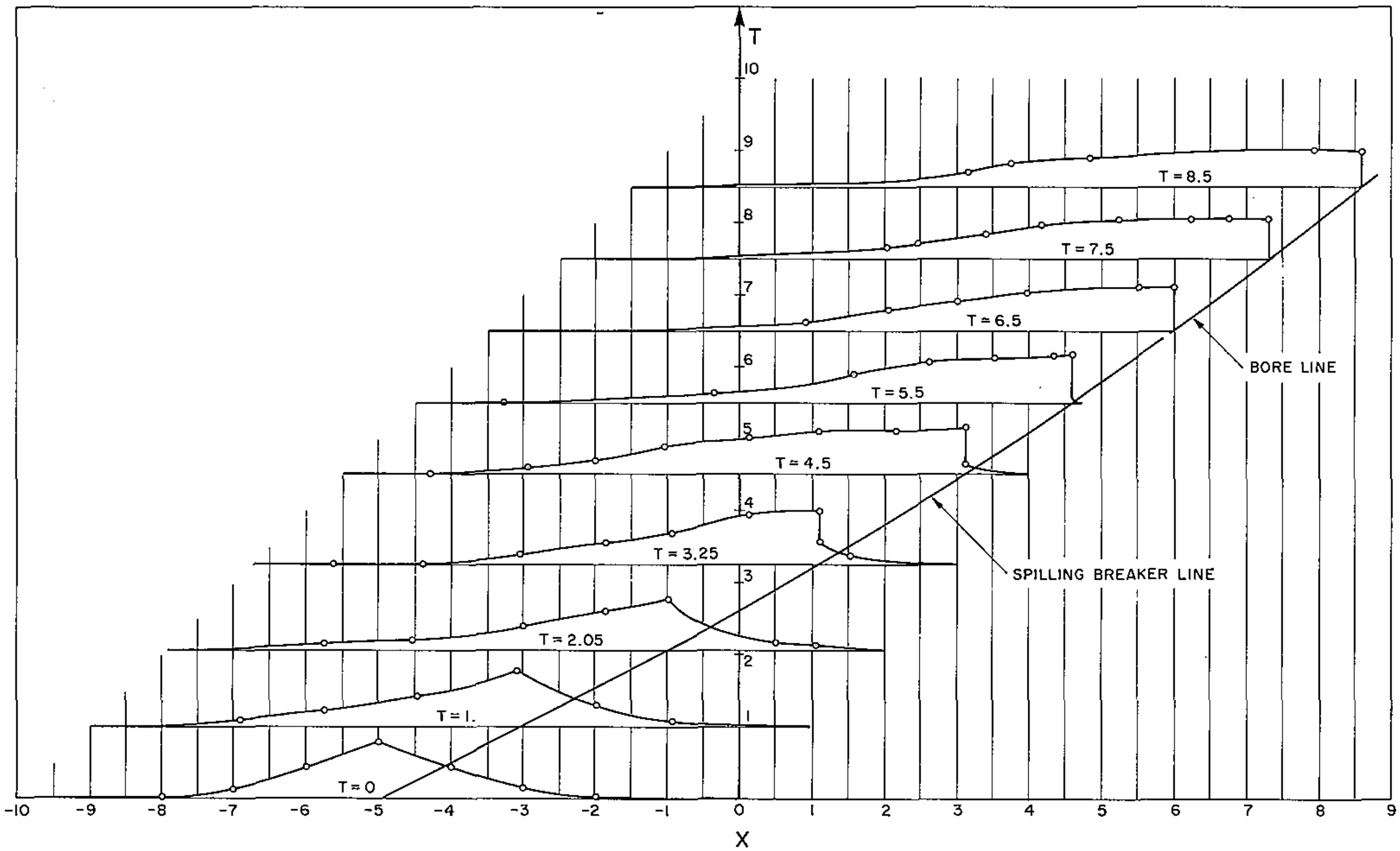


FIGURE XVII-15

TIME-HISTORY OF THE WAVE PROFILES BREAKING ON A 1/10 BOTTOM SLOPE and η. In the case where x and t are specified, a priori, only two unknowns remain: u and η. For this purpose the long wave equations can be treated directly by a finite difference process, where the intervals Δx and Δt, i.e., the locations of x and t, are specified independently from the characteristic lines.

For this purpose, a square mesh or a staggered mesh can be used (see Figure XVII-14).

XVII-5.3.1 <u>The Square Mesh Method</u>

u and η at point (4) can be deduced from the value of u and η at points (1) and (2) directly as follows (see Figure XVII-14). From the continuity equation one has:

$$\eta_4 = \eta_2 + \frac{\Delta t}{2\,\Delta x}\left[u\,(d+\eta)\Big|_3 - u\,(d+\eta)\Big|_1\right]$$

and from the momentum equation

$$u_4 = u_2 - \frac{\Delta t}{2\,\Delta x}\,[g\,(\eta_3 - \eta_1) + u_2\,(u_3 - u_1)]$$

The calculation will proceed by calculating u_5 and η_5 from points (2), (3), (6) and so on.

A similar method can be applied by making use of the long wave equation under the characteristic form presented in Section XVII-3.2.1.

XVII-5.3.2 <u>The Stability Criterion</u>

The error which is made in transforming the differential equation into a finite difference form may result in a cumulative error

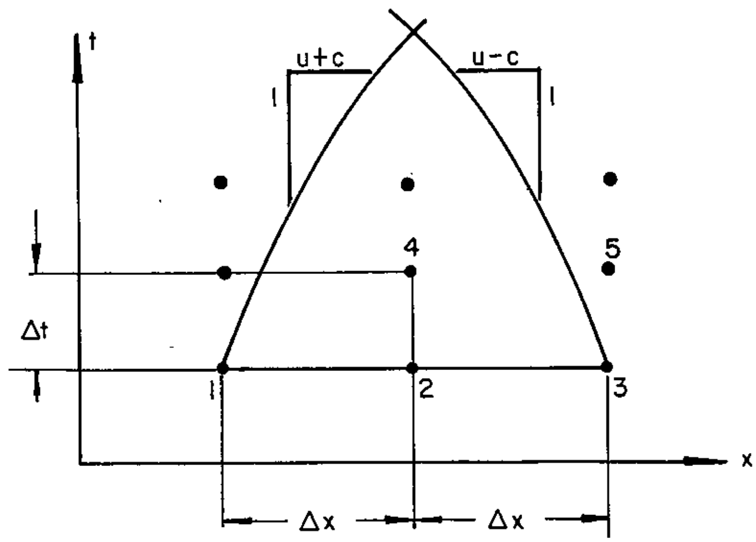


FIGURE XVII-14
NOTATION FOR APPLICATION OF THE SQUARE
MESH METHOD

which may blow up as $t = \Sigma \Delta t$ increases. The finite difference method is stable provided the point (x, t) under consideration is within the domain of dependence of the points which are used for its determination.

Consequently, since in general $(u + c)_1 > (u - c)_3$ (see Figure XVII-14), the criterion for stability is

$$\Delta t < \frac{\Delta x}{u + c}$$

Also, the variation of u and c with respect to x and t has to be gentle enough in order that the finite difference terms have a value close to the differential.

XVII-6 ON SOME EXACT STEADY STATE SOLUTIONS

In general, unsteady state solution is a typical characteristic of the long wave theories, i.e., the wave profile changes its shape as the wave proceeds. This fact is inherent in the long wave equations. However, the insertion of bottom friction and a vertical acceleration term, even approximated, into the long wave equations permits the finding of some special steady state solutions. They are the "monoclinal" wave, the solitary wave, and the cnoidal wave. Only the first two cases are analyzed in this book.

It is recalled that the assumption of a steady state solution results in the search of a function such as $\left.\begin{matrix}\eta \\ u\end{matrix}\right\} = f(x - Ct)$ where C is a constant. Consequently:

$$\frac{\partial}{\partial t} = -C\frac{\partial}{\partial x}$$

XVII-6.1 MONOCLINAL WAVES

Let us consider the long wave equation with a bottom friction term such as (see Section XVII-3.1.4):

$$\left\{\begin{aligned} &\frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} = 0 \\ &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = gS - \frac{g}{C_h^2}\frac{u|u|}{h} \end{aligned}\right.$$

Inserting the relationship $\frac{\partial}{\partial t} = -C\frac{\partial}{\partial x}$ in the continuity equation gives

$(u - C) \frac{\partial h}{\partial x} + h \frac{\partial u}{\partial x} = 0$ which integrates as $(u - C)h = A$ (constant). From the momentum equation $(u - C) \frac{\partial u}{\partial x} + g \frac{\partial h}{\partial x} = g \left[S - \frac{u^2}{C_h^2 h} \right]$. Eliminating $\frac{\partial u}{\partial x}$ between these two equations gives:

$$\left[g - \frac{(u - C)^2}{h} \right] \frac{\partial h}{\partial x} = g \left[S - \frac{u^2}{C_h^2 h} \right]$$

such that

$$\frac{\partial h}{\partial x} = \frac{S - \frac{(C h + A)^2}{C_h^2 h^3}}{1 - \frac{A^2}{g h^3}}$$

This differential equation has a number of solutions, some without physical significance, the solution depending upon whether the water depth is larger or smaller than the critical depth, as shown by Figure XVII-15. These waves are called monoclinal waves or uniformly progressive flow. This theory is particularly suitable for the study of flood waves in rivers.

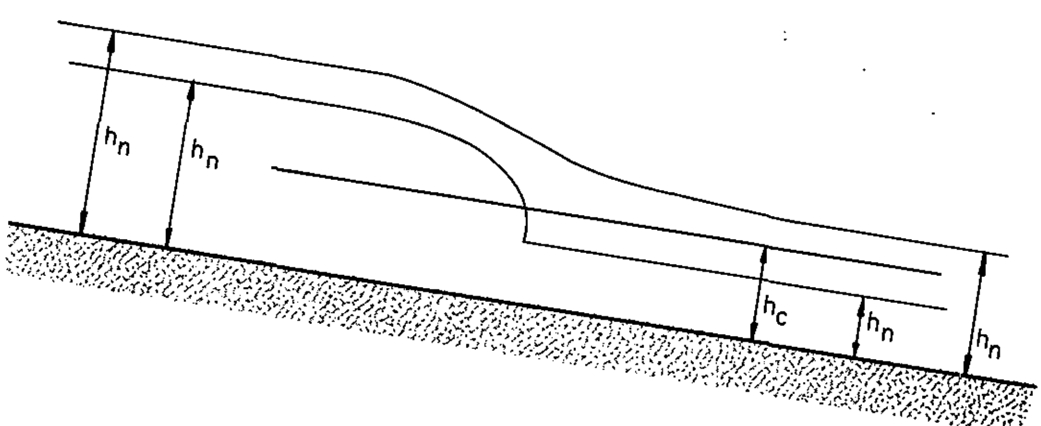


FIGURE XVII-15
MONOCLINAL WAVE

XVII-6.2 SOLITARY WAVE THEORY

Let us consider the long wave equation for a horizontal bottom with a vertical acceleration term as it has been established in Sections XVII-3.3 and XVII-3.4.1.

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + g \frac{\partial \eta}{\partial x} + \frac{d + \eta}{3} \frac{\partial^3 \eta}{\partial t^2 \partial x} = 0$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial [u (d + \eta)]}{\partial x} = 0$$

and inserting the relationship $\frac{\partial}{\partial t} = - C \frac{\partial}{\partial x}$ gives:

$$\frac{\partial}{\partial x} \left[- C u + \frac{u^2}{2} + g \eta + \frac{C^2 h}{3} \frac{\partial^2 \eta}{\partial x^2} \right] = 0$$

and

$$\frac{\partial}{\partial x} \left[- C \eta + (d + \eta) u \right] = 0$$

The quantities between brackets are independent from x and consequently equal to constants. It is seen that these constants are nil since u and η tend to zero when $x \to \infty$.

Eliminating u gives

$$\frac{C^2 \eta}{d + \eta} = \frac{C^2 \eta^2}{2 (d + \eta)^2} + g \eta + \frac{C^2 (d + \eta)}{3} \frac{\partial^2 \eta}{\partial x^2}$$

or, by considering η small with respect to d and developing the square root as $(1 + \alpha)^{\frac{1}{2}} \cong 1 + \frac{1}{2} \alpha$, one has

$$C = \left[g\,(d + \eta)\left(1 + \frac{\eta}{4\,d} + \frac{d^2}{6\,\eta}\,\frac{\partial^2 \eta}{\partial x^2}\right)\right]^{\frac{1}{2}}$$

When the free surface curvature is negligible, the value for C becomes

$$C = \sqrt{g\,d}\,\left[1 + \frac{3}{4}\,\frac{\eta}{d}\right]$$

which can also be obtained directly by application of the momentum theorem.

In this case, since one has assumed C to be a constant and since the wave profile remains unchanged with time, one must have:

$$1 + \frac{\eta}{4\,d} + \frac{d^2}{6\,\eta}\,\frac{\partial^2 \eta}{\partial x^2} = \text{constant} = \frac{H}{2\,d}$$

H being a constant, which is specified in the following. This equation can be integrated as follows:

$$\frac{\partial^2 \eta}{\partial x^2} = \frac{3\,\eta}{2\,d^3}\,[2\,H - 3\,\eta]$$

and since $\frac{\partial \eta}{\partial x} = 0$ when $\eta \to 0$,

$$2\,\frac{\partial \eta}{\partial x}\,\frac{\partial^2 \eta}{\partial x^2}\,dx = \frac{3\,\eta}{d^3}\,(2\,H - 3\,\eta)\,d\,\eta$$

or

$$\left[\frac{d\,\eta}{d\,x}\right]^2 = \frac{3\,\eta^2}{d^3}\,(H - \eta)$$

It is seen that $\eta = H$ gives $\frac{d\,\eta}{d\,x} = 0$ and corresponds to the top of the wave;

consequently H is the wave height. The equation can still be written:

$$\frac{d\eta}{dx} = \left(\frac{3}{d^3}\right)^{\frac{1}{2}} \eta\,(H - \eta)^{\frac{1}{2}}$$

and after separation of the variables, it can be integrated in the form

$$\int \frac{d\eta}{\eta\sqrt{H - \eta}} = \int \left(\frac{3}{d^3}\right)^{\frac{1}{2}} dx$$

which gives for the wave profile:

$$\eta = \frac{H}{\cosh^2\left[\frac{\sqrt{3}}{2}\left(\frac{H}{d}\right)^{\frac{1}{2}} \frac{x}{d}\right]}$$

as shown in Figure XVII-16.

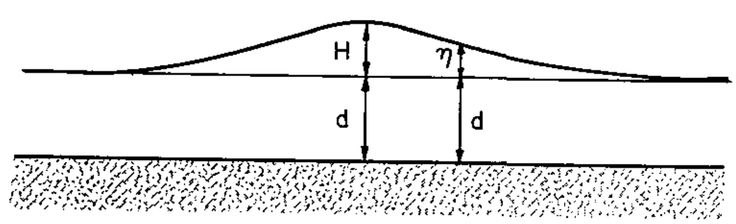


FIGURE XVII-16
SOLITARY WAVE

XVII-1 Determine the value of the hydraulic radius R_H for a circular conduit as a function of the maximum water depth.

XVII-2 A trapezoidal channel has a width at the bottom of 40 feet and a bank slope of 45^o. The bottom slope $S = 0.002$, and the Manning coefficient $n = 0.03$. Calculate the curve for a normal depth y_n as a function of the discharge Q_n up to $Q = 1000\ ft^3/sec$.

Calculate the curve giving the critical discharge Q_c as a function of the critical depth y_c and determine the value of the discharge for which critical depth equals normal depth.

XVII-3 Is it possible to define a channel which has a constant hydraulic radius R_H whatever the water depth h?

Answer:

$$R_H = \frac{A}{P} = \frac{dA}{dP} = \frac{b\,dh}{\sqrt{db^2 + dh^2}}$$

By integrating and taking $b = R_H$ when $h = 0$, one finds:

$$h = R\left[\ell n\left(b + \sqrt{b^2 - R^2}\right) - \ell n\, R\right]$$

XVII-4 Demonstrate that the free surface slope of a steady gradually varied flow in an open channel is equal to the sum of the energy slope and the slope due to the velocity change $\frac{d}{dx}\left(\frac{V^2}{2g}\right)$.

XVII-5 Demonstrate that $\frac{dy}{dx} = S \frac{1 - (Q/Q_n)^2}{1 - (Q/Q_c)^2}$ where S is the bottom slope, Q the discharge, Q_n the normal discharge, Q_c the critical discharge. Also demonstrate that in the case where the Manning formula is used for large rectangular channels,

$$\frac{dy}{dx} = S \frac{1 - (y_n/y)^{10/3}}{1 - (y_c/y)^3}$$

Demonstrate also that in a rectangular channel of variable width ℓ,

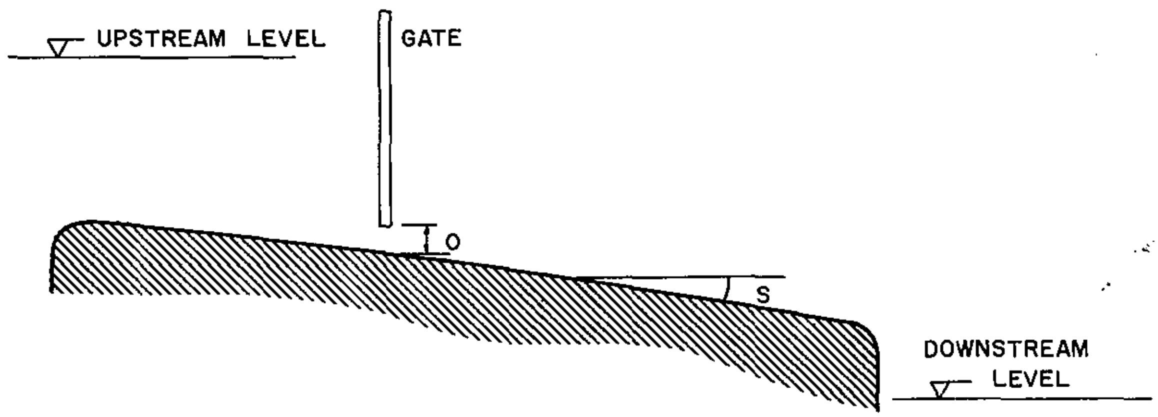
$$\frac{dy}{dx} = \frac{S - S_f + \frac{Q^2 y}{g A^3} \frac{d\ell}{dx}}{1 - \frac{Q^2 \ell}{g A^3}}$$

XVII-6 From the equation

$$\frac{dy}{dx} = S \frac{y^3 - y_n^3}{y^3 - y_c^3}$$

and depending upon the sign of S and the relative value of y, y_n and y_c, determine all the possible kinds of backwater curves which may be encountered in a rectangular channel.

XVII-7 Determine all the kinds of water surface profiles which are encountered in the case of the following figure in the two cases where the normal depth is larger than the critical depth and at the opposite, where the critical depth is larger than the normal depth.



The opening of the gate o will be considered as a variable n in such a way that o can be smaller and/or larger than h_c and h_n.

XVII-8 Establish under which conditions there are two normal depths and two critical depths in a circular closed conduit. Determine all the possible backwater curves which may exist in such a conduit.

XVII-9 Establish the required condition for establishing a hydraulic jump on a sudden bottom drop corresponding to various assumptions on the exact location of the jump with respect to the sudden bottom drop.

XVII-10 Consider a vertical jet hitting a horizontal large circular plate at its center. The plate is ended by a weir providing a quasi-constant water depth h. Determine the location of the circular hydraulic jump as a function of the discharge Q of the jet and h. The bottom friction forces will be neglected.

XVII-11 Consider the energy diagram such as shown on Figure XV-5. Indicate the modification which needs to be made to such a figure in the case of unsteady motion.

XVII-12 Demonstrate that the long wave equations can be transformed as

$$u + 2c = \text{constant}$$
$$u - 2c = \text{constant}$$

along a line of slope $\dfrac{dx}{dt} = u \pm c \pm \dfrac{gS}{\dfrac{\partial}{\partial x}(u \pm 2c)}$.

<u>Answer:</u>

The characteristic equation $\dfrac{d}{dt}(u \pm 2c) = -gS$ can still be written:

$$\frac{\partial}{\partial t}(u \pm 2c) \pm \left[u \pm c \pm \frac{gS}{\frac{\partial}{\partial x}(u \pm 2c)}\right] \frac{\partial}{\partial x}(u \pm 2c) = 0$$

i.e.,

$$\frac{d}{dt}(u \pm 2c) = 0$$

along lines of the defined slopes.

XVII-13 Transform the differential equations which are used in the method of characteristics for the study of the propagation of two-dimensional long waves into a finite difference system corresponding to intervals Δx, Δt forming a square mesh. Do the same in the case of a staggered mesh.

XVII-14 Demonstrate that the equation of Barre de St. Venant can be written:

$$\alpha \frac{\partial^2 \phi}{\partial x^2} + 2\beta \frac{\partial^2 \phi}{\partial x \, \partial t} + \gamma \frac{\partial^2 \phi}{\partial t^2} = F$$

where

$$\alpha = [Q^2 - \frac{g A^3}{\ell}]$$

$$\beta = QA$$

$$\gamma = A^3$$

$$F = A^3 g \left[\frac{A}{\ell^2} \frac{d\ell}{dx} - S - \frac{\ell Q|Q|}{C_h^2 A^3}\right]$$

and give the definition for ϕ.

Answer:

Continuity:

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

$$Q = \frac{\partial \phi}{\partial t}, \qquad A = -\frac{\partial \phi}{\partial x}$$

$$d\phi = Q\,dt - A\,dx$$

Then $u = \frac{Q}{A}$ and $h = \frac{A}{\ell}$ (ℓ: width) are expressed as functions of ϕ in the momentum equation:

$$\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} = -gS - \frac{g}{C_h^2 R_H} u|u|$$

CHAPTER XVIII

WAVE MOTION AS A RANDOM PROCESS

XVIII-1 HARMONIC ANALYSIS

XVIII-1.1 THE CONCEPT OF REPRESENTATION OF SINUSOIDAL WAVES IN THE FREQUENCY DOMAIN

Consider a sinusoidal wave profile given by

$$\eta = A \cos (mx + kt + \epsilon)$$

where A is the amplitude, $m = \frac{2\pi}{L}$ and $k = \frac{2\pi}{T}$ where L and T are wave length and period respectively, and ϵ is a phase angle. If this wave is observed at a point as a function of time and the origin is chosen such that the initial phase position is zero, it can be written

$$\eta = A \cos kt$$

This wave can be described as a wave of amplitude A and frequency k (in radians). It can be represented as in Figure XVIII-1 which is referred to as an amplitude spectrum.

Suppose a wave of arbitrary shape, but having a period T (frequency $\frac{2\pi}{T}$) is to be represented on an amplitude spectrum such as Figure XVIII-1. Recalling that almost any periodic function can be

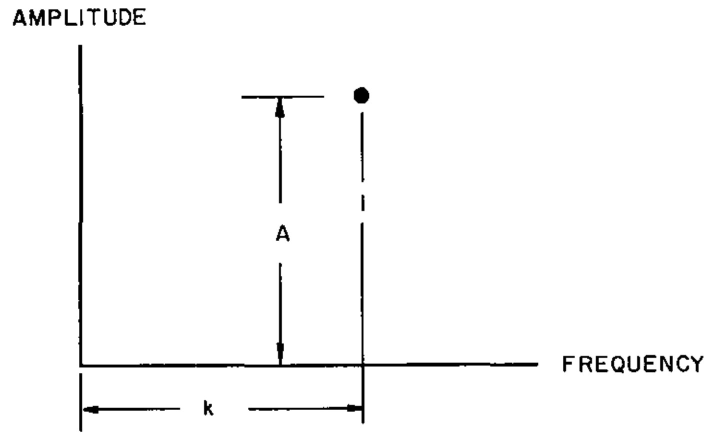


FIGURE XVIII-1
AMPLITUDE SPECTRUM OF A
SINUSOIDAL WAVE

represented as a Fourier series of fundamental period T and harmonics,

$$\eta = \sum A_n \cos (nkt + \epsilon_n)$$

This wave would have an amplitude frequency spectrum appearing as in Figure XVIII-2.

If a water surface is described by a combination of a number of waves having different periods (frequencies) it could be represented as a diagram such as Figure XVIII-2 but having amplitude and frequency components at many different places.

For example, the sum of two sinusoids of different periods (T_1, T_2) would appear in Figure XVIII-3.

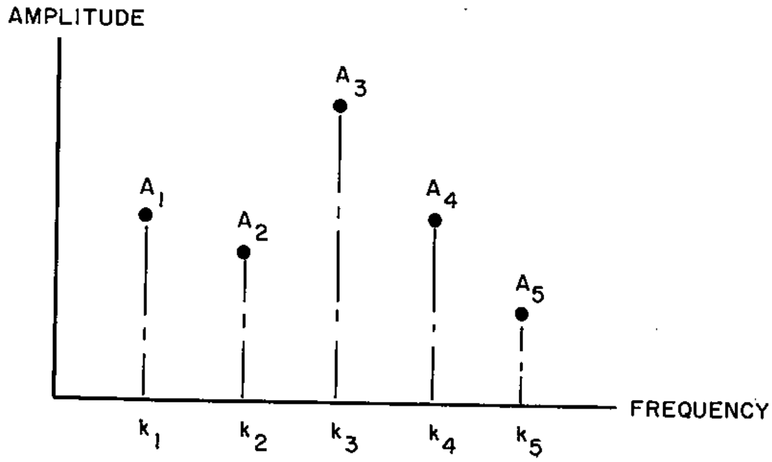


FIGURE XVIII-2
AMPLITUDE SPECTRUM OF A PERIODIC WAVE OF ARBITRARY SHAPE

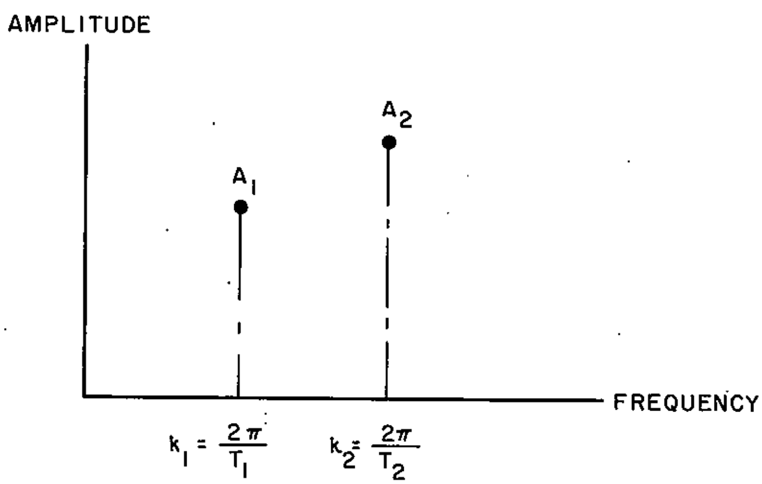


FIGURE XVIII-3
AMPLITUDE SPECTRUM OF TWO SINUSOIDS

XVIII-1.2 REVIEW OF FOURIER ANALYSIS

XVIII-1.2.1 The immediate temptation is to take a wave record such as Figure XVIII-4. Assume that it can be represented in the interval $0 < t < T_1$ by a Fourier series:

$$\eta(t) = \frac{A_o}{2} + \sum\left[A_n \cos\frac{2\pi n t}{T_1} + B_n \sin\frac{2\pi n t}{T_1}\right]$$

and evaluate A_n and B_n. The limits on the integer n have not yet been assigned. They should be $1 < n < \infty$ but would obviously have to be chosen much smaller than the upper limit and it would be hoped that for some comparatively large but finite value of n the series of the above equation would approach the observed water surface of Figure XVIII-4.

In order to determine the coefficients A_n and B_n in the

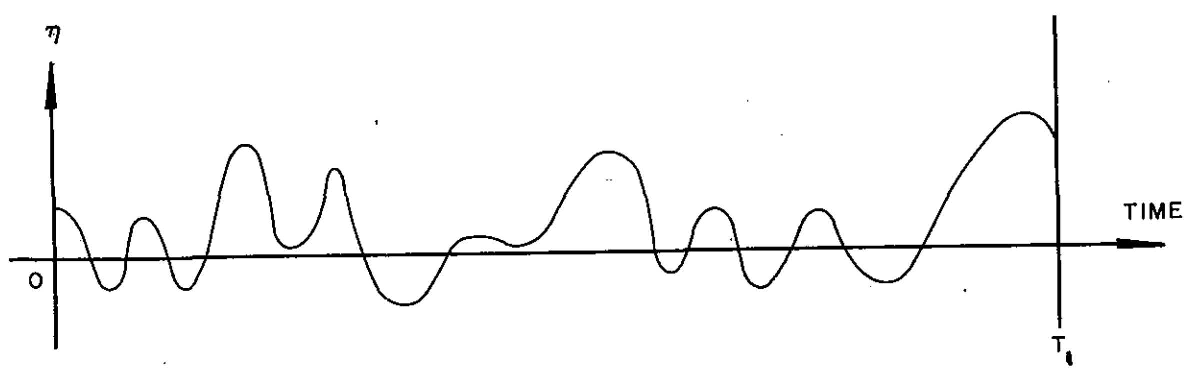


FIGURE XVIII-4
A TYPICAL WAVE RECORD AT SEA

right hand side of the above equation, it is recalled that

$$\int_0^{T_1} \cos\frac{2\pi n t}{T_1} \sin\frac{2\pi m t}{T_1}\, dt = 0 \quad \text{for all } m \text{ and } n$$

$$\int_0^{T_1} \cos\frac{2\pi n t}{T_1} \cos\frac{2\pi m t}{T_1}\, dt = 0 \quad \text{for } m \neq n$$

$$= \frac{T_1}{2} \quad \text{for } m = n$$

$$\int_0^{T_1} \sin\frac{2\pi n t}{T_1} \sin\frac{2\pi m t}{T_1}\, dt = 0 \quad \text{for } m \neq n$$

$$= \frac{T_1}{2} \quad \text{for } m = n$$

Use is made of these properties by multiplying both sides of the previous equation for $\eta(t)$ by $\cos\frac{2\pi m t}{T_1}$ and integrating between 0 and T_1.

$$\int_0^{T_1} \eta(t) \cos\frac{2\pi m t}{T_1}\, dt = \frac{A_o}{2}\int_0^{T_1} \cos\frac{2\pi m t}{T_1}\, dt$$

$$+\sum_n A_n \int_0^{T_1} \cos\frac{2\pi n t}{T_1} \cos\frac{2\pi m t}{T_1}\, dt + \sum_n B_n \int_0^{T_1} \sin\frac{2\pi n t}{T_1} \cos\frac{2\pi m t}{T_1}\, dt$$

$$= \frac{A_m T_1}{2}$$

so that

$$A_m = \frac{2}{T_1}\int_0^{T_1} \eta(t) \cos\frac{2\pi m t}{T_1}\, dt$$

and similarly

$$B_m = \frac{2}{T_1} \int_o^{T_1} \eta(t) \sin \frac{2\pi m t}{T_1} dt$$

XVIII-1.2.2 Sufficient and necessary conditions that the function $\eta(t)$ can be represented by this set of equations cannot be stated precisely and fully within the scope of this book. (The reader is referred to many standard textbooks on Fourier Analysis.) It is sufficient for the present purpose to write one of the necessary conditions, which is

$$\int_o^{T_1} |\eta(t)|\, dt \neq \infty$$

This condition will be satisfied for wave records with T_1 finite.

The coefficients A_n and B_n can be represented on two figures such as Figure XVIII-2 (one for A_n and one for B_n). The points for each coefficient will occur at multiples of $k_n = \frac{2\pi}{T_1}$ where T_1 is the length of the record.

In order for the Fourier coefficients A_n and B_n to be representative of a sea state for all time, the record length T_n must be very long. The limits on n will be very large and the practical details of such a Fourier representation are seen to be enormous. A comparatively short record of the sea will yield values for A_n and B_n which would only be representative of that particular record at that particular time and place. A slightly different record taken, say, a few minutes later or a few feet away would yield completely different A_n and B_n. This simplified Fourier analysis is unstable, hence a much more stable description of the sea-state must be found.

XVIII-1.3 FOURIER ANALYSIS IN EXPONENTIAL FORM

The expressions for the Fourier series representatively given in the previous section will be more easily handled in their exponential form. The transformation is as follows:

$$\cos nkt = \frac{1}{2}\,[e^{inkt} + e^{-inkt}]$$

$$\sin nkt = \frac{1}{2i}\,[e^{inkt} - e^{-inkt}]$$

so that the previous expression for $\eta(t)$ given at the beginning of the previous section may be written:

$$\eta(t) = \frac{A_o}{2} + \frac{1}{2}\sum_{1}^{\infty} (A_n - iB_n)\, e^{inkt} + \frac{1}{2}\sum_{1}^{\infty} (A_n + iB_n)\, e^{-inkt}$$

Now it follows from the definition of A_n and B_n that

$$A_n = A_{-n} \quad \text{since} \quad \cos(-nkt) = \cos nkt$$

$$B_n = -B_{-n} \quad \text{since} \quad \sin(-nkt) = -\sin nkt$$

so that the last term of the above equation can be written as:

$$\frac{1}{2}\sum_{1}^{\infty} (A_n + iB_n)\, e^{-inkt} \equiv \frac{1}{2}\sum_{-1}^{-\infty} (A_n - iB_n)\, e^{inkt}$$

Inserting this equation into the previous one gives

$$\eta(t) = \sum_{-\infty}^{\infty} A_n' \, e^{inkt}$$

where A_n' is given by

$$A_n' = \frac{1}{2}(A_n - iB_n)$$

Hence

$$A_n' = \frac{1}{T_1} \int_0^{T_1} \eta(t) \, e^{-inkt} dt \quad \text{for } n = 0, \pm 1, \pm 2 \ldots$$

It is seen that A_n' is a complex function of frequency (nk) for a given record of $\eta(t)$ of length T_1 containing information on phase and amplitude of the components making up that particular record. This operation is called a Fourier transform.

XVIII-1.4 RANDOM FUNCTIONS

There still remains the problem of deriving a more stable description of a sea state and a method to handle longer wave records. As the record length tends to infinity, the condition $\int_0^{T_1} |\eta(t)| \, dt \neq \infty$ will obviously be violated. The Fourier transform giving the value of A_n' will no longer be valid. Some method has to be devised to handle a random process under the general concept of harmonic analysis. A random phenomenon is one in which the fluctuations of the quantity under observation as a function of time cannot be precisely predicted. No two water level records will ever be identical. They will, however, have certain identifiable statistical properties. In a random sea where the variety of wave forms is infinite, characterization by wave form is contrary to the inherent feature of the process. Characteristics that are common to all possible samples of that sea-state are required.

XVIII-1.5 AUTO-CORRELATION

Consider an observer who records a water surface elevation at time $t = t_1$ at a fixed point. What can he say about the water surface elevation at time $t = t_1 + \Delta t$ where $\Delta t = 0.1$ sec., 1.0 sec., 10 sec., or 100 sec.? If the observer is watching a sinusoidal wave train he can say quite a lot about the times $t_1 + \Delta t$ but if the process is random he cannot be sure of the future. The best he can do would be to give some estimate of the expected value of the sea surface elevation. (The expected value is defined as the average from an infinite number of observations.) The problem is to correlate the water surface elevation at time t with its value at time $t + \Delta t$. The correlation function

$$R(\tau) = \lim_{T_1 \to \infty} \frac{1}{T_1} \int_o^{T_1} \eta(t)\, \eta(t+\tau)\, dt$$

predicts the expected (average) value of the product of two values of the water surface which are separated in time at the same place by τ. To confirm that $\eta(t)$ and $\eta(t+\tau)$ are taken from the same record, the expression for $R(\tau)$ is called the <u>auto-correlation function</u> of $\eta(t)$. (Occassionally the cross-correlation between two different signals may be required.)

Some properties of the auto-correlation as defined by $R(\tau)$ are developed:

1. The auto-correlation function is even, i.e., $R(\tau) = R(-\tau)$, or

$$\lim_{T_1 \to \infty} \frac{1}{T_1} \int_0^{T_1} \eta(t)\ \eta(t+\tau)\, dt$$

$$= \lim_{T_1 \to \infty} \frac{1}{T_1} \int_0^{T_1} \eta(t)\ \eta(t-\tau)\, dt$$

2. The value of the auto-correlation at $\tau = 0$ is the mean square water surface fluctuation since

$$R(\tau = 0) = \lim_{T_1 \to \infty} \frac{1}{T_1} \int_0^{T_1} \eta^2(t)\, dt$$

3. The value of the auto-correlation as $\tau \to \infty$ if the observed phenomenon contains no periodic or drifting components is zero. The demonstration of this property is beyond the scope of this book. It can be noted, however, that if a process is random the correlation between an observation at time t and an observation taken at time τ later would tend to become infinitessimally small as the time τ became large.

XVIII-1.6 AUTO-CORRELATION TO ENERGY (OR VARIANCE) DENSITY SPECTRUM

XVIII-1.6.1 Dirac Delta Function

The concept of a spectrum (amplitude and phase) in the frequency domain can be introduced. The idea of a density spectrum is similar to the idea of a probability density. The energy density spectrum really expresses the rate of change of variance or derivative of the signal as a function of frequency. However, the square of the amplitude is used instead of the amplitude. The density spectrum corresponding to a simple sinusoidal wave (Figure XVIII-3) would appear as a spike of infinite height at the frequency $\frac{2\pi}{T}$ since the variation of the square amplitude as a function of frequency is infinity. However, the spike has by definition a finite area:

$$\int_{-\infty}^{\infty} S(k)\, dk = \frac{1}{2} A^2$$

Such a function is referred to as a Dirac delta function. A periodic wave of arbitrary shape which has the amplitude spectrum of Figure XVIII-2 will show a comb of Dirac delta functions, each one of infinite height at the frequencies k_1, k_2, k_3, ... but having finite areas equal to $A_1^2/2$, $A_2^2/2$, $A_3^2/2$, etc.

XVIII-1.6.2 Variance Spectrum and Power Spectrum

A random function which contains no periodic components will have no infinite density spikes in the frequency domain. Consider the Fourier transform of the auto-correlation function

$$\Phi(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau)\, e^{-ik\tau}\, d\tau$$

Then by definition:

$$R(\tau) = \int_{-\infty}^{\infty} \Phi(k)\, e^{ik\tau}\, dk$$

When $\tau = 0$

$$R(0) = \int_{-\infty}^{\infty} \Phi(k)\, dk = \lim_{T_1 \to \infty} \frac{1}{T} \int_{0}^{T_1} \eta(t)\, \eta(t)\, dt$$

$$= \overline{\eta^2(t)}$$

It is seen that the total area under the curve $\Phi(k)$ is the mean square value of $\eta(t)$. The area of any element $\Phi(k)\, dk$ represents the mean square contribution of the variance of $\eta(t)$ in the interval $\pm \frac{1}{2} \delta k$. $\Phi(k)$ is the variance density frequency spectrum of $\eta(t)$. It is often miscalled the energy spectrum or the power spectrum or simply the spectrum of the process $\eta(t)$. It can be shown that the variance spectrum may be called the energy spectrum for deep water linear waves. It is never a "power" spectrum as used in the above definition, but it is always a variance spectrum for any water surface record – linear or non-linear.

The reason for the term energy spectrum follows from the definition,

$$\int_{-\infty}^{\infty} \Phi(k)\, dk = \overline{\eta^2}(t)$$

If $\eta(t)$ is written as,

$$\eta(t) = \sum_{n=1}^{\infty} A_n \cos(knt + \epsilon_n)$$

then upon squaring and averaging the right hand side of the above equation

$$\overline{\eta^2} = \frac{1}{2} \sum A_n^2$$

It is now recalled that the average energy per unit length and per unit of crest in a linear (small amplitude) wave can be written as

$$E_{av.} = \frac{1}{8} \rho g H^2$$

where H is the wave height $(2 A_n)$. A wave of height $2 A_n$ and frequency k_n in the confused sea contributes $\frac{1}{2} \rho g A_n^2$ to the energy of that sea. It is seen from the above equations that $\rho g \int \Phi(k)\, dk$ represents the energy in the sea (for linear waves). The description of a sea in terms of its variance spectrum $\Phi(k)\, dk$ is a purely statistical definition. If physical meaning such as "energy spectrum" is used, it should be born in mind that the "variance spectrum" and "energy spectrum" are only equivalent (after a factor of ρg is inserted) for deep water small amplitude wave motions.

XVIII-1.6.3

Advantage can be taken of the even property of $R(\tau)$ to reduce the Fourier transform to the cosine transform.

$$\Phi(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau) \cos k\tau \, d\tau$$

and from the symmetry $\Phi(k) = \Phi(-k)$ (by inspection) an alternative definition for the spectrum is

$$S(k) = \frac{1}{\pi}\int_0^\infty R(\tau)\cos k\tau\, d\tau$$

Some authors use the frequency spectrum in cycles/sec. $(\frac{k}{2\pi})$. The relationship is simply:

$$S(f) = 2\int_0^\infty R(\tau)\cos 2\pi f\tau\, d\tau$$

One advantage of this definition is that the factor $\frac{1}{2\pi}$ in the Fourier transform is not required. It is also pointed out that in order to transform a spectrum $S(k)$ to the equivalent $S(f)$ it must be recalled that this is a density spectrum and the identity $S(f)\,df \equiv S(k)\,dk$ must be maintained so that $S(k) = \frac{1}{2\pi}S(f)$. Some authors have also advocated the use of "period" spectrum defined by

$$S(T)\,dT \equiv S(f)\,df \quad \text{where} \quad f = \frac{1}{T}$$

This has some advantage in giving a "feel" for the effects of wave period but in fact there is no reason why the same "feel" cannot be developed for frequency (f or $k = 2\pi f$) once familiarity is achieved.

The spectrum and the auto-correlation function have many desirable properties when used to describe random processes such as sea-states. They (particularly the spectrum) have stable statistical properties in that sample records taken from the same sea-state yield closely reproducible spectra. One further obvious usefulness is the condensation of say a 30-minute record of water surface into 50-100 points in a frequency domain. Attempts have been made to fit empirical

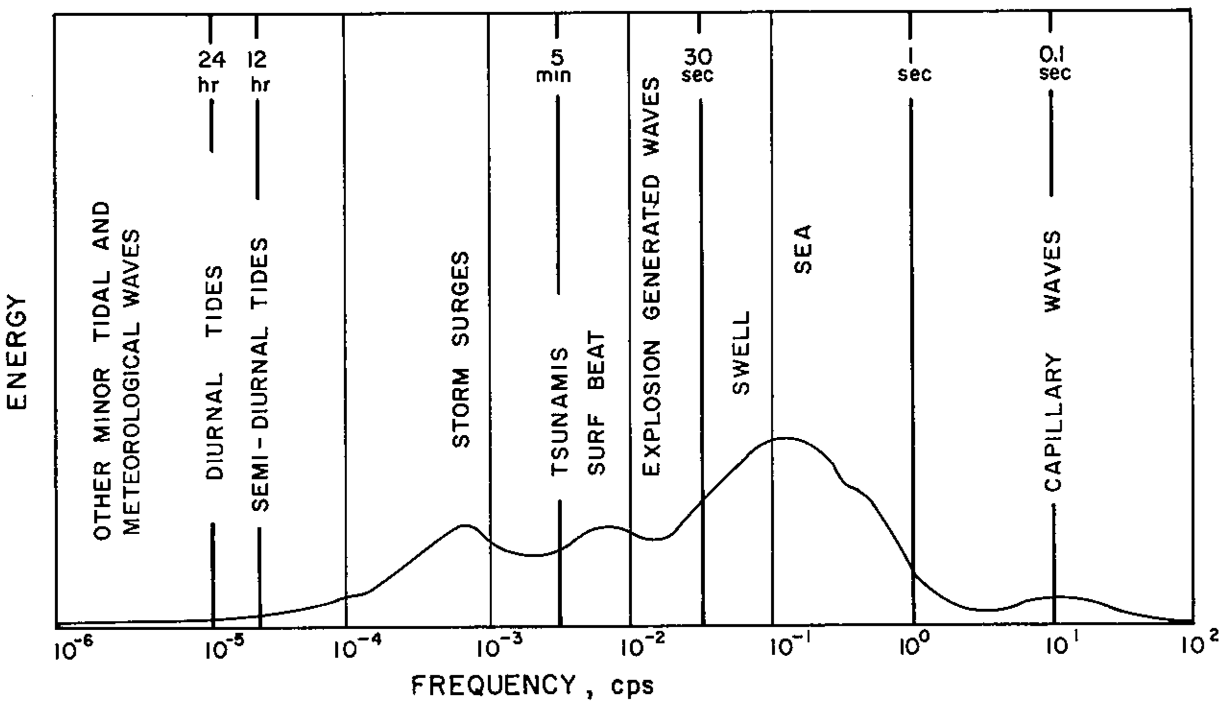


FIGURE XVIII-5
ENERGY SPECTRUM AT SEA

curves to wind spectra.

A typical spectrum of sea-surface phenomena is shown in Figure XVIII-5. This figure illustrates the complete range of the wave spectrum from infra-tidal effects to capillary waves.

XVIII-2 PROBABILITY FOR WAVE MOTIONS

XVIII-2.1 THE CONCEPTS OF PROBABILITY DISTRIBUTIONS AND PROBABILITY DENSITY

A probability distribution is normally defined when the proportion of values of the considered variable which are less than a particular value are plotted against that value. The most familiar shape for such a distribution is the typical "S" shape, as is shown in Figure XVIII-6.

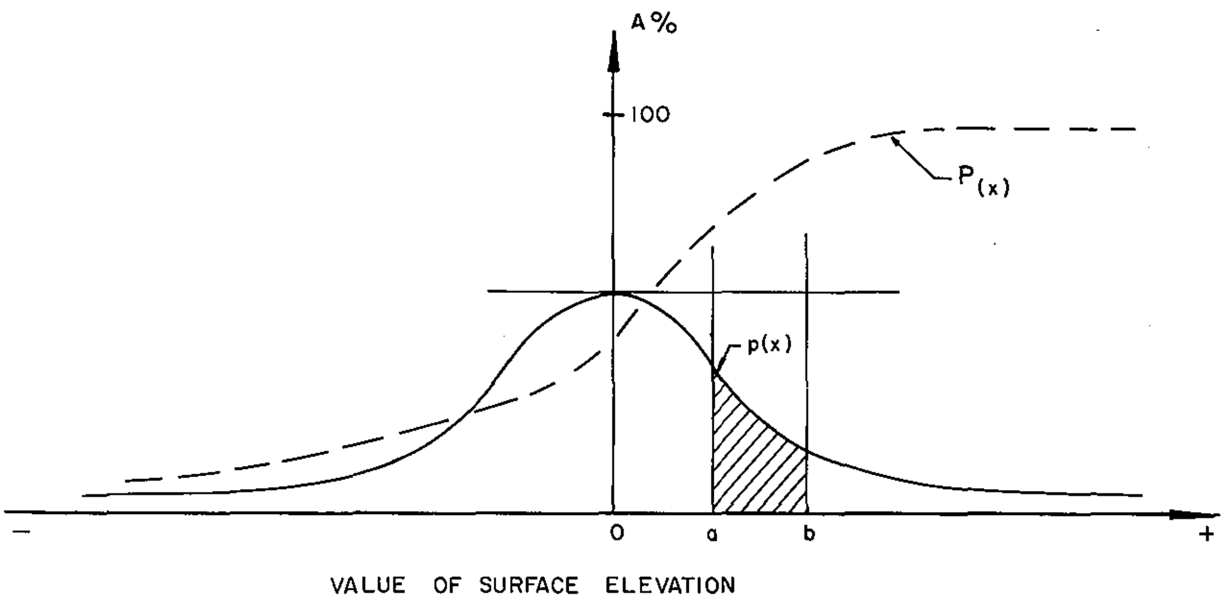


FIGURE XVIII-6
PROBABILITY DISTRIBUTION OF FREE SURFACE ELEVATION

The derivative of the probability distribution is called the probability density. The area under the probability density which lies between two values a, b defines the probability that the result of the event being observed lies between a and b. The total probability of the event having all possible outcomes must be unity.

Suppose the water surface elevation of the sea is being considered. Then

$$\int_{-\infty}^{\infty} p(\eta)\, d\eta = 1$$

where $p(\eta)$ is the probability density of the water surface elevation η. The probability distribution of η, say $P(\eta)$, would be (see Figure XVIII-6)

$$P(\eta) = \int_{-\infty}^{\eta} p(\eta)\, d\eta$$

XVIII-2.2 THE PROBABILITY DENSITY OF THE WATER SURFACE ORDINATES

There is no theoretical justification for the form of the probability density of a sea state. It is usually assumed to be Gaussian (normal) having zero mean value (measured from the still water level), i.e.,

$$p(\eta)\, d\eta = \frac{1}{\sqrt{2\pi \overline{\eta^2}}}\, e^{-\frac{\eta^2}{2\overline{\eta^2}}}\, d\eta$$

where $\overline{\eta^2}$ is the mean square surface fluctuation.

The form of this equation may be varied. If η is measured in units of $\sqrt{\overline{\eta^2}}$ $(=k_\eta)$ then the above equation becomes

$$p\left(\frac{\eta}{k_\eta}\right) d\left(\frac{\eta}{k_\eta}\right) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\eta}{k_\eta}\right)^2} d\left(\frac{\eta}{k_\eta}\right)$$

It is noted here that $\overline{\eta^2} = k_\eta^2$ is the mean square surface fluctuation and is equal to the area under the spectrum. Some authors use different definitions for the total area under the spectrum involving factors of 2, 4 or sometimes 8 or 16 for various reasons. Mathematically speaking, there is a preference for retaining

$$\int_0^\infty S(f)\, df = \overline{\eta^2} = k_\eta^2$$

and this definition is normal for spectral applications in most other fields. In other words, the total area under the spectrum is the variance of the process being studied.

XVIII-2.3 PROBABILITY FOR WAVE HEIGHTS

A one-dimensional wave train is assumed to be described by the function

$$\eta(t) = \sum A_n \cos(k_n t + \epsilon_n)$$

where the range of k_n is distributed over the frequencies contained in the spectrum, the ϵ_n's are arbitrary phase positions and the A_n's are governed by the spectrum as a function of k_n.

If the spectrum is spread over a narrow range of frequencies such that the mid-frequency is k_m and the actual values of k_n only vary from this by a small amount, the above equation can be written,

$$\eta(t) = \sum A_n \cos(k_n t - k_m t + \epsilon_n + k_m t)$$

$$= A_c \cos k_m t + A_s \sin k_m t$$

where

$$A_c = \sum A_n \cos(k_n t - k_m t + \epsilon_n)$$

$$A_s = \sum A_n \sin(k_n t - k_m t + \epsilon_n)$$

In the equation for $\eta(t)$ the assumption of a narrow range of k_n means that A_c and A_s vary very slowly with time.

Now let $R = \sqrt{A_c^2 + A_s^2}$. Also, from the central limit theorem of probability the values of A_c and A_s from the definitions of the above equations will be normally distributed so long as the sums are taken over a sufficiently large number of terms and

$$\overline{A_c^2} = \overline{A_s^2} = \overline{\eta^2} = \int_0^\infty S(f)\, df$$

Therefore, the probability that A_c, A_s lies within the element $dA_c\, dA_s$ is given by the following probability density function

$$p(A_c, A_s)\, dA_c\, dA_s = \frac{1}{2\pi\, \overline{\eta^2}} \exp\left[-\frac{A_c^2 + A_s^2}{2\overline{\eta^2}}\right] dA_c\, dA_s$$

so long as the variables A_c and A_s are statistically independent. This follows from the fact that $\overline{A_c A_s} = 0$. Now, put

$$A_c = R \cos\alpha$$

$$A_s = R \sin\alpha$$

and transform $p(A_c, A_s)$ by means of the identity $p(A_c, A_s)\, dA_c\, dA_s \equiv p(R, \alpha)\, R\, dR\, d\alpha$ to,

$$p(R, \alpha)\, dR\, d\alpha = \frac{1}{2\pi} \frac{R}{\overline{\eta^2}}\, e^{-R^2/2\overline{\eta^2}}\, d\alpha\, dR$$

in which the variables can be separated to give

$$p(R, \alpha)\, dR\, d\alpha = p(R)\, dR \cdot p(\alpha)\, d\alpha$$

$$= \frac{R}{\overline{\eta^2}}\, e^{-R^2/2\overline{\eta^2}}\, dR \cdot \frac{1}{2\pi}\, d\alpha$$

so that

$$p(R)\, dR = \frac{R}{\overline{\eta^2}}\, e^{-R^2/2\overline{\eta^2}}\, dR$$

$$p(\alpha)\, d\alpha = \frac{1}{2\pi}\, d\alpha$$

In fact R can be considered as the wave amplitude $\frac{H}{2}$ and α can be considered as the wave phase position ϵ_n. The last equation indicates that the phase position of the waves has a uniform distribution between zero and 2π. $p(R)\,dR$ can be further simplified since for a sine wave the mean square value is equal to one half of the square of the amplitude.

Therefore, if $H = 2R$, $\overline{\eta^2} = \frac{1}{2}\overline{R^2} = \frac{1}{8}\overline{H^2}$ so that

$$p(H)\,dH = \frac{2H}{\overline{H^2}}\, e^{-H^2/\overline{H^2}}\, dH$$

which is the well-known Rayleigh distribution.

In spite of the many apparent assumptions in the derivation of this equation, it has been found to be an extremely good fit for observed wave height distributions in wind generated seas.

XVIII-2.4 PROBABILITY FOR WAVE PERIODS

There is at present no simple development of the probability distribution of wave periods from the spectrum. The probability of wave periods (based on zero-crossings or on the crossing of $\eta(t)$ with the still water level) should be related to the auto-correlation function in some manner but the exact form has not been developed. In order to introduce some estimate of the "mean period" based on the variance density spectrum, the concept of spectrum moments is introduced. The moment of the spectrum of order n is given by

$$M_n = \int_0^\infty f^n\, S(f)\, df$$

This equation is in direct analogy to the moment of a probability density

$(\int_{-\infty}^{\infty} x^n \, p(x) \, dx)$.

The expected (average) value for the time between successive zero crossings of a process having the spectrum $S(f)$ is given by

$$\frac{1}{2}\,\tilde{\bar{T}} = \frac{1}{2}\left[\frac{M_o}{M_2}\right]^{\frac{1}{2}} = \frac{1}{2}\left[\frac{\int_o^{\infty} S(f)\, df}{\int_o^{\infty} f^2\, S(f)\, df}\right]^{\frac{1}{2}}$$

where $\tilde{\bar{T}}$ is defined as the mean apparent wave period (the wave period will be twice the expected time between successive zero crossings). The proof of the above equation can be found in more advanced textbooks. It is considered as beyond the scope of the current chapter. In a similar manner the expected period between successive maxima and minima (points of zero gradient, $\frac{\partial \eta}{\partial t} = 0$) is found to be given by the ratio

$$\frac{1}{2}\left[\frac{M_2}{M_4}\right]^{\frac{1}{2}}$$

and this is seen to be quite different from $\frac{\tilde{\bar{T}}}{2}$ for a general spectrum shape $S(f)$. The two are identical if $S(f)$ is considered as a delta function having only one frequency.

The relationship of $\tilde{\bar{T}}$ to the auto-correlation function is best illustrated by recalling a well-known statistical theorem. "The moment of a probability distribution of order n is given by the n^{th} derivative of the characteristic function of the probability density at the origin." The characteristic function of a probability density is its Fourier transform. The Fourier transform of a spectrum is the auto-correlation function (see Section XVIII-1.6) so that M_o, M_2 and M_4 are simply the zero, second and fourth derivatives of the auto-correlation function at $\tau = 0$.

XVIII-2.5 PROBABILITY FOR SUB-SURFACE VELOCITY AND ACCELERATIONS

The probability distributions of sub-surface velocities and accelerations in the case of small wave motions (linear assumption) are usually assumed to have a normal distribution. In order to define a normal distribution it is only necessary to be able to define the variance. This can be done for velocities and accelerations in terms of the surface variance spectrum and the use of hydrodynamic potential theory.

The water surface is described by the sum of an infinite number of sinusoids having the variance spectral distribution $S(f)$. For the case of any linear random process it can be demonstrated that the effect of a linear operation on the process can be described in terms of the spectrum of the process multiplied by the square of the generating transform. A sea state is described by

$$\eta(t) = \sum A_n \sin(2\pi f_n t + \epsilon_n)$$

The corresponding sub-surface velocities and accelerations are (see Section XVI-3.4),

$$u(t) = \sum 2\pi f_n A_n \frac{\cosh m_n(d+z)}{\sinh m_n d} \sin(2\pi f_n t + \epsilon_n)$$

and

$$\frac{\partial u(t)}{\partial t} = \sum 4\pi^2 f_n^2 A_n \frac{\cosh m_n(d+z)}{\sinh m_n d} \cos(2\pi f_n t + \epsilon_n)$$

The square of the modulus of the operating transform for the process $u(t)$ is seen to be,

$$\left[2\pi f_n \frac{\cosh m_n (d+z)}{\sinh m_n d}\right]^2$$

So that,

$$S_u(f)\, df = \left[\frac{2\pi f \cosh m(d+z)}{\sinh m d}\right]^2 S_\eta(f)\, df$$

where $S_u(f)$ is the velocity spectrum at elevation z and $S_\eta(f)$ is the surface spectrum.

By definition of the spectrum,

$$\overline{u^2} = \int_0^\infty S_u(f)\, df$$

$$= \int_0^\infty \left[\frac{2\pi f \cosh m(d+z)}{\sinh m d}\right]^2 S_\eta(f)\, df$$

It must be recalled in the transformation function that m is a function of f given by the wave equation (see Section XVI-3.3)

$$(2\pi f)^2 = m g \tanh m d$$

Once $\overline{u^2}$ is determined the probability density distribution is given by

$$p(u)\, du = \frac{1}{\sqrt{2\pi \overline{u^2}}}\, e^{-\frac{u^2}{\overline{u^2}}}\, du$$

In a similar manner the probability density of the acceleration and sub-surface pressures can be derived.

XVIII-3 DISCUSSION OF NON-LINEAR PROBLEMS

XVIII-3.1 EFFECTS OF NON-LINEARITY ON PROBABILITY DISTRIBUTIONS

It is well-known and readily observable that high waves tend to have shorter peaked crests and longer flatter troughs than the simple sinusoidal representation would indicate. Since the mean water level must remain the same it would be expected that the probability density of the surface ordinates will not be the same for large sea states as for lower ones. The positive ordinates will be larger but less frequent whereas negative ordinates will be smaller but more frequent. A skewness is introduced into the probability density (see Figure XVIII-7).

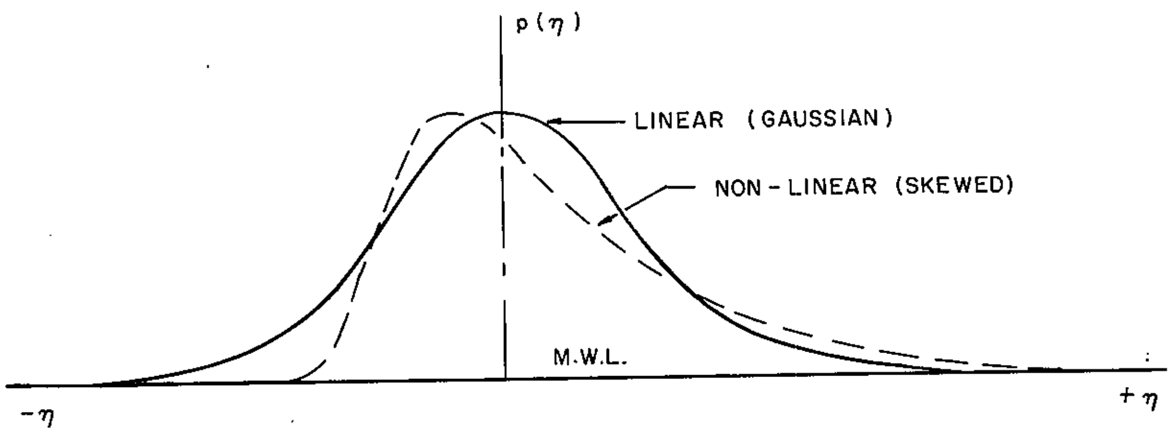


FIGURE XVIII-7
AN ILLUSTRATION OF NON-LINEAR EFFECTS ON THE PROBABILITY DISTRIBUTION OF FREE SURFACE ELEVATION

The effect of skewness in water surface ordinate probability distributions is reflected to some extent in the probability distributions of wave heights. However this effect is not very marked and does not appear to be very important. It must be recalled that as the crests become more peaked the troughs become flatter. As far as wave heights are concerned these effects are compensatory and wave height probabilities for high seas are still very close to the Rayleigh distribution.

The effects on wave period distributions of large seas has not received much study. In random seas the concept of "wave period" does not really exist. The only thing which may be mentioned is that the times between successive up-down crossings of the mean water level will be shorter than the times between successive down-up crossings. The estimates of average time between successive zero crossings and successive maxima and minima as presented in the section on wave period probability will not be very reliable.

XVIII-3.2 EFFECTS OF NON-LINEARITIES ON SPECTRA AND SPECTRAL OPERATIONS

Non-linear problems can appear in wave spectrum studies in many forms. Clearly the mechanical process of harmonic analysis as described in Section XVIII-1 is not affected by non-linear problems. It is the interpretation of the resultant spectrum which is difficult. The high frequency part of the spectrum is an aggregate of the small ripples mixed with the sea and the harmonics of some of the larger low frequency wave components. It would be preferable to separate the harmonics since they travel at the phase speed of the "fundamental" wave, whereas the shorter waves travel much slower.

The spectrum destroys all details of phase position and also all differences between "crests" and "troughs". The normal wave spectrum is obviously an insufficient descriptor of a high (non-linear) sea state.

One of the major advantages of the spectral description of a sea state became apparent in the section on sub-surface velocities and accelerations. The velocity, acceleration, pressure, etc. spectra are easily determined from the water surface spectrum. The transfer functions for non-linear sea states will not be so simple. One other form of non-linearity arises in operations on the spectrum. For example, the prediction of wave force spectra from sea surface spectra when the function is of the form $f = a u^2 + b \dot{u}$ is one involving a non-linear operation on the water surface spectrum. These problems are not yet solved.

XVIII-1 Sketch the auto-correlation functions for a sine wave and a random noise. Suggest a method for detecting a sinusoidal component in a random process.

Answer:

The method for detecting periodic components consists of determining the auto-correlation function at an arbitrarily large lag time τ.

XVIII-2 Sketch the variance density spectrum of a periodic wave. Why is the description of a periodic motion in density spectrum form not very useful?

Answer:

The density spectrum of a periodic function is in general an infinite series of delta functions. A periodic phenomenon is best described in terms of a simple Fourier series rather than a density spectrum. The relative amplitudes of the harmonics cannot be represented in a "density" domain.

XVIII-3 If $f(x) = x^2$ and $p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} x^2}$, determine $p[f(x)]$

and sketch.

Answer:

For each value of x there is only one value of $p(x)$. It is required after the transformation of x to $y = f(x)$ $(= x^2)$ that the probability $p(y)$ is also unique. In particular, $p(x)$ in the range $x - \frac{\delta x}{2} < x < x + \frac{\delta x}{2}$ is now represented by $p(y)$ in the y plane in the range $y - \frac{\delta y}{2} < y < y + \frac{\delta y}{2}$. This requirement is only met if the identity $p(x)\,dx \equiv p(y)\,dy$ is satisfied for all x and y. This yields the relationship $p(y) = p[f(x)] = p(x)\frac{dx}{dy}$. However, in this particular problem one further point arises. $p(y)$ can only exist for $y = x^2$ for positive values of y. In short, for each value of y, two values of x, $\pm x$ can satisfy the relationship $y = x^2$, Hence, $p(y)\,dy \equiv p(-x)\,dx + p(x)\,dx$ is the identity required and since $p(x)$ is an even function,

$$p(y)\,dy = 2\,p(x)\,dx \qquad \text{when} \qquad y = x^2$$

therefore

$$p(y) = 2\,p(x)\frac{dx}{dy}$$

$$= \frac{1}{(2\pi y)^{\frac{1}{2}}}\, e^{-\frac{y}{2}} \qquad \text{for} \qquad y > 0$$

$$= 0 \qquad \text{for} \qquad y < 0$$

XVIII-4 Show that the auto-correlation function of any stationary random variable is even.

Answer:

$$R(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{t=\theta}^{t=T} f(t)\, f(t+\tau)\, dt$$

Substitute $t = t - \tau$:

$$r(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{t=-\tau}^{t=T-\tau} f(t-\tau)\, f(t)\, dt$$

does not alter the average properties. Now the limits can be shifted without loss of generality since the process is a stationary one. Hence

$$R(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{o}^{T} f(t-\tau)\, f(t)\, dt$$

$$= R(-\tau) \quad \text{by definition.}$$

XVIII-5 In the determination of the spectrum of a random process, the auto-correlation is first computed followed by a Fourier transformation in the form of a cosine transformation. What would you expect to find if a sine transform on the auto-correlation was performed? Why?

Answer:

$R(\tau)$ can be represented by the Fourier series:

$$R(\tau) = a_o + \sum_{n=-\infty}^{\infty} a_n \cos n\omega t + \sum_{n=-\infty}^{\infty} b_n \sin n\omega t$$

The sine transformation corresponds to the operation

$$\int R(\tau) \sin m\omega\tau \, d\tau$$

From the general theory of Fourier analysis, this integral only exists for any specified value of m when $R(\tau)$ has sinusoidal components containing the argument $m\omega\tau$. Since $R(\tau)$ is an even function, the assumed Fourier series

$$R(\tau) = a_o + \sum a_n \cos n\omega t + \sum b_n \sin n\omega t$$

cannot contain any sinusoidal (asymmetric) components. Hence

$$R(\tau) = a_o + \sum a_n \cos n\omega t$$

and it follows that the sine transformation is identically zero.

XVIII-6 A random variable x has the exponential probability density

$$p(x) = a \exp(-b|x|)$$

where a and b are constants. Determine the relationship between a and b and the probability distribution function $P(x)$. Sketch $p(x)$ and $P(x)$.

Answer:

$$b = 2a, \quad P(x) = \frac{1}{2} e^{-2a|x|} \quad \text{for} \quad x < 0$$

$$= \frac{1}{2} + \frac{1}{2}\left[1 - e^{-2ax}\right] \quad \text{for} \quad x > 0$$

XVIII-7 On the assumption of linear wave motion, derive the function for the variance of the sub-surface bottom pressure in water depth d for a sea state given by

$$\eta(t) = \sum A_n \cos(\tau\pi f_n t + \epsilon_n)$$

Answer:

$$\eta(t) = \sum A_n \cos(2\pi f_n t + \epsilon_n)$$

From linear theory for periodic waves

$$P = \rho g \left[z + \frac{1}{\cosh kd} \eta \right]$$

Therefore

$$S_p(f) = (\rho g)^2 \frac{1}{\cosh^2 kd} S_\eta(f)$$

$$\sigma_p = \int_o^\infty S_p(f) = (\rho g)^2 \int_o^\infty \frac{1}{\cosh^2 kd} S_\eta(f)$$

XVIII-8 The probability density distribution of wave heights in a confused sea is given by:

$$p(H)\,dH = \frac{H}{2\,\overline{H^2}} e^{-H^2/\overline{H^2}}\,dH$$

Establish the relationships between the most probable wave height, the average wave height, the significant wave height (the mean of the highest 1/3 of the wave), and the highest wave (probability 1/100).

Answer:

			Probability
$H_{average}$	=	1.25 x H m.p.	0.460
$H_{most\ probable}$	=	1 x H m.p.	0.606
$H_{significant}$	=	2 x H m.p.	0.135
$H_{maximum}$	=	3 x H m.p.	0.01

NOTATION

A or ΔA	Cross section or element of cross section
C	Wave (or phase) velocity
C_D	Drag coefficient
C_M	Inertial coefficient
C_f	Resistance coefficient on a boundary due to shearing stress
C_h	Chezy coefficient
C_x	Drag coefficient
D	Diameter of a pipe
E	Modulus of elasticity Specific energy
F	Force
F'	Force on a body due to added mass
F_e	External forces
F_i	Internal forces
H	Total head, sum of kinetic head, piezometric head, and pressure head $\left(H = \frac{V^2}{2g} + \frac{p}{\omega} + z\right)$ Also wave height
ΔH	Head loss

I	Specific force
K	Bulk modulus of elasticity Also hydraulic conductivity of a porous medium Also $A\,C_h\sqrt{R_H}$: "conveyance" of a channel
M	Mass of a body
M'	Added mass
OX, OY	Horizontal axes (usually at the still water level)
OZ	Vertical axis (usually positive upwards)
P	Wetted perimeter Probability distribution function
Q (or q)	Discharge
R	Total force on a boundary due to shearing stress Also radius of a cylinder or of a sphere Also the reflection coefficient in wave theory
R (or r)	Radius Also the universal gas constant
R (or r), θ	Coordinates of a point in a cylindrical system of coordinates
R_H	Hydraulic radius
R_ℓ	Reynolds number
$R(\tau)$	Auto-correlation function
S	Wave spectrum Also bottom slope = $\tan\theta$

S (or ΔS)	Element of streamline
S_c	Critical bottom slope
S_f	Slope of the energy line
T (or ΔT)	Interval of time or period (wave theory)
$\tilde{\bar{T}}$	Average time interval between the crossing of free surface elevation with the still water level
U	Group velocity Also average velocity in a pipe $(U = Q/A)$ Also velocity of an immersed body in a fluid
U_E	Rate of propagation of energy
$U_R = \frac{\eta_o}{L}\left(\frac{L}{d}\right)^3$	Ursell parameter
U_o	Velocity outside the boundary layer
$\vec{V}$	Velocity vector
$\bar{\vec{V}}\ (u, v, w)$	Average velocity vector with respect to time and its components in a turbulent flow
$\bar{\bar{\vec{V}}}\ (\bar{\bar{u}}, \bar{\bar{v}}, \bar{\bar{w}})$	Average velocity vector with respect to space and its components in a flow through porous medium
$\vec{V}'\ (u', v', w')$	Velocity vector for the turbulent fluctuations and its components
W	Velocity of propagation of a bore
X, Y, Z	Volume or body force (gravity) along OX, OY, OZ respectively $(X = 0,\ Y = 0,\ Z = -\rho g z)$

$a = \frac{\partial u}{\partial x}$, $b = \frac{\partial v}{\partial y}$, $c = \frac{\partial w}{\partial z}$	Coefficients of linear deformation along OX, OY, and OZ respectively
$c = \sqrt{g\,(d+\eta)}$	Definition
d	Depth (wave theory)
e	Thickness of a pipe wall
$f = \frac{g}{C_h^2}$	Friction coefficient
$f = \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$, $g = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$, $h = \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$	Coefficients of angular deformation along OX, OY, and OZ respectively
g	Gravity acceleration
h	Depth (channel) Also water depth Also coefficient of angular deformation
$h^* = \frac{h}{E}$	Reduced water depth
h_c	Critical depth
h_n	Normal depth

h_1, h_2	Conjugate water depths
i	$\sqrt{-1}$
k	Coefficient of permeability for porous medium Also coefficient of Von Karman in the theory of turbulence, universal constant $\cong 0.4$ Also $\frac{2\pi}{T}$ in the wave theory and periodic motions where T is the period
$k = \frac{C_p}{C_v}$	C_p is the specific head at constant pressure and C_v the specific heat at constant volume
k_s	Characteristic number for roughness size in pipes and wall boundaries
l	Prandtl's mixing length
ℓ	Width of a channel at the free surface
m	Mass Also $\frac{2\pi}{L}$, L: wave length (wave theory)
n (or Δn)	Element perpendicular to an area or to a streamline Also element of an equipotential line
p	Pressure Also probability density function
Δp	Pressure difference over a finite interval
p^*	$p + \rho g z$: pressure and gravity force
$\bar{p}$	Average pressure with respect to time in a turbulent flow

$\bar{\bar{p}}$	Average pressure with respect to space in a flow through porous medium
$q^* = \dfrac{g}{E\sqrt{2gE}}$	Reduced discharge
r_o	Radius of a pipe
$s = \dfrac{kD}{\nu}$	Strouhal number
t	Time
u	Local velocity in a pipe or in a boundary layer
$u^* = \sqrt{\dfrac{\tau_o}{\rho}}$	Shear velocity
u, v, w	Components of the velocity vector $\vec{V}$ along the three coordinate axes OX, OY, and OZ respectively
v_r	Component of the velocity vector along a radius in a cylindrical system of coordinates
v_θ	Component of the velocity vector perpendicular to a radius in a cylindrical system of coordinates
x, y, z	Coordinates of a point along OX, OY, and OZ respectively
x_o, y_o, z_o	Coordinates of a point at time t_o
Γ	Circulation of velocity
Φ	Dissipation function
Φ_m	Dissipation function due to viscous force

Φ_t	Dissipation function due to turbulent fluctuations
α	Correction factor for the kinetic energy term in a pipe Also (tang α) slope
δ	Diameter of a particle of a porous medium Also boundary layer thickness (general definition)
δ_o	Boundary layer thickness at $x = x_o$
δ^*	Displacement thickness for a boundary layer
ϵ	Coefficient of Boussinesq for shearing stress due to turbulent exchange Also void coefficient for a porous medium
η	Elevation of the free surface around the still water level
θ	tan θ: bottom slope Also momentum thickness for a boundary layer
λ	Second coefficient of viscosity for gas
μ	Coefficient of viscosity
ν	Kinematic coefficient of viscosity $(\nu = \frac{\mu}{\rho})$

$$\left.\begin{aligned} \xi &= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \\ \eta &= \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \zeta &= \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \end{aligned}\right\}$$ Coefficients of rotation along OX, OY, and OZ rexpectively

ρ	Density
σ	Normal stress
$[\sigma]$	Normal stress for a turbulent flow
τ	Shearing stress Also time interval
$[\tau]$	Shearing stress for a turbulent flow
τ_o	Shearing stress at the wall
ϕ	Potential function: $\vec{V} = -\overrightarrow{\text{grad}}\,\phi \quad \left(u = -\frac{\partial \phi}{\partial x},\ v = -\frac{\partial \phi}{\partial y},\ w = -\frac{\partial \phi}{\partial z}\right)$
$\phi\ (k)$	Variance spectrum
ψ	Stream function: $u = \frac{\partial \psi}{\partial y},\ v = -\frac{\partial \psi}{\partial x}$
ω	Angular rotation
$\bar{\omega}$	Specific weight $(\bar{\omega} = \rho\, g)$
$\frac{\partial A}{\partial *}$	Partial derivative (with respect to *)
$\frac{dA\ (x,\ y,\ z,\ t)}{dt}$	Total derivative (with respect to t) $= \frac{\partial A}{\partial t} + u\frac{\partial A}{\partial x} + v\frac{\partial A}{\partial y} + w\frac{\partial A}{\partial z}$
$\vec{A} \cdot \vec{B}$	Scalar product $= \|A\|\ \|B\|\ \cos\angle A, B$
$\vec{A} \times \vec{B}$	Vector product $= 1\ \|A\|\ \|B\|\ \sin\angle A, B$ where 1 is a vector perpendicular to the plane AB

∇	Del $\equiv \vec{i}\dfrac{\partial}{\partial x} + \vec{j}\dfrac{\partial}{\partial x} + \vec{k}\dfrac{\partial}{\partial z}$ where i, j, k are unit vectors
$\overrightarrow{\text{grad}}$ A or ∇A	Gradient of A or vector of components $\dfrac{\partial A}{\partial x}$, $\dfrac{\partial A}{\partial y}$, $\dfrac{\partial A}{\partial z}$, i.e., total variation of A with respect to space $\overrightarrow{\text{grad}}\, A = \overrightarrow{\dfrac{\partial A}{\partial x}} + \overrightarrow{\dfrac{\partial A}{\partial y}} + \overrightarrow{\dfrac{\partial A}{\partial z}} = \vec{i}\dfrac{\partial A}{\partial x} + \vec{j}\dfrac{\partial A}{\partial y} + \vec{k}\dfrac{\partial A}{\partial z}$
$\overrightarrow{\text{div}}\ \vec{A}$ or $\nabla\cdot\vec{A}$	Divergence of A: scalar sum of $\dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z}$
$\overrightarrow{\text{curl}}\ \vec{A}$ or $\overrightarrow{\text{rot}}\ \vec{A}$ or $\nabla \times \vec{A}$	Rotation of A. Vector of components 2ξ, 2η, 2ζ curl $\vec{A} = 2(\vec{\xi} + \vec{\eta} + \vec{\zeta}) = 2(\vec{i}\xi + \vec{j}\eta + \vec{k}\zeta)$

$$\text{or}\quad \text{curl}\ \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$\nabla^2 A = \text{div.}\,(\overrightarrow{\text{grad}}\, A)$	Laplacian, scalar sum of $\dfrac{\partial^2 A}{\partial x^2} + \dfrac{\partial^2 A}{\partial y^2} + \dfrac{\partial^2 A}{\partial z^2} = \dfrac{\partial^2 A}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial A}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 A}{\partial \theta^2} + \dfrac{\partial^2 A}{\partial z^2}$ when $\nabla^2 A = 0$, A is a harmonic function

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