Binomial Heaps

Last Week…

- Priority Queue Abstract Data Type

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  Object: Set S of elements with “keys” (“priority”) that can be compared

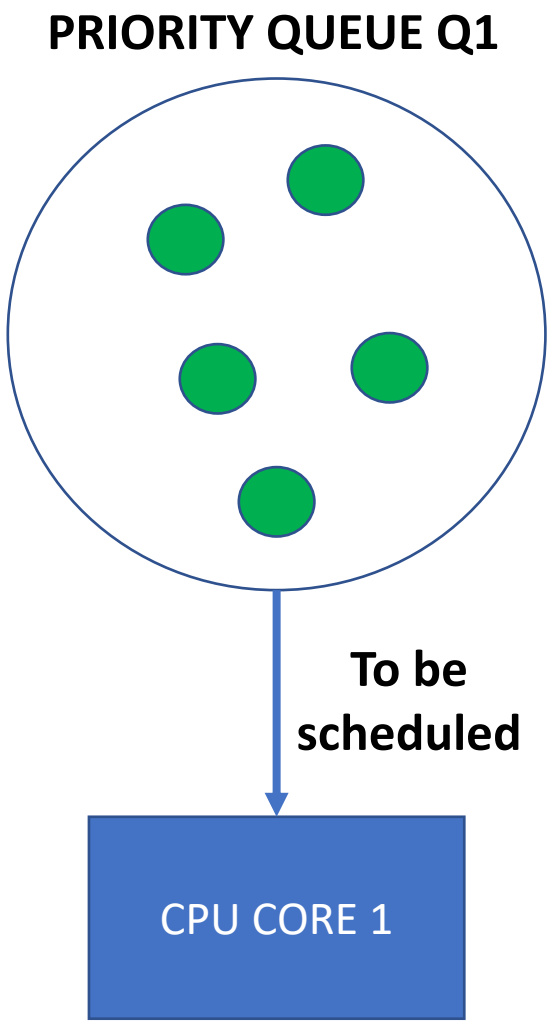
Last Week…

- Priority Queue Abstract Data Type

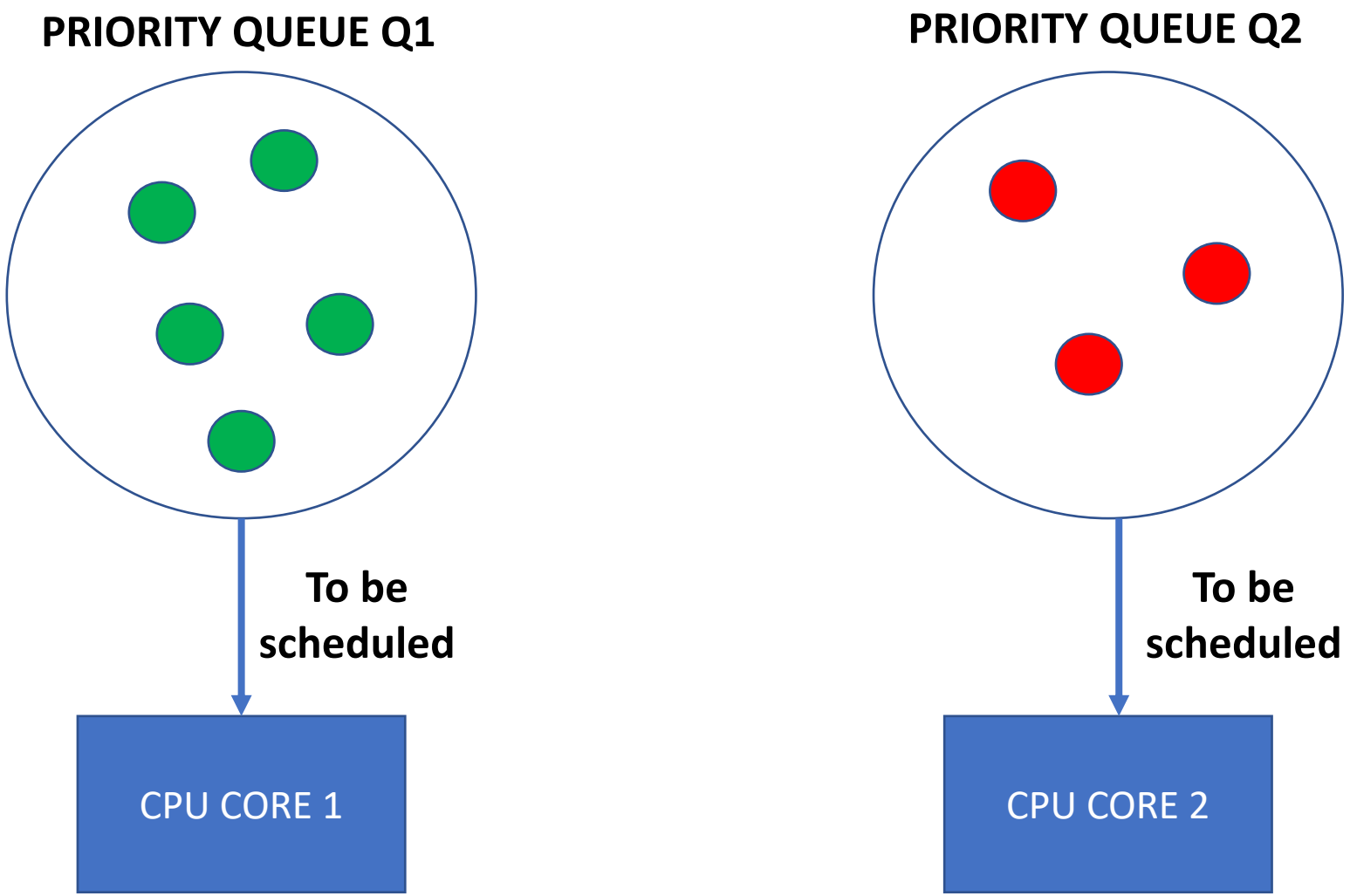
  Object: Set S of elements with “keys” (“priority”) that can be compared

  Operations: **Insert**(S,x), **Max**(S), **Extract_Max**(S)

Application: Job Scheduling by OS



Consider the following scenario



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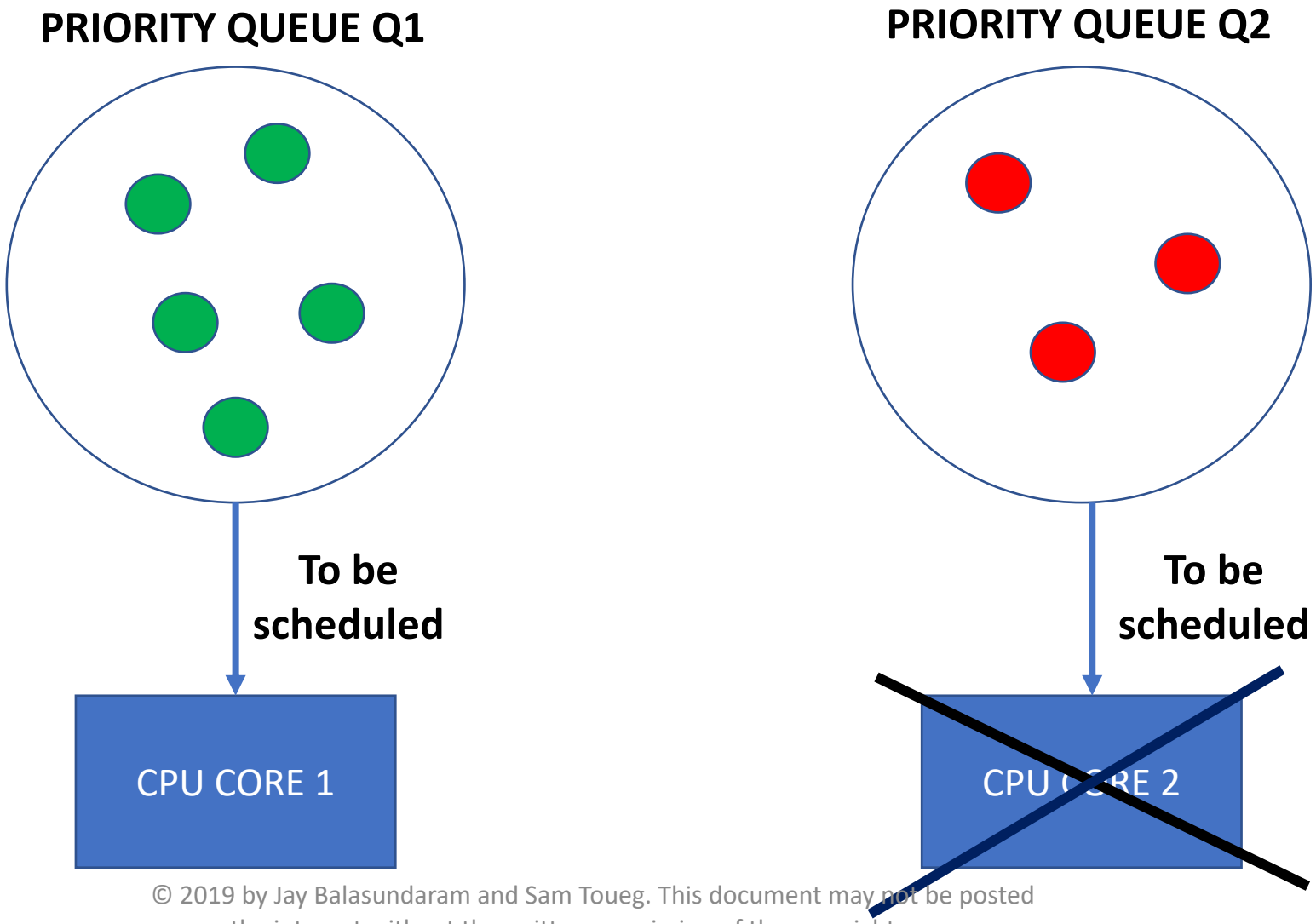

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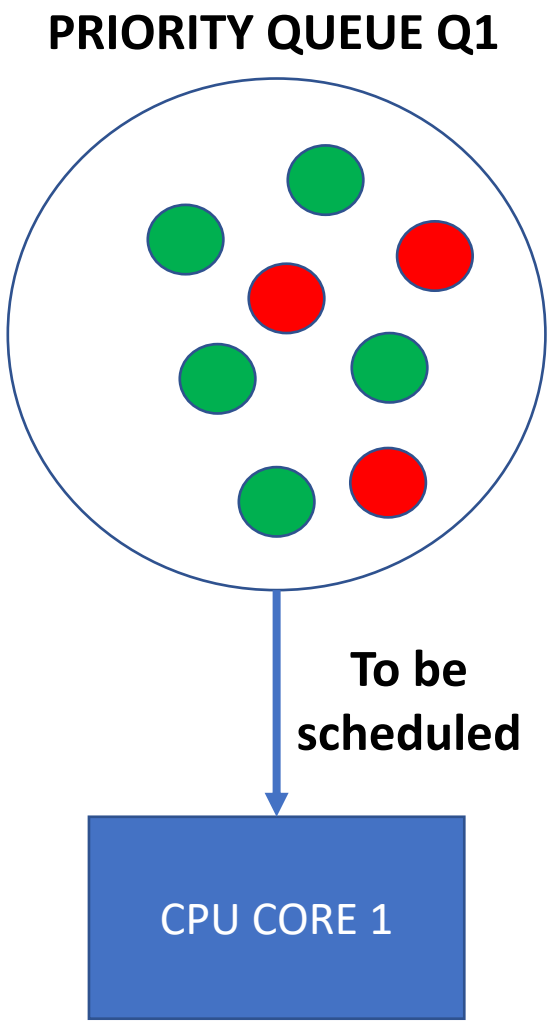

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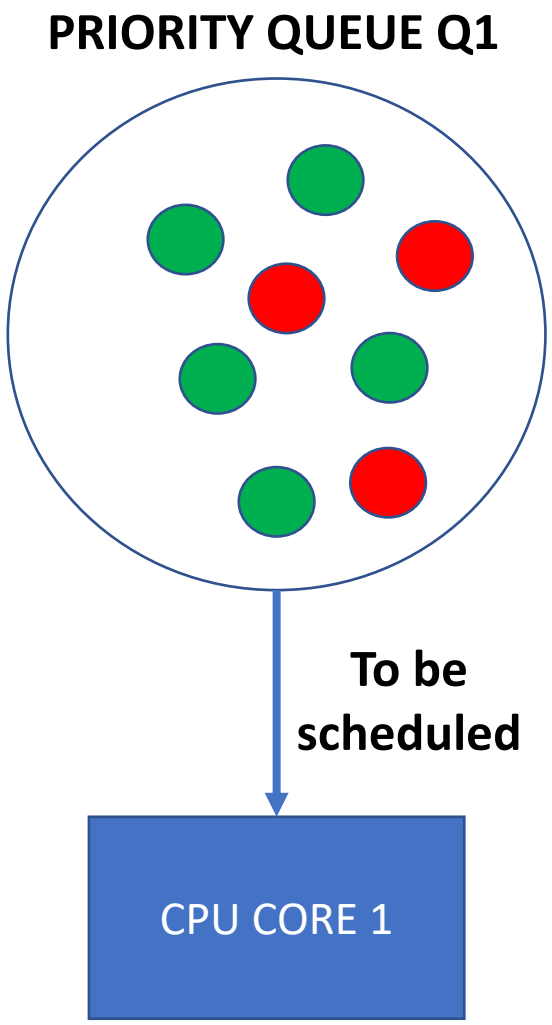

Goal: Implement **Union**(Q1, Q2)

Abstract Data Types

Abstract Data Types		Insert	Min	Extract_Min	Union
Priority Queues		✓	✓	✓	X

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Mergeable Priority Queues					

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Priority Queues		✓	✓	✓	X
Mergeable Priority Queues		✓	✓	✓	✓

Data Structures

Abstract Data Types	Data Structures	Insert	Min	Extract_Min	Union
Priority Queues	**Min-Heap**	✓	✓	✓	X
Mergeable Priority Queues		✓	✓	✓	✓

Data Structures

Abstract Data Types	Data Structures	Insert	Min	Extract_Min	Union
Priority Queues	**Min-Heap**	**O(log n)**	✓	✓	X
Mergeable Priority Queues		✓	✓	✓	✓

Data Structures

Abstract Data Types	Data Structures	Insert	Min	Extract_Min	Union
Priority Queues	Min-Heap	O(log n)	O(1)	✓	X
Mergeable Priority Queues		✓	✓	✓	✓

Data Structures

Abstract Data Types	Data Structures	Insert	Min	Extract_Min	Union
Priority Queues	Min-Heap	O(log n)	O(1)	O(log n)	X
Mergeable Priority Queues		✓	✓	✓	✓

Data Structures

Abstract Data Types	Data Structures	Insert	Min	Extract_Min	Union
Priority Queues	Min-Heap	O(log n)	O(1)	O(log n)	X
Mergeable Priority Queues	Min Binomial Heap	✓	✓	✓	✓

Data Structures

Abstract Data Types	Data Structures	Insert	Min	Extract_Min	Union
Priority Queues	Min-Heap	O(log n)	O(1)	O(log n)	X
Mergeable Priority Queues	Min Binomial Heap	O(log n)	O(log n)	O(log n)	O(log n)

Min Binomial Heaps

Visualizing Binomial Heaps

Elements are stored in a sequence of Binomial Trees.

Binomial Trees

B_k tree: defined recursively

Binomial Trees

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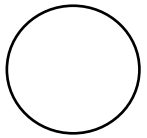
k = 0

Binomial Trees

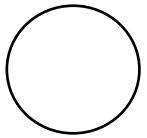
B_k tree: defined recursively

k = 0 B_0 :



Binomial Trees

B_k tree: defined recursively

k = 0 B_0 : 

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Binomial Trees

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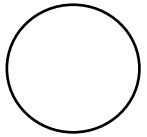
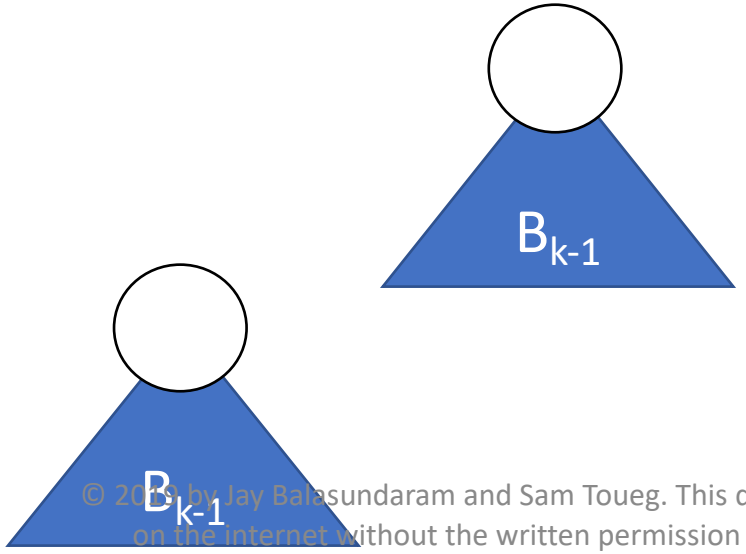


k > = 1



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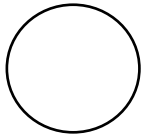
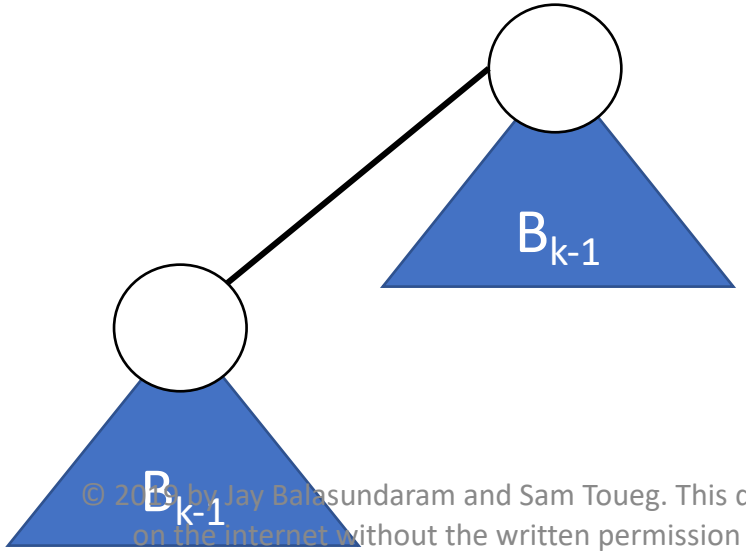


k > = 1 B_k :



Binomial Trees (for k = 0, 1, 2, 3)

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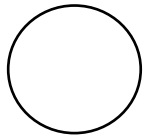
B_0



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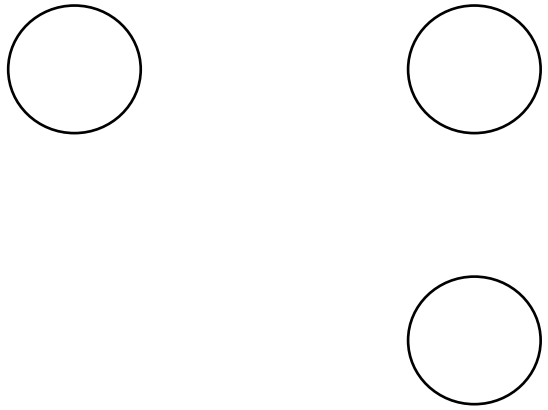
B_0



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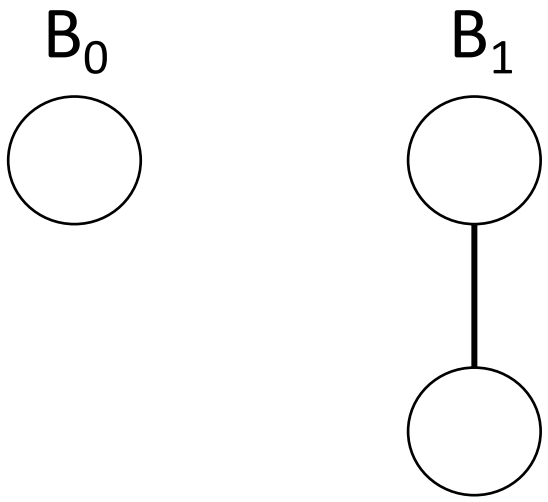

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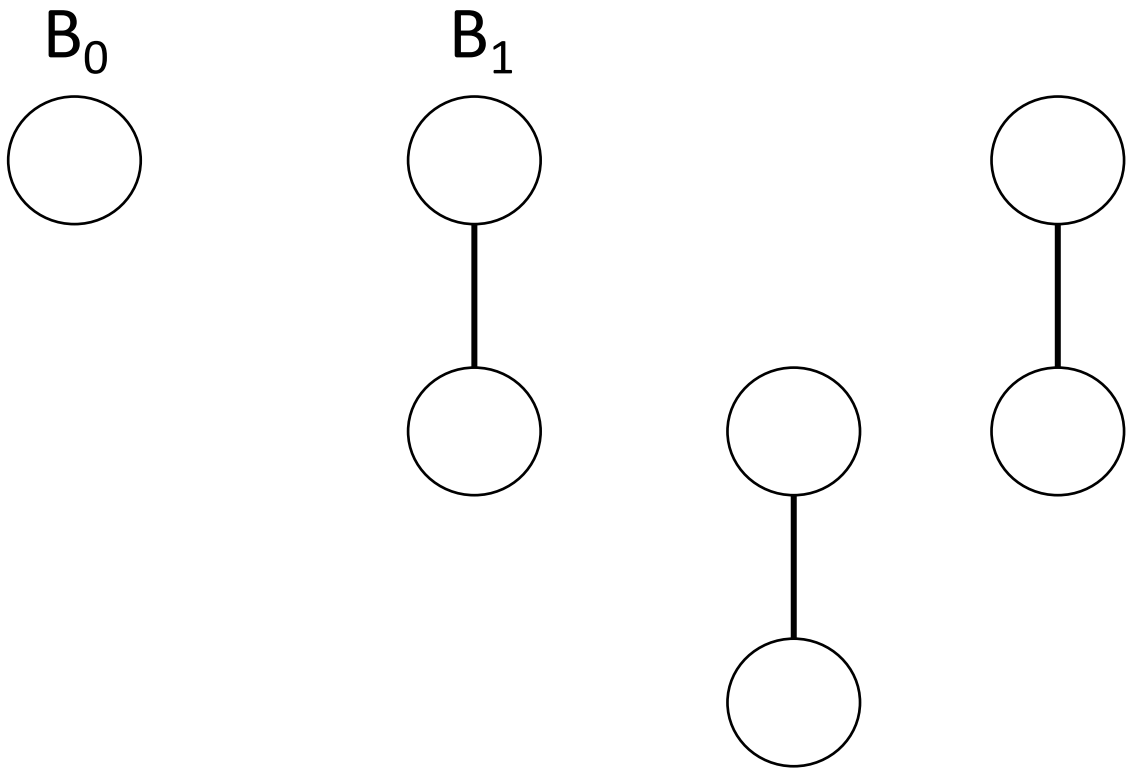

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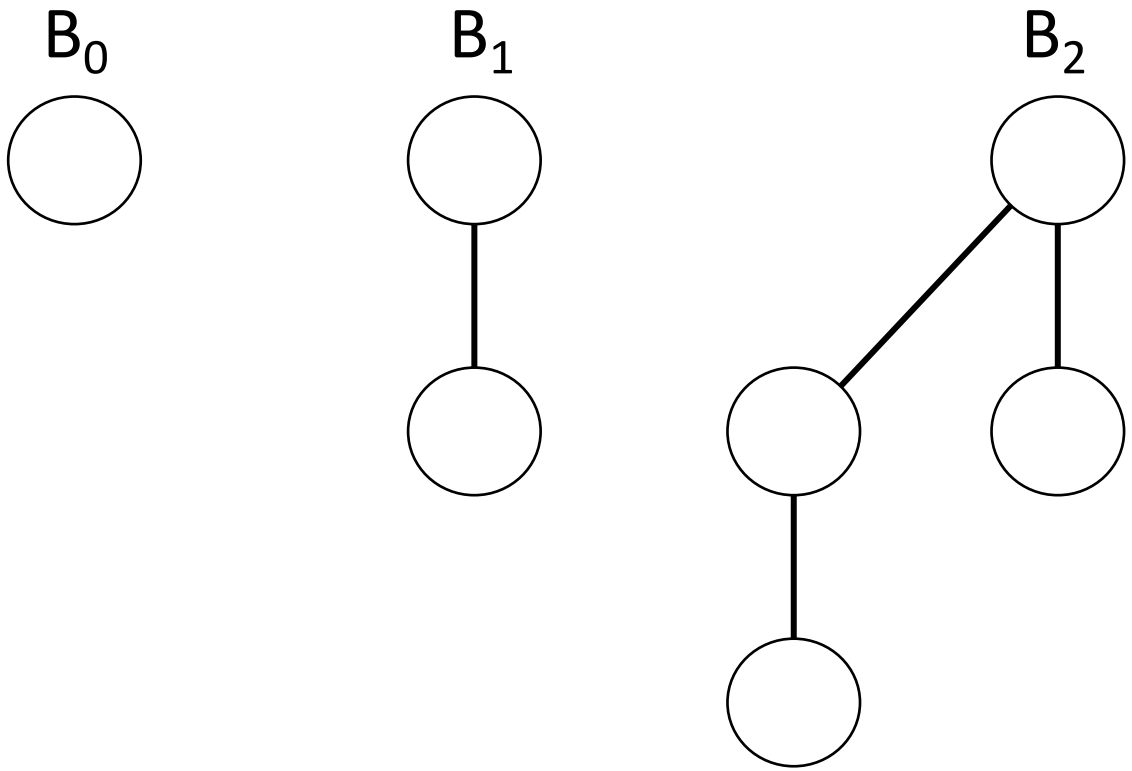

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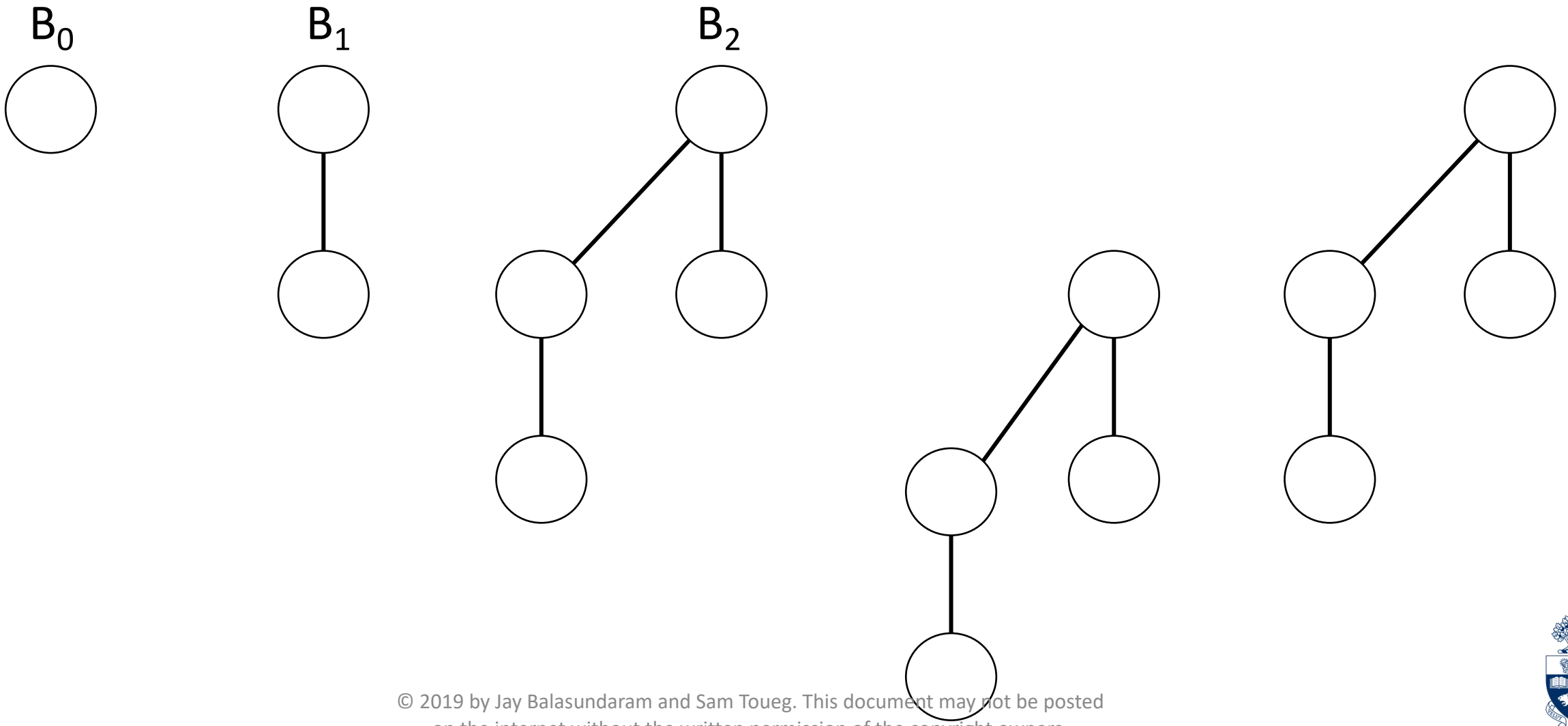

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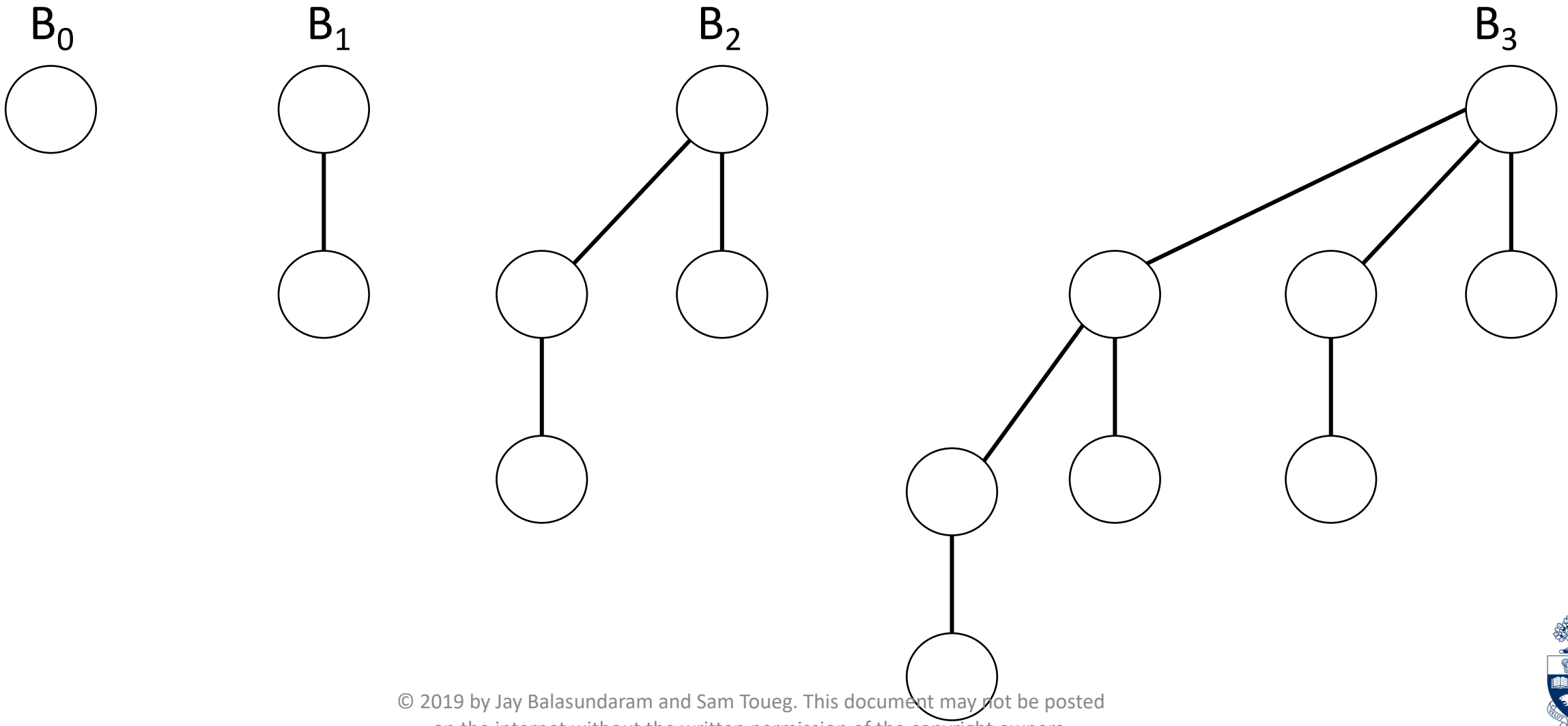
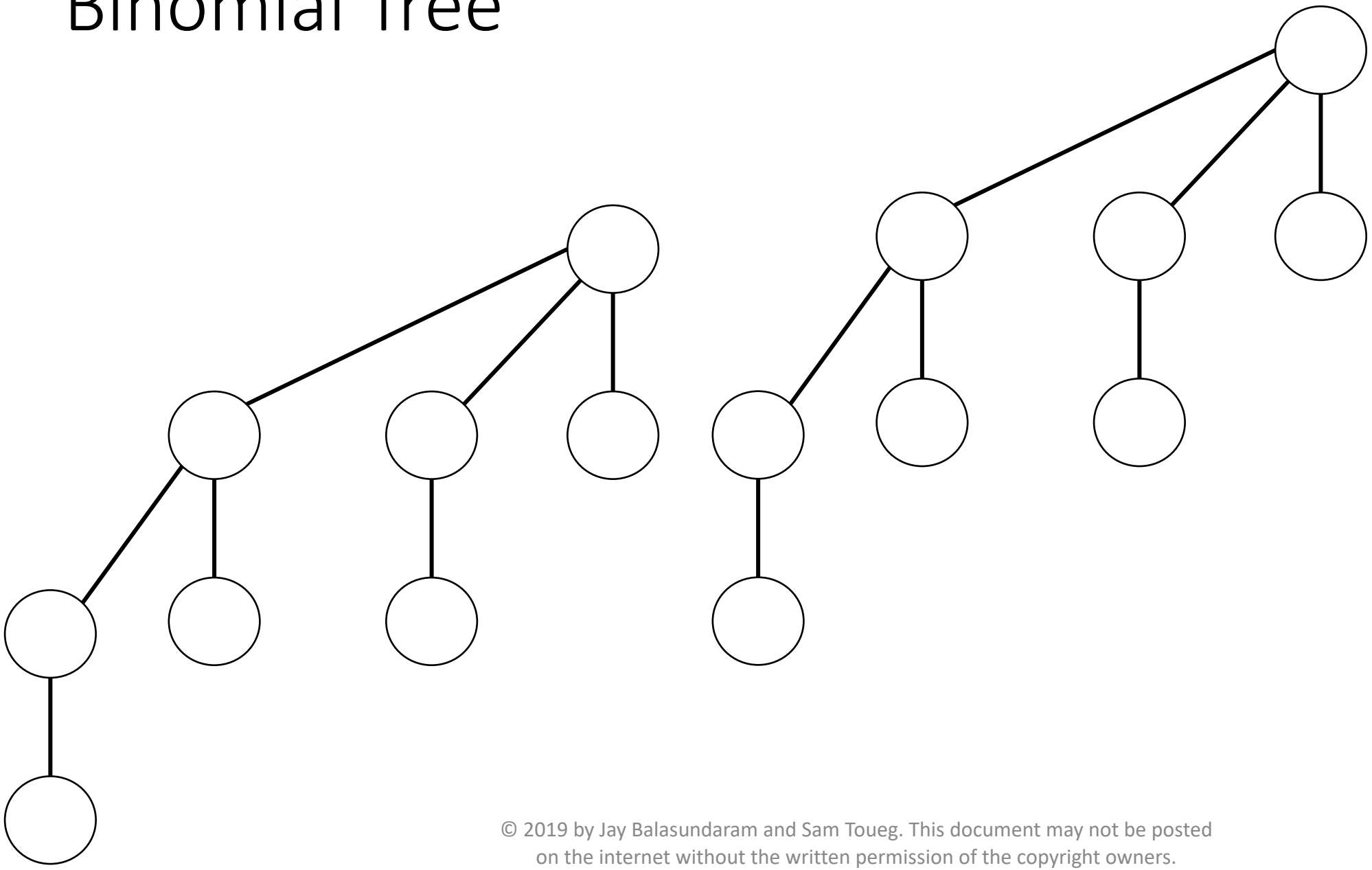
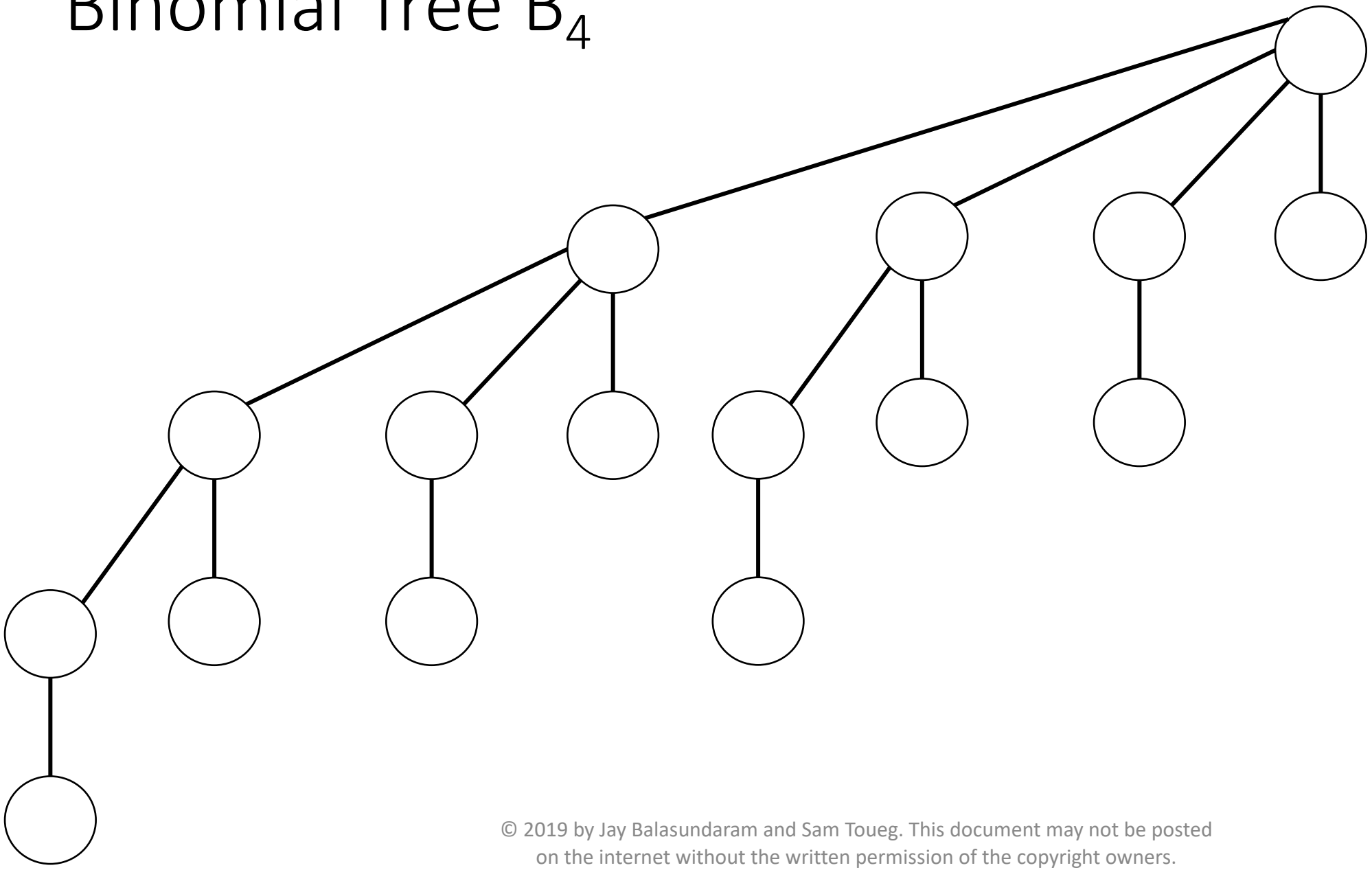


Binomial Tree



Binomial Tree B_4



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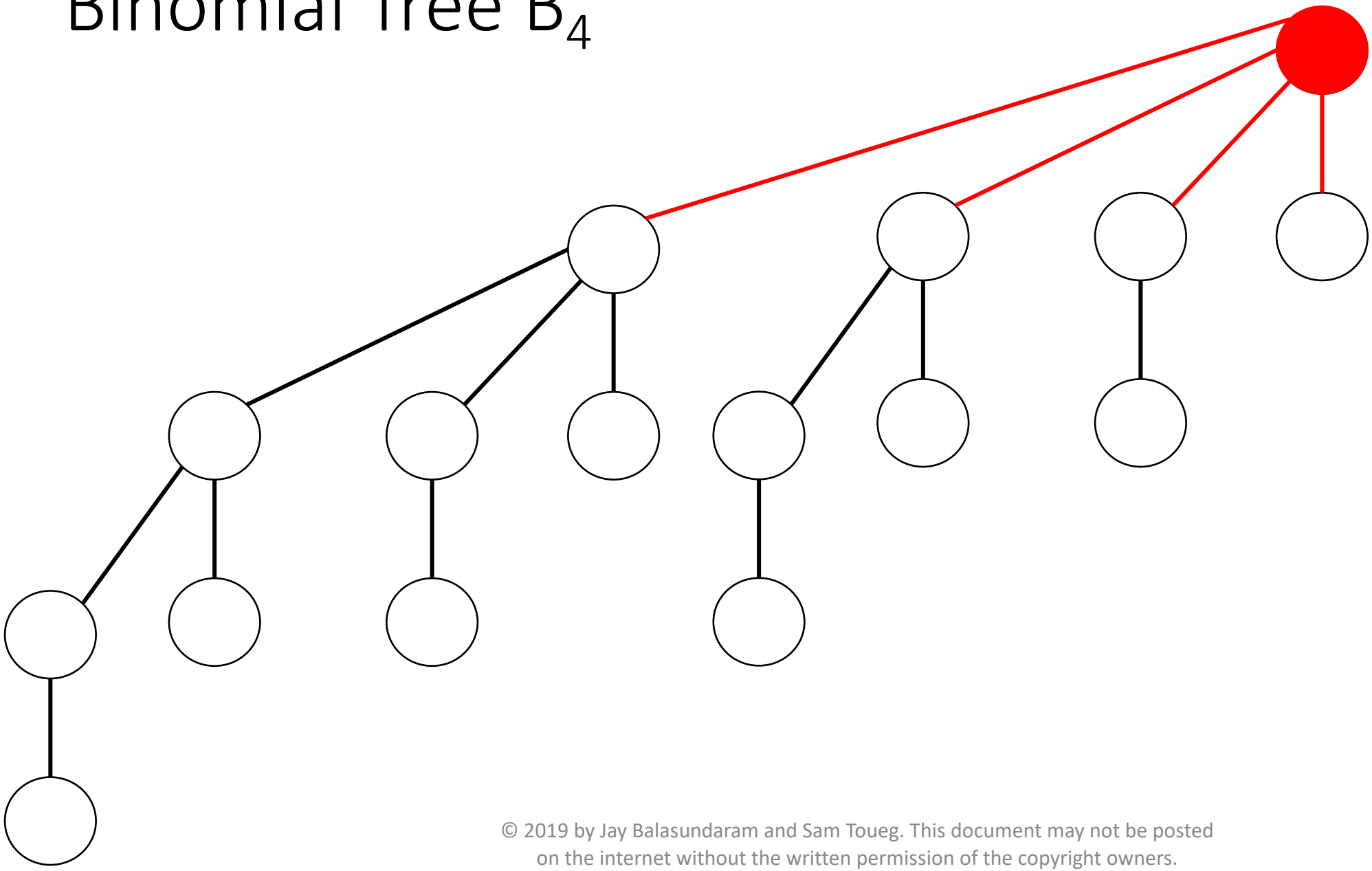

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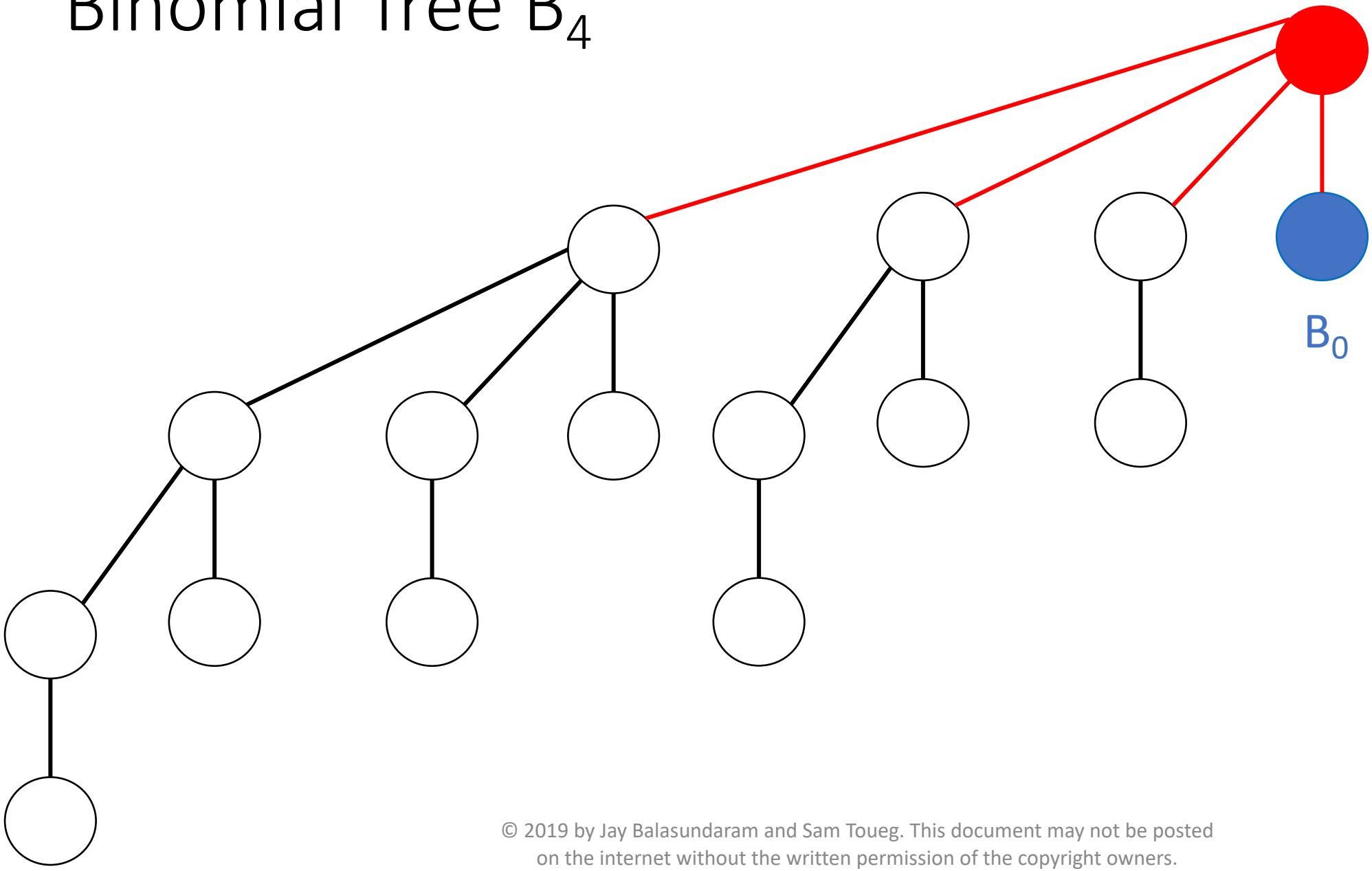

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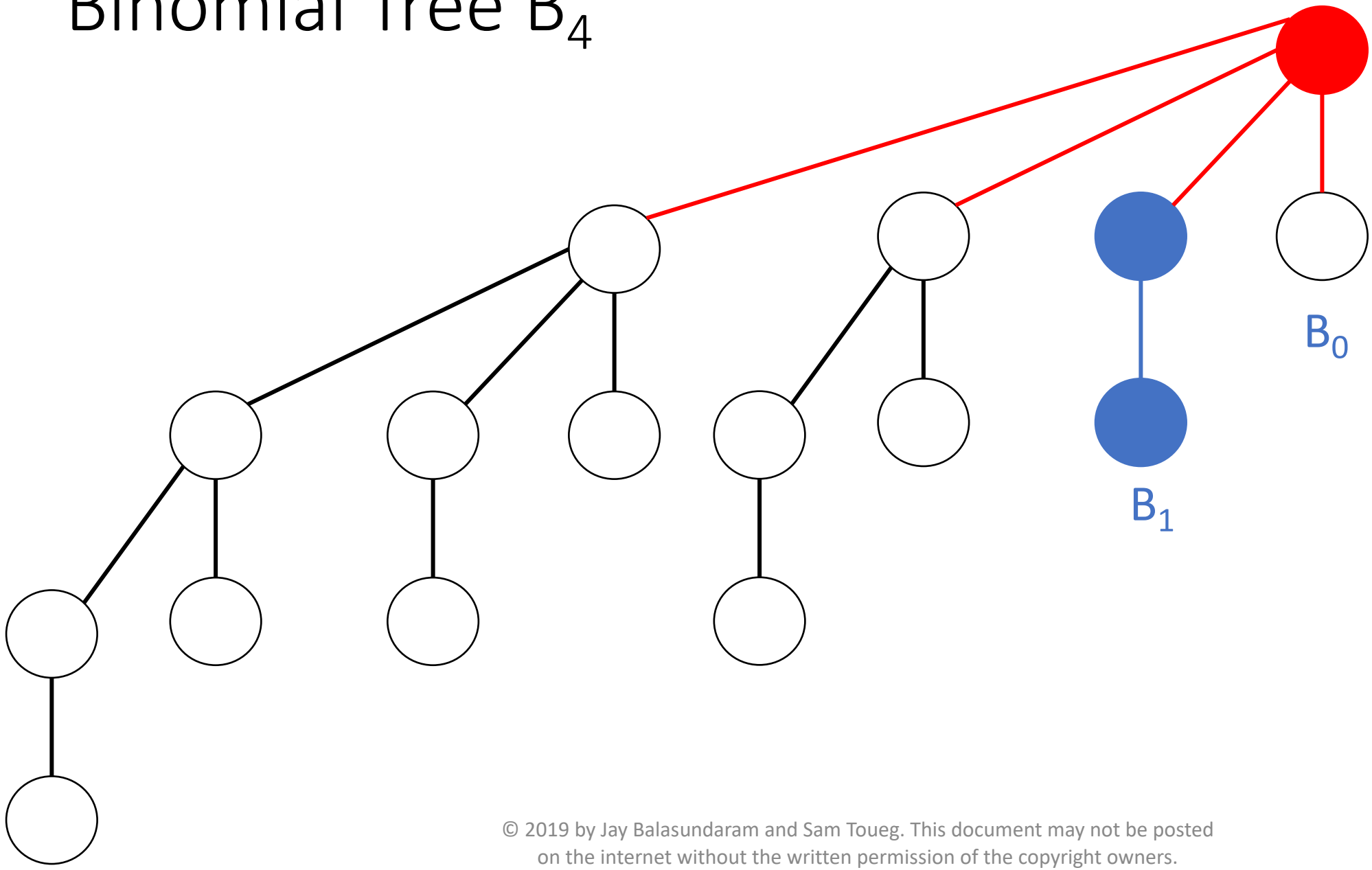

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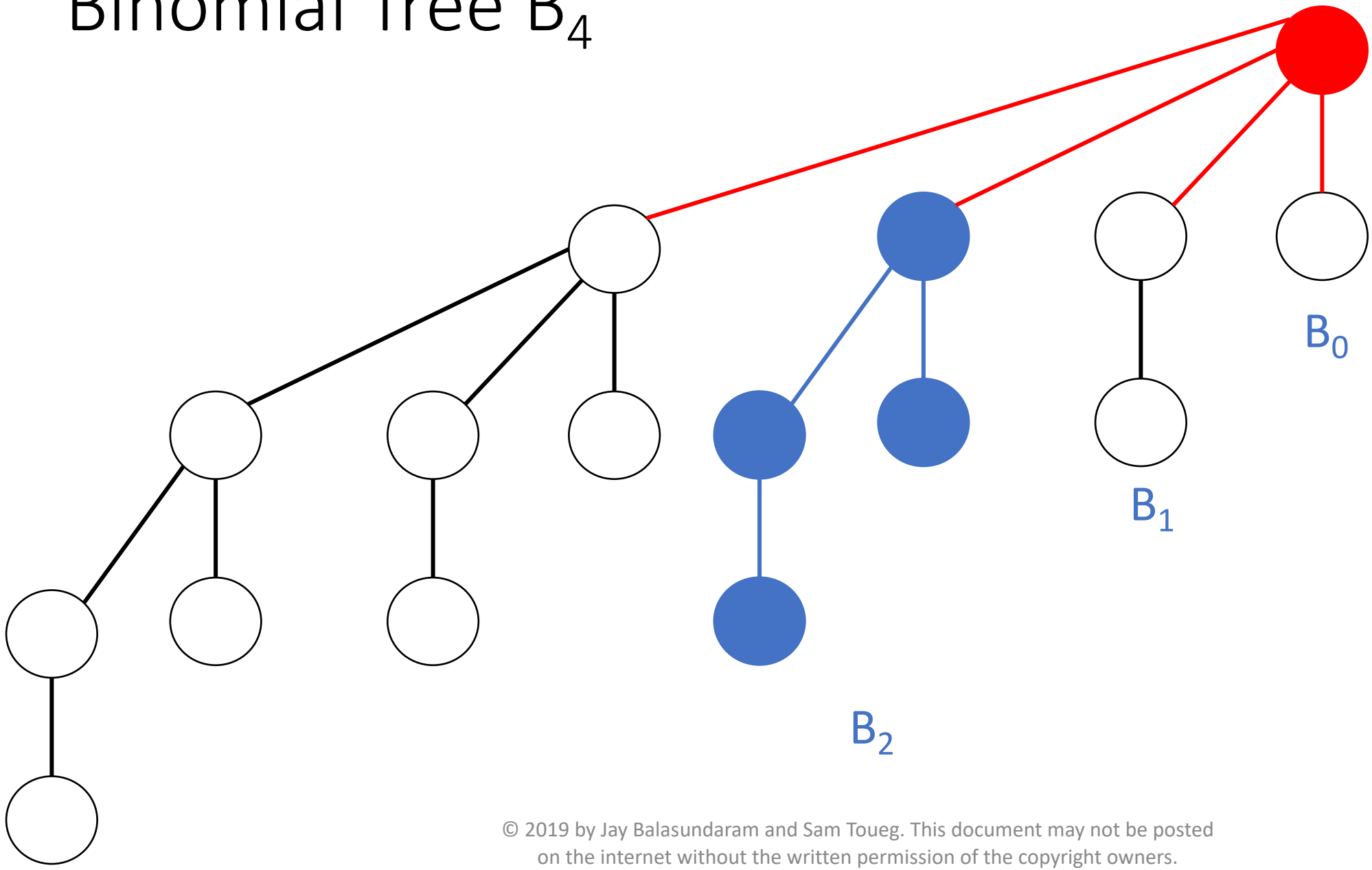

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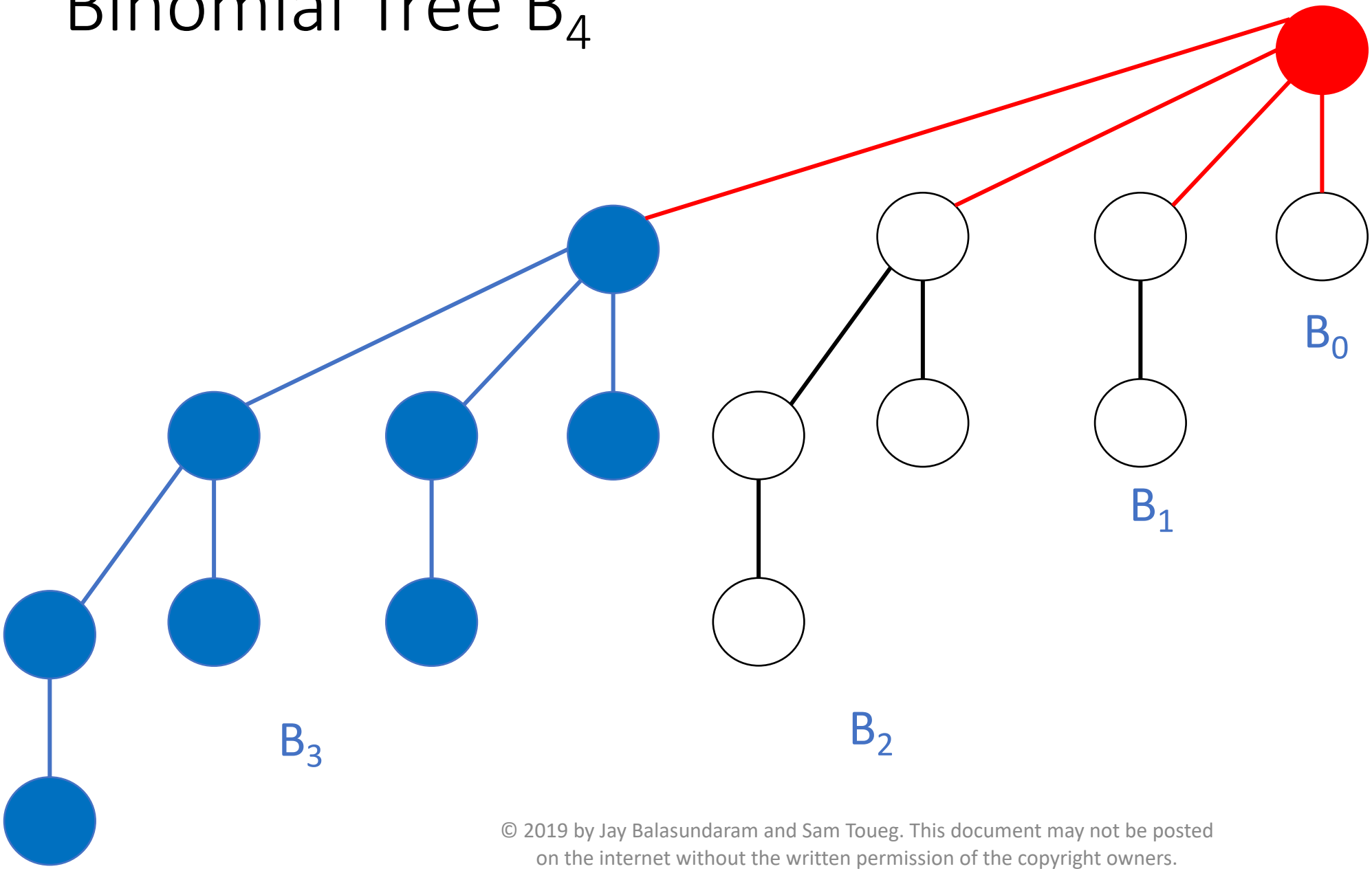
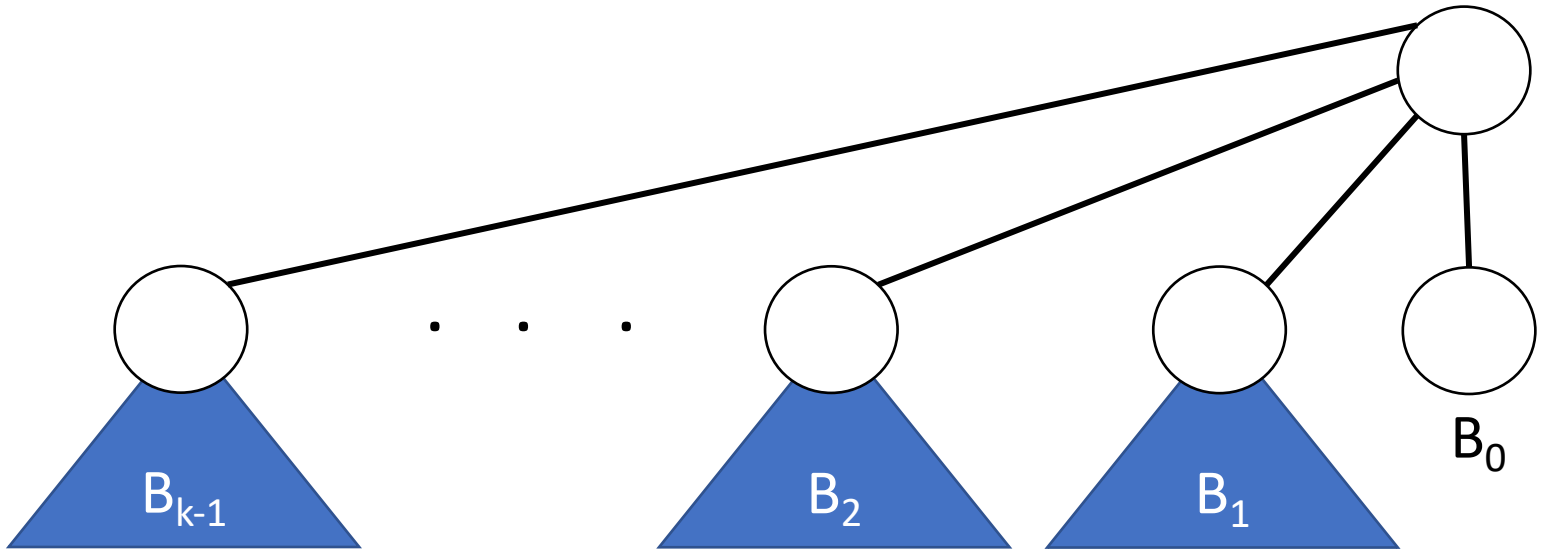


Binomial Tree B_k



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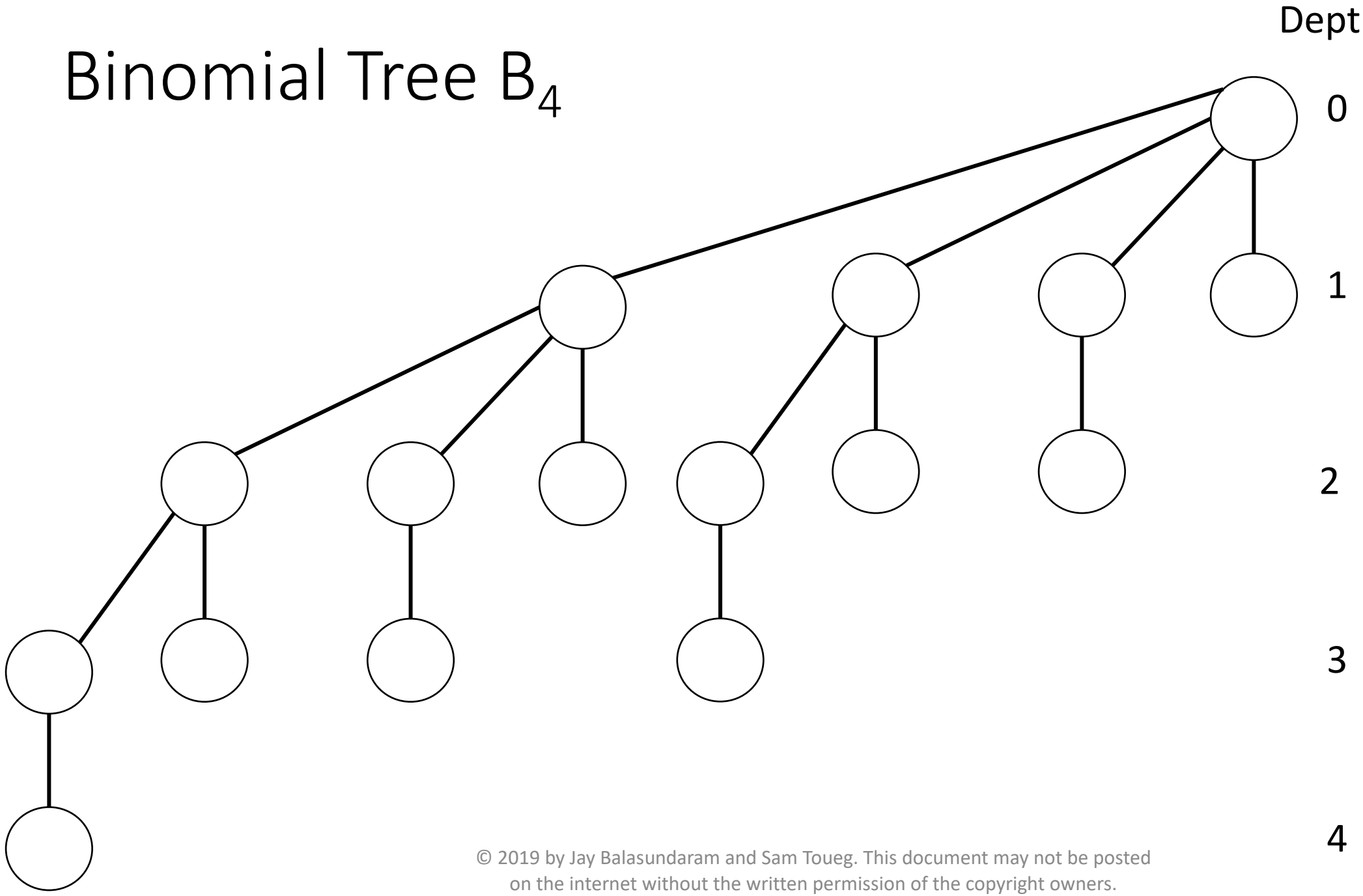

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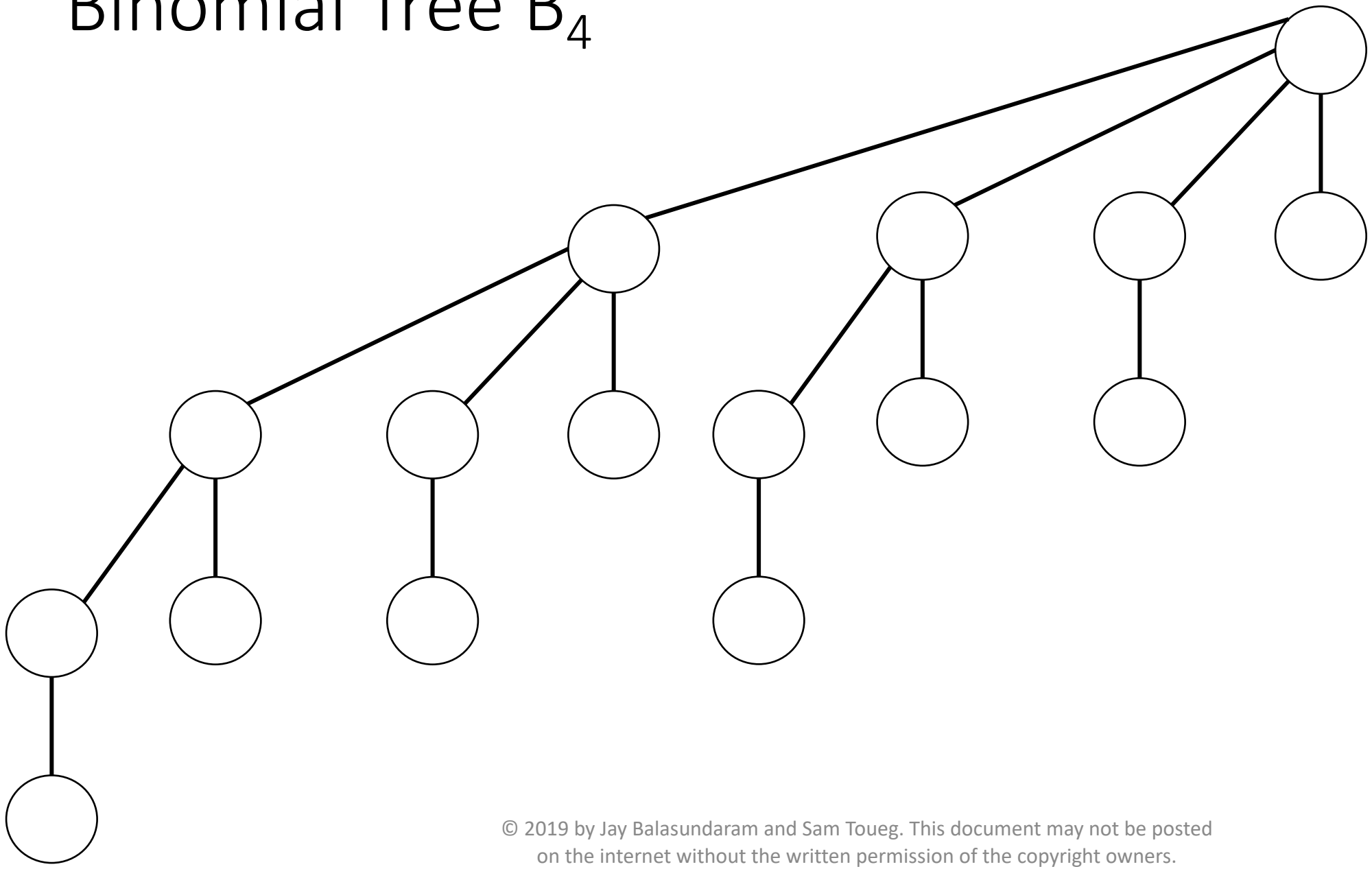
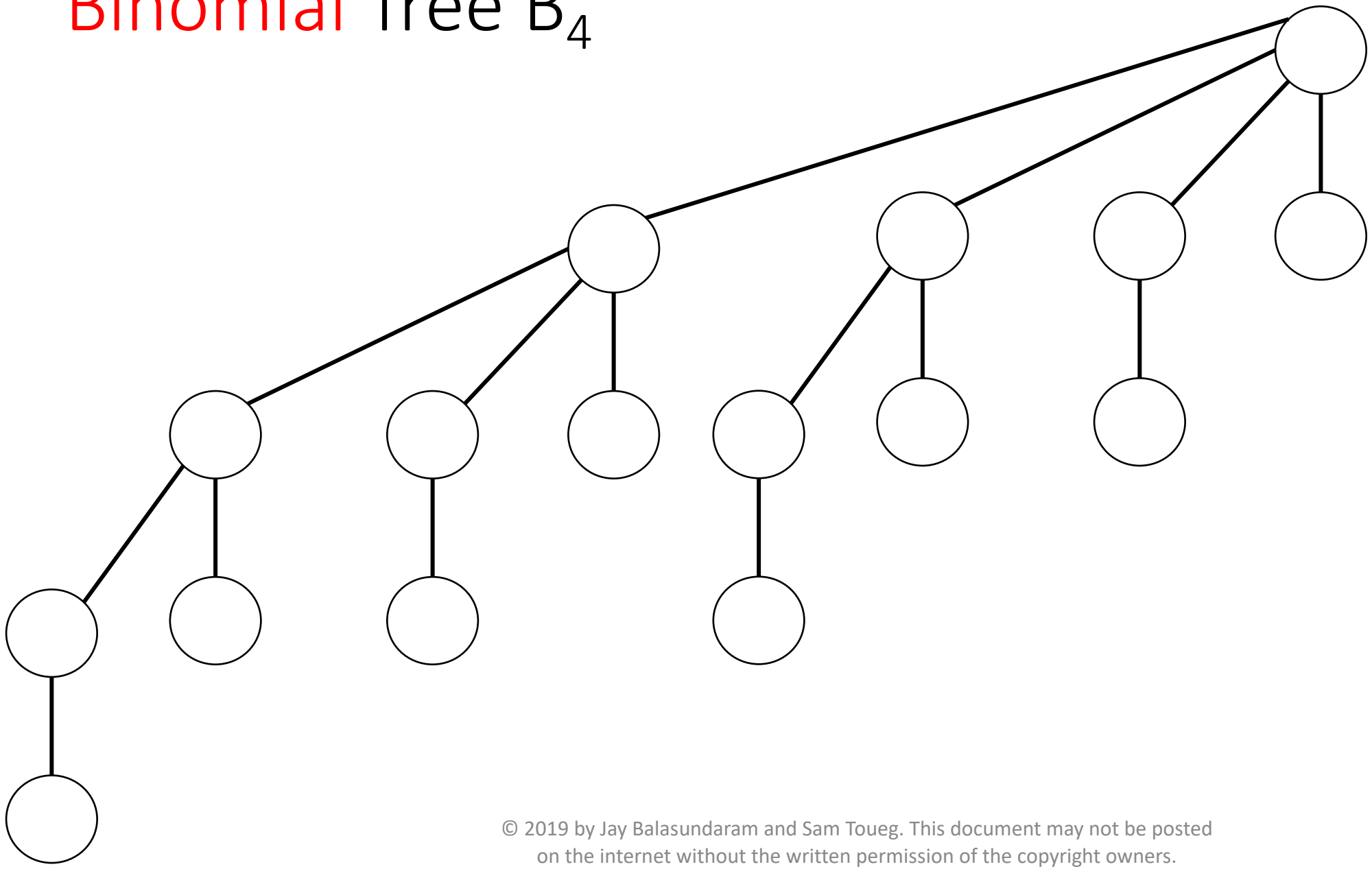


Depth	# of nodes
0	1
1	4
2	6
3	4
4	1

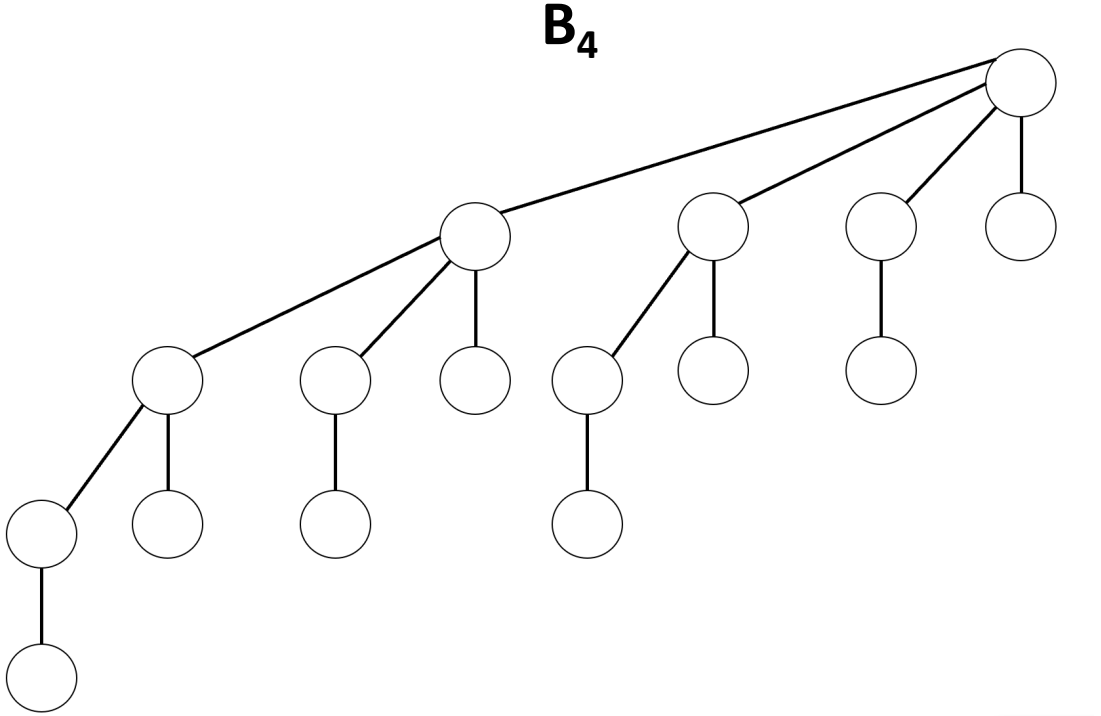
Binomial Tree B_4



Depth	# of nodes
0	$\binom{4}{0} = 1$
1	$\binom{4}{1} = 4$
2	$\binom{4}{2} = 6$
3	$\binom{4}{3} = 4$
4	$\binom{4}{4} = 1$

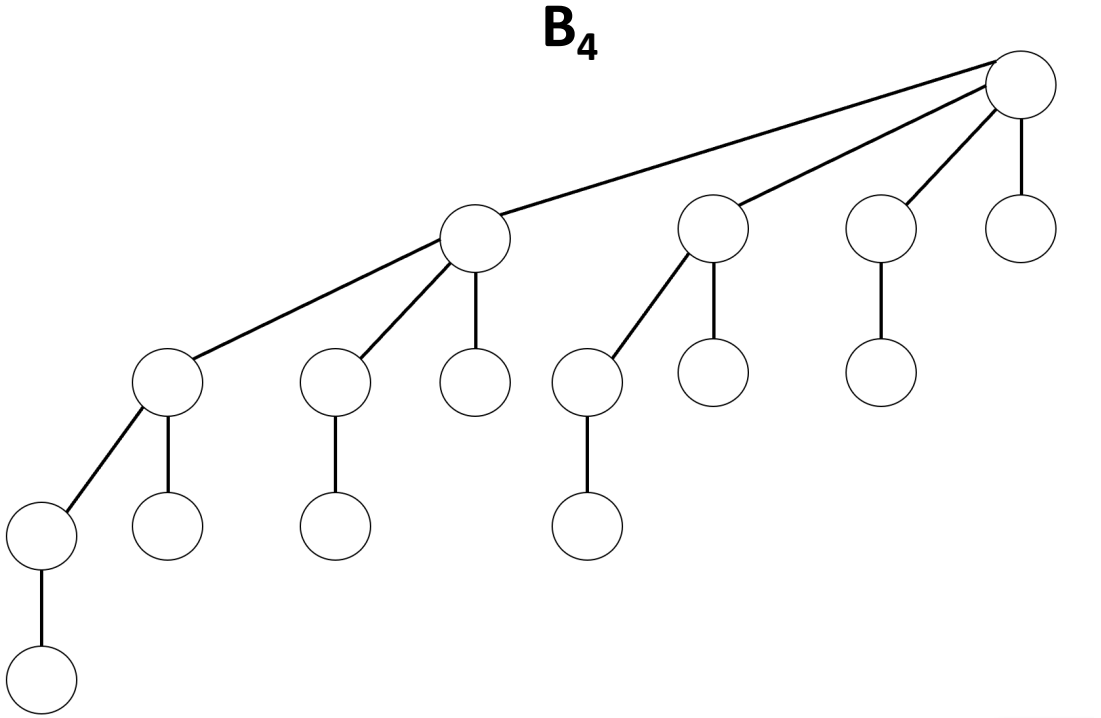
Binomial Tree properties

- B_k tree has:



Binomial Tree properties

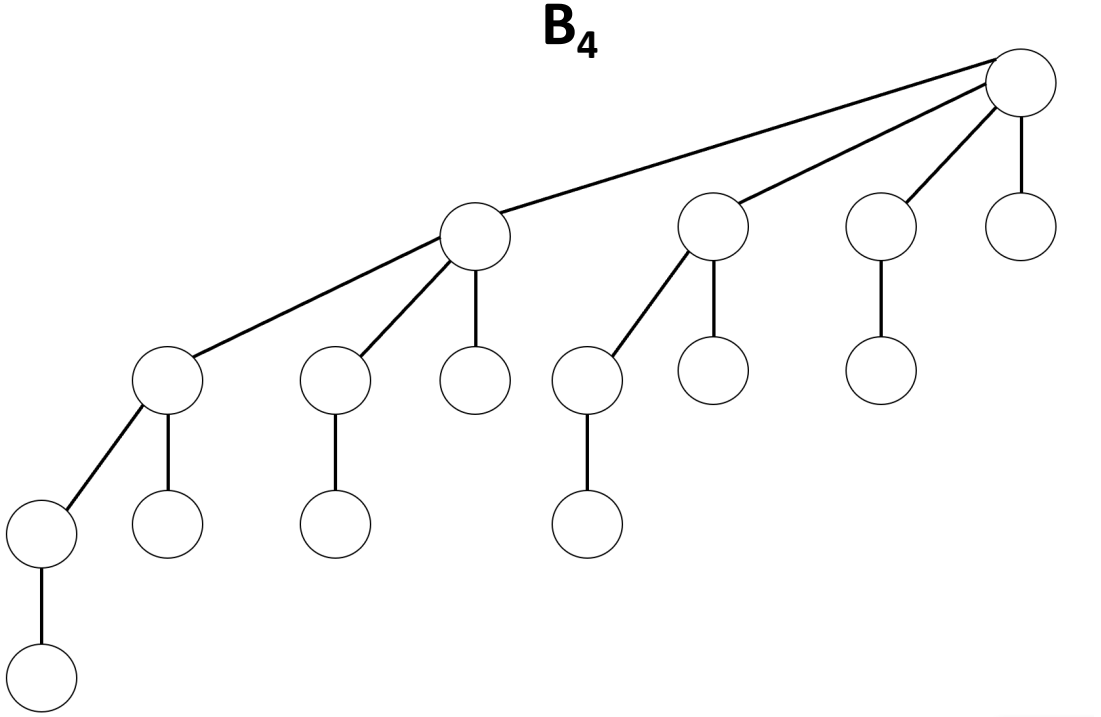
- B_k tree has:
  - Height k



Height = 4

Binomial Tree properties

- B_k tree has:
  - Height k
  - 2^k nodes

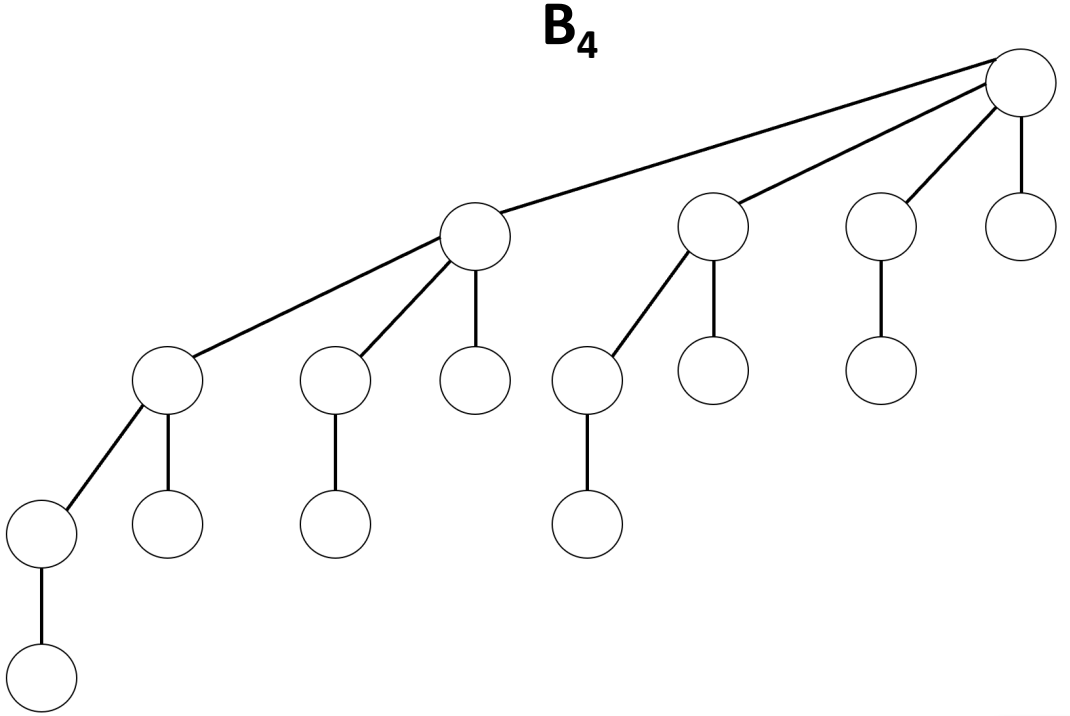


Height = 4

2^4 = 16 nodes

Binomial Tree properties

- B_k tree has:
  - Height k
  - 2^k nodes
  - $\binom{k}{d}$ nodes at depth d



Height = 4

2^4 = 16 nodes

$\binom{4}{2}$ = 6 nodes at level 2

Binomial Forest of size n (denoted F_n)

A sequence of B_k trees with *strictly decreasing* k's and a total of n nodes.

Example: Binomial Forest F_7 of n = 7 nodes

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$n = 7 = < 1\ 1\ 1 >_2 = 2^2 + 2^1 + 2^0$

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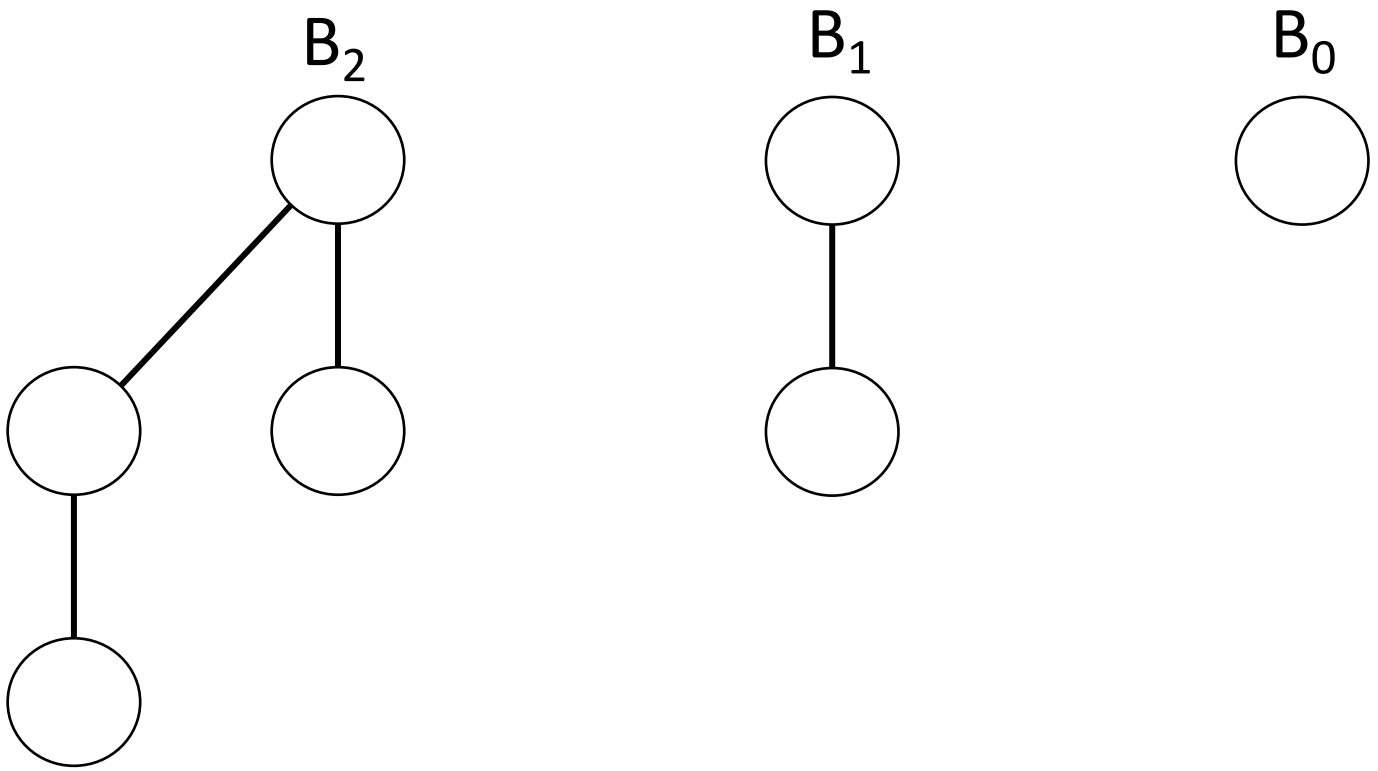
F_7 :



Example: Binomial Forest F_9 of n = 9 nodes

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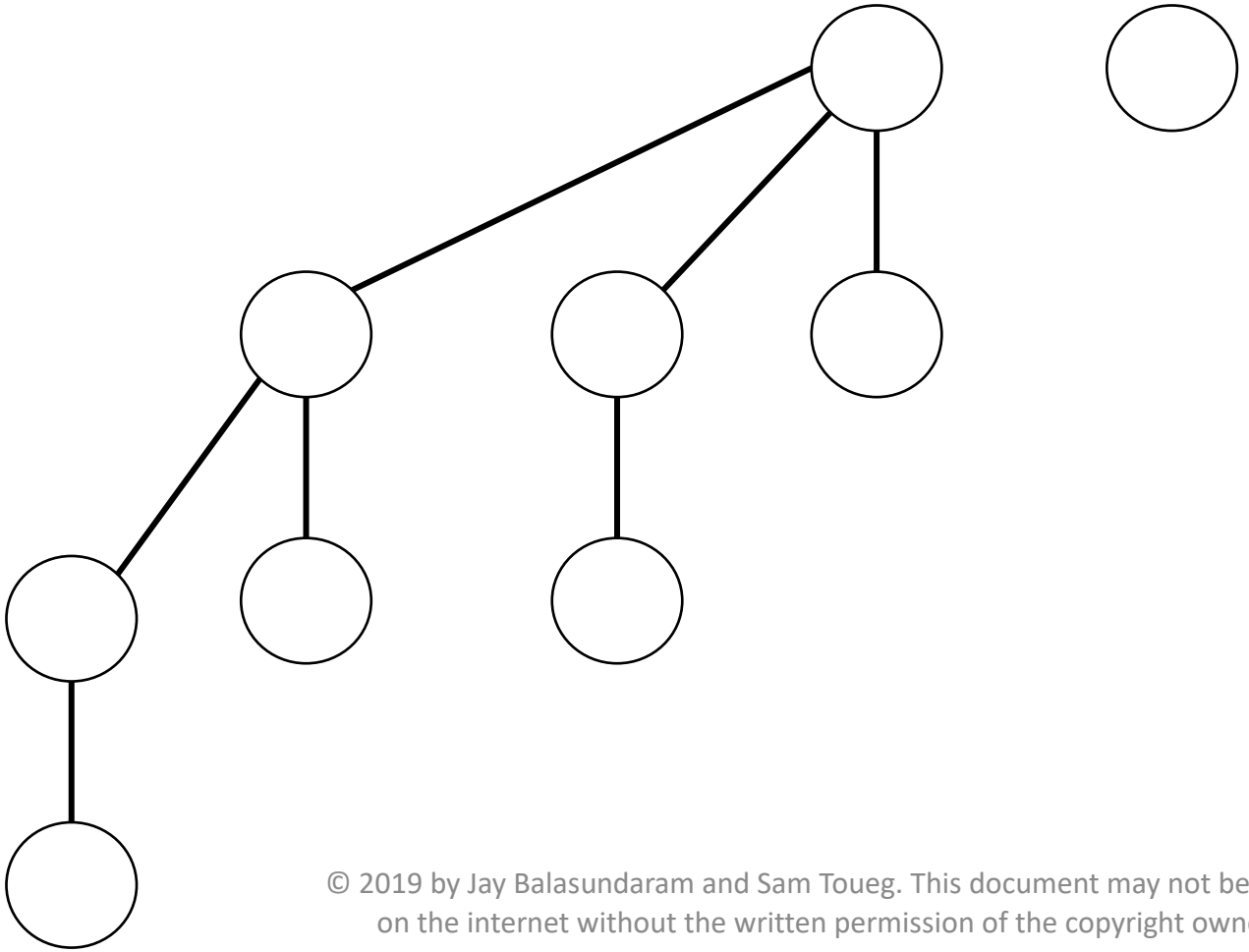
F_9 : $\qquad$ B_3 $\qquad$ B_0



Binomial Forest F_n

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- F_n has n nodes.

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- Let α(n) = # of 1's in binary representation of n
  - F_n has α(n) trees

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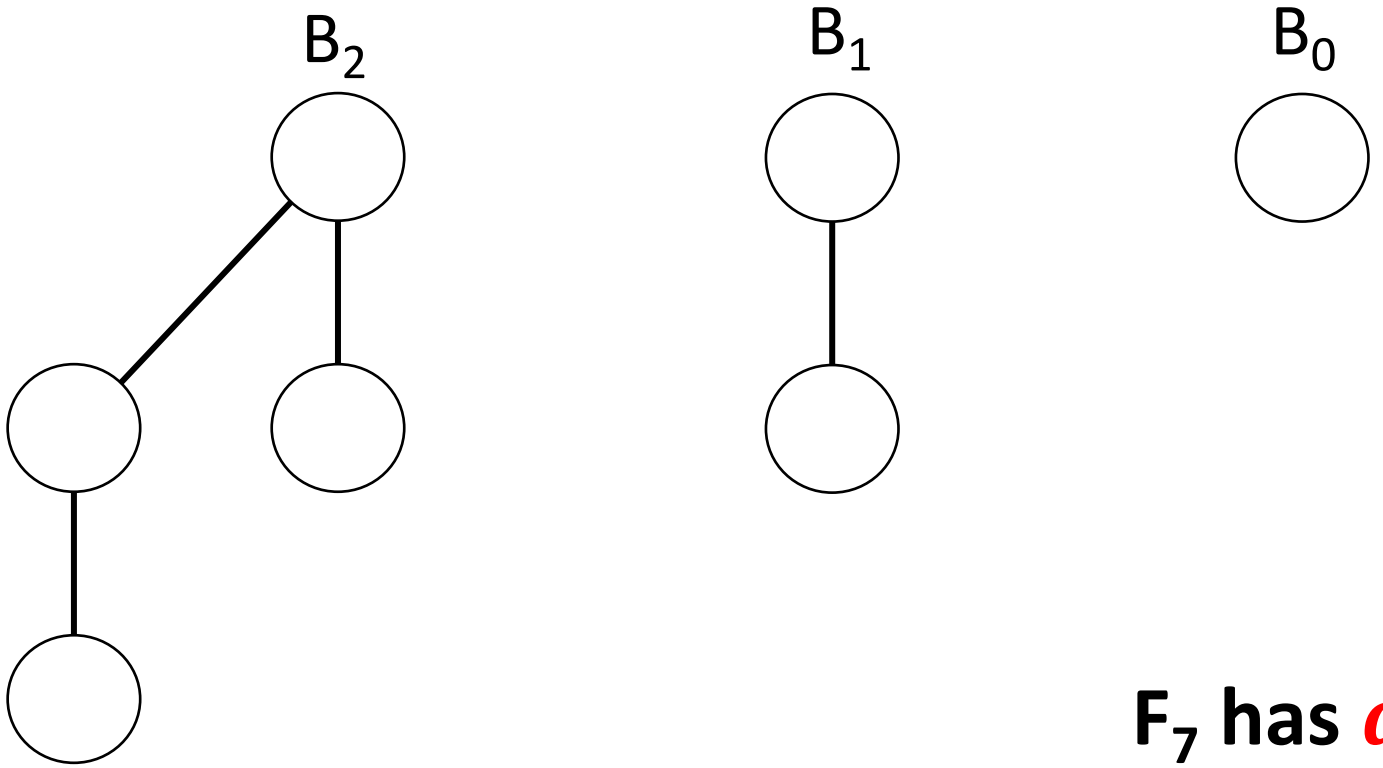
F_7 :

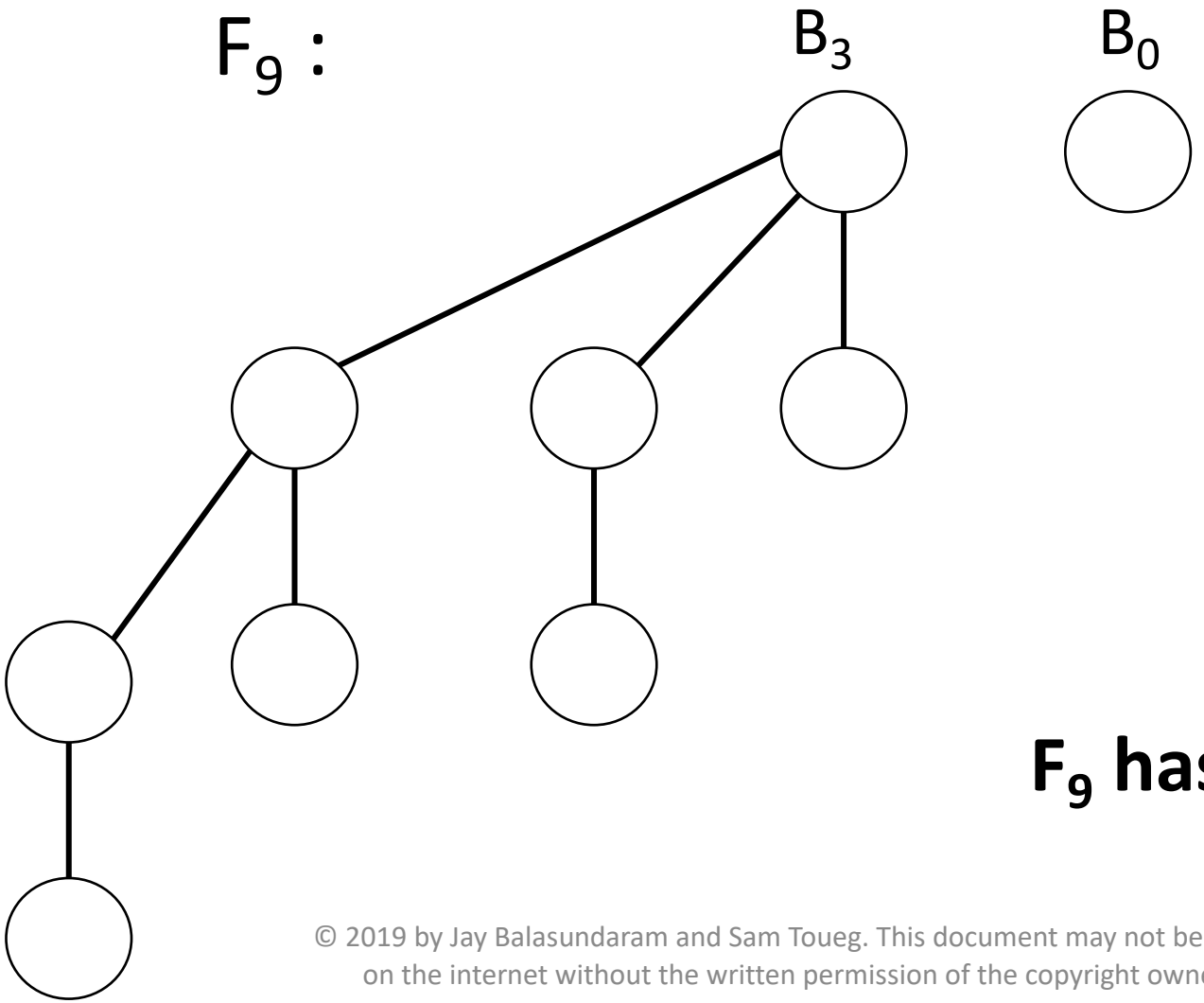


F_7 has $\alpha(7) = 3$ trees

Example: Binomial Forest F_9 of n = 9 nodes

n = 9 = $< 1\ 0\ 0\ 1 >_2$

F_9 :



F_9 has $\alpha(9) = 2$ trees

Binomial Forest F_n

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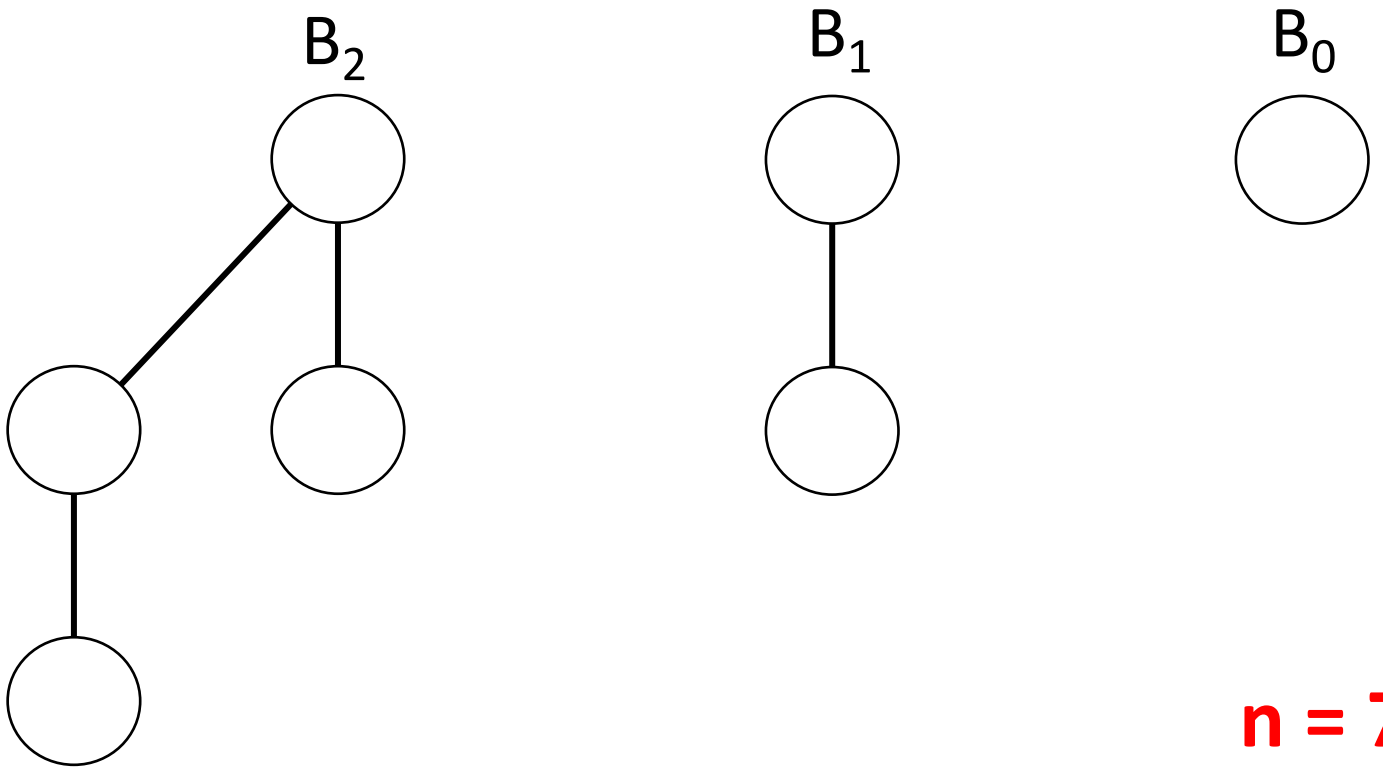
  - F_n has n - α(n) edges

Example: Binomial Forest F_7 of n = 7 nodes

n = 7 = < 1 1 1 >$_2$

F_7 :



n = 7, $\alpha(7) = 3$

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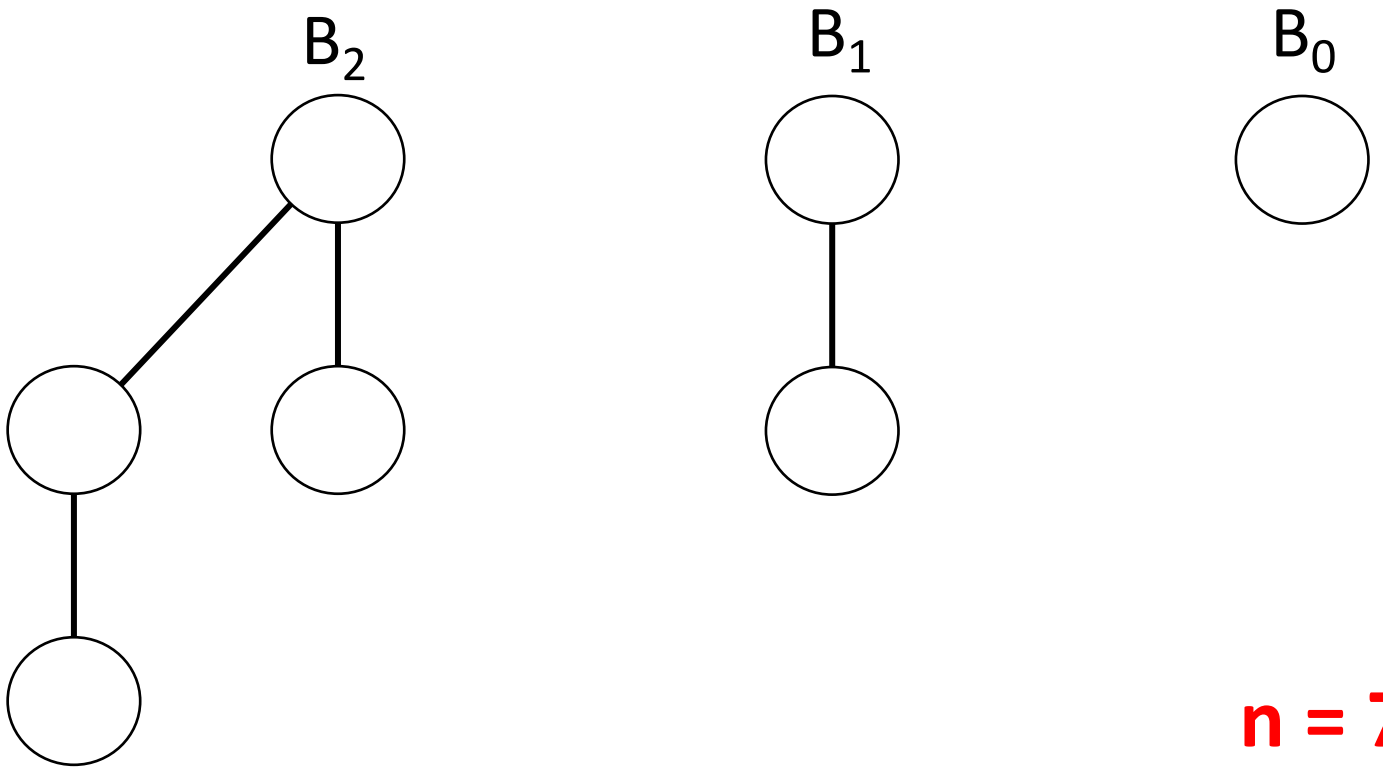
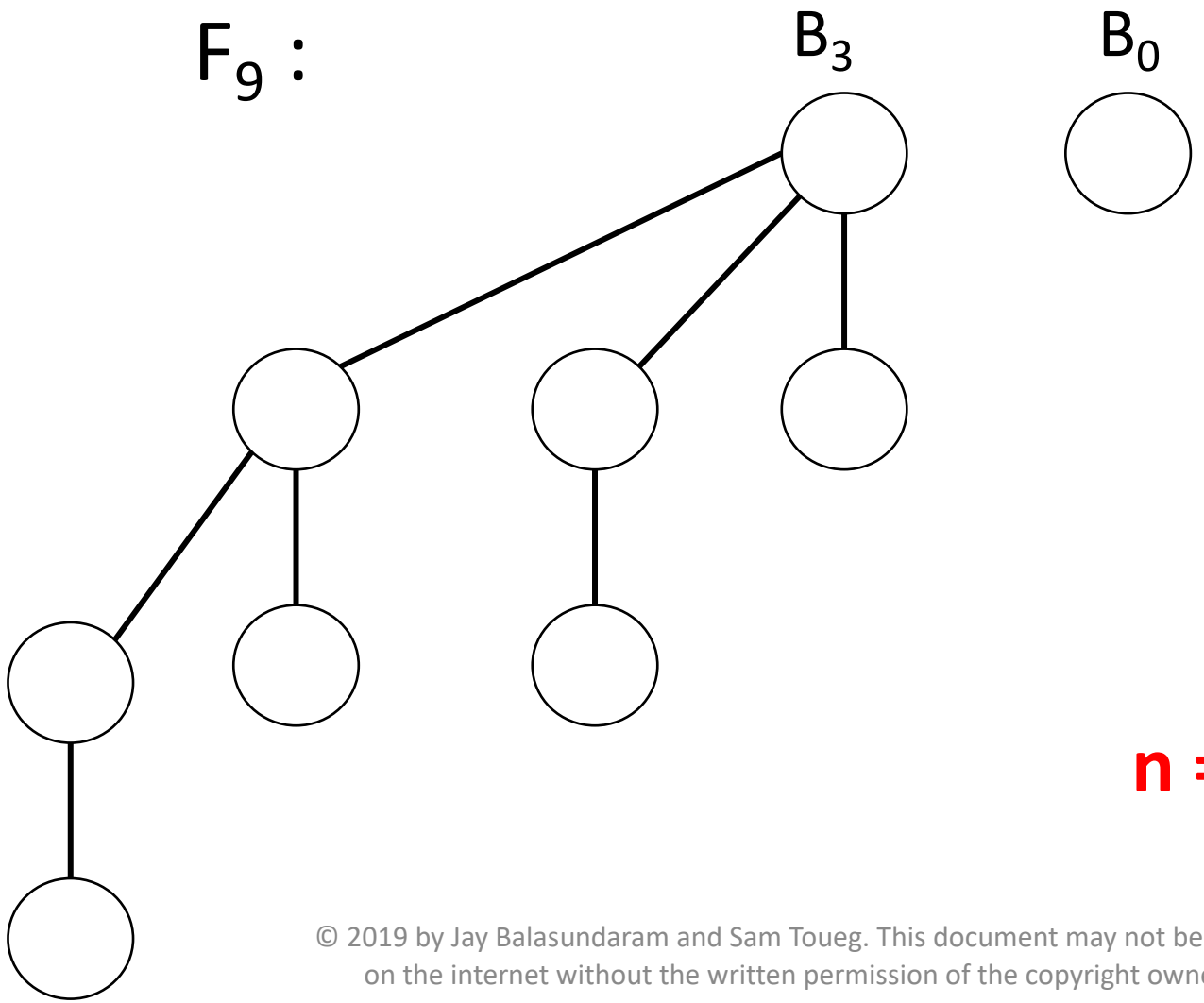


n = 7, $\alpha(7) = 3$

F_7 has 7-$\alpha(7)$ = 4 edges

Example: Binomial Forest F_9 of n = 9 nodes

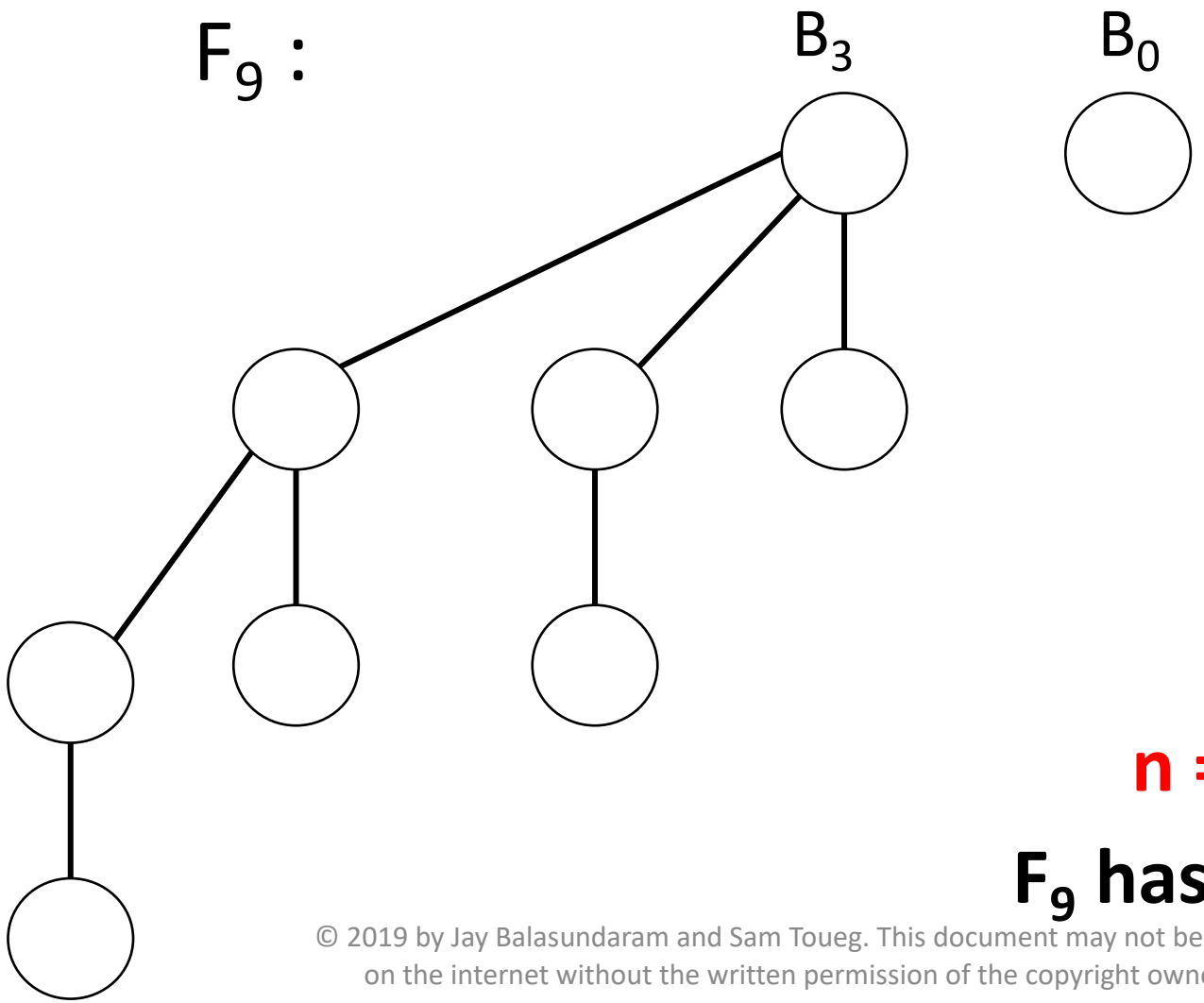
n = 9 = < 1 0 0 1 $>_2$



n = 9, $\alpha(9) = 2$

Example: Binomial Forest F_9 of n = 9 nodes

n = 9 = < 1 0 0 1 $>_2$



n = 9, $\alpha(9) = 2$

F_9 has 9-$\alpha(9)$ = 7 edges

Binomial Forest F_n

- F_n has n nodes.
- n = $<b_t, b_{t-1}, \ldots, b_0>_2$. Note that t = $\lfloor \log_2 n \rfloor$
- F_n : <all trees B_i such that bit b_i = 1>

- Let α(n) = # of 1's in binary representation of n
  - F_n has α(n) trees
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A Min Binomial Heap of n elements is a Binomial Forest F_n such that

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1. Each node of F_n stores one element

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   i.e. each tree satisfies the Min-Heap property

Example of a Min Binomial heap

S = {10, 13, 1, 3, 8, 18, 7} n = 7 elements

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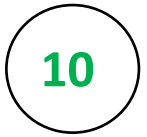
B_0



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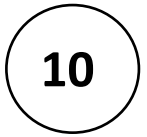

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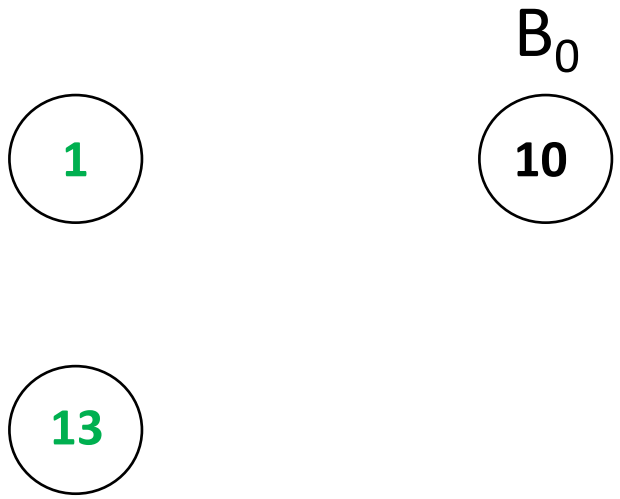

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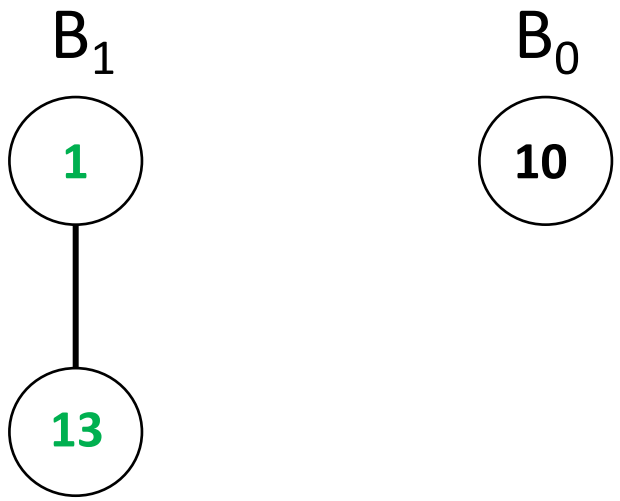

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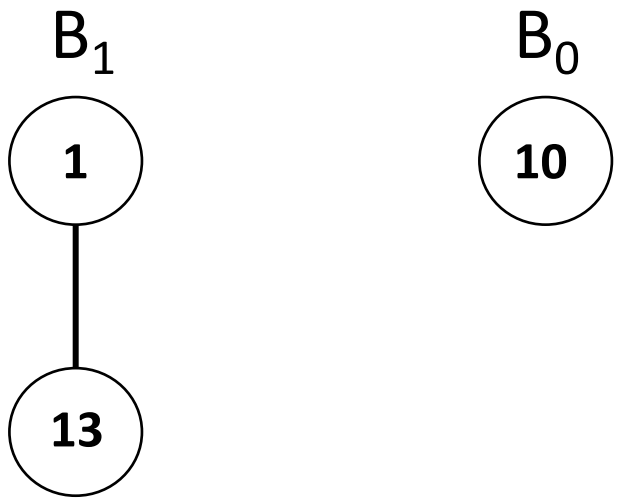

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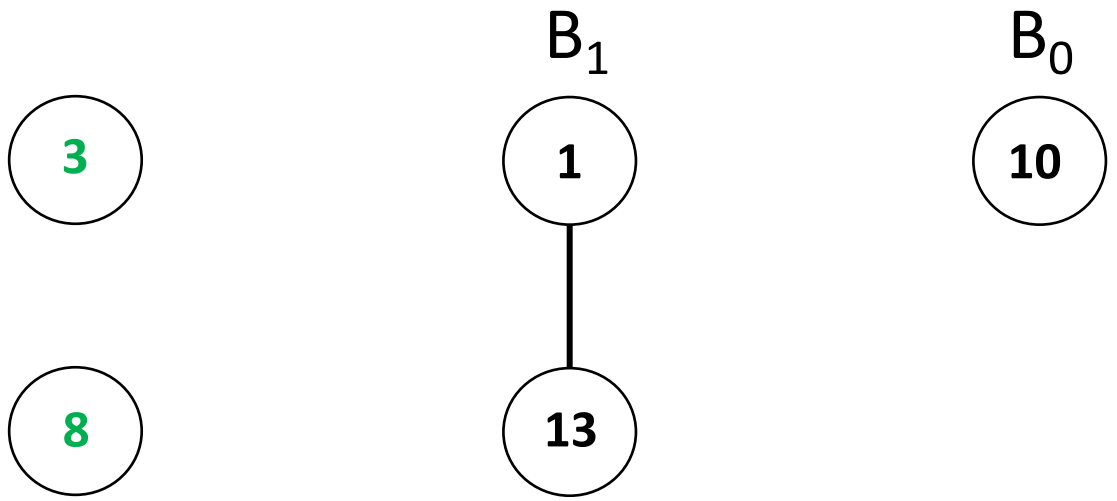

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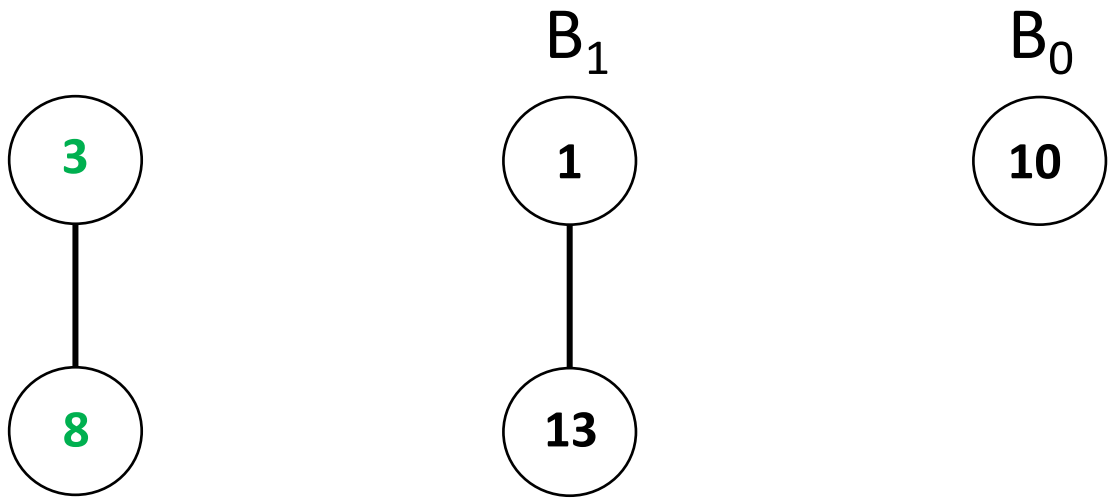

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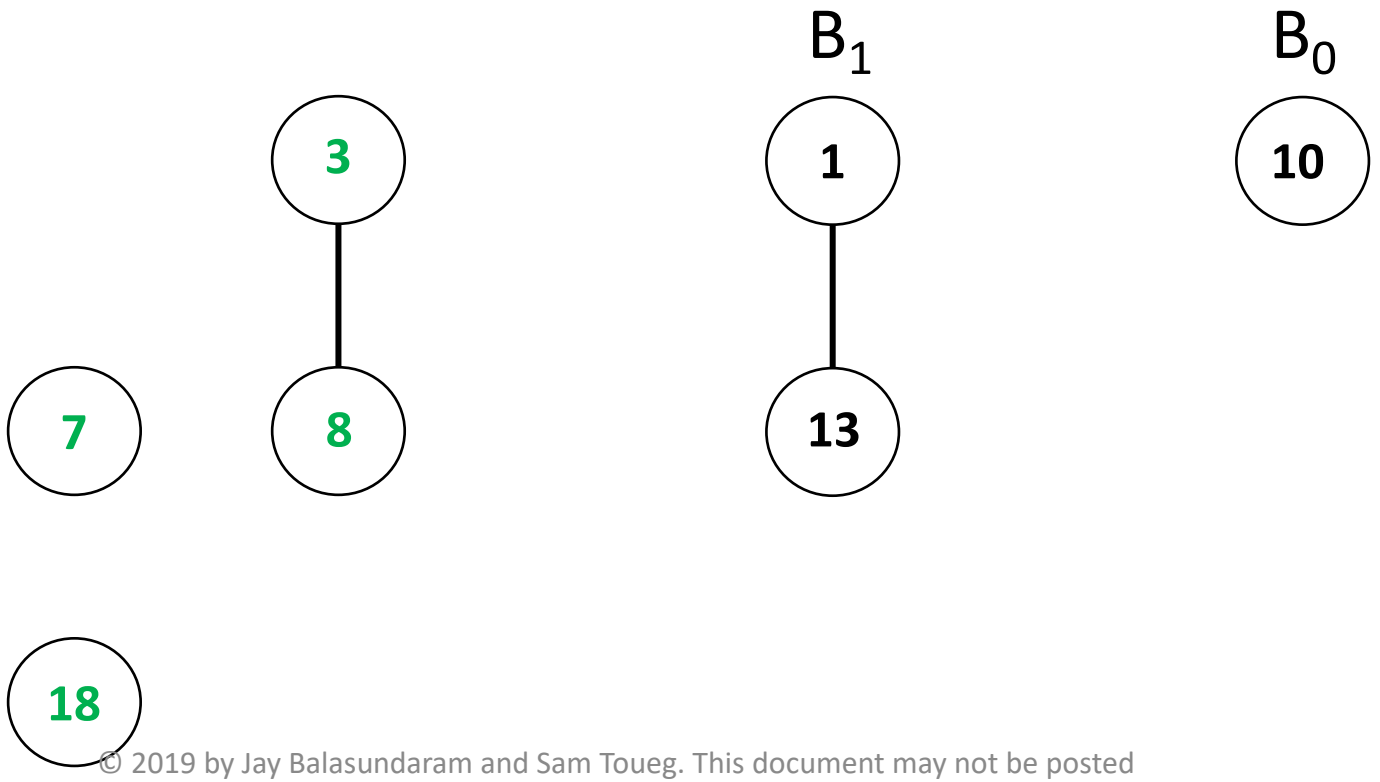

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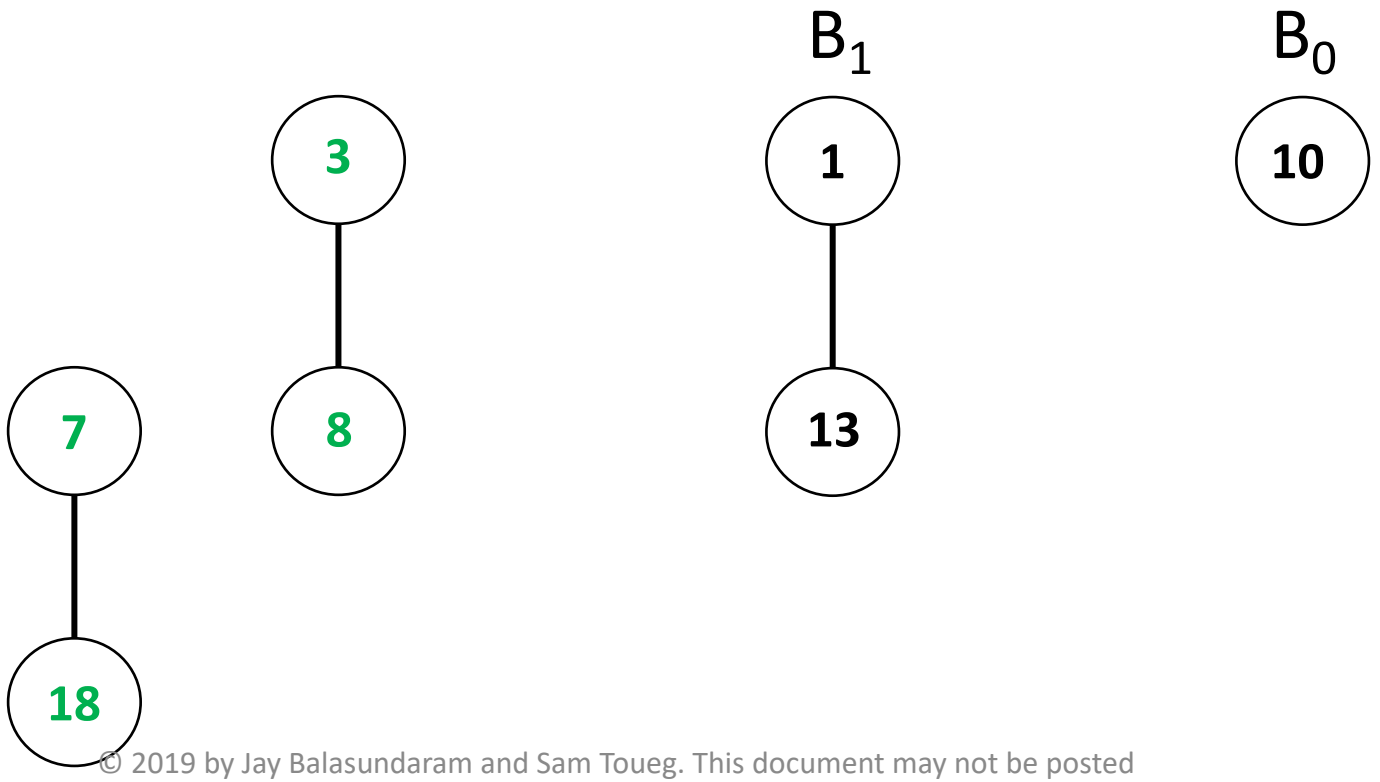

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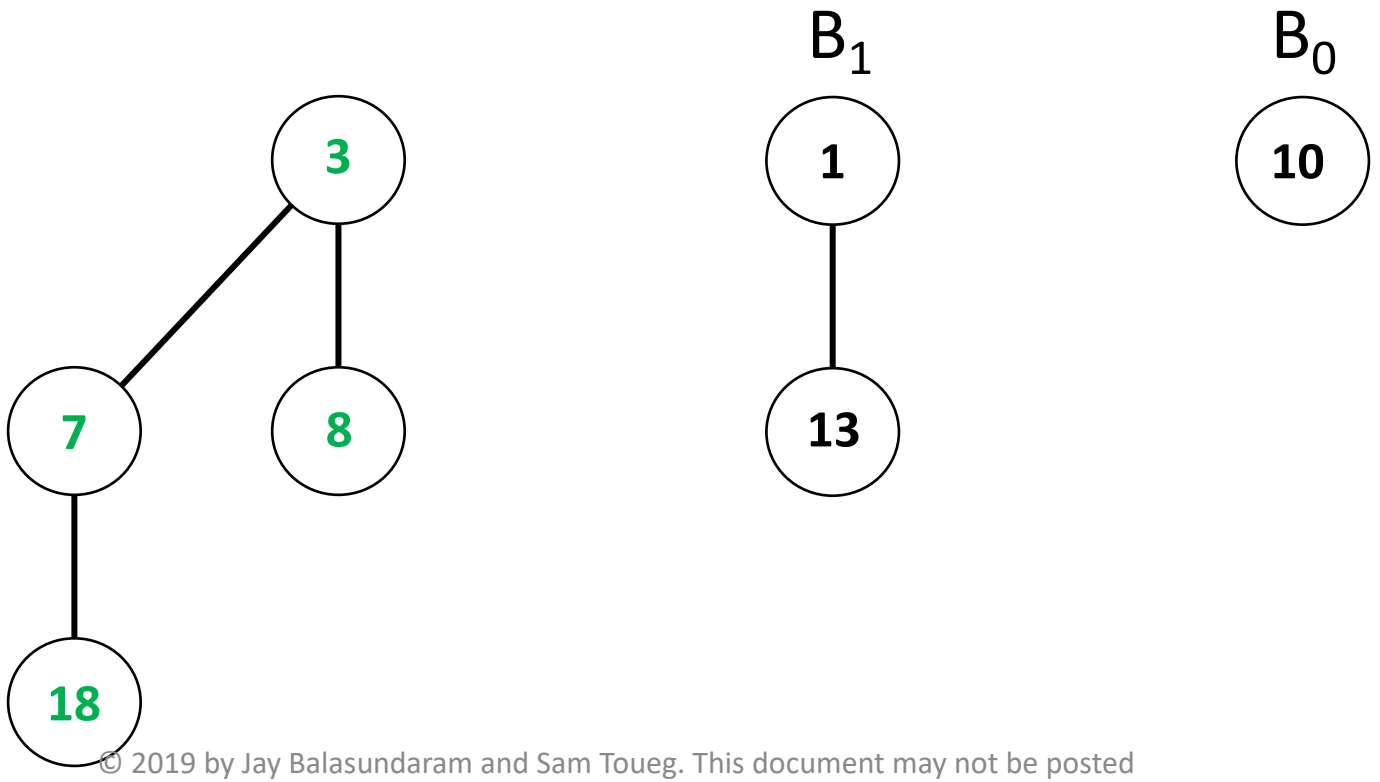

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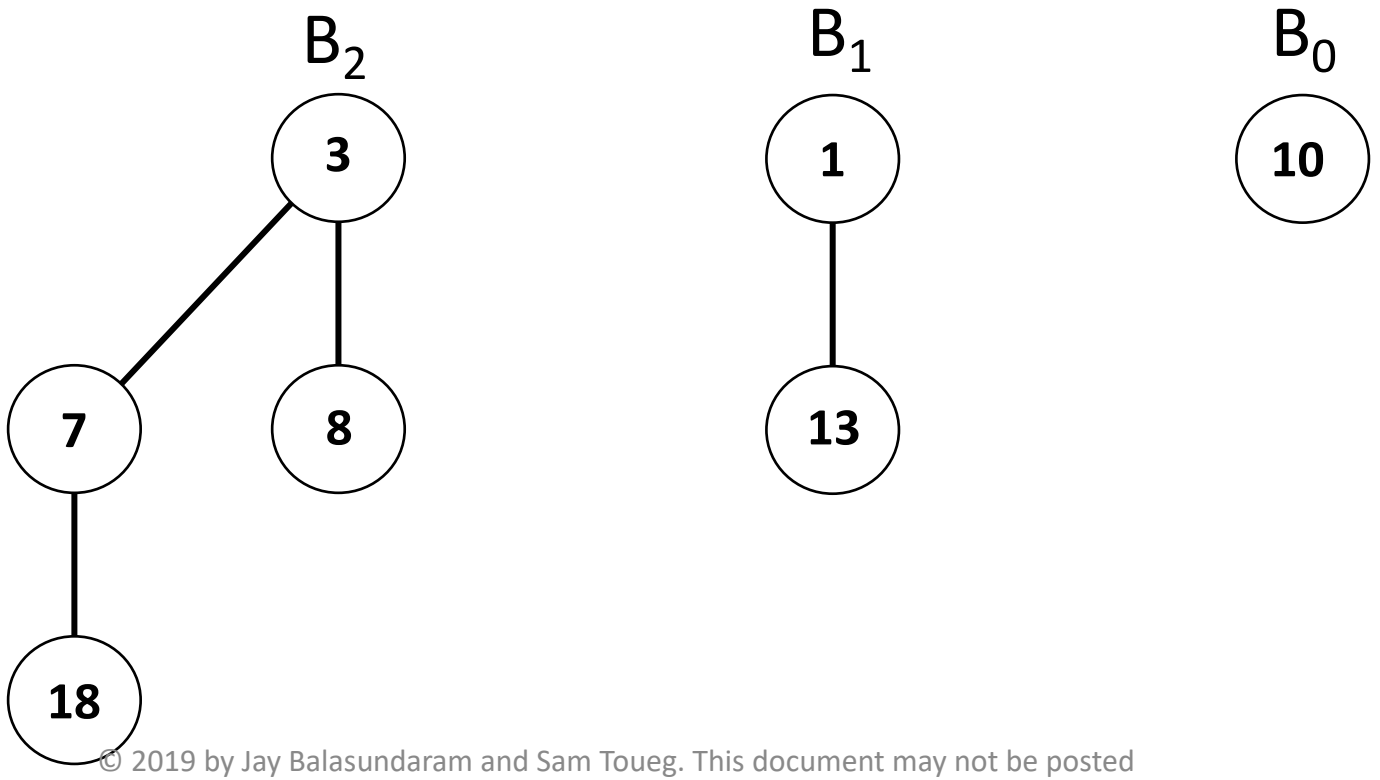

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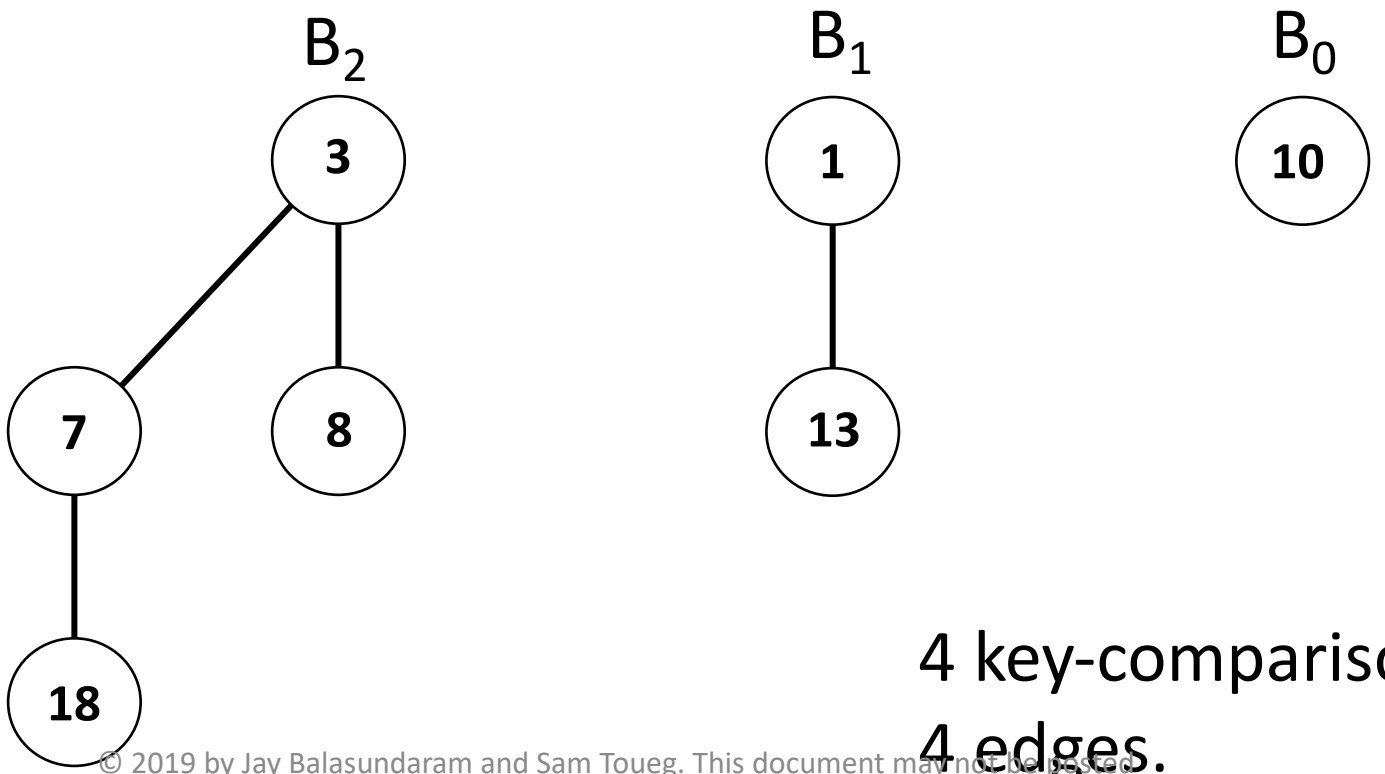

4 key-comparisons.

4 edges.

FACTS:

- One key-comparison per Binomial Heap edge.

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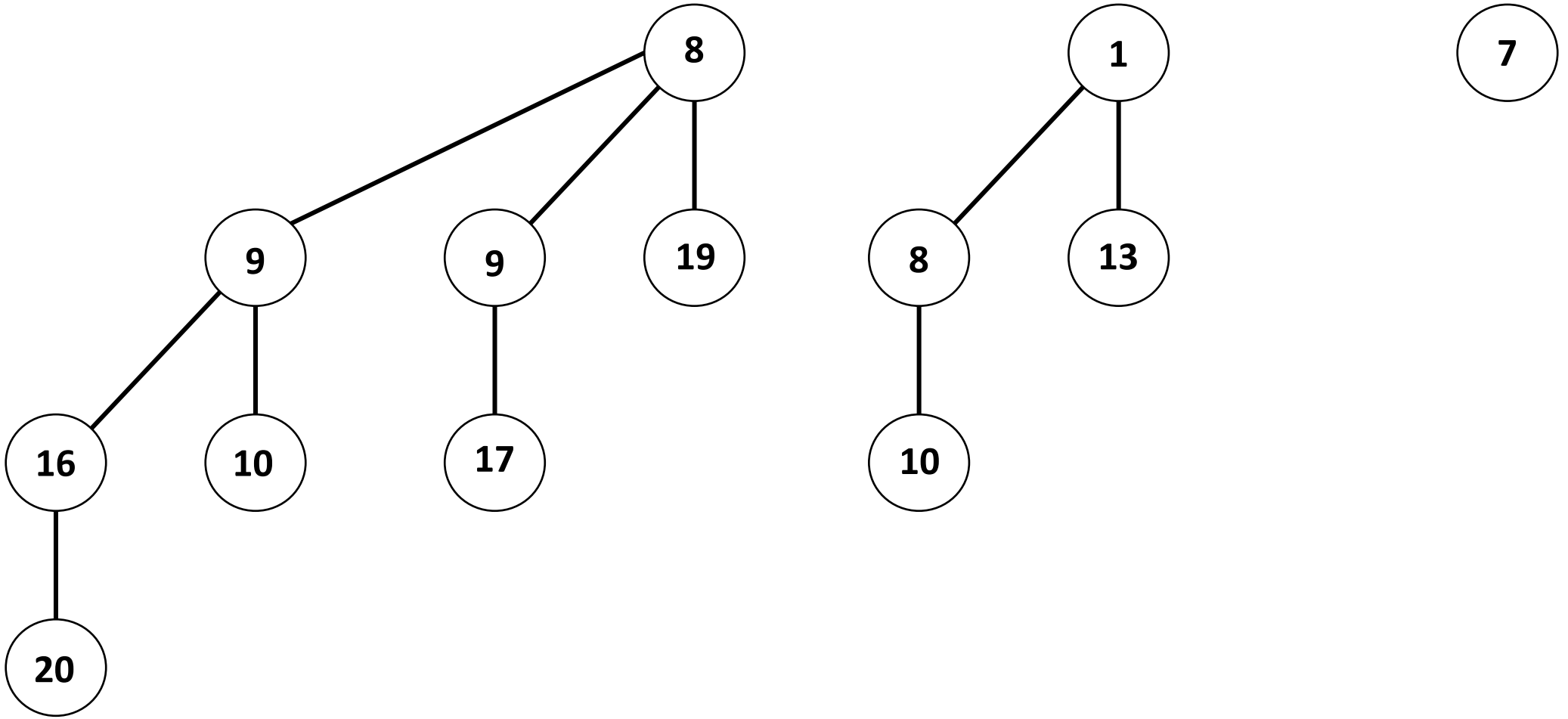
- A Binomial Heap for n elements can be built in O(n) key-comparisons.

Storing Binomial Heaps in Memory

Binomial Heap (Visually)



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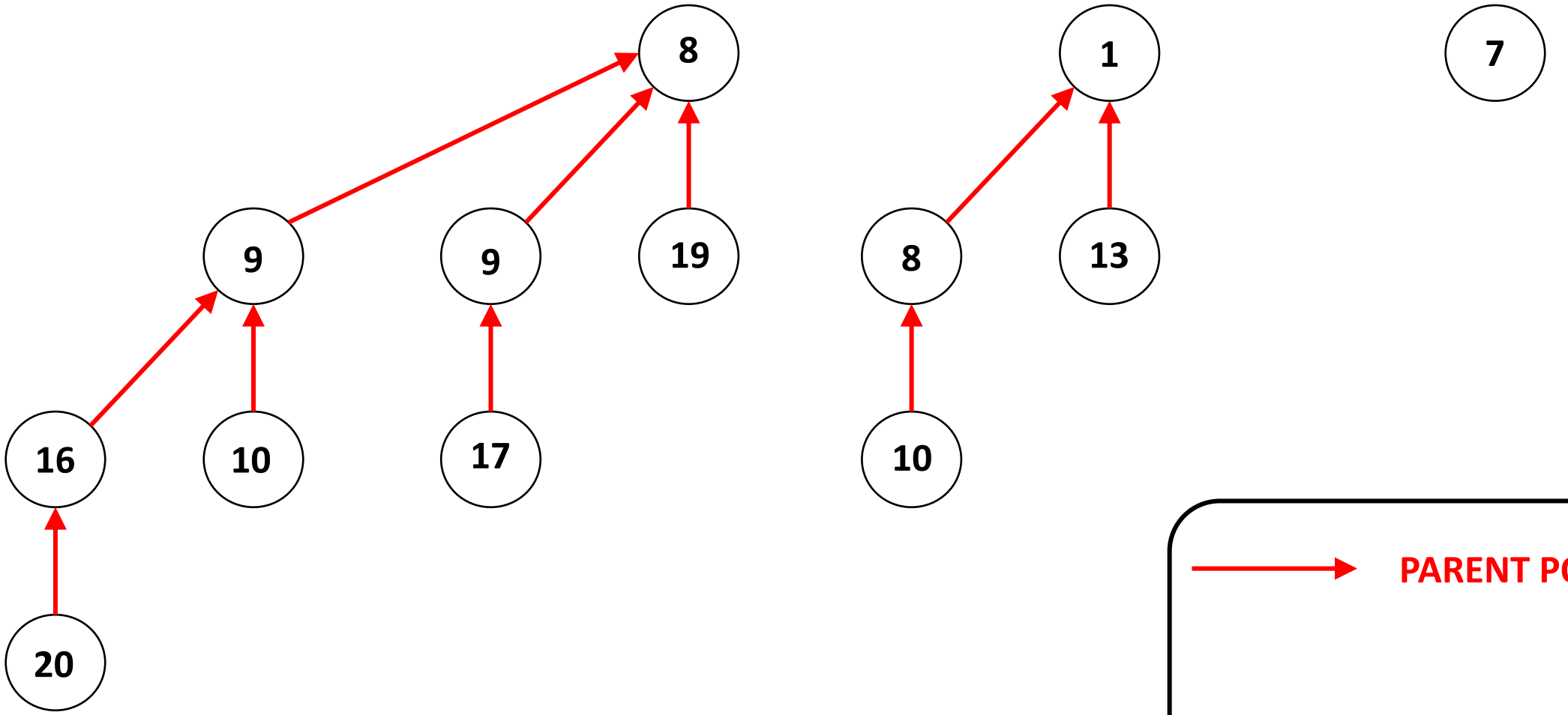

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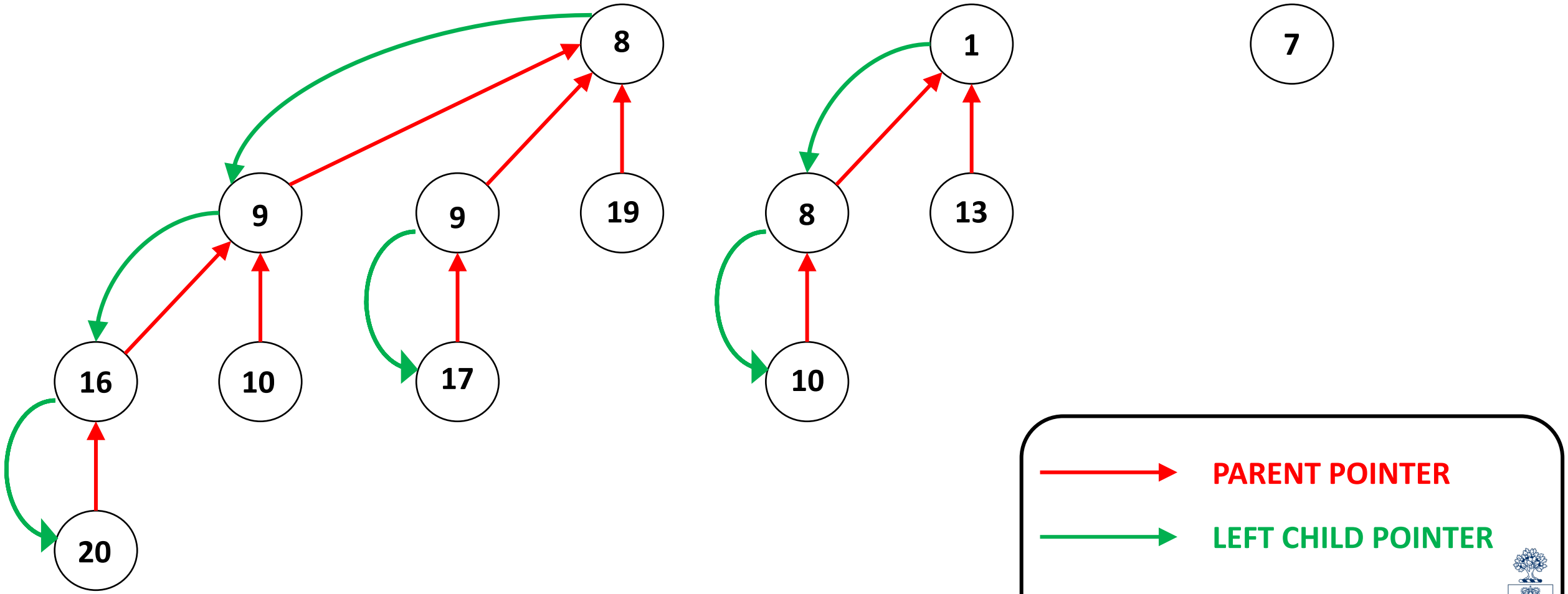

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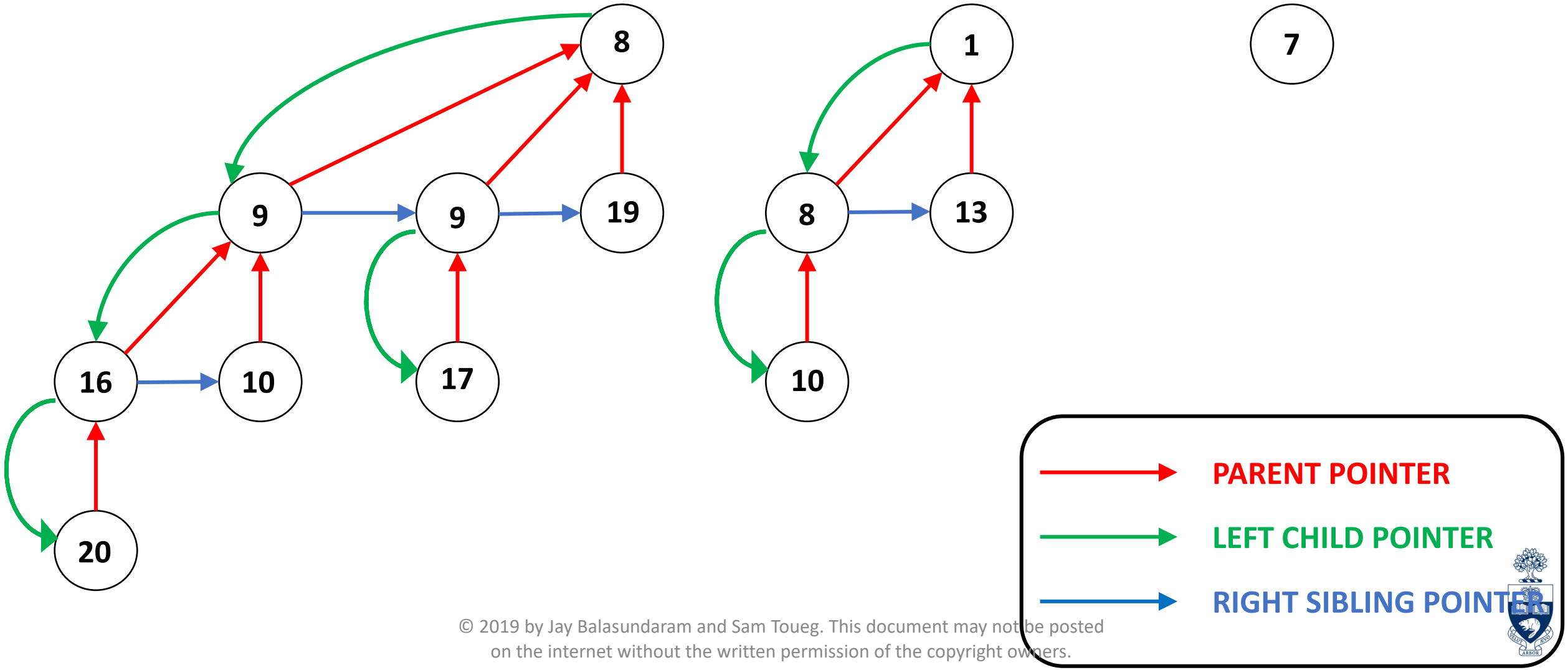

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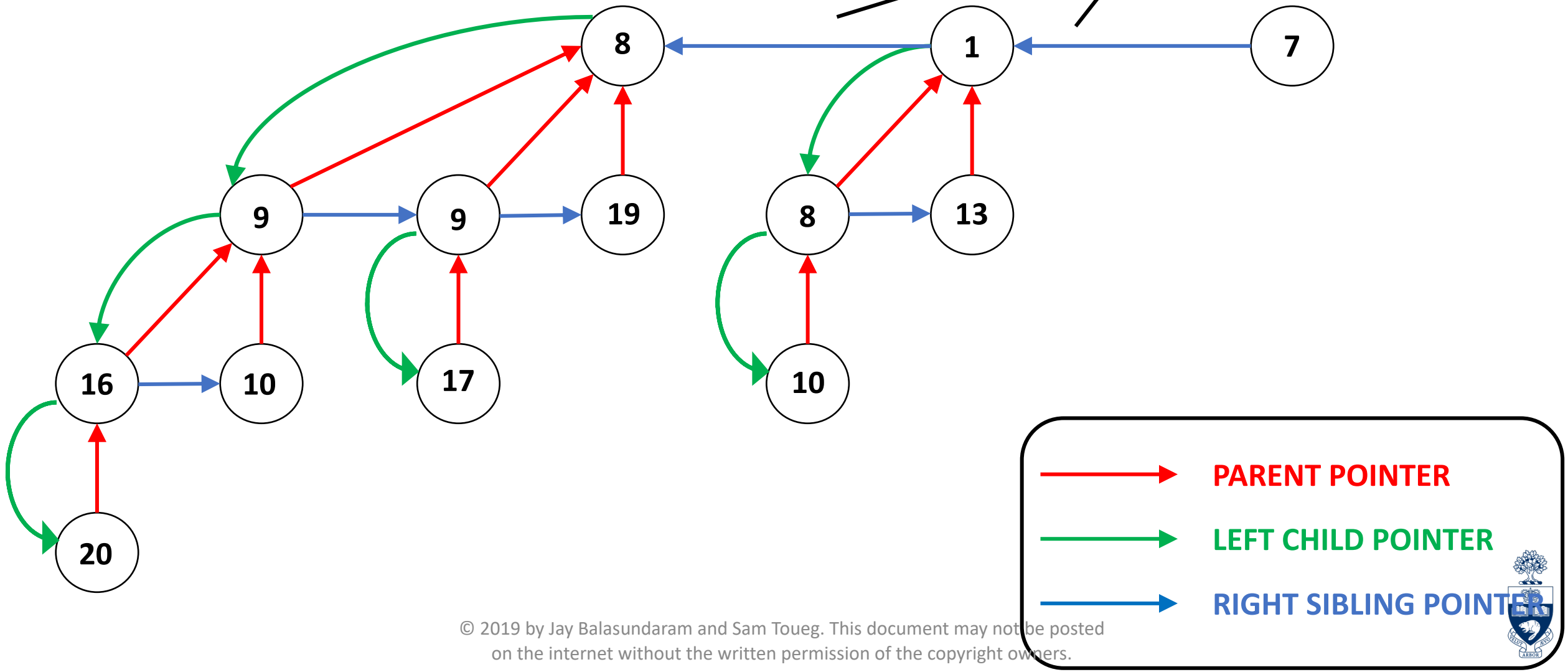

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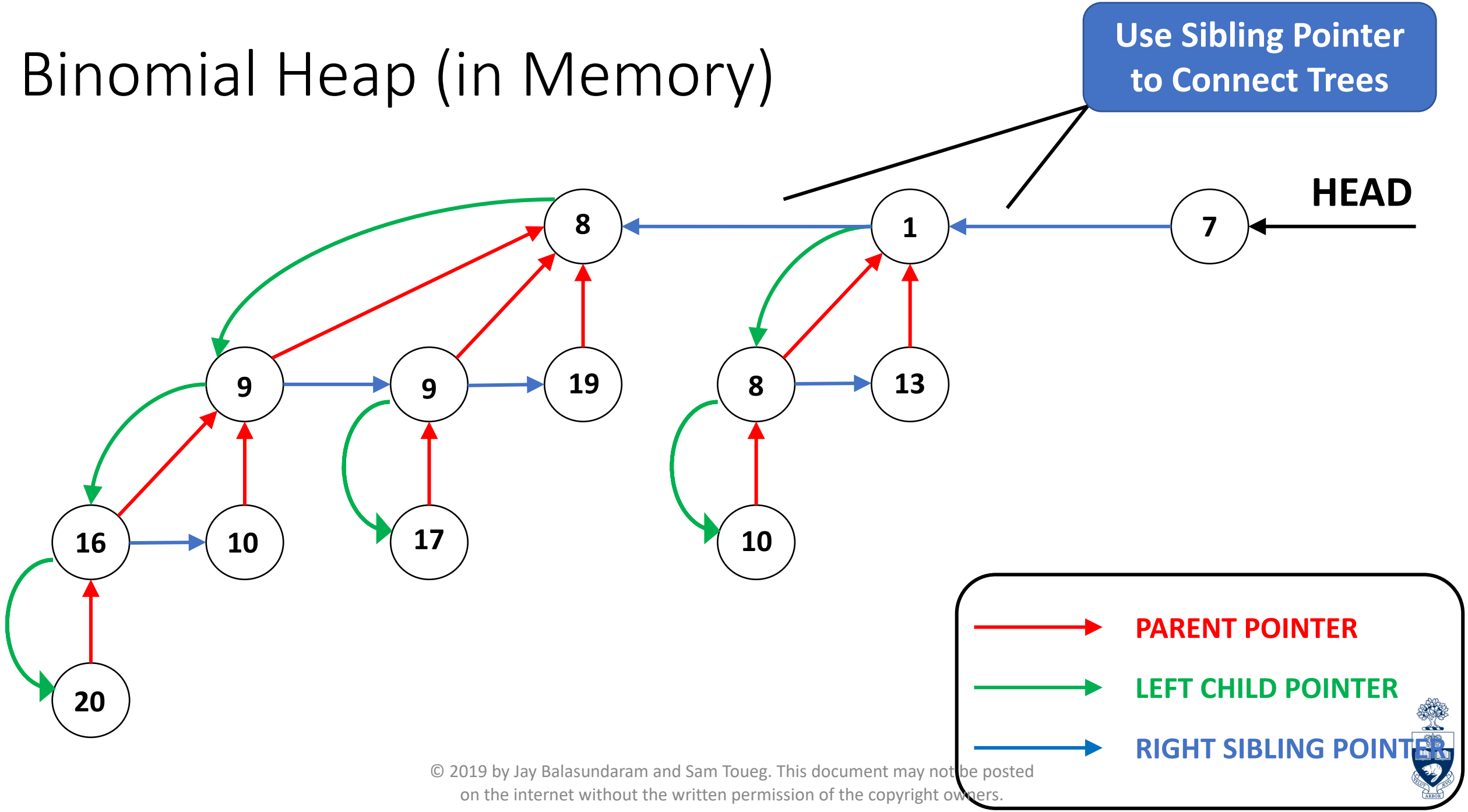

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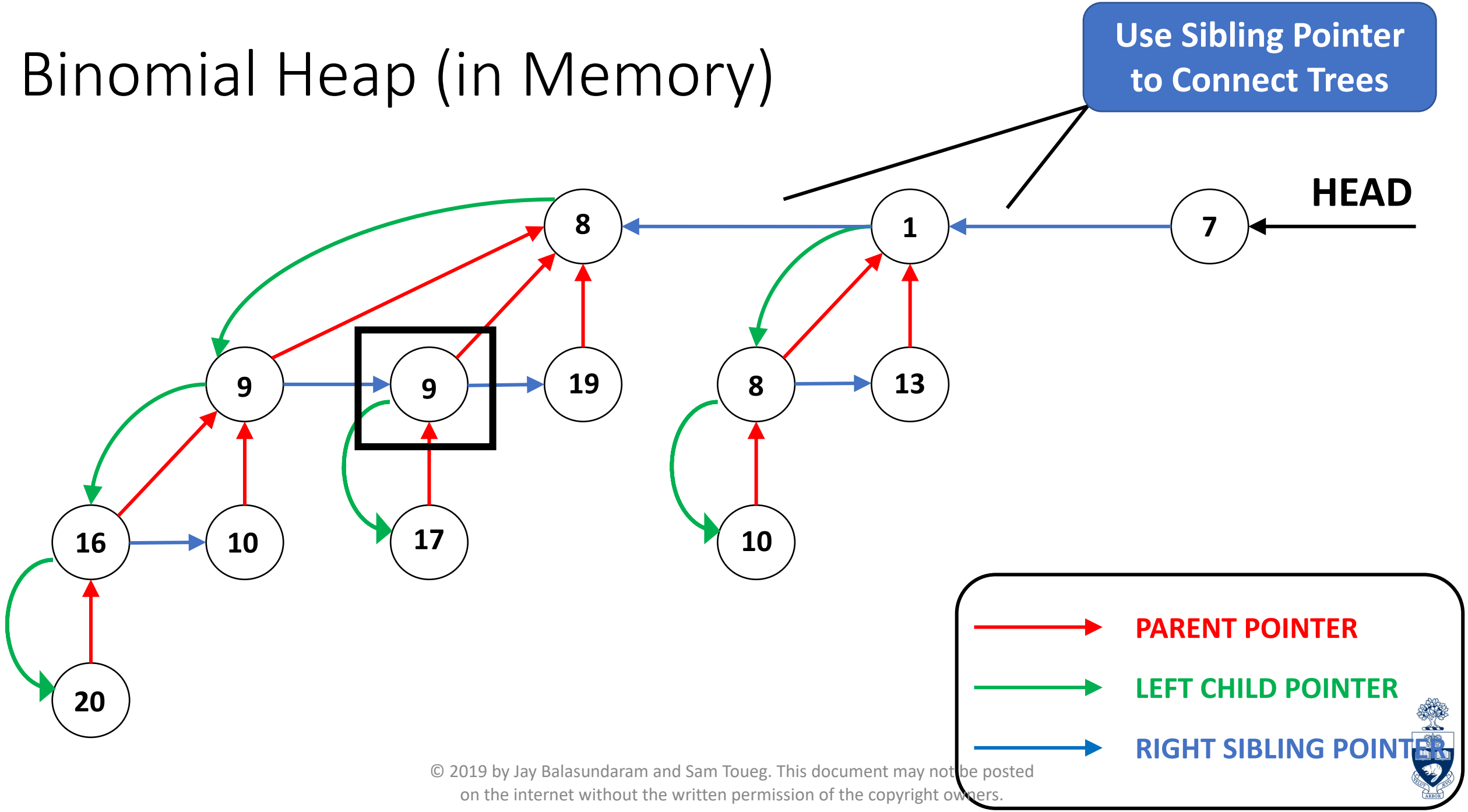

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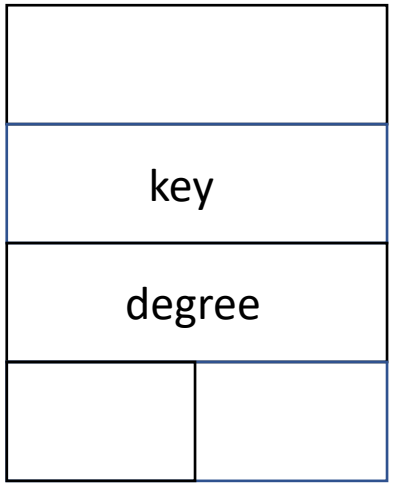

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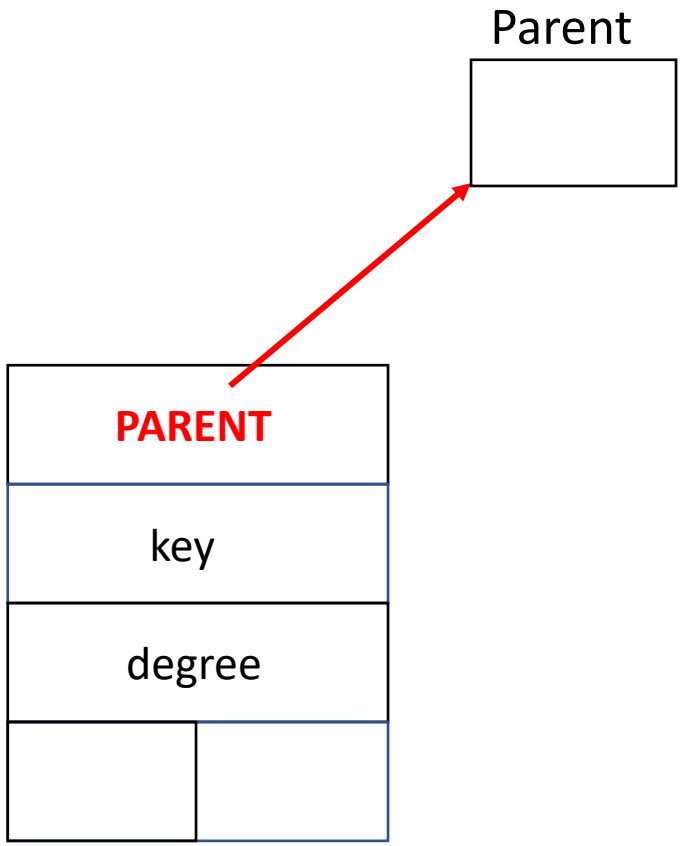

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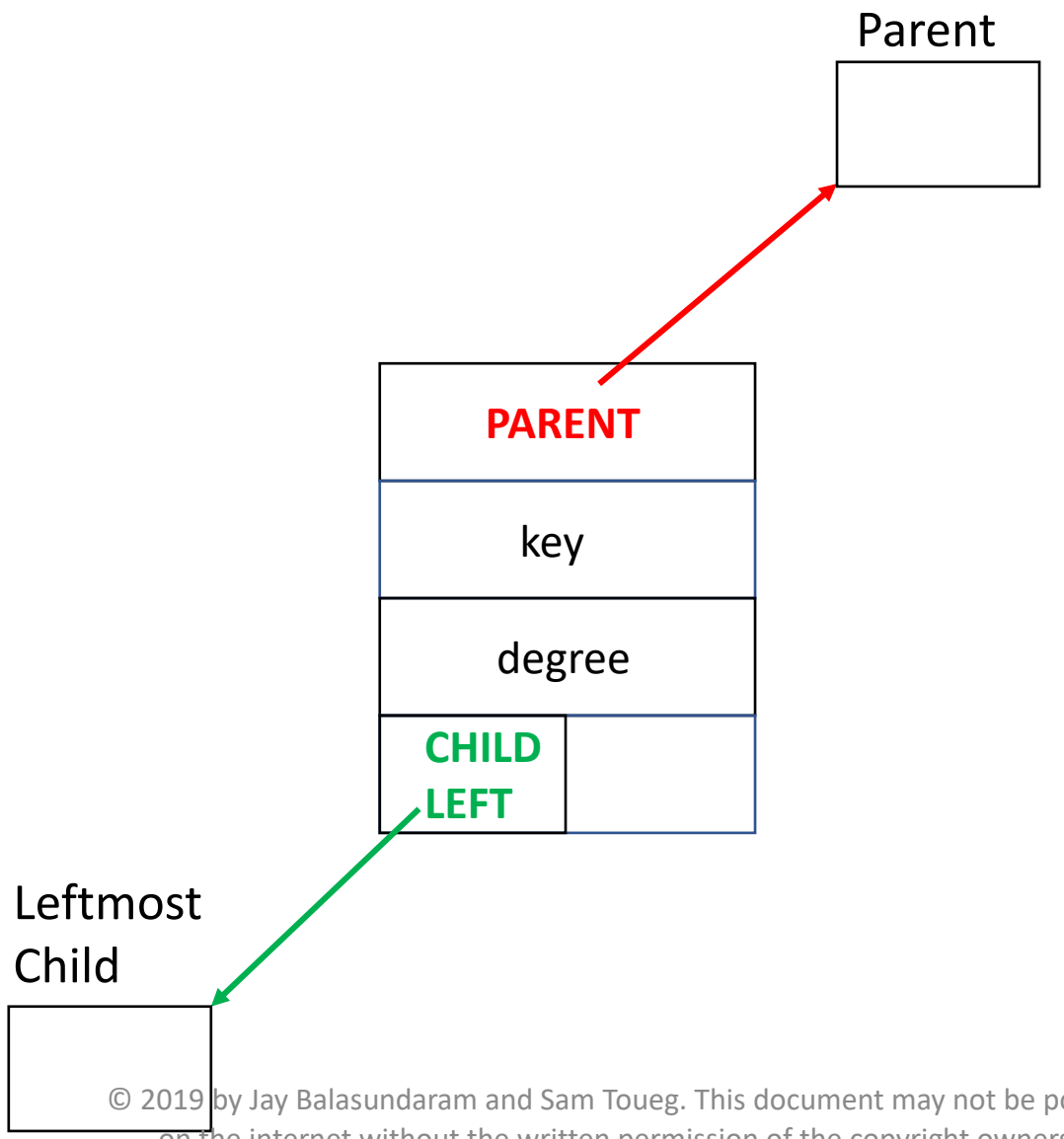

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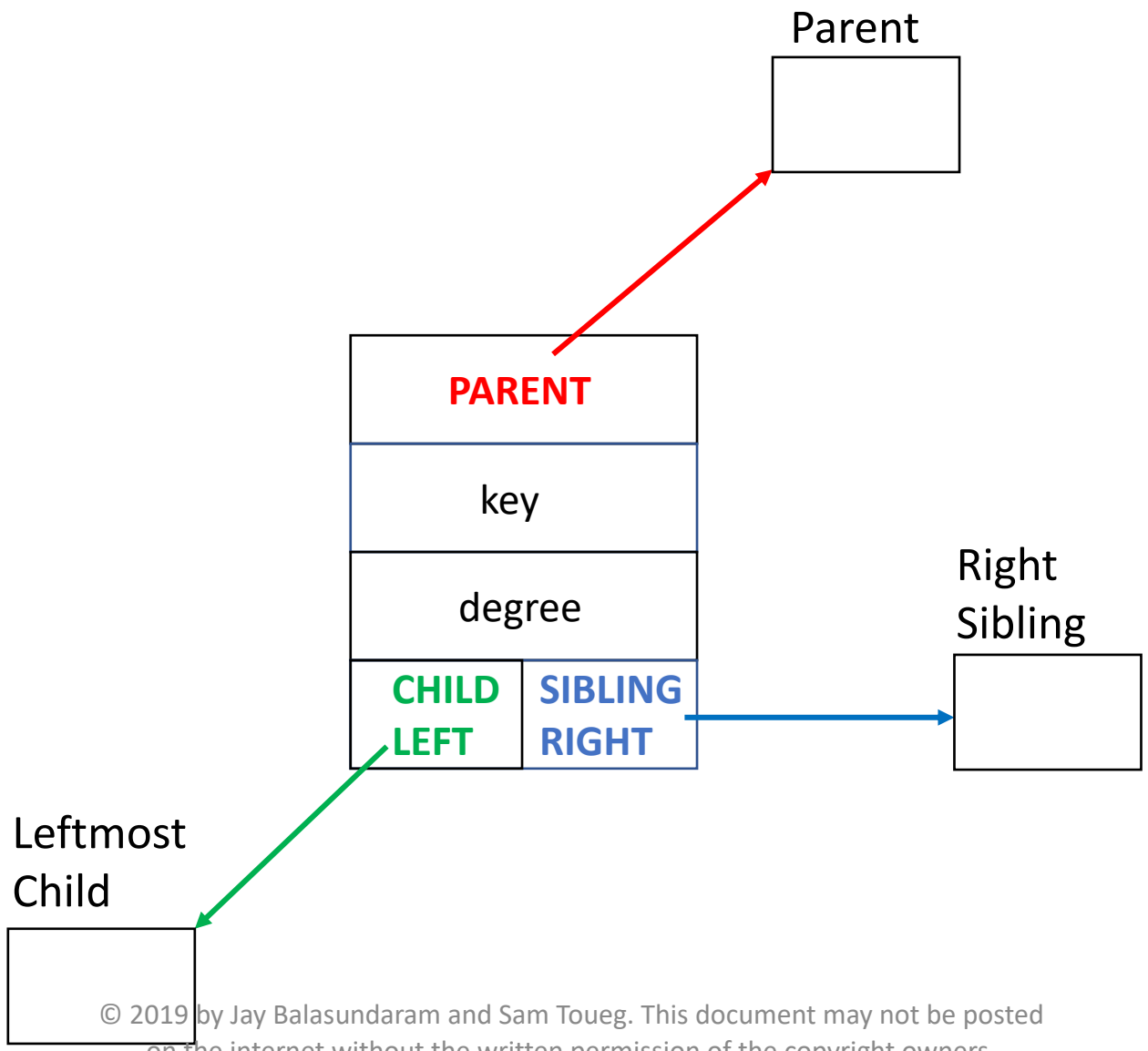
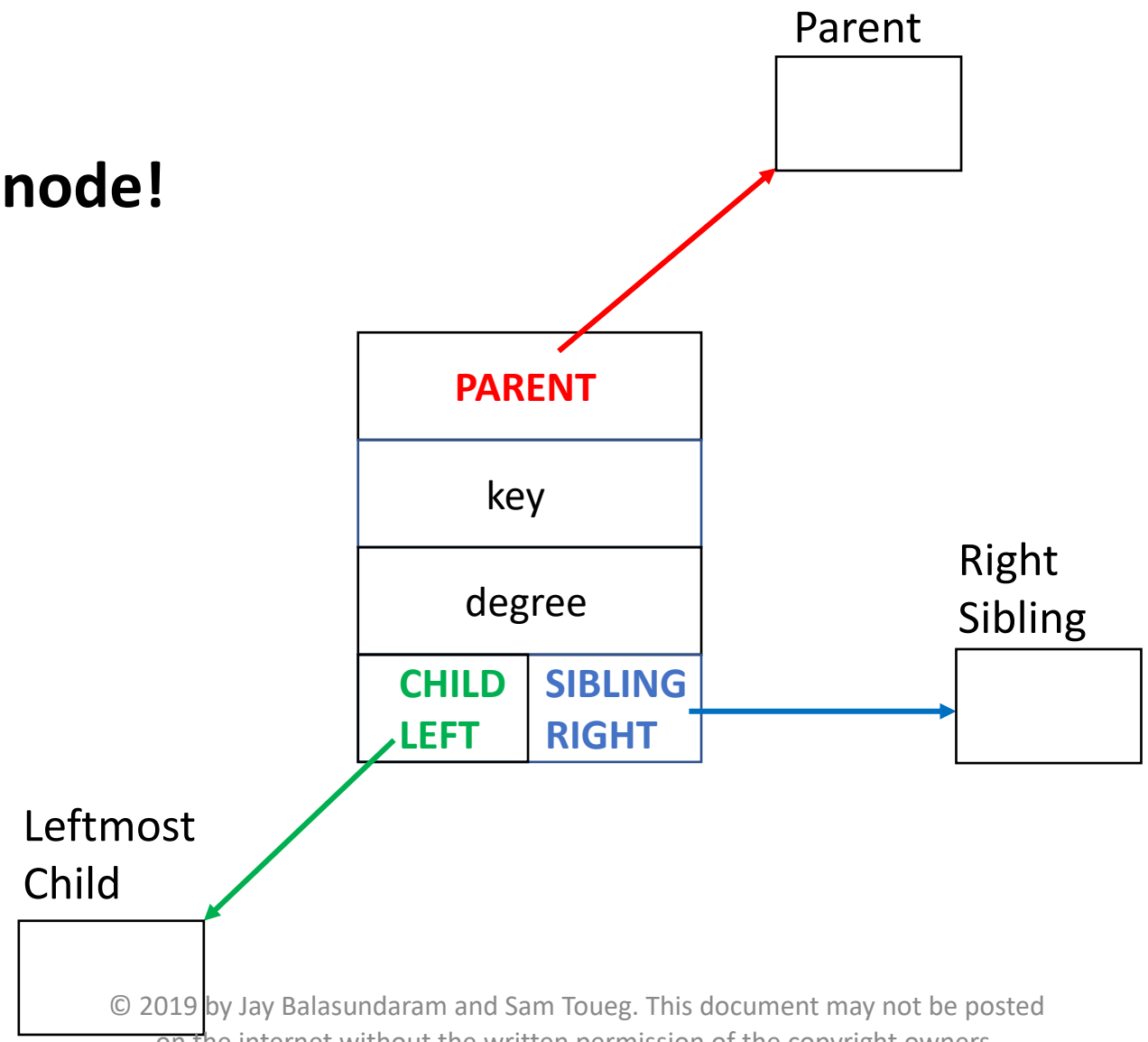
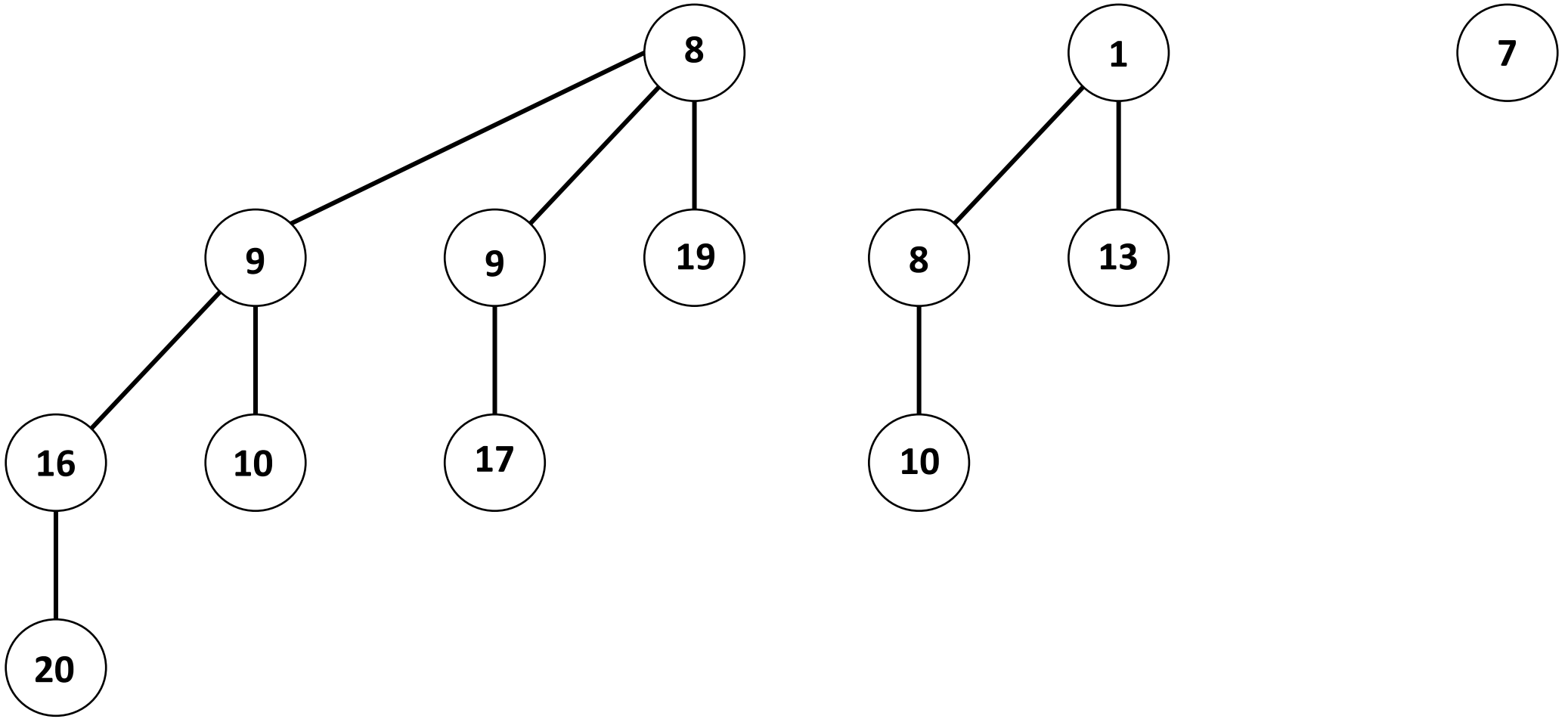


Binomial Heap (in Memory)

3 pointers per node!



Binomial Heap (Visually)



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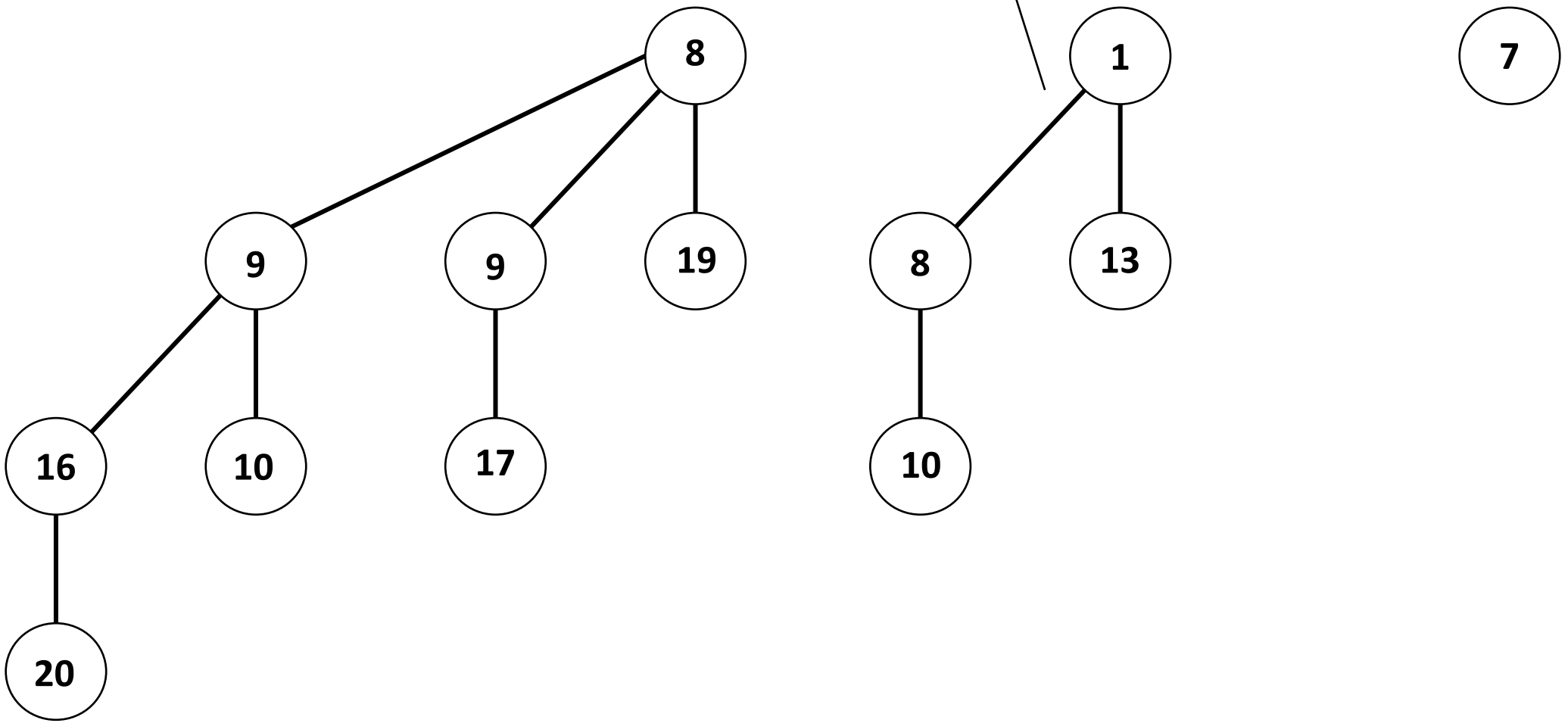

Binomial Heap Operations

Must implement the following operations:

- **Union(T, Q)**
- **Insert(T, x)**
- **Min(T)**
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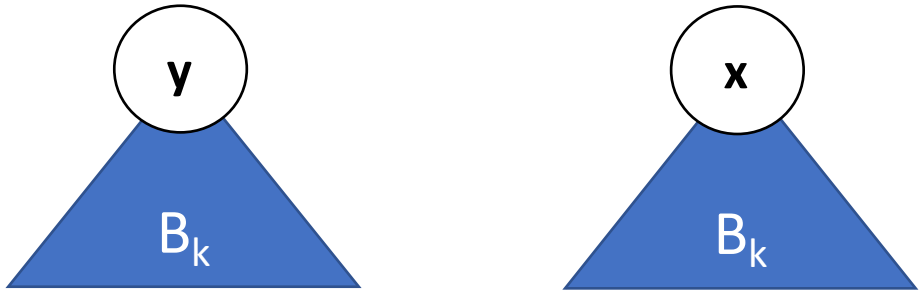

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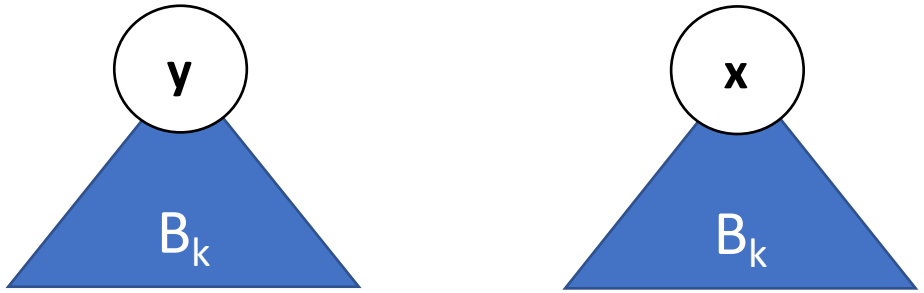

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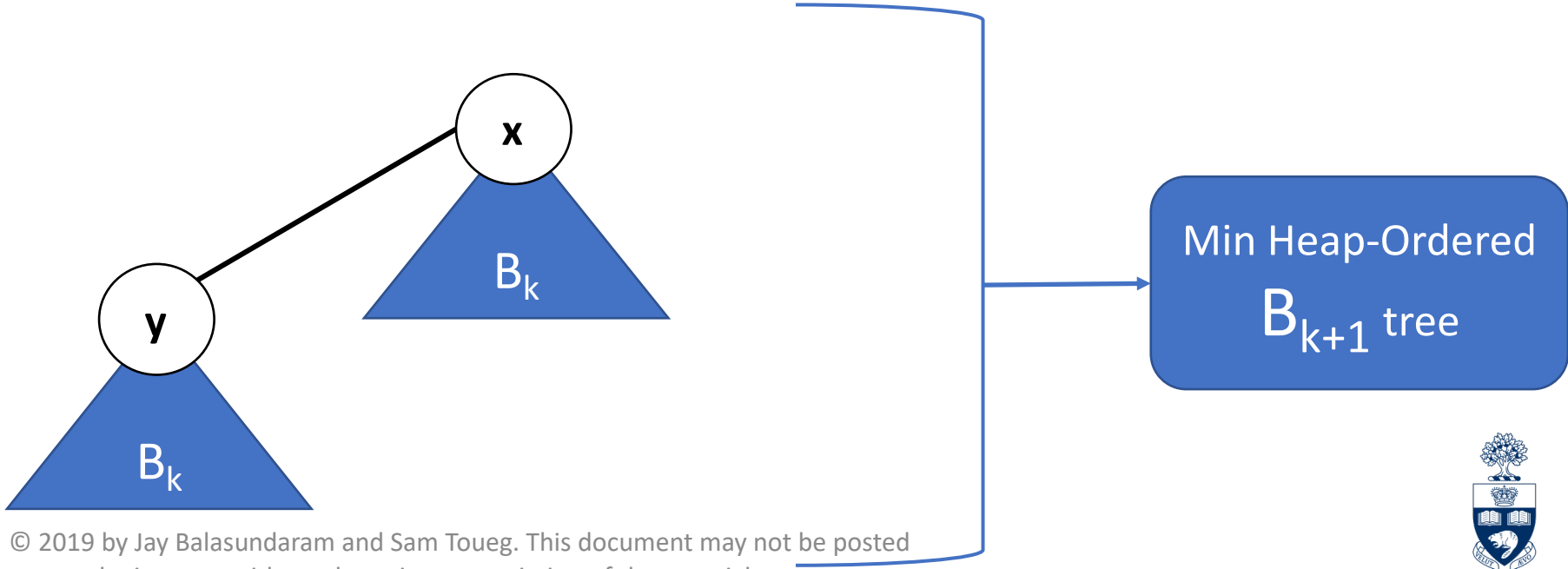

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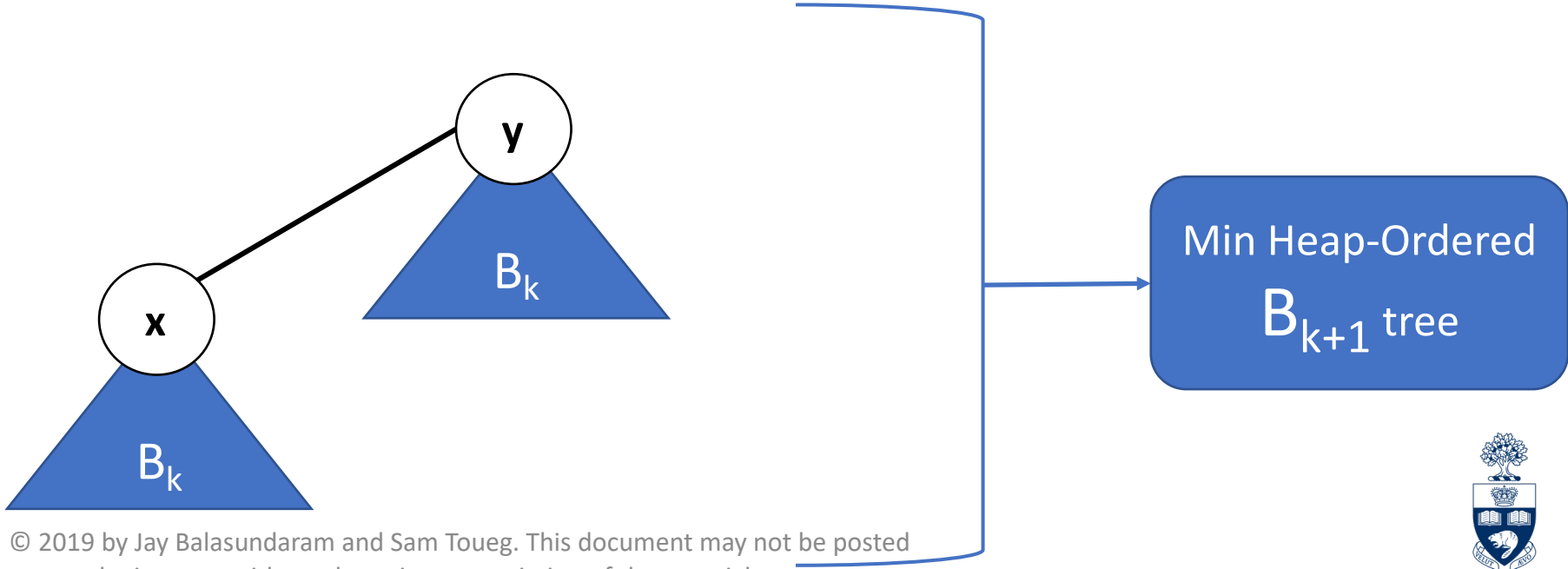

High Level Ideas

Cannot merge two (binary) min-heaps So easily !

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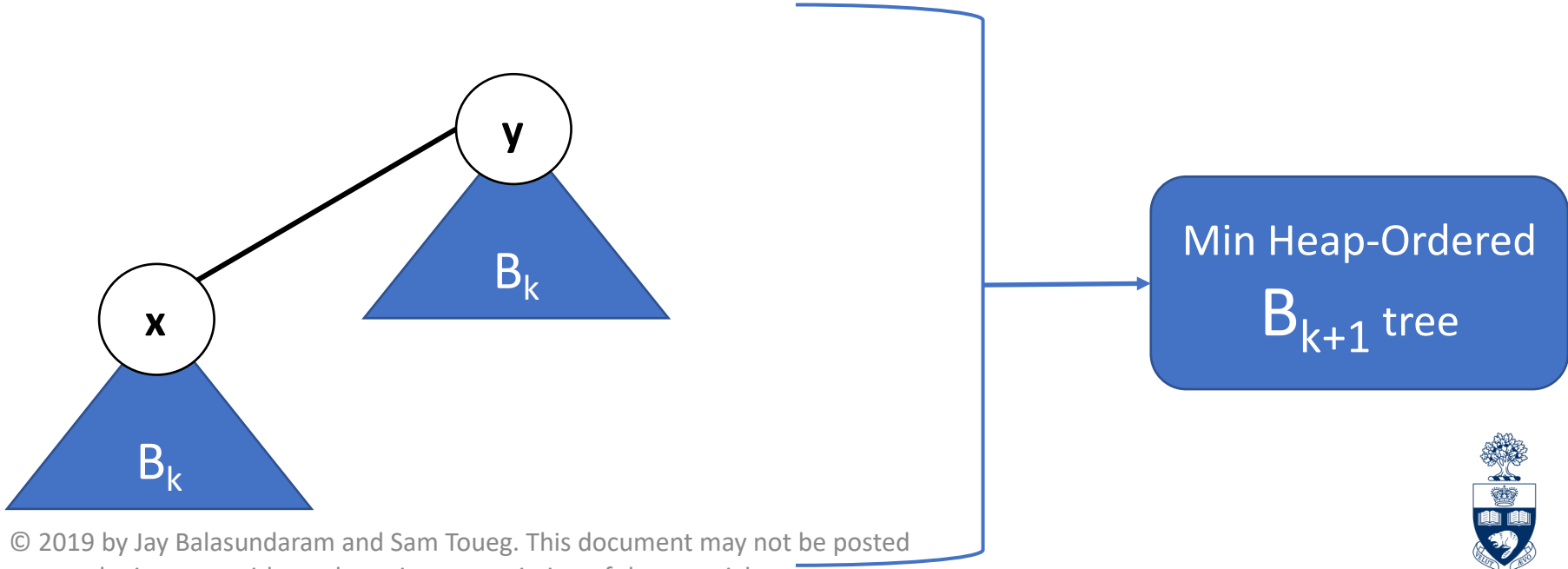

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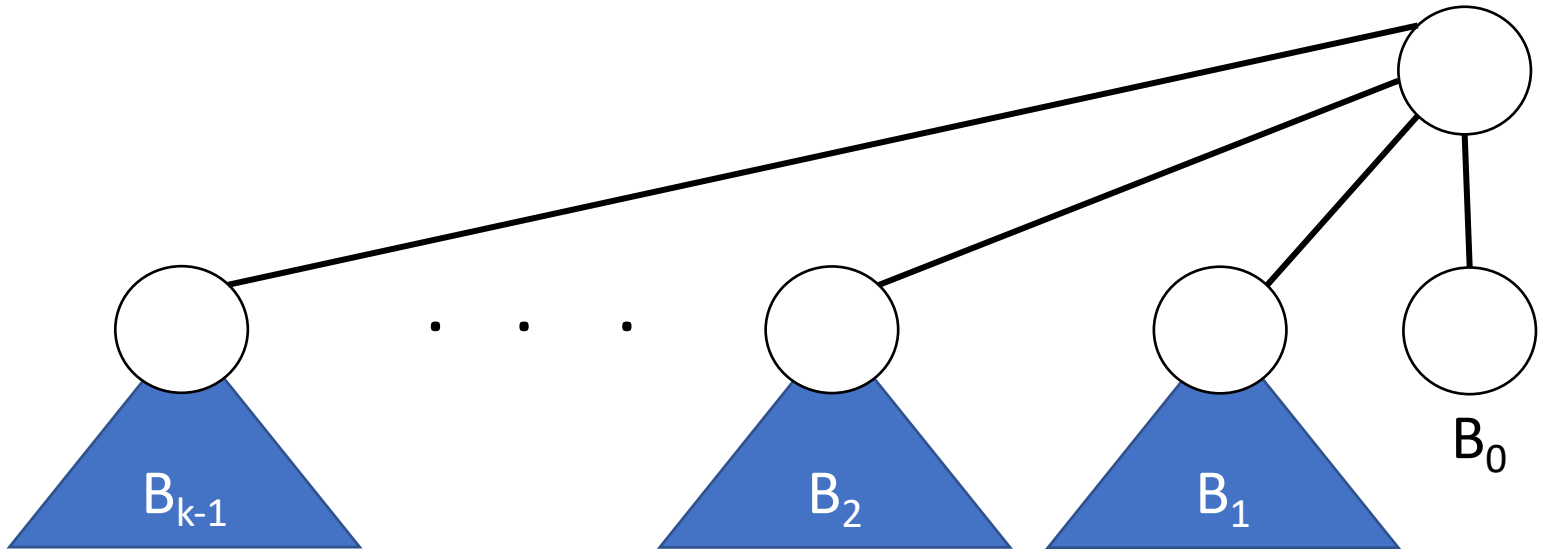

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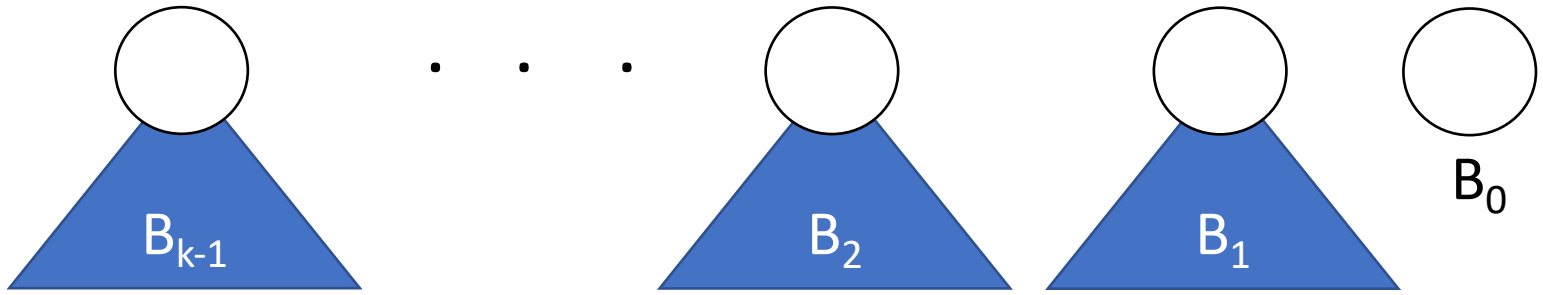

Union(T, Q)

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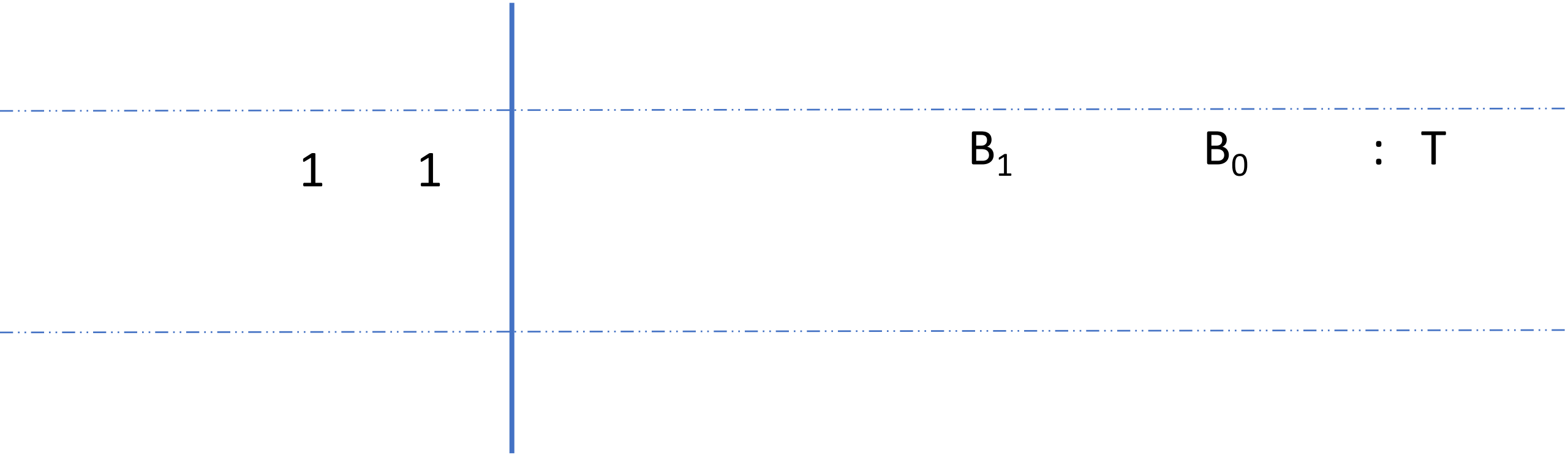
T is a Binomial Heap of size n = 3 = < 1 1>$_2$



Union(T, Q)

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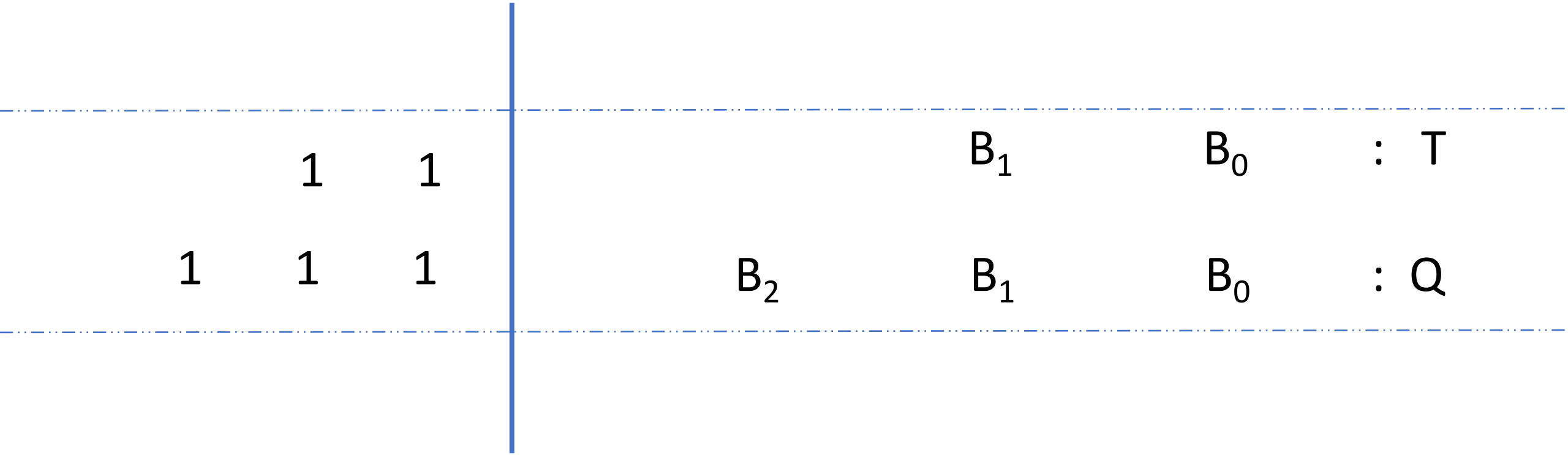
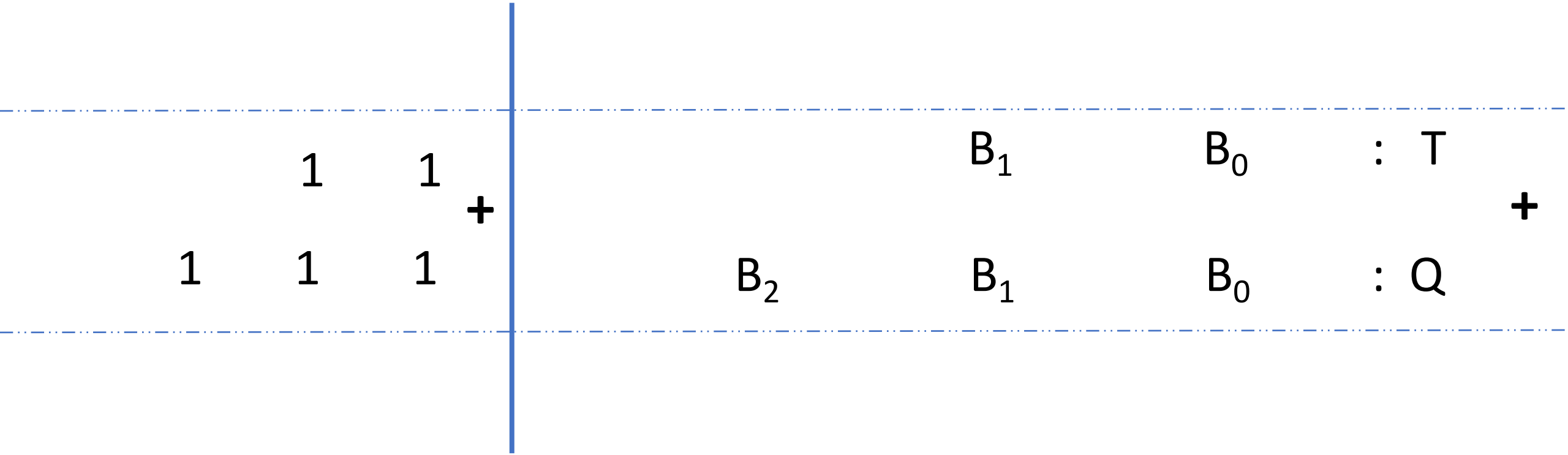
Q is a Binomial Heap of size n = 7 = < 1 1 1>$_2$



Union(T, Q)

T is a Binomial Heap of size n = 3 = $< 1\ 1 >_2$

Q is a Binomial Heap of size n = 7 = $< 1\ 1\ 1 >_2$



Union(T, Q)

T is a Binomial Heap of size n = 3 = < 1 1>$_2$

Q is a Binomial Heap of size n = 7 = < 1 1 1>$_2$

		1				B_1		**Carry**
		1	1			B_1	B_0	: T
				+				+
1		1	1		B_2	B_1	B_0	: Q
			0				X	**Result**

Union(T, Q)

T is a Binomial Heap of size n = 3 = < 1 1>$_2$

Q is a Binomial Heap of size n = 7 = < 1 1 1>$_2$

1	1			B_2	B_1			**Carry**
	1	1			B_1	B_0	: T	
			+					+
1	1	1		B_2	B_1	B_0	: Q	
	1	0			B_1	X		**Result**

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T is a Binomial Heap of size n = 3 = $< 1\ 1 >_2$

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1	1	1			B_3	B_2	B_1			**Carry**
		1	1				B_1	B_0	: T	
				+						+
	1	1	1			B_2	B_1	B_0	: Q	
1	0	1	0		B_3	X	B_1	X		**Result**

Union(T, Q)

T is a Binomial Heap of size n = 3 = < 1 1>$_2$

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1	1	1			B_3	B_2	B_1			**Carry**
		1	1				B_1	B_0	: T	
				+						+
	1	1	1			B_2	B_1	B_0	: Q	
1	0	1	0		B_3	X	B_1	X		**Result**

How many new edges were added?

Union(T, Q)

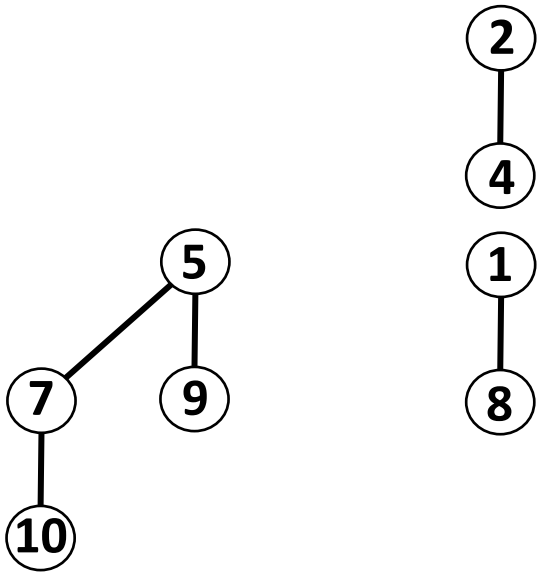
T is a Binomial Heap of size n = 3 = < 1 1>$_2$

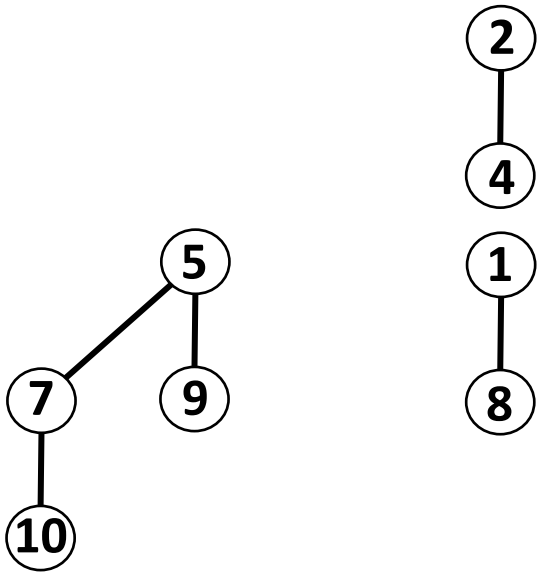
Q is a Binomial Heap of size n = 7 = < 1 1 1>$_2$

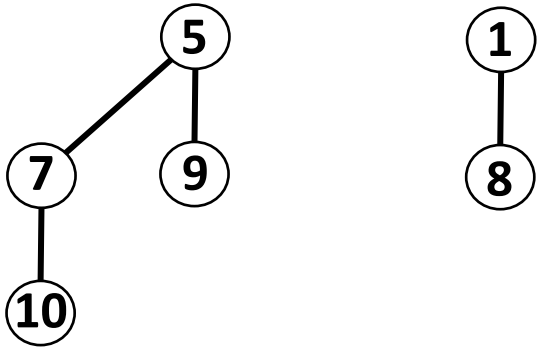
1	1	1			B_3	B_2	B_1			**Carry**
		1	1				B_1	B_0	: T	
				+						+
	1	1	1			B_2	B_1	B_0	: Q	
1	0	1	0		B_3	X	B_1	X		**Result**

How many new edges were added?

3







MERGE

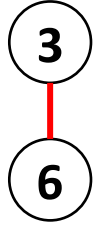
3

6

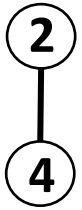
: T

+

: Q

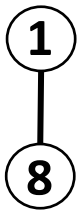


Carry

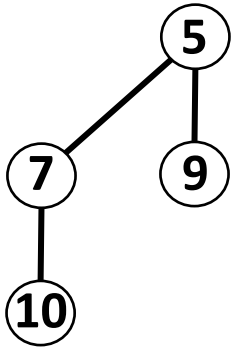


3 : T

\+

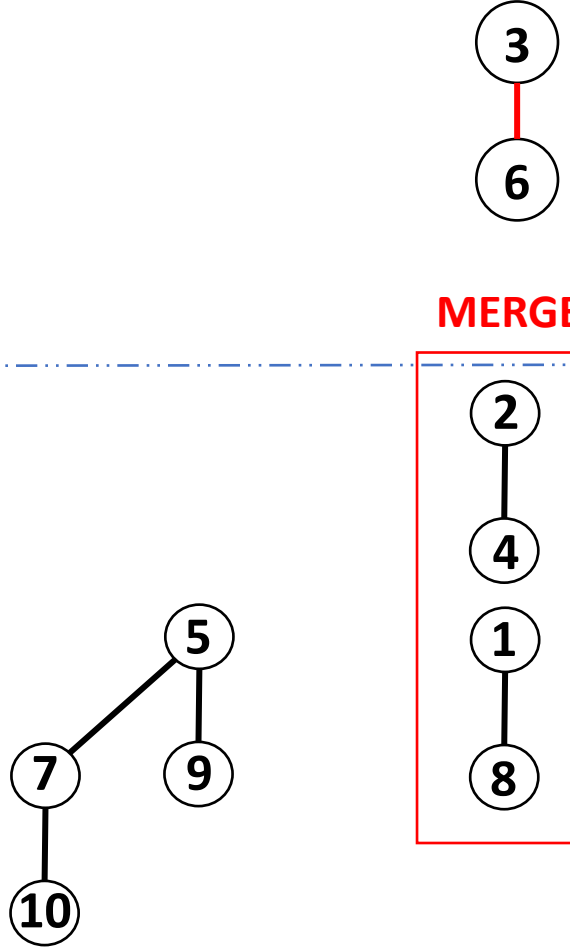


6 : Q



X

Result



Carry

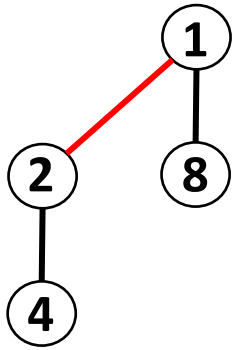
3 : T

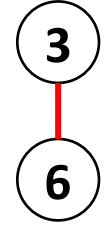
\+

6 : Q

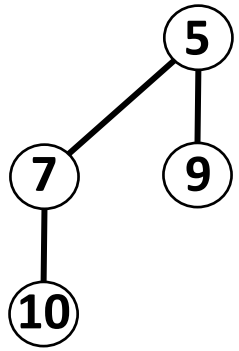
X

Result





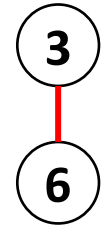
Carry



3 : T

\+

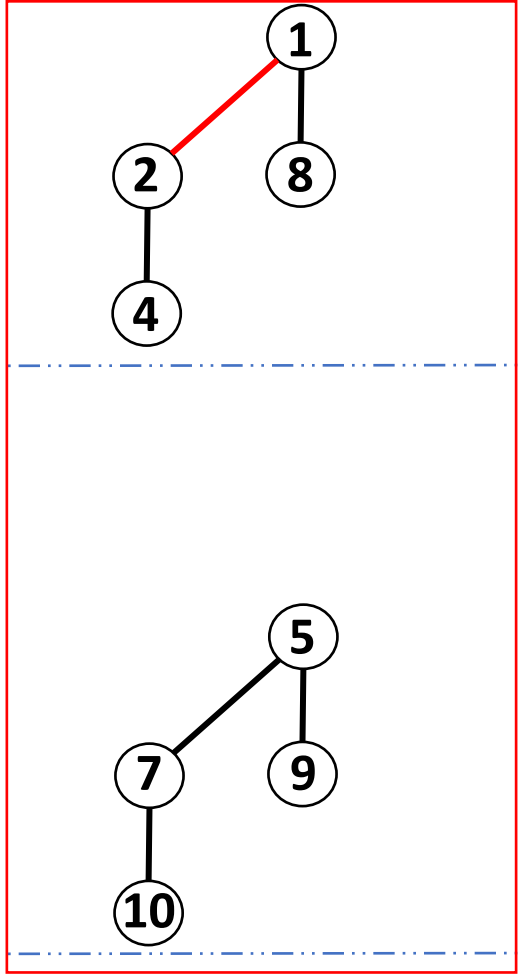
6 : Q

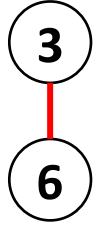


X

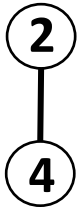
Result

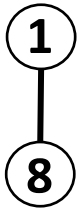
MERGE





Carry

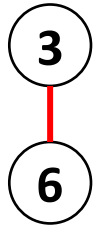




3 : T

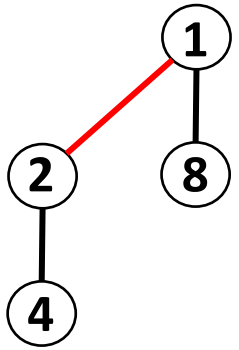
\+

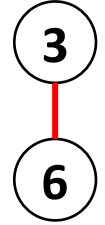
6 : Q



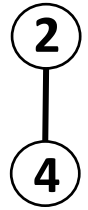
X

Result





Carry



3 : T

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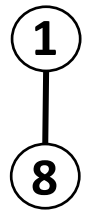
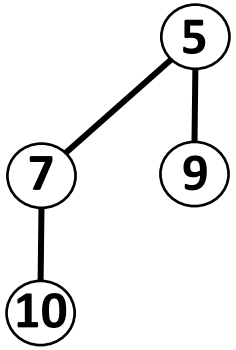


6 : Q



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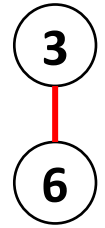
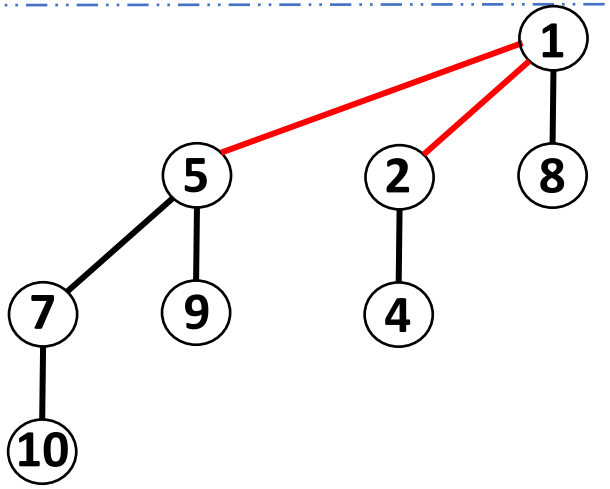
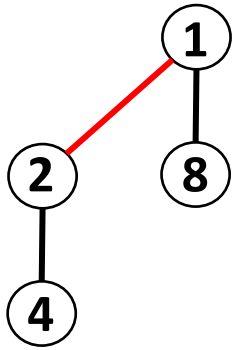
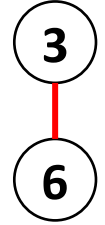


X

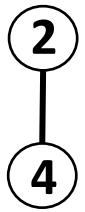
Result





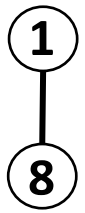


Carry

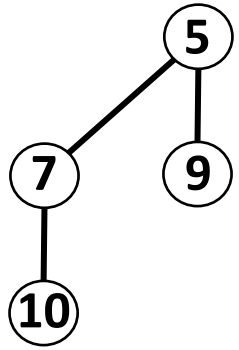


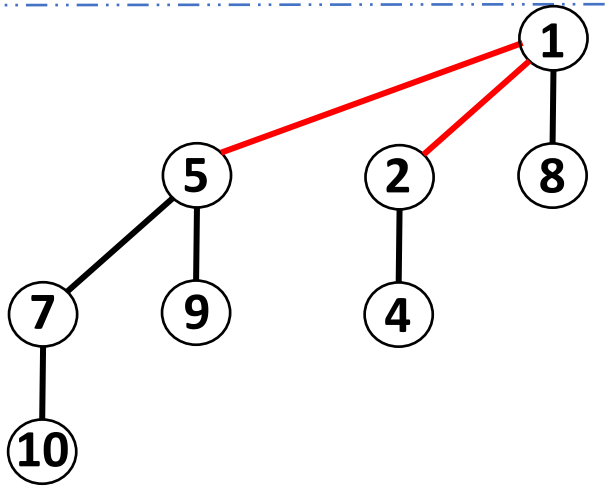
3 : T

\+



6 : Q





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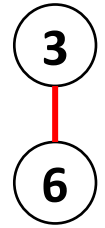
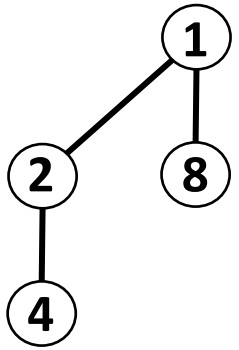


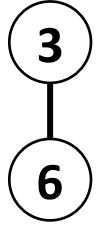
X

Result

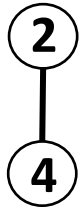
3 new edges.

3 key-comparisons.



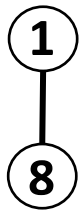


Carry

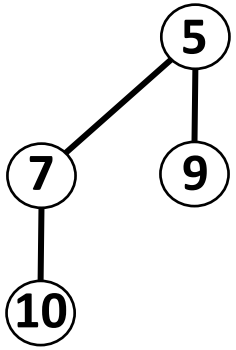


3 : T

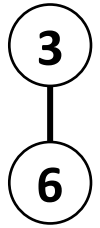
\+



6 : Q

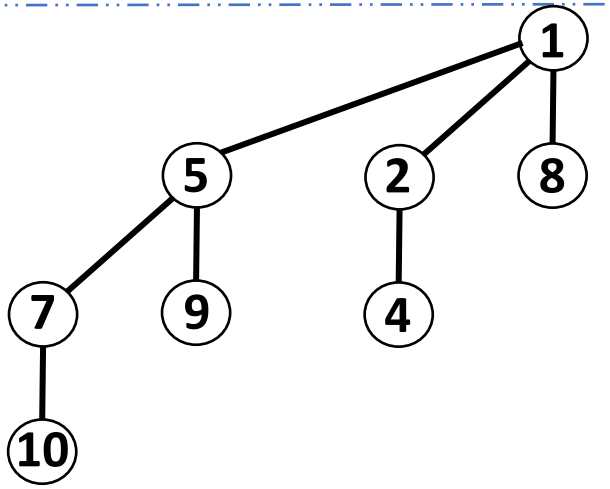


X



X

Result



Worst-Case Complexity of **Union(T, Q)**

Say |T| <= n and |Q| <= n (i.e. each contains at most n elements)

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Say |T| <= n and |Q| <= n (i.e. each contains at most n elements)

$\Rightarrow$ Each of T, Q have O(log n) B_k trees.

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$\Rightarrow$ Each of T, Q have O(log n) B_k trees.

$\Rightarrow$ **Union**(T, Q) takes at most O(log n) key-comparisons