Graphs Algorithms I

Breadth First Search

```
BFS(G, s)                                              /*  G = (V, E) and s ∈ V  */
    color[s] ← grey ;  d[s] ← 0 ;  p[s] ← NIL
    For each v ∈ V – {s}   do
            color[v] ← white
            d[v] ← ∞
            p[v] ← NIL

    Q ← empty ; ENQ(Q, s)                  /*  Q: nodes that are discovered but not yet explored  */
    While Q is not empty   do
            u ← DEQ(Q)                                 /*  Explore u  */
            For each (u, v) ∈ E   do                   /*  Explore edge (u,v)  */
                    If color[v] = white   then do      /*  If v is first discovered  */
                            color[v] ← grey
                            d[v] ← d[u] + 1
                            p[v] ← u
                            ENQ(Q, v)
                    End If
            End For
            color[u] ← black                           /*  Done exploring u  */
    End While
End BFS
```

Breadth First Search

Proof of Correctness

Execute BFS(s) on a graph G

For every node v of G:

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Length of shortest path (Distance) : $\delta(s,v)$

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(Distance)

Lemma 0: d[v] δ(s,v)

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Length of shortest path : δ(s,v)
(Distance)

Lemma 0: d[v] $\geq$ δ(s,v)

We would like to prove the following:

Main Theorem:

After BFS(s), for every $v \in V$,

$$d[v] = \delta(s,v)$$

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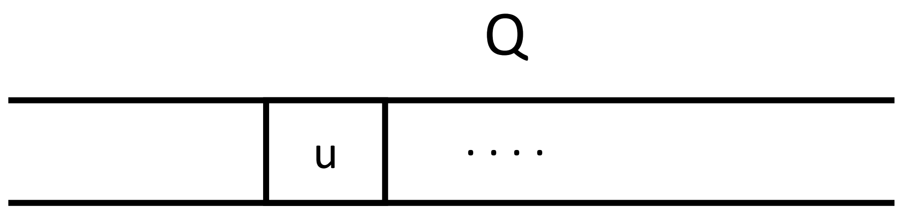
That is: the discovery path to v is a shortest path to v

We first prove the following:

Lemma 1:

If u enters Q



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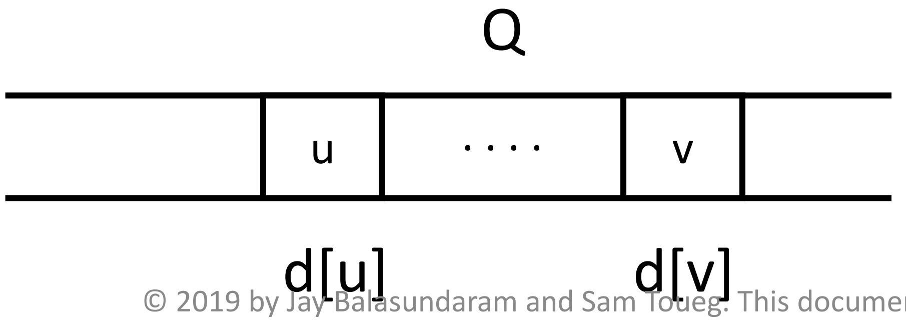
If u enters Q **before** v enters Q during the execution of BFS(s), then:



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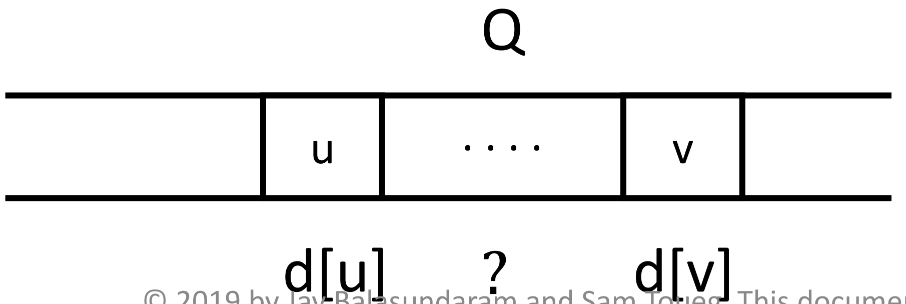

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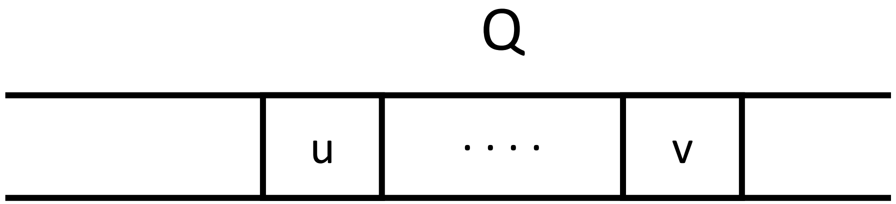

d[u] $\geq$ d[v]

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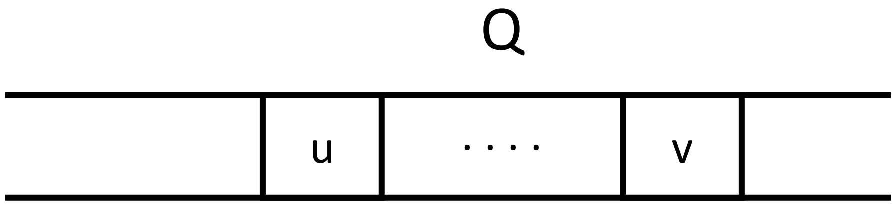

$d[u] = d[v]$

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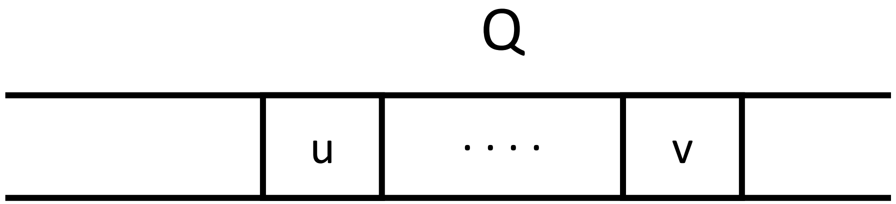

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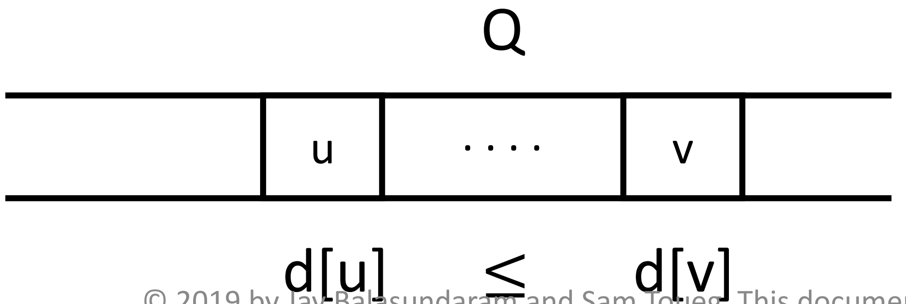

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Suppose, for contradiction, that Lemma 1 is false.

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Suppose, for contradiction, that Lemma 1 is false.

- Let v be the **first** node that enter Q such that d[u] > d[v] for some node u that entered Q before v.

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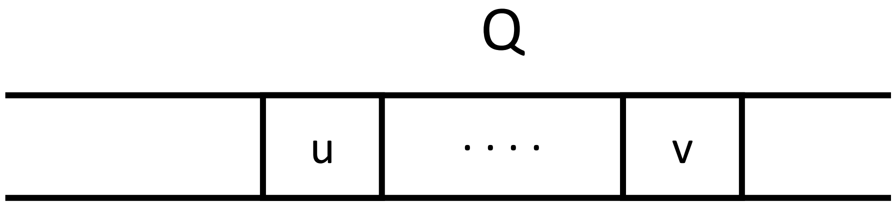
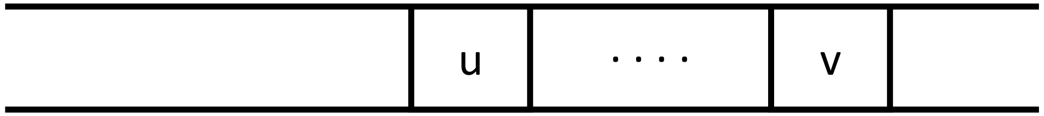


$d[u] > d[v]$

Proof of Lemma 1 contd:

Q



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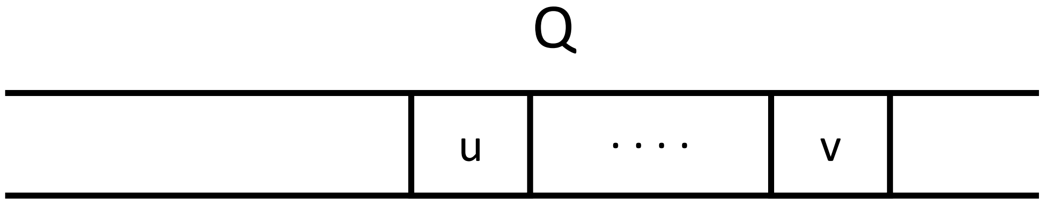
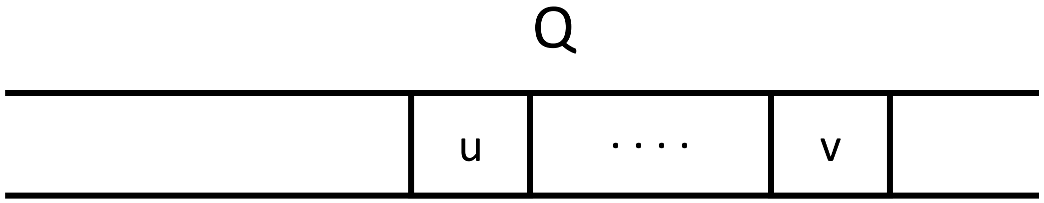
Proof of Lemma 1 contd:



$d[u] > d[v]$

- $v \neq s$

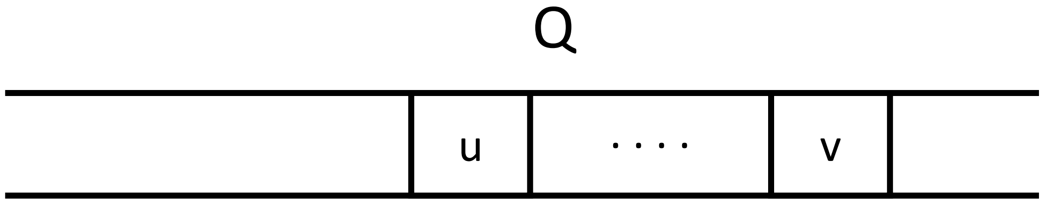
Proof of Lemma 1 contd:



d[u] > d[v]

- v ≠ s because no vertex u enters Q before s

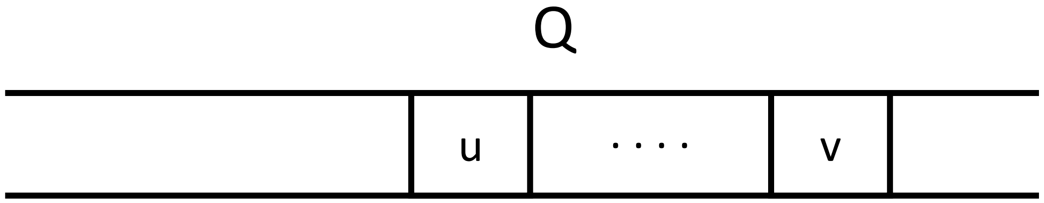
Proof of Lemma 1 contd:



d[u] > d[v]

- $v \neq s$ because no vertex u enters Q before s
- $u \neq s$

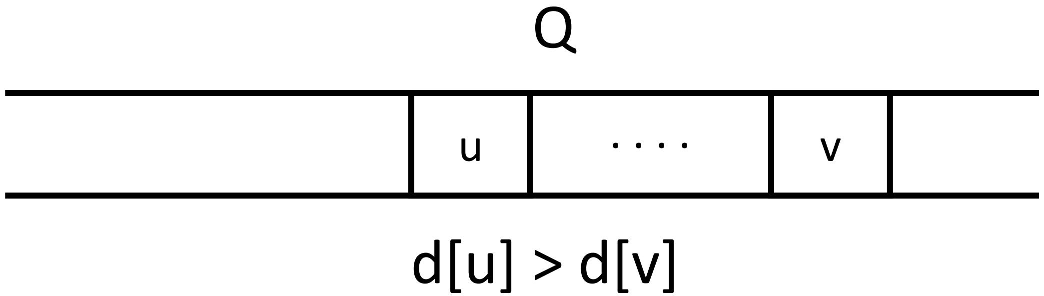
Proof of Lemma 1 contd:



d[u] > d[v]

- v $\neq$ s because no vertex u enters Q before s
- u $\neq$ s because d[s] = 0 and d[v] $\geq$ 0

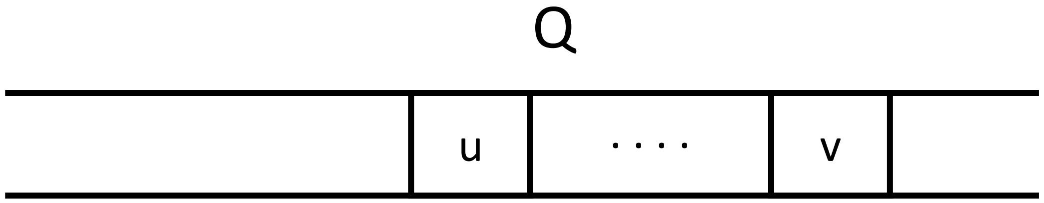
Proof of Lemma 1 contd:



- $v \neq s$ because no vertex u enters Q before s
- $u \neq s$ because $d[s] = 0$ and $d[v] \geq 0$

$\Rightarrow$ u and v entered Q during the exploration of some nodes, say u' and v' respectively

Proof of Lemma 1 contd:



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$\Rightarrow d[u] =$

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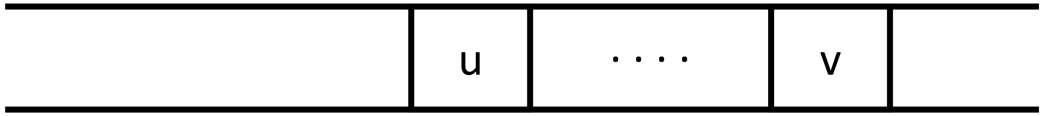

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$\Rightarrow d[u] = d[u'] + 1$

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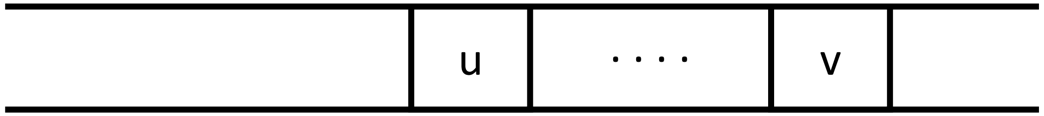

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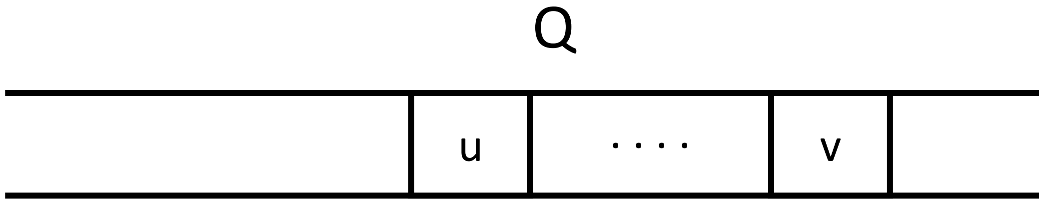

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- $u \neq s$ because $d[s] = 0$ and $d[v] \geq 0$

  ⇒ u and v entered Q during the exploration of some nodes, say u’ and v’ respectively

  ⇒ $d[u] = d[u'] + 1$ and $d[v] = d[v'] + 1$

- Since $d[u] \neq d[v]$,

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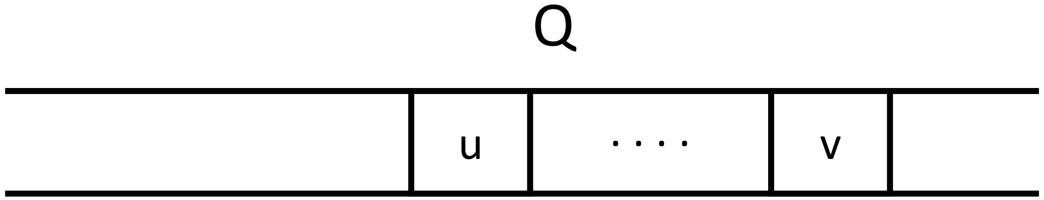

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- Since $d[u] \neq d[v]$, $d[u'] \neq d[v']$

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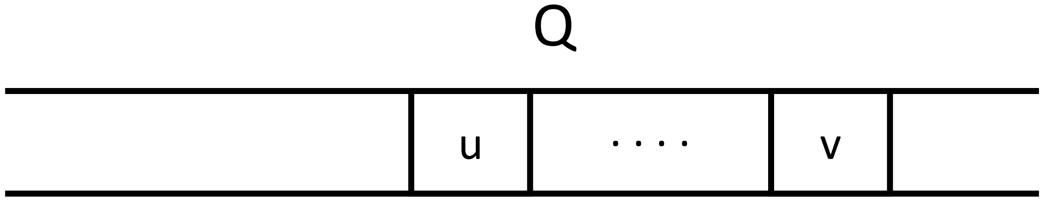

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  $\Rightarrow d[u] = d[u'] + 1$ and $d[v] = d[v'] + 1$

- Since $d[u] \neq d[v]$, $d[u'] \neq d[v'] \Rightarrow u' \neq v'$ i.e. u' and v' are **distinct** nodes

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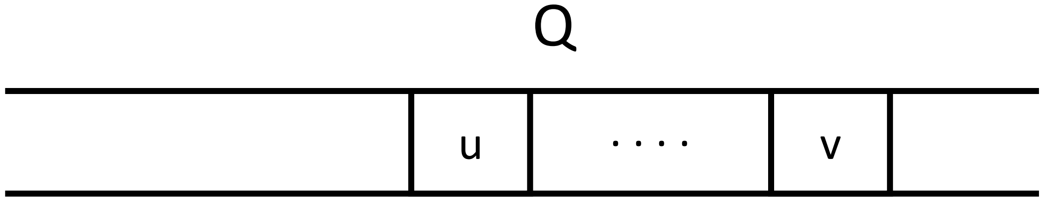

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- u was discovered before v

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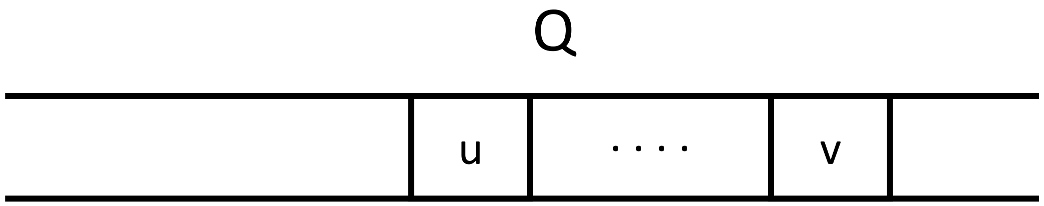

d[u] > d[v]

- v $\neq$ s because no vertex u enters Q before s
- u $\neq$ s because d[s] = 0 and d[v] $\geq$ 0

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- Since d[u] $\neq$ d[v],   d[u'] $\neq$ d[v'] $\Rightarrow$ u' $\neq$ v'   i.e. u' and v' are distinct nodes
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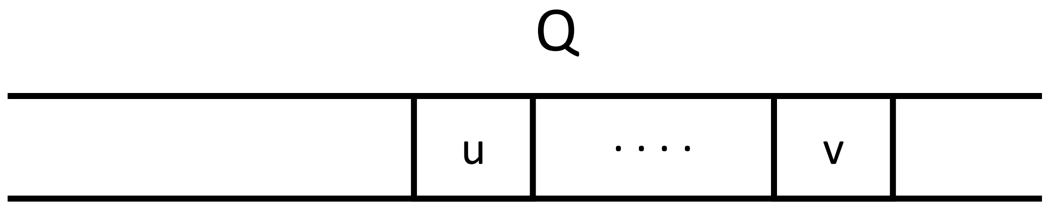

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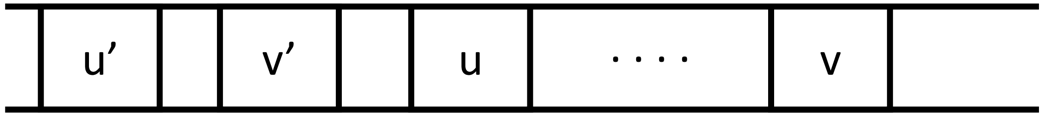
  $\Rightarrow$ d[u] = d[u'] + 1 and d[v] = d[v'] + 1

- Since d[u] $\neq$ d[v], d[u'] $\neq$ d[v'] $\Rightarrow$ u' $\neq$ v' i.e. u' and v' are distinct nodes
- u was discovered before v $\Rightarrow$ u' was discovered before v'

  $\Rightarrow$ u' entered Q before v'

Proof of Lemma 1 contd:

Q



d[u] > d[v]

- v $\neq$ s because no vertex u enters Q before s
- u $\neq$ s because d[s] = 0 and d[v] $\geq$ 0

  $\Rightarrow$ u and v entered Q during the exploration of some nodes, say u’ and v’ respectively

  $\Rightarrow$ d[u] = d[u’] + 1 and d[v] = d[v’] + 1

- Since d[u] $\neq$ d[v], d[u’] $\neq$ d[v’] $\Rightarrow$ u’ $\neq$ v’ i.e. u’ and v’ are distinct nodes
- u was discovered before v $\Rightarrow$ u’ was discovered before v’

  $\Rightarrow$ u’ entered Q before v’

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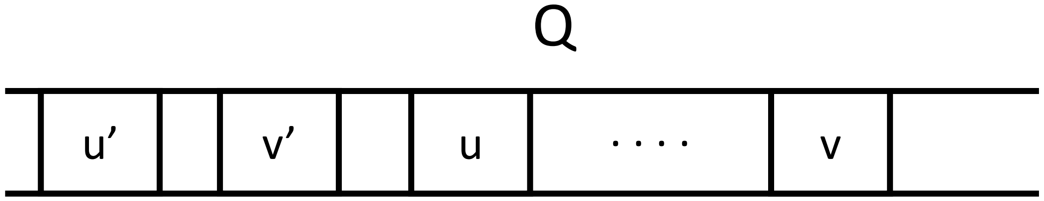

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  $\Rightarrow$ u and v entered Q during the exploration of some nodes, say u’ and v’ respectively

  $\Rightarrow$ $d[u] = d[u'] + 1$ and $d[v] = d[v'] + 1$

- Since $d[u] \neq d[v]$, $d[u'] \neq d[v']$ $\Rightarrow$ $u' \neq v'$ i.e. u’ and v’ are distinct nodes
- u was discovered before v $\Rightarrow$ u’ was discovered before v’

  $\Rightarrow$ u’ entered Q before v’

  $\Rightarrow$ $d[u']$ $d[v']$

Proof of Lemma 1 contd:

Q

u’		v’		u	. . . .	v	

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- u was discovered before v $\Rightarrow$ u’ was discovered before v’

  $\Rightarrow$ u’ entered Q before v’

  $\Rightarrow$ $d[u'] \leq d[v']$ By definition of v

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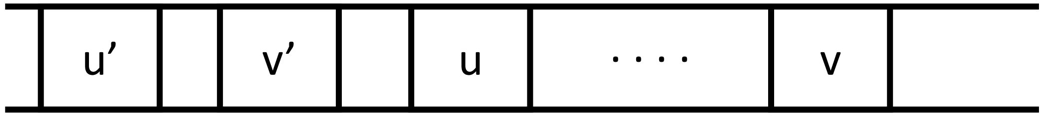

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- u was discovered before v $\Rightarrow$ u’ was discovered before v’

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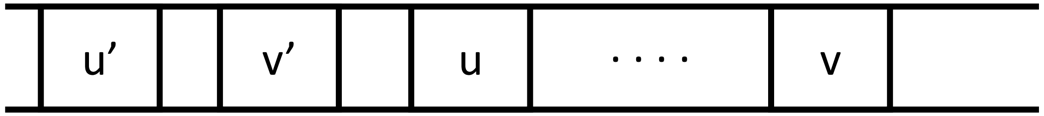

d[u] > d[v]

- v ≠ s because no vertex u enters Q before s
- u ≠ s because d[s] = 0 and d[v] ≥ 0

⇒ u and v entered Q during the exploration of some nodes, say u’ and v’ respectively

⇒ d[u] = d[u’] + 1 and d[v] = d[v’] + 1

- Since d[u] ≠ d[v], d[u’] ≠ d[v’] ⇒ u’ ≠ v’ i.e. u’ and v’ are distinct nodes
- u was discovered before v ⇒ u’ was discovered before v’

⇒ u’ entered Q before v’

⇒ d[u’] ≤ d[v’] By definition of v

⇒ d[u’] + 1 ≤ d[v’] + 1

⇒ d[u] ≤ d[v]

Proof of Lemma 1 contd:

Q

u’		v’		u	· · · ·	v	

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- u was discovered before v $\Rightarrow$ u’ was discovered before v’

$\Rightarrow$ u’ entered Q before v’

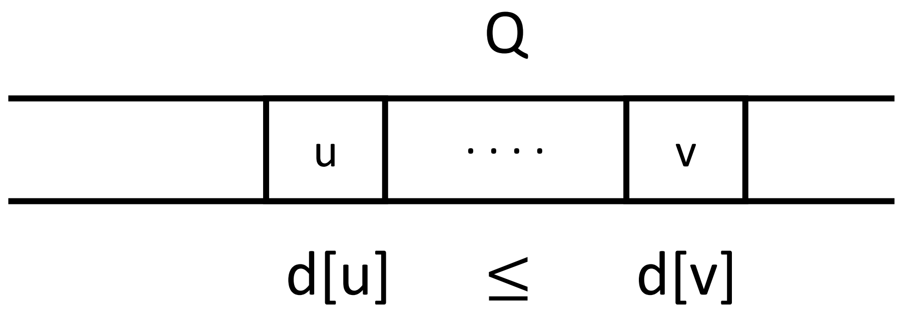
$\Rightarrow d[u'] \leq d[v']$ By definition of v

$\Rightarrow d[u'] + 1 \leq d[v'] + 1$

$\Rightarrow d[u] \leq d[v]$ Contradiction !

So we proved the following:

Lemma 1: If u enters Q **before** v enters Q then d[u] ≤ d[v]



Main Theorem:

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Suppose, for contradiction, that there exists x ∈ V such that $d[x] \neq \delta(s,x)$.

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Proof of Main Theorem:

Suppose, for contradiction, that there exists x ∈ V such that $d[x] \neq \delta(s,x)$. Clearly $x \neq s$.

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Proof of Main Theorem:

Suppose, for contradiction, that there exists $x \in V$ such that $d[x] \neq \delta(s,x)$. Clearly $x \neq s$.

- Let v be the **closest** node from s such that $d[v] \neq \delta(s,v)$

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Proof of Main Theorem:

Suppose, for contradiction, that there exists x ∈ V such that $d[x] \neq \delta(s,x)$. Clearly $x \neq s$.

- Let v be the **closest** node from s such that $d[v] \neq \delta(s,v)$
- By Lemma 0, d[v] $\delta(s,v)$

Main Theorem:

After BFS(s), for every v ∈ V, d[v] = δ(s,v)

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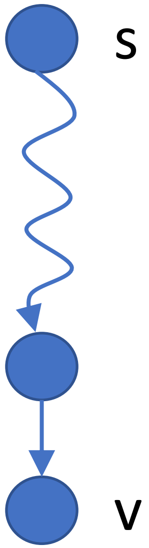

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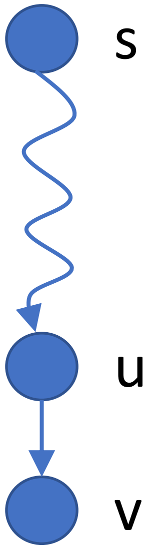

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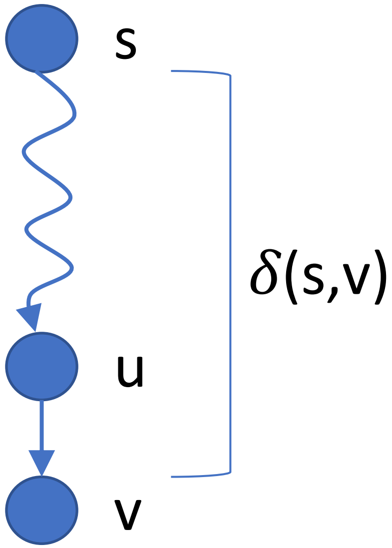

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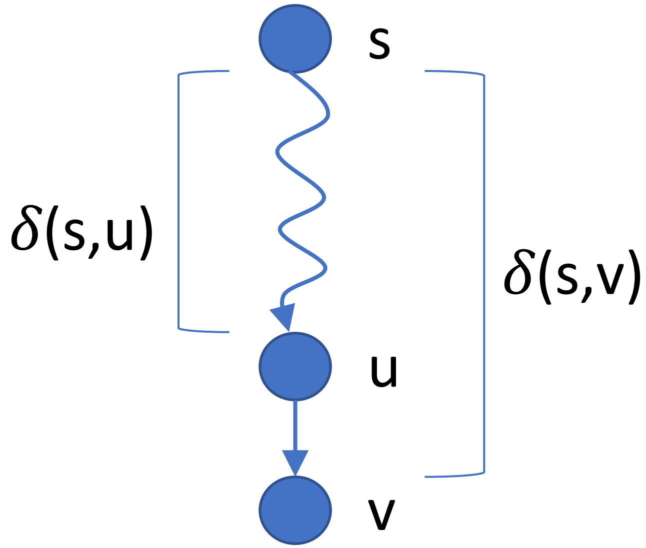

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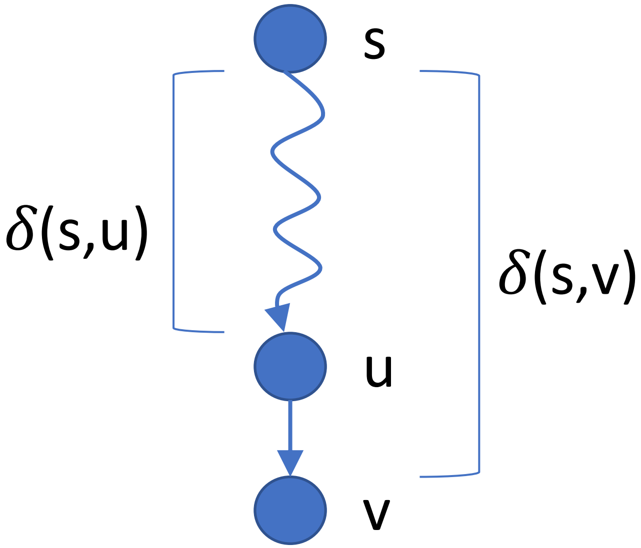

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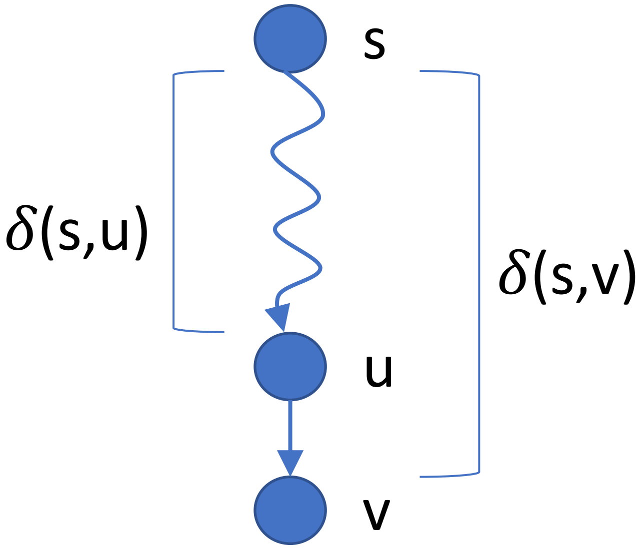

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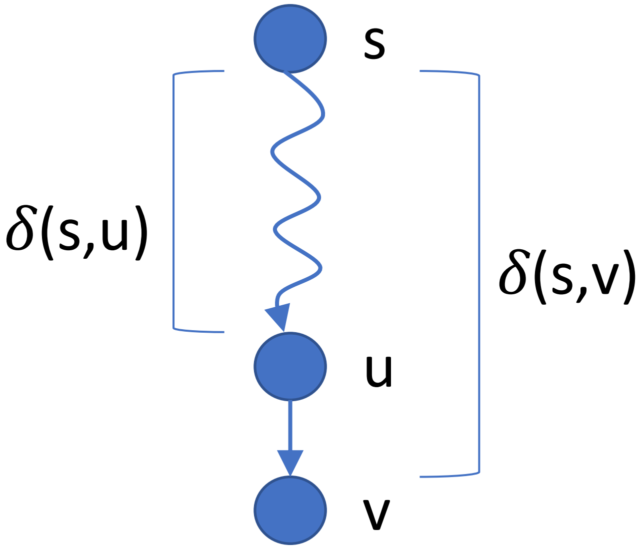

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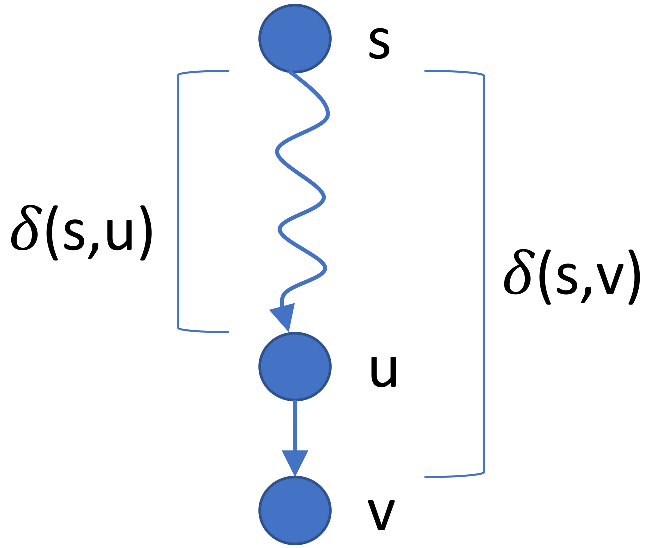

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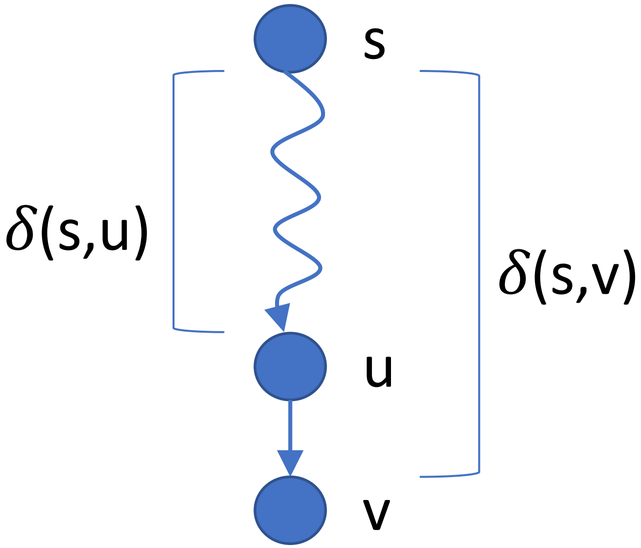

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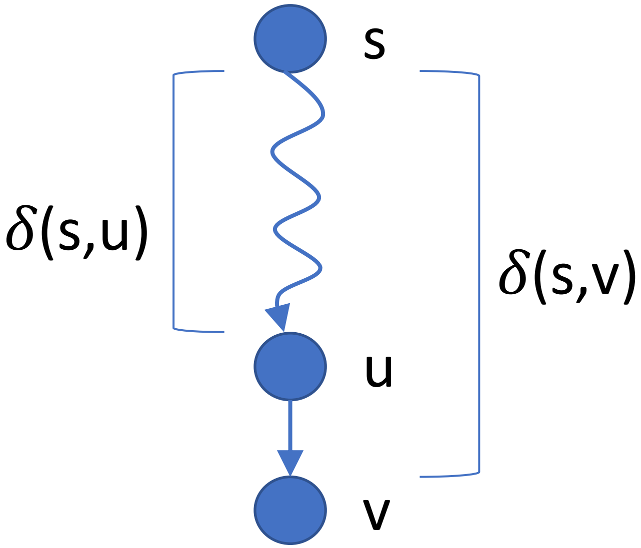

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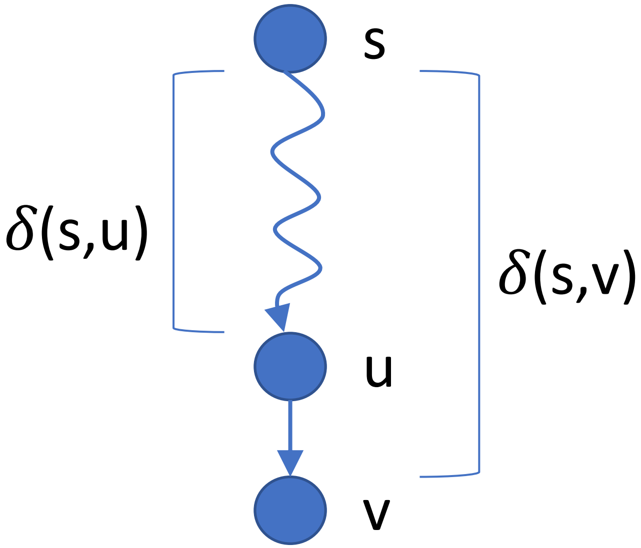

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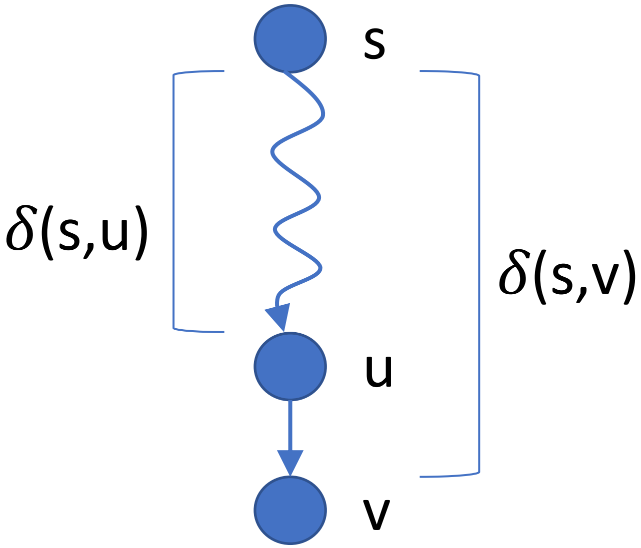

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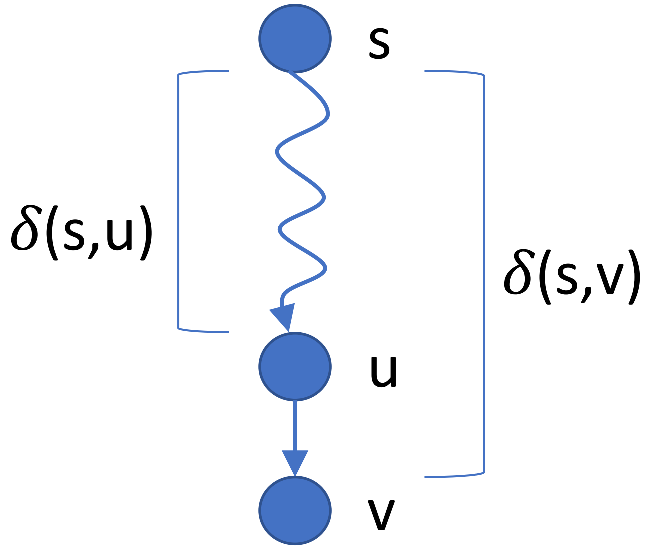

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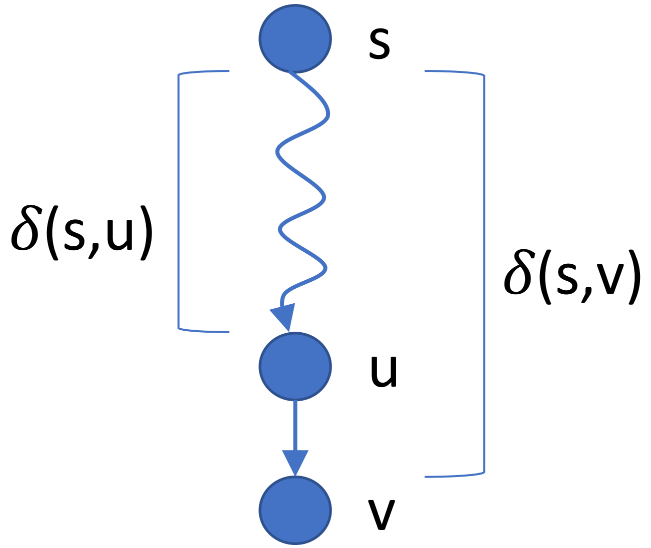

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Case 2. v is black

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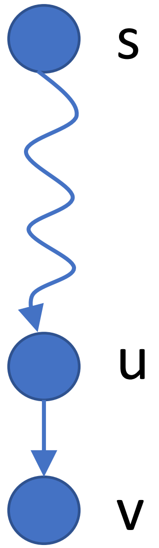

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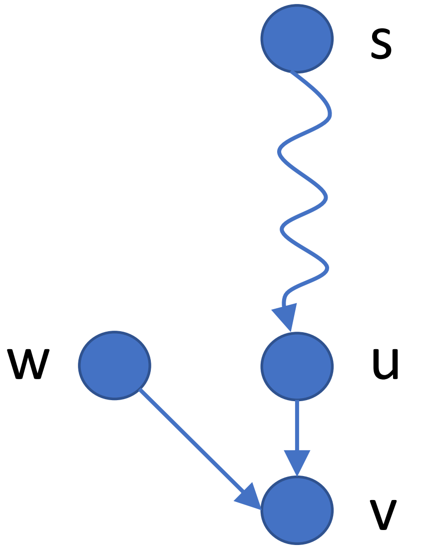
⇒ Some node w discovered v before u is explored



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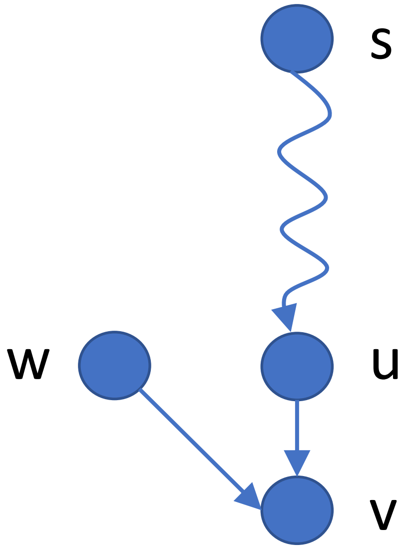

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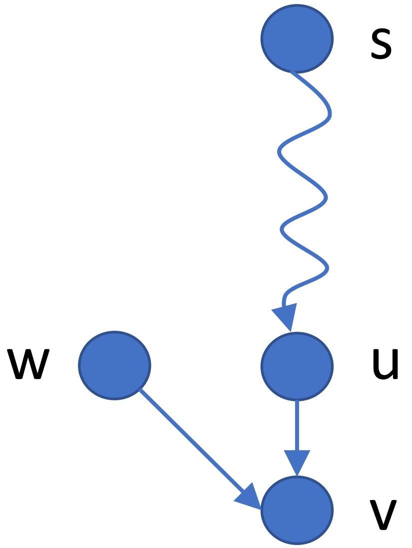
⇒ w is explored before u and d[v] = d[w] + 1



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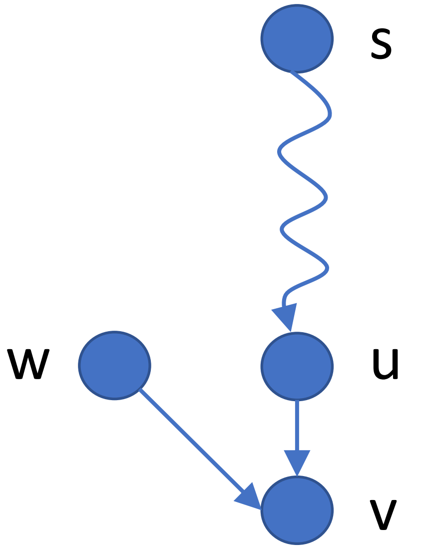

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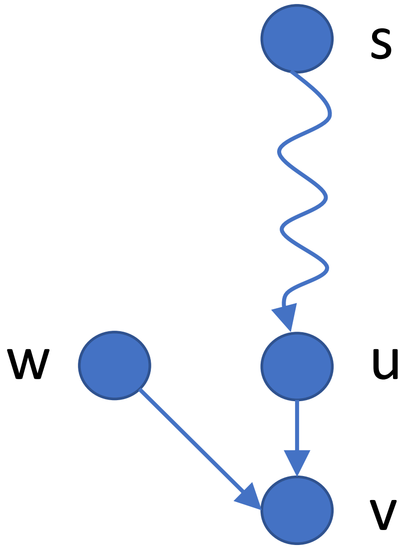

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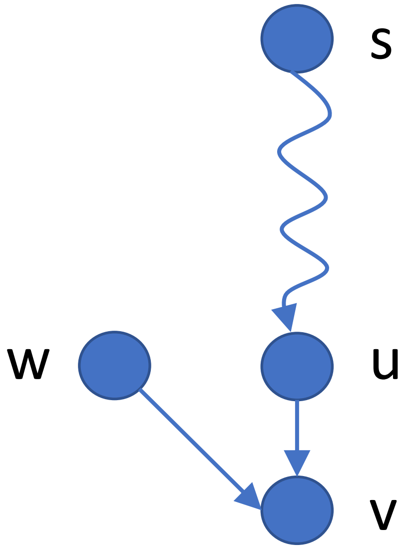
⇒ d[w]   d[u]

By Lemma 1



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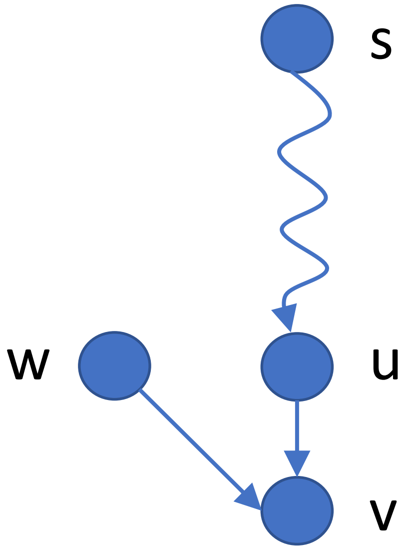

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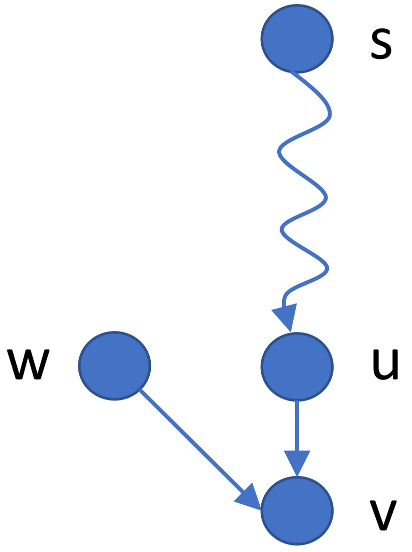

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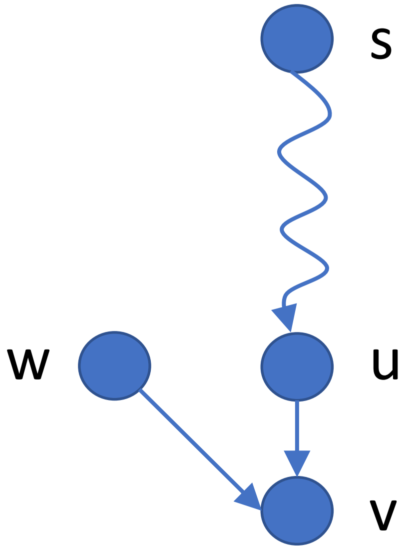

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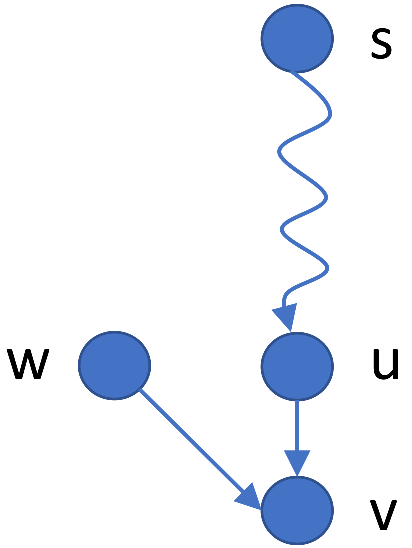

We just proved

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So the BFS(s) discovery path from s to v is a shortest path from s to v in G