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Operations:

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Find(z): Given (a ptr to) z, find set S that contains z and return the representative of S

Example

S_j : Set represented by j

	S_1	S_2	S_3	S_4	S_5
Initially :	{1}	{2}	{3}	{4}	{5}

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Union(S_1, S_5) :	{1, 5}	{2}	{3, 4}	X	X

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Union(S_1, S_3) :	{1, 5, 3, 4}	{2}	X	X	X

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Union(S_1, S_3) :	{1, 5, 3, 4}	{2}	X	X	X
Find(4) = S_1					
Find(2) =					

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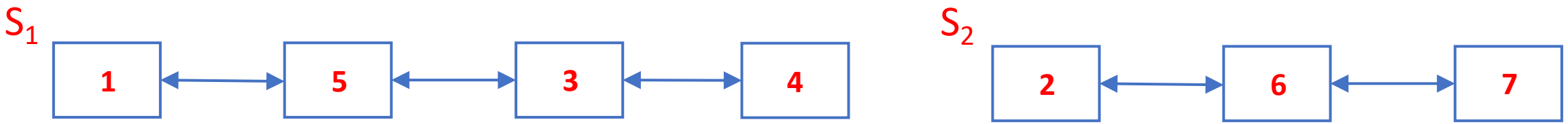
Goal: a data structure that minimizes the total cost of executing such sequences

Data Structures for Disjoint Sets

1. Linked Lists

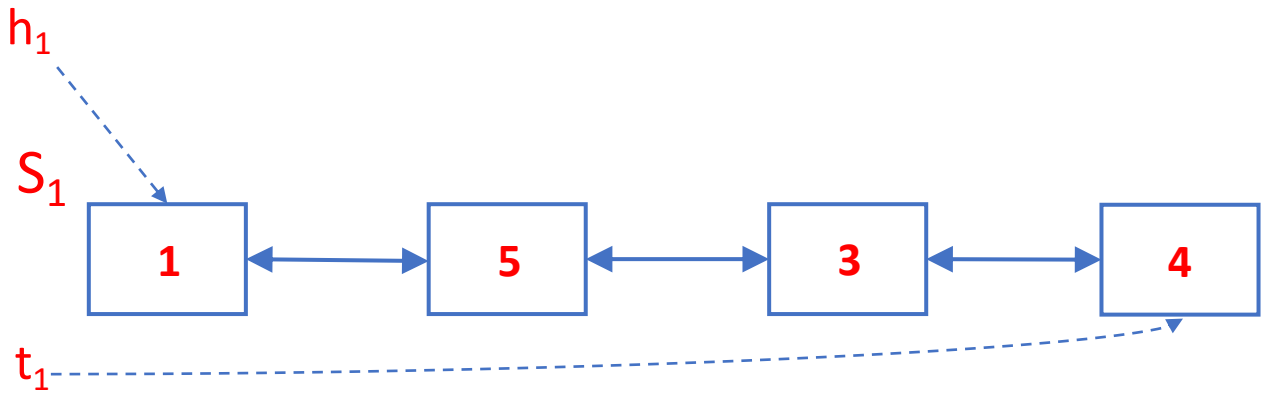
1. Linked Lists

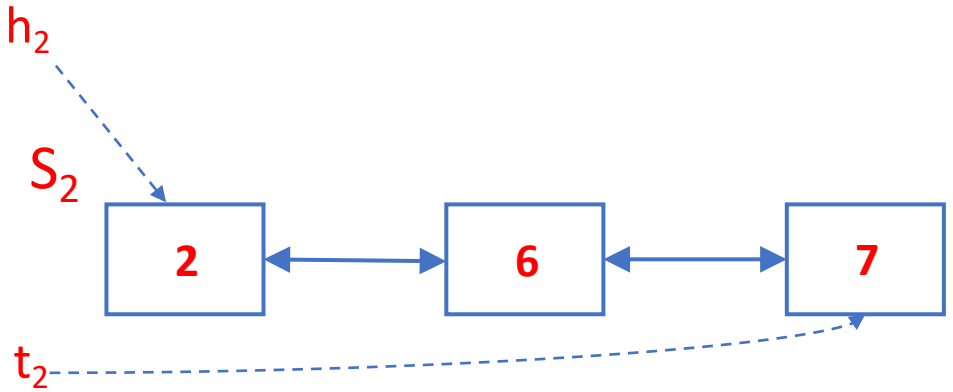
- One list per set



1. Linked Lists

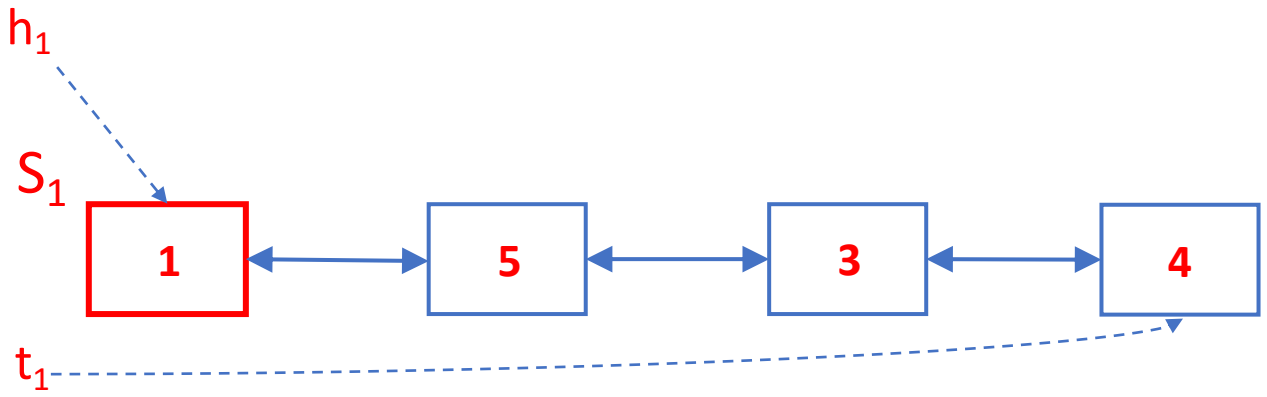
- One list per set
- Store a head pointer and tail pointer for each set

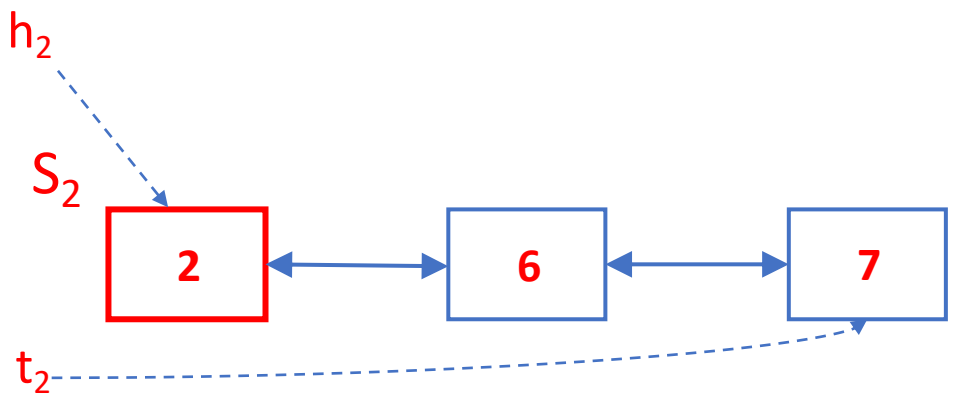




1. Linked Lists

- One list per set
- Store a head pointer and tail pointer for each set
- First element of the set is the representative





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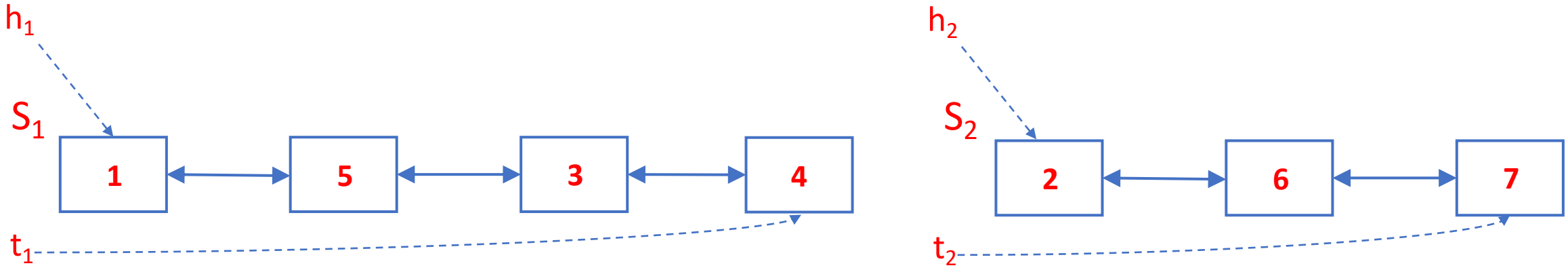

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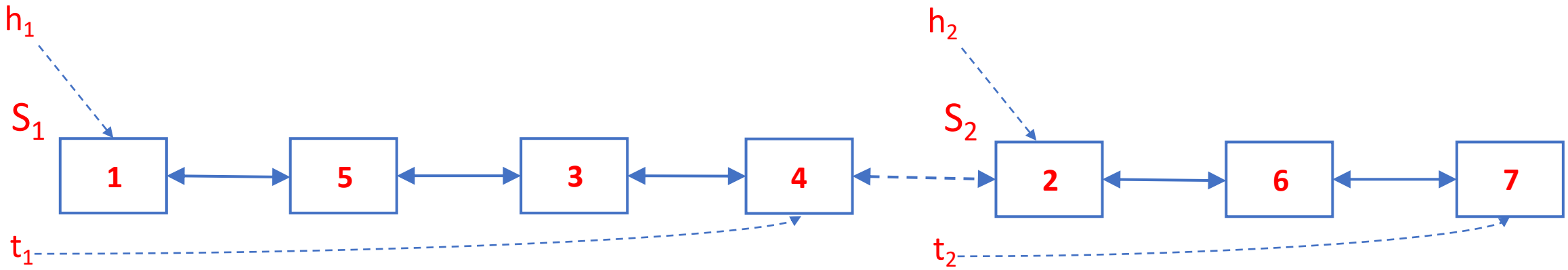

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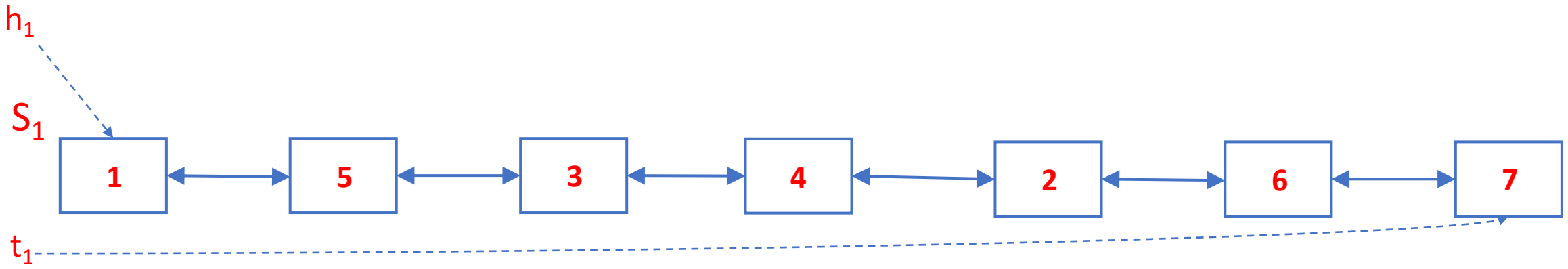

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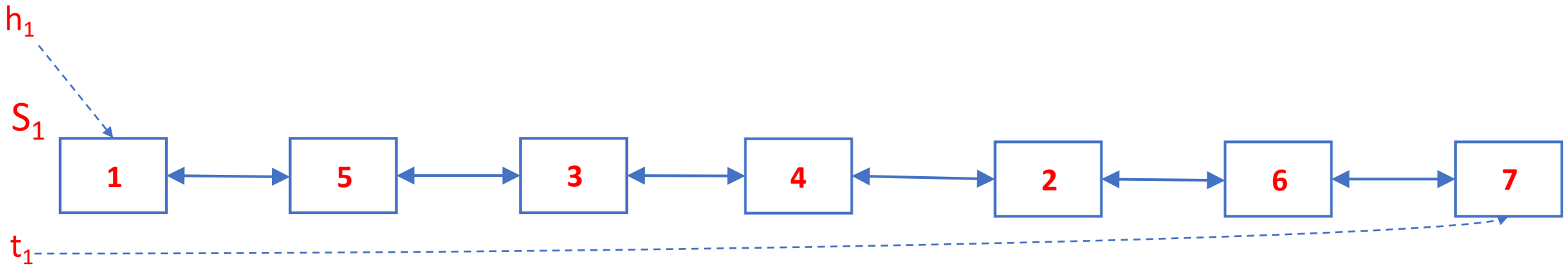

Each **Union** : O(1)

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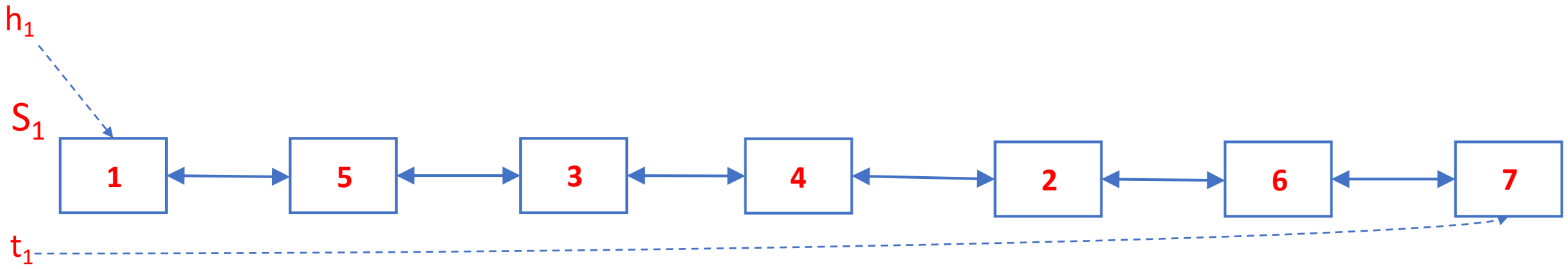

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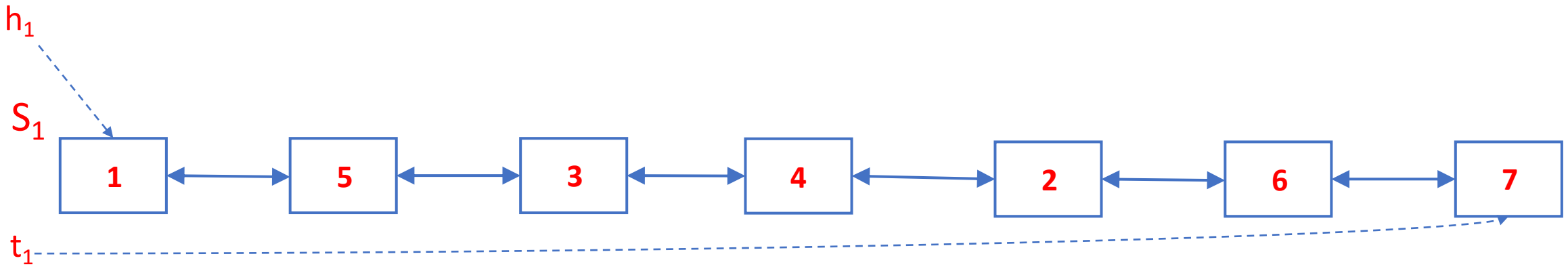

Each **Union** : O(1)

Each **Find** : O(n)

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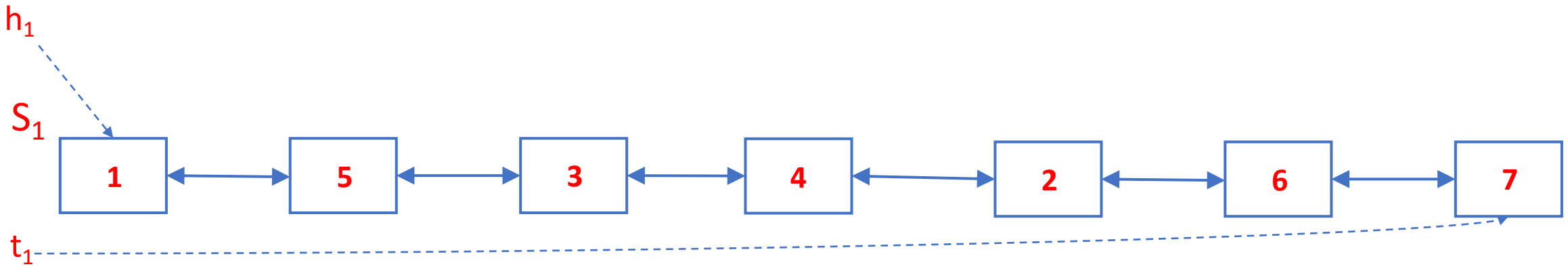

σ : sequence of n – 1 **Unions** mixed with m ≥ n **Finds**

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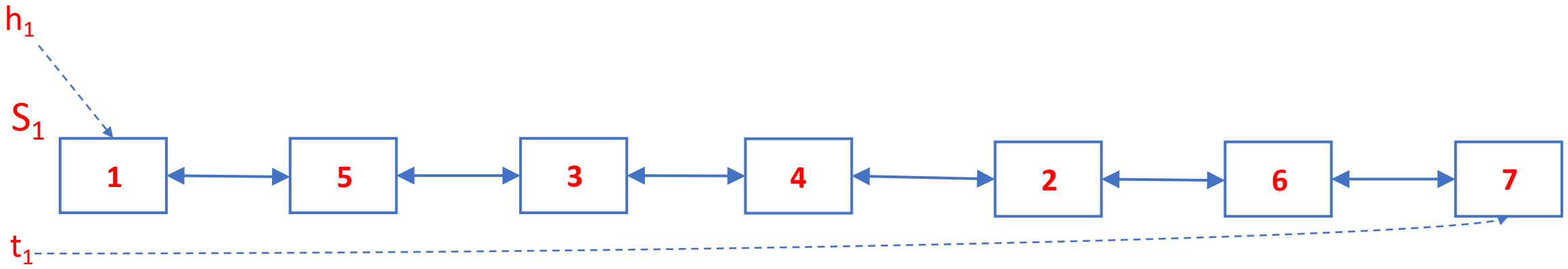

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Worst-case cost of σ :

Each **Union** : O(1)

Each **Find** : O(n)

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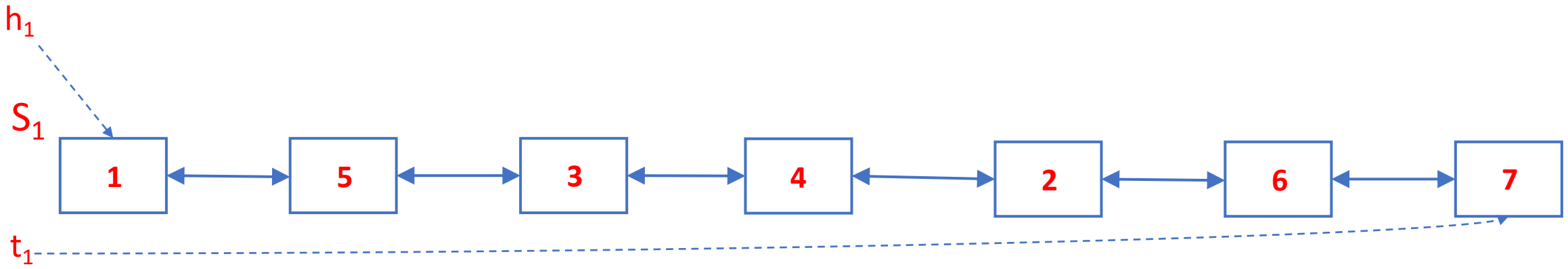

Each **Union** : O(1)

Each **Find** : O(n)

σ : sequence of n – 1 **Unions** mixed with m ≥ n **Finds**

Worst-case cost of σ : O(n. 1

Per Union

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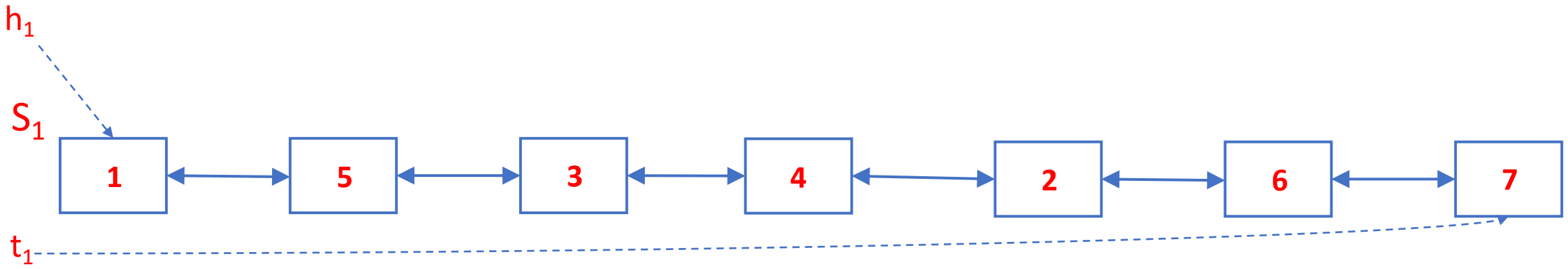

Each **Union** : O(1)

Each **Find** : O(n)

σ : sequence of n – 1 **Unions** mixed with m ≥ n **Finds**

Worst-case cost of σ : O(n. 1 + m. n)

Per **Union** Per **Find**

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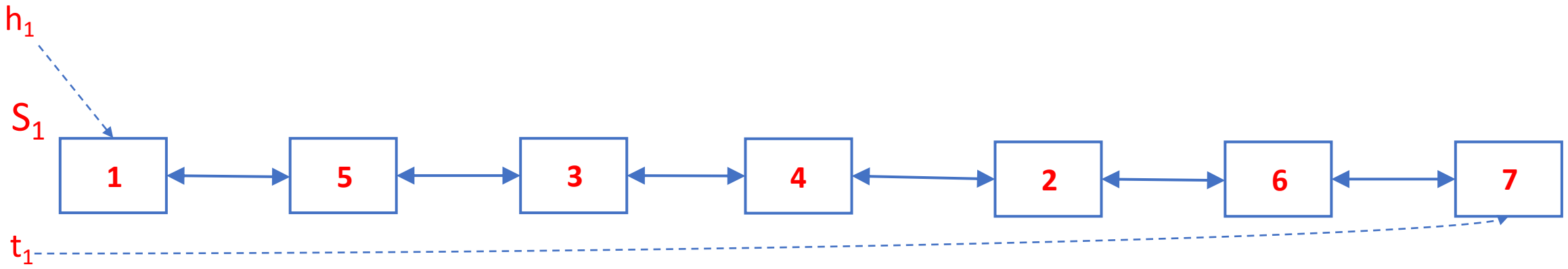


Each **Union** : O(1)

Each **Find** : O(n)

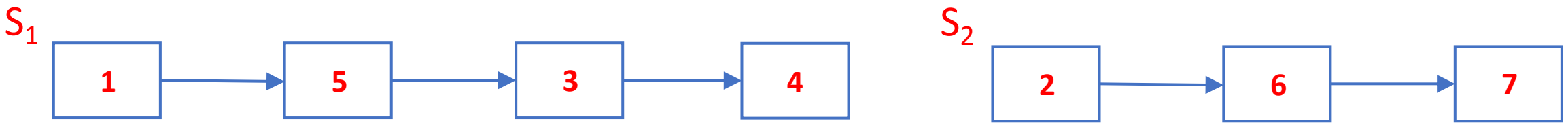
σ : sequence of n – 1 **Unions** mixed with m ≥ n **Finds**

Worst-case cost of σ : O(n. 1 + m. n) = O(m. n)

Per **Union** Per **Find**

2. Augmented Linked Lists

- Each element also points to its representative



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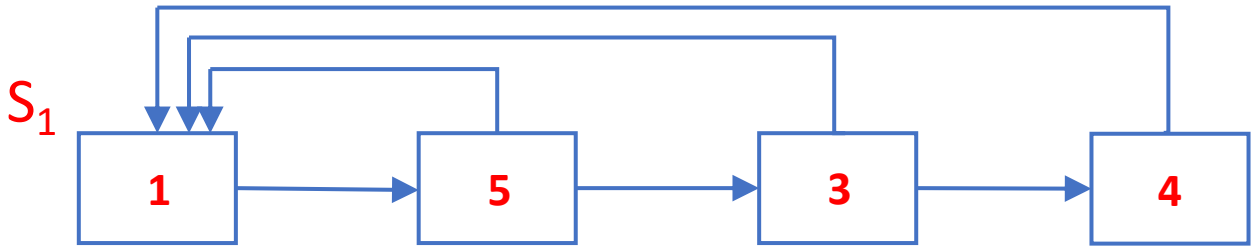

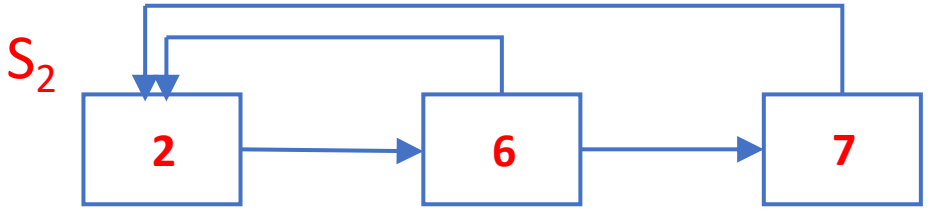

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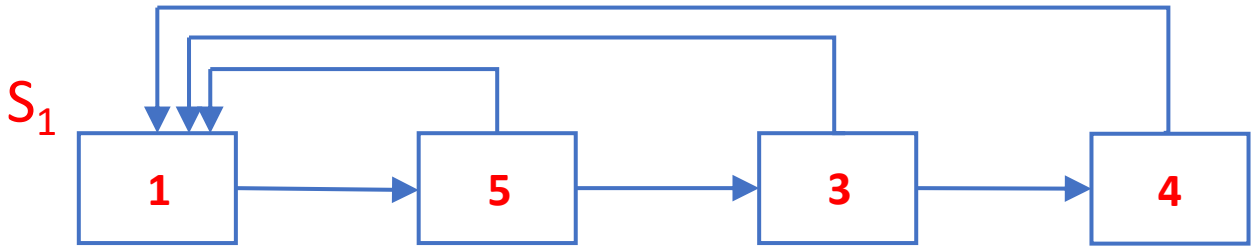

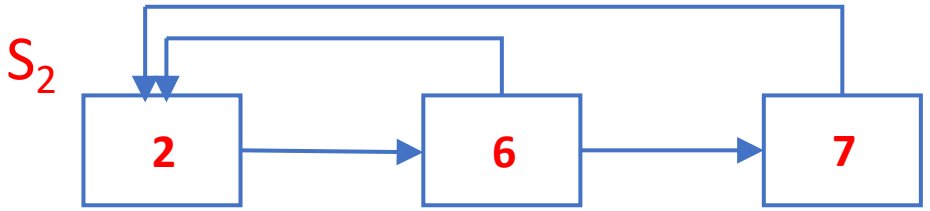

Each **Find** : O(1)

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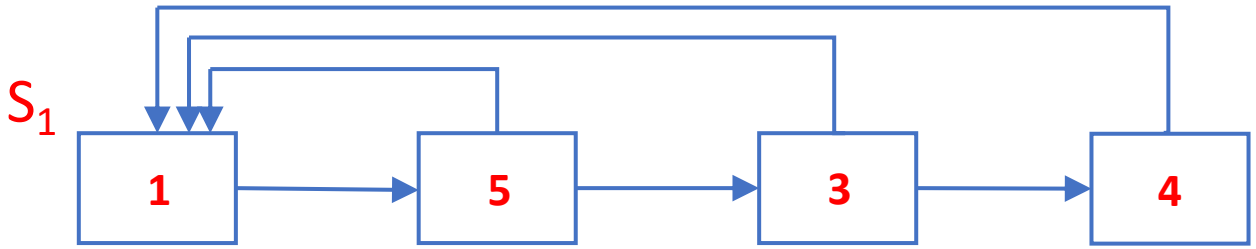

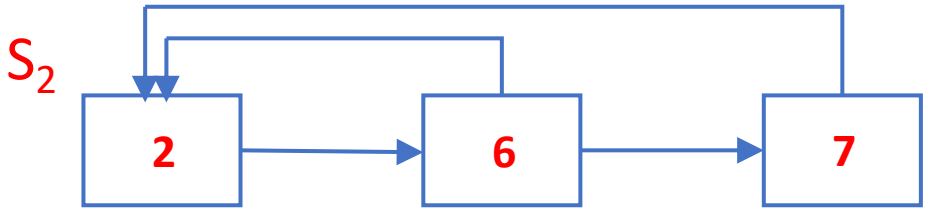

Each **Find** : O(1)

Union

2. Augmented Linked Lists

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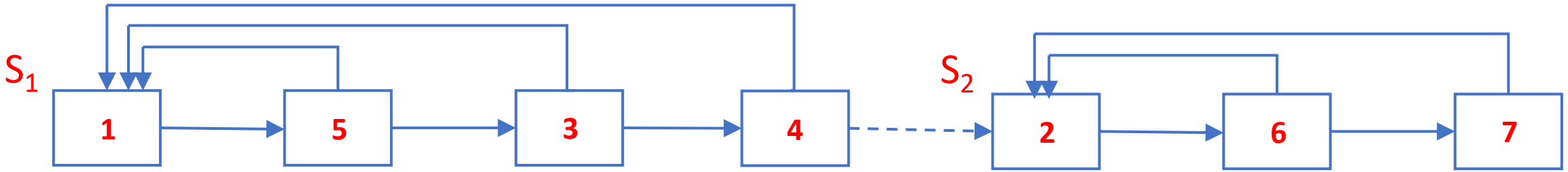

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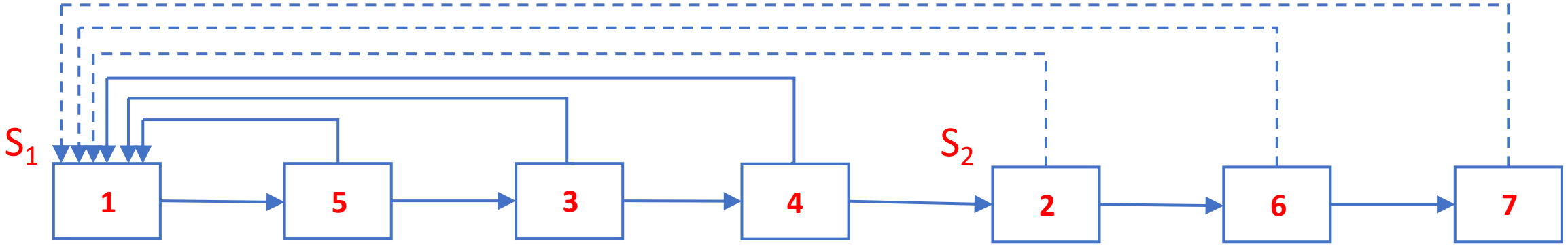

Each **Find** : O(1)

Union : must redirect O(n) pointers to the new representative

2. Augmented Linked Lists

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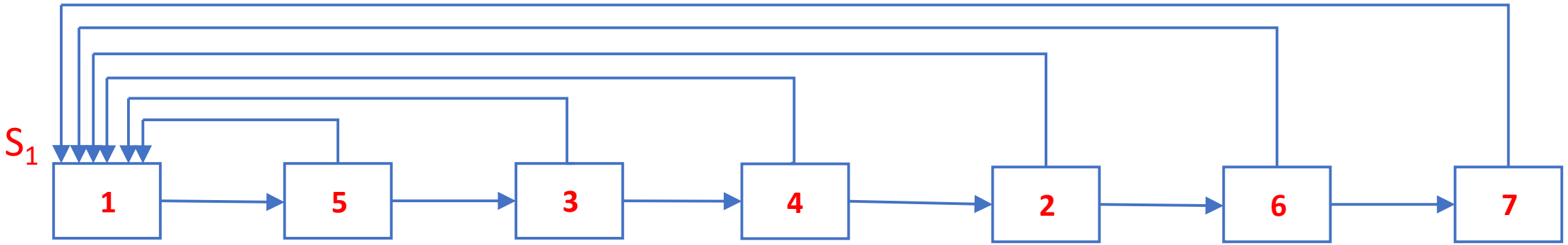

Each **Find** : O(1)

Each **Union** : O(n)

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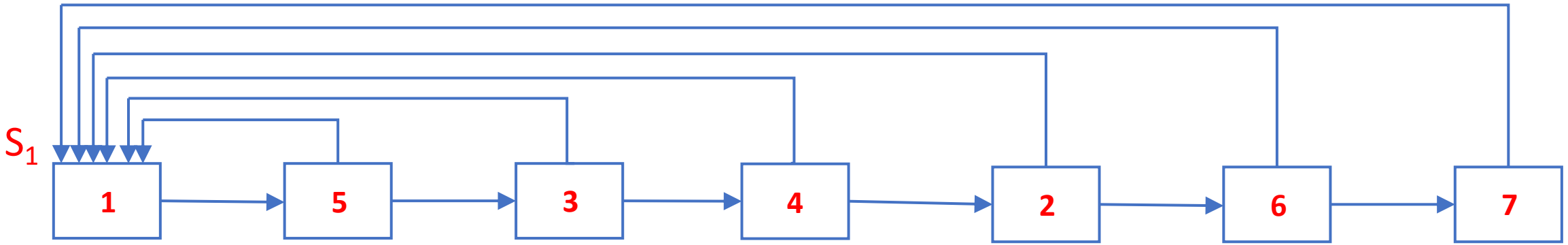


Each **Find** : O(1)

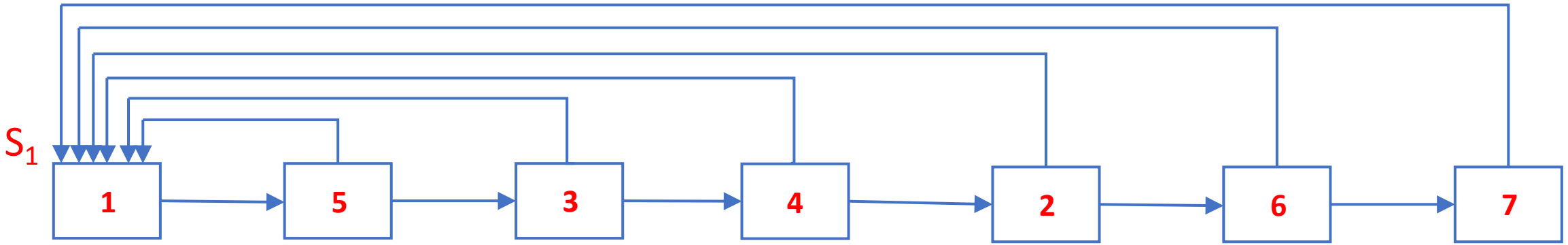
Each **Union** : O(n)

Worst-case cost of σ : O(n . n + m . 1)

Per **Union** Per **Find**

2. Augmented Linked Lists

- Each element also points to its representative



Each **Find** : O(1)

Each **Union** : O(n)

Worst-case cost of σ : $O(n \cdot n + m \cdot 1) = O(n^2 + m)$

Per **Union** Per **Find**

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m Finds **n-1 Unions**

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Proof :

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Proof : Go to the tutorial ☺

3. Forest structure for Union-Find

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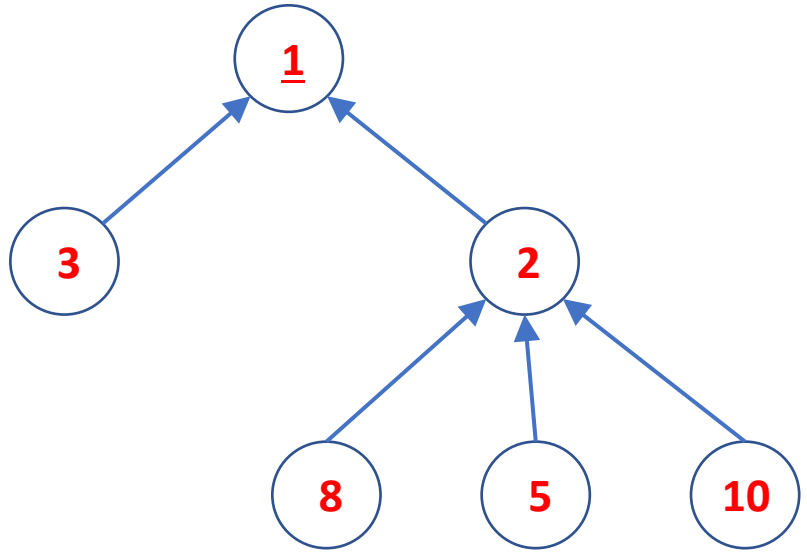
- Each set is represented by a tree

$S_1 = \{\underline{1}, 3, 2, 8, 5, 10\}$



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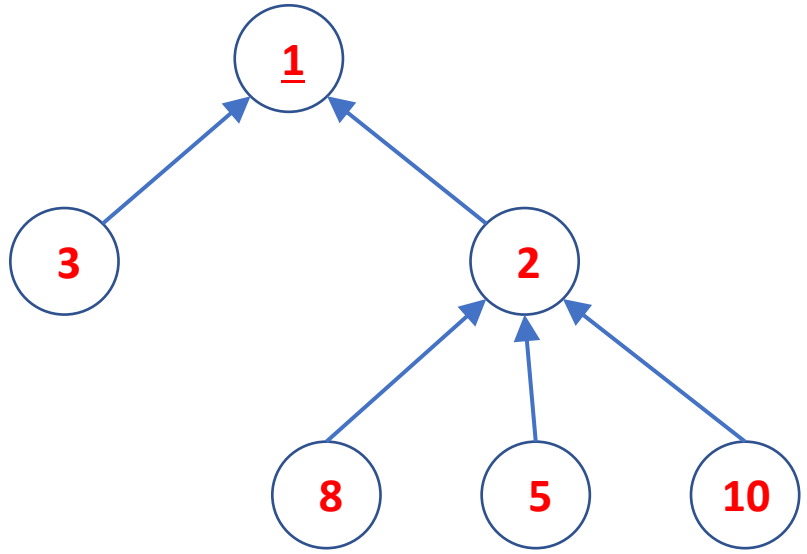

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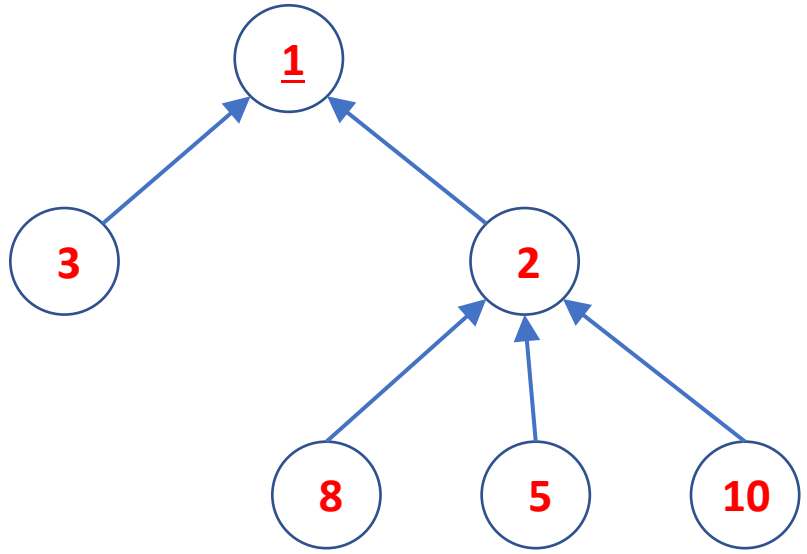

3. Forest structure for Union-Find

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- Each non-root node points to its parent
- The root contains the set representative

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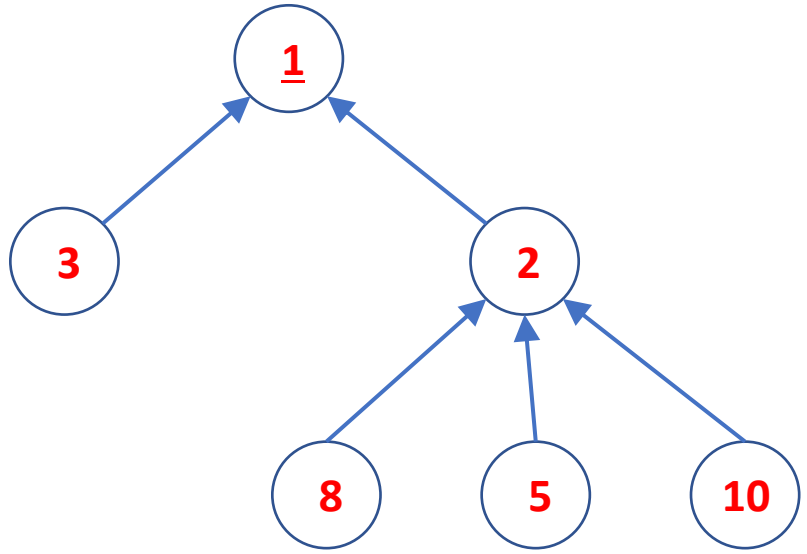


Example with n = 10

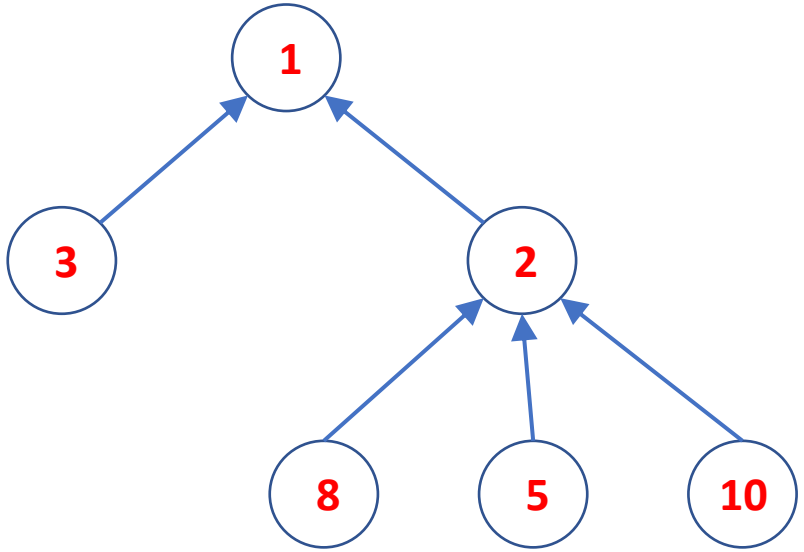
S = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

Example with $n = 10$ $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$S_1 = \{1, 3, 2, 8, 5, 10\}$ $S_6 = \{6\}$ $S_9 = \{9, 7, 4\}$

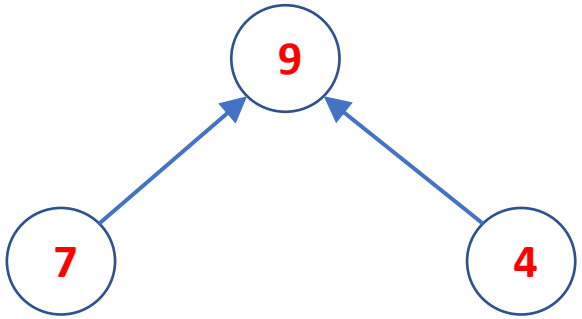
Example with $n = 10$ $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$S_1 = \{1, 3, 2, 8, 5, 10\}$



$S_6 = \{6\}$

$S_9 = \{9, 7, 4\}$



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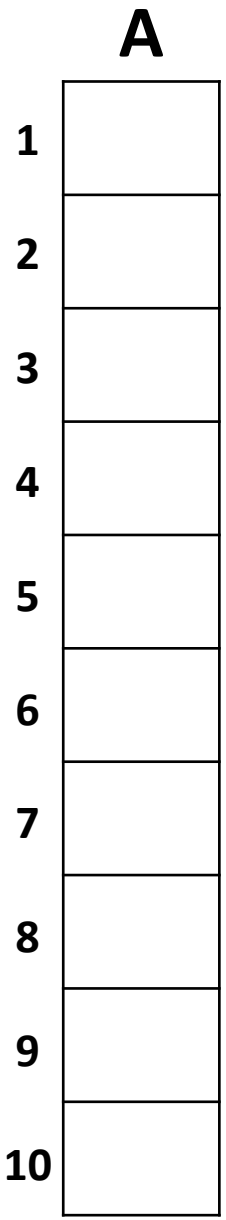

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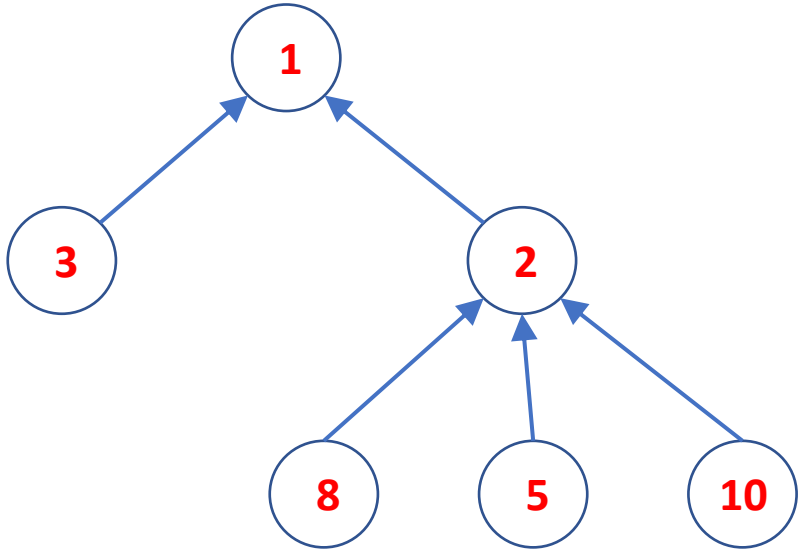

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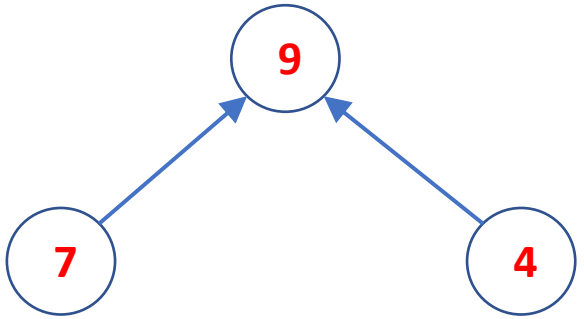
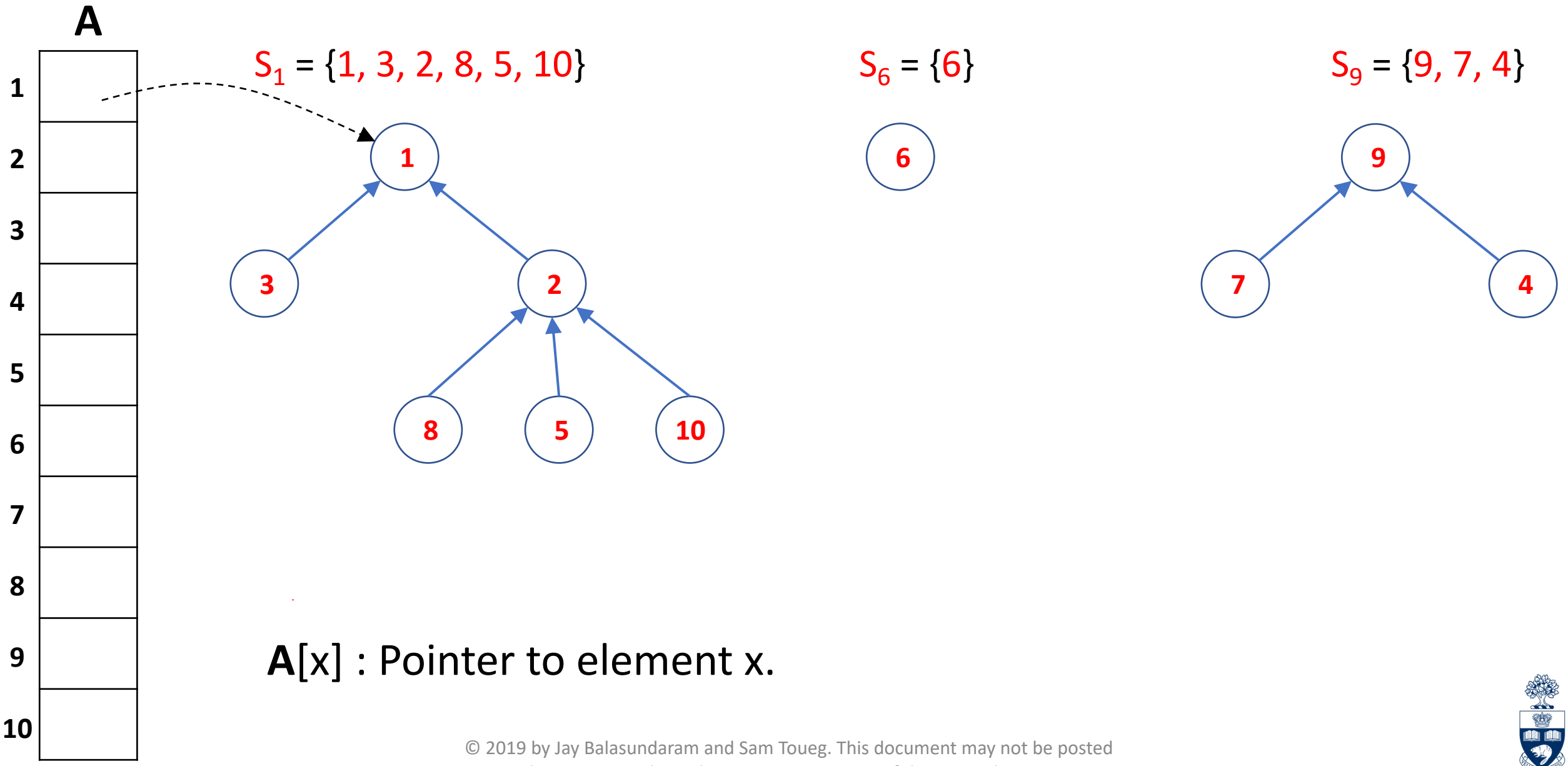


A[x] : Pointer to element x.

Example with n = 10



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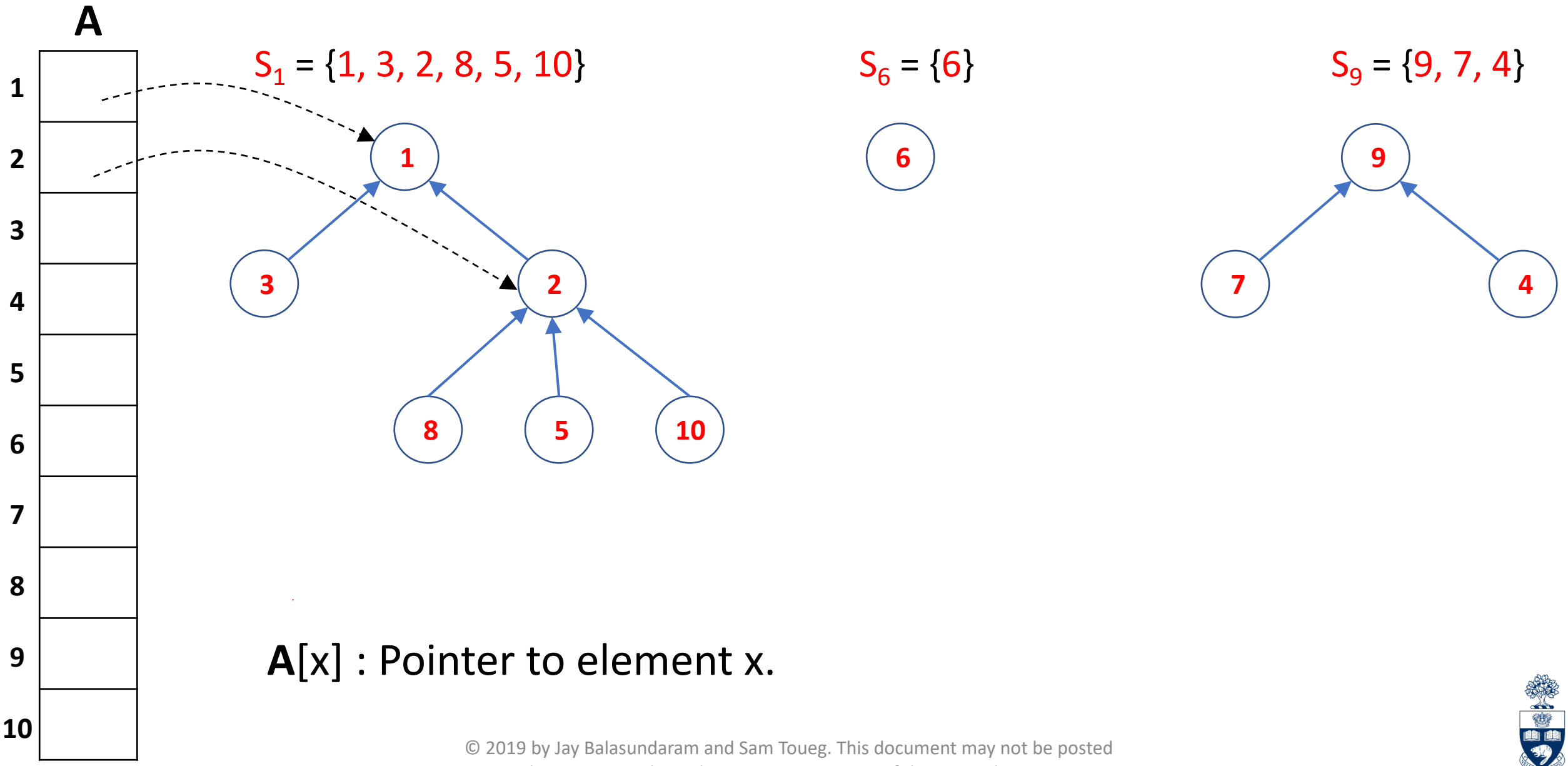

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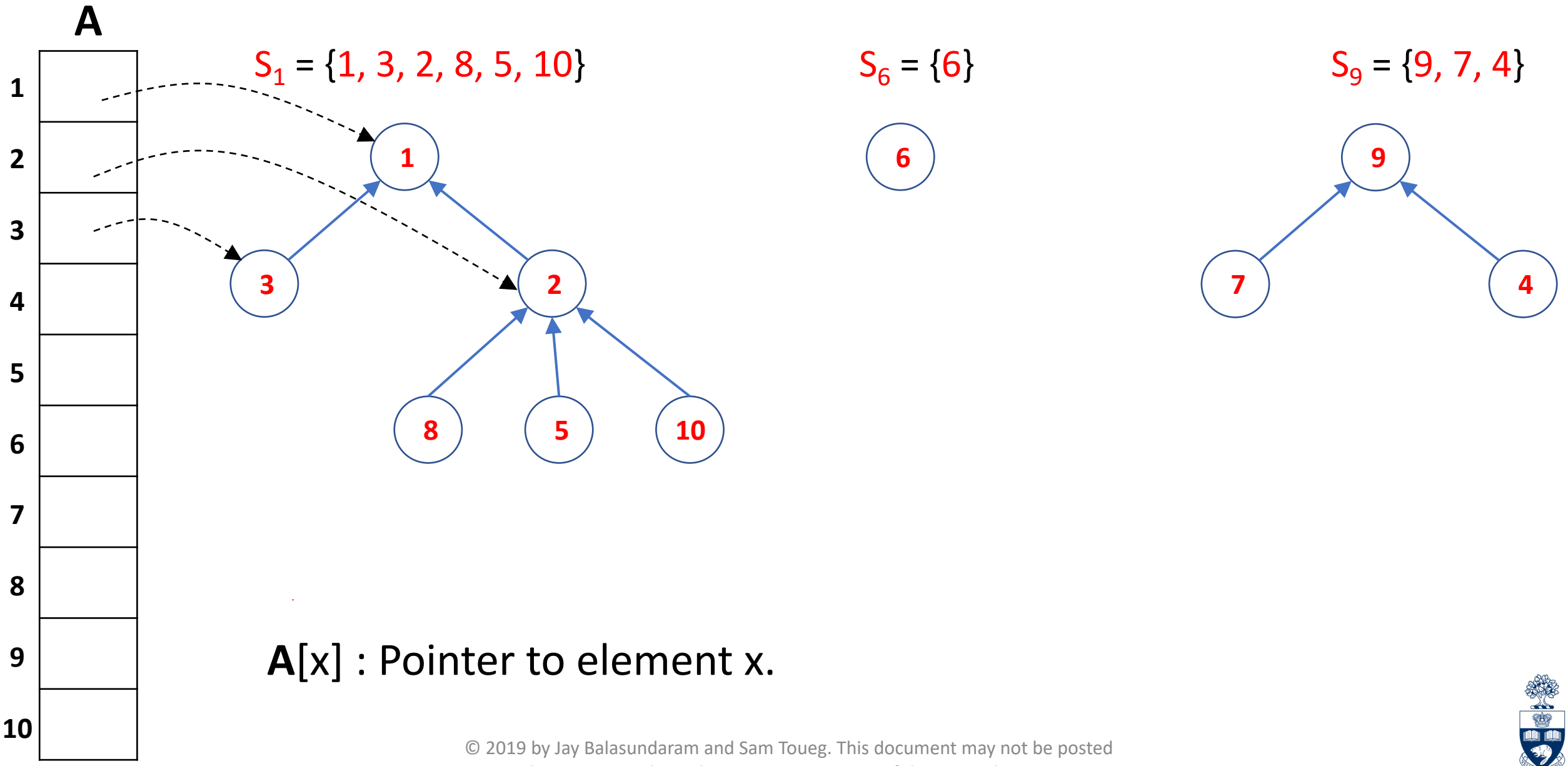

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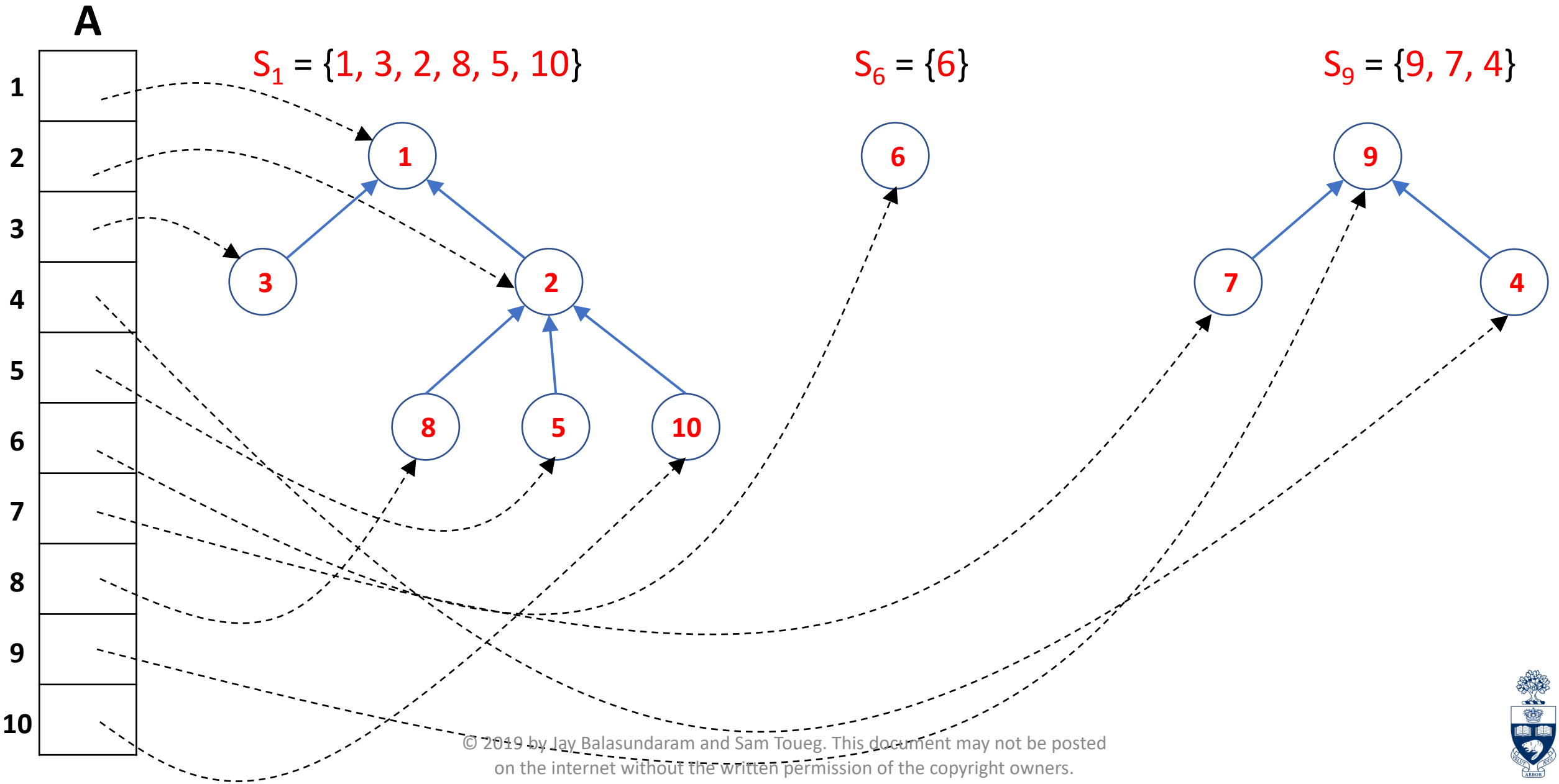
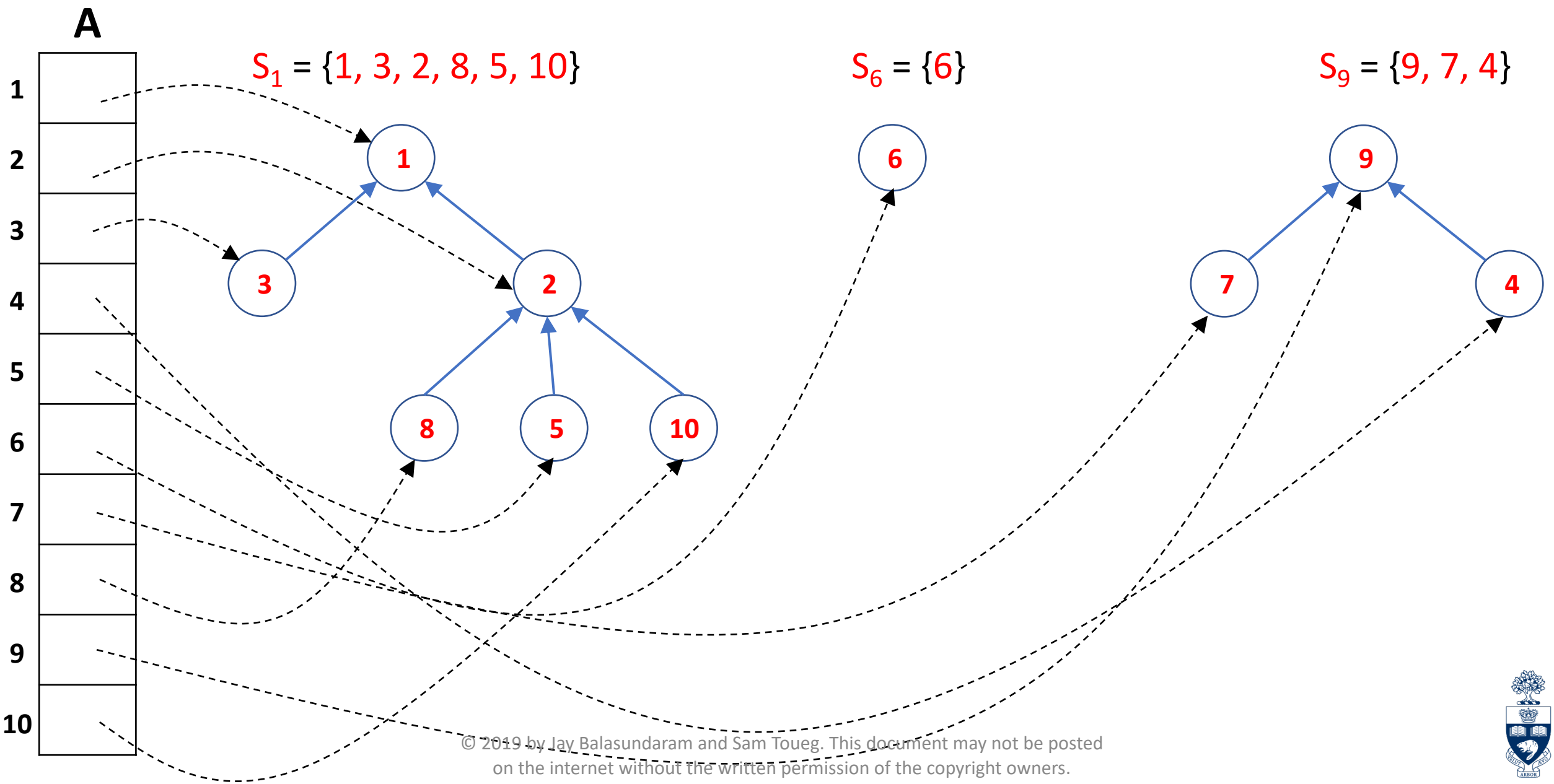
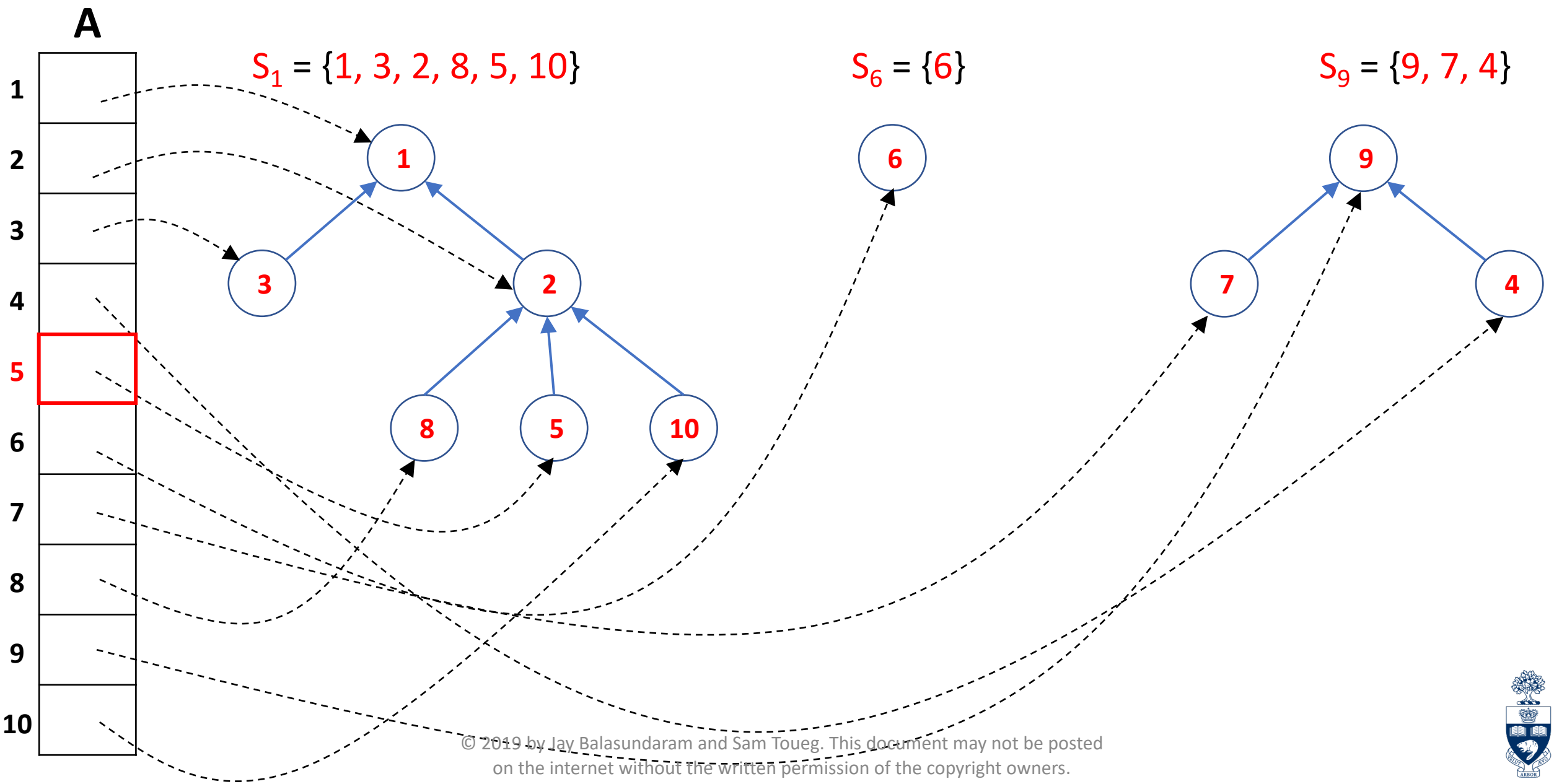


Find(5)



Find(5)



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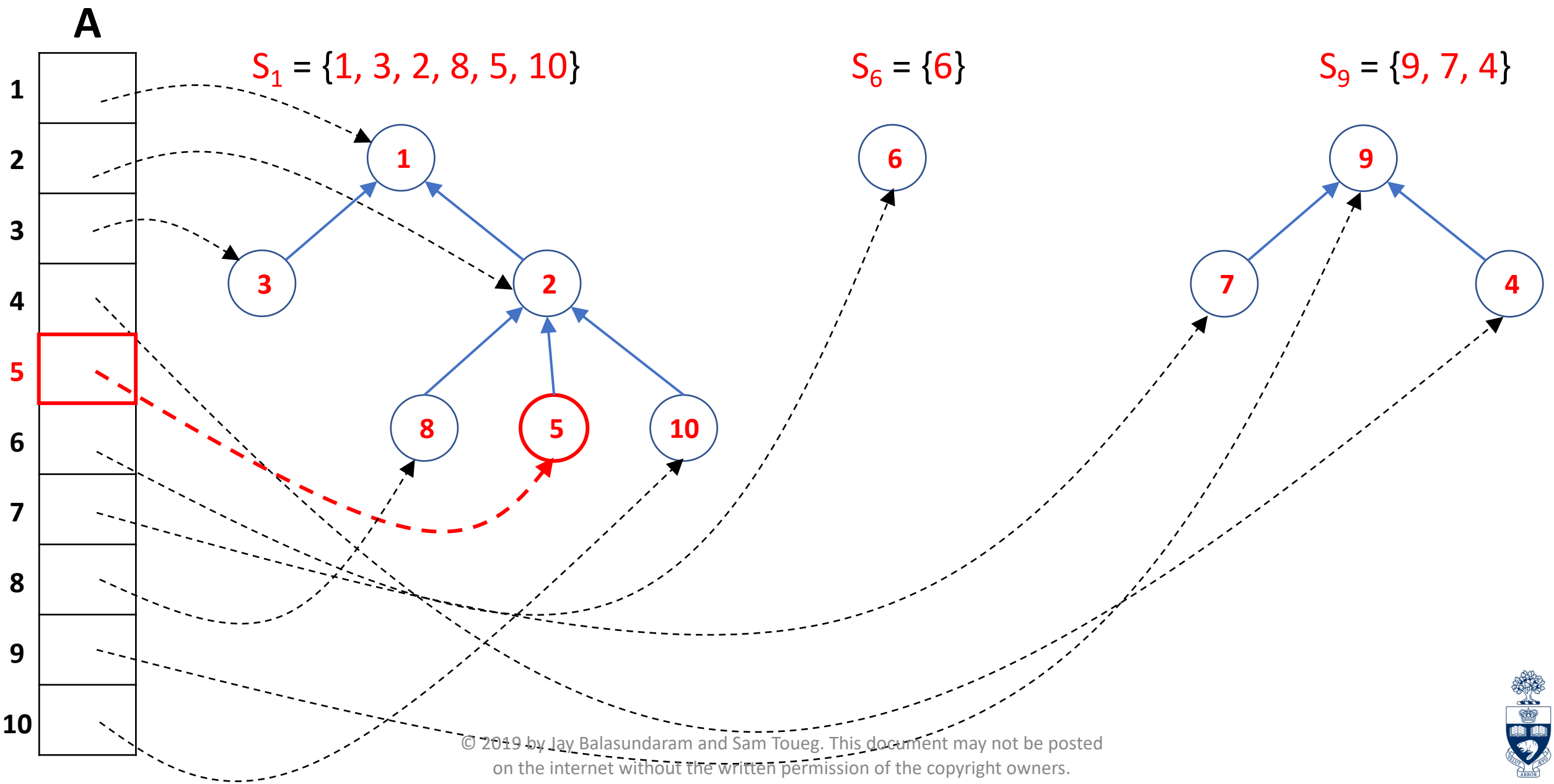

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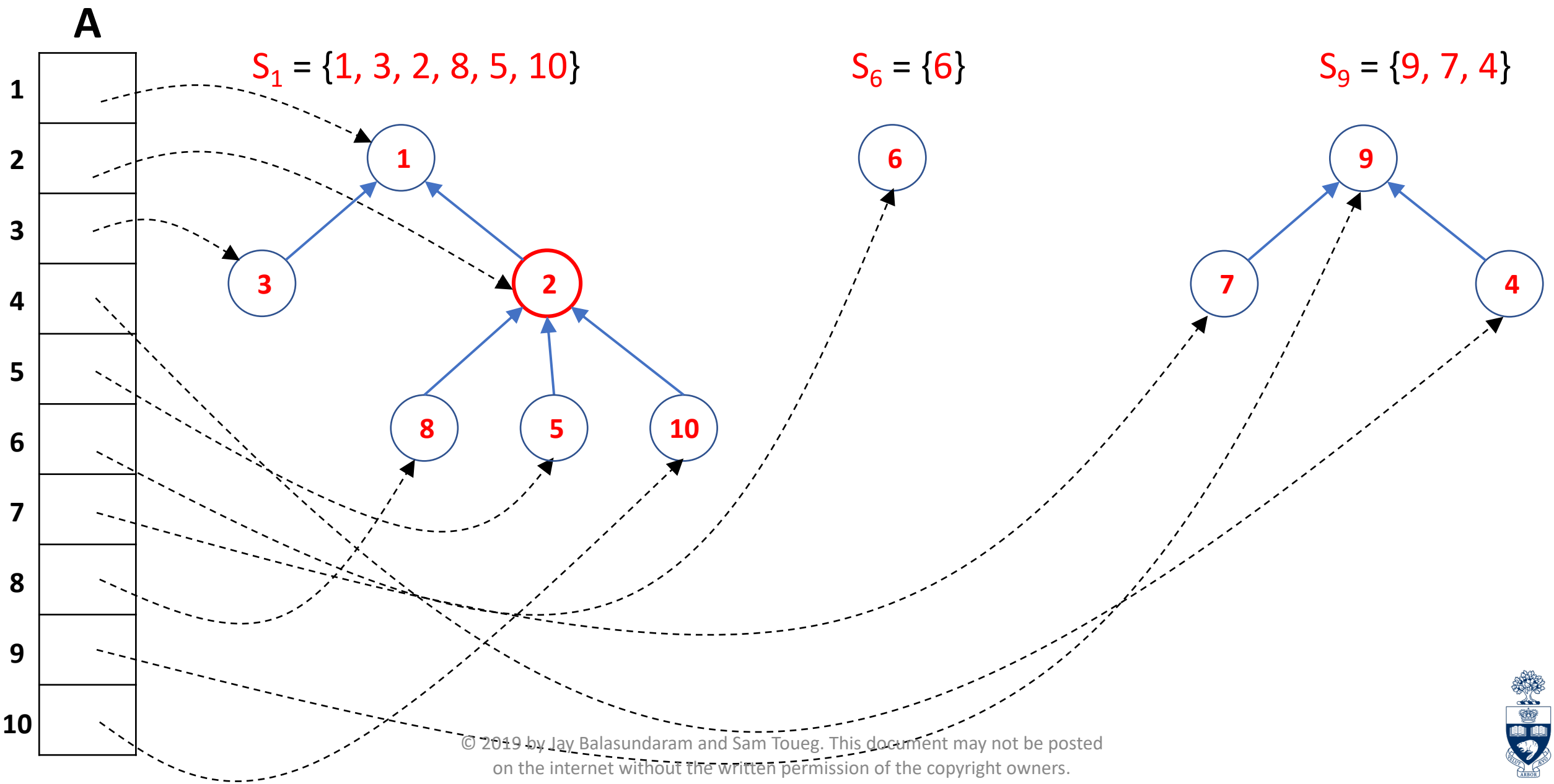

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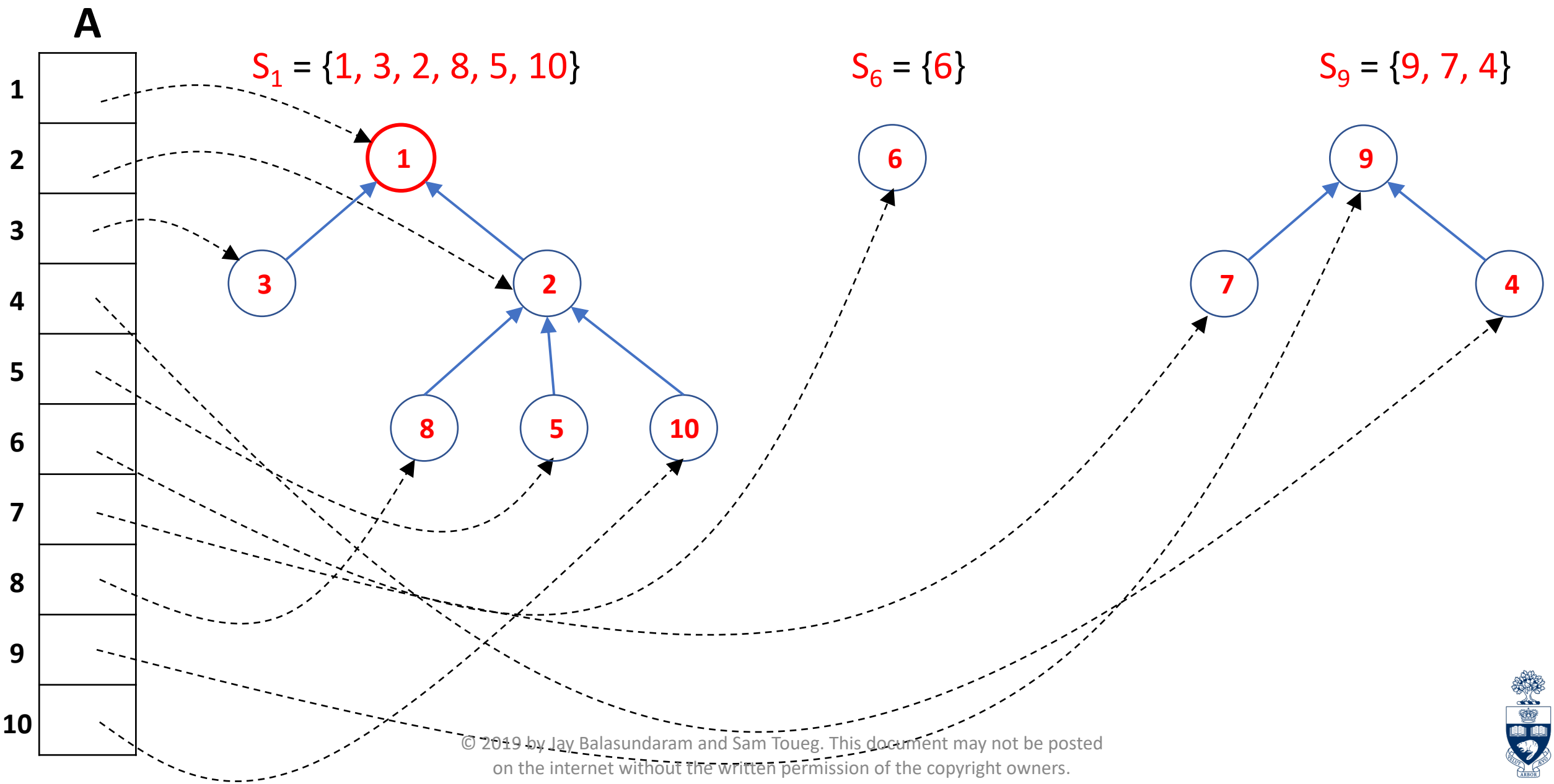
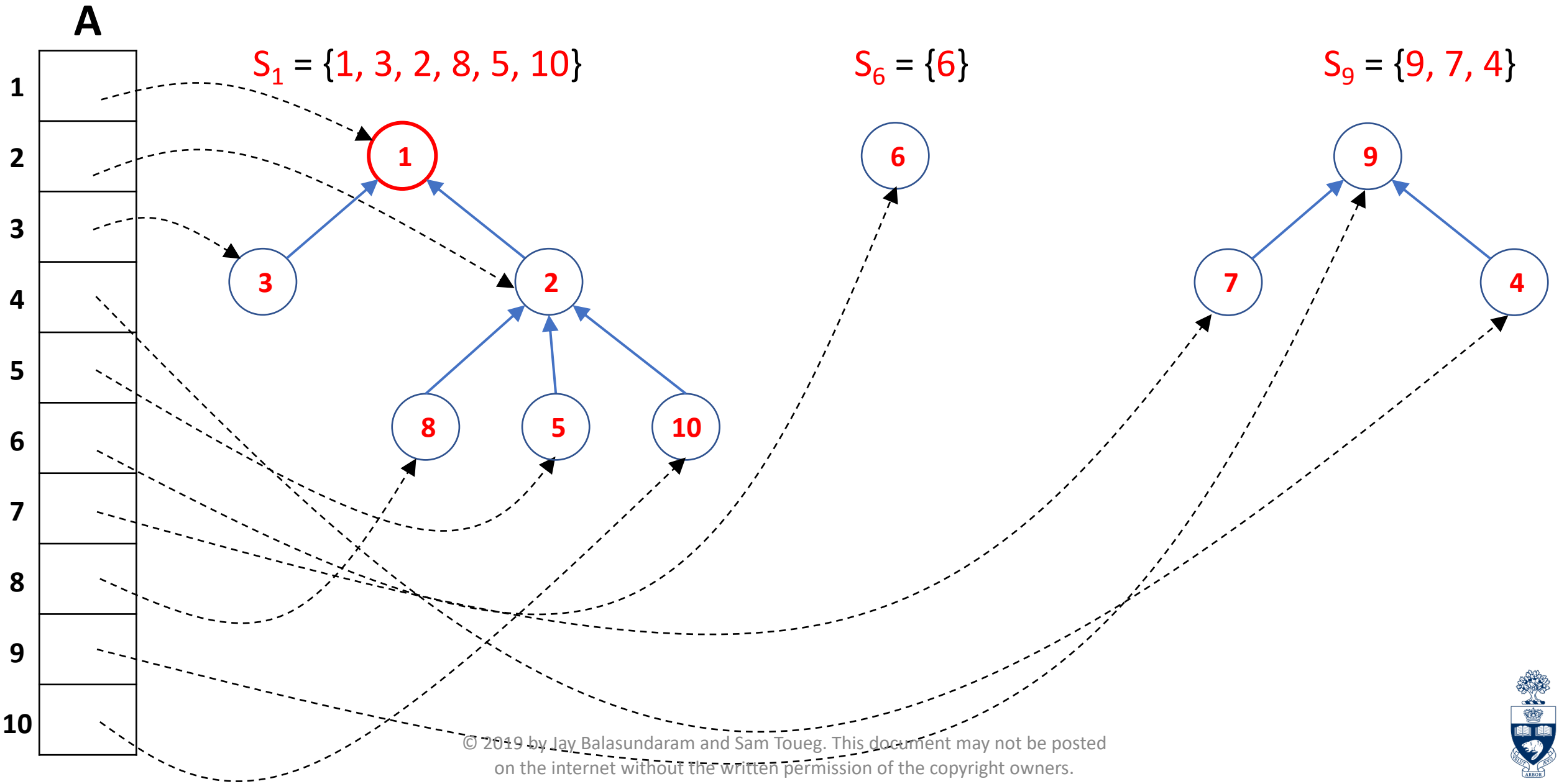
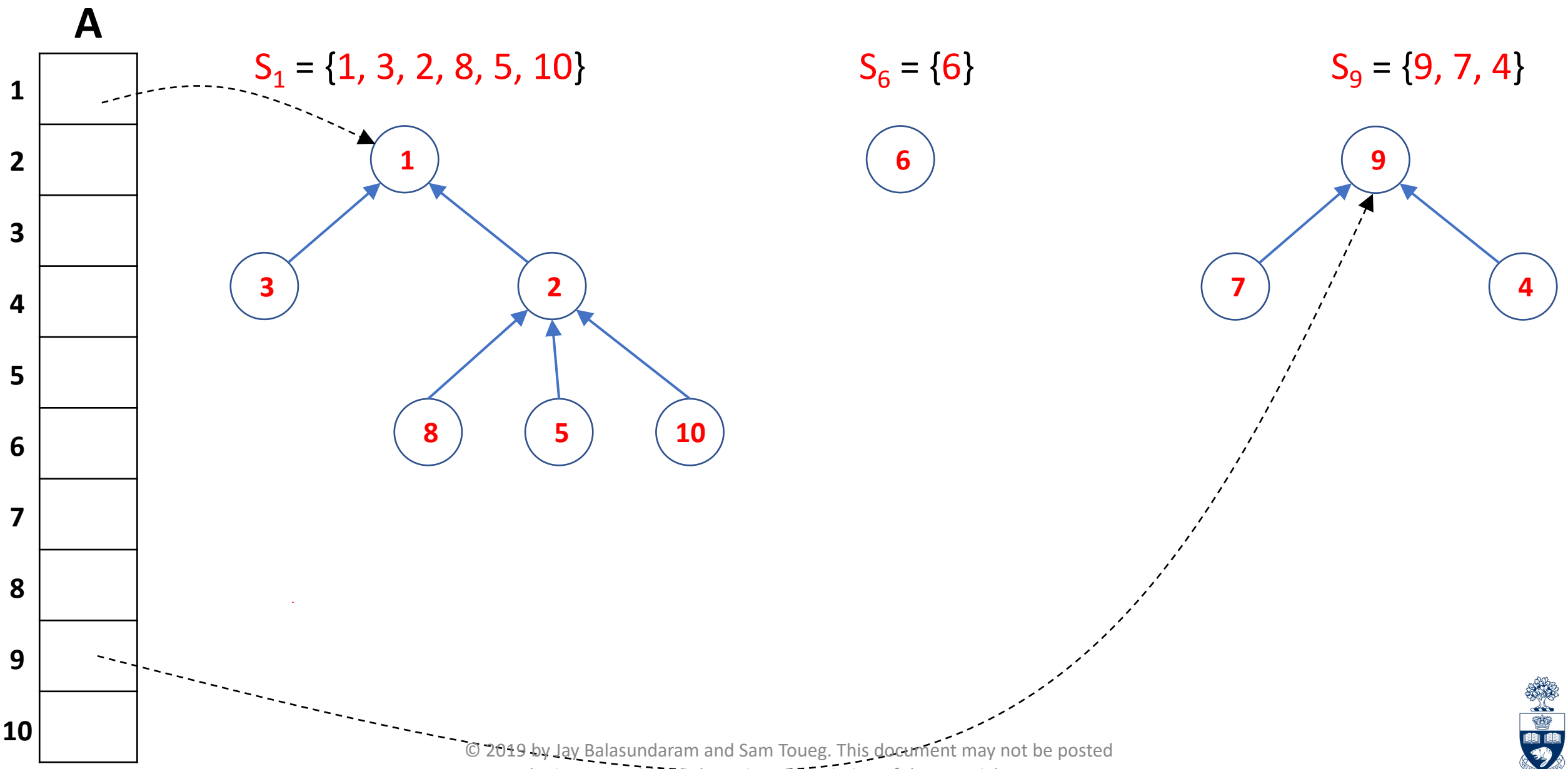


Find(5) : Return (ptr to) 1



Union(1, 9)



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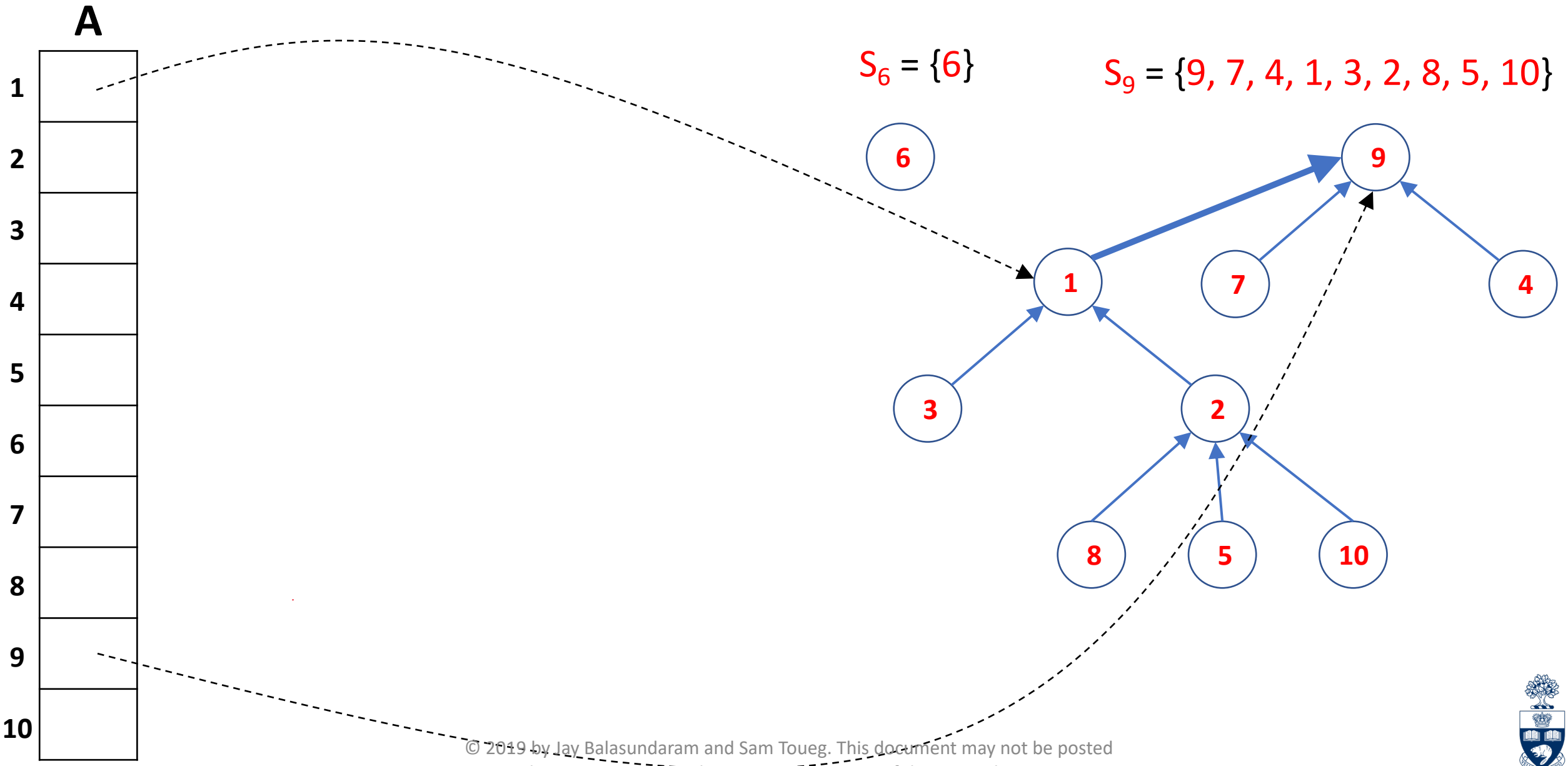
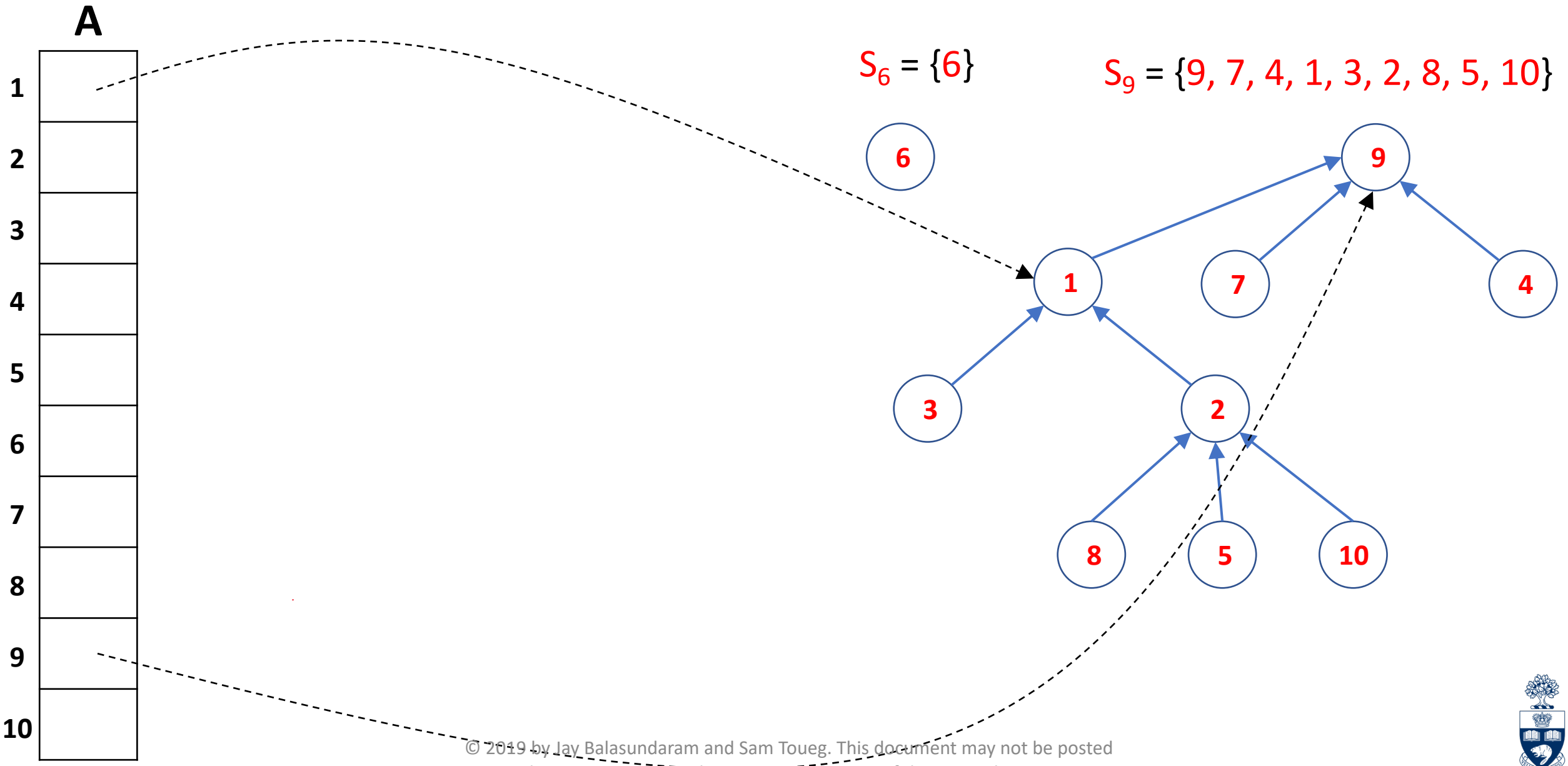


Union(1, 9) : Return (ptr to) 9



Operations: Example with n = 5

Initially :

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Initially :

Union(4, 3) :

Operations: Example with n = 5

Initially :

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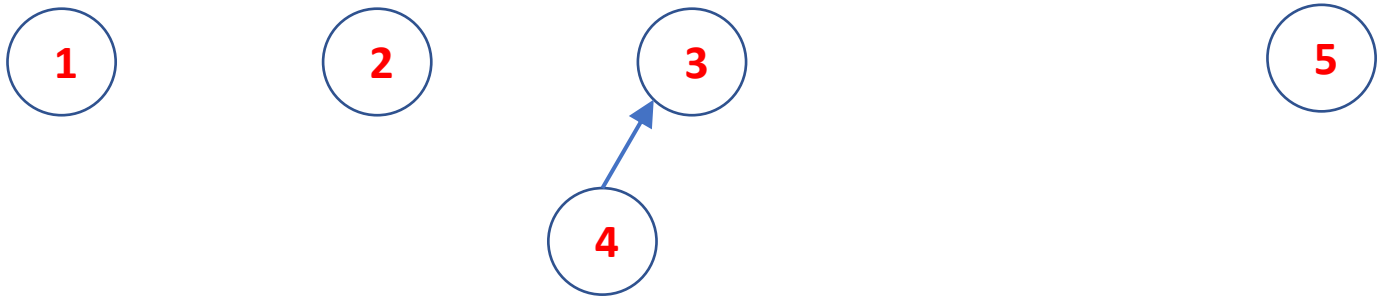

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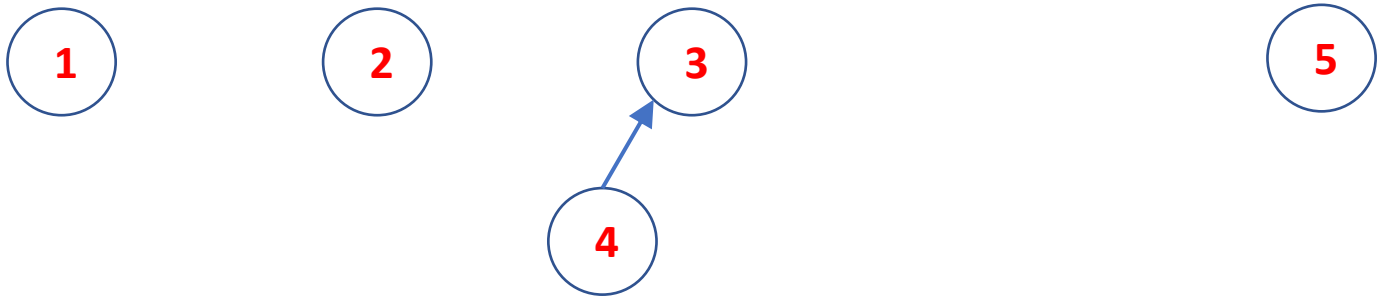

Union(5, 1) :

Operations: Example with n = 5

Initially :

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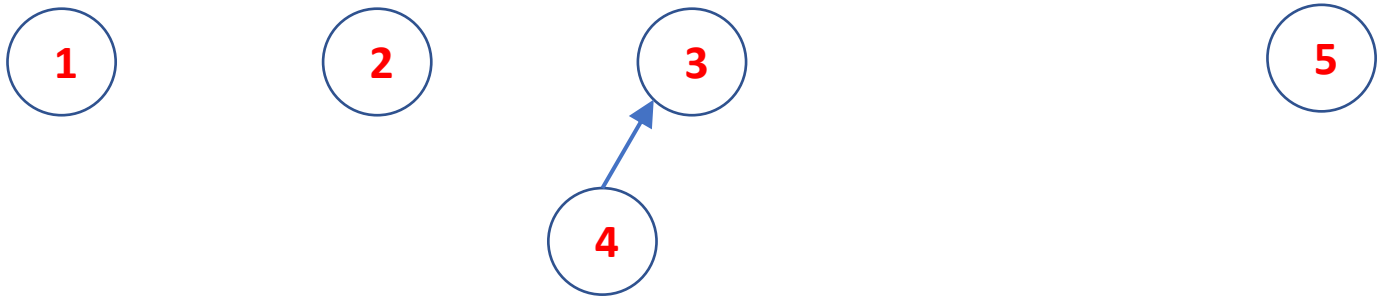

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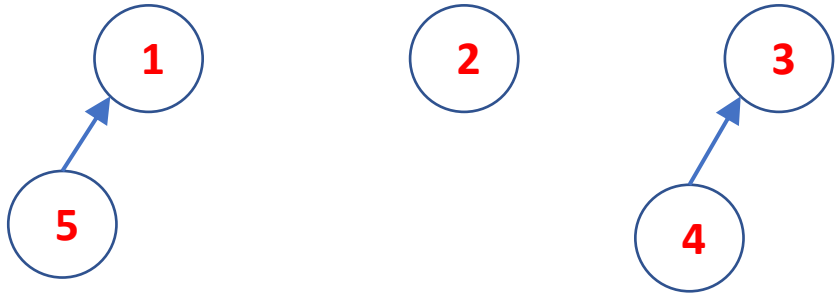

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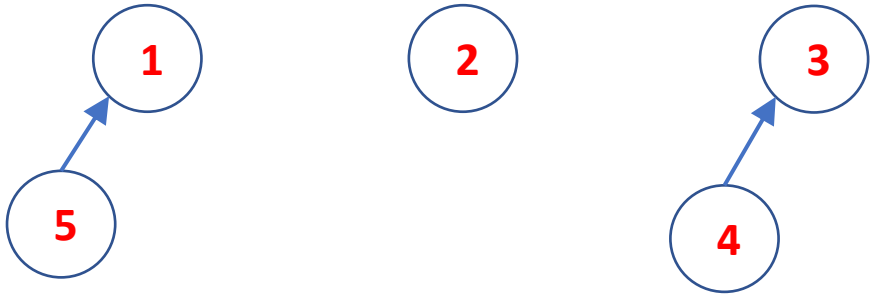

Operations: Example with n = 5

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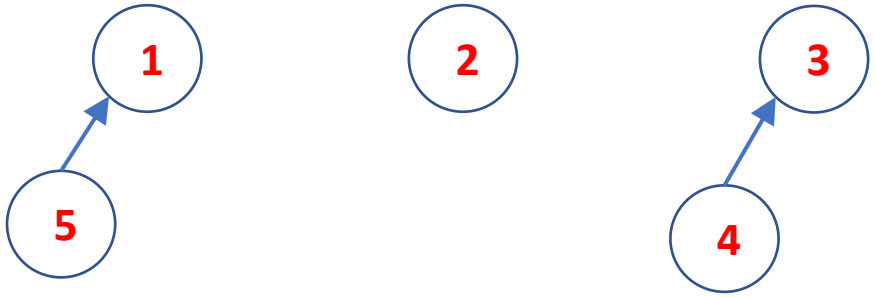
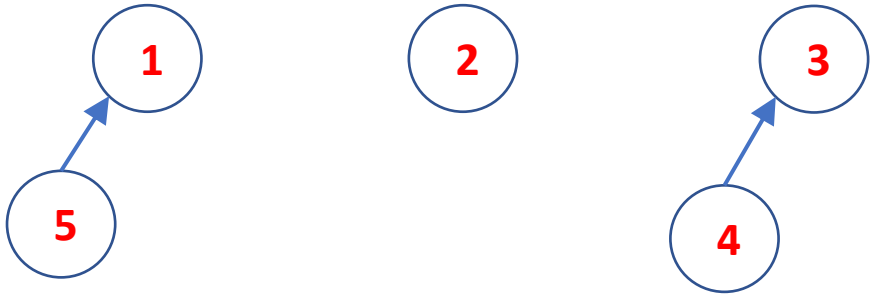


Union(3, 1) :

Operations: Example with n = 5

Union(5, 1) :



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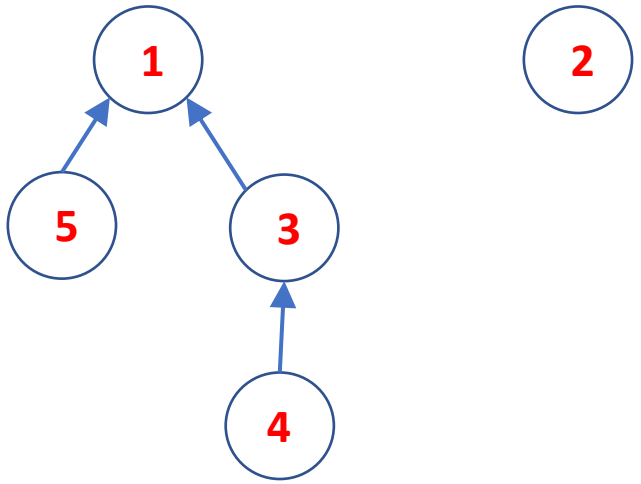

Operations: Example with n = 5

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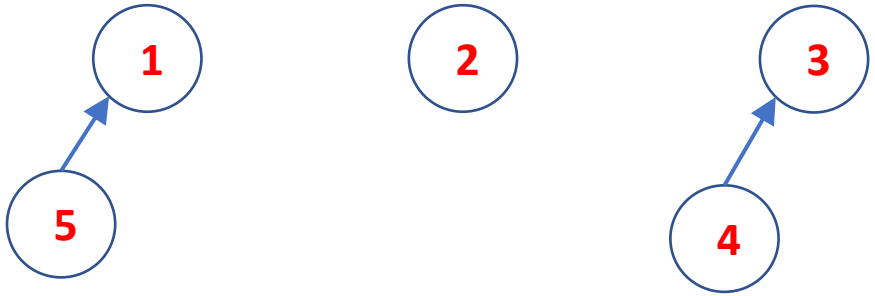

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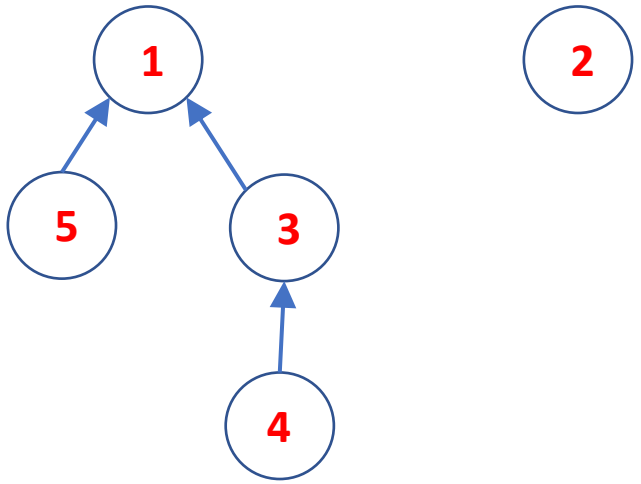
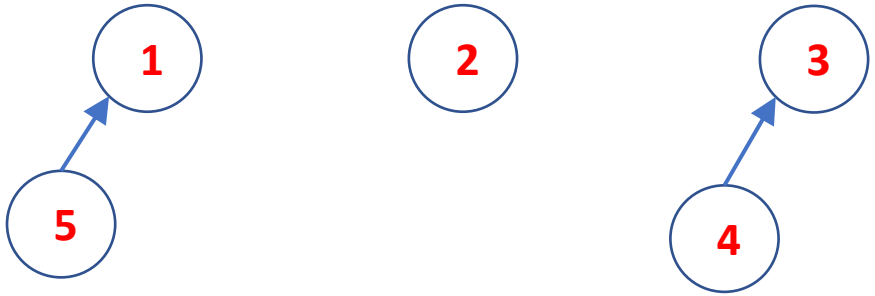


Union(2, 1) :

Operations: Example with n = 5

Union(5, 1) :



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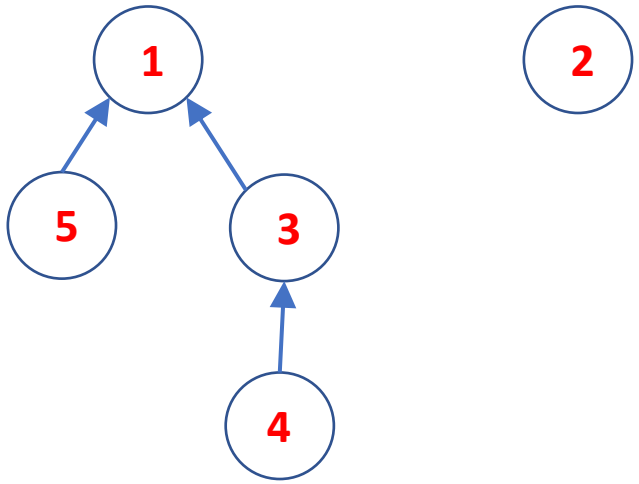
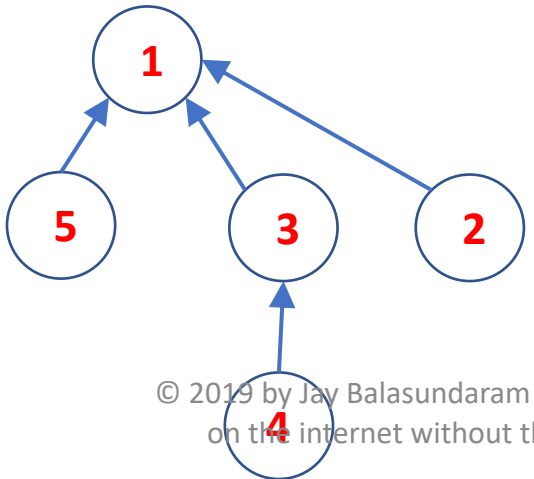


Union(2, 1) :



Operations

- **Find**(x): Follow path from x up to root, return ptr to the root

- **Union**(S_x, S_y): Make root of S_x the child of root of S_y

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  Cost is O(1)

Disjoint Forest: Analysis of Time Complexity

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- Cost of **Union** : O(1)

Disjoint Forest: Analysis of Time Complexity

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σ : Sequence of $n - 1$ **Unions** mixed with $m \geq n$ **Finds**

Disjoint Forest: Analysis of Time Complexity

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σ : Sequence of $n - 1$ **Unions** mixed with $m \geq n$ **Finds**

What is the cost of σ, in the worst-case?

Disjoint Forest: Analysis of Time Complexity

Initially : 

Disjoint Forest: Analysis of Time Complexity

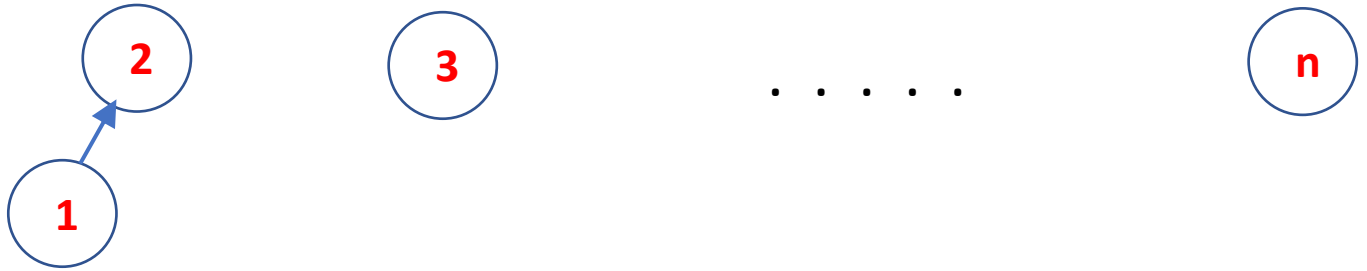
Initially : 

Union(1, 2) :

Disjoint Forest: Analysis of Time Complexity

Initially :

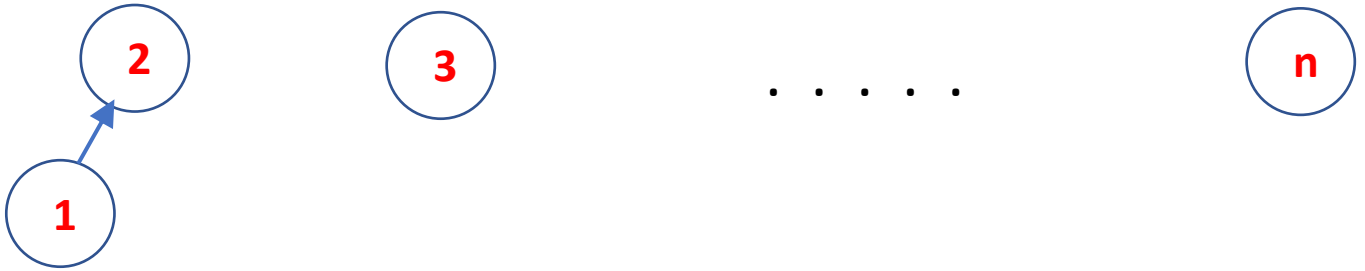
Union(1, 2) :



Disjoint Forest: Analysis of Time Complexity

Initially :

Union(1, 2) :

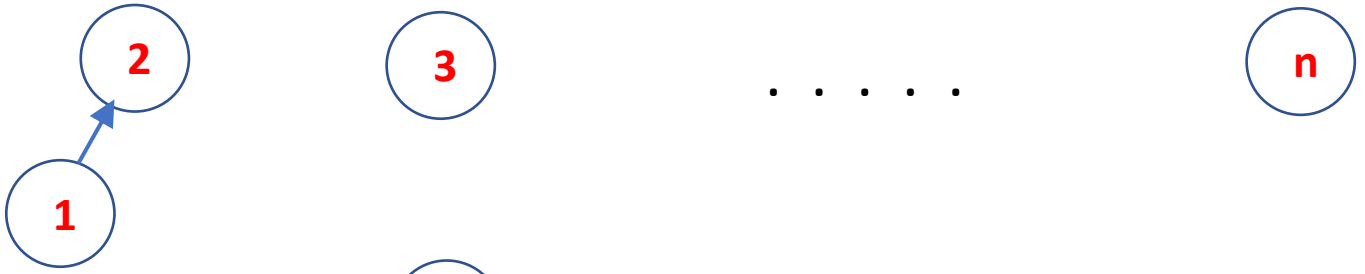


Union(2, 3) :

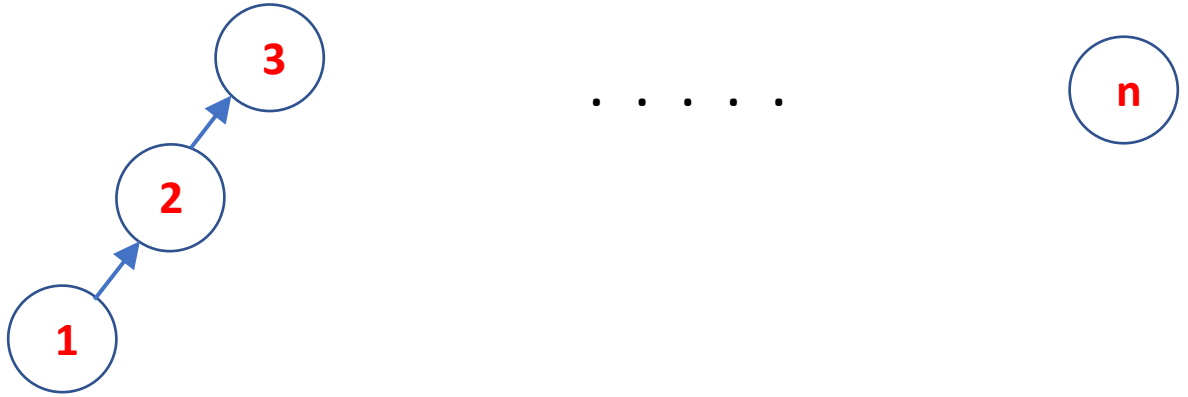
Disjoint Forest: Analysis of Time Complexity

Initially :

Union(1, 2) :



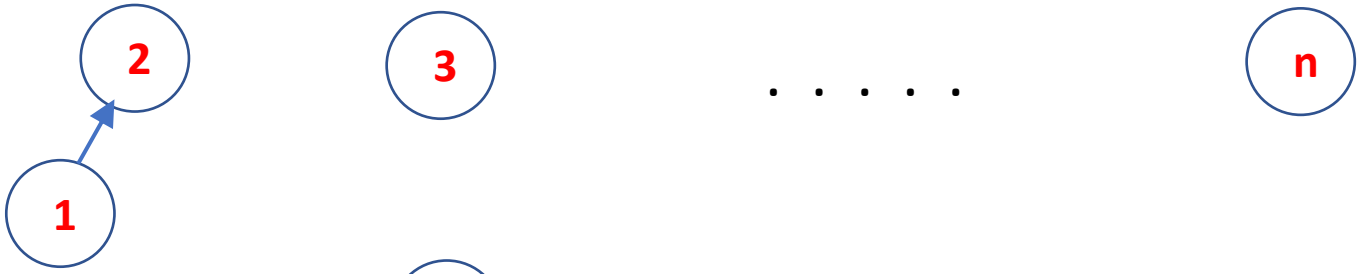
Union(2, 3) :



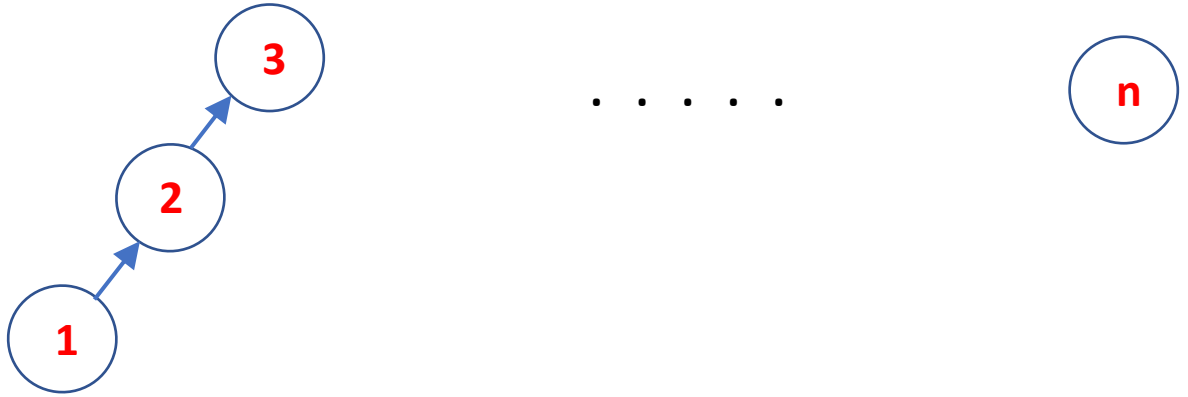
Disjoint Forest: Analysis of Time Complexity

Initially :

Union(1, 2) :



Union(2, 3) :



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Disjoint Forest: Analysis of Time Complexity

Initially :

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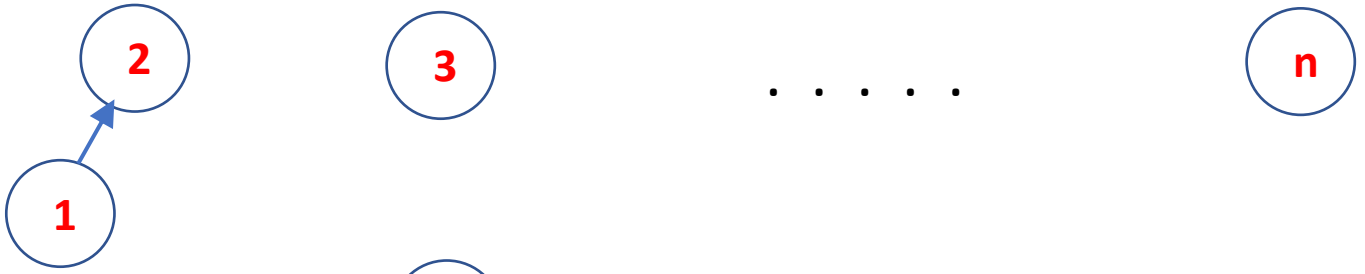

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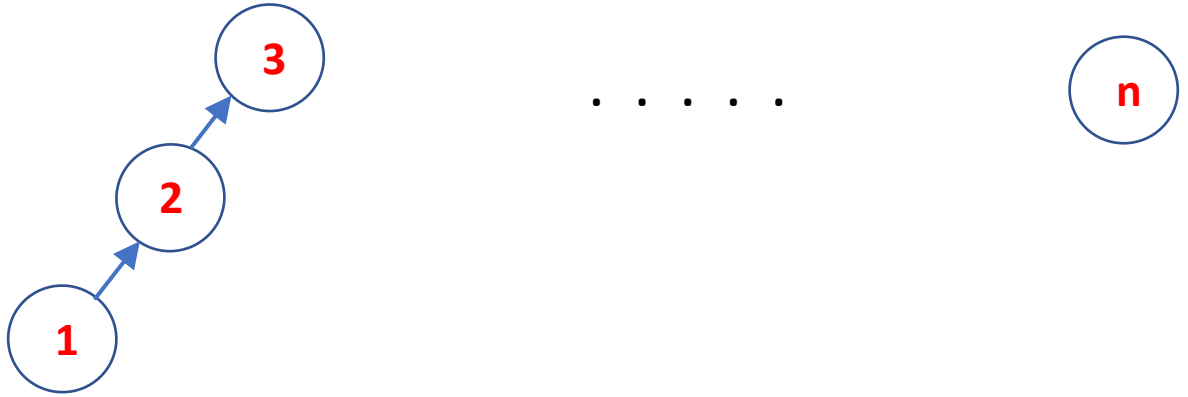

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Union(n-1, n) :

Disjoint Forest: Analysis of Time Complexity

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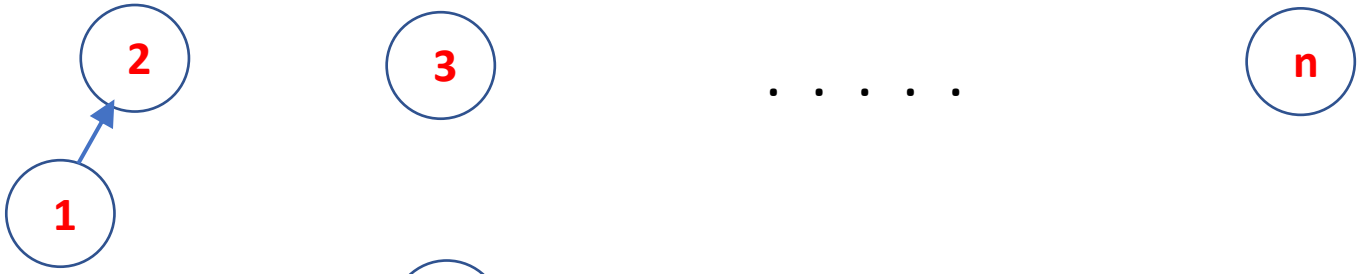
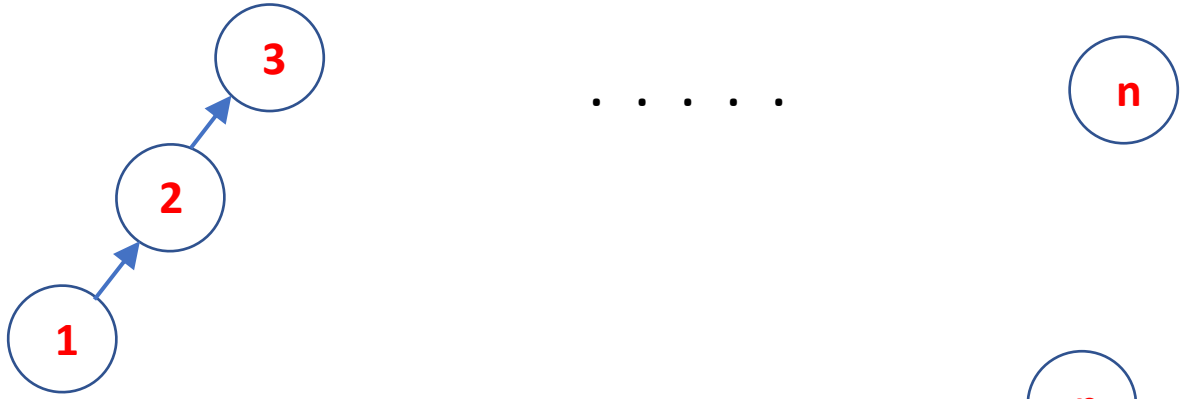
Union(1, 2) :



Union(2, 3) :



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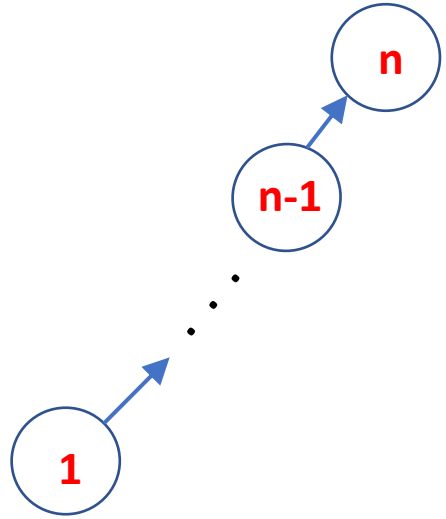
Union(n-1, n) :



Disjoint Forest: Analysis of Time Complexity

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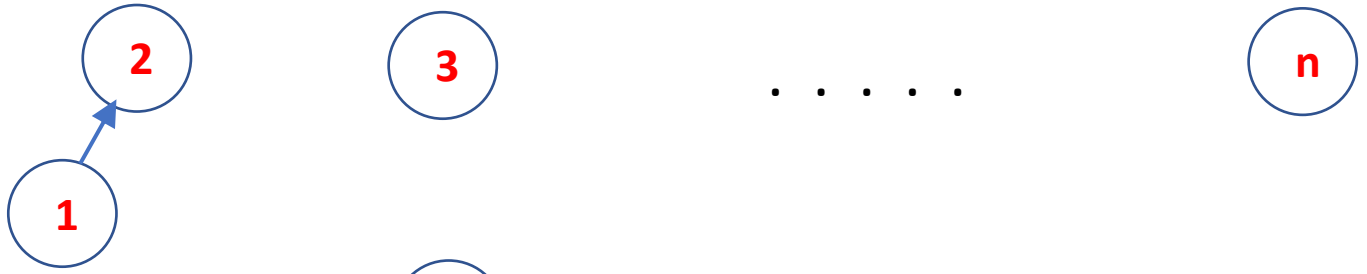
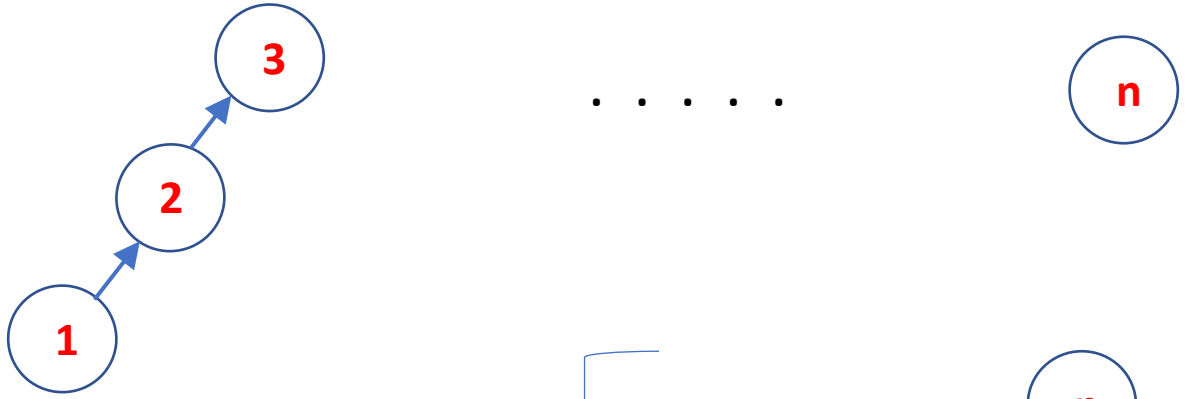


Union(2, 3) :



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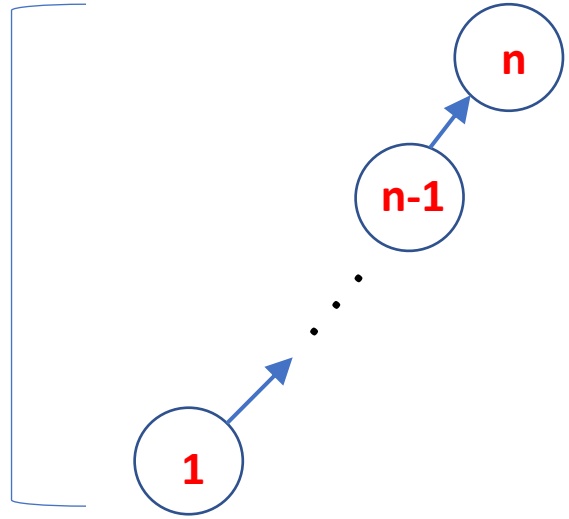
Union(n-1, n) :

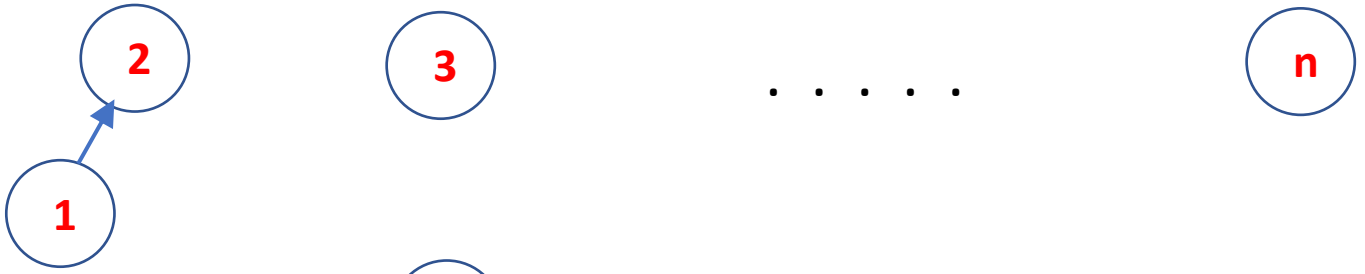
height = n-1



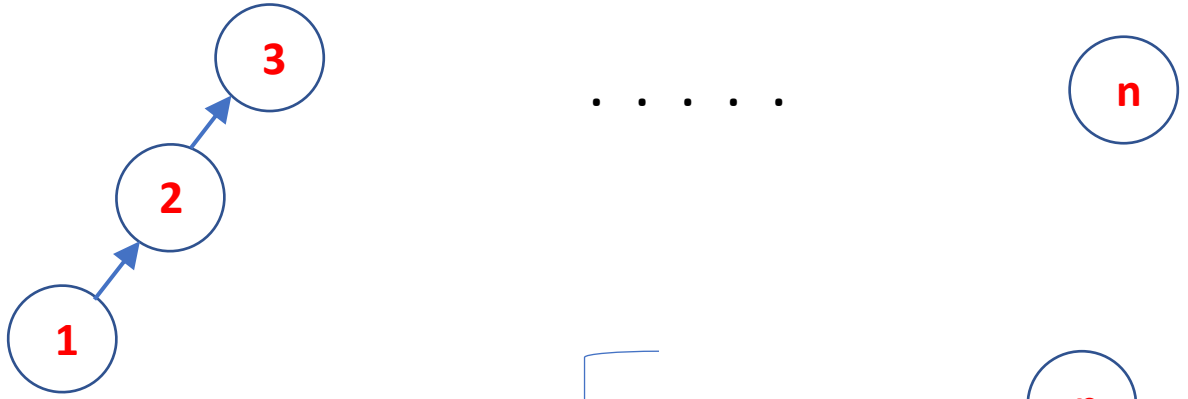
Disjoint Forest: Analysis of Time Complexity

Initially :

Union(1, 2) :



Union(2, 3) :



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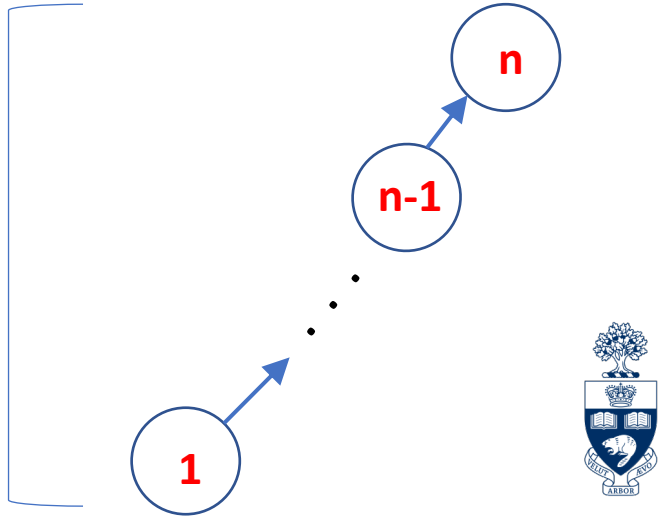
Union(n-1, n) :



height = n-1

Cost of m **Find**(1) : Ω(m.n) !

Disjoint Forest: Analysis of Time Complexity

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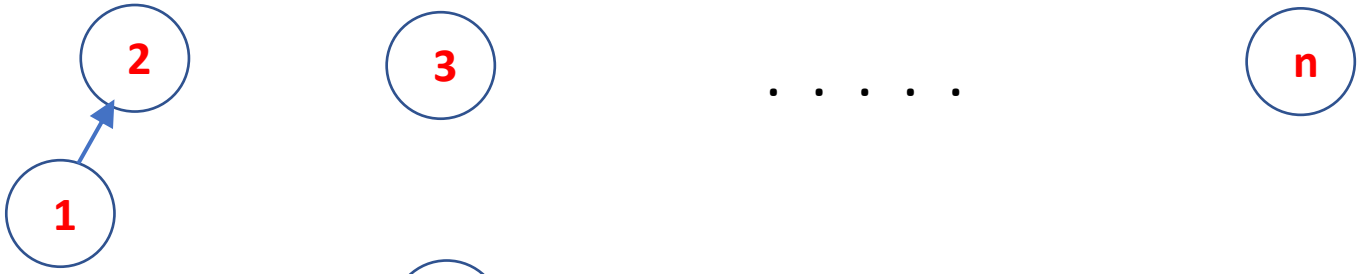

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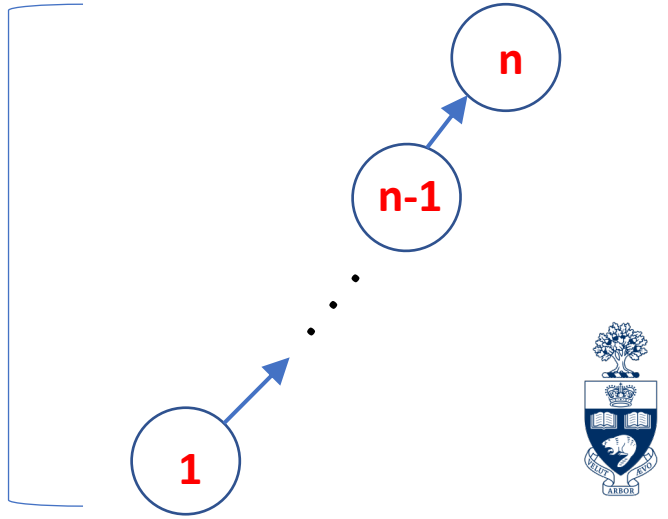
Union(n-1, n) :



height = n-1

Cost of m **Find**(1) : $\Omega(m.n)$!

$\Rightarrow$ Cost of σ, in the worst-case, is $\Omega(m.n)$

Disjoint Forest: Analysis of Time Complexity

Worst-case cost of executing σ is $\Omega(m.n)$

Cost of each **Find** : O(1 + length of the **Find** path)

To reduce cost of σ, reduce length of **Find** paths

Disjoint Forest: Analysis of Time Complexity

Worst-case cost of executing σ is $\Omega(m.n)$

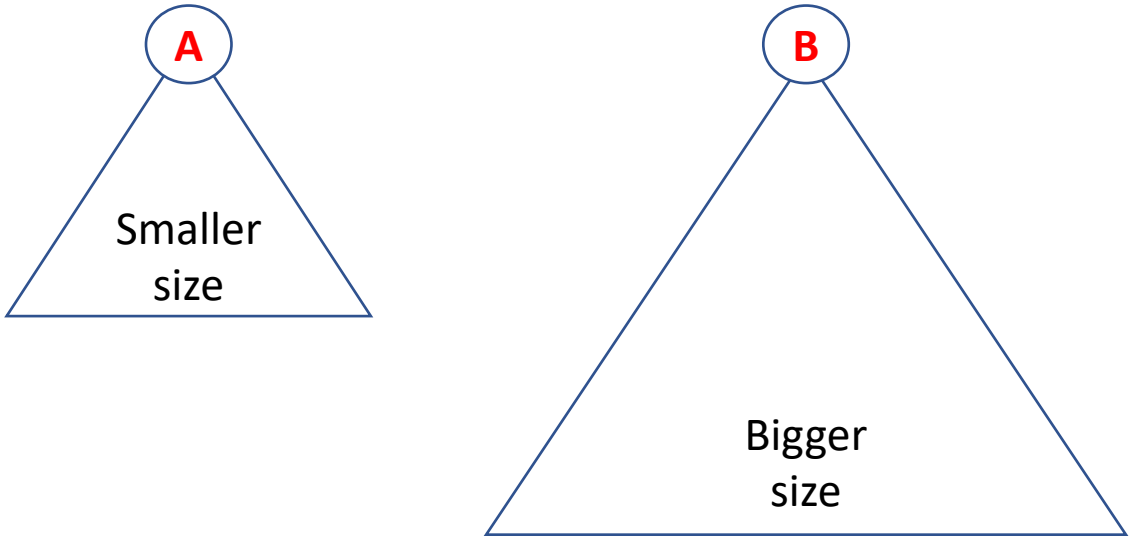
Cost of each **Find** : O(1 + length of the **Find** path)

To reduce cost of σ, reduce length of **Find** paths

$\Rightarrow$ reduce height of the trees formed during the execution of σ

Heuristic 1: **W**eighted **U**nion (**WU**) by Size

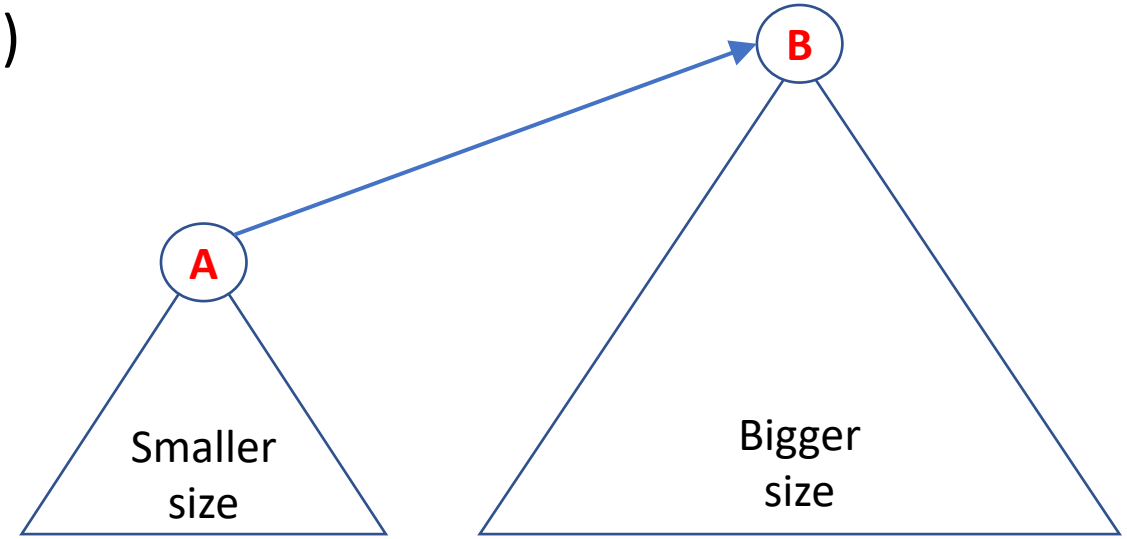
Union(A, B)



size: # of nodes in the tree

Heuristic 1: **W**eighted **U**nion (**WU**) by Size

Union(A, B)



size: # of nodes in the tree

WU rule (by size): Smaller size tree becomes the child of the bigger size tree

Heuristic 1: **W**eighted **U**nion (**WU**) by Size

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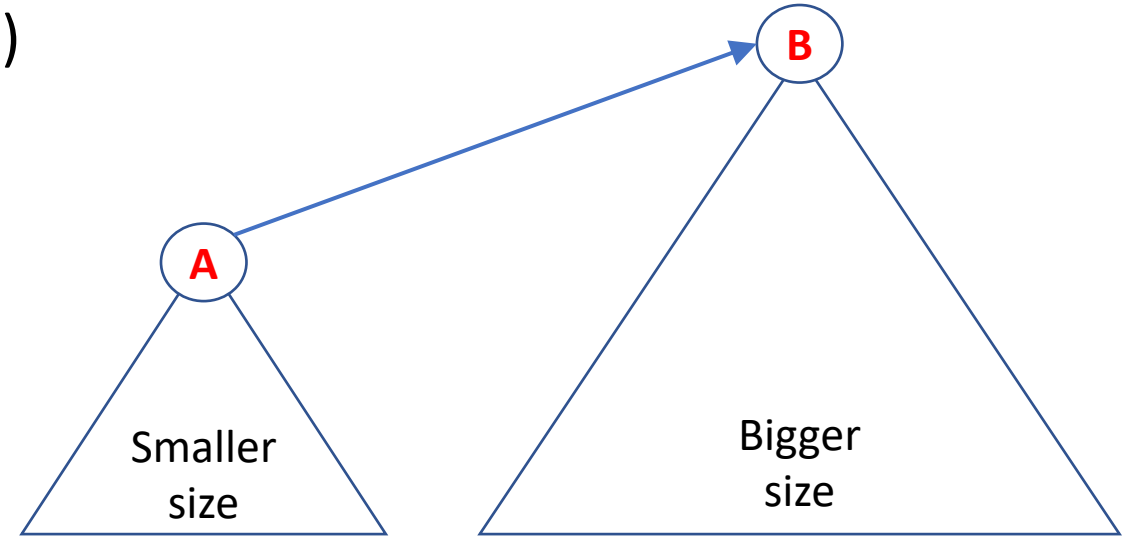

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- Any tree T created during the execution of σ has height at most $\log_2 n$

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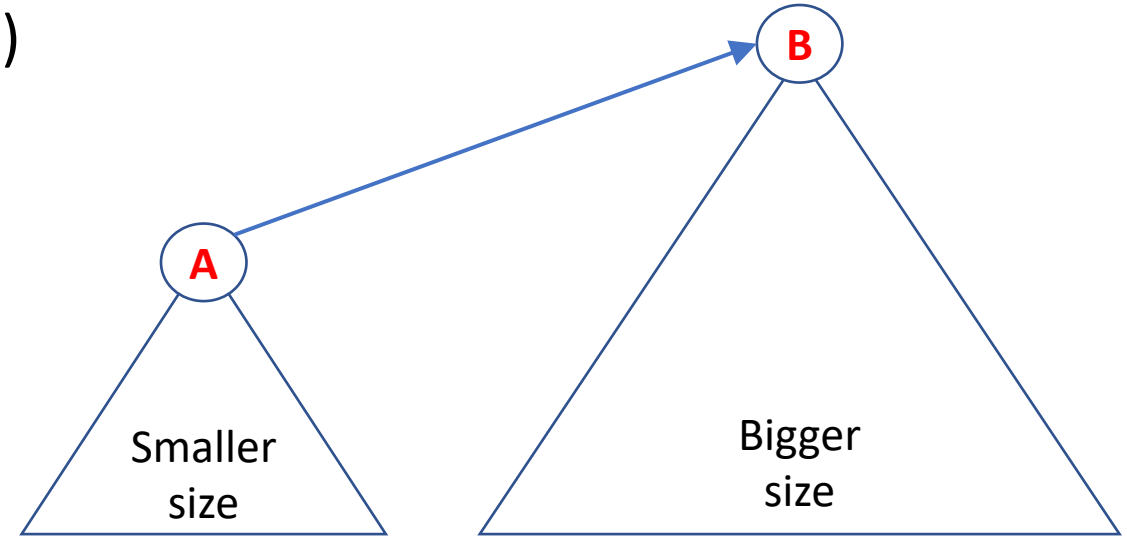

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With **WU**:

- Any tree T created during the execution of σ has height at most $\log_2 n$
- The worst-case cost of executing σ is O( )

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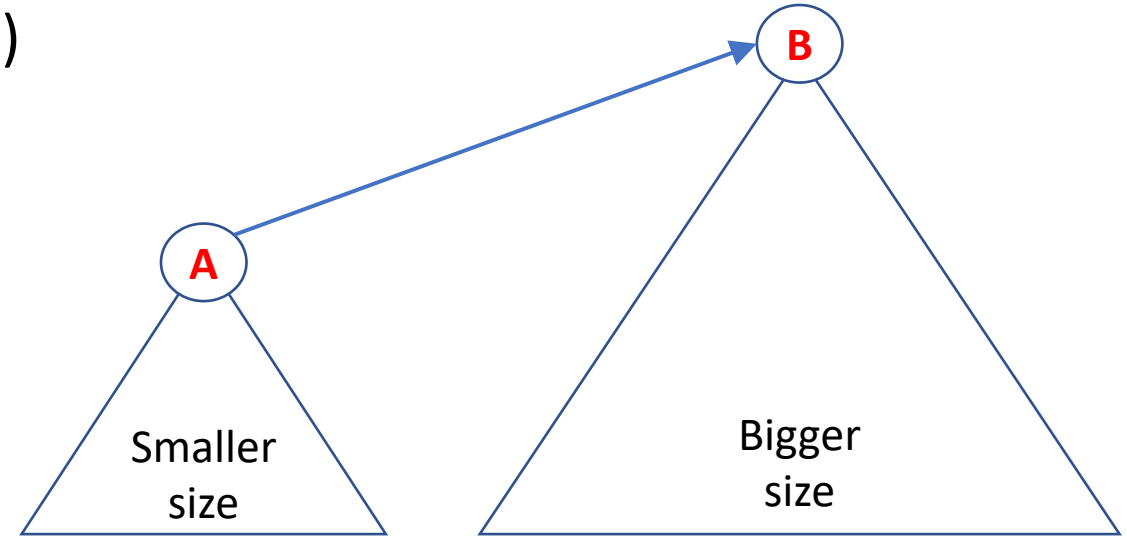

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n-1 U's

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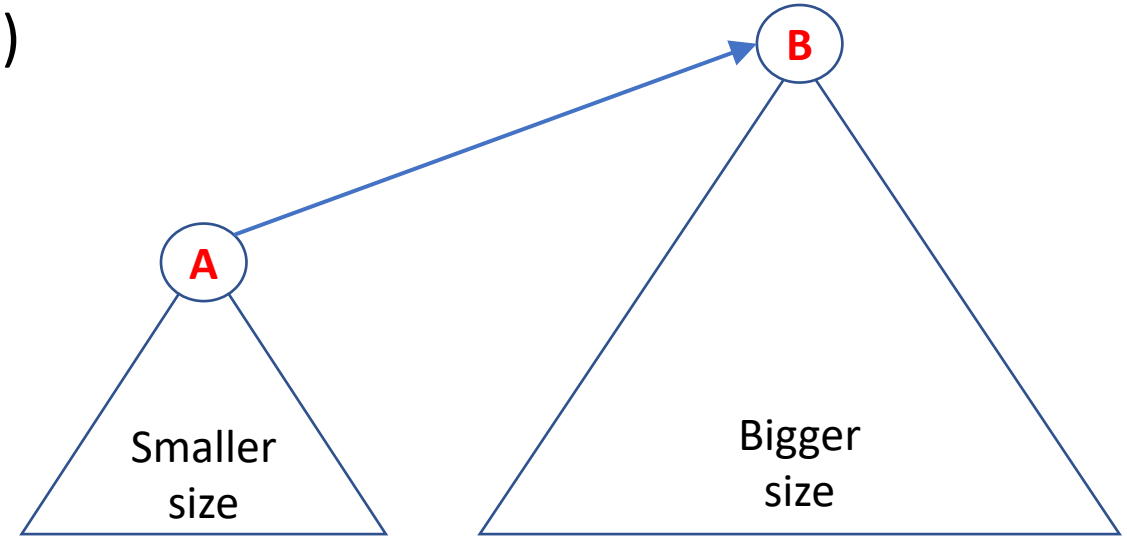

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WU rule (by size): Smaller size tree becomes the child of the bigger size tree

With **WU**:

- Any tree T created during the execution of σ has height at most $\log_2 n$
- The worst-case cost of executing σ is $O(n + m \log n)$

n-1 U's m F's

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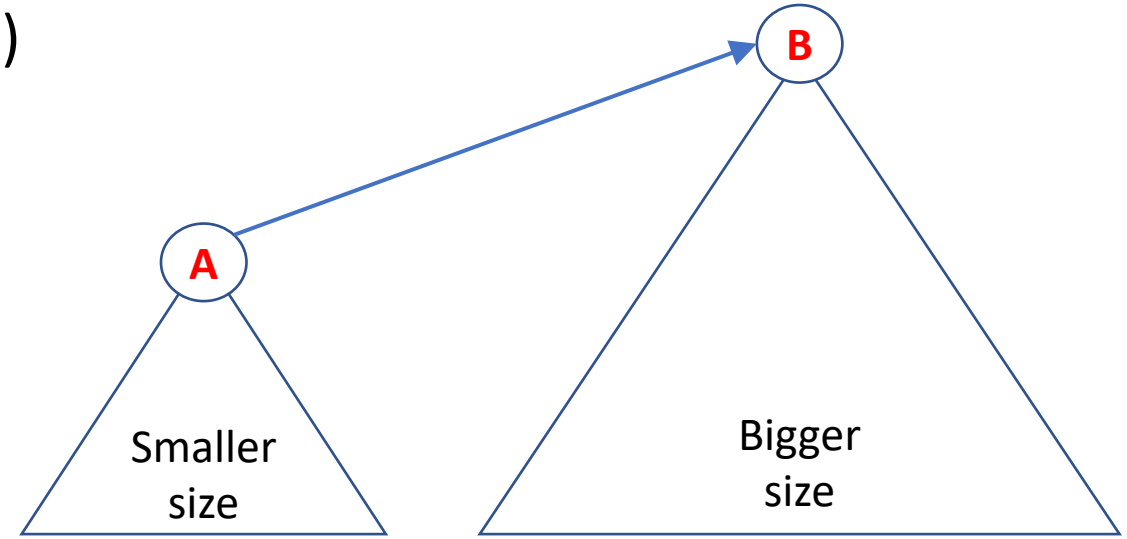
Union(A, B)



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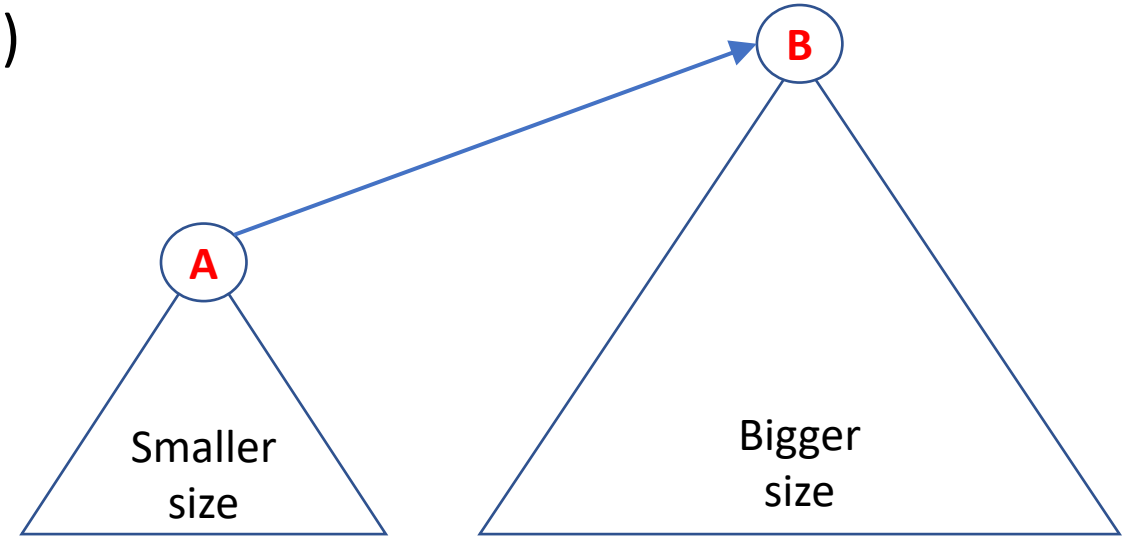

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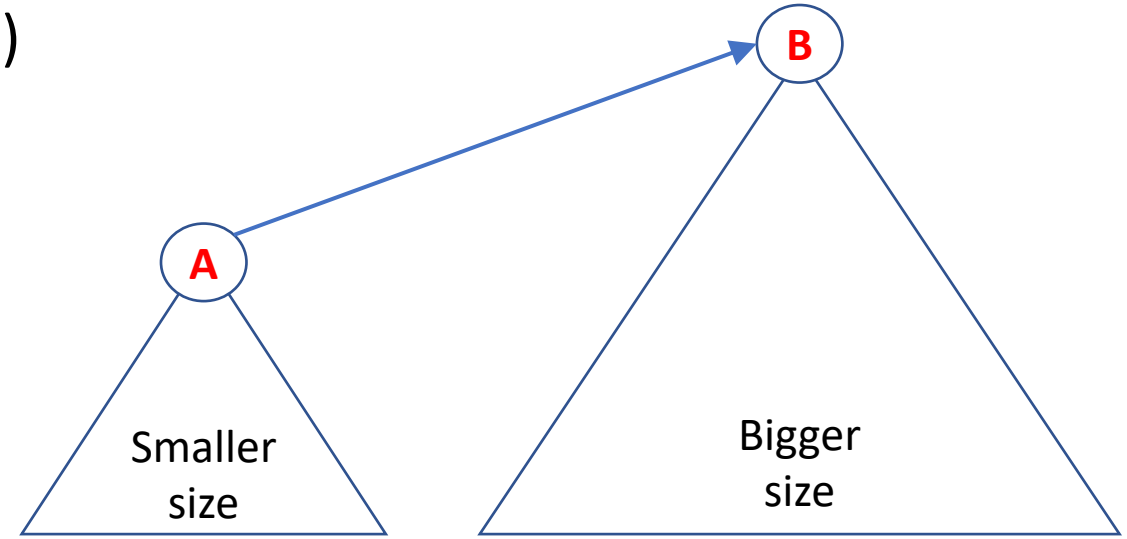

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WU rule (by size): Smaller size tree becomes the child of the bigger size tree

With **WU**:

- Any tree T created during the execution of σ has height at most $\log_2 n$
- The worst-case cost of executing σ is $O(m \log n)$

We now prove this!

Lemma: With **WU**, any tree T of height h created during the execution of σ has at least 2^h nodes, i.e., $|T| \geq 2^h$

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Since $|T| \leq n$, $2^h \leq |T| \leq n \Rightarrow$ $h \leq \log_2 n$

Lemma: With **WU**, any tree T of height h created during the execution of σ has at least 2^h nodes, i.e., $|T| \geq 2^h$

Proof Sketch: Induction on h

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- h = 0. Any tree of height zero has at least $2^0 = 1$ nodes

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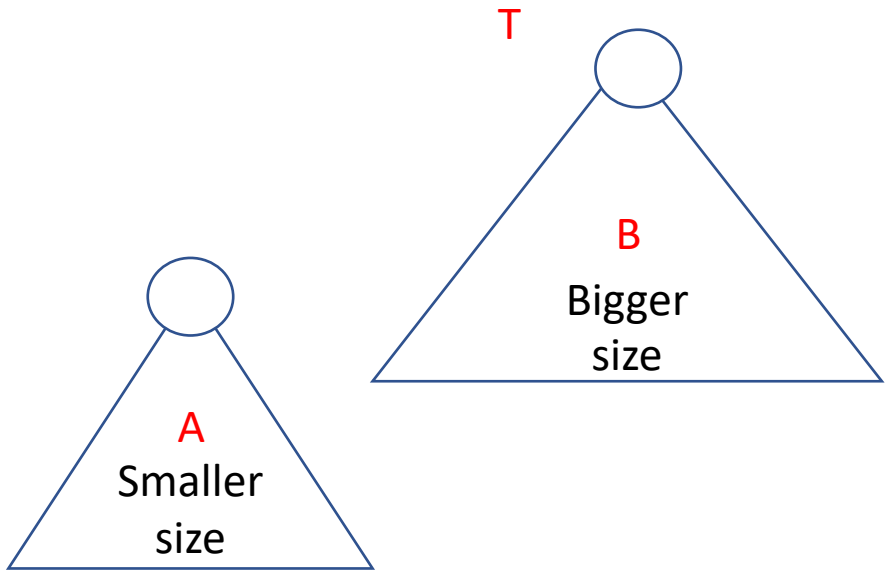

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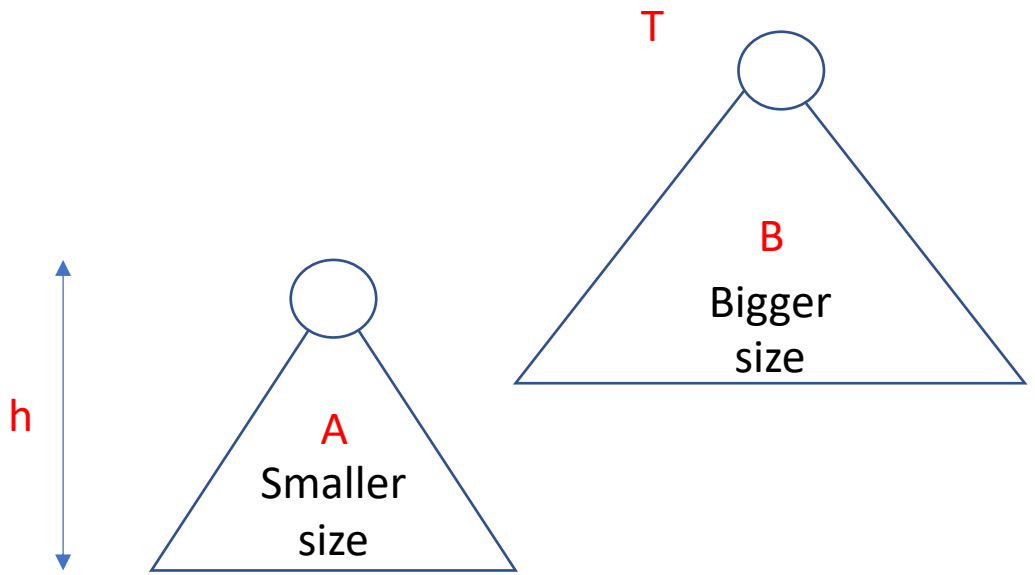

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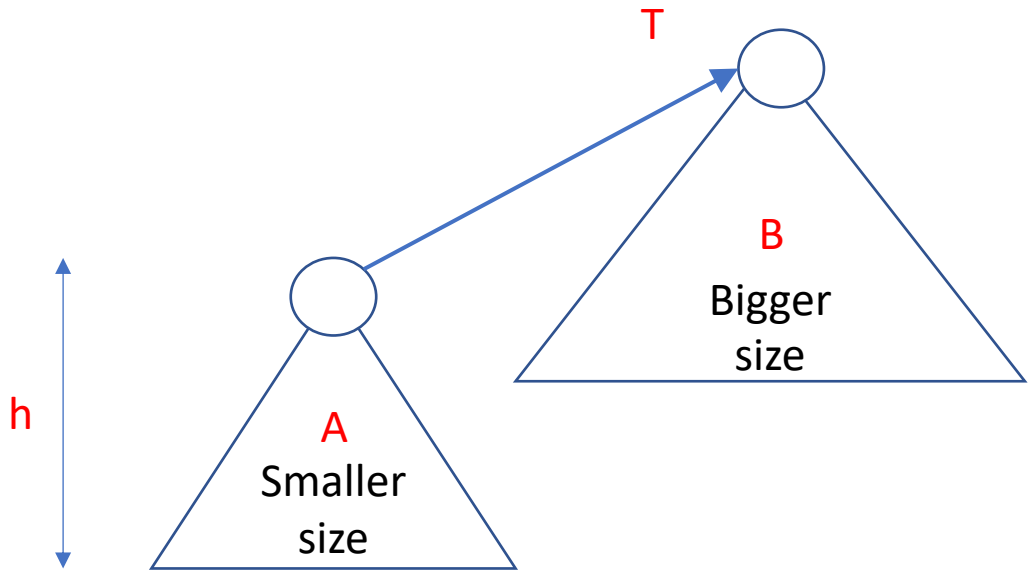

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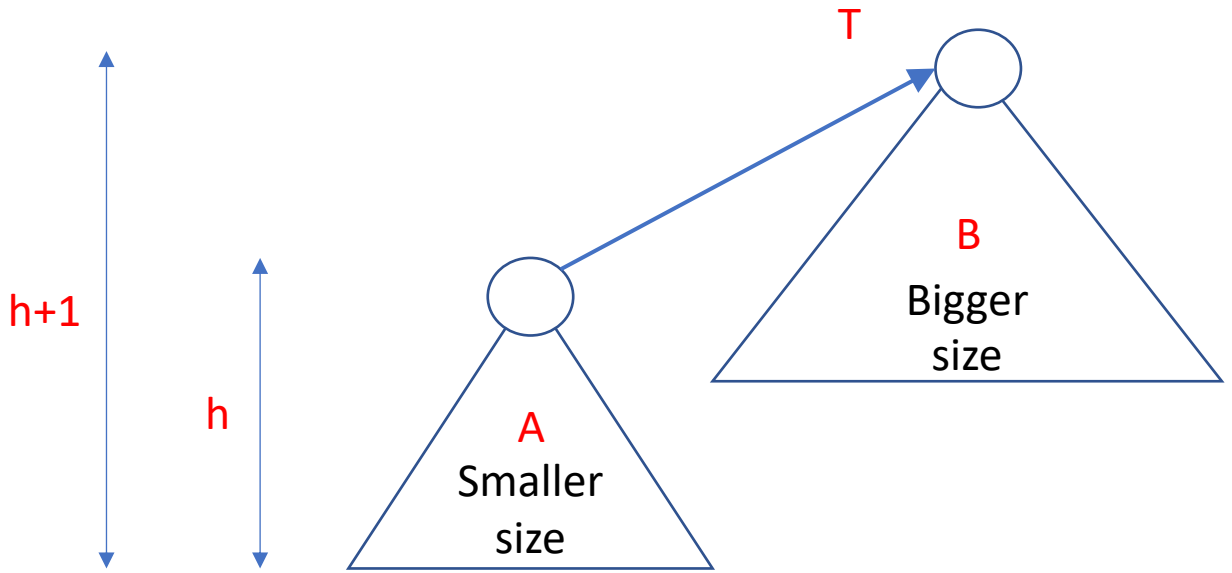

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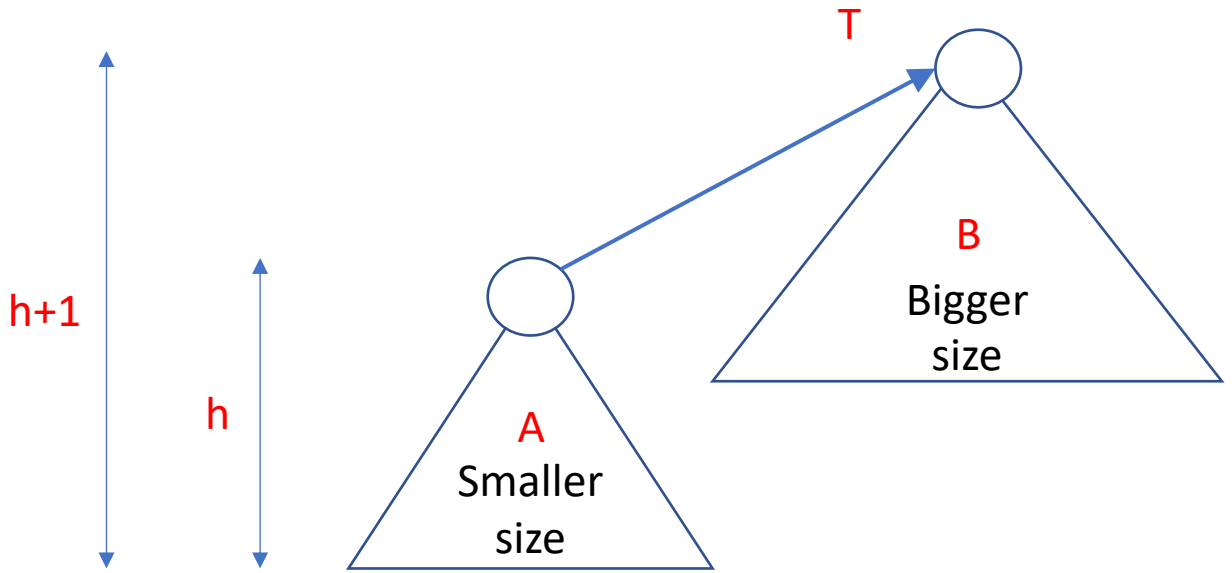

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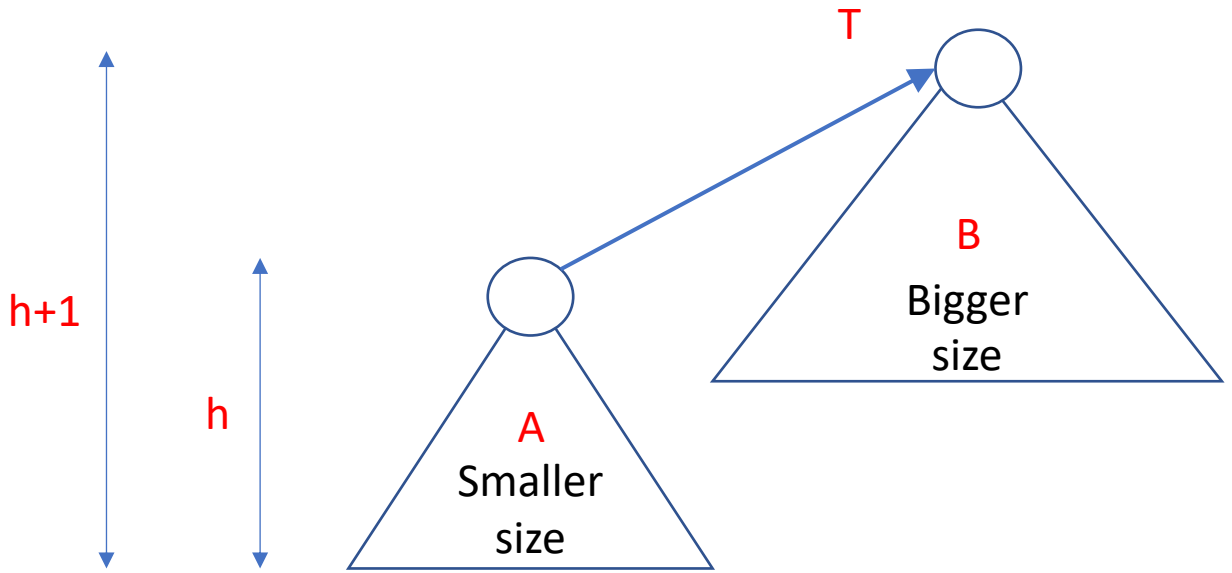

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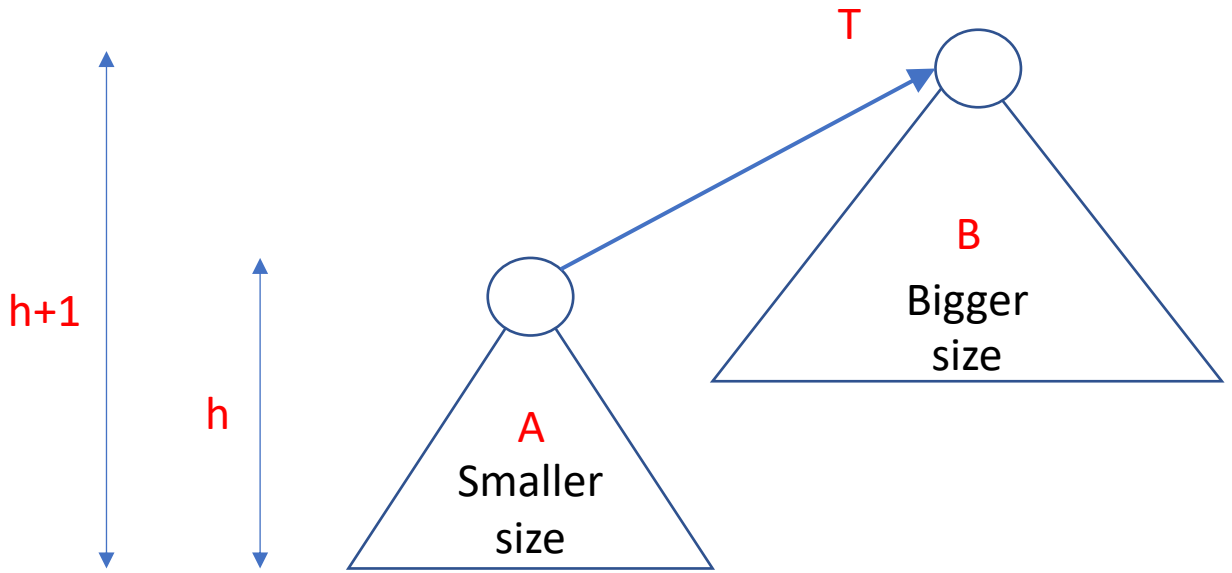

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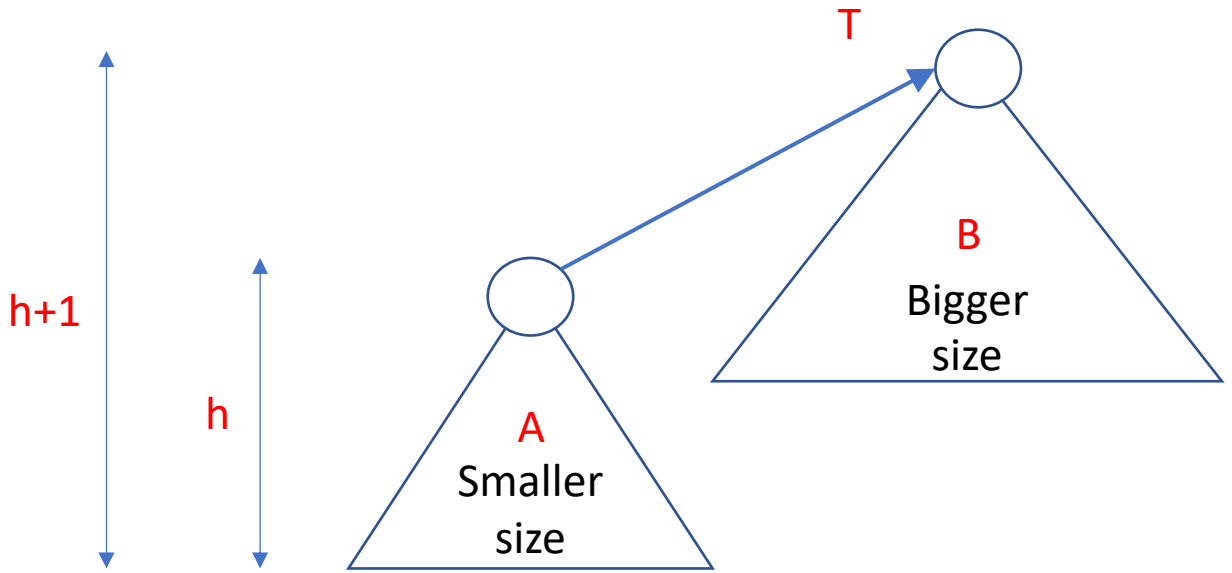

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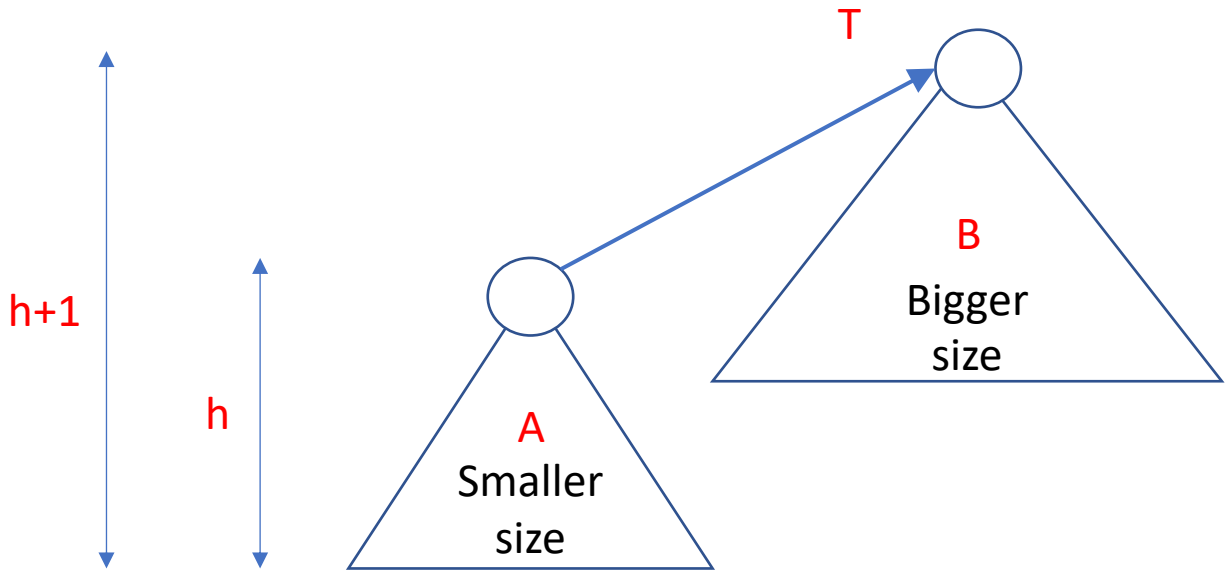

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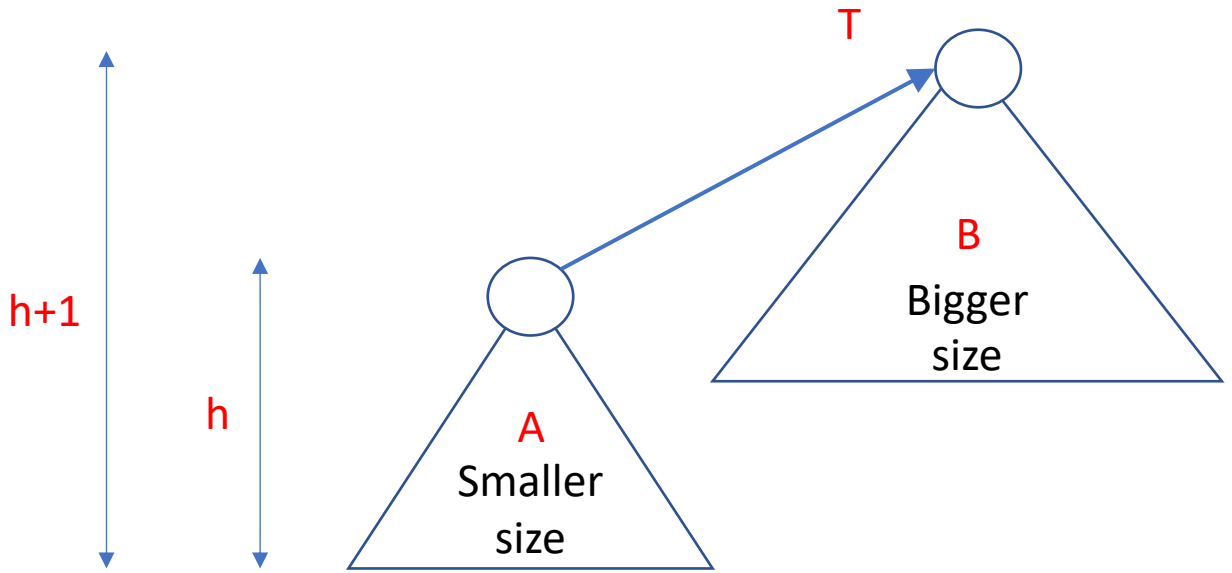

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By **WU** : $|B| \geq |A|$

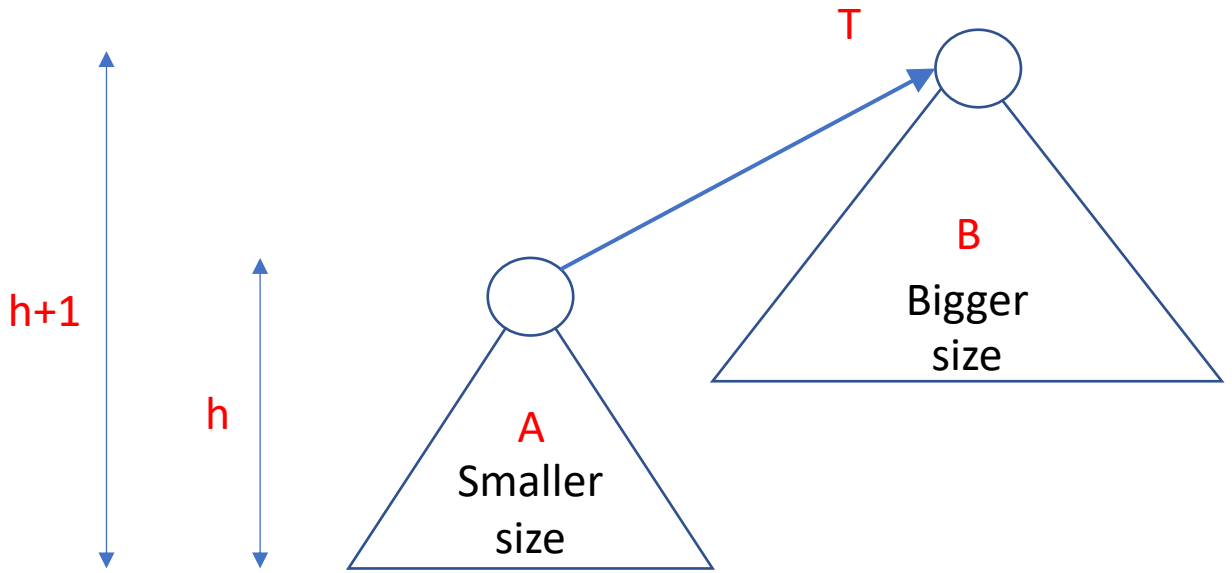
$|B| \geq 2^h$

$|T| = |A| + |B| \geq 2^h + 2^h = 2^{h+1}$ nodes

Lemma: With **WU**, any tree T of height h created during the execution of σ has at least 2^h nodes, i.e., $|T| \geq 2^h$

Proof Sketch: Induction on h

- h = 0. Any tree of height zero has at least $2^0 = 1$ nodes
- Suppose the lemma holds for $h \geq 0$ (I.H.). We show it holds for h+1



By I.H. : $|A| \geq 2^h$

By **WU** : $|B| \geq |A|$
$|B| \geq 2^h$

$|T| = |A| + |B| \geq 2^h + 2^h = 2^{h+1}$ nodes $\Rightarrow$ T has at least 2^{h+1} nodes