Bloom Filters

Burton Howard Bloom [1970]

Course Website: Bloom Filters Survey by A. Broder and M. Mitzenmacher

Bloom Filter

- Space-efficient “Probabilistic Dictionary”

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- Maintain the "fingerprints" of the elements of a set S

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Can have False Positives!

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- Can accomplish this using a **BF** of size $\approx$ 10 MB, with False Positive rate just 2%

Bloom Filter

Bloom Filter

- Array **BF**[0 … m-1] of m bits, initially all 0’s

BF	
0	0
0	1
0	
0	.
0	. .
0	
0	m - 1

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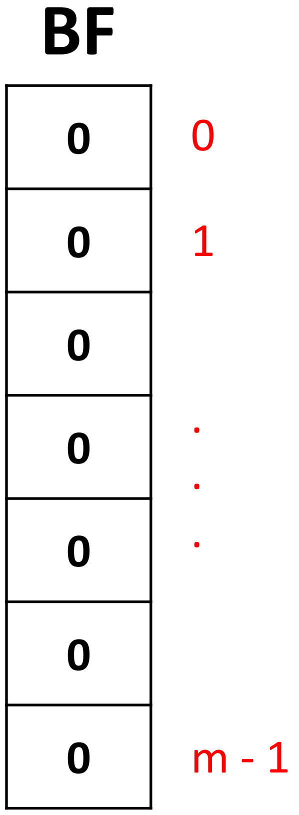
  $h_i : U \rightarrow \{0, 1, \ldots, m\text{-}1\}$



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  h_i satisfying **SUHA**

SUHA: Every element is equally likely to hash into any of the m slots of **BF,** independent of where the other elements have hashed to.

BF

0	0
0	1
0	
0	.
0	.
0	.
0	m - 1

Example BF[0 … 7] with $t = 2$: h_1 and h_2

BF	
0	0
0	1
0	2
0	3
0	4
0	5
0	6
0	7

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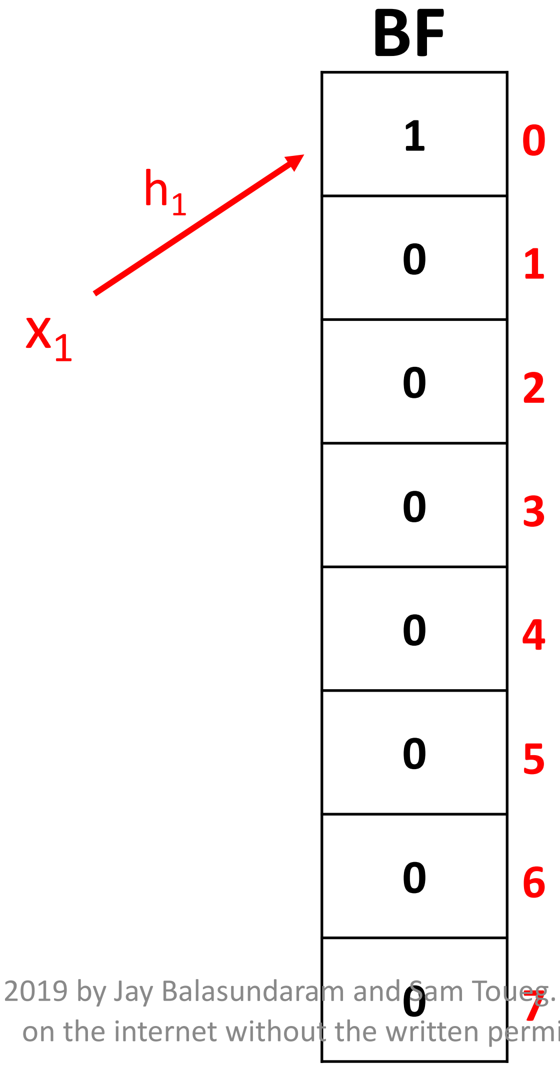

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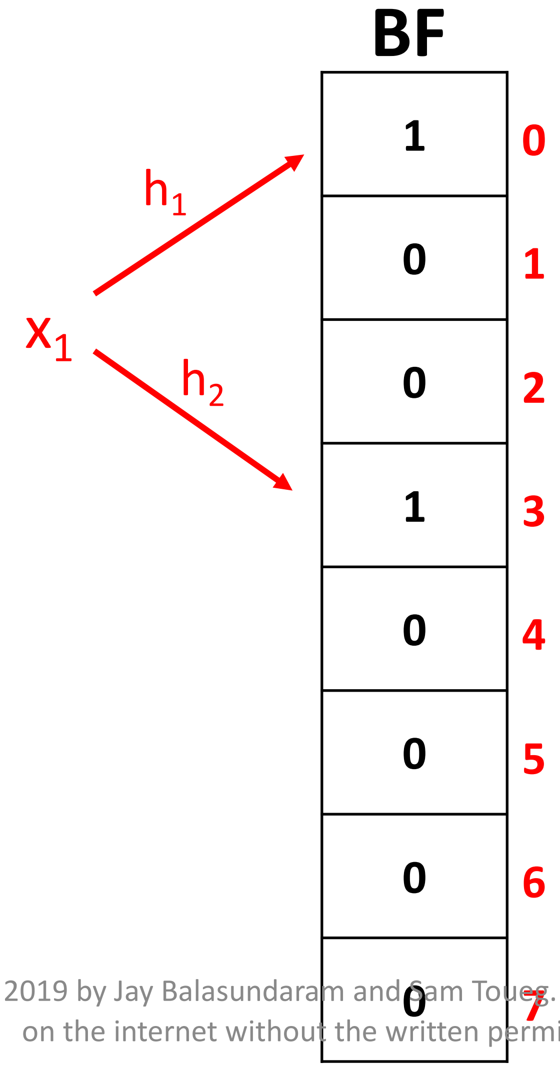

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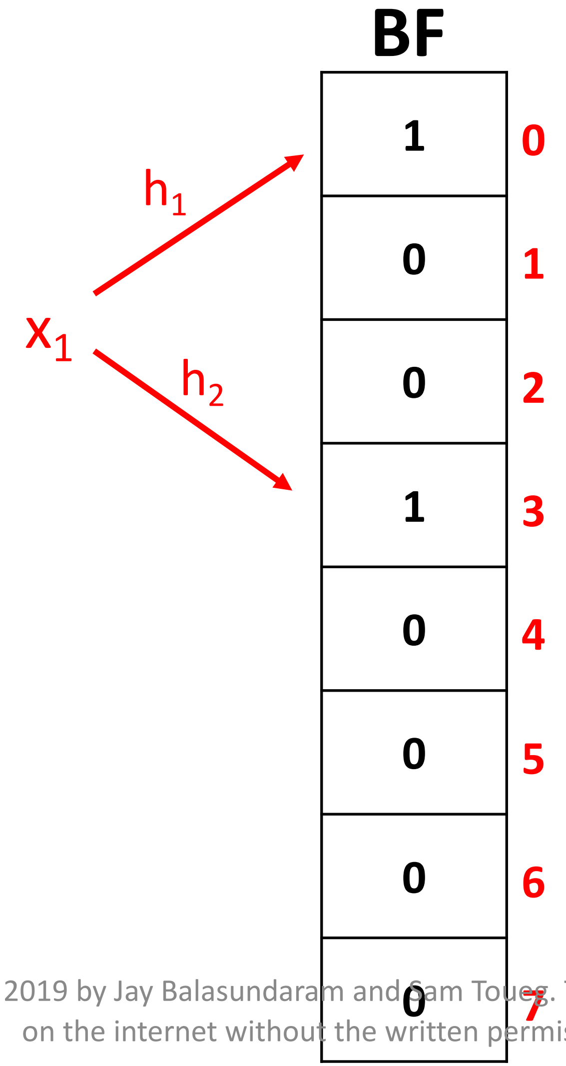

BF_Insert(x_2)

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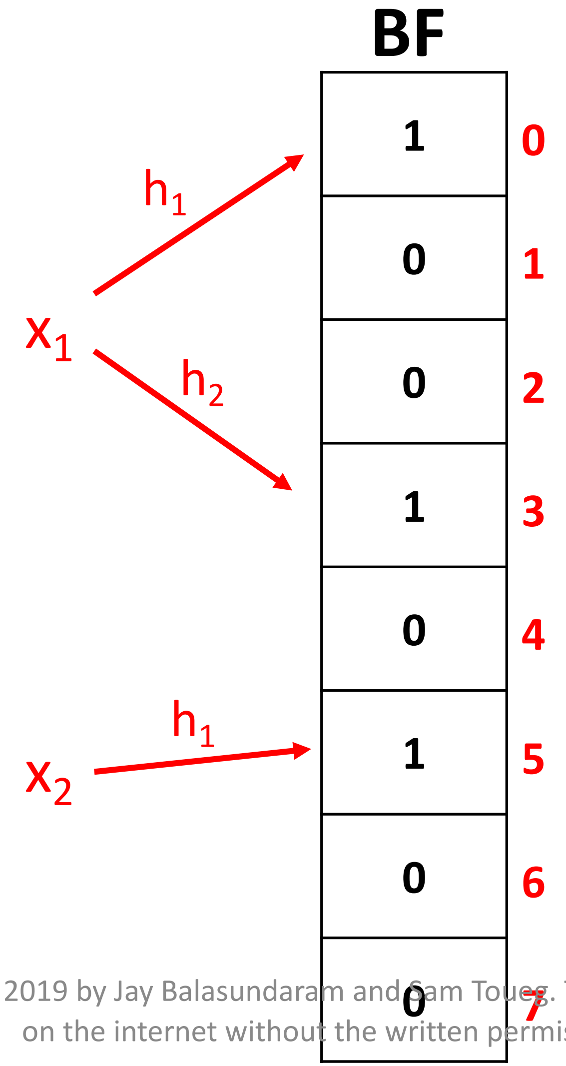

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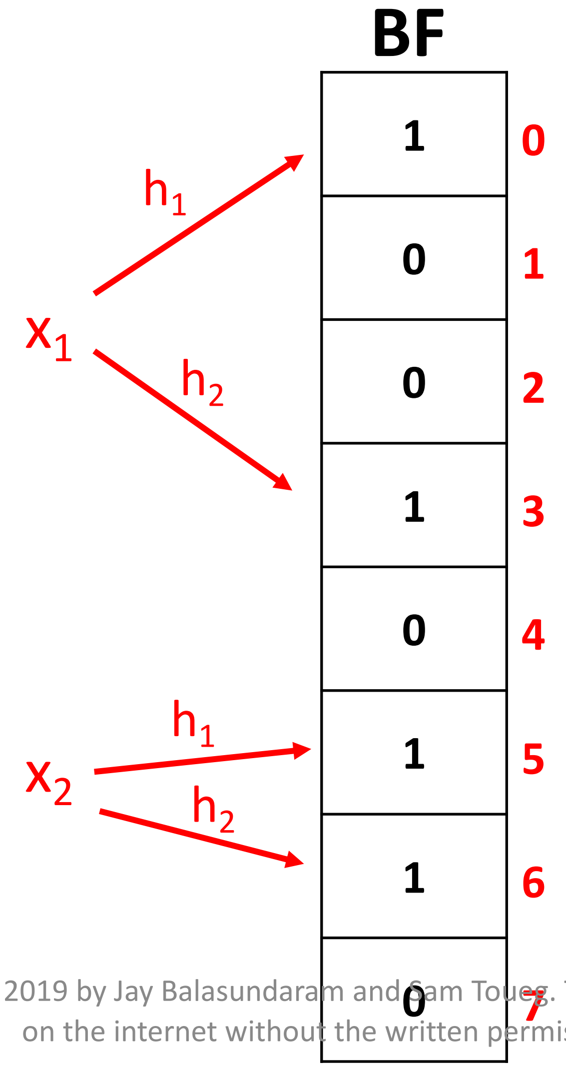

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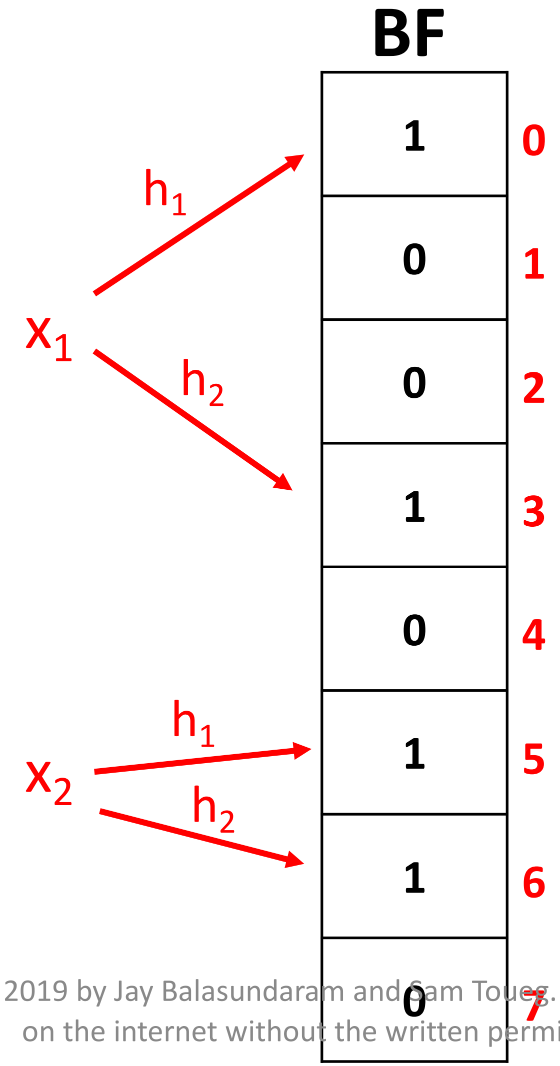

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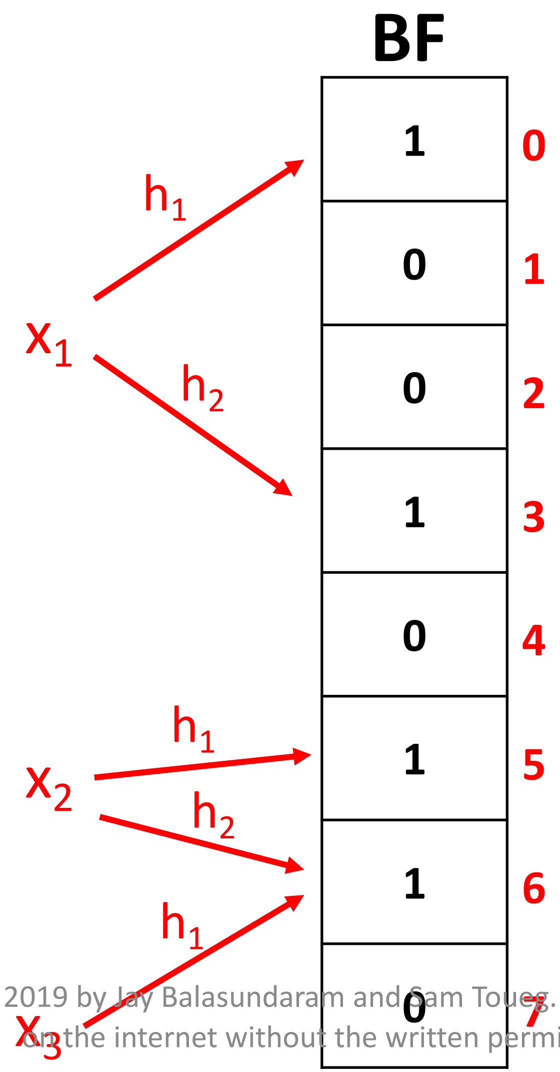

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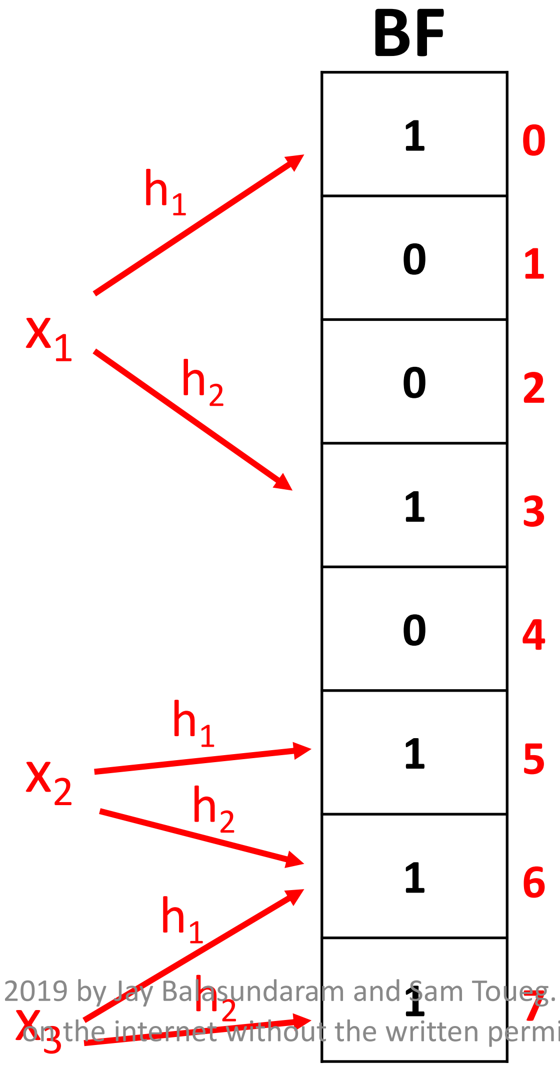

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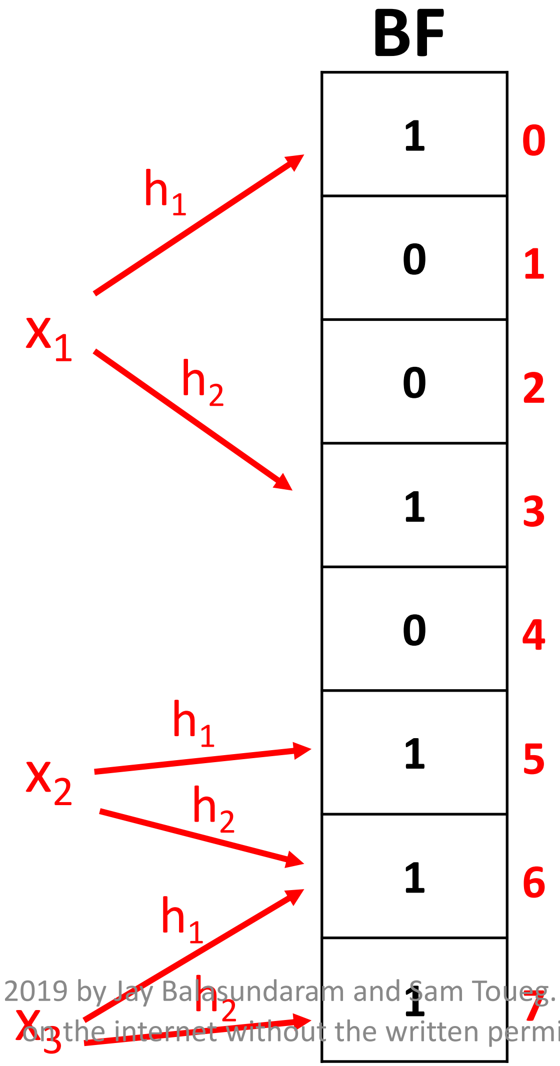

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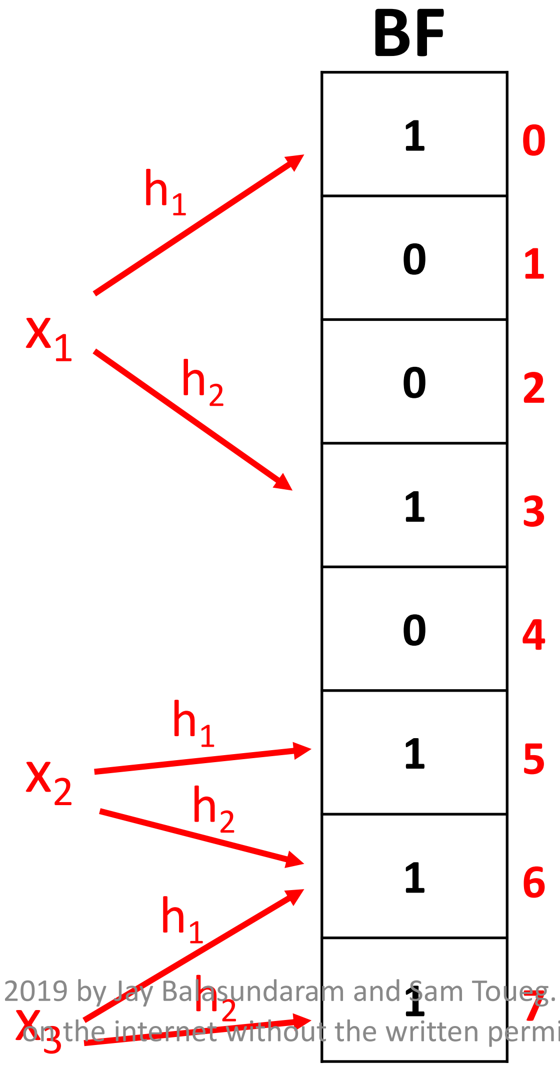

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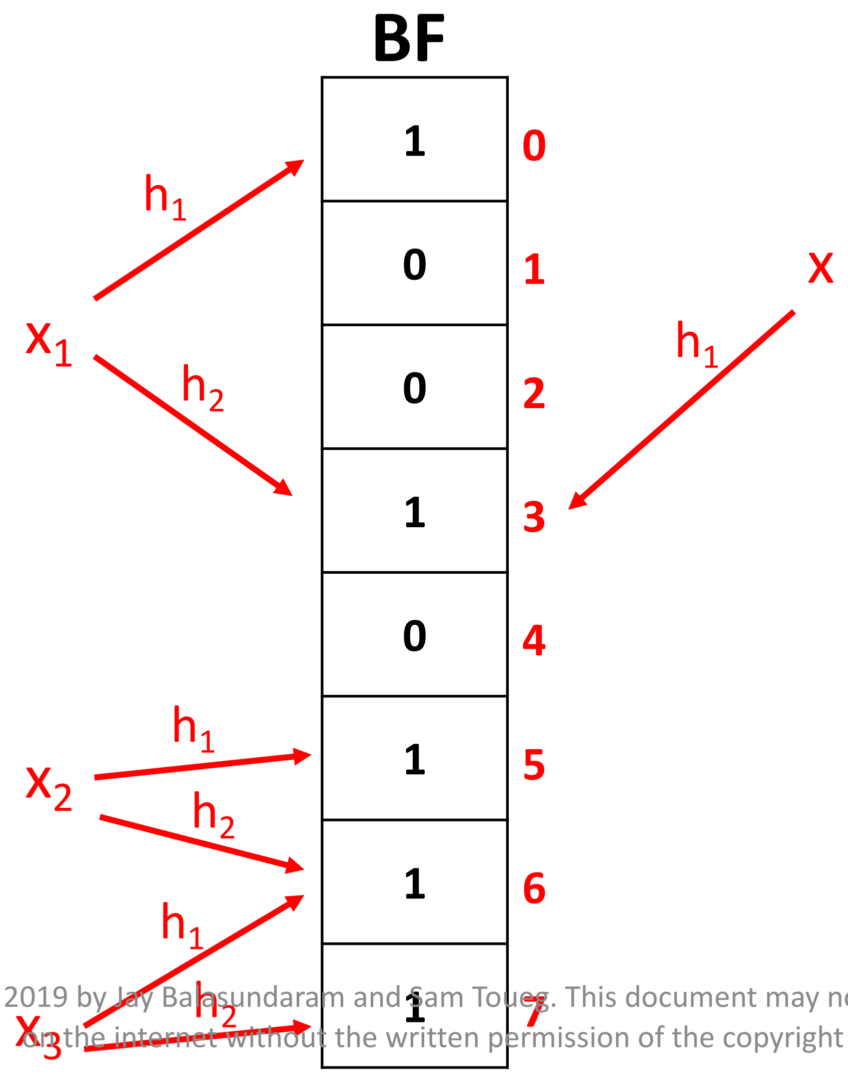

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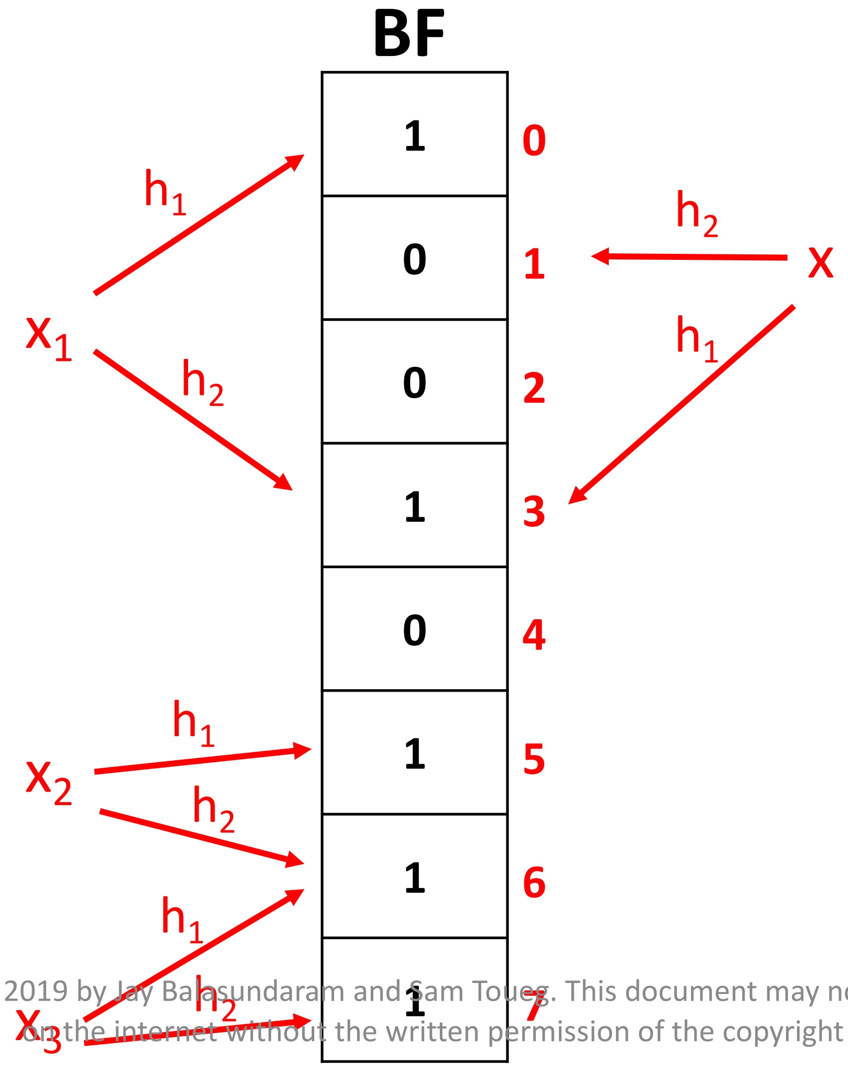

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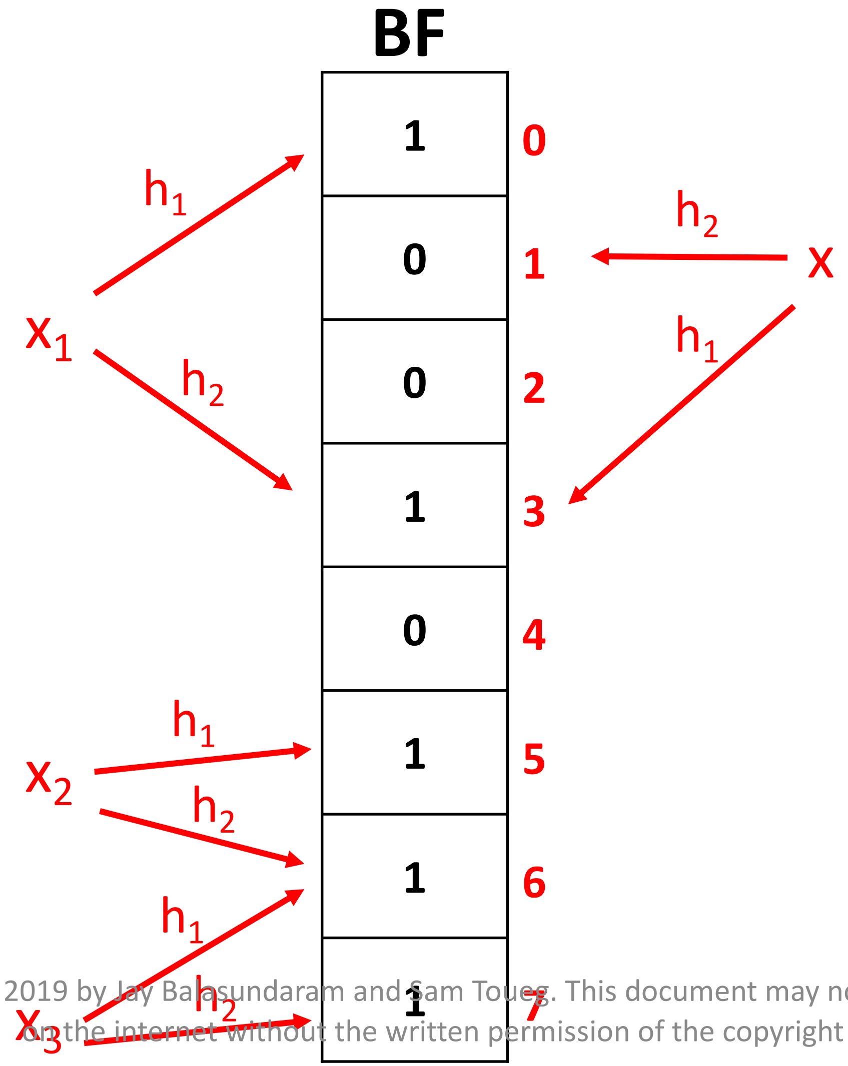

SEARCHES

BF_Search(x) = “No”

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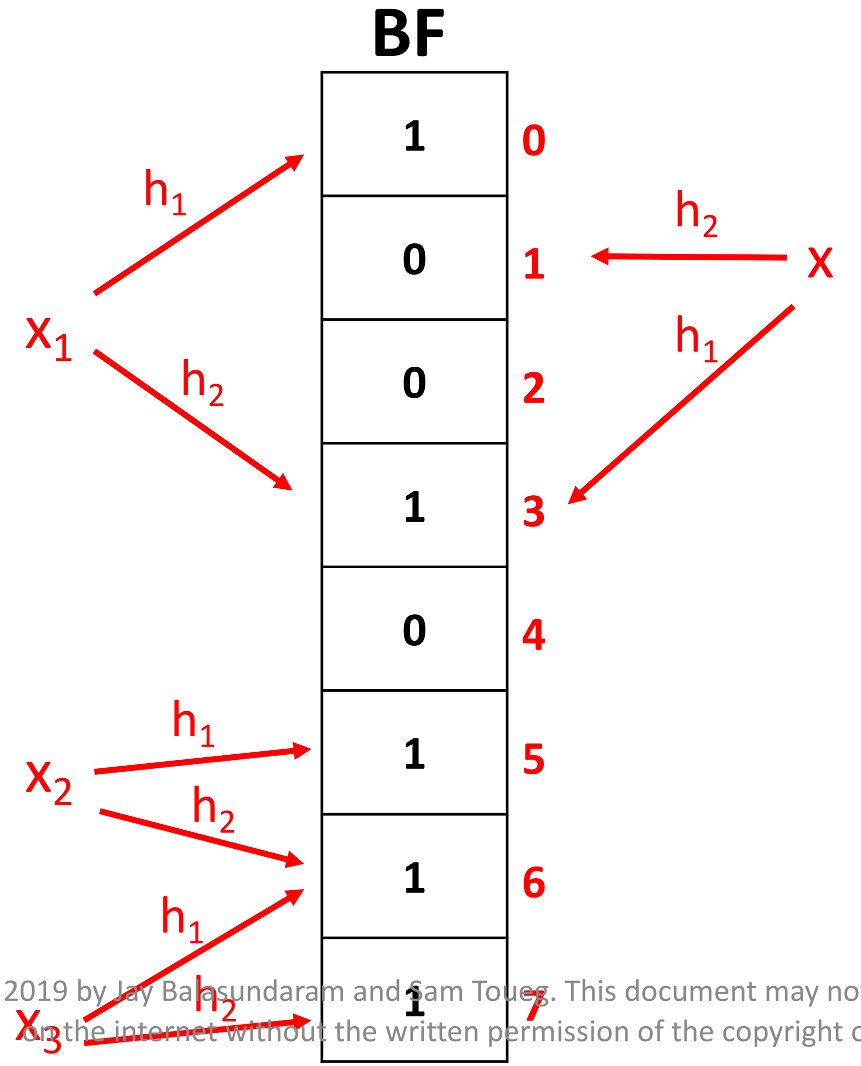

SEARCHES

BF_Search(x) = "No"

$\Rightarrow x \notin x_1, x_2, x_3$

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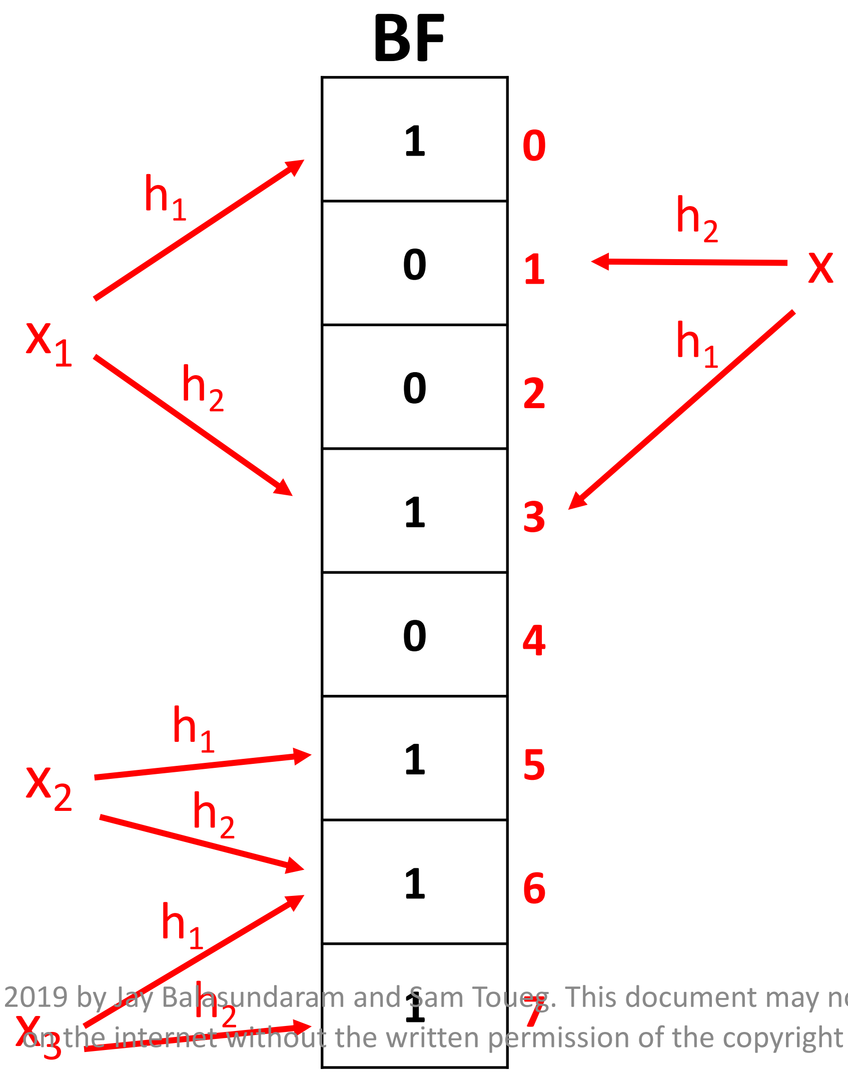

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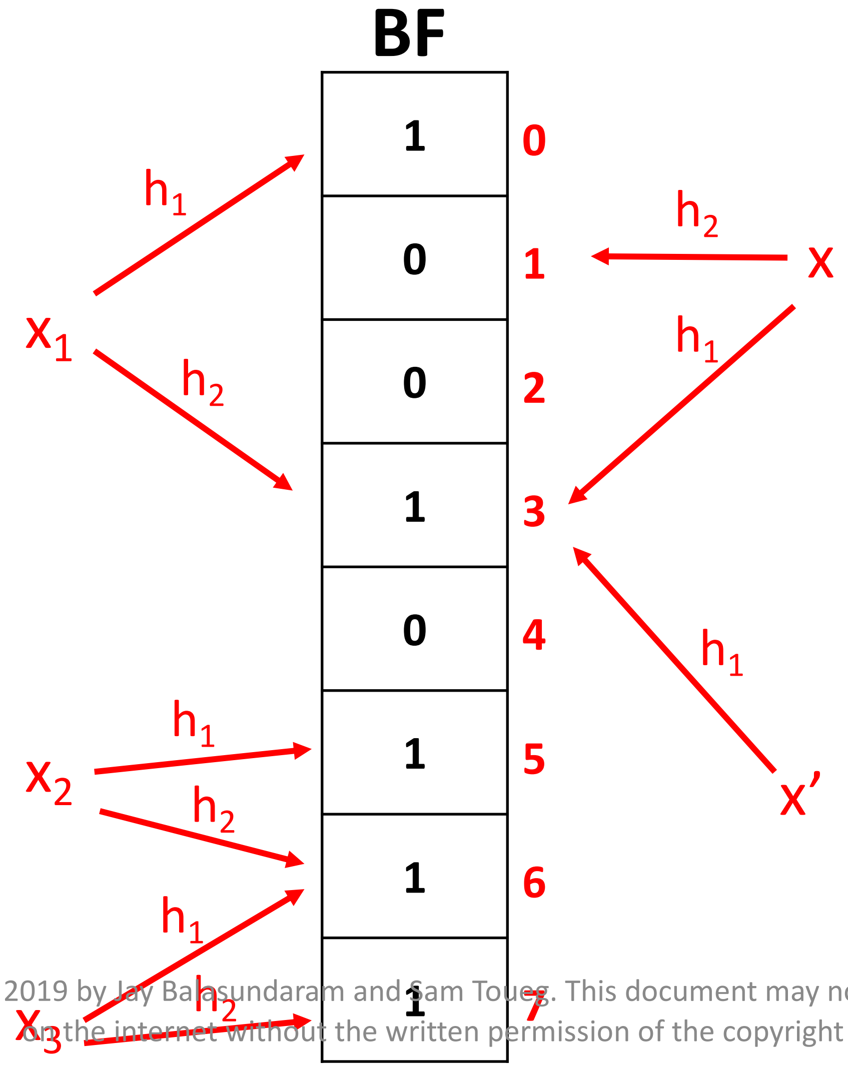

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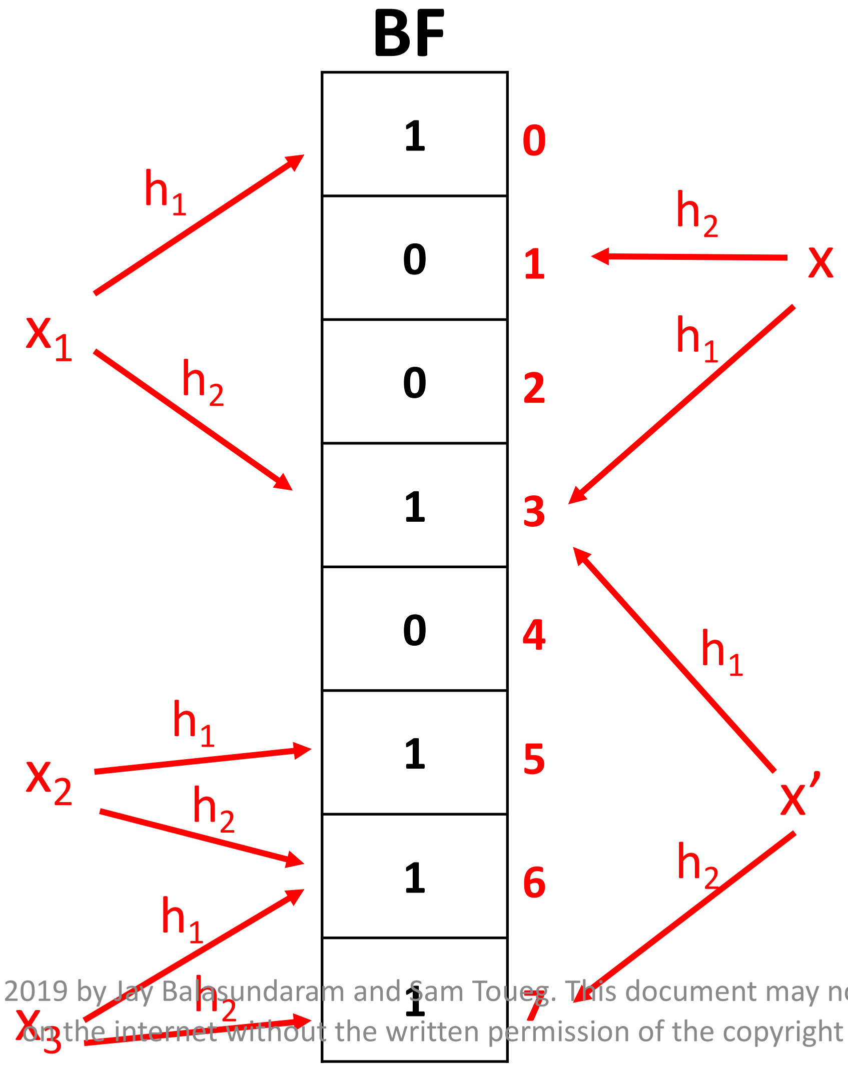

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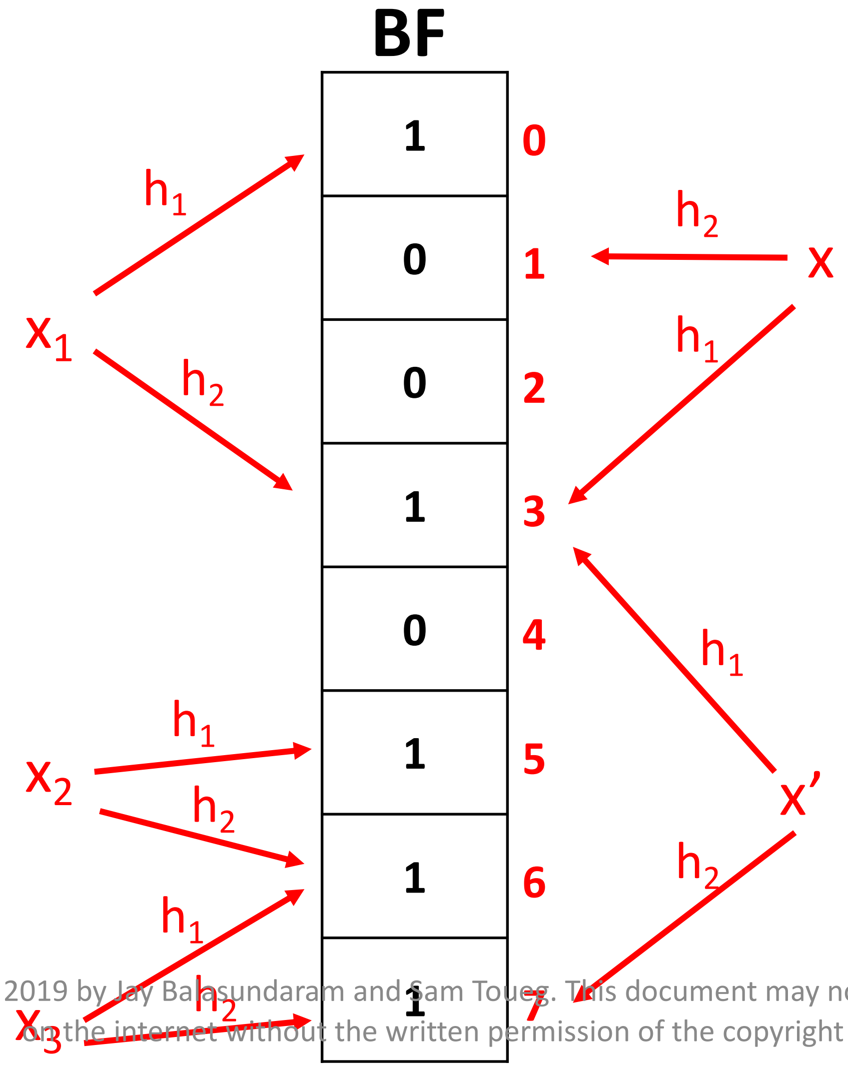

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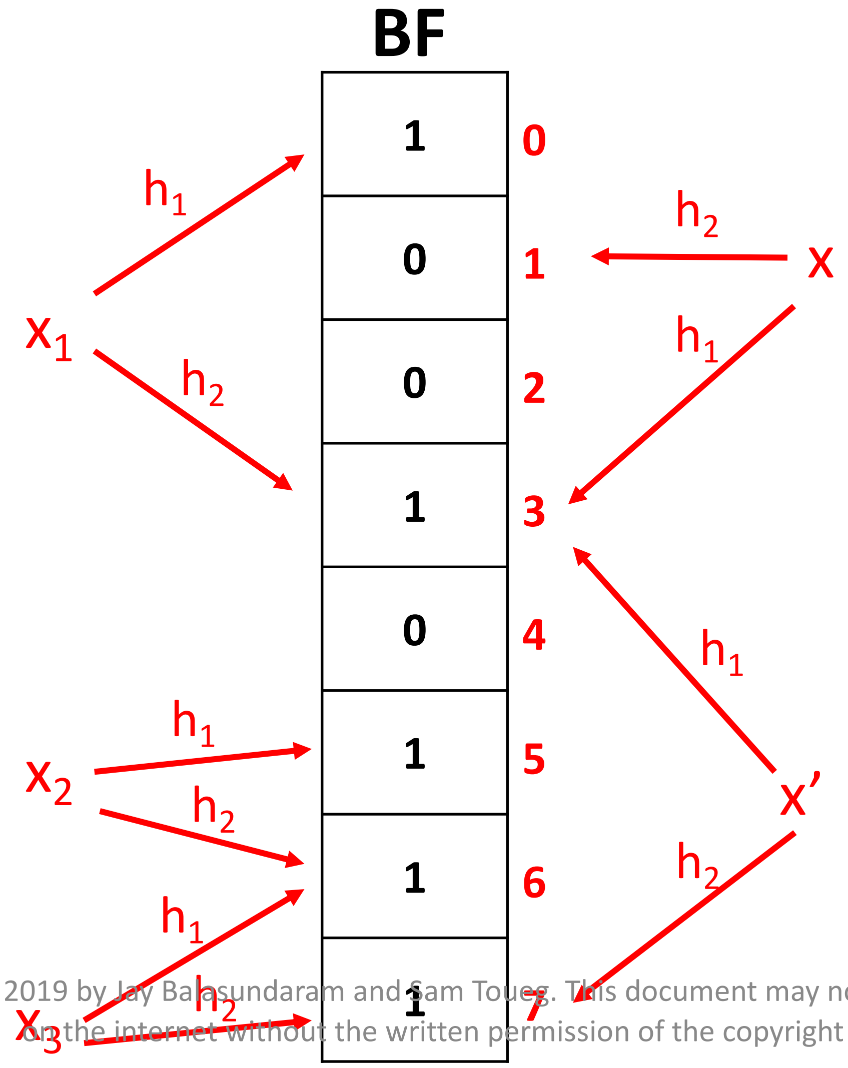

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False Positive !

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for $i = 1$ to t :

    BF[$h_i(x)$] = 1

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Probability of False Positive

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Setup:

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We would like to compute:

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We would like to compute:

- Pr[false positive] = Pr [**BF_Search**(x) = "Probably Yes"]

We first compute:

- For an arbitrary index i of **BF**, Pr[**BF**[i] = 0] after inserting x_1, x_2, …, x_n

Probability of False Positive

Consider an arbitrary index i of the **BF**

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After inserting $x_1, x_2, \ldots, x_n$,

$\Pr\big[\mathbf{BF}[i] = 0\big] =$

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$$\Pr\left[\mathbf{BF}[i] = 0\right] = \Pr\left[\bigcap_{k=1}^{n} \bigcap_{j=1}^{t} h_j(x_k) \neq i\right]$$

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By SUHA and independence of h_i s : these events are mutually independent!

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$$= \prod_{k=1}^{n} \prod_{j=1}^{t} \Pr\big[h_j(x_k) \neq i\big]$$

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$$= \prod_{k=1}^{n} \prod_{j=1}^{t} \underbrace{\Pr\big[h_j(x_k) \neq i\big]}_{1-1/m}$$

Because of SUHA !

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Consider an arbitrary index i of the **BF**

After inserting $x_1, x_2, \ldots, x_n$,

$$\Pr\Big[\mathbf{BF}[i] = 0\Big] = \Pr\Big[\bigcap_{k=1}^{n} \bigcap_{j=1}^{t} h_j(x_k) \neq i\Big]$$

$$= \prod_{k=1}^{n} \prod_{j=1}^{t} \underbrace{\Pr\Big[h_j(x_k) \neq i\Big]}_{1 - 1/m}$$

Because of SUHA !

$$= (1 - 1/m)^{nt}$$

Probability of False Positive

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$\approx (e^{-1/m})^{nt} = e^{-nt/m}$

$\Pr\big[\textbf{BF}[i] = 1\big] = 1 - \Pr\big[\textbf{BF}[i] = 0\big]$

Probability of False Positive

Consider an arbitrary index i of the **BF**

After inserting $x_1, x_2, \ldots, x_n$,

$$\Pr\left[\mathbf{BF}[i] = 0\right] = (1 - 1/m)^{nt} \qquad \left[1 - y \approx e^{-y} \quad \text{(for small } y = 1/m\text{)}\right]$$

$$\approx (e^{-1/m})^{nt} = e^{-nt/m}$$

$$\Pr\left[\mathbf{BF}[i] = 1\right] = 1 - \Pr\left[\mathbf{BF}[i] = 0\right]$$

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Lemma 1: After inserting x_1, x_2, …, x_n into a **BF** of size m with t hash functions,

Probability of False Positive

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Probability of False Positive

Lemma 1: After inserting $x_1, x_2, \ldots, x_n$ into a **BF** of size m with t hash functions,

for every index i, $\Pr\left[\mathbf{BF}[i] = 1\right] \approx \underbrace{1 - e^{-nt/m}}_{q}$

Probability of a False Positive

Do **BF_Search**(x) for $x \notin x_1, x_2, \ldots, x_n$

Pr[false positive] = Pr [**BF_Search**(x) = “Probably Yes”]

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Not independent !

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By Lemma 1 !

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By Lemma 1 !

$$\approx q^t$$

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By Lemma 1 !

$$\approx q^t = \left(1 - e^{-nt/m}\right)^t$$

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$\frac{m}{n}$

# of elements

Probability of a False Positive

Pr[false positive] ≈ $\left(1 - e^{-nt/m}\right)^{t}$

$$\frac{m}{n}$$

# of bits in BF

# of elements

Probability of a False Positive

Pr[false positive] $\approx \left(1 - e^{-nt/m}\right)^t$

# of bits in BF

$\frac{m}{n}$ (# of bits per element)

# of elements

Probability of a False Positive

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Find derivative of $\left(1 - e^{-nt/m}\right)^t$ w.r.t t and set it to 0

Probability of a False Positive

Pr[false positive] ≈ $\left(1 - e^{-nt/m}\right)^t$

Fix the ratio $\frac{m}{n}$ (# of bits per element)

Find t (i.e. # of hash functions) which minimizes Pr[false positive]

Optimal $t = (\log_e 2)\,\frac{m}{n} = 0.69\,\frac{m}{n}$

Probability of a False Positive

Pr[false positive] $\approx \left(1 - e^{-nt/m}\right)^{t}$ $= 0.62^{\frac{m}{n}}$

with optimal t

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Optimal $t = (\log_e 2)\,\frac{m}{n} = 0.69\,\frac{m}{n}$

Example

- Want a Bloom Filter for a set S of $n = 10$ Million URLs

Example

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- Can allocate 8 bits per URL

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Much less than space required to store 1 full URL

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