Amortized Analysis

Dynamic Tables

Dynamic Table

T : table

Operations: Insert, Delete items of T

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Load Factor $\alpha(T)$ =

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$$\text{Load Factor } \alpha(T) = \frac{\text{\# items currently stored in T}}{\text{size(T)}}$$

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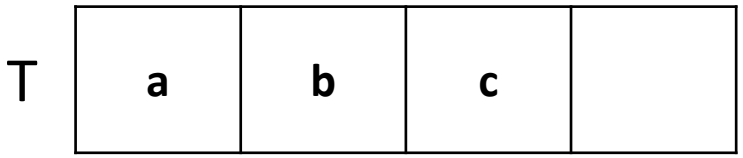

size(T) = 4

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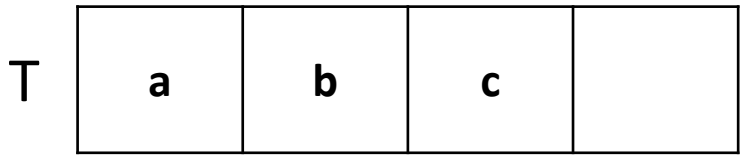

$\alpha(T) = 3/4$

size(T) = 4

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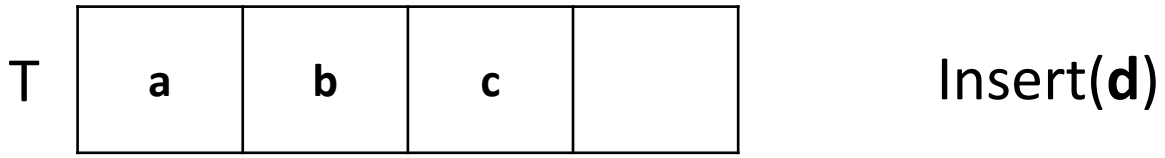

Insert(**d**)

size(T) = 4

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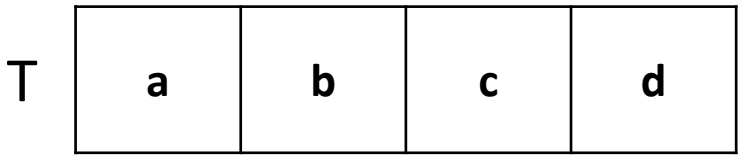
$$\text{Load Factor } \alpha(T) = \frac{\text{\# items currently stored in T}}{\text{size(T)}}$$



size(T) = 4

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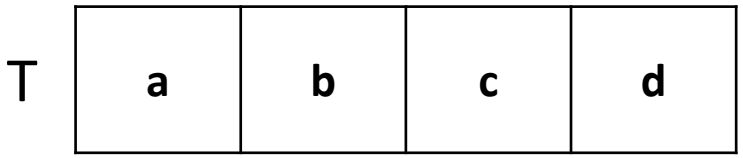
$$\text{Load Factor } \alpha(T) = \frac{\text{\# items currently stored in T}}{\text{size(T)}}$$



size(T) = 4

Table is full, i.e. $\alpha(T) = 1$

Dynamic Table

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Operations: Insert, Delete items of T

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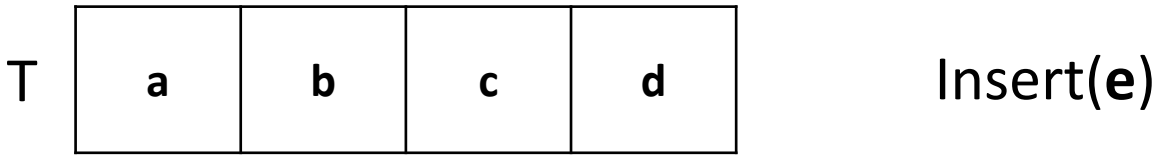

size(T) = 4

Table is full, i.e. $\alpha(T) = 1$

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Operations: Insert, Delete items of T

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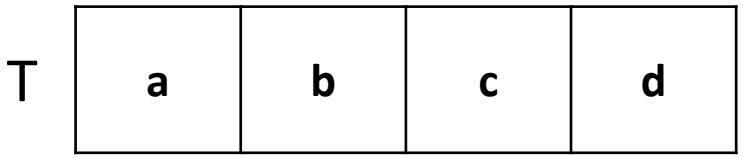

Insert(**e**)

size(T) = 4

Table is full, i.e. $\alpha(T) = 1$

Problem: Insert element when T is full

Table Expansion

Table Expansion

(1) Allocate new table T larger than T

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Typically, size(new T) = 2 size(T)

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(3) Insert the new element into the new T

Table Expansion

(1) Allocate new table T larger than T

Typically, size(new T) = 2 size(T)

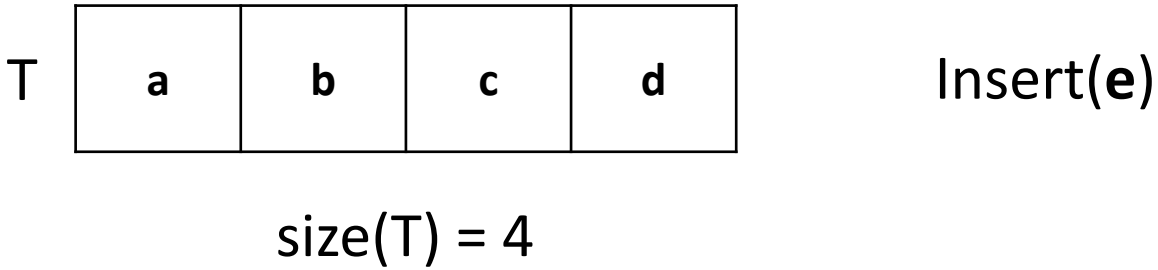
(2) Copy all items of T into the new T

(3) Insert the new element into the new T

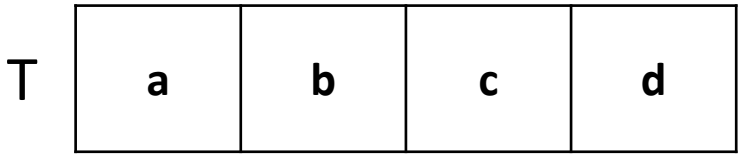
With this scheme, $\alpha(T)$ remains $\geq 1/2$

(i.e. no more than half the space of T is wasted)

Dynamic Table

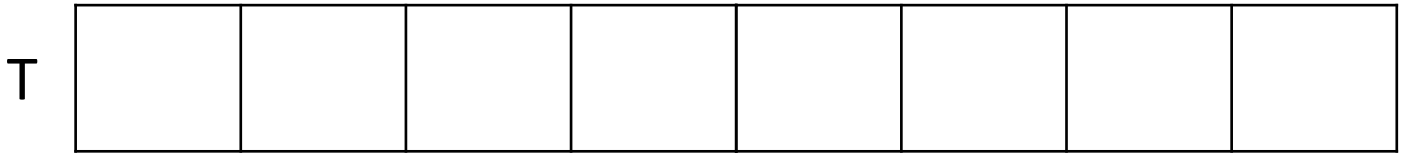


Dynamic Table



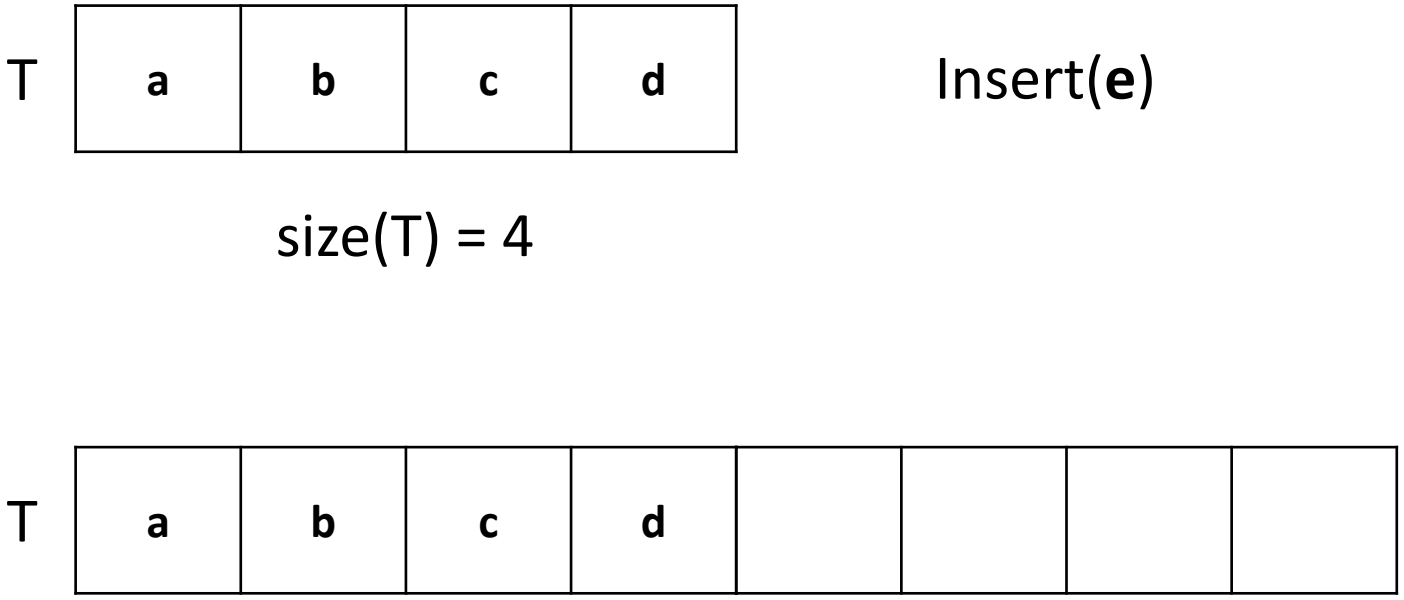
Insert(**e**)

size(T) = 4

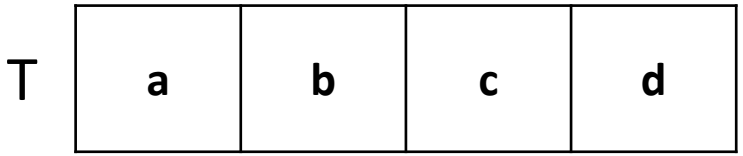


size(T) = 8

Dynamic Table



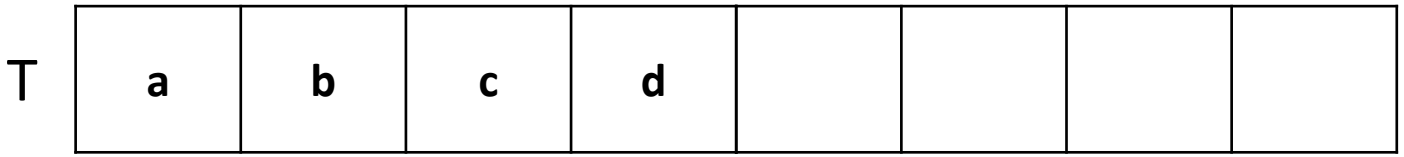
Dynamic Table



Insert(**e**)

size(T) = 4

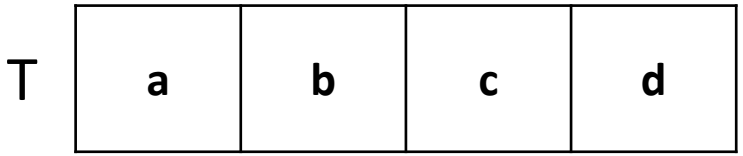
Cost of table expansion



Cost of Insert(**e**) = 4

size(T) = 8

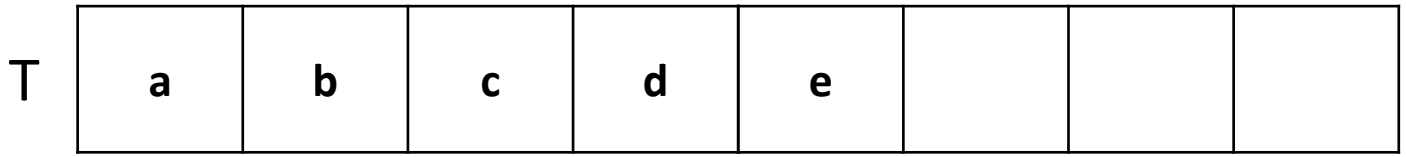
Dynamic Table



Insert(**e**)

size(T) = 4

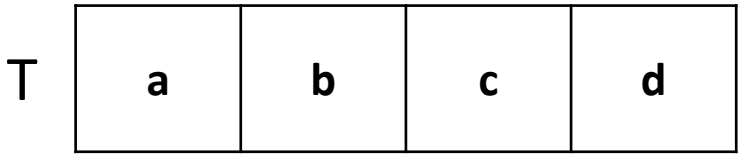
Cost of table expansion



Cost of Insert(**e**) = 4

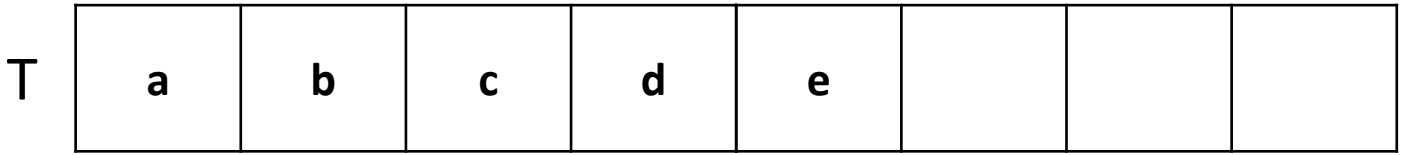
size(T) = 8

Dynamic Table



Insert(**e**)

size(T) = 4



size(T) = 8

Cost of table expansion

Cost of inserting e

Cost of Insert(**e**) = 4 + 1

Amortized Analysis

Starting from empty table T of size 1,

What is the total cost of n successive Inserts into T?

Aggregate Analysis

Example: n = 25

Aggregate Analysis

Example: n = 25

Total cost =

Aggregate Analysis

Example: n = 25

	Cost of inserting elements	Cost of table expansion
Total cost =		

Aggregate Analysis

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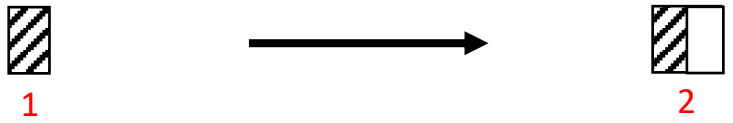
	Cost of inserting elements	Cost of table expansion
Total cost =	25	

Aggregate Analysis

Example: n = 25

	Cost of inserting elements	Cost of table expansion
Total cost =	25	



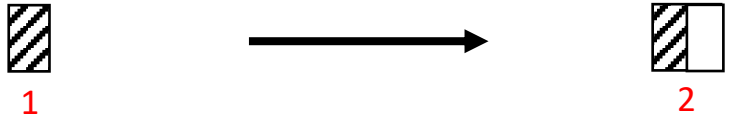
Aggregate Analysis

Example: n = 25

Cost of inserting elements

Cost of table expansion

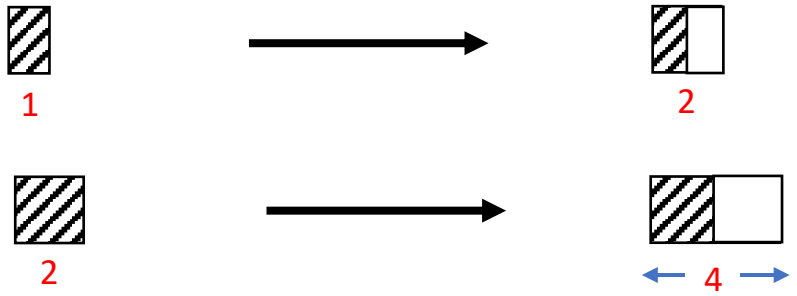
Total cost = 25 + 1



Aggregate Analysis

Example: n = 25

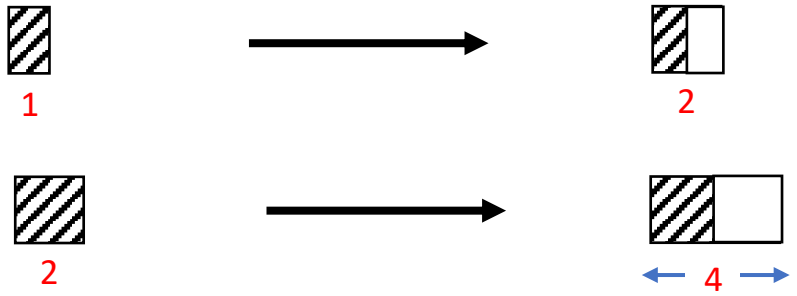
	Cost of inserting elements		Cost of table expansion
Total cost =	25	+	1



Aggregate Analysis

Example: n = 25

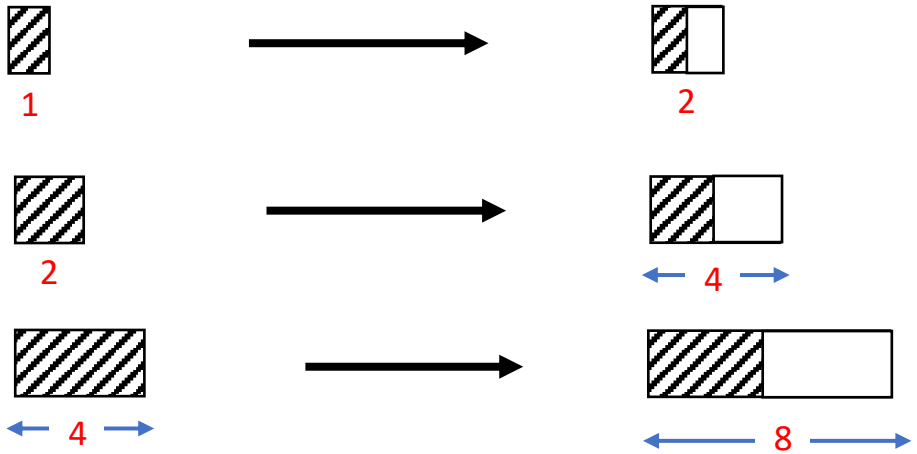
	Cost of inserting elements		Cost of table expansion
Total cost =	25	+	1
		+	2



Aggregate Analysis

Example: n = 25

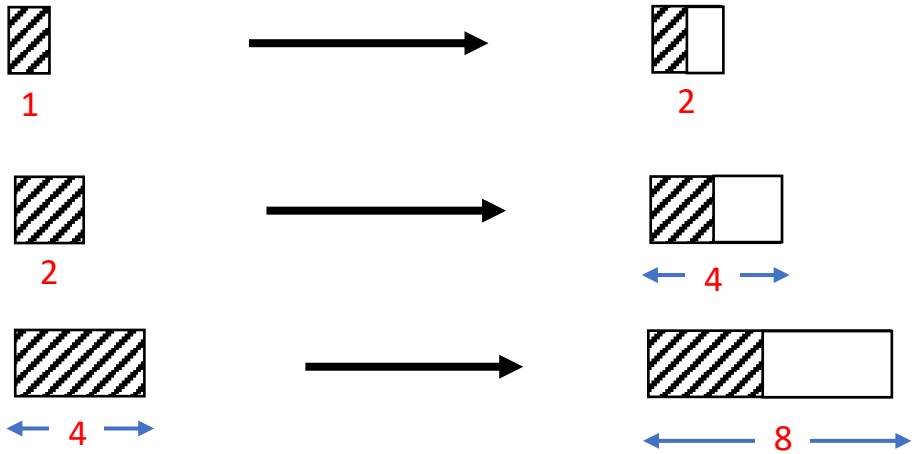
	Cost of inserting elements		Cost of table expansion
Total cost =	25	+	1
		+	2



Aggregate Analysis

Example: n = 25

	Cost of inserting elements		Cost of table expansion
Total cost =	25	+	1
		+	2
		+	4



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Example: n = 25

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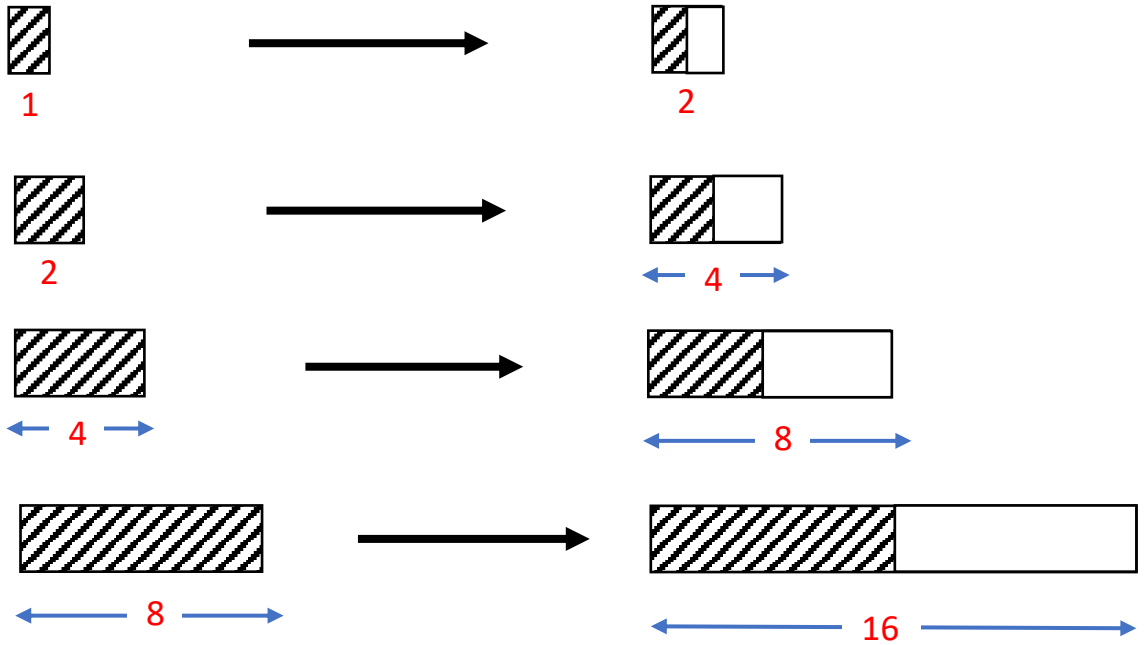
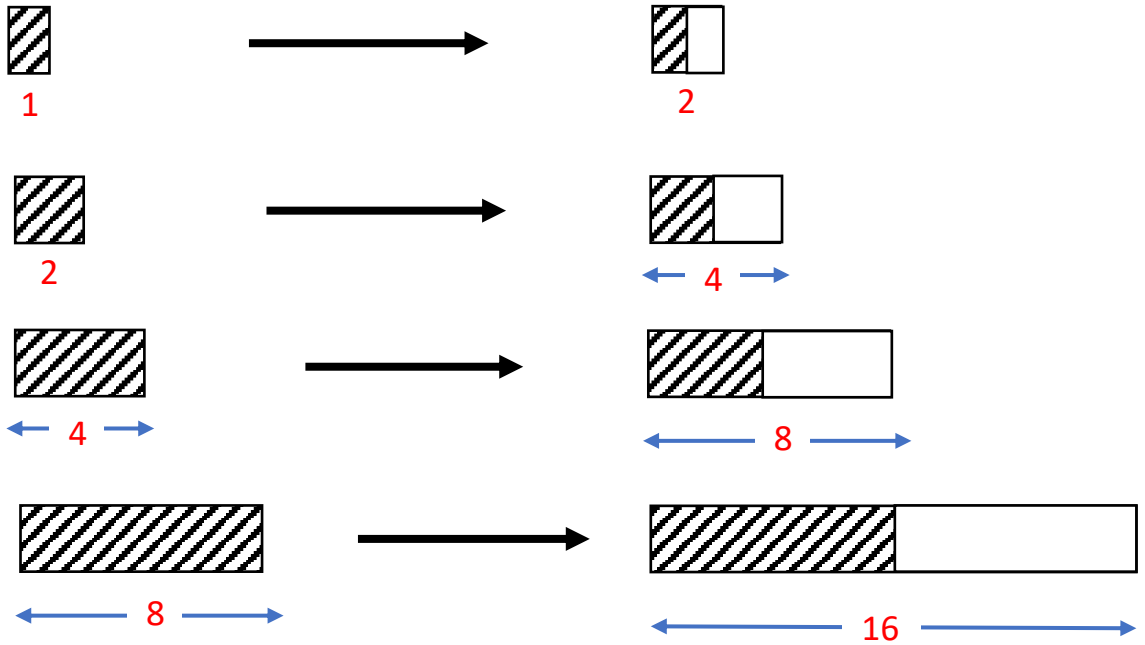
Total cost =	25	+	1
		+	2
		+	4



Aggregate Analysis

Example: n = 25

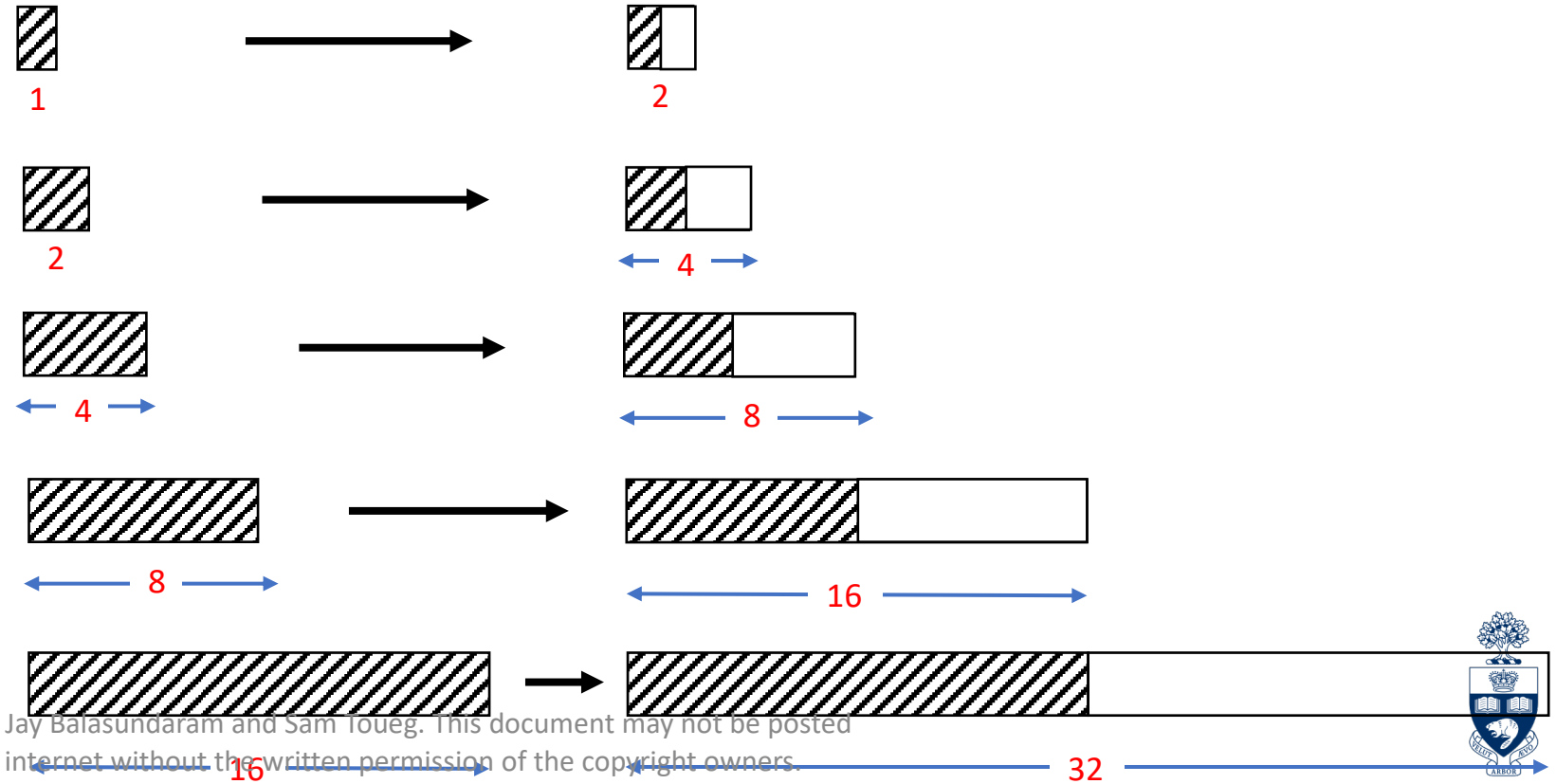
	Cost of inserting elements	Cost of table expansion
Total cost =	25	+ 1
		+ 2
		+ 4
		+ 8



Aggregate Analysis

Example: n = 25

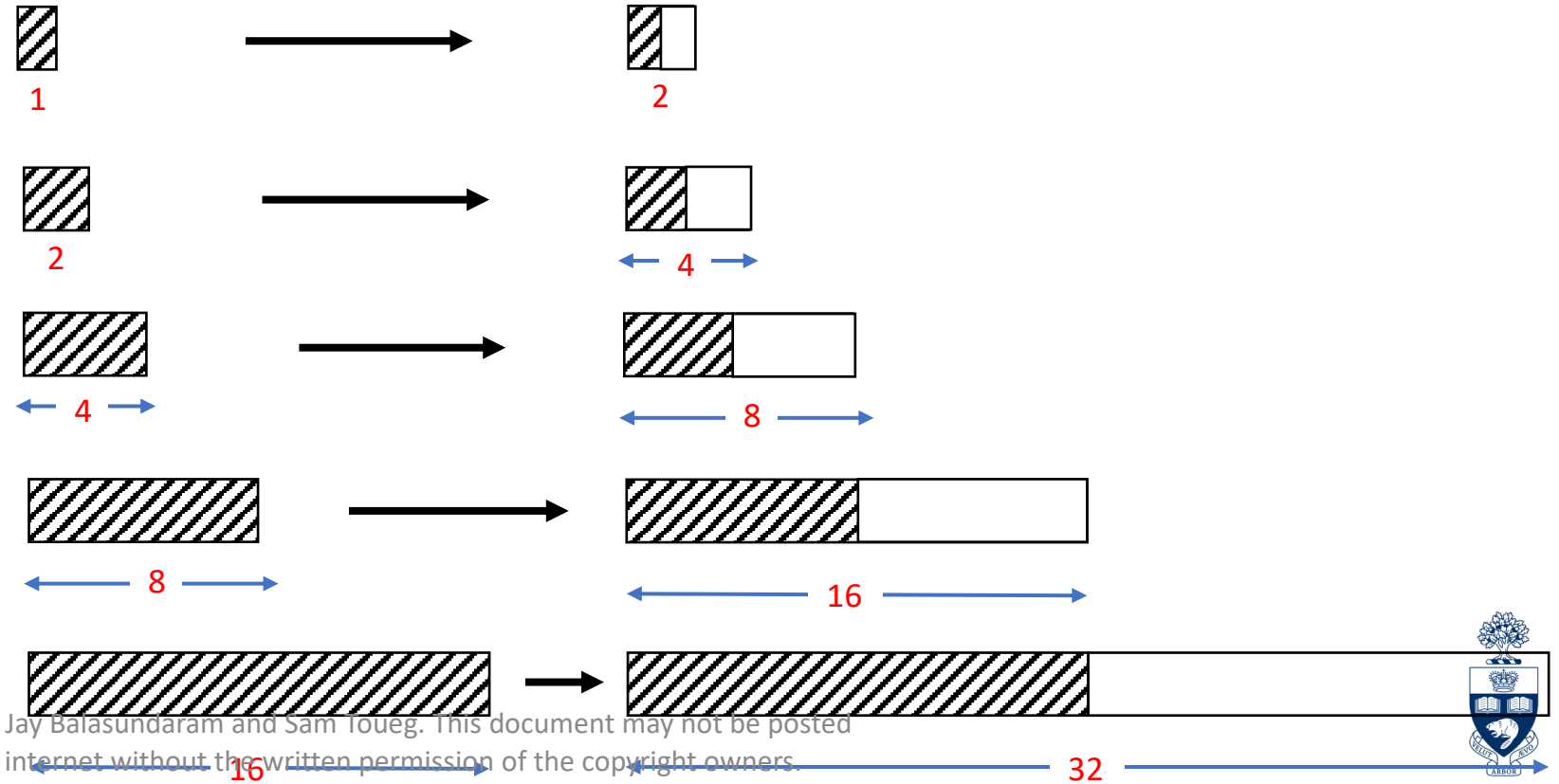
	Cost of inserting elements	Cost of table expansion
Total cost =	25	+ 1
		+ 2
		+ 4
		+ 8
		+ 16
		+ 32



Aggregate Analysis

Example: n = 25

	Cost of inserting elements	Cost of table expansion
Total cost =	25	+ 1
		+ 2
		+ 4
		+ 8
		+ 16



Aggregate Analysis

Total cost of 25 Inserts = 25 + all powers of 2 smaller than 25

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Total cost of n Inserts = n + all powers of 2 smaller than n

Aggregate Analysis

Total cost of 25 Inserts = 25 + all powers of 2 smaller than 25

Total cost of n Inserts = n + all powers of 2 smaller than n

Total cost of n Inserts $\leq n + \sum_{k=0} 2^k$

Aggregate Analysis

Total cost of 25 Inserts = 25 + all powers of 2 smaller than 25

Total cost of n Inserts = n + all powers of 2 smaller than n

$$\text{Total cost of } n \text{ Inserts} \leq n + \sum_{k=0}^{k=\lfloor \log_2 n \rfloor} 2^k$$

Aggregate Analysis

Total cost of 25 Inserts = 25 + all powers of 2 smaller than 25

Total cost of n Inserts = n + all powers of 2 smaller than n

$$\text{Total cost of } n \text{ Inserts} \leq n + \sum_{k=0}^{k=\lfloor \log_2 n \rfloor} 2^k = n + (2^{\lfloor \log_2 n \rfloor + 1} - 1)$$

Aggregate Analysis

Total cost of 25 Inserts = 25 + all powers of 2 smaller than 25

Total cost of n Inserts = n + all powers of 2 smaller than n

$$\text{Total cost of } n \text{ Inserts} \leq n + \sum_{k=0}^{k=\lfloor \log_2 n \rfloor} 2^k = n + (2^{\lfloor \log_2 n \rfloor + 1} - 1) \leq n + 2 \times 2^{\lfloor \log_2 n \rfloor}$$

Aggregate Analysis

Total cost of 25 Inserts = 25 + all powers of 2 smaller than 25

Total cost of n Inserts = n + all powers of 2 smaller than n

$$\text{Total cost of } n \text{ Inserts} \leq n + \sum_{k=0}^{k=\lfloor \log_2 n \rfloor} 2^k = n + (2^{\lfloor \log_2 n \rfloor + 1} - 1) \leq n + 2 \times 2^{\lfloor \log_2 n \rfloor}$$

$$\leq n + 2n$$

Aggregate Analysis

Total cost of 25 Inserts = 25 + all powers of 2 smaller than 25

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$$\leq n + 2n$$

$$= 3n$$

Aggregate Analysis

Total cost of 25 Inserts = 25 + all powers of 2 smaller than 25

Total cost of n Inserts = n + all powers of 2 smaller than n

$$\text{Total cost of } n \text{ Inserts} \leq n + \sum_{k=0}^{k=\lfloor \log_2 n \rfloor} 2^k = n + (2^{\lfloor \log_2 n \rfloor + 1} - 1) \leq n + 2 \times 2^{\lfloor \log_2 n \rfloor}$$

$$\leq n + 2n$$

$$= 3n$$

Amortized cost per Insert $\leq$ $3n/n = 3$

Aggregate Analysis

Total cost of 25 Inserts = 25 + all powers of 2 smaller than 25

Total cost of n Inserts = n + all powers of 2 smaller than n

$$\text{Total cost of } n \text{ Inserts} \leq n + \sum_{k=0}^{k=\lfloor \log_2 n \rfloor} 2^k = n + (2^{\lfloor \log_2 n \rfloor + 1} - 1) \leq n + 2 \times 2^{\lfloor \log_2 n \rfloor}$$

$$\leq n + 2n$$

$$= 3n$$

Amortized cost per Insert $\leq$ $3n/n = 3$ $\Rightarrow$ Amortized cost per Insert is $O(1)$

Accounting method

Recall that if:

c_i : actual cost of i^{th} operation

$\hat{c}_i$: cost charged for the i^{th} operation [i.e. amortized cost of i^{th} operation]

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We require $\sum_{i=1}^{n} \hat{c}_i \geq \sum_{i=1}^{n} c_i$ for all sequence of n operations

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Equivalently, $\sum_{i=1}^{n} \hat{c}_i - \sum_{i=1}^{n} c_i \geq 0$

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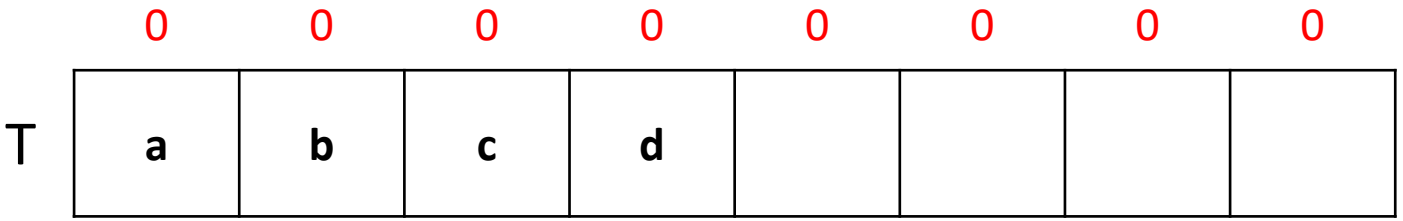
Equivalently, $\sum_{i=1}^{n} \hat{c}_i - \sum_{i=1}^{n} c_i \geq 0$

Total credit in the data structure

Accounting method

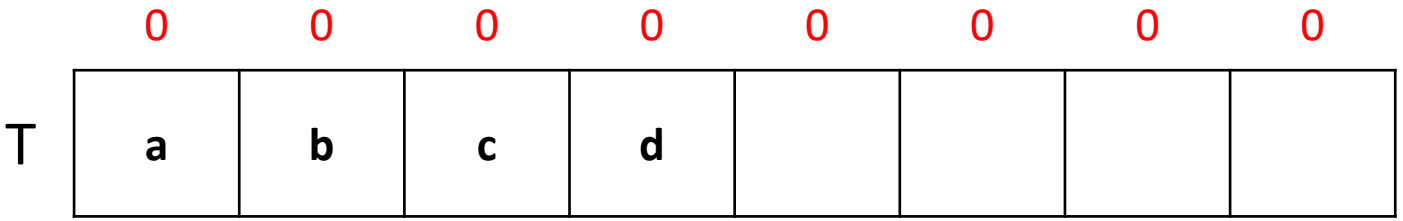
Insert(x)



Accounting method

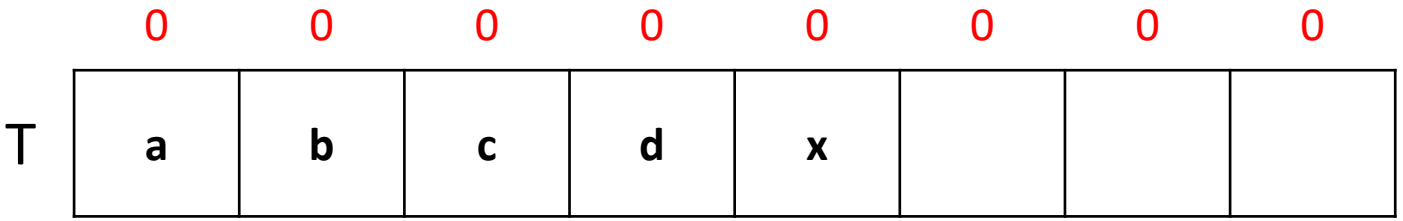
Insert(**x**)

Insert(**x**) is charged



Accounting method

Insert(x)

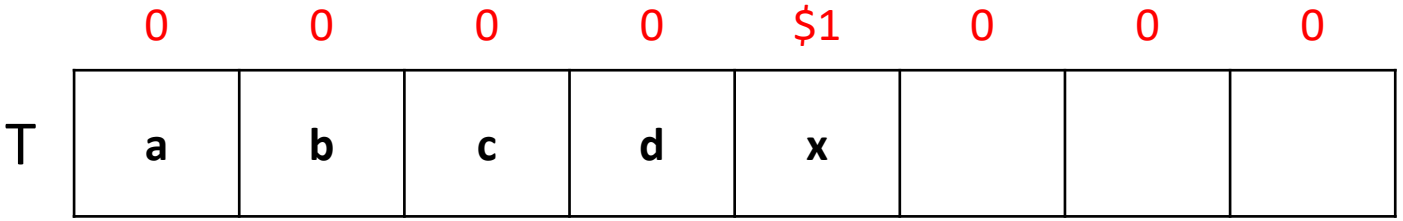


Insert(x) is charged

$1 for inserting **x** (actual cost)

Accounting method

Insert(x)

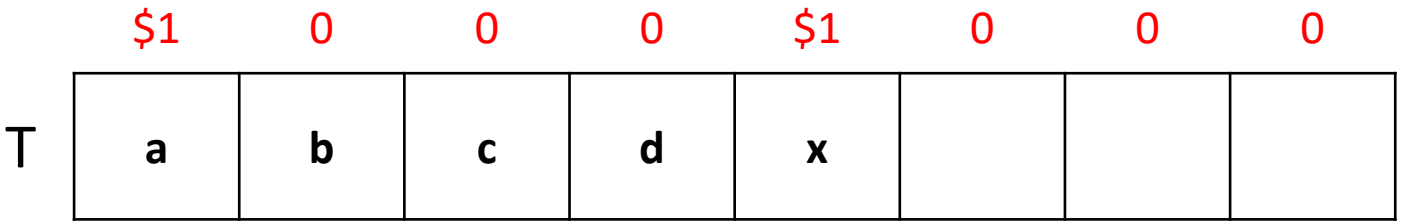


Insert(x) is charged

$1 for inserting **x** (actual cost)
+ $1 credit on **x** (for copying **x** over)

Accounting method

Insert(x)

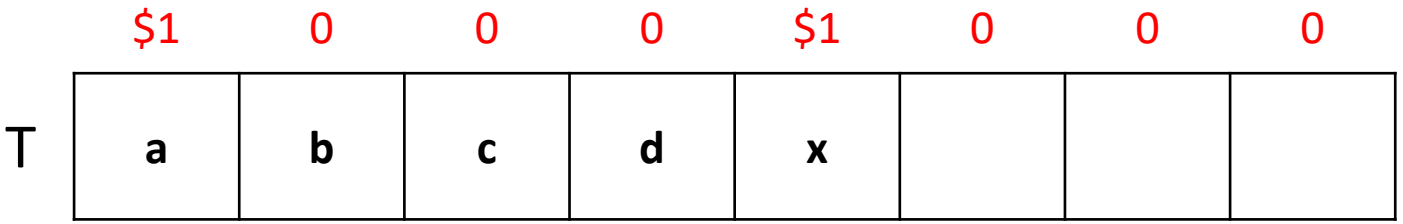


Insert(x) is charged

$1 for inserting **x** (actual cost)
+ $1 credit on **x** (for copying **x** over)
+ $1 credit on **a** (for copying **a** over)

Accounting method

Insert(x)



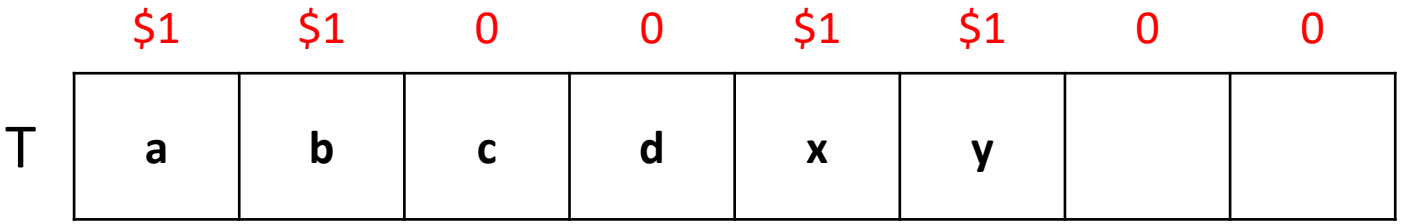
Insert(x) is charged $3

$1 for inserting **x** (actual cost)
+ $1 credit on **x** (for copying **x** over)
+ $1 credit on **a** (for copying **a** over)

Accounting method

Insert(**y**)

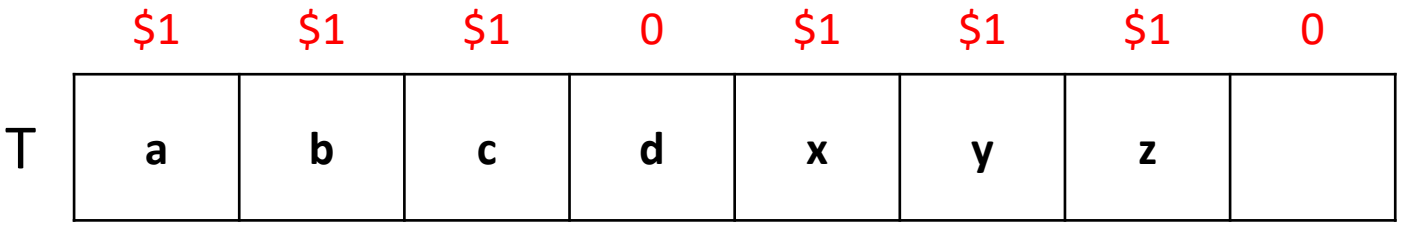
Insert(**y**) is charged $3



Accounting method

Insert(z)

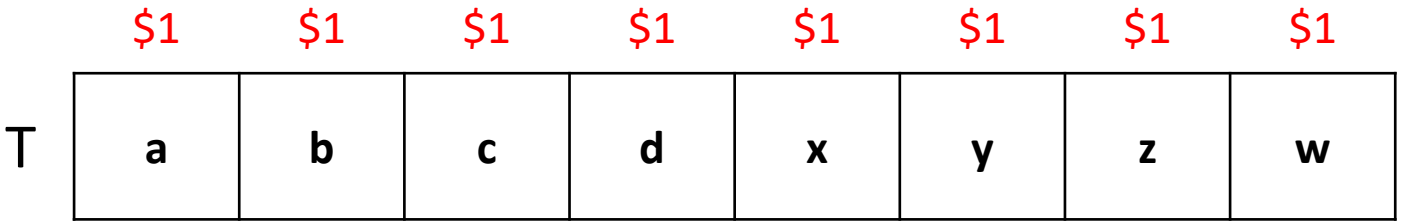
Insert(z) is charged $3



Accounting method

Insert(**w**)

Insert(**w**) is charged $3



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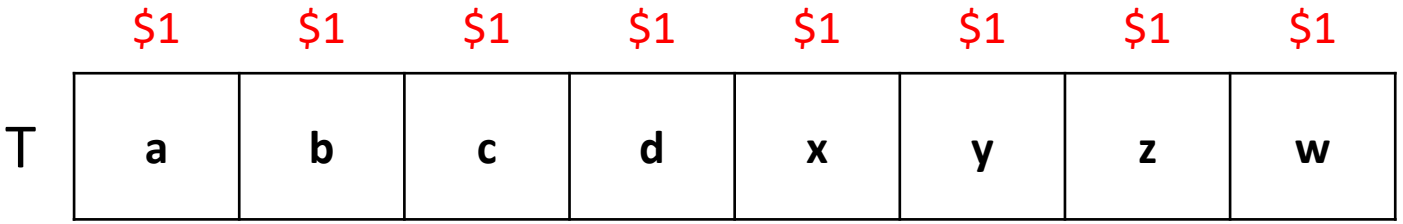
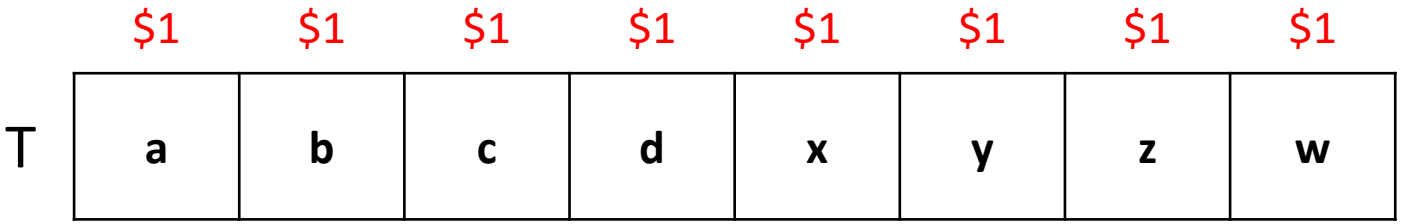


When table full,

Total Credit = # elements in the Table

Accounting method



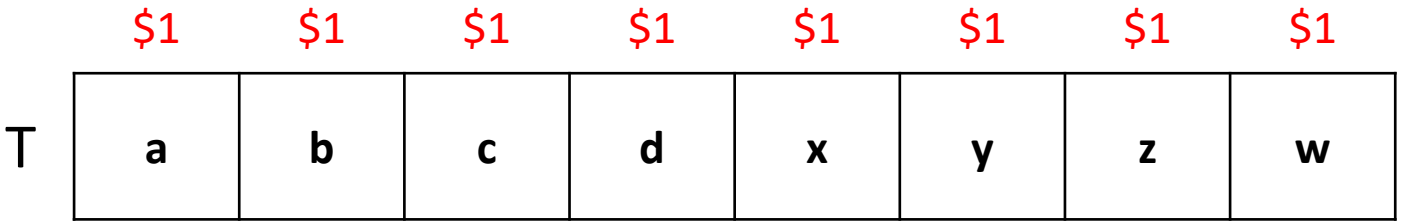
When table full,

Total Credit = # elements in the Table

On next Insert, use credits to move elements into new table

Accounting method

Insert(v)



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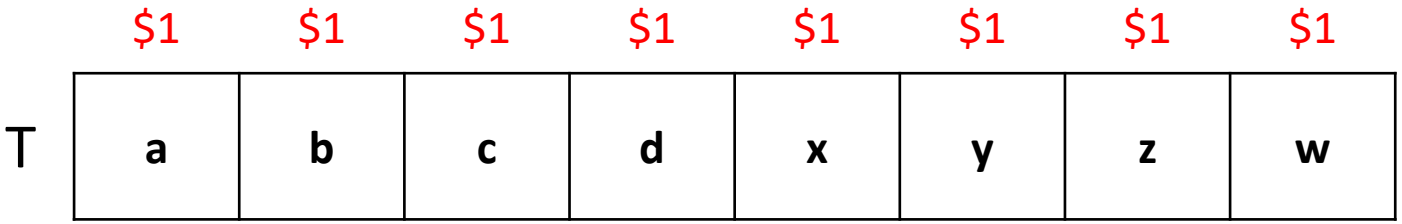
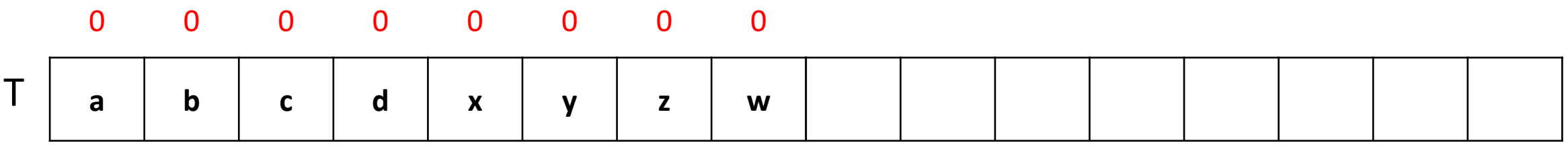
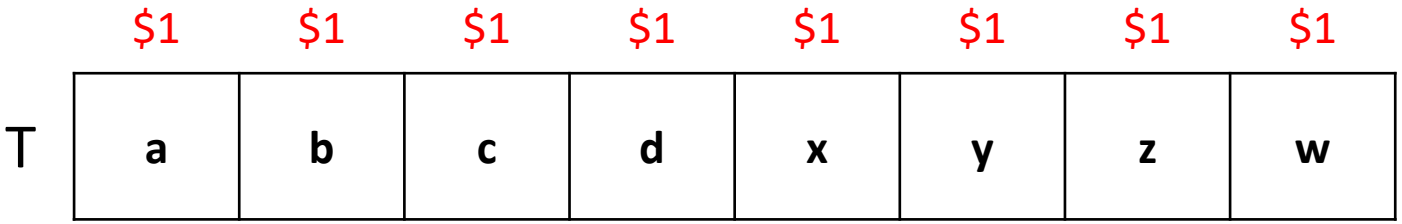


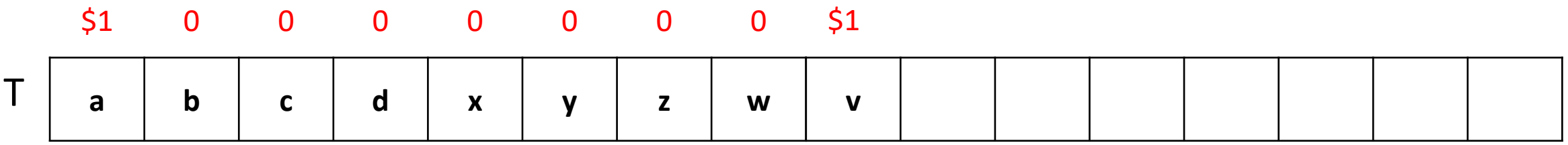


Accounting method

Insert(**v**)

Insert(**v**) is charged $3



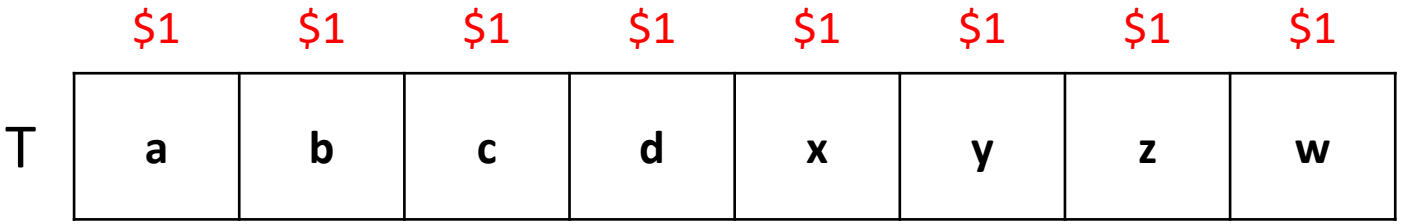


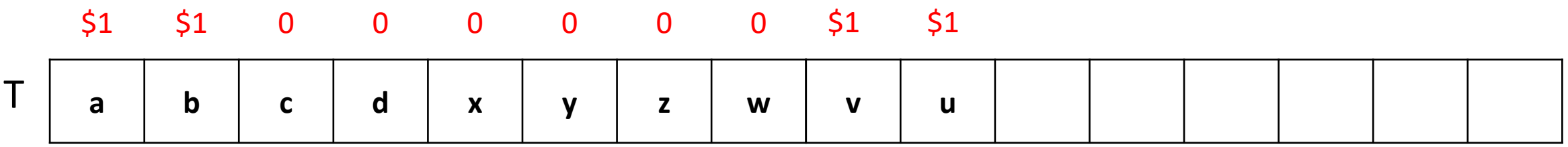
Accounting method

Insert(v)

Insert(v) is charged $3

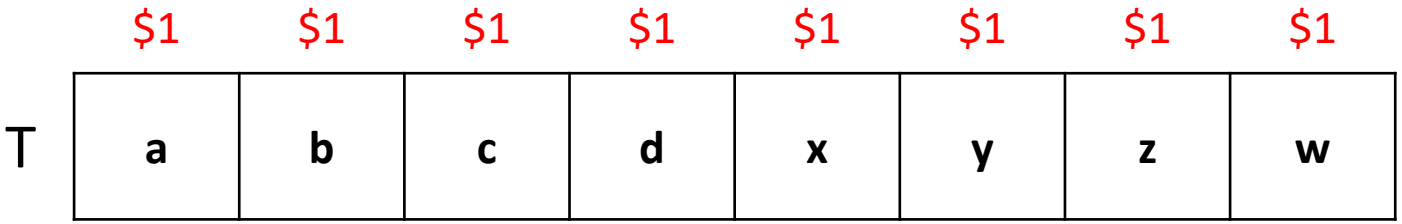
Insert(u) is charged $3





Accounting method

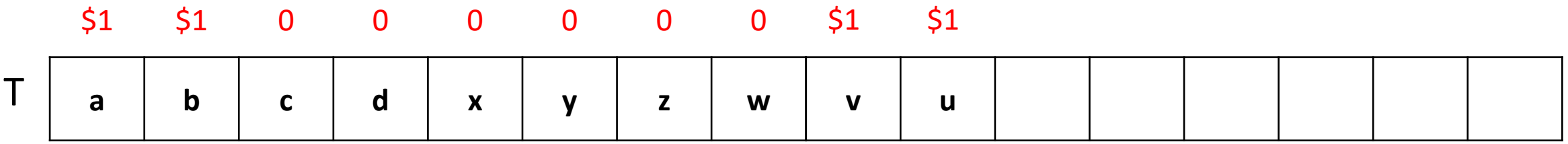
Insert(v)



Insert(v) is charged $3

Insert(u) is charged $3

and so on…



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σ : Sequence of n Inserts, starting from empty table T of size 1

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Charging \$3 per Insert in σ
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Charging \$3 per Insert in σ ensures total credit is always ≥ 0

$\Rightarrow$ Amortized cost per Insert is O(1)

Insert and Delete

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Problem: After deleting items, α(T) decreases

Insert and Delete

Problem: After deleting items, $\alpha(T)$ decreases $\Rightarrow$ Memory waste increases

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(1) Allocating new smaller table,

(2) Copying all the items to the new table

Insert and Delete

Problem: After deleting items, $\alpha(T)$ decreases ⇒ Memory waste increases

How small ?

If $\alpha(T)$ becomes too small, reduce memory waste by

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Insert and Delete

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We want:

(1) $\alpha(T) \geq$ constant c (to reduce memory waste)

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How small ?

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We want:

(1) $\alpha(T) \geq$ constant c (to reduce memory waste)

(2) Amortized cost per operation (insert/delete) is O(1)

Insert and Delete: Naïve approach

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Insert : If α(T) = 1, and Insert occurs, size(new T) = 2 size(T)

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Approach ensures $\alpha(T) \geq 1/2$

σ : Arbitrary sequence of n Inserts and Deletes, starting from empty table T of size 1

What is the amortized cost per operation in σ ?

Naïve approach : Bad sequence σ of $n = 2^k$ operations

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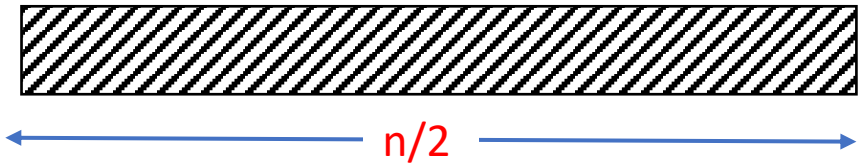
σ : n/2 Inserts,

T



Naïve approach : Bad sequence σ of $n = 2^k$ operations

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Cost : $\geq n/2$

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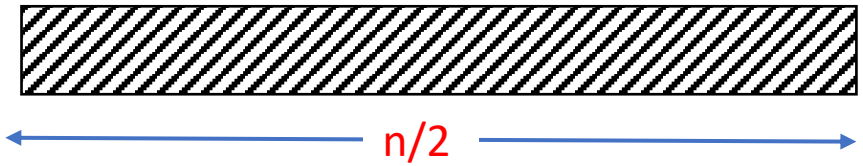

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σ : n/2 Inserts, Insert,

Cost : $\geq n/2$

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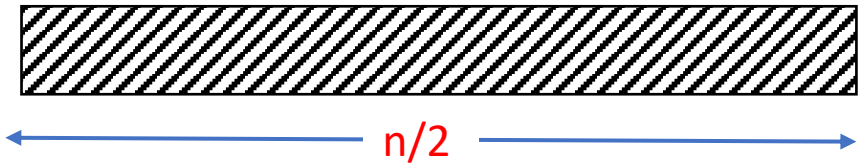

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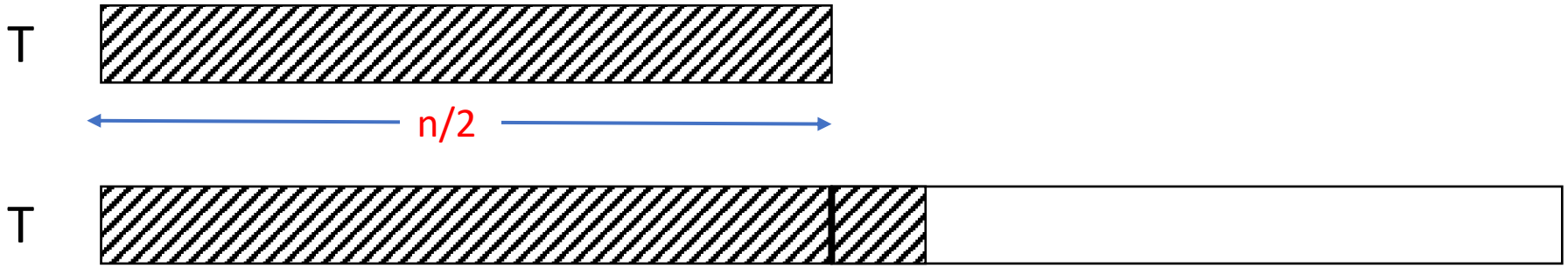

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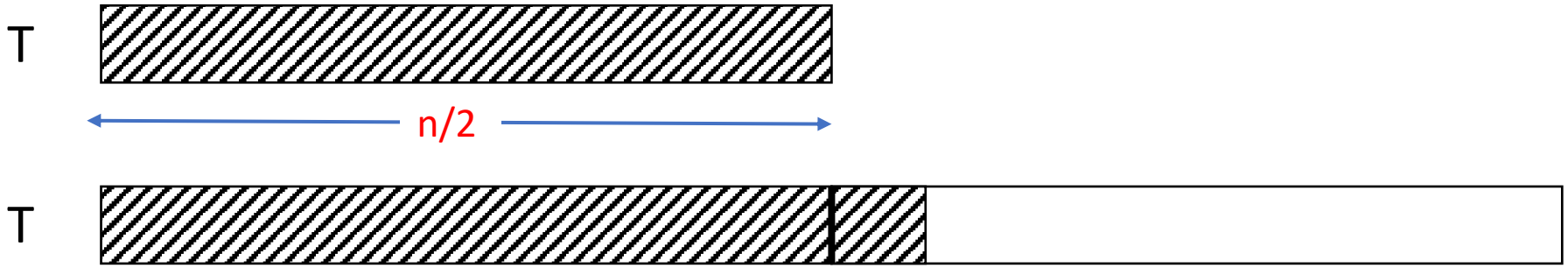

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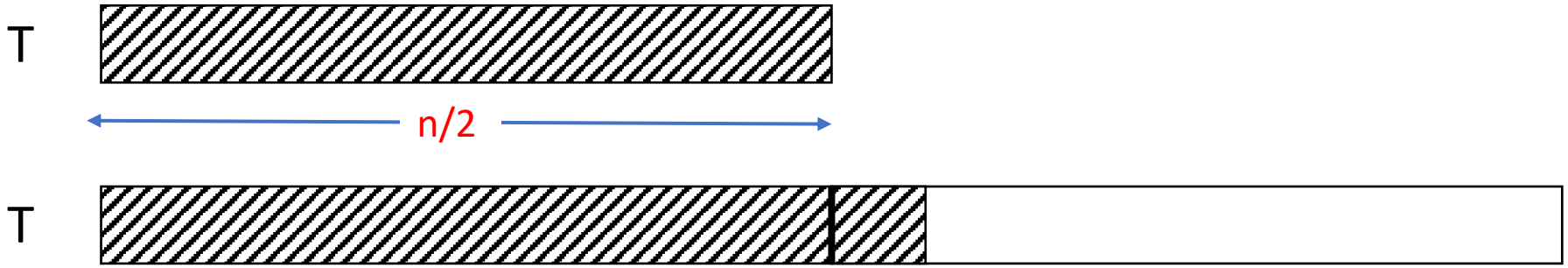

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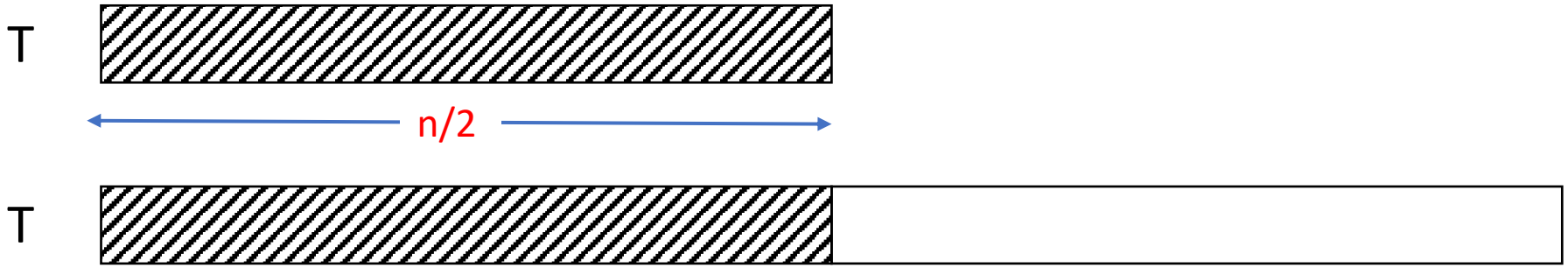

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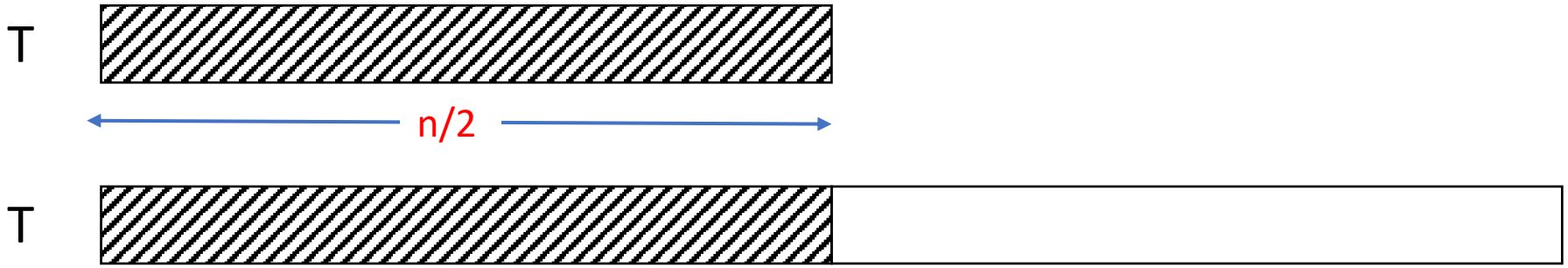

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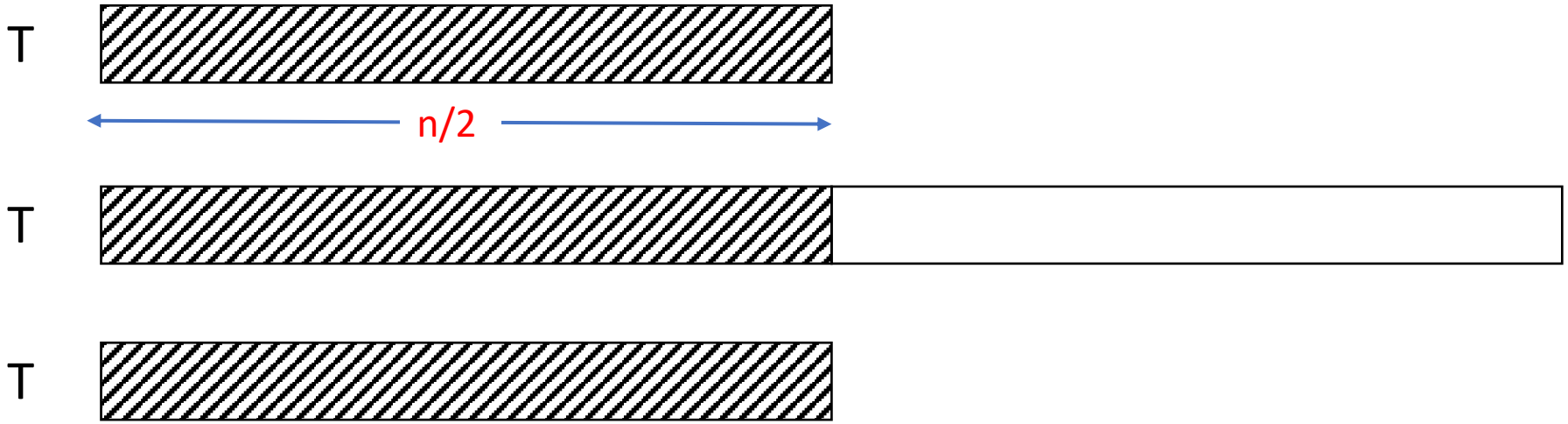

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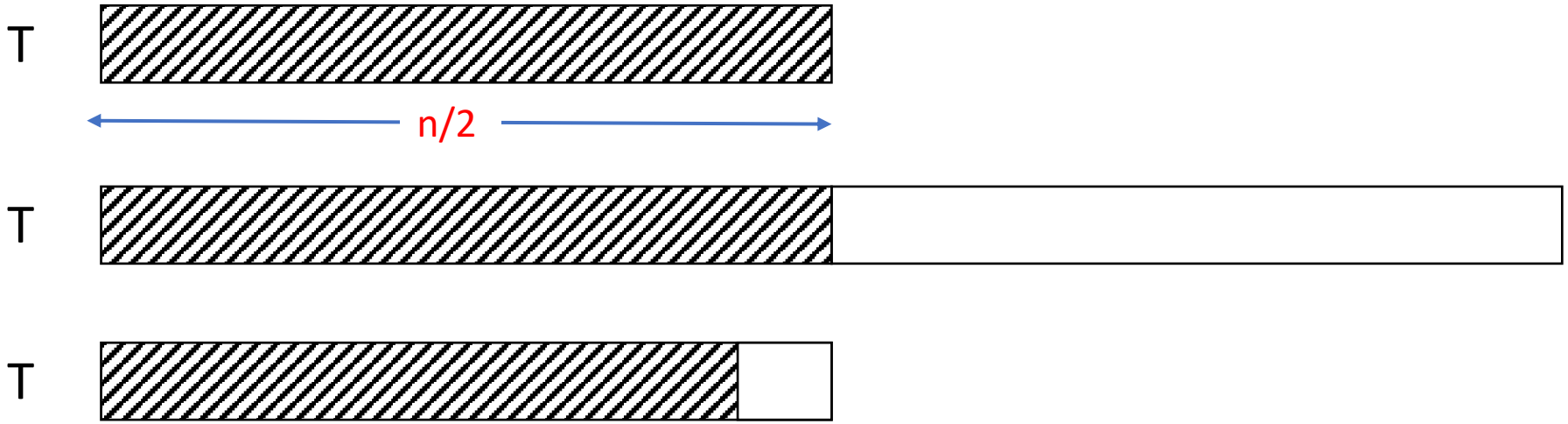

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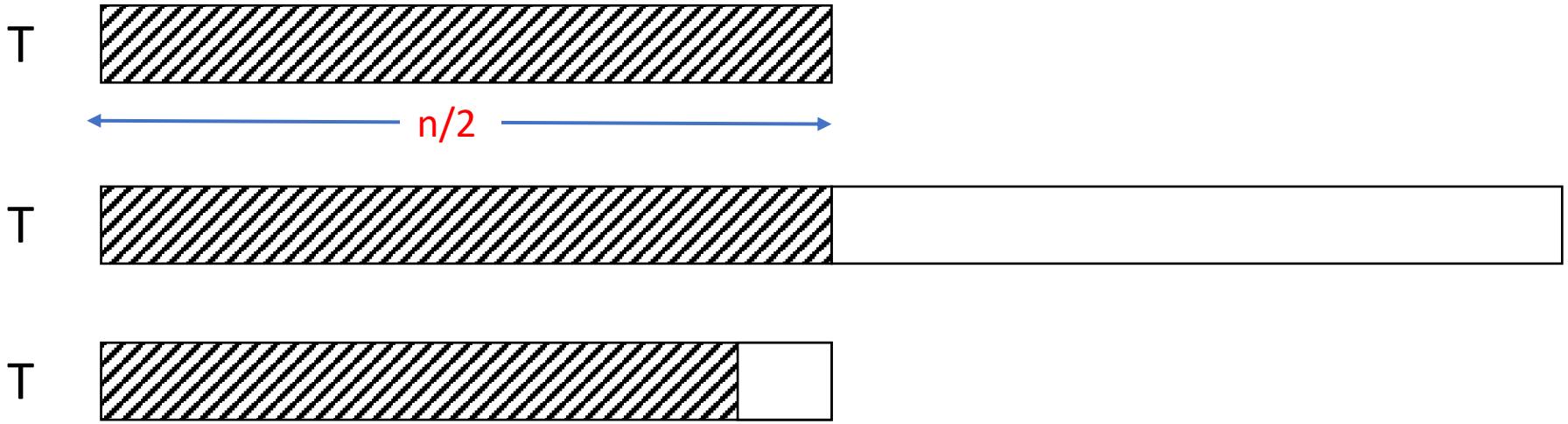

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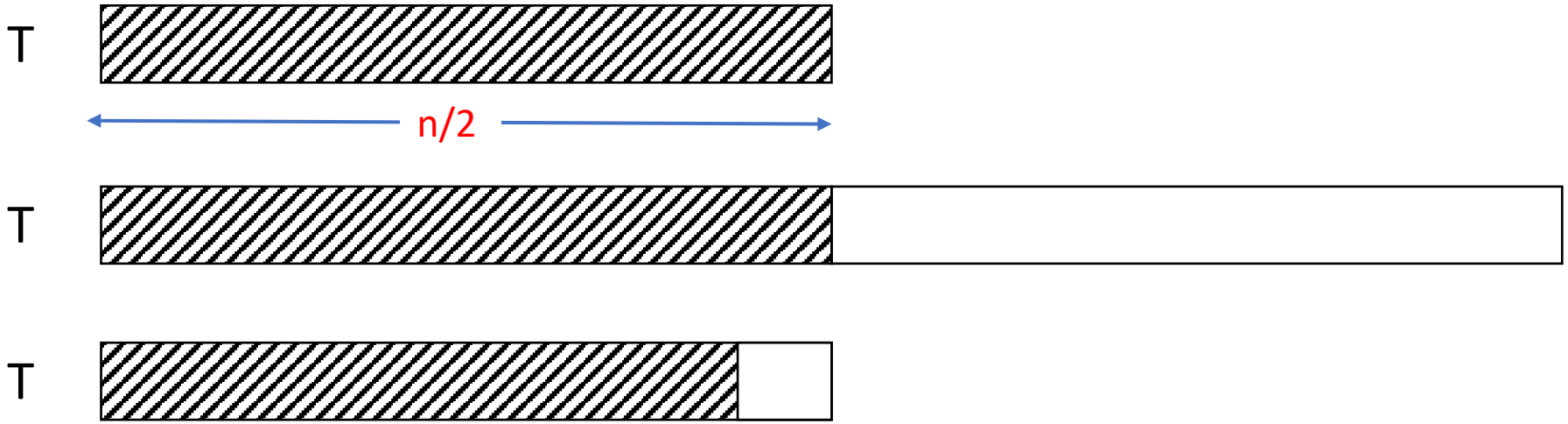

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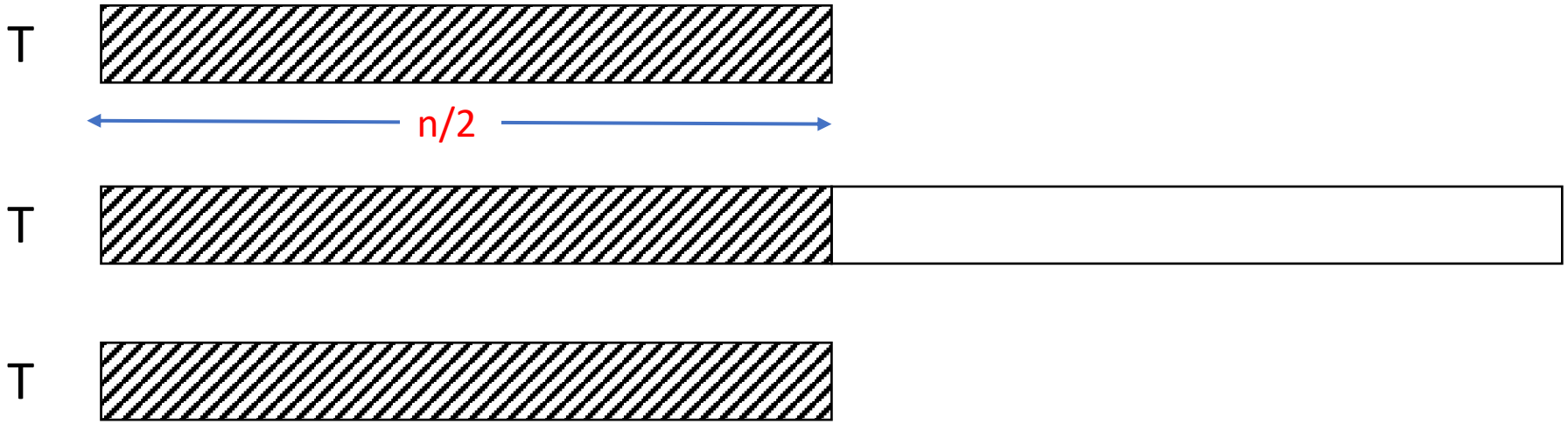

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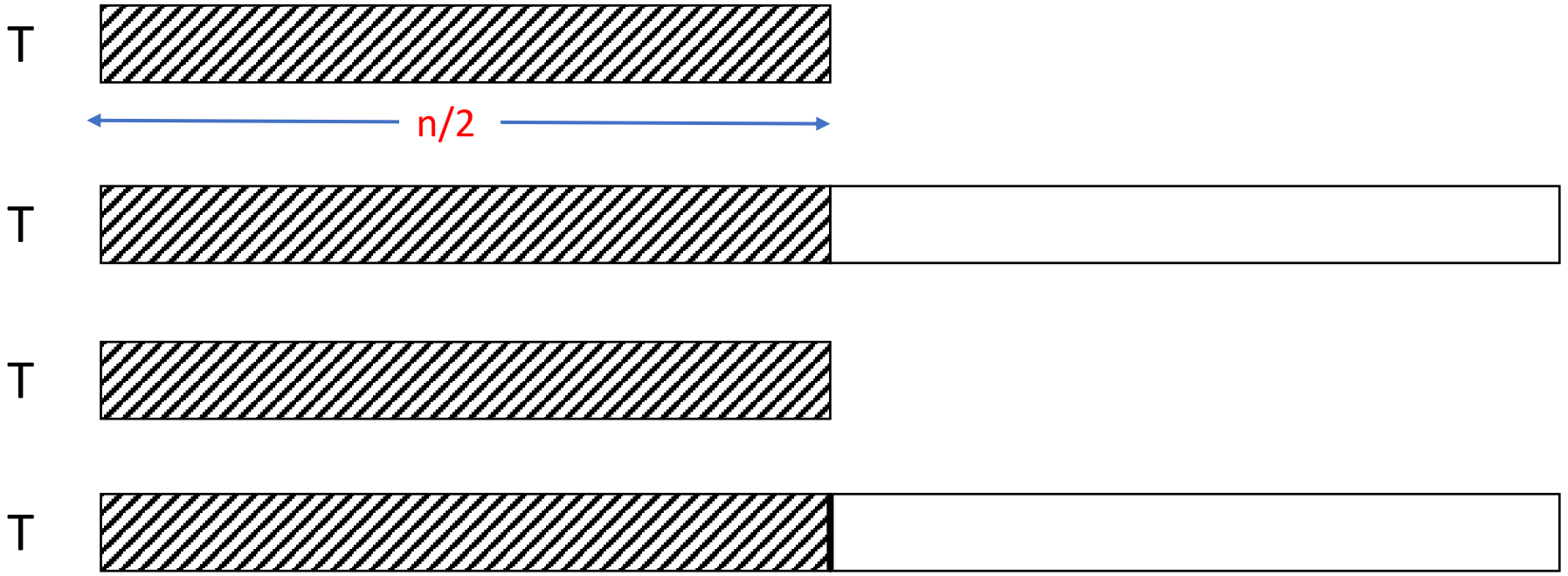

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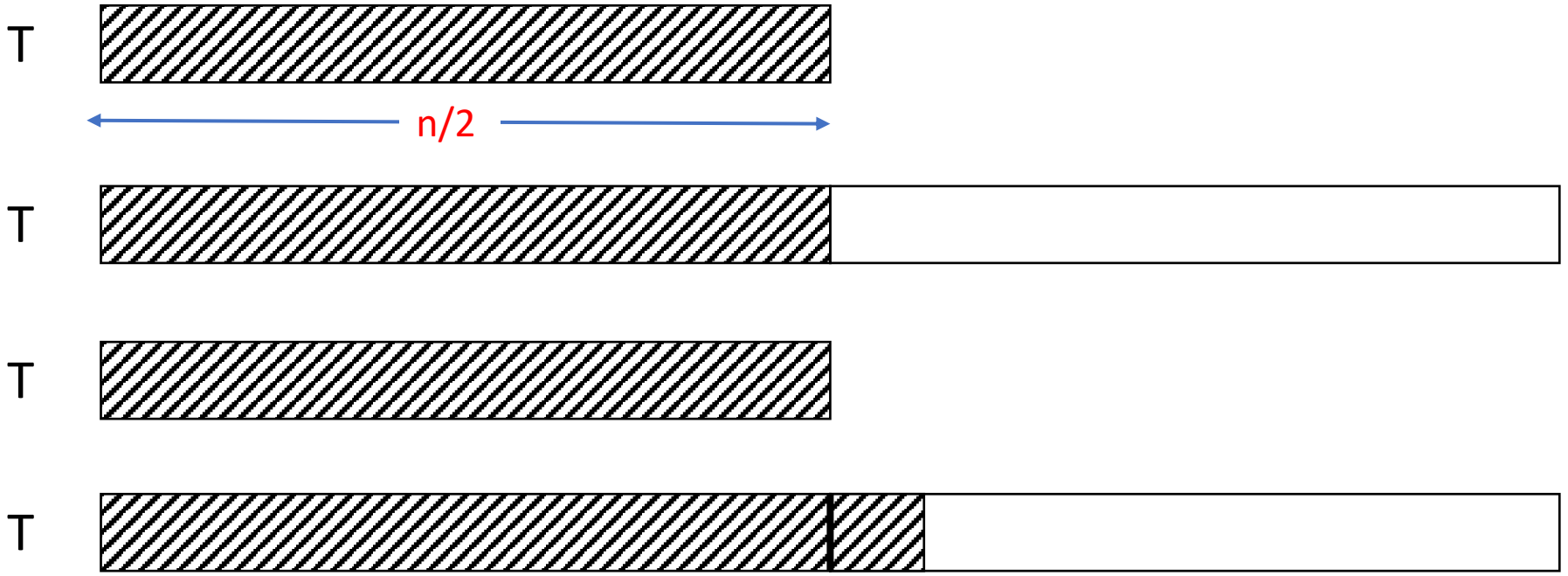

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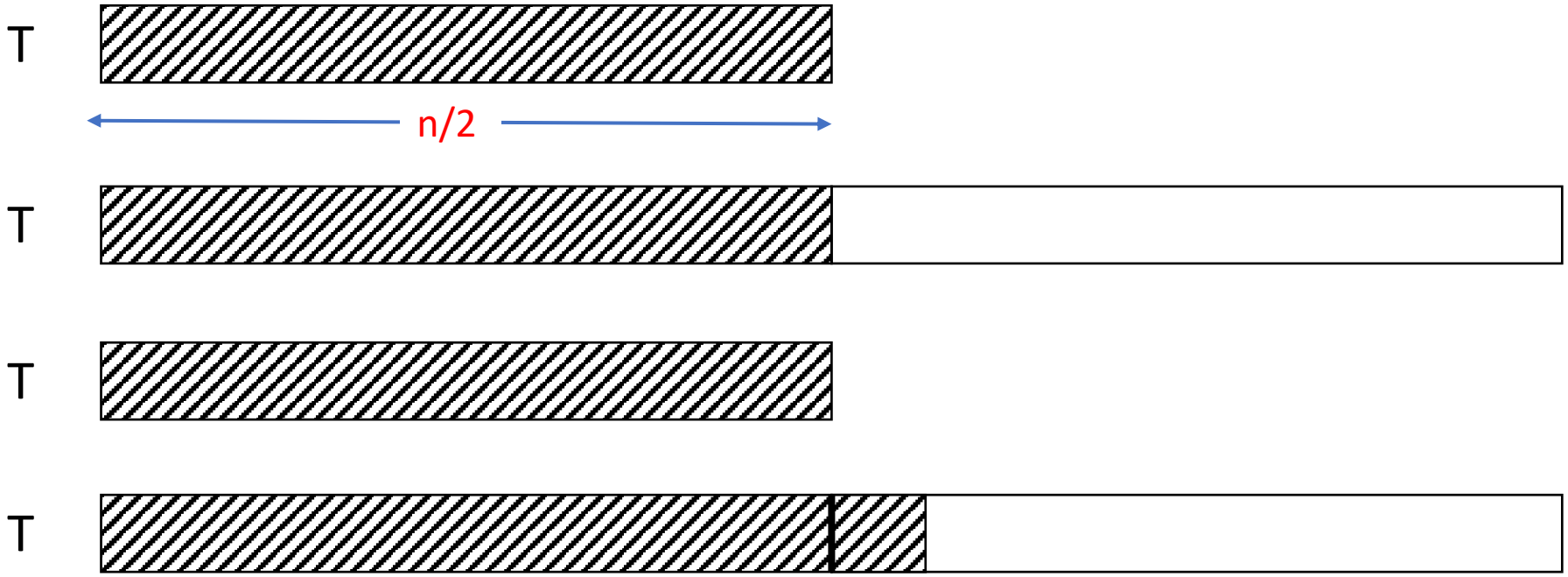

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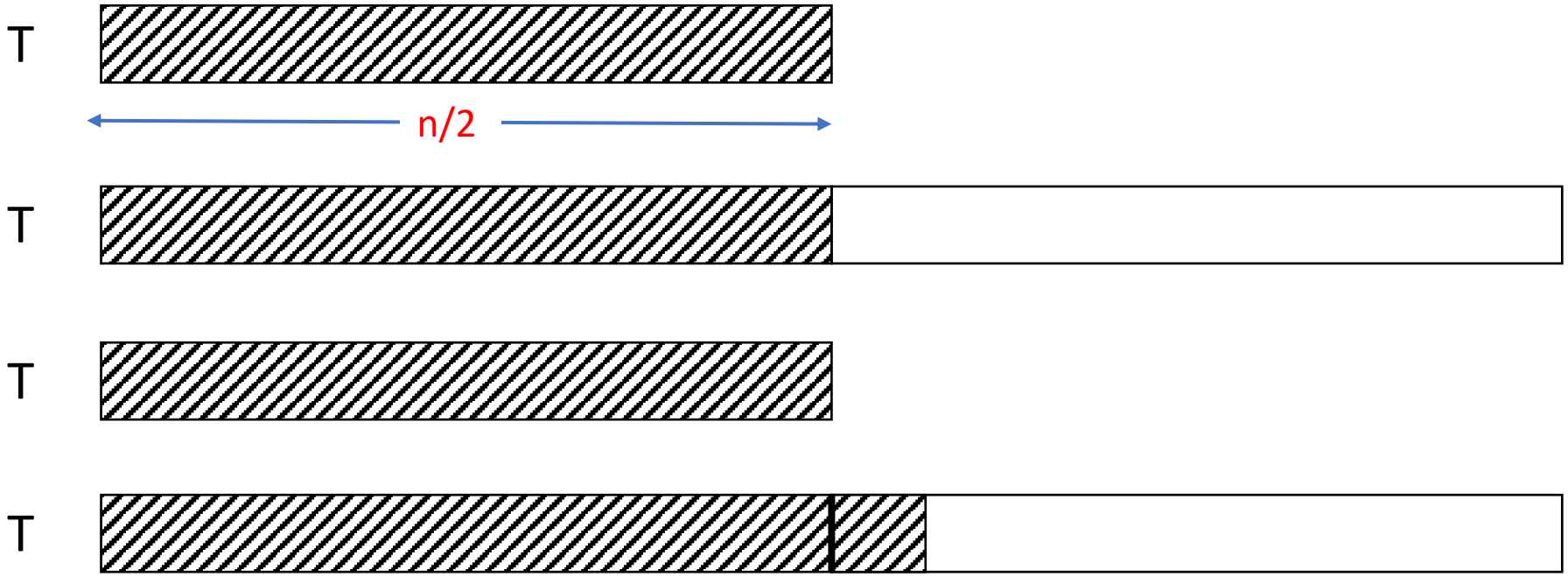

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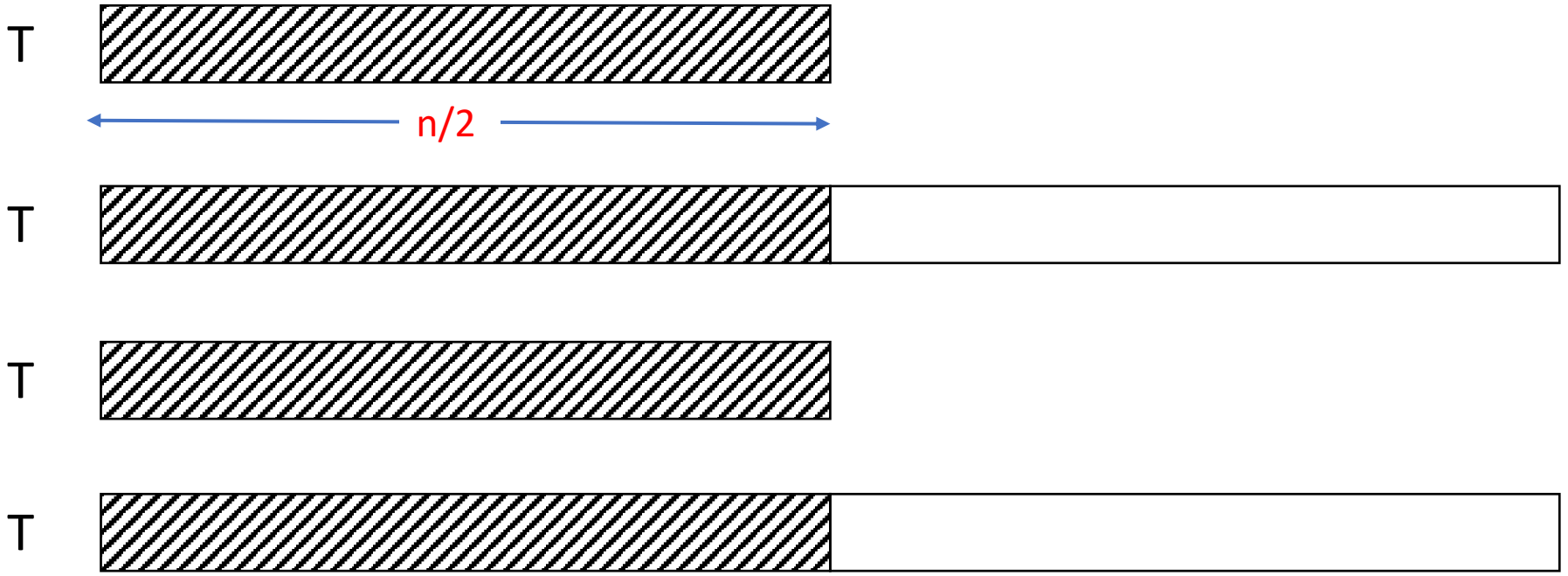

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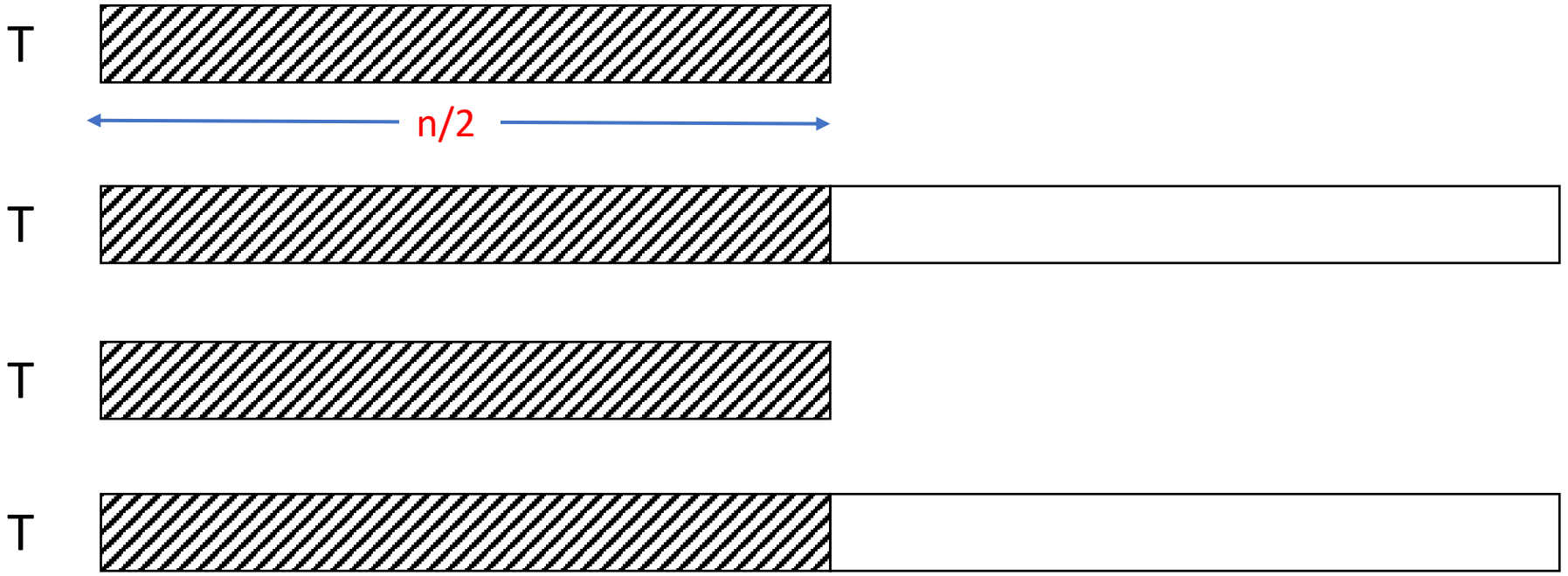

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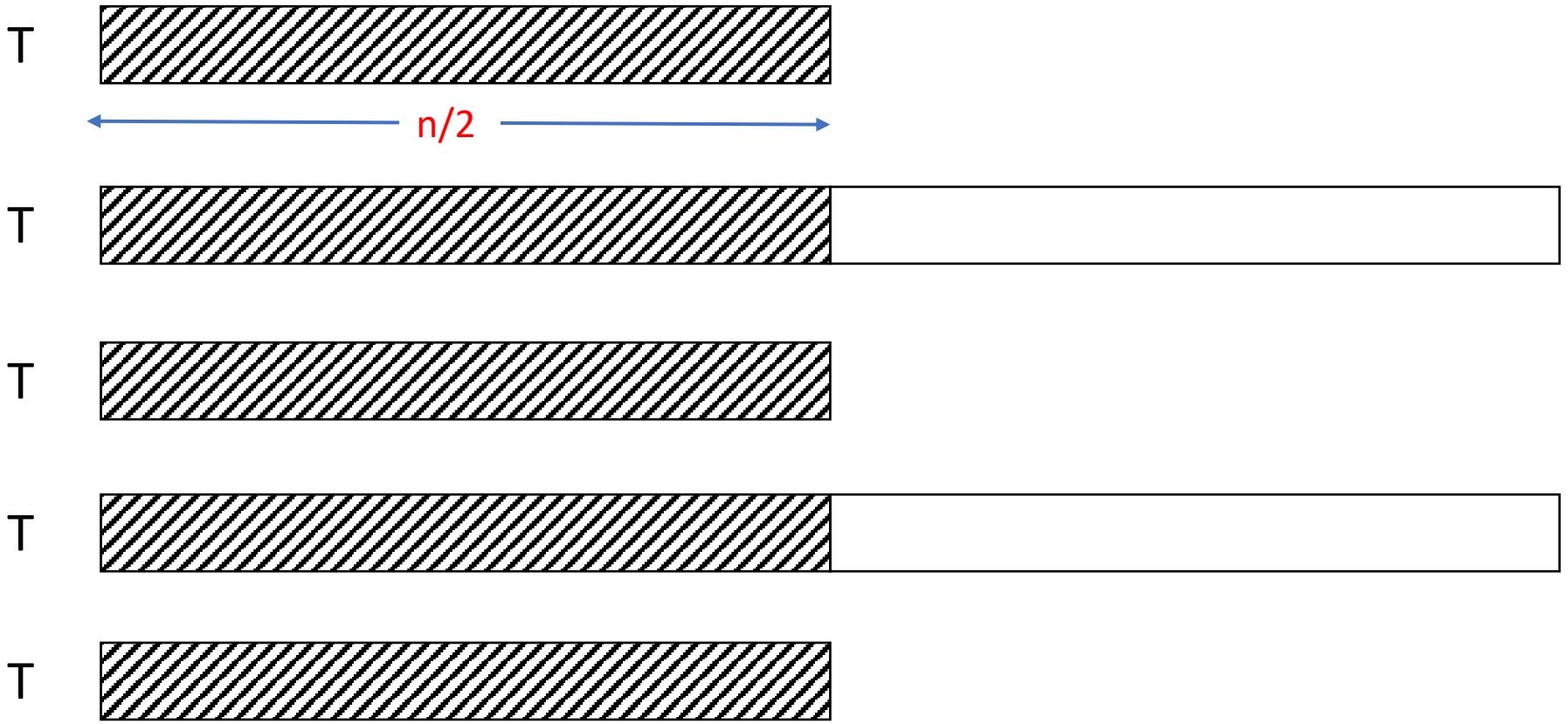

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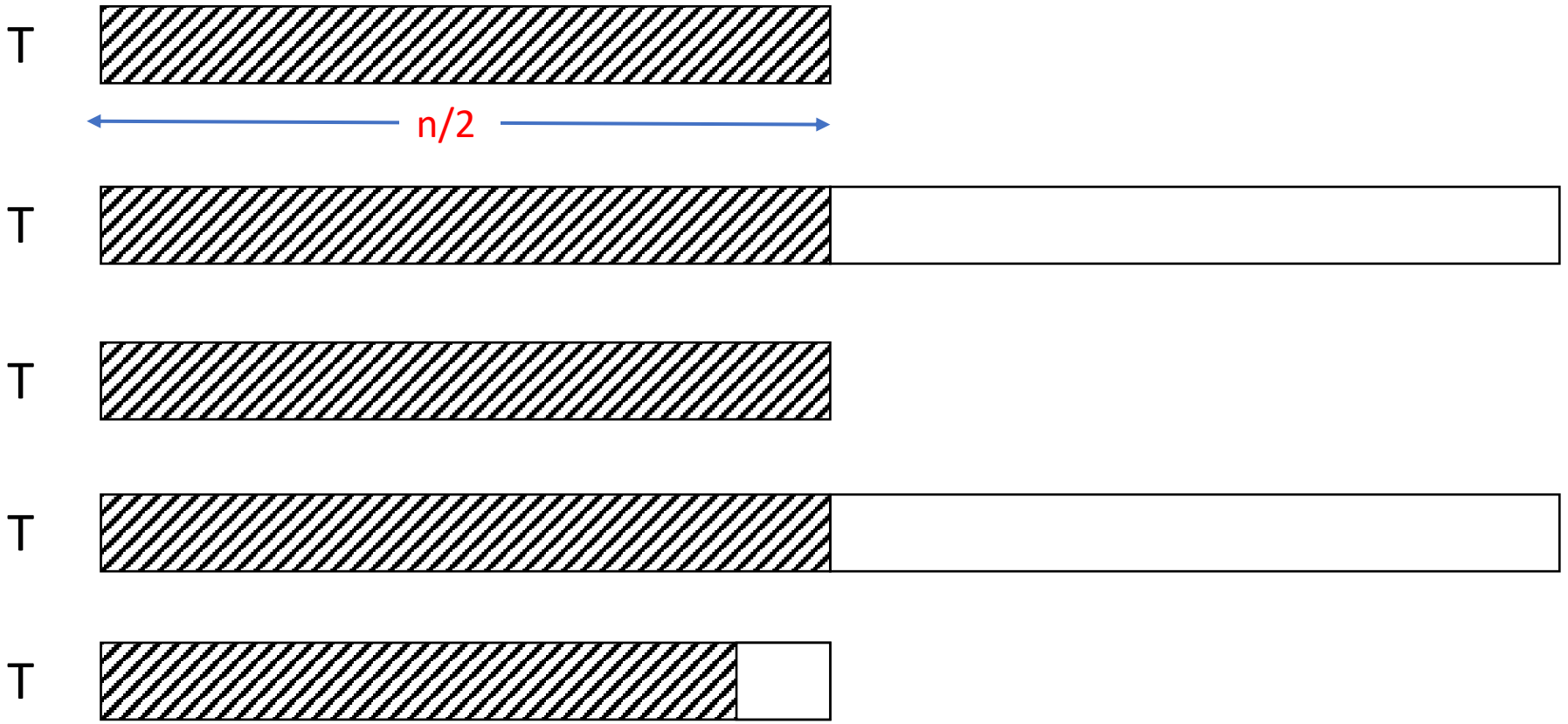

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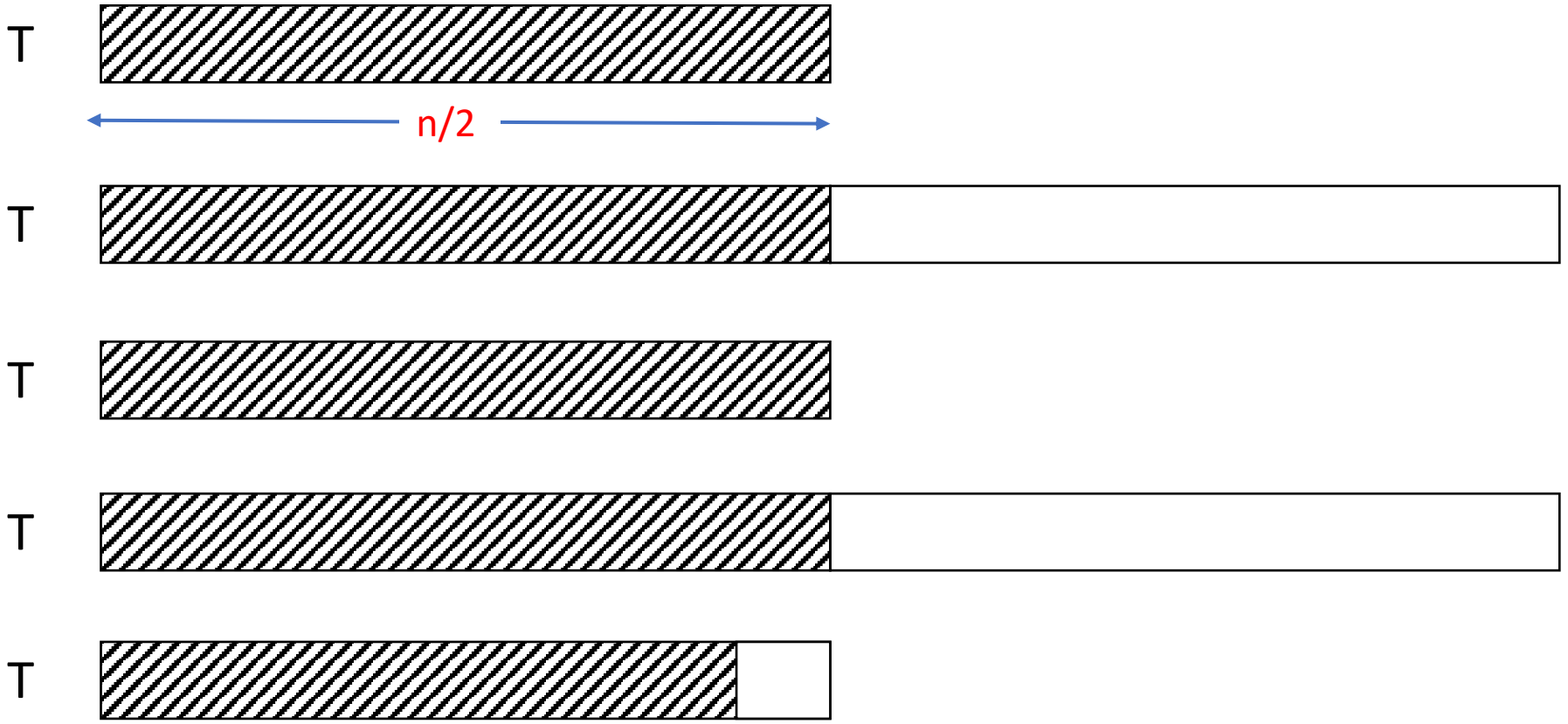

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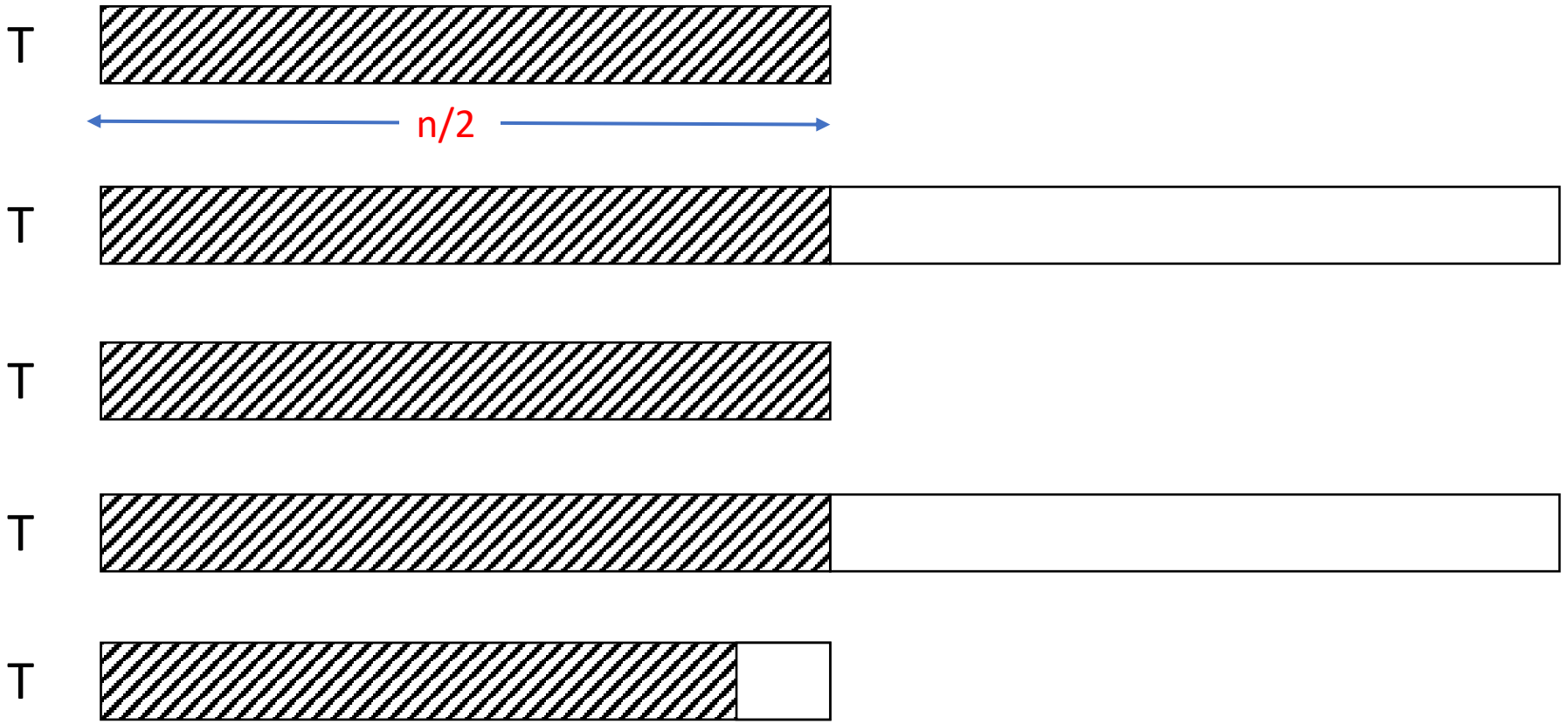
σ : n/2 Inserts, Insert, Delete, Delete, Insert, Insert, Delete, Delete, …

Cost : $\geq n/2$ $\geq n/2$ $\geq n/2$ $\geq n/2$ $\geq n/2$



Naïve approach : Bad sequence σ of $n = 2^k$ operations

σ : $\underbrace{n/2 \text{ Inserts}}$, $\underbrace{\text{Insert}}$, $\underbrace{\text{Delete, Delete}}$, $\underbrace{\text{Insert, Insert}}$, $\underbrace{\text{Delete, Delete}}$, …

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Total cost $\geq (n/4)(n/2)$

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Total cost $\geq (n/4)(n/2)$

Total cost is $\Omega(n^2)$

Naïve approach : Bad sequence σ of $n = 2^k$ operations

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Total cost $\geq (n/4)(n/2)$

Total cost is $\Omega(n^2)$

Amortized cost per operation is $\Omega(n)$

Insert and Delete: Good Approach

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Approach ensures $\alpha(T) \geq 1/4$

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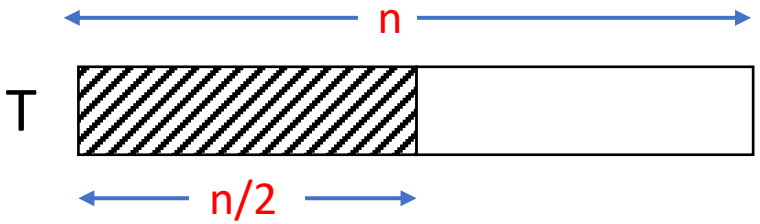
Approach ensures $\alpha(T) \geq 1/4$

σ : Arbitrary sequence of n Inserts and Deletes, starting from empty table T of size 1

What is the amortized cost per operation in σ ?

Amortized Analysis

Say a new T is just created,
(Total credit : $0)



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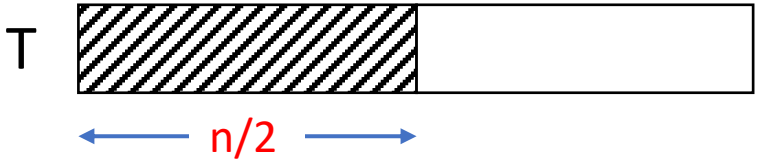
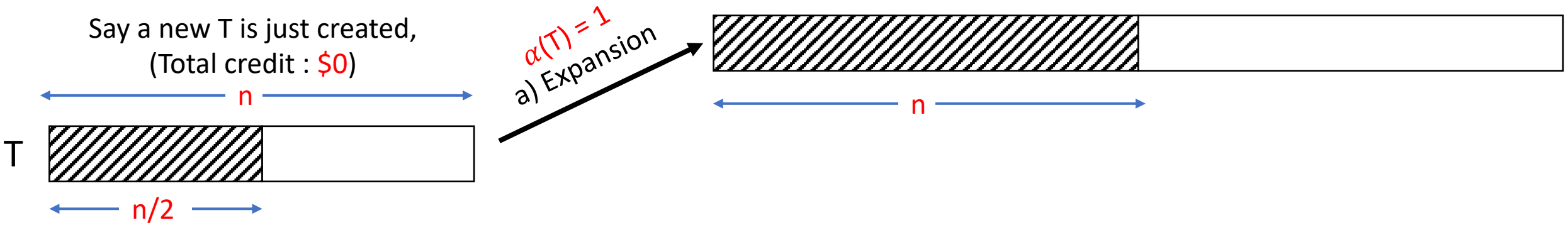


A sequence of Inserts, Deletes applied to T may cause:

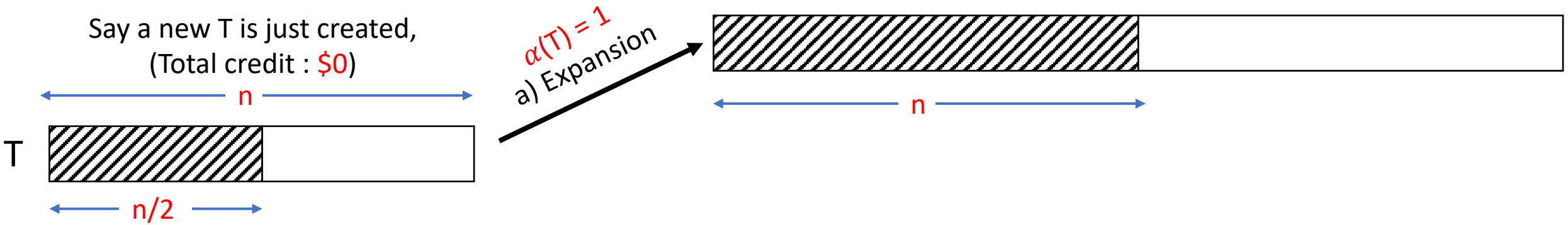
Amortized Analysis



A sequence of Inserts, Deletes applied to T may cause:

a) Expansion.

Amortized Analysis

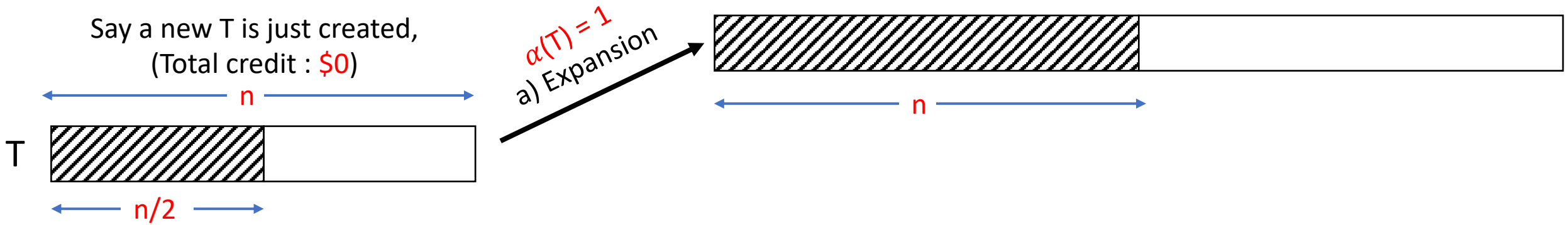


A sequence of Inserts, Deletes applied to T may cause:

a) Expansion. In this case:

- The seq contains $\geq n/2$ Inserts.

Amortized Analysis

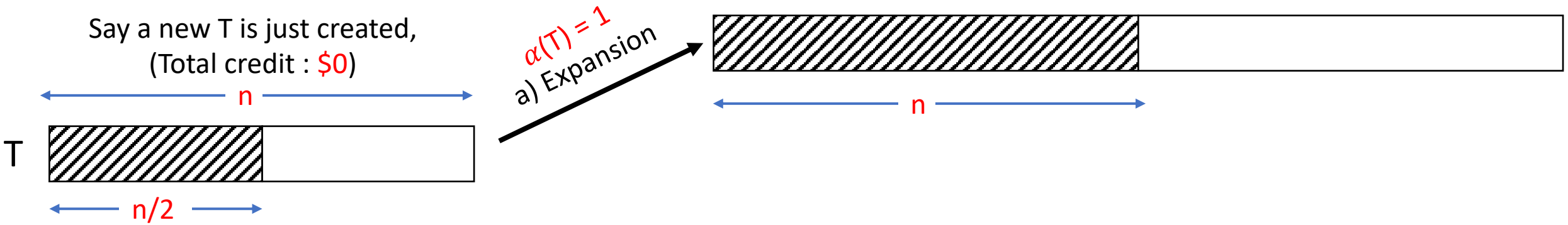


A sequence of Inserts, Deletes applied to T may cause:

a) Expansion. In this case:

- The seq contains $\geq n/2$ Inserts.
- Cost of copying items into new T : n

Amortized Analysis



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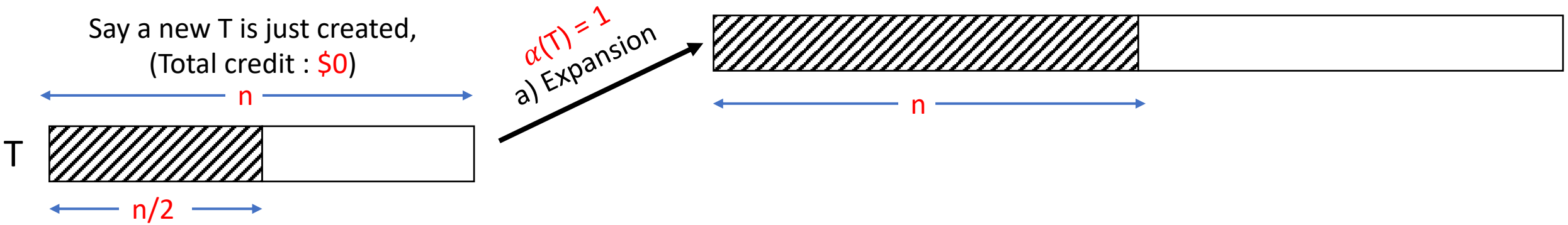
Charging Scheme:

Charge each Insert $3

$1 actual cost + $2 credit for future expansion

Amortized Analysis



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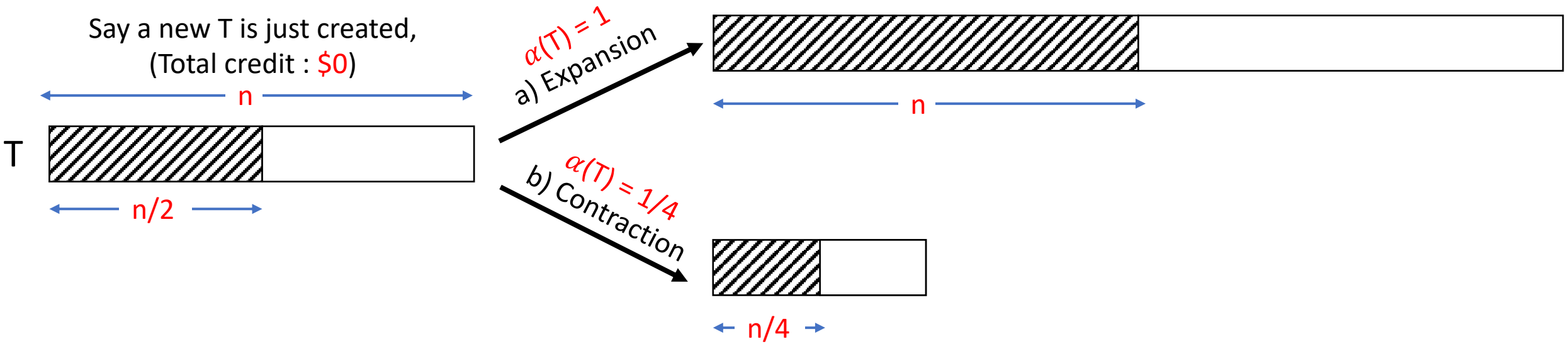
$n/2$ Inserts $\Rightarrow$ $(n/2)$ ($2) = $n credit,
which covers cost of table expansion

Amortized Analysis



A sequence of Inserts, Deletes applied to T may cause:

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- Cost of copying items into new T : n

a) Contraction.

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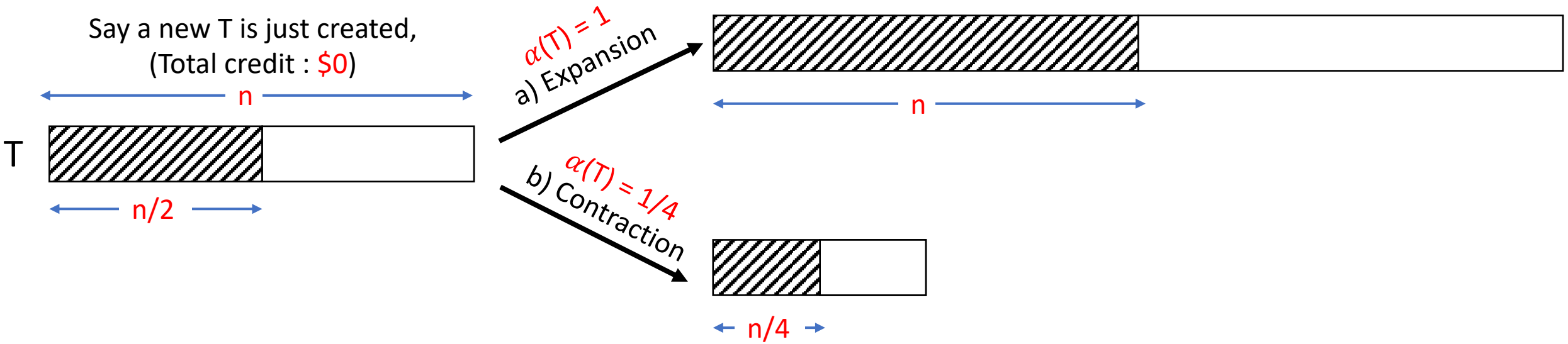
Amortized Analysis



A sequence of Inserts, Deletes applied to T may cause:

a) Expansion. In this case:

- The seq contains ≥ n/2 Inserts.
- Cost of copying items into new T : n

a) Contraction. In this case:

- The seq contains ≥ n/4 Deletes.

Charging Scheme:

Charge each Insert $3

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n/2 Inserts ⇒ (n/2) ($2) = $n credit,
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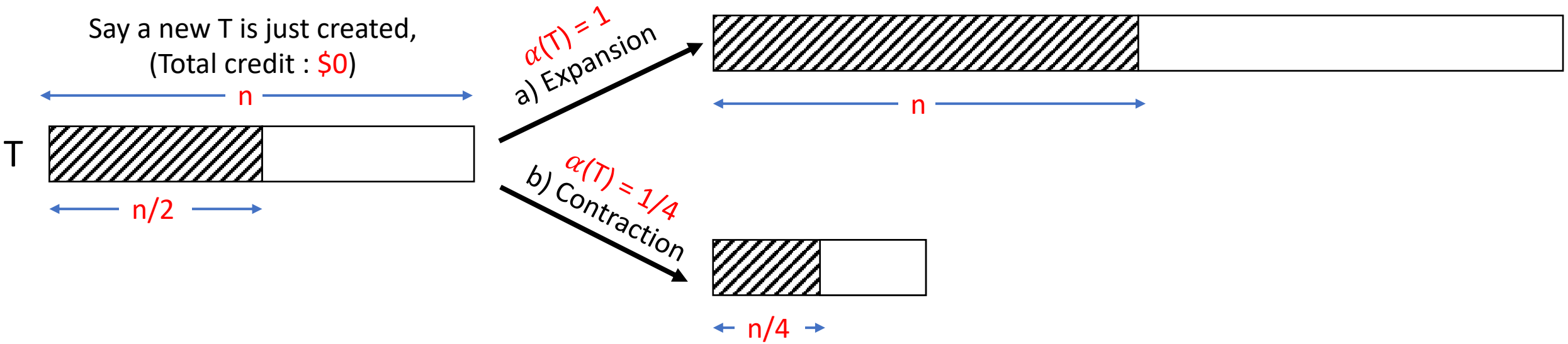
Amortized Analysis



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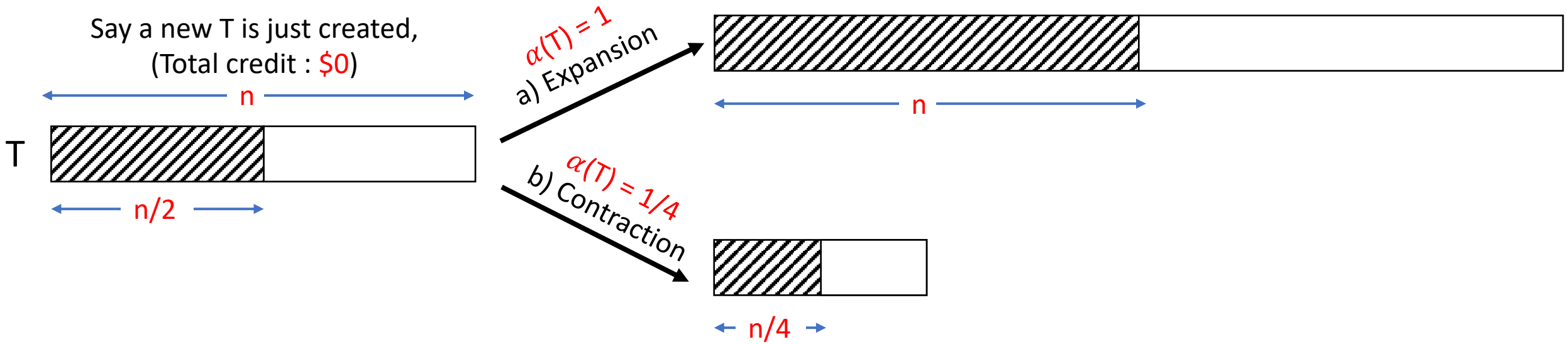
Amortized Analysis



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a) Contraction. In this case:

- The seq contains ≥ n/4 Deletes.
- Cost of copying items into new T : n/4

Charging Scheme:

Charge each Insert $3

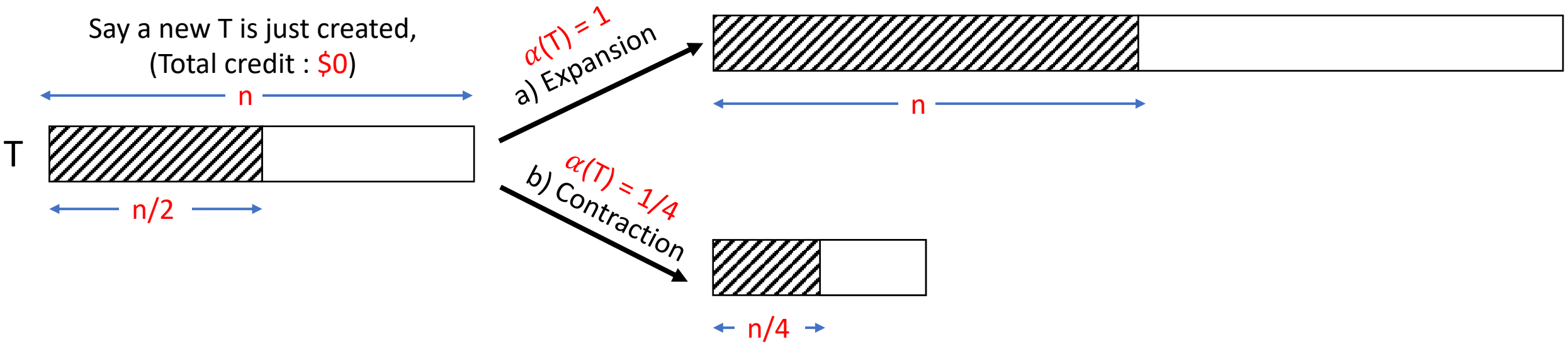
$1 actual cost + $2 credit for future expansion

n/2 Inserts ⇒ (n/2) ($2) = $n credit,
which covers cost of table expansion

Charge each Delete $2

$1 actual cost + $1 credit for future contraction

Amortized Analysis



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- The seq contains ≥ n/2 Inserts.
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a) Contraction. In this case:

- The seq contains ≥ n/4 Deletes.
- Cost of copying items into new T : n/4

Charging Scheme:

Charge each Insert $3

$1 actual cost + $2 credit for future expansion

n/2 Inserts ⇒ (n/2) ($2) = $n credit,
which covers cost of table expansion

Charge each Delete $2

$1 actual cost + $1 credit for future contraction

n/4 Deletes ⇒ (n/4) ($1) = $n/4 credit,
which covers cost of table contraction