Graph Algorithms II

Depth First Search

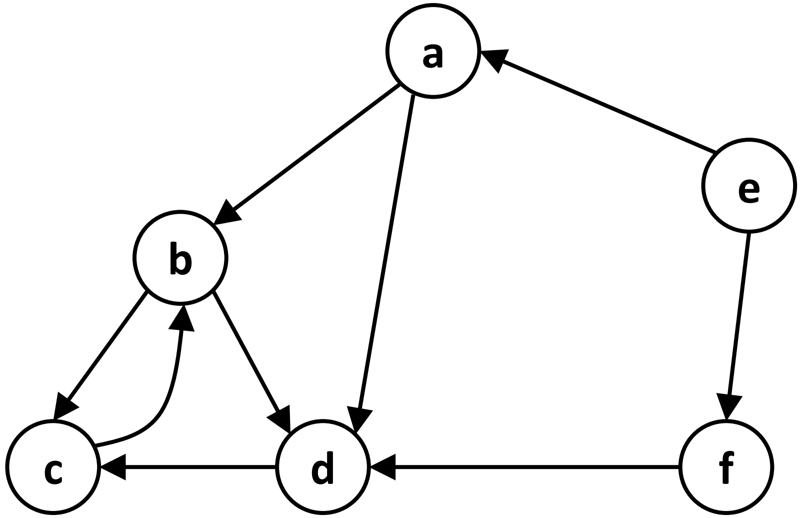
Depth First Search

- Brief review
- An application: detecting cycles

Depth First Search

- Brief review
- An application: detecting cycles

DFS(G)



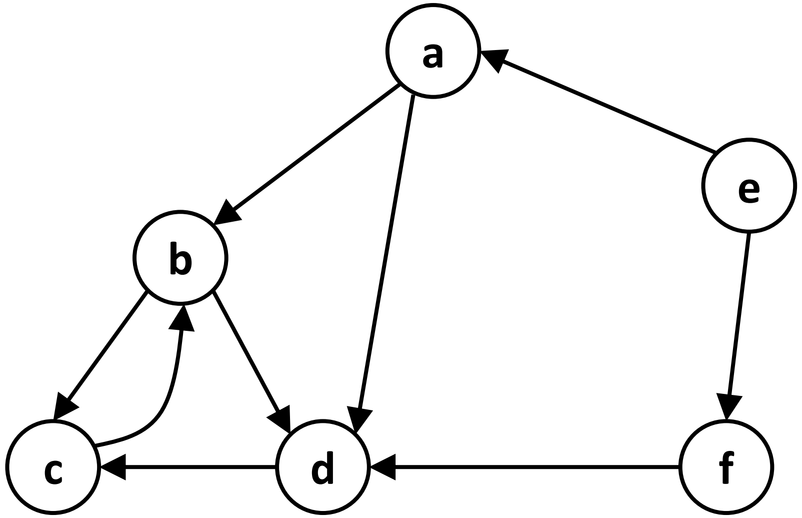
Adj List of G :

a → b → d
b → d → c
c → b
d → c
e → a → f
f → d

Colors of nodes:

W	W	W	W	W	W
e	a	b	c	d	f

DFS(G)



Adj List of G :

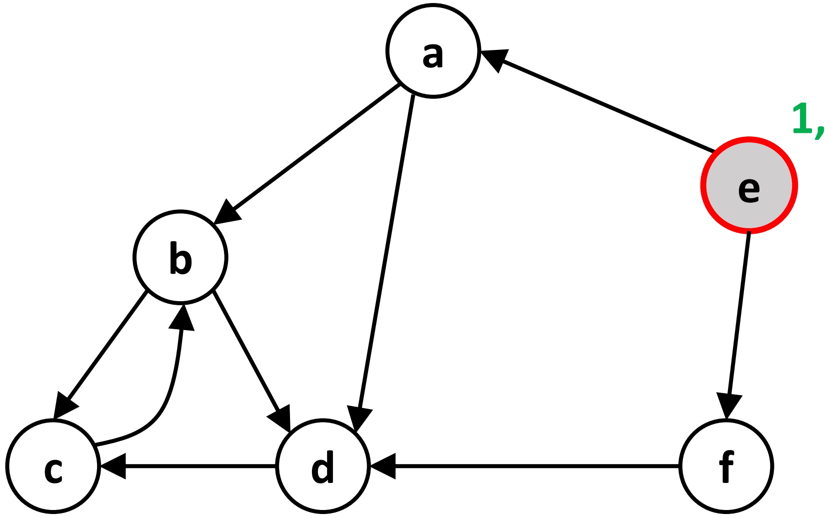
- **a** → **b** → **d**
- **b** → **d** → **c**
- **c** → **b**
- **d** → **c**
- **e** → **a** → **f**
- **f** → **d**

Colors of nodes:

W	W	W	W	W	W
e	**a**	**b**	**c**	**d**	**f**

DFS-Explore(G, **e)**

DFS(G)



Adj List of G :

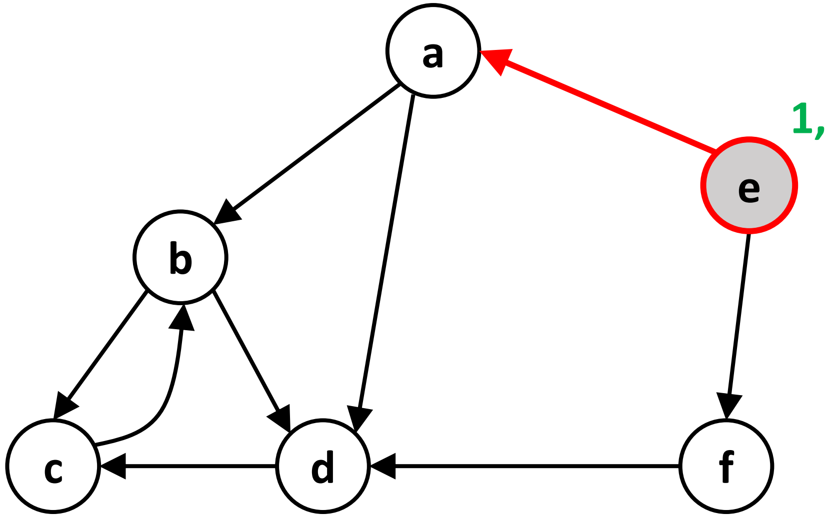
- a → b → d
- b → d → c
- c → b
- d → c
- e → a → f
- f → d

Colors of nodes:

e	a	b	c	d	f
G	W	W	W	W	W

DFS-Explore(G, **e**)

DFS(G)



Adj List of G :

a → b → d
b → d → c
c → b
d → c
e → a → f
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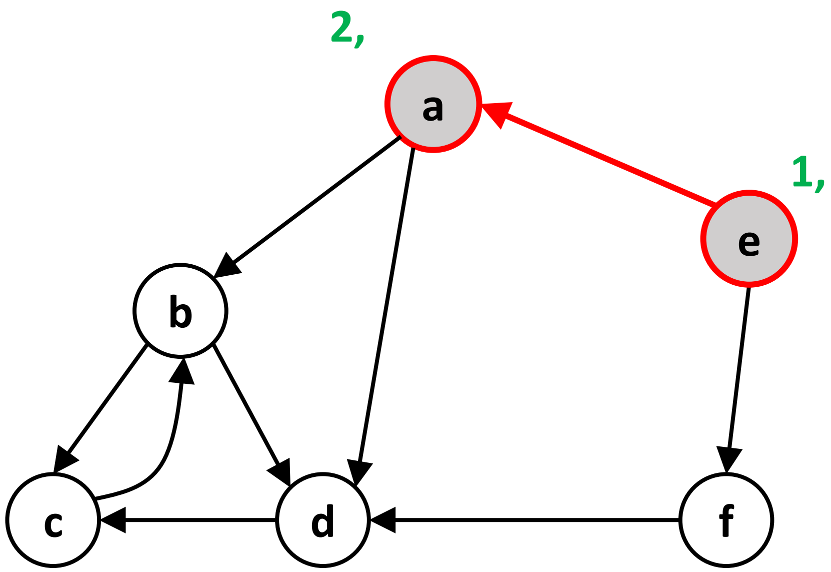
Colors of nodes:

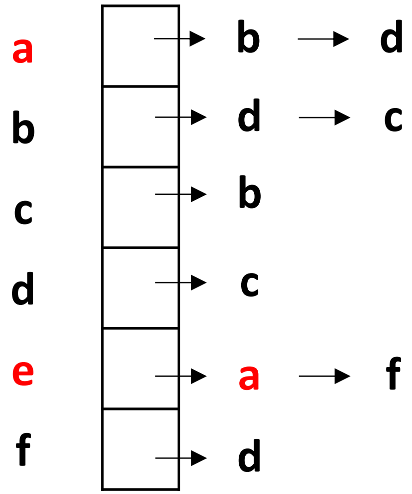
G	W	W	W	W	W
e	a	b	c	d	f

DFS-Explore(G, **e)**

DFS(G)



Adj List of G :



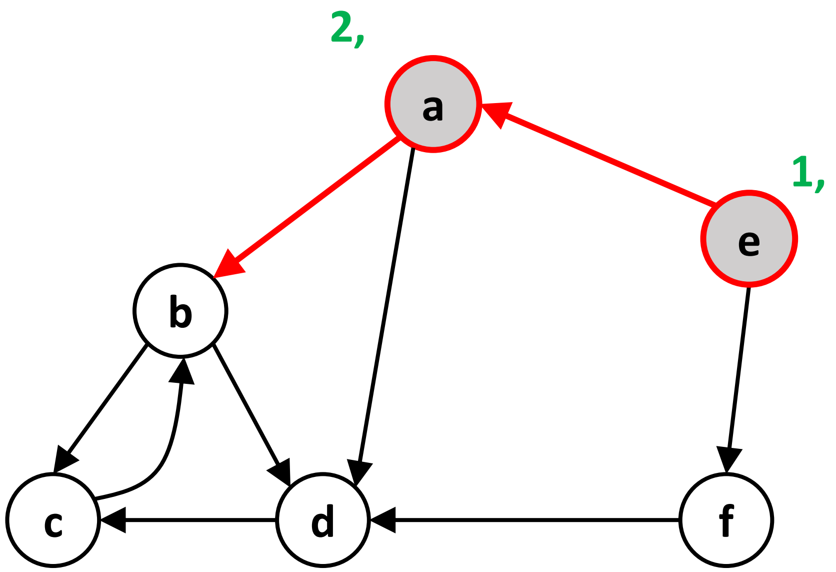
Colors of nodes:

G	G	W	W	W	W
e	a	b	c	d	f

⬆

DFS-Explore(G, **e)**

DFS(G)



Adj List of G :

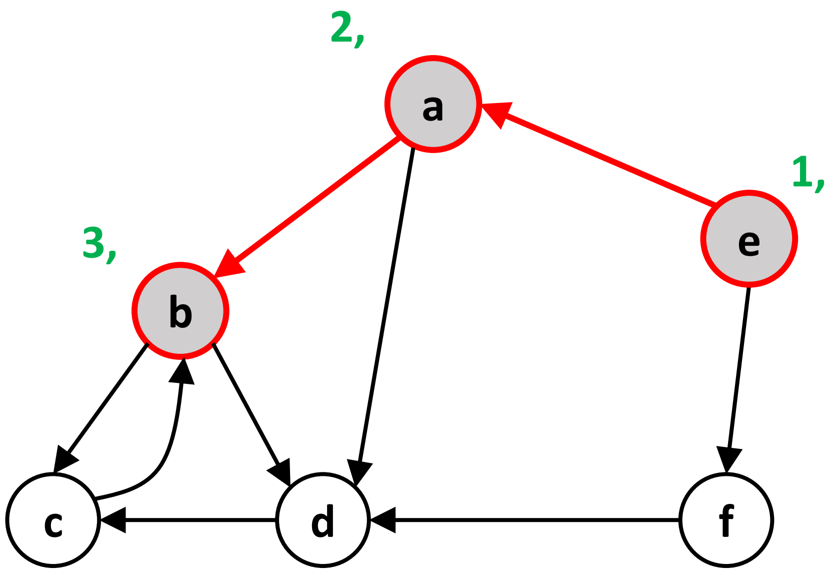
- a → b → d
- b → d → c
- c → b
- d → c
- e → a → f
- f → d

Colors of nodes:

e	a	b	c	d	f
G	G	W	W	W	W

DFS-Explore(G, e)

DFS(G)



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c → b
d → c
e → a → f
f → d

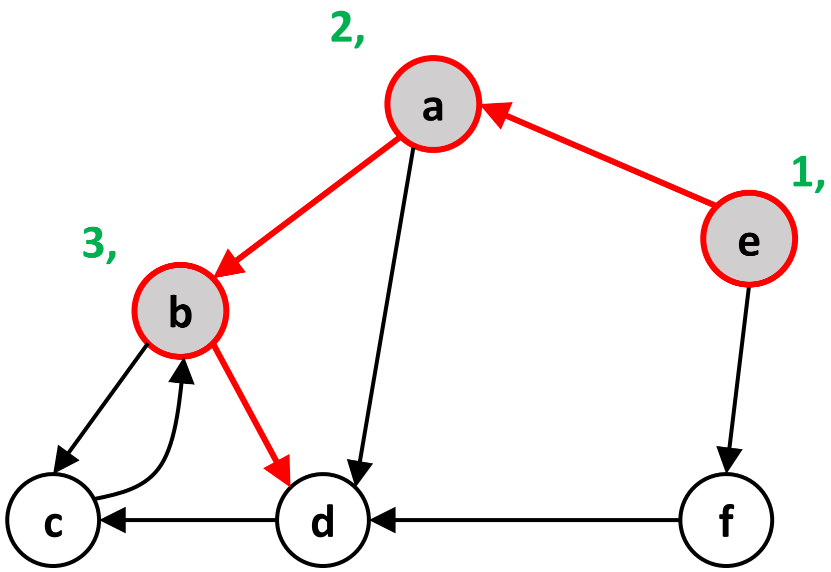
Colors of nodes:

G	G	G	W	W	W
e	a	b	c	d	f

↑

DFS-Explore(G, e)

DFS(G)



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d → c
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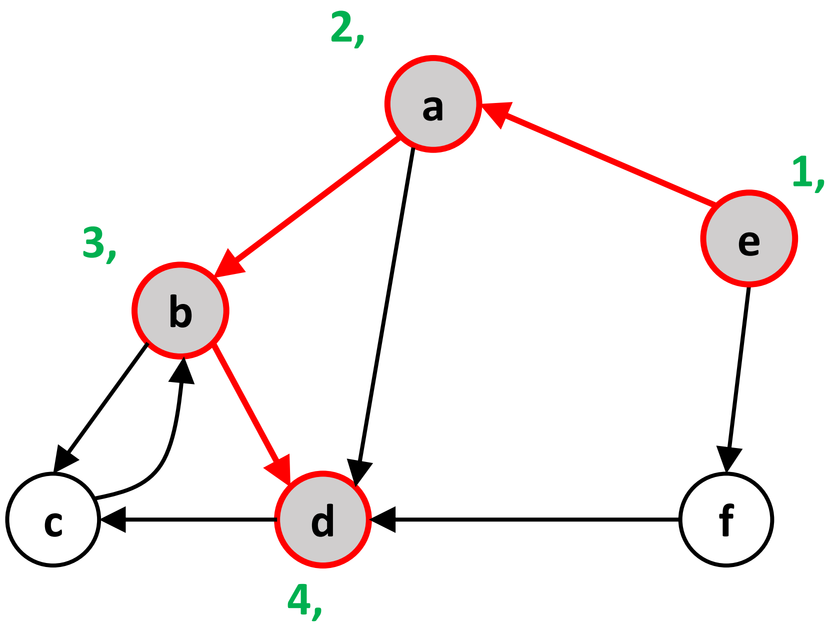
Colors of nodes:

e	a	b	c	d	f
G	G	G	W	W	W

↑

DFS-Explore(G, **e)**

DFS(G)



Adj List of G :

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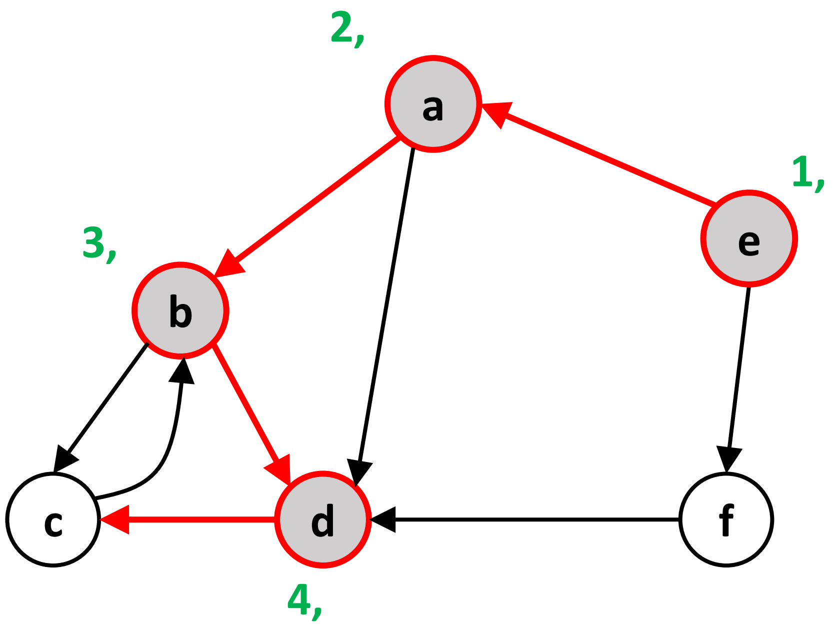
b	→ d → c
c	→ b
d	→ c
e	→ a → f
f	→ d

Colors of nodes:

G	G	G	W	G	W
e	a	b	c	d	f

DFS-Explore(G, **e)**

DFS(G)



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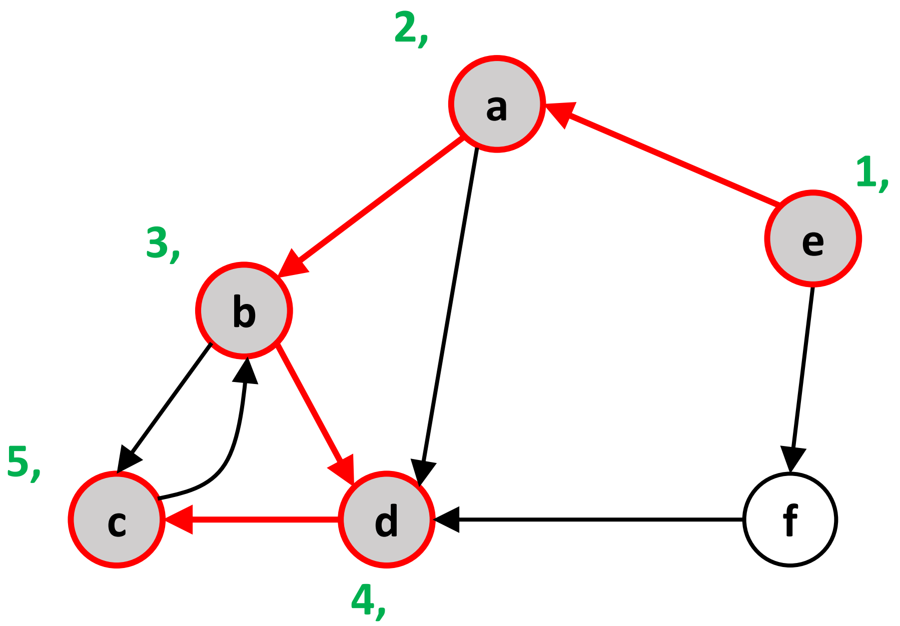
Colors of nodes:

G	G	G	W	G	W
e	a	b	c	d	f

DFS-Explore(G, e)

DFS(G)



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c → b
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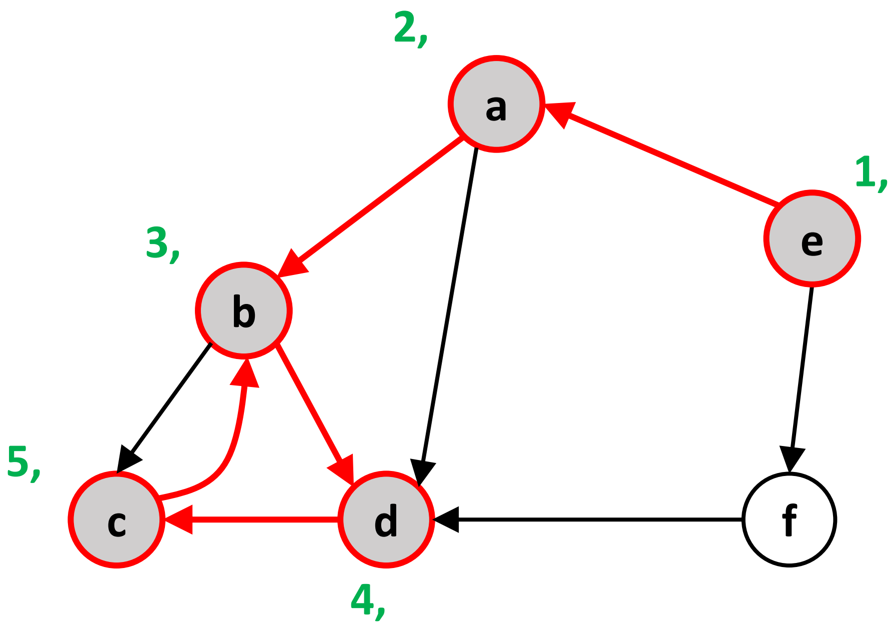
Colors of nodes:

G	G	G	G	G	W
e	a	b	c	d	f

⬆

DFS-Explore(G, e)

DFS(G)



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Colors of nodes:

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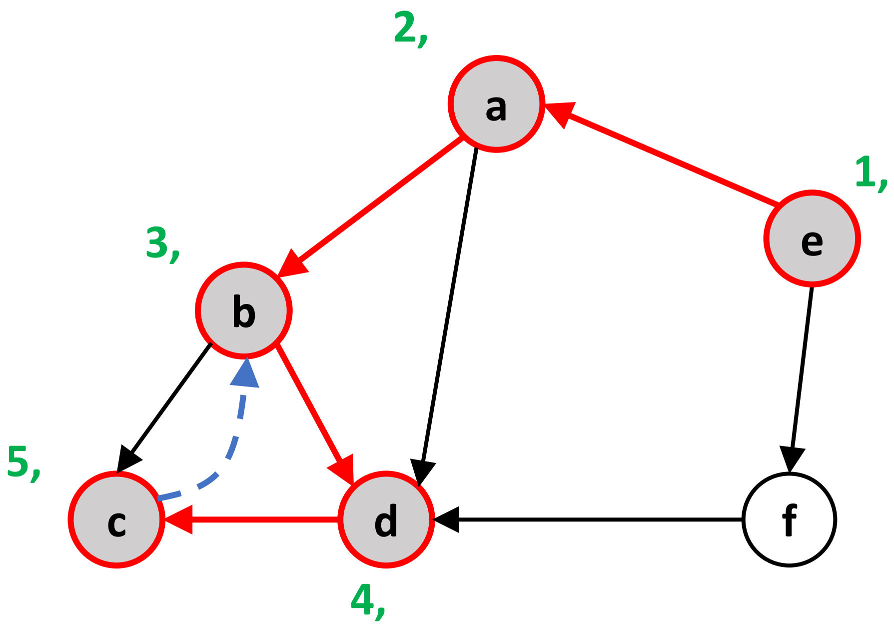
e	a	b	c	d	f

DFS-Explore(G, **e)**

DFS(G)



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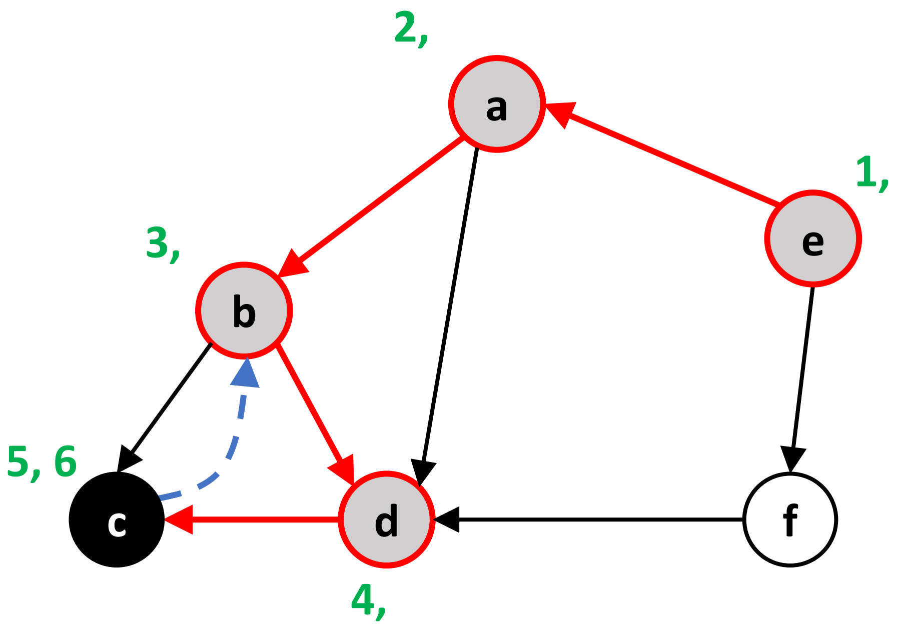
d → c
e → a → f
f → d

Colors of nodes:

G	G	G	G	G	W
e	a	b	c	d	f

DFS-Explore(G, e)

DFS(G)



Adj List of G :

a	→	b	→ d
b	→	d	→ c
✓ c	→	b	
d	→	c	
e	→	a	→ f
f	→	d	

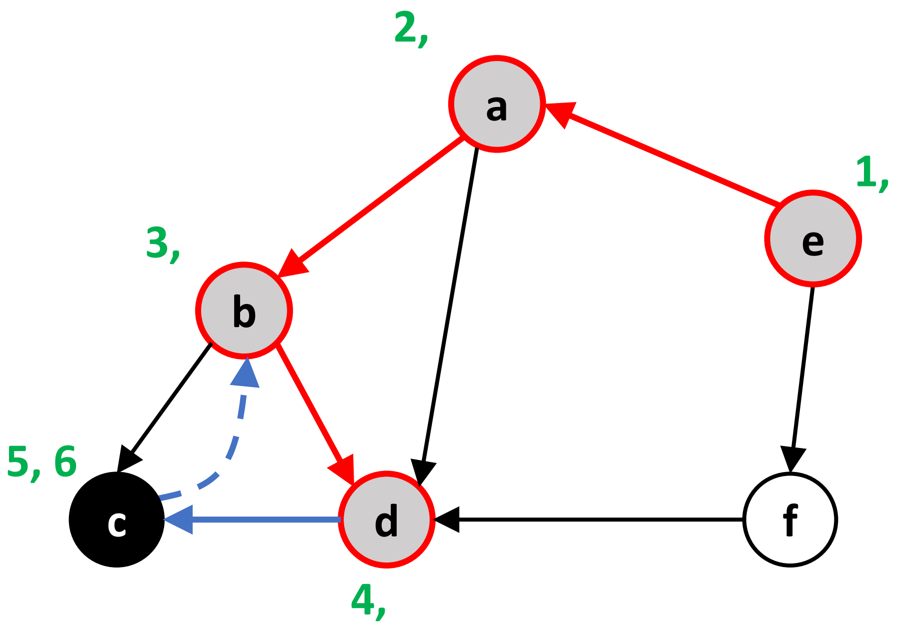
Colors of nodes:

e	a	b	c	d	f
G	G	G	B	G	W

⬆

DFS-Explore(G, **e**)

DFS(G)



Adj List of G :

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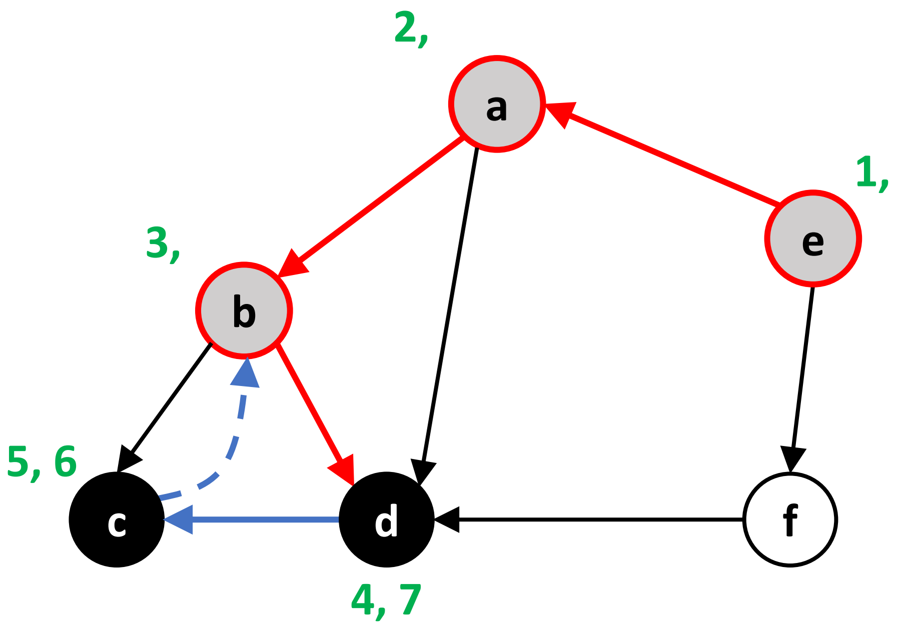
b → d → c
✓ c → b
d → c
e → a → f
f → d

Colors of nodes:

G	G	G	B	G	W
e	a	b	c	d	f

DFS-Explore(G, e)

DFS(G)



Adj List of G :

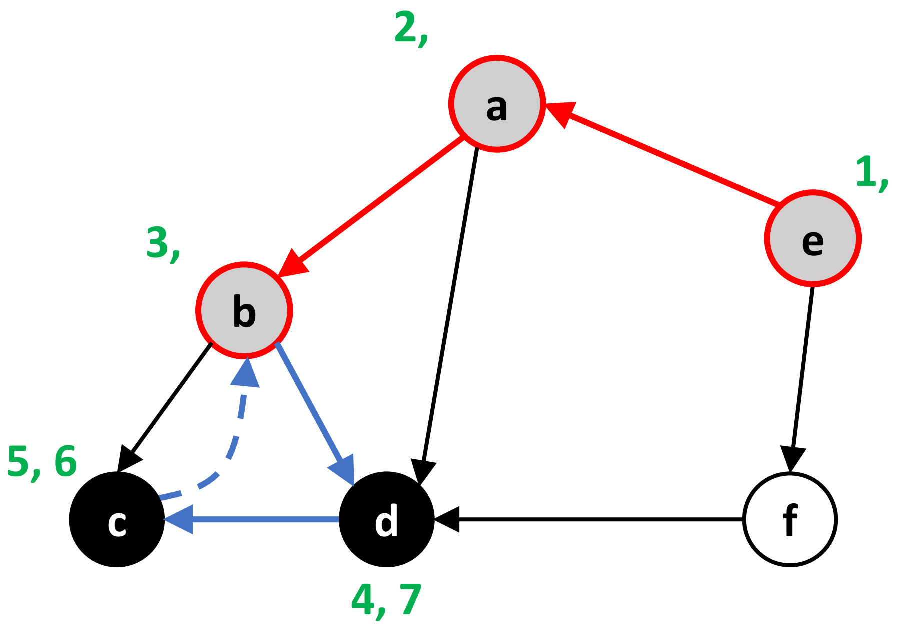
a	→ b	→ d
b	→ d	→ c
✓ c	→ b	
✓ d	→ c	
e	→ a	→ f
f	→ d	

Colors of nodes:

e	a	b	c	d	f
G	G	G	B	B	W

DFS-Explore(G, e**)**

DFS(G)



Adj List of G :

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Colors of nodes:

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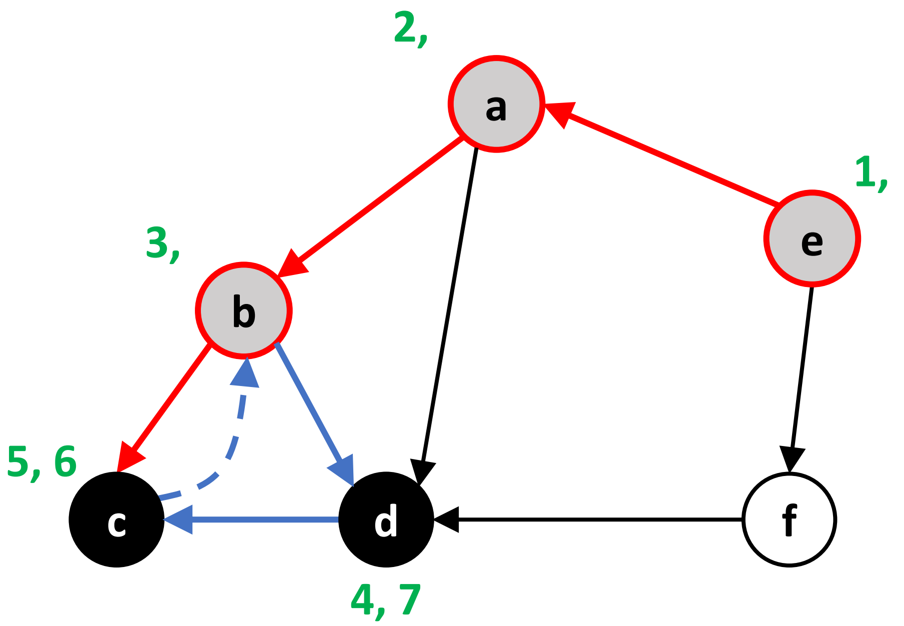
DFS-Explore(G, e)

DFS(G)



Adj List of G :

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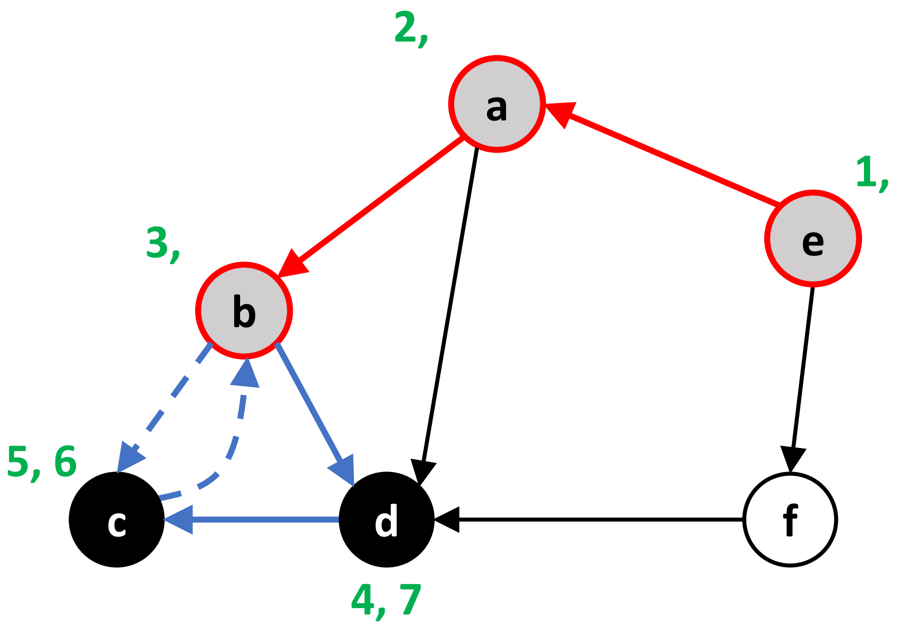
DFS-Explore(G, e)

DFS(G)



Adj List of G :

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Colors of nodes:

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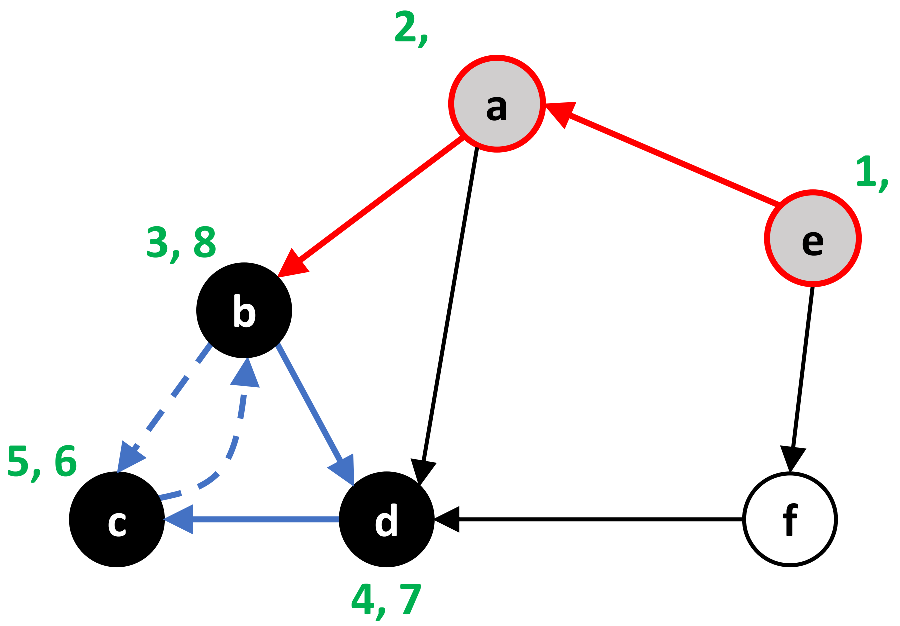
e	a	b	c	d	f

↑

DFS-Explore(G, e)

DFS(G)



Adj List of G :

	a	→ b	→ d
✓	b	→ d	→ c
✓	c	→ b	
✓	d	→ c	
	e	→ a	→ f
	f	→ d	

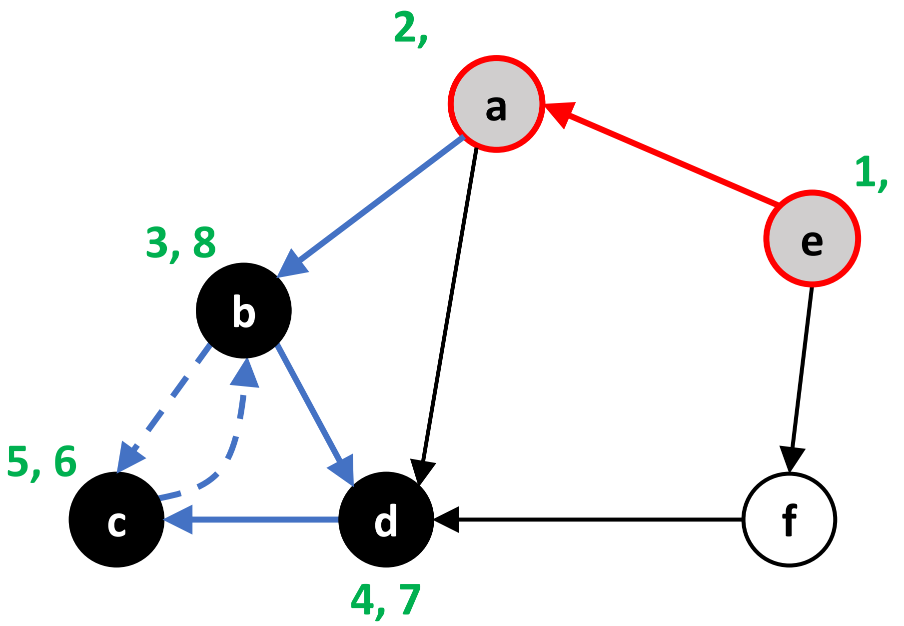
Colors of nodes:

G	G	B	B	B	W
e	a	b	c	d	f

↑

DFS-Explore(G, e)

DFS(G)



Adj List of G :

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- ✓ b → d → c
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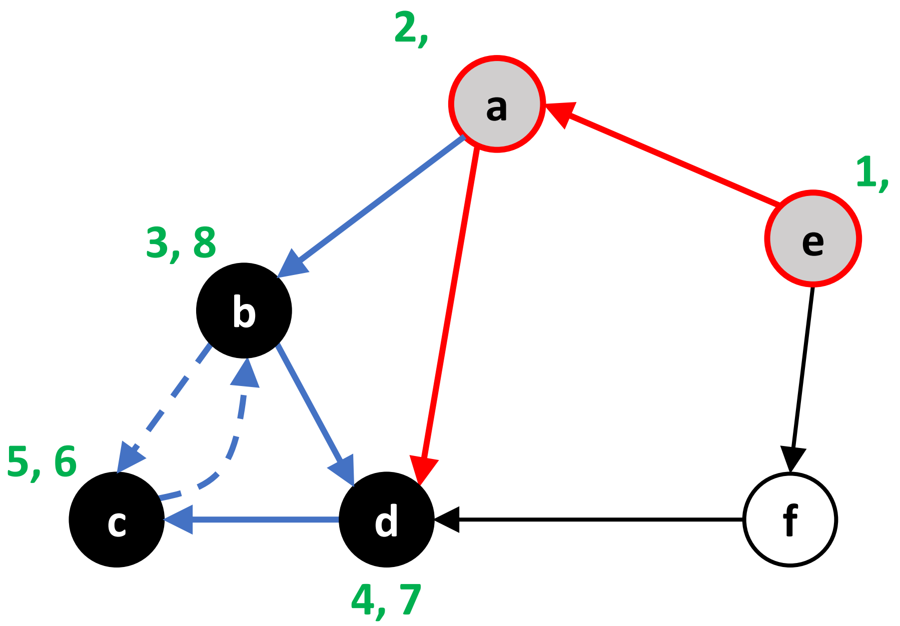
DFS-Explore(G, e)

DFS(G)



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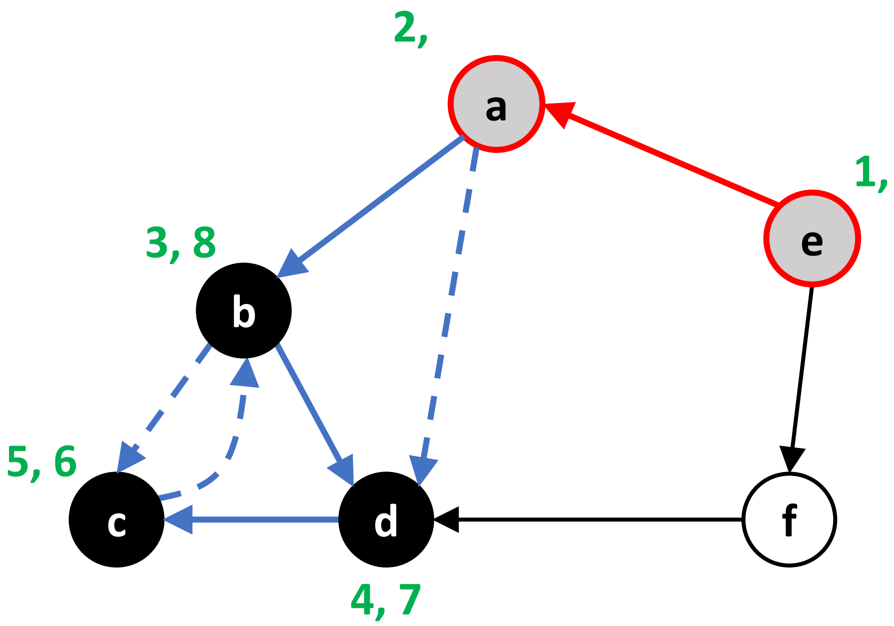
⬆

DFS-Explore(G, e)

DFS(G)



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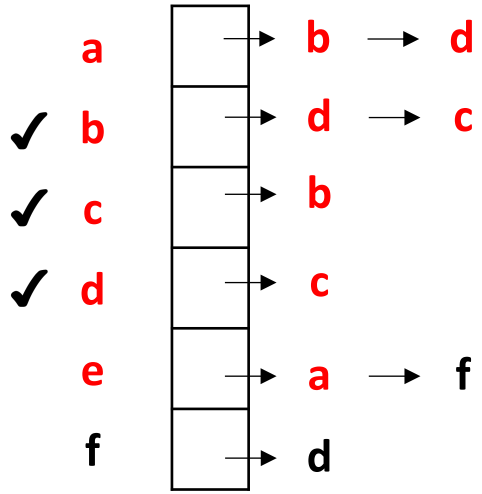
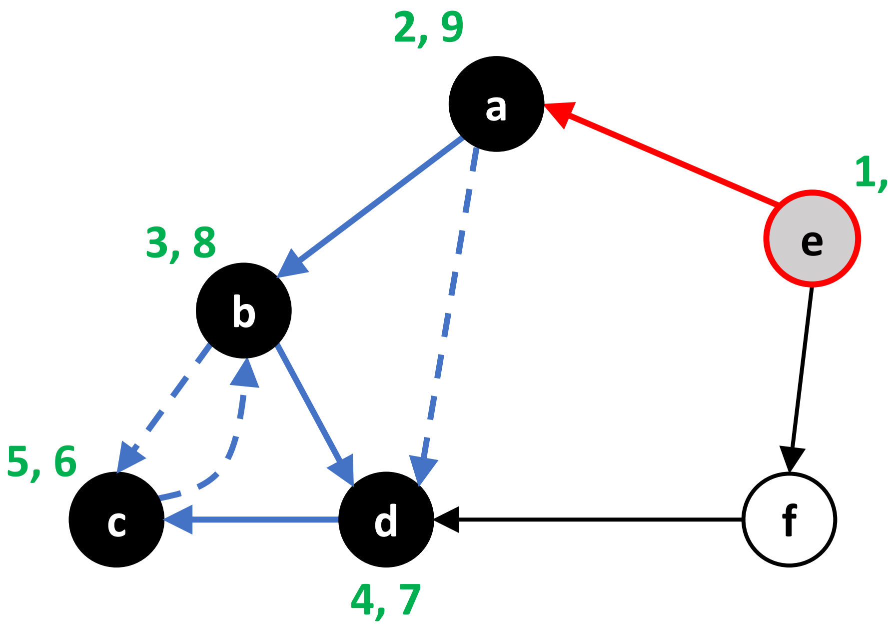


Colors of nodes:

e	a	b	c	d	f
G	G	B	B	B	W

DFS-Explore(G, e)

DFS(G)



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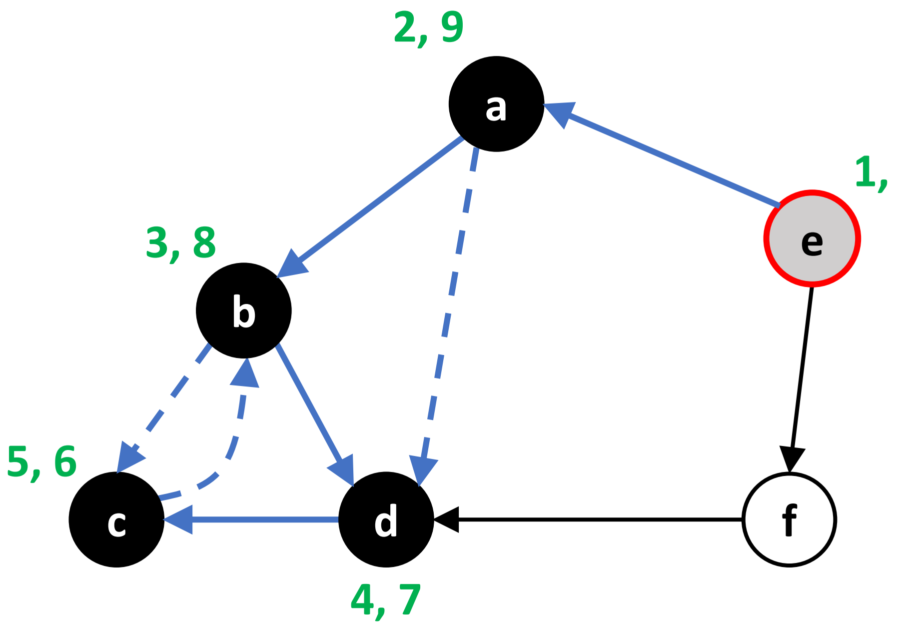
Colors of nodes:

G	B	B	B	B	W
e	a	b	c	d	f

⬆

DFS-Explore(G, e)

DFS(G)



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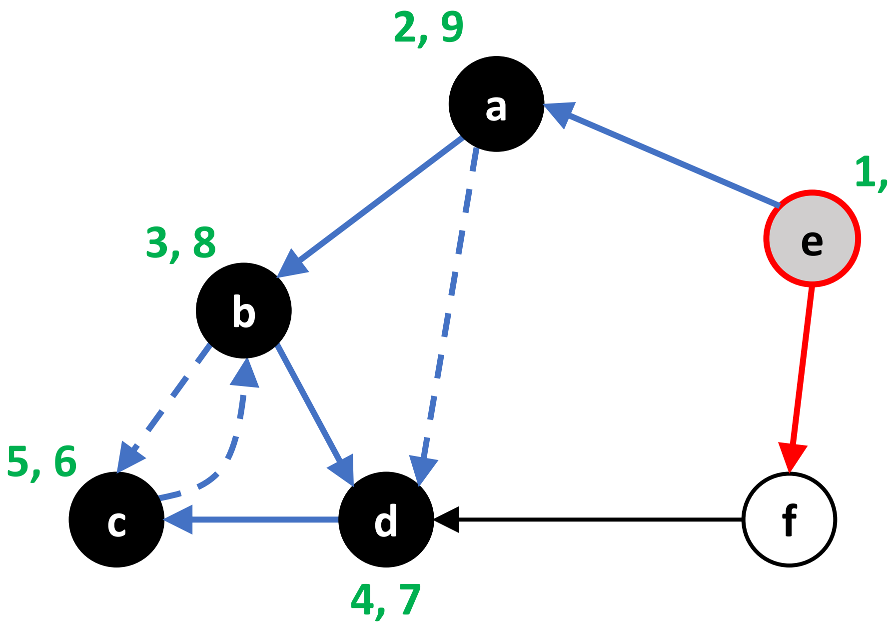
Colors of nodes:

G	B	B	B	B	W
e	a	b	c	d	f

DFS-Explore(G, **e)**

DFS(G)



Adj List of G :

✓ a → b → d
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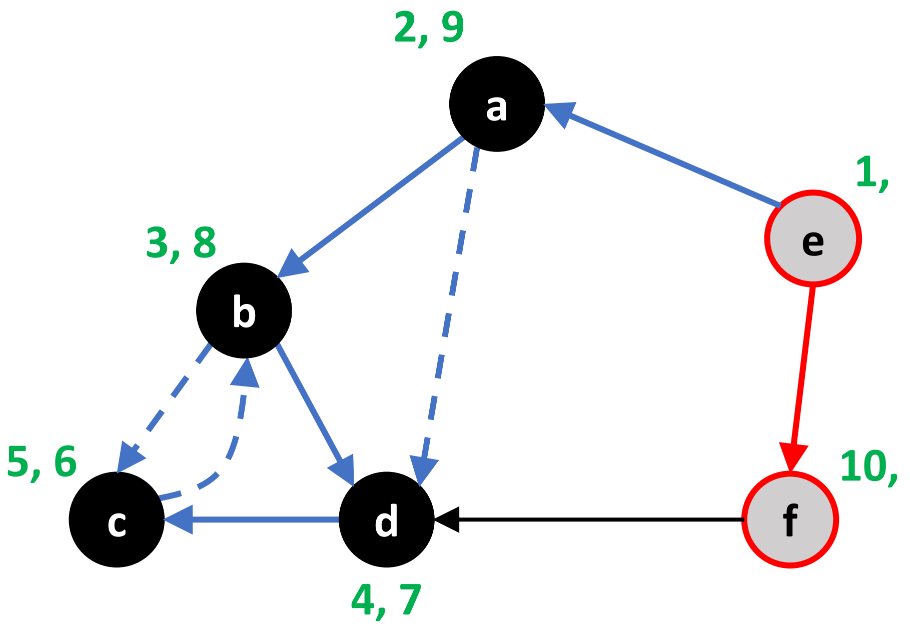
Colors of nodes:

G	B	B	B	B	W
e	a	b	c	d	f

DFS-Explore(G, e)

DFS(G)



Adj List of G :

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- ✓ c → b
- ✓ d → c
- e → a → f
- f → d

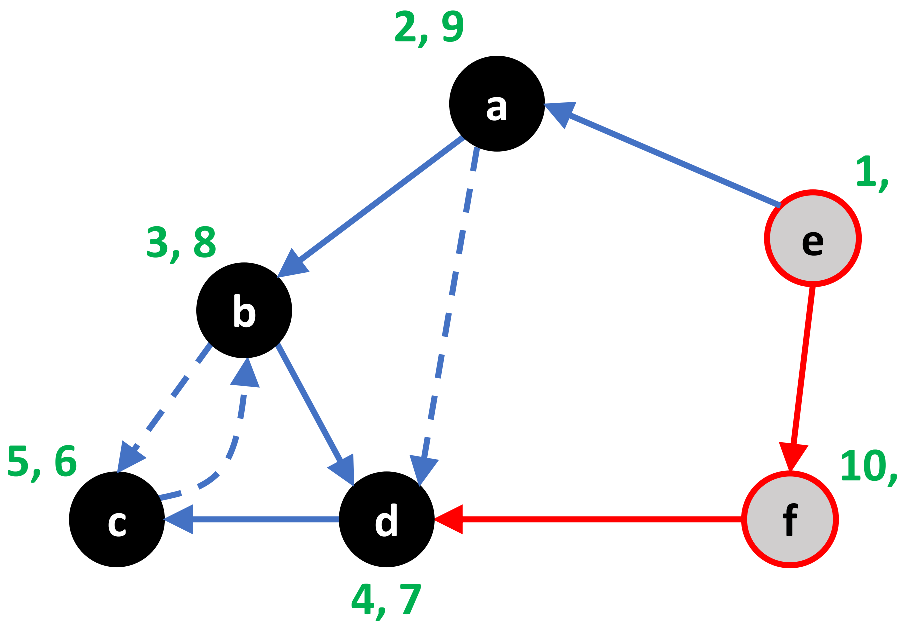
Colors of nodes:

G	B	B	B	B	G
e	a	b	c	d	f

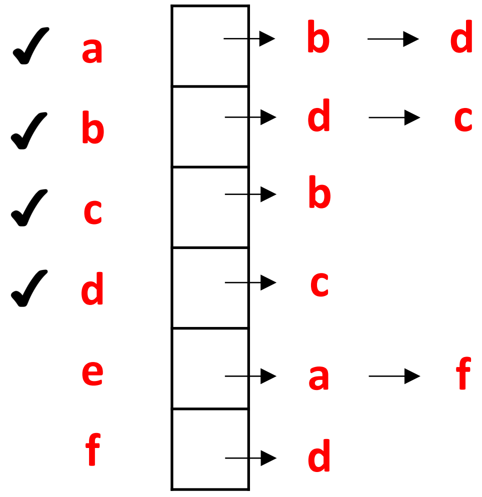
⬆

DFS-Explore(G, **e)**

DFS(G)



Adj List of G :

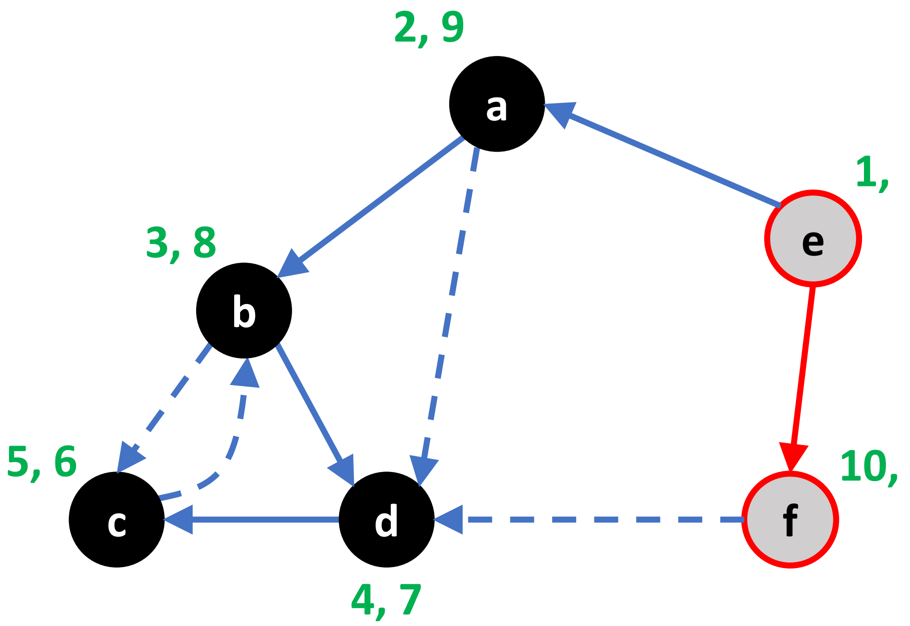


Colors of nodes:

G	B	B	B	B	G
e	**a**	**b**	**c**	**d**	**f**

DFS-Explore(G, e)

DFS(G)



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✓ a → b → d
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✓ c → b
✓ d → c
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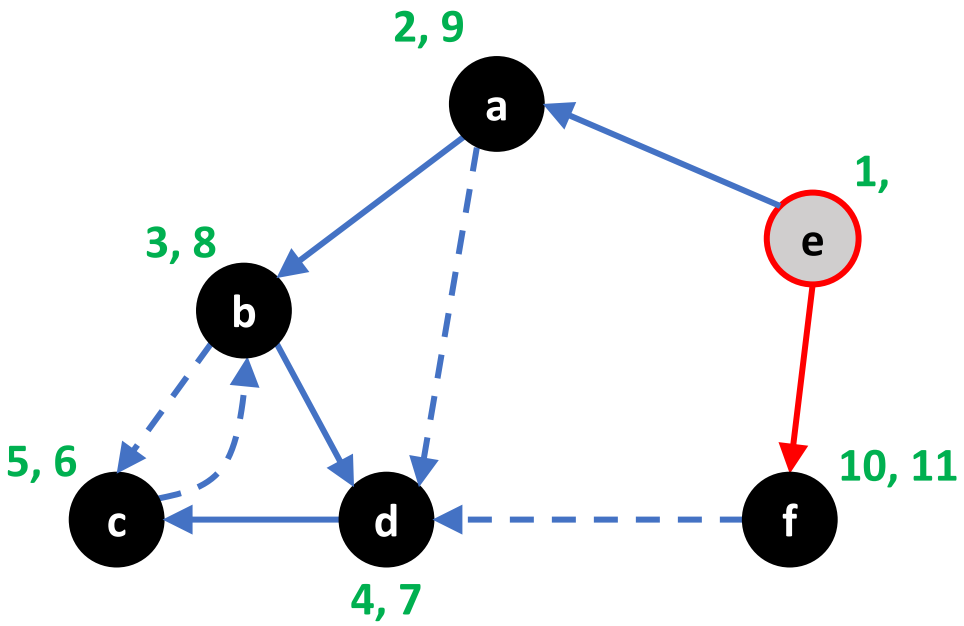
f → d

Colors of nodes:

G	B	B	B	B	G
e	a	b	c	d	f

DFS-Explore(G, e**)**

DFS(G)



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✓ b → d → c
✓ c → b
✓ d → c
e → a → f
✓ f → d

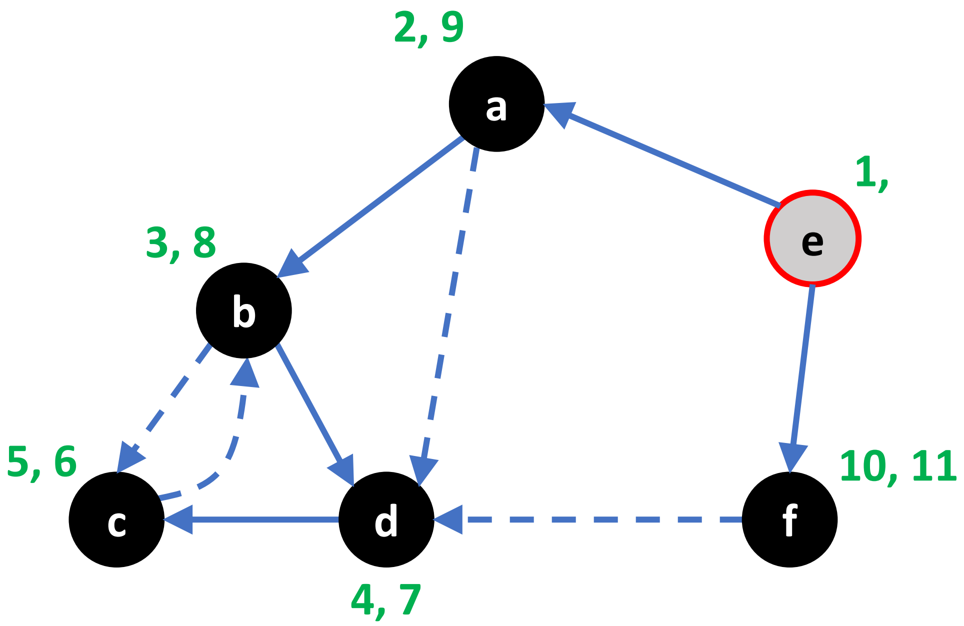
Colors of nodes:

G	B	B	B	B	B
e	**a**	**b**	**c**	**d**	**f**

⬆

DFS-Explore(G, **e)**

DFS(G)



Adj List of G :

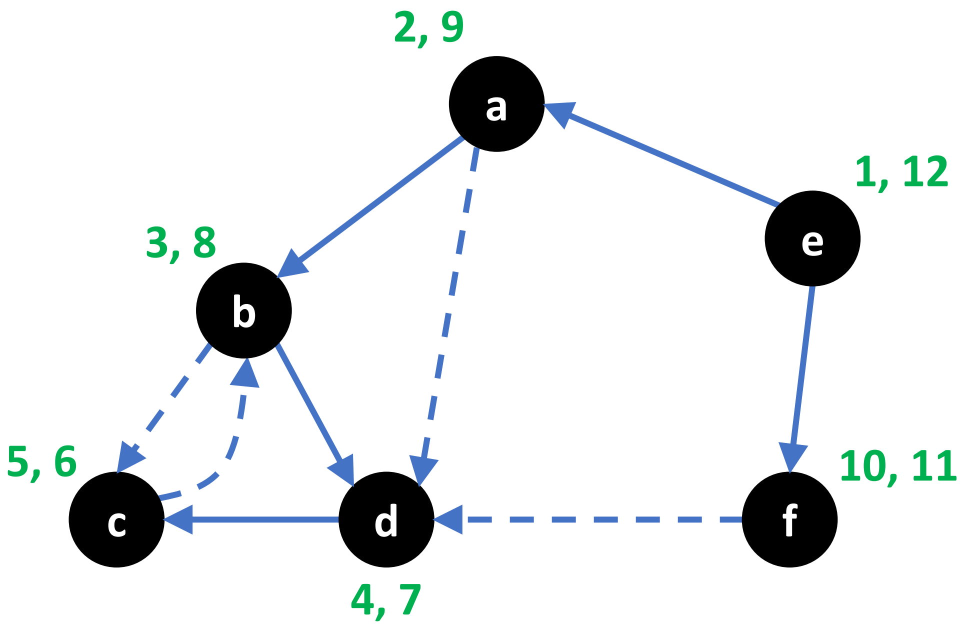
✓ a	b	d
✓ b	d	c
✓ c	b	
✓ d	c	
e	a	f
✓ f	d	

Colors of nodes:

e	a	b	c	d	f
G	B	B	B	B	B

DFS-Explore(G, e)

DFS(G)



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✓ c → b
✓ d → c
✓ e → a → f
✓ f → d

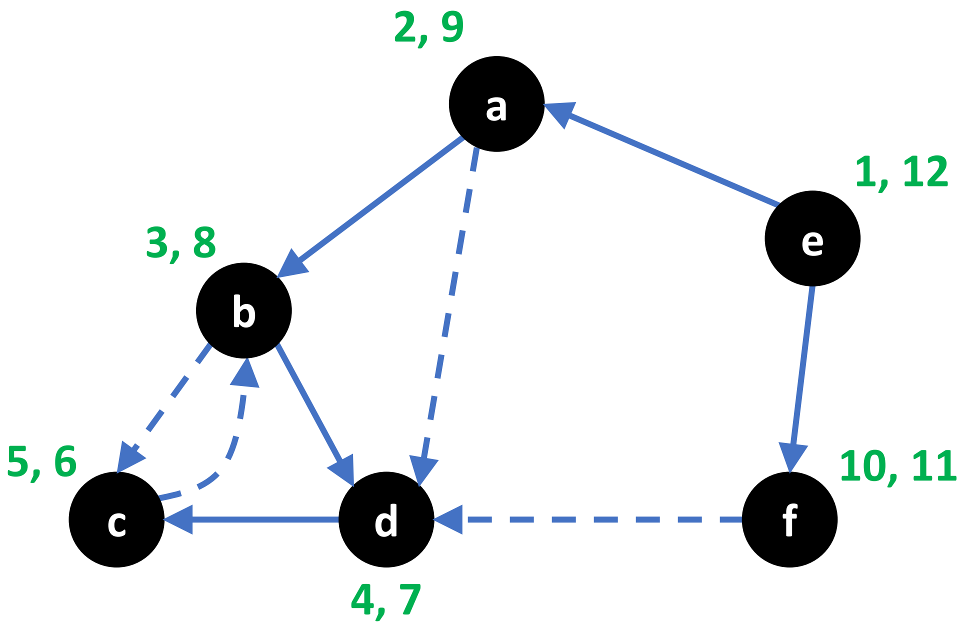
Colors of nodes:

B	B	B	B	B	B
e	**a**	**b**	**c**	**d**	**f**

⬆

DFS-Explore(G, **e)**

DFS(G)



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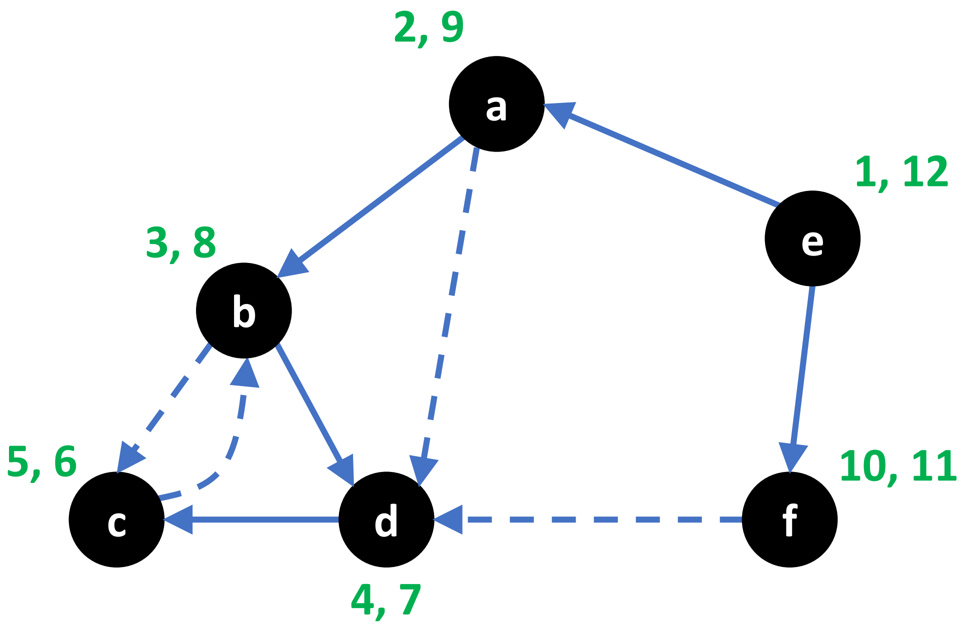
Colors of nodes:

B	B	B	B	B	B
e	a	b	c	d	f

DFS(G)



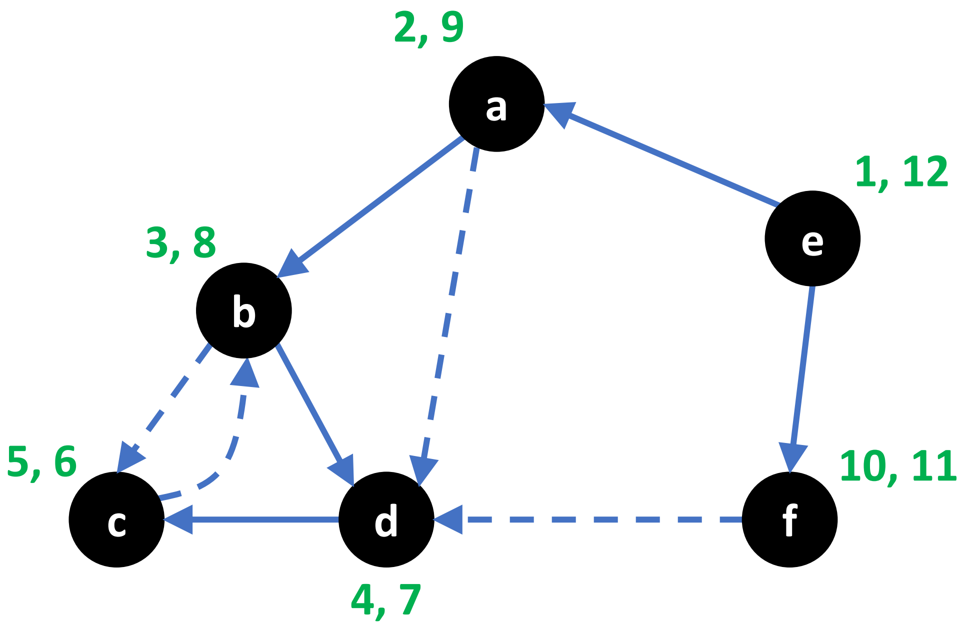
Adj List of G :

- ✓ a → b → d
- ✓ b → d → c
- ✓ c → b
- ✓ d → c
- ✓ e → a → f
- ✓ f → d

Colors of nodes:

e	a	b	c	d	f
B	B	B	B	B	B

DFS(G)



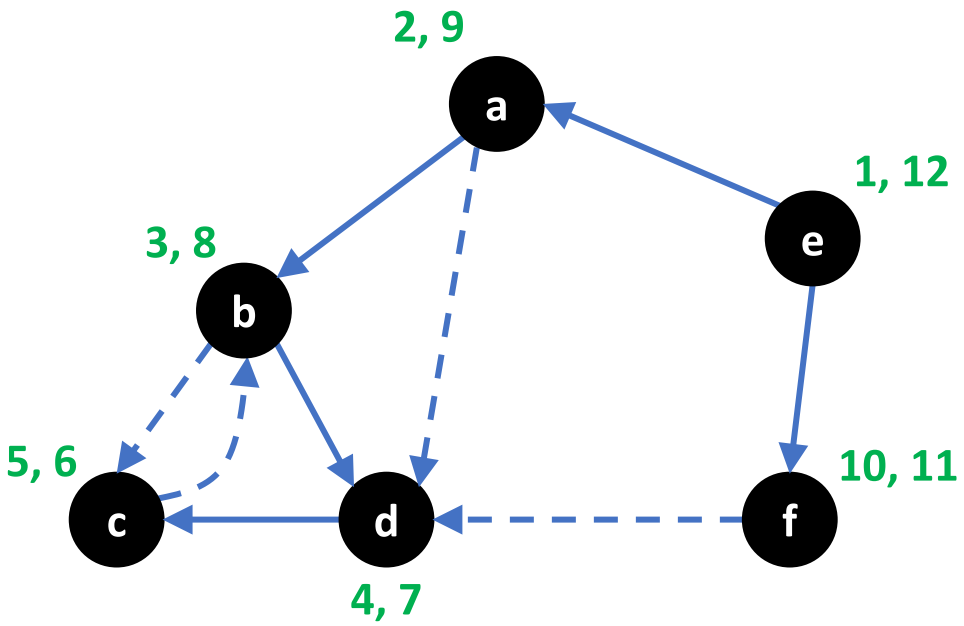
Adj List of G :

✓ a → b → d
✓ b → d → c
✓ c → b
✓ d → c
✓ e → a → f
✓ f → d

Colors of nodes:

B	B	B	B	B	B
e	a	b	c	d	f

DFS(G)



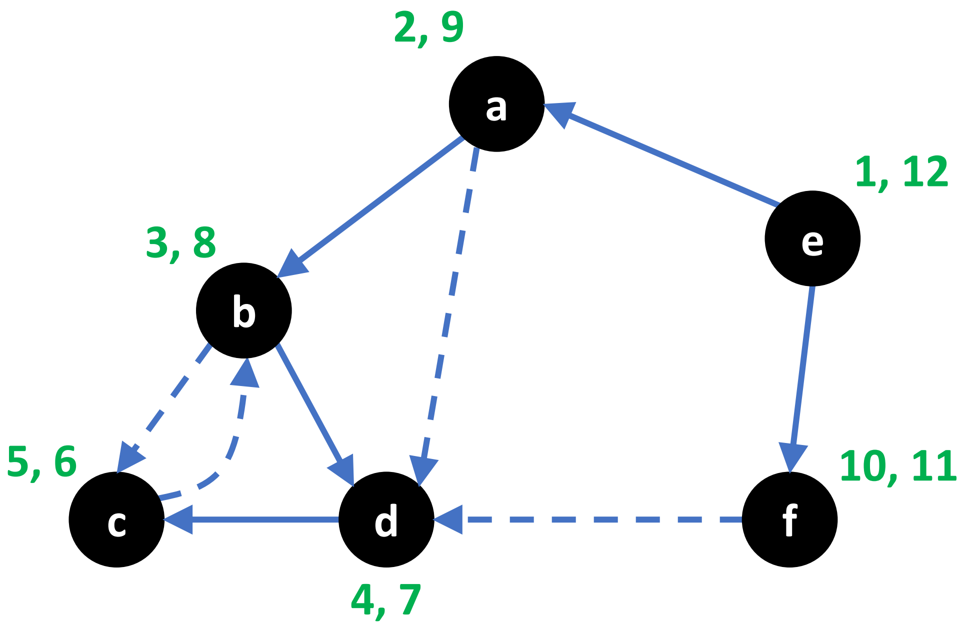
Adj List of G :

✓	a	→ b → d
✓	b	→ d → c
✓	c	→ b
✓	d	→ c
✓	e	→ a → f
✓	f	→ d

Colors of nodes:

e	a	b	c	d	f
B	B	B	B	B	B

DFS(G)



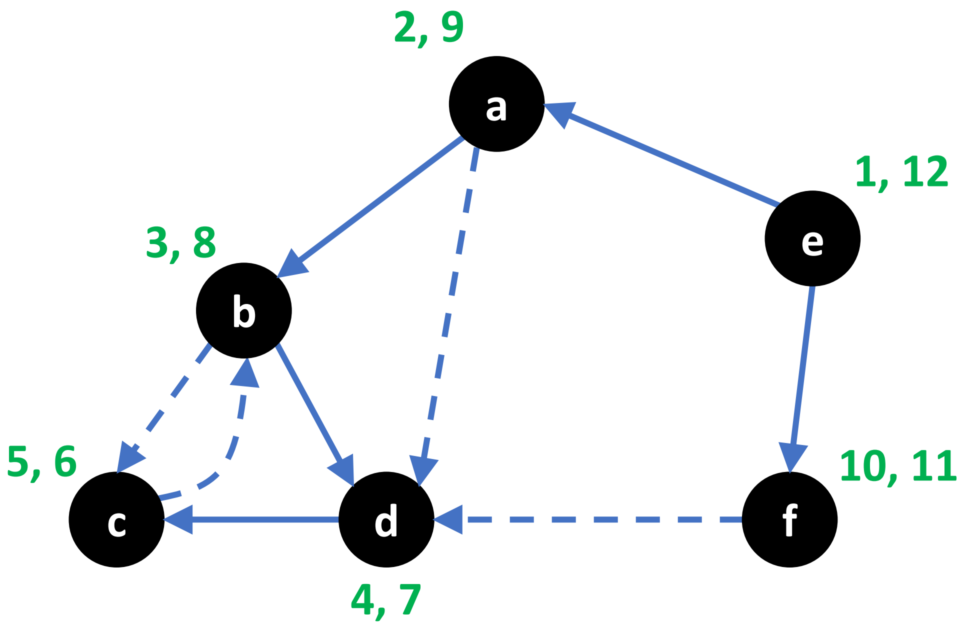
Adj List of G :

✓ a	→ b	→ d
✓ b	→ d	→ c
✓ c	→ b	
✓ d	→ c	
✓ e	→ a	→ f
✓ f	→ d	

Colors of nodes:

B	B	B	B	B	B
e	a	b	c	d	f

DFS(G)



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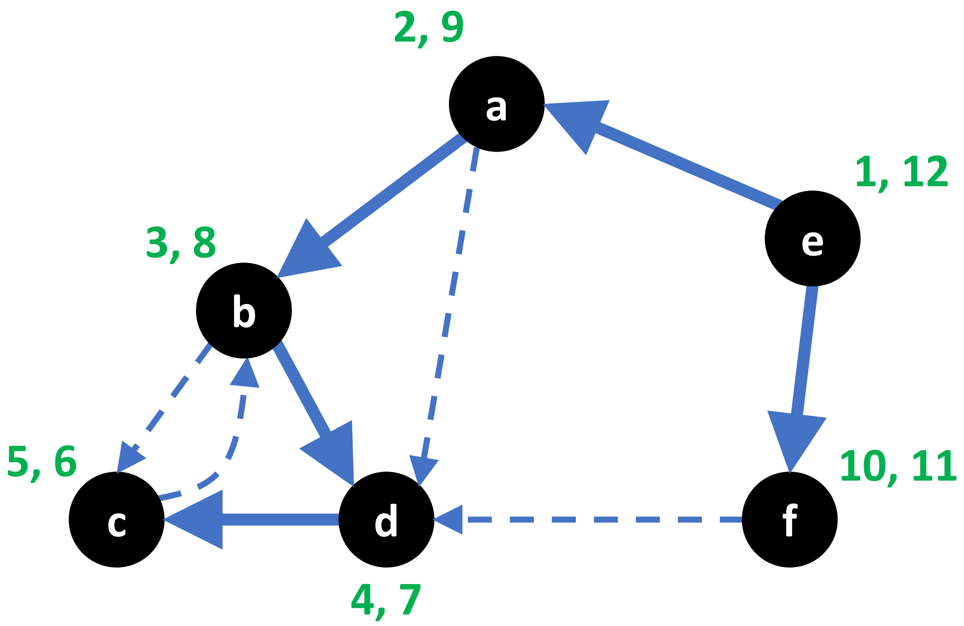
✓ c → b
✓ d → c
✓ e → a → f
✓ f → d

Colors of nodes:

B	B	B	B	B	B
e	a	b	c	d	f

DFS(G)

DFS Forest



Adj List of G :

✓ a → b → d

✓ b → d → c

✓ c → b

✓ d → c

✓ e → a → f

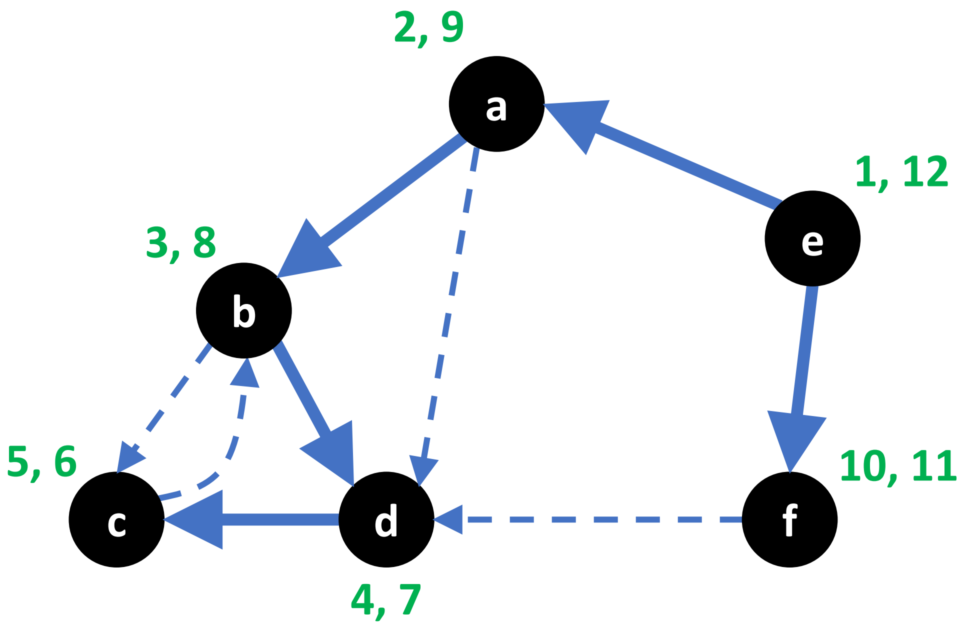
✓ f → d

Colors of nodes:

B	B	B	B	B	B
e	**a**	**b**	**c**	**d**	**f**

DFS(G)

Edge Classification by DFS



An edge (u, v) ∈ E is a

Tree edge ⇔ u is the **parent** of v in the DFS forest

Adj List of G :

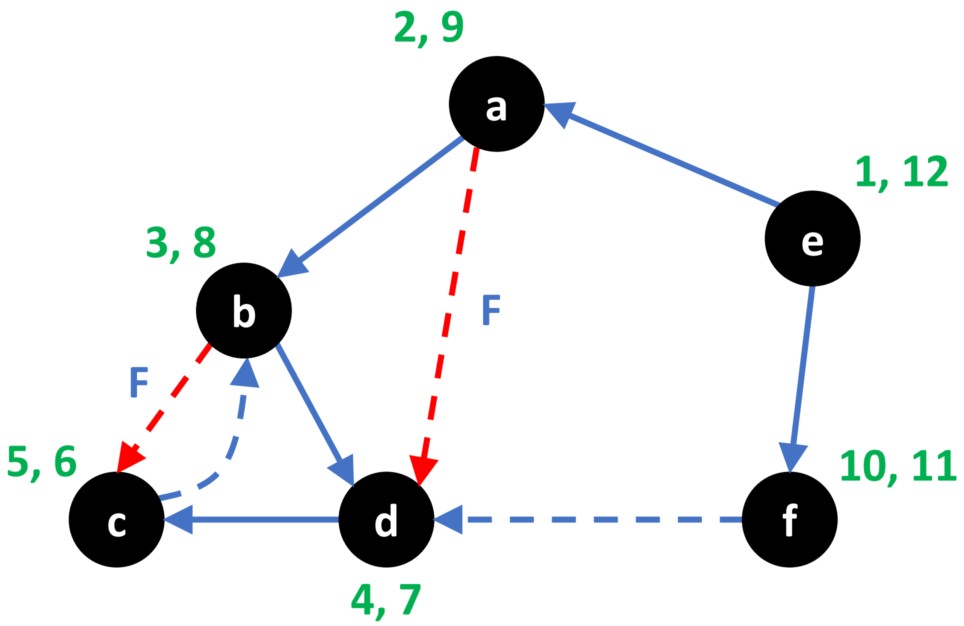
- ✓ a → b → d
- ✓ b → d → c
- ✓ c → b
- ✓ d → c
- ✓ e → a → f
- ✓ f → d

Colors of nodes:

B	B	B	B	B	B
e	a	b	c	d	f

DFS(G)

Edge Classification by DFS



A **non-tree** edge (u, v) ∈ E is a

Forward edge ⇔ u is an **ancestor** of v in the DFS forest

Adj List of G :

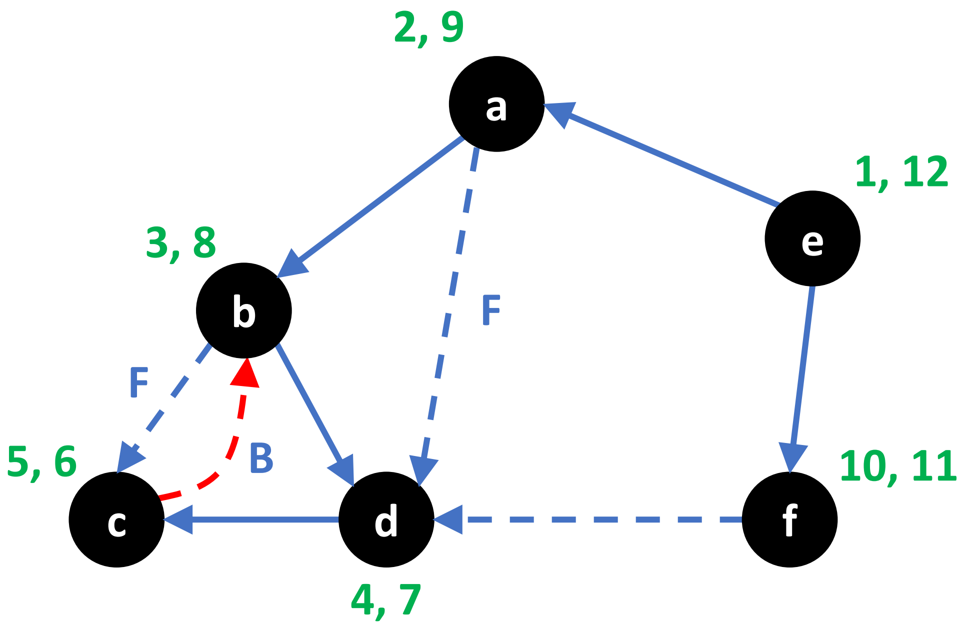
- ✓ a → b → d
- ✓ b → d → c
- ✓ c → b
- ✓ d → c
- ✓ e → a → f
- ✓ f → d

Colors of nodes:

e	a	b	c	d	f
B	B	B	B	B	B

DFS(G)

Edge Classification by DFS



A **non-tree** edge (u, v) ∈ E is a

Back edge ⇔ u is a **descendent** of v in the DFS forest

Adj List of G :

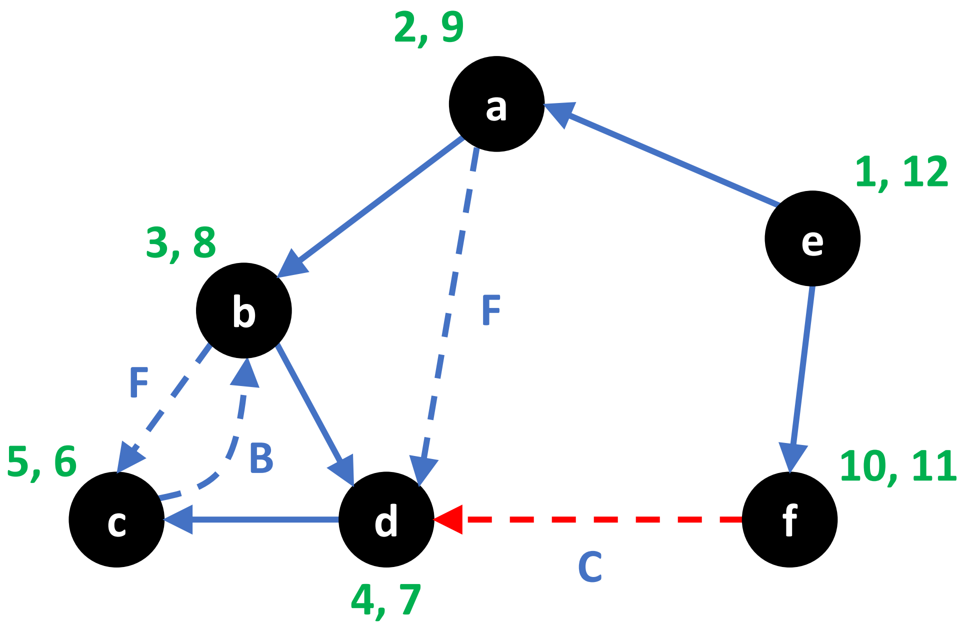
- ✓ a → b → d
- ✓ b → d → c
- ✓ c → b
- ✓ d → c
- ✓ e → a → f
- ✓ f → d

Colors of nodes:

B	B	B	B	B	B
e	a	b	c	d	f

DFS(G)

Edge Classification by DFS

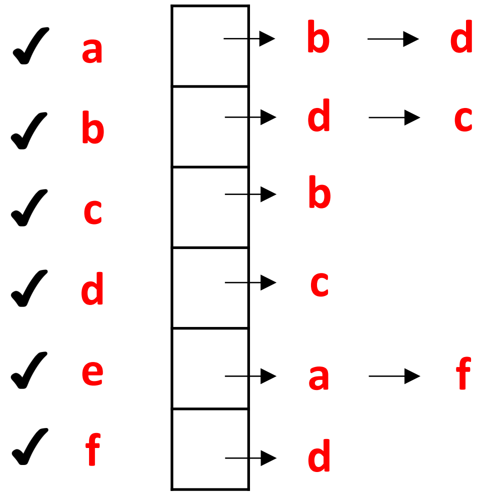


A **non-tree** edge (u, v) ∈ E is a

Cross edge ⇔ u is **neither ancestor nor descendent** of v in the DFS forest

⇔ not a forward or back edge

Adj List of G :



Colors of nodes:

B	B	B	B	B	B
e	a	b	c	d	f

Edge Classification by DFS

A DFS of a directed graph $G = (V, E)$ classifies its edges as follows.

$(u,v) \in E$ is a :

1. **Tree edge** $\Leftrightarrow$ u is the **parent** of v in the DFS forest

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Non-tree edges

Edge Classification by DFS

A DFS of a directed graph G = (V, E) classifies its edges as follows.

(u,v) ∈ E is a :

1. **Tree edge** ⇔ u is the **parent** of v in the DFS forest
2. **Forward edge** ⇔ u is an **ancestor** of v in the DFS forest
3. **Back edge** ⇔ u is a **descendant** of v in the DFS forest
4. **Cross edge** ⇔ u is **neither ancestor nor descendent** of v in the forest

Non-tree edges

Claim 1 :

u is an ancestor of v in a DFS of G

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Claim 2 :

For any 2 nodes u and v,

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For any 2 nodes u and v, we CANNOT have d[u]

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Claim 3 :

If $(u,v) \in E$

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Claim 2 :

Exploration of v

For any 2 nodes u and v, we CANNOT have $d[u] < d[v] < f[u] < f[v]$

Exploration of u

Claim 3 :

If $(u,v) \in E$ then

v is discovered **before** we finish exploring u.

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Claim 2 :

Exploration of v

For any 2 nodes u and v, we CANNOT have $d[u] < d[v] < f[u] < f[v]$

Exploration of u

Claim 3 :

If $(u,v) \in E$ then $d[v] < f[u]$

$$u \longrightarrow v$$

v is discovered **before** we finish exploring u.

Using DFS for Cycle Detection

White-Path-Theorem [Theorem 22.9 CLRS]

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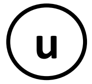

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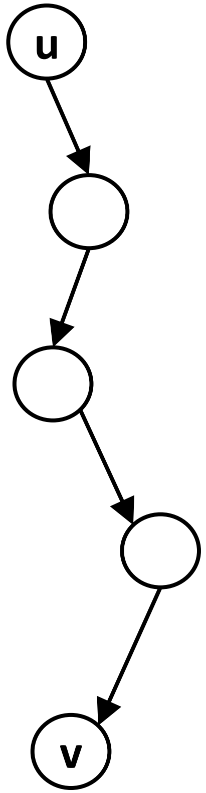


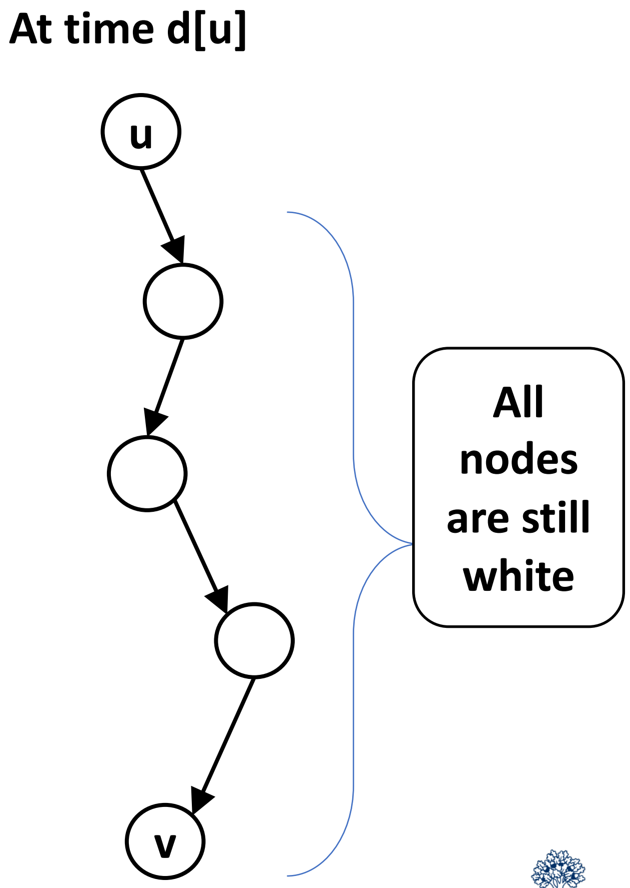
White-Path-Theorem [Theorem 22.9 CLRS]

For all graphs G and all DFS of G:

v becomes a descendant of u

$$\Downarrow$$

At the time d[u] the DFS discovers u,
there is a path from u to v in G
that consists entirely of white nodes



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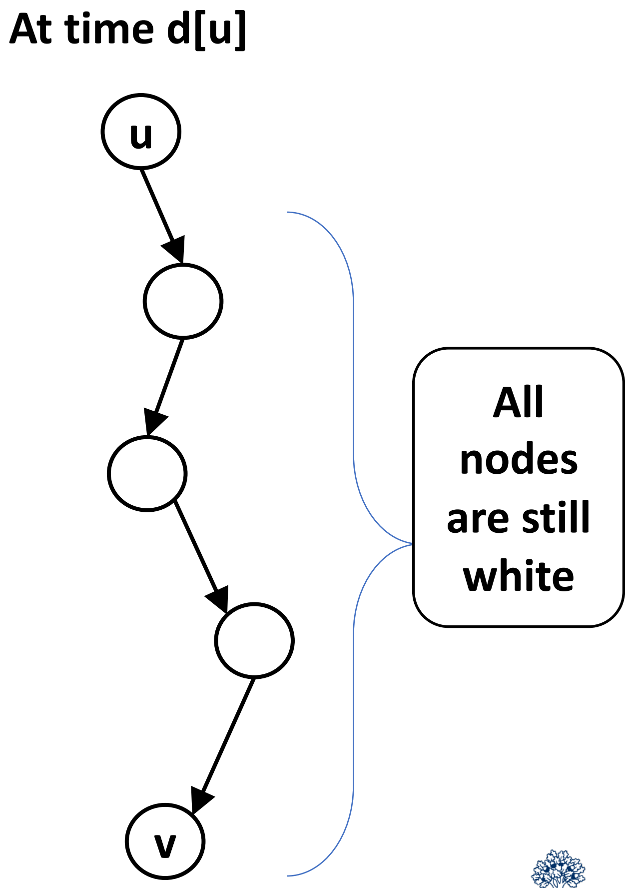

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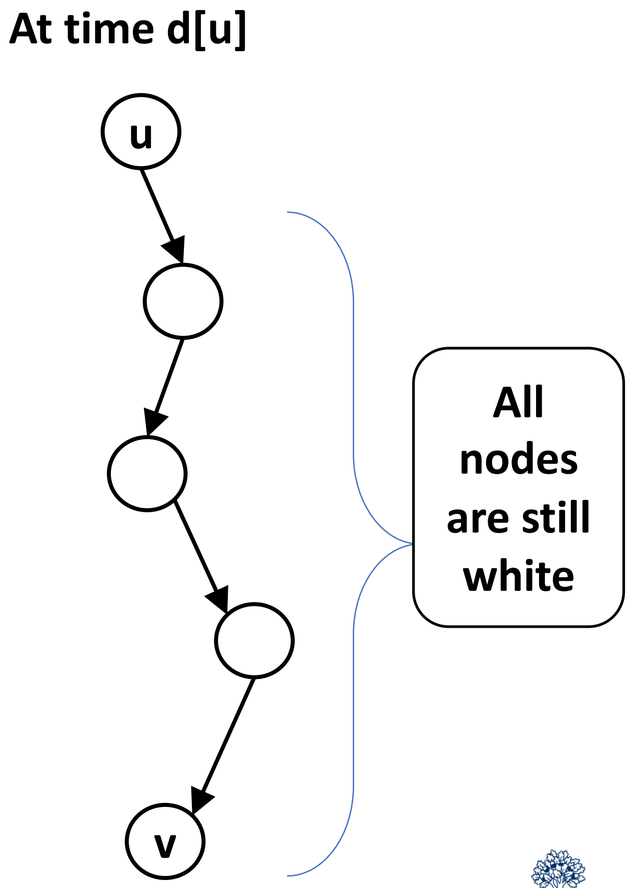

Proof: consider any G and any DFS of G

Proof: consider any G and any DFS of G

$\Rightarrow$: Suppose v is a descendant of u in this DFS.

Proof: consider any G and any DFS of G

⇒ : Suppose v is a descendant of u in this DFS. We must show:

At time d[u]

u

Proof: consider any G and any DFS of G

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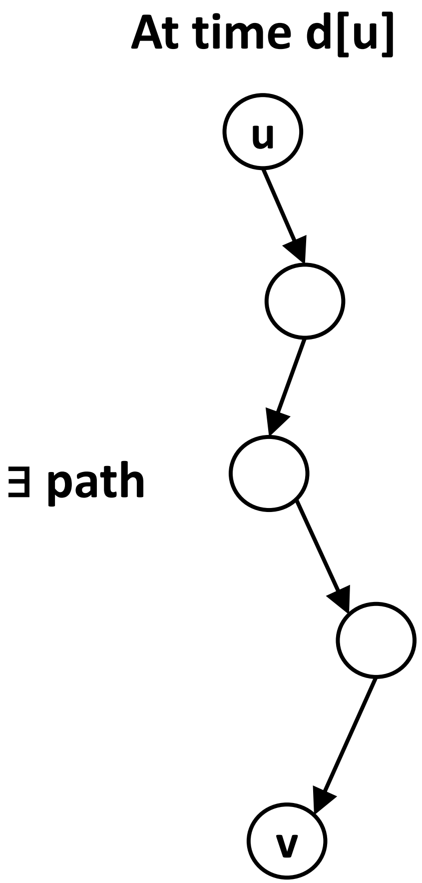

Proof: consider any G and any DFS of G

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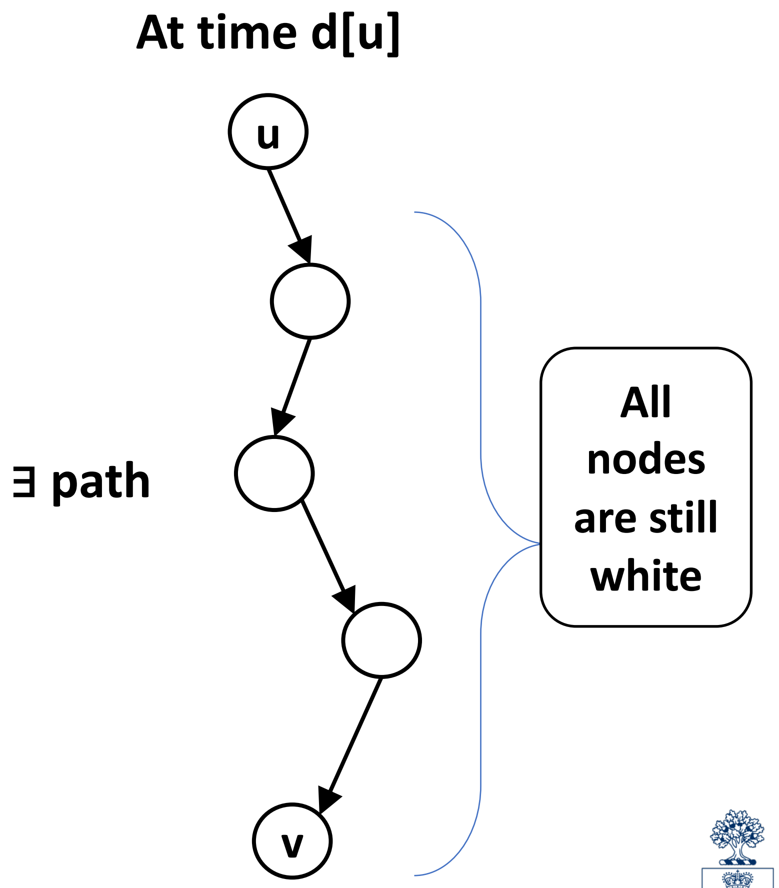
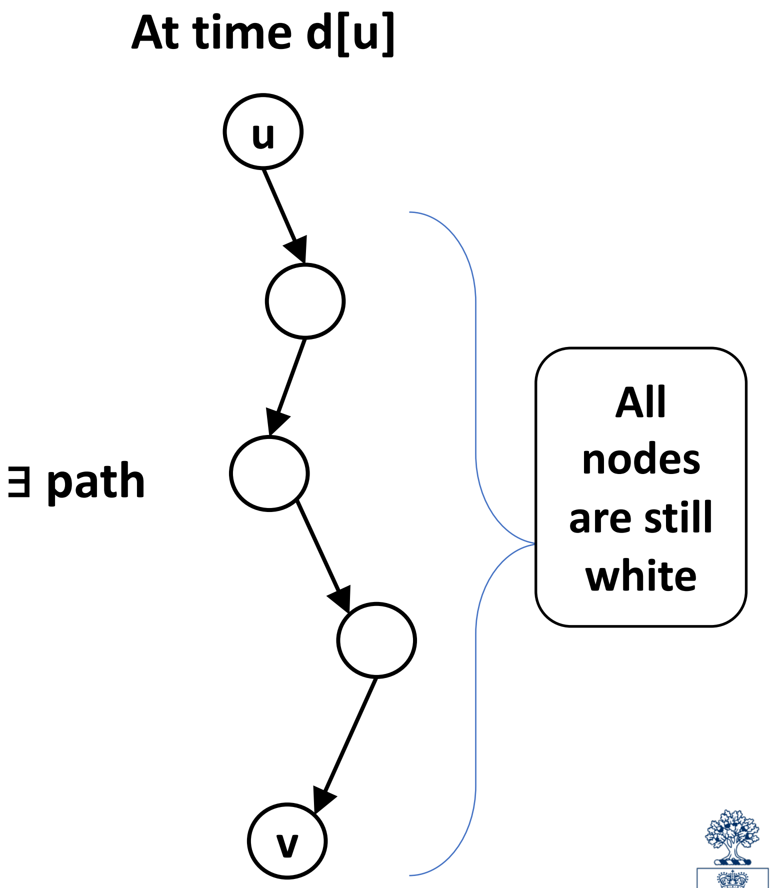


Proof: consider any G and any DFS of G

$\Rightarrow$: Suppose v is a descendant of u in this DFS. We must show:

Let $u \rightarrow u_1 \rightarrow u_2 \rightarrow \ldots u_j \rightarrow u_k \rightarrow v$
be the DFS **discovery path** from u to v

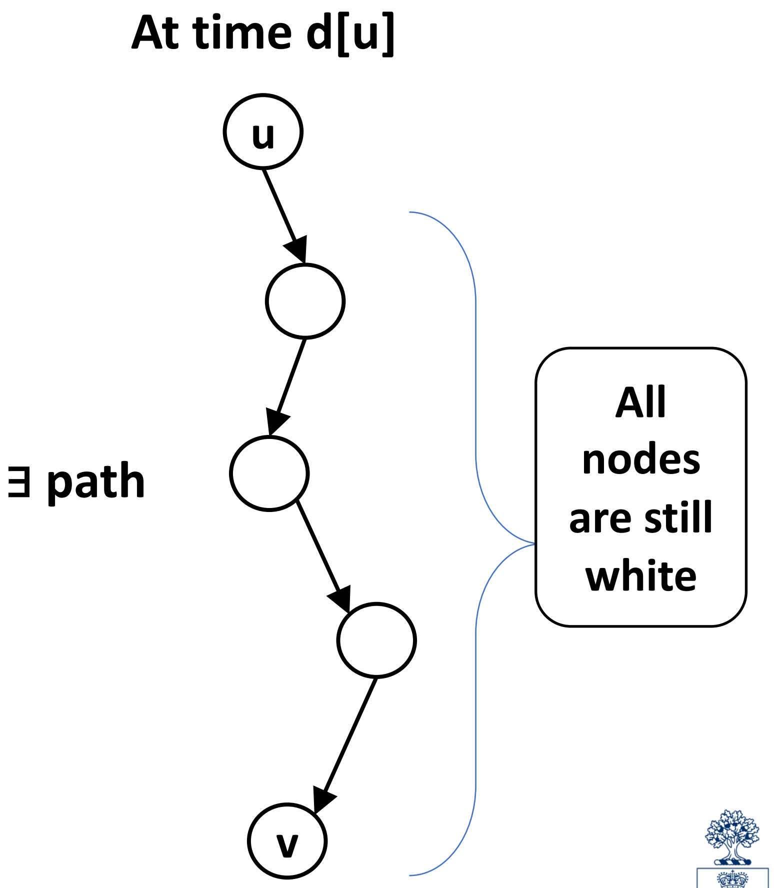


Proof: consider any G and any DFS of G

$\Rightarrow$: Suppose v is a descendant of u in this DFS. We must show:

Let $u \rightarrow u_1 \rightarrow u_2 \rightarrow . . . u_j \rightarrow u_k \rightarrow v$
be the DFS **discovery path** from u to v

At the time d[u] when u is discovered,
all the other nodes on that path
have **not** been discovered yet.



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Let $u \rightarrow u_1 \rightarrow u_2 \rightarrow . . . u_j \rightarrow u_k \rightarrow v$
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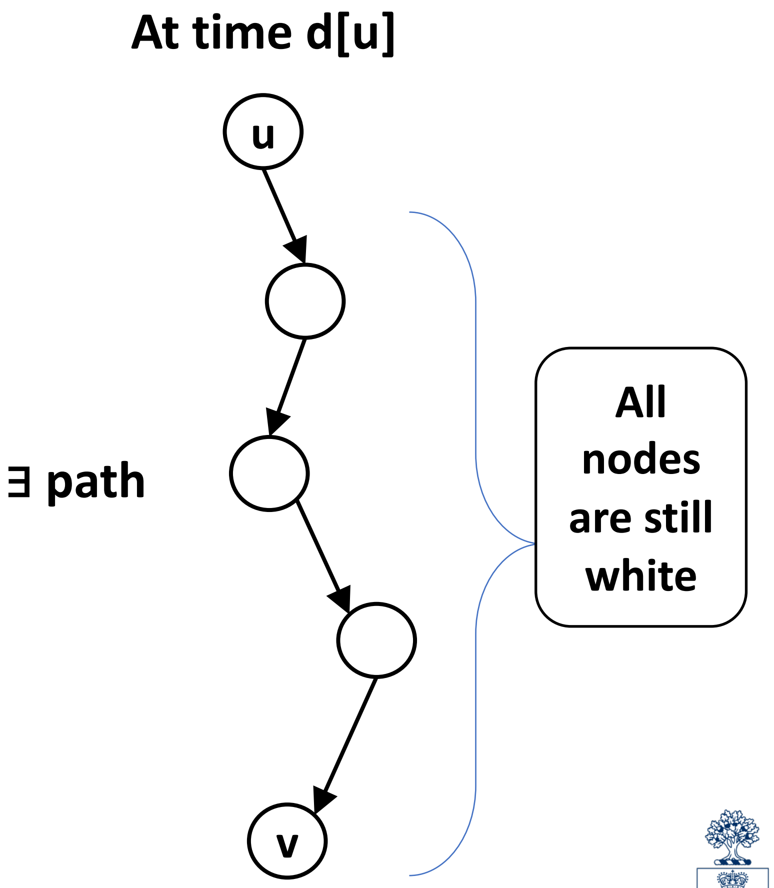
At the time d[u] when u is discovered,
all the other nodes on that path
have **not** been discovered yet.

Hence, they are all white.



Proof: consider any G and any DFS of G

$\Leftarrow$: Suppose that at the time d[u] when u is discovered,

Proof: consider any G and any DFS of G

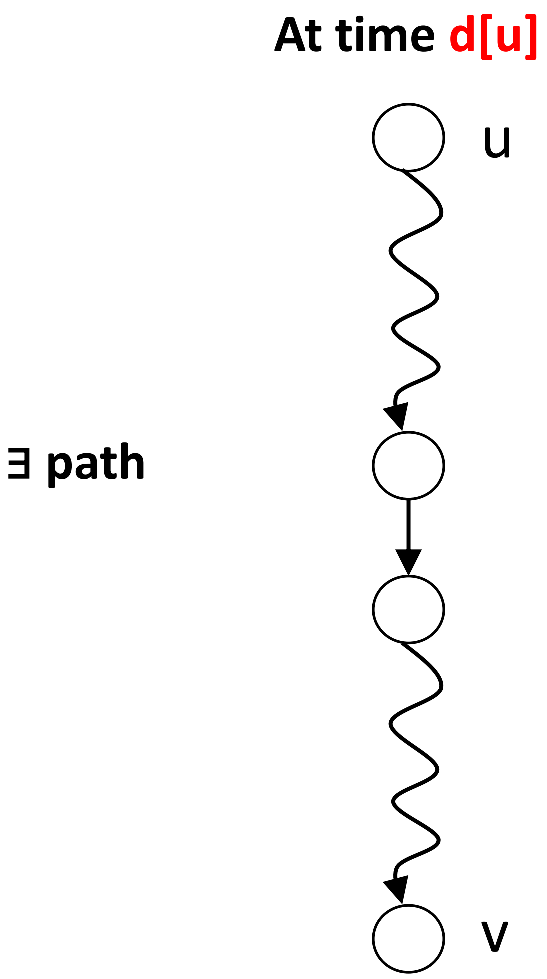
$\Leftarrow$: Suppose that at the time d[u] when u is discovered,

At time d[u]

u

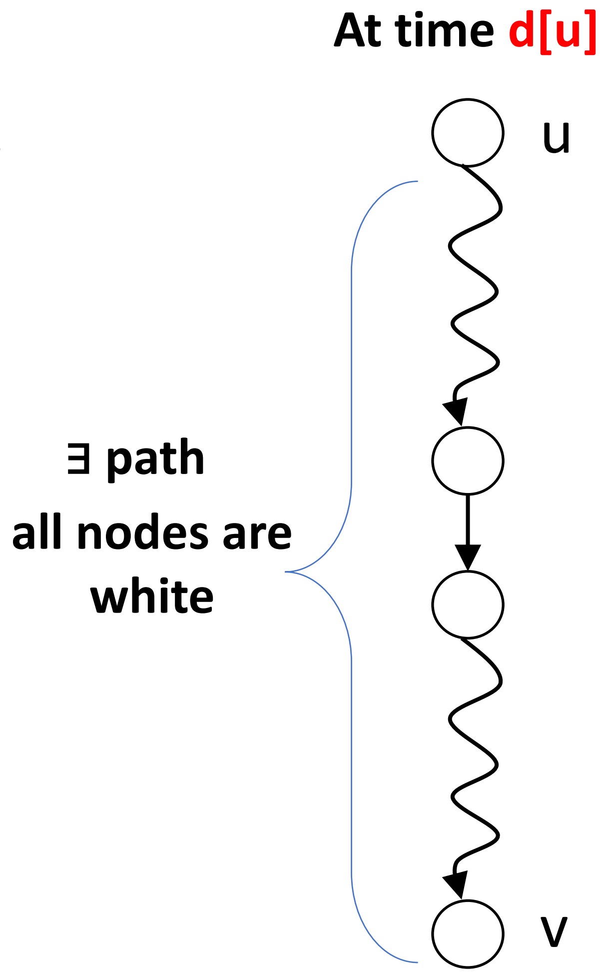
Proof: consider any G and any DFS of G

$\Leftarrow$: Suppose that at the time d[u] when u is discovered, there is a path from u to v



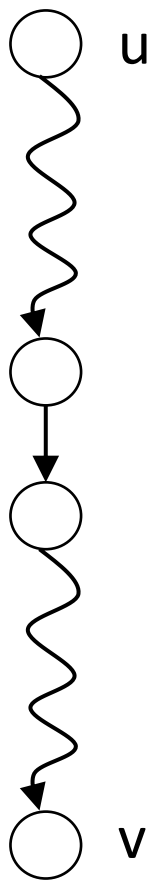
Proof: consider any G and any DFS of G

$\Leftarrow$: Suppose that at the time d[u] when u is discovered, there is a path from u to v consisting entirely of white nodes.



Proof: consider any G and any DFS of G

⇐ : Suppose that at the time d[u] when u is discovered,
there is a path from u to v consisting entirely of white nodes.

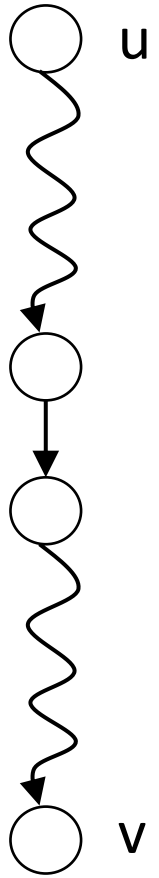


Claim: v become descendant of u.

Proof: consider any G and any DFS of G

⇐ : Suppose that at the time d[u] when u is discovered,
there is a path from u to v consisting entirely of white nodes.

Claim: **All** nodes in that path (including v) become descendants of u.



Proof: consider any G and any DFS of G

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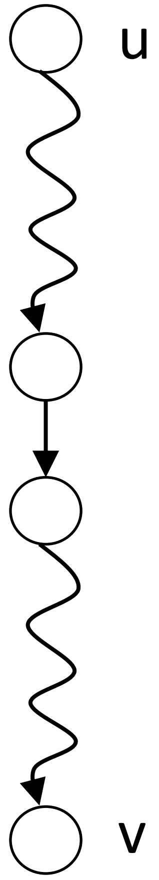
Claim: **All** nodes in that path (including v) become descendants of u.

Suppose for contradiction, this claim is false.



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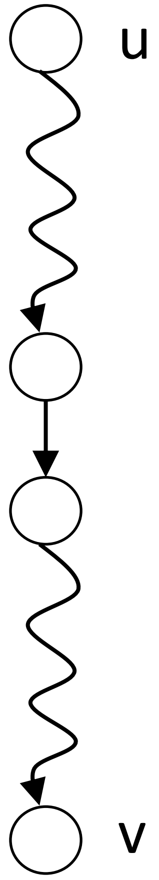


Claim: **All** nodes in that path (including v) become descendants of u.

Suppose for contradiction, this claim is false.

That is, **some** node on that path does **not** become a descendant of u.

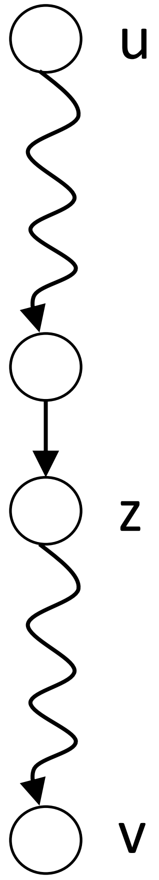
Proof: consider any G and any DFS of G

$\Leftarrow$: Suppose that at the time d[u] when u is discovered, there is a path from u to v consisting entirely of white nodes.

Claim: **All** nodes in that path (including v) become descendants of u.

Suppose for contradiction, this claim is false.

Let z be the **closest** node to u in that path that does **not** become a descendant of u.



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<u>Claim:</u> **All** nodes in that path (including v) become descendants of u.

Suppose for contradiction, this claim is false.

Let z be the **closest** node to u in that path
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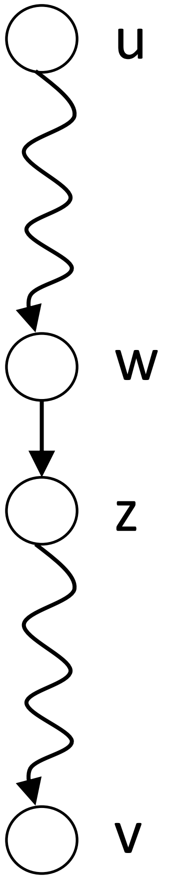
Let w be the node before z in that path.



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⇐ : Suppose that at the time d[u] when u is discovered, there is a path from u to v consisting entirely of white nodes.

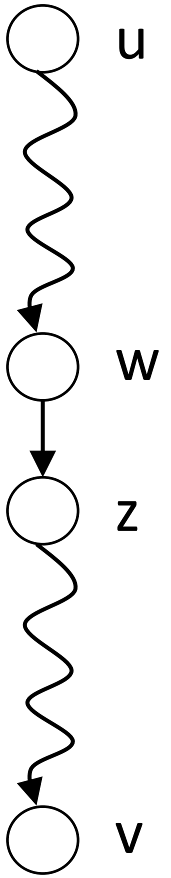
<ins>Claim:</ins> **All** nodes in that path (including v) become descendants of u.

Suppose for contradiction, this claim is false.

Let z be the **closest** node to u in that path that does **not** become a descendant of u.

Let w be the node before z in that path.

By the definition of z, w becomes a descendant of u



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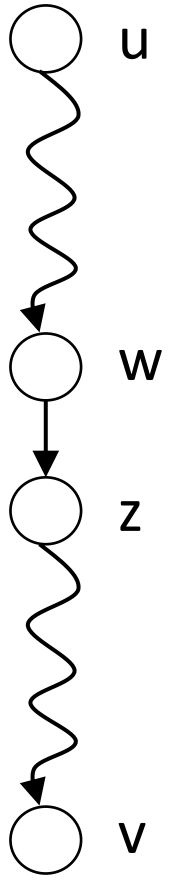
<u>Claim:</u> **All** nodes in that path (including v) become descendants of u.

Suppose for contradiction, this claim is false.

Let z be the **closest** node to u in that path that does **not** become a descendant of u.

Let w be the node before z in that path.

By the definition of z, w becomes a descendant of u, or w = u.



Proof: consider any G and any DFS of G

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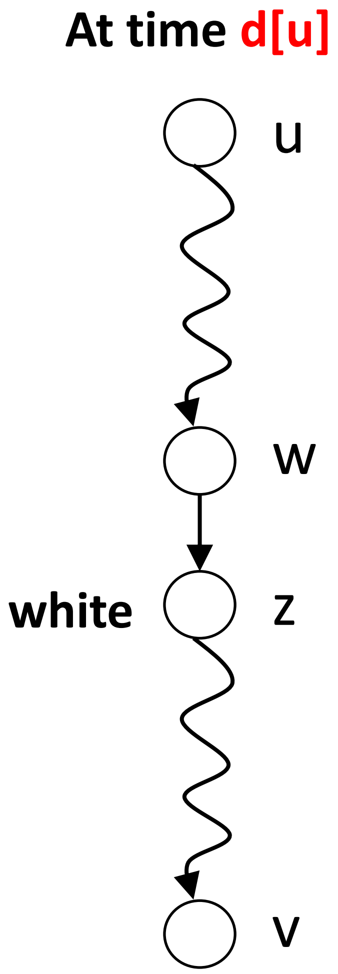

Proof: consider any G and any DFS of G

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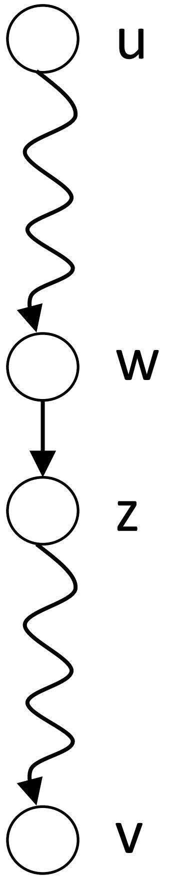
Suppose for contradiction, this claim is false.

Let z be the **closest** node to u in that path
that does **not** become a descendant of u.

Let w be the node before z in that path.

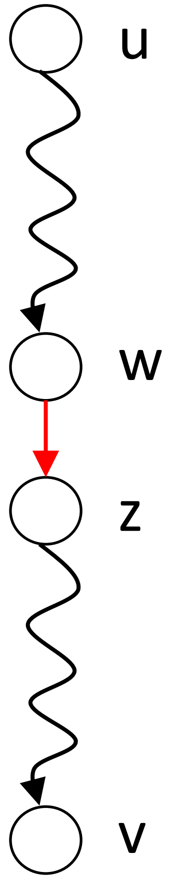
By the definition of z, w becomes a descendant of u, or w = u.

(1) d[u] < d[z] (when u is discovered, z is white)



Proof: consider any G and any DFS of G

$\Leftarrow$: Suppose that at the time d[u] when u is discovered, there is a path from u to v consisting entirely of white nodes.



Claim: **All** nodes in that path (including v) become descendants of u.

Suppose for contradiction, this claim is false.

Let z be the **closest** node to u in that path that does **not** become a descendant of u.

Let w be the node before z in that path.

By the definition of z, w becomes a descendant of u, or w = u.

(1) d[u] < d[z] (when u is discovered, z is white)

Proof: consider any G and any DFS of G

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Suppose for contradiction, this claim is false.

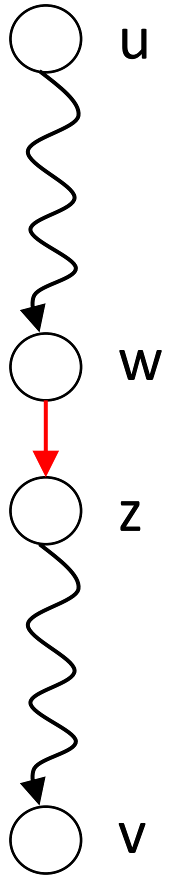
Let z be the **closest** node to u in that path that does **not** become a descendant of u.

Let w be the node before z in that path.

By the definition of z, w becomes a descendant of u, or w = u.

(1) d[u] < d[z] (when u is discovered, z is white)

(z is discovered before we finish exploring w)



Proof: consider any G and any DFS of G

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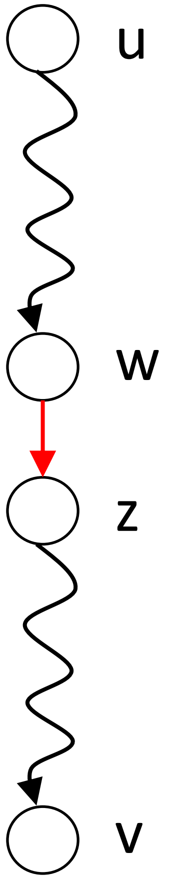

Claim: **All** nodes in that path (including v) become descendants of u.

Suppose for contradiction, this claim is false.

Let z be the **closest** node to u in that path that does **not** become a descendant of u.

Let w be the node before z in that path.

By the definition of z, w becomes a descendant of u, or w = u.

(1) $d[u] < d[z]$ (when u is discovered, z is white)

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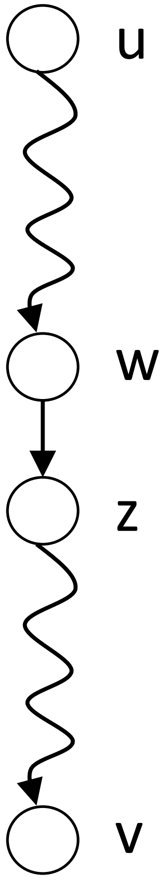

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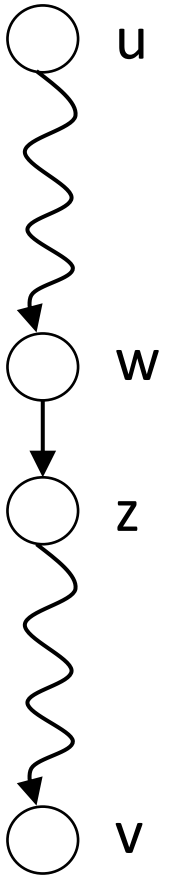

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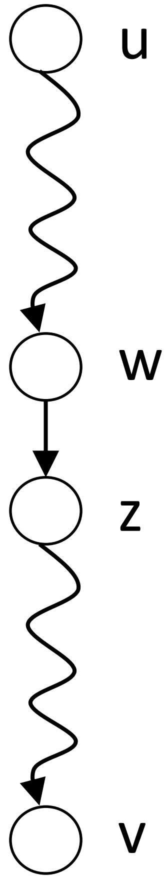

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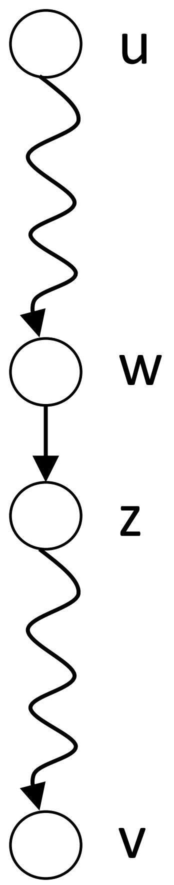

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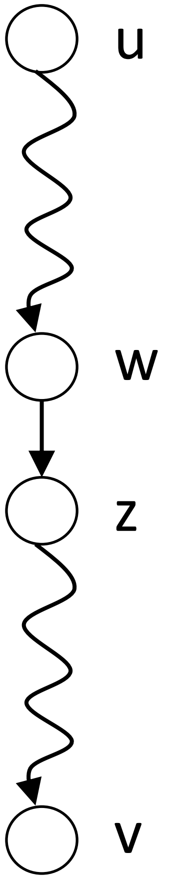

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(1) & (2) & (3) $\Rightarrow$

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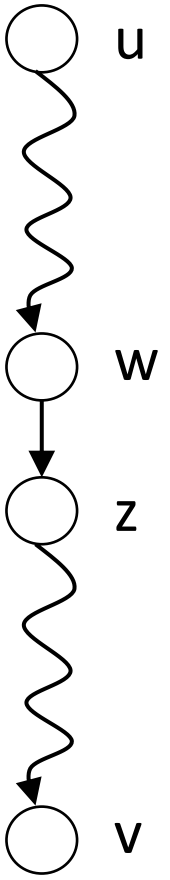

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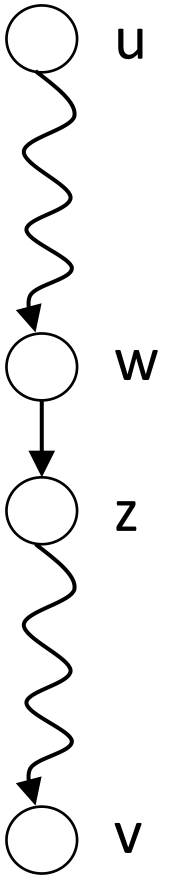

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(1) & (2) & (3) $\Rightarrow$ $d[u] < d[z] < f[u]$ $\Rightarrow$ $d[u] < d[z] < f[z] < f[u]$ (By Claim 2)

Proof: consider any G and any DFS of G

$\Leftarrow$: Suppose that at the time $d[u]$ when u is discovered,
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<u>Claim:</u> **All** nodes in that path (including v) become descendants of u.

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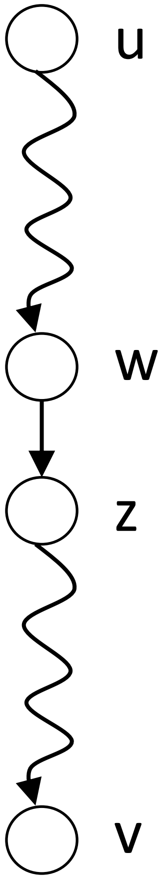

(1) & (2) & (3) $\Rightarrow$ $d[u] < d[z] < f[u]$ $\Rightarrow$ $d[u] < d[z] < f[z] < f[u]$ (By Claim 2)

$\Rightarrow$ z is a descendant of u (By Claim 1)

Proof: consider any G and any DFS of G

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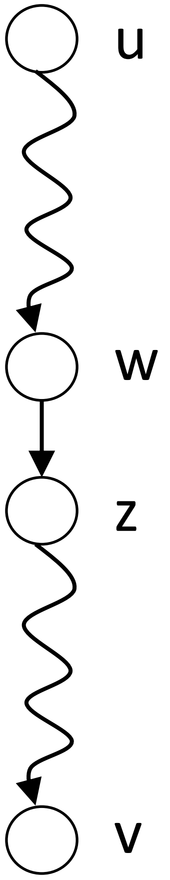

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$\Rightarrow$ z is a descendant of u (By Claim 1) Contradiction!

White-Path-Theorem [Theorem 22.9 CLRS]

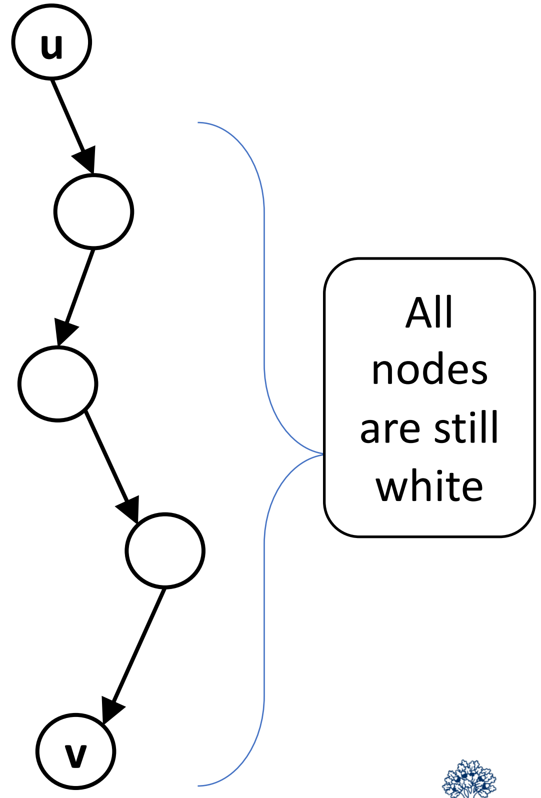
For all graphs G and all DFS of G:

v becomes a descendant of u

$\Updownarrow$

At the time d[u] the DFS discovers u,
there is a path from u to v in G
that consists entirely of white nodes

At time d[u]



Application of White-Path-Theorem [Theorem 22.11 CLRS]

For all directed graphs G and all DFS of G:

G has a cycle

$$\Updownarrow$$

DFS of G has a back edge

Proof: consider any directed G and any DFS of G

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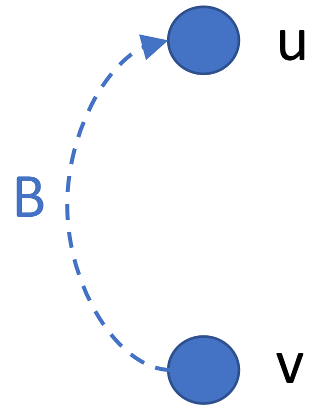
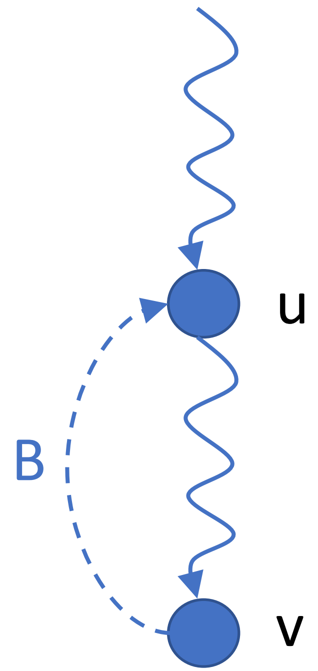
$\Leftarrow$: Suppose DFS of G has a back edge (v, u).



Proof: consider any directed G and any DFS of G

$\Leftarrow$: Suppose DFS of G has a back edge (v, u).

Then v is a descendant of u in the DFS.

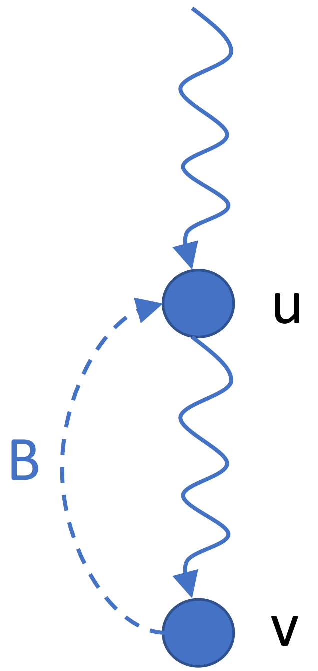


Proof: consider any directed G and any DFS of G

$\Leftarrow$: Suppose DFS of G has a back edge (v, u).

Then v is a descendant of u in the DFS.

Hence, u $\rightsquigarrow$ v is a path in the DFS forest.



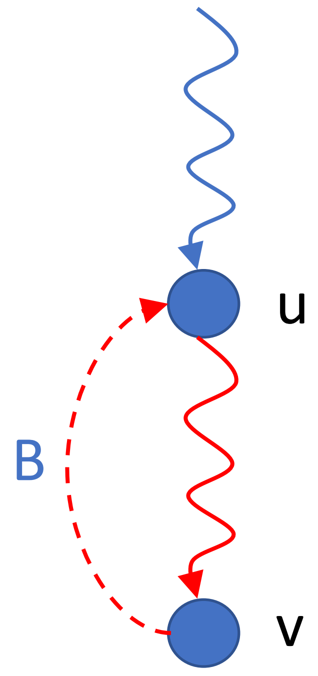
Proof: consider any directed G and any DFS of G

⇐ : Suppose DFS of G has a back edge (v, u).

Then v is a descendant of u in the DFS.

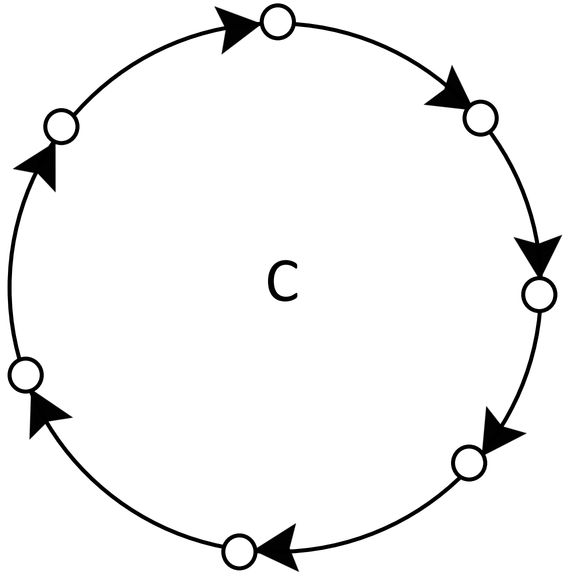
Hence, $u \rightsquigarrow v$ is a path in the DFS forest.

Hence, G has a cycle $u \rightsquigarrow v \to u$.



Proof: consider any directed G and any DFS of G

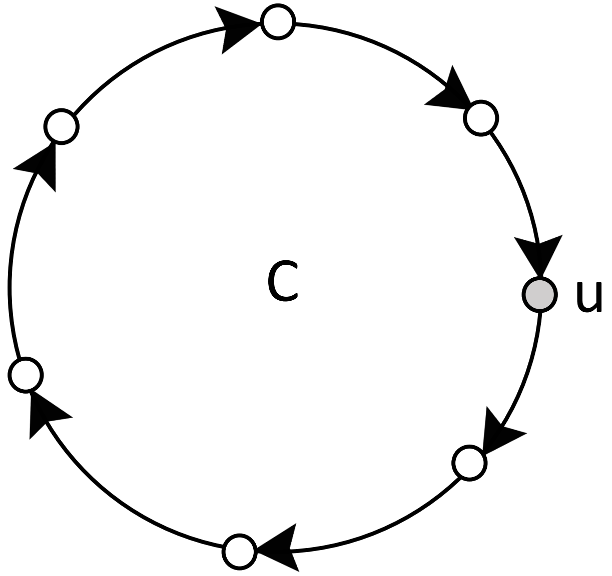
⇒ : Suppose G has a cycle C.



Proof: consider any directed G and any DFS of G

$\Rightarrow$: Suppose G has a cycle C.

Let u be the **first** node in C that the DFS discovers.

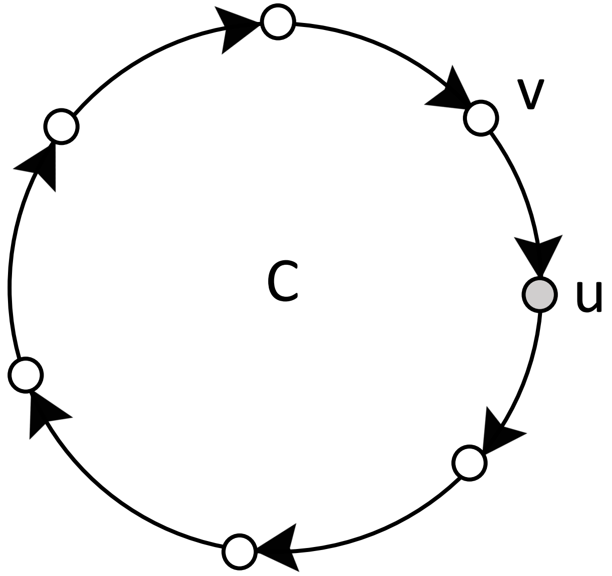


Proof: consider any directed G and any DFS of G

⇒ : Suppose G has a cycle C.

Let u be the **first** node in C that the DFS discovers.

Let v be the node before u in C.



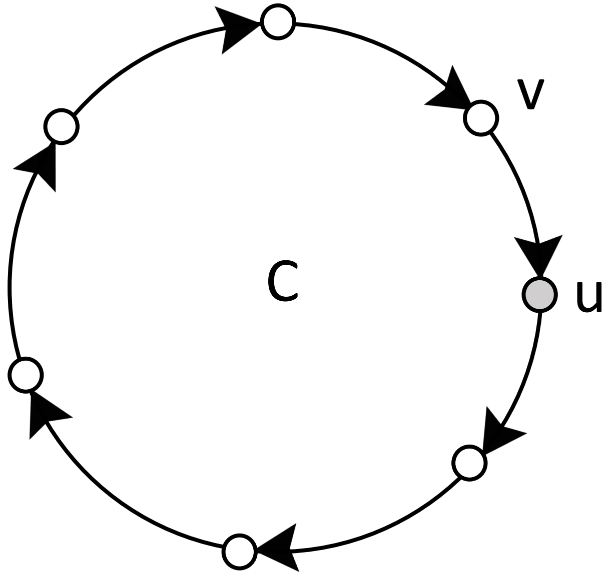
Proof: consider any directed G and any DFS of G

$\Rightarrow$: Suppose G has a cycle C.

Let u be the **first** node in C that the DFS discovers.

Let v be the node before u in C.

At time d[u] when DFS discovers u, all nodes in path u $\rightsquigarrow$ v in C are



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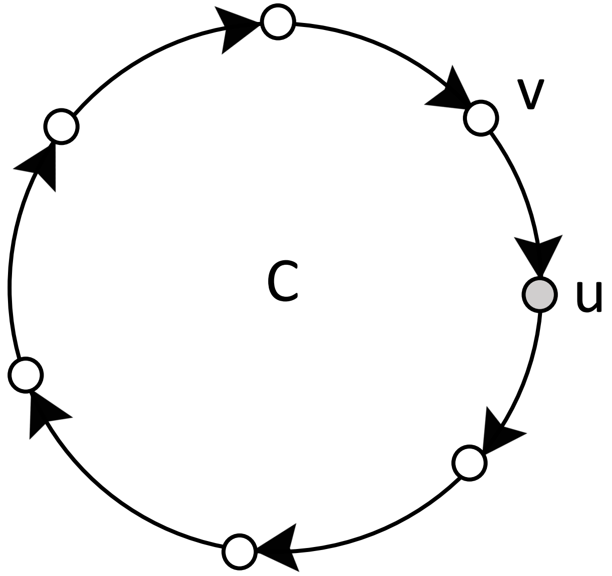
⇒ :   Suppose G has a cycle C.

Let u be the **first** node in C that the DFS discovers.

Let v be the node before u in C.

At time d[u] when DFS discovers u, all nodes in path u $\rightsquigarrow$ v in C are (white, grey or black?)



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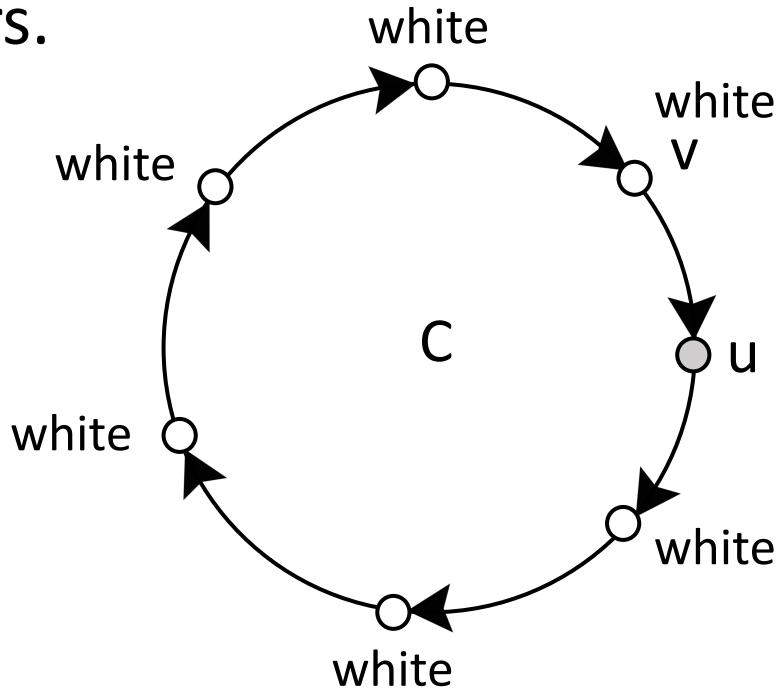
$\Rightarrow$: Suppose G has a cycle C.

Let u be the **first** node in C that the DFS discovers.

Let v be the node before u in C.

At time d[u] when DFS discovers u, all nodes in path u $\rightsquigarrow$ v in C are still white, including v.



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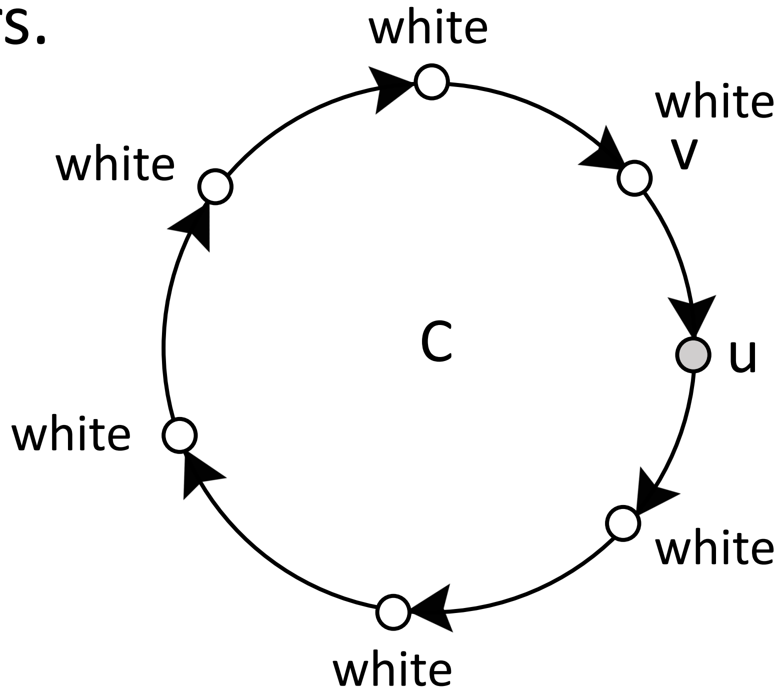
By the WPT:



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Let u be the **first** node in C that the DFS discovers.

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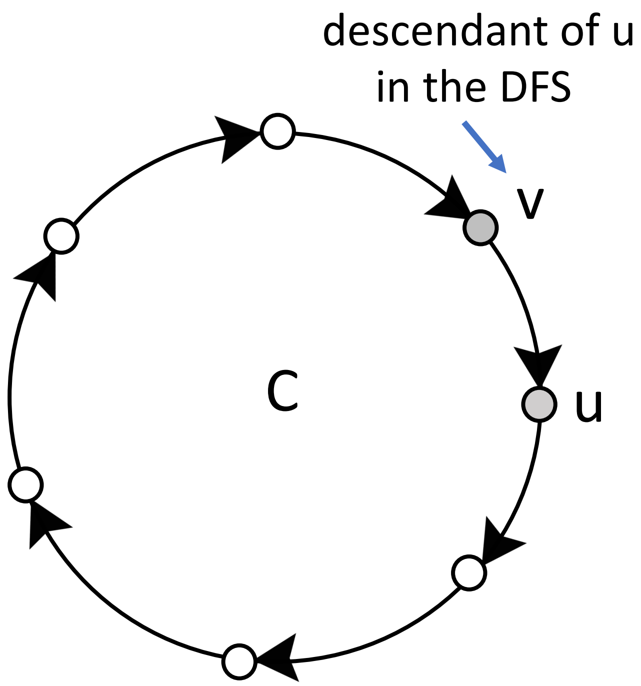
At time d[u] when DFS discovers u, all nodes in path u $\rightsquigarrow$ v in C are still white, including v.

By the WPT, v becomes a descendant of u in the DFS.



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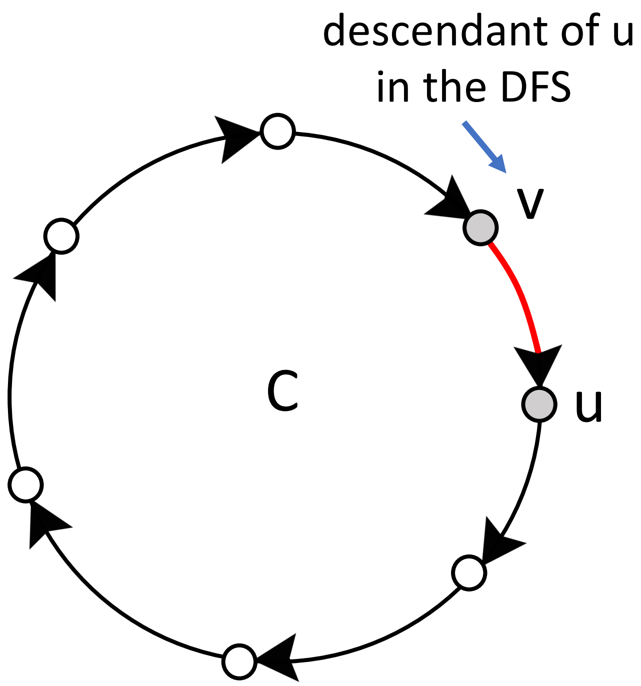
By the WPT, v becomes a descendant of u in the DFS.

When v is explored, the edge (v, u) is explored.



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$\Rightarrow$: Suppose G has a cycle C.

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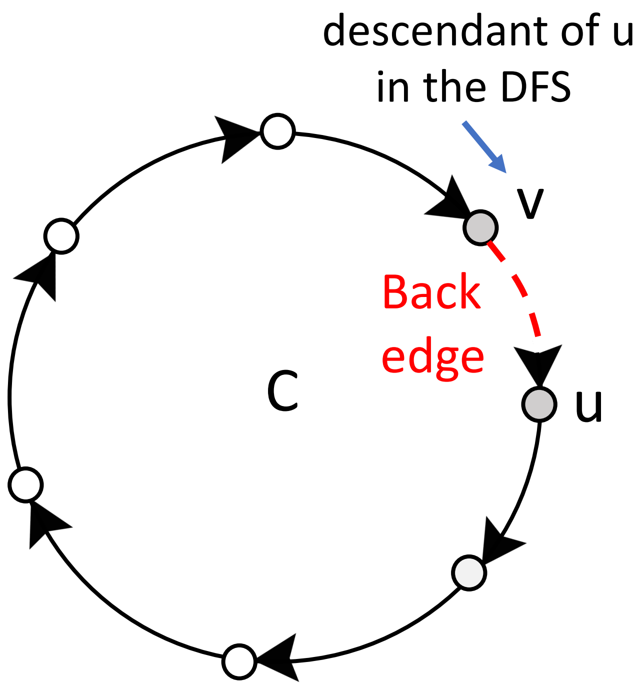
Let v be the node before u in C.

At time d[u] when DFS discovers u, all nodes in path u $\rightsquigarrow$ v in C are still white, including v.

By the WPT, v becomes a descendant of u in the DFS.

When v is explored, the edge (v, u) is explored.
Since v is a descendant of u, (v, u) becomes a back edge.



Proof: consider any directed G and any DFS of G

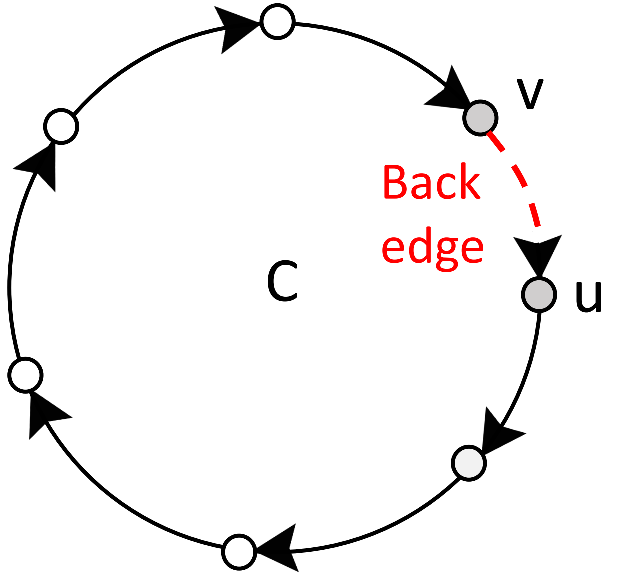
$\Rightarrow$: Suppose G has a cycle C.

Let u be the **first** node in C that the DFS discovers.

Let v be the node before u in C.

At time d[u] when DFS discovers u, all nodes in path u $\rightsquigarrow$ v in C are still white, including v.

By the WPT, v becomes a descendant of u in the DFS.



When v is explored, the edge (v, u) is explored.
Since v is a descendant of u, (v, u) becomes a back edge.

Hence, the DFS has a back edge.

Application of White-Path-Theorem [Theorem 22.11 CLRS]

For all directed graphs G and all DFS of G:

G has a cycle

$\Updownarrow$

DFS of G has a back edge