Disjoint Set - Union/Find

Disjoint Set – Union/Find

- n distinct elements named 1, 2, …, n
- Initially, each element is in its own set

$$S_1 = \{1\}, \quad S_2 = \{2\}, \dots, \quad S_n = \{n\}$$

- Each set has a representative element
- S_x : Set represented by element x

Operations:

Union(S_x, S_y): Create set $S = S_x \cup S_y$ and return the representative of S

Find(z): Given (a ptr to) z, find set S that contains z and return the representative of S

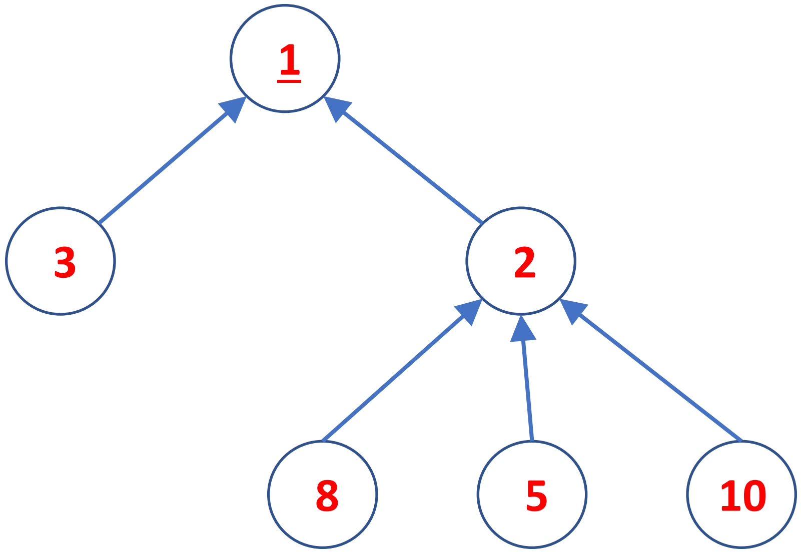
σ : Sequence of $n - 1$ **Unions** mixed with $m \geq n$ **Finds**

Goal: a data structure that minimizes the total cost of executing such sequences

Forest structure for Union-Find

- Each set is represented by a tree
- The root contains the set representative

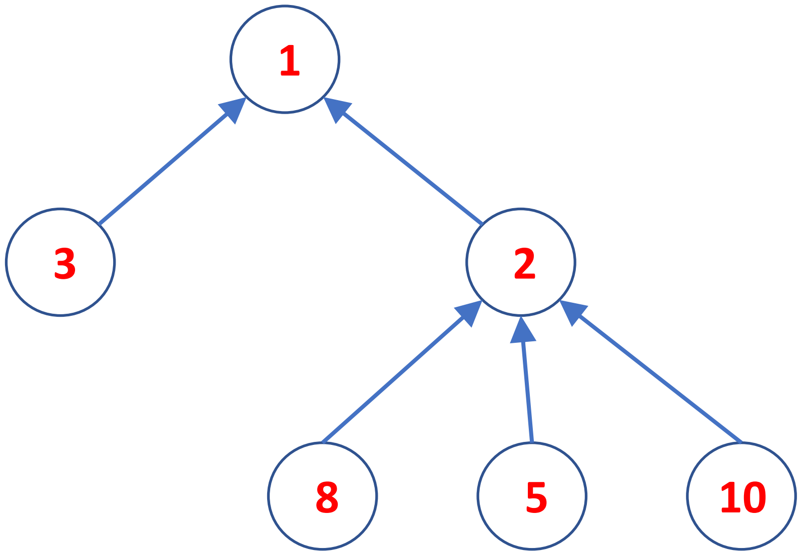
$S_1 = \{\underline{1}, 3, 2, 8, 5, 10\}$



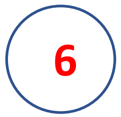
Example with n = 10

$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

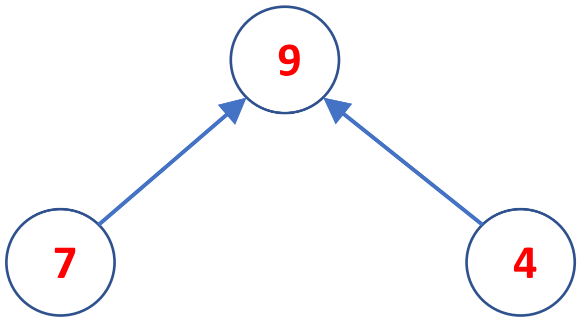
$S_1 = \{1, 3, 2, 8, 5, 10\}$



$S_6 = \{6\}$



$S_9 = \{9, 7, 4\}$



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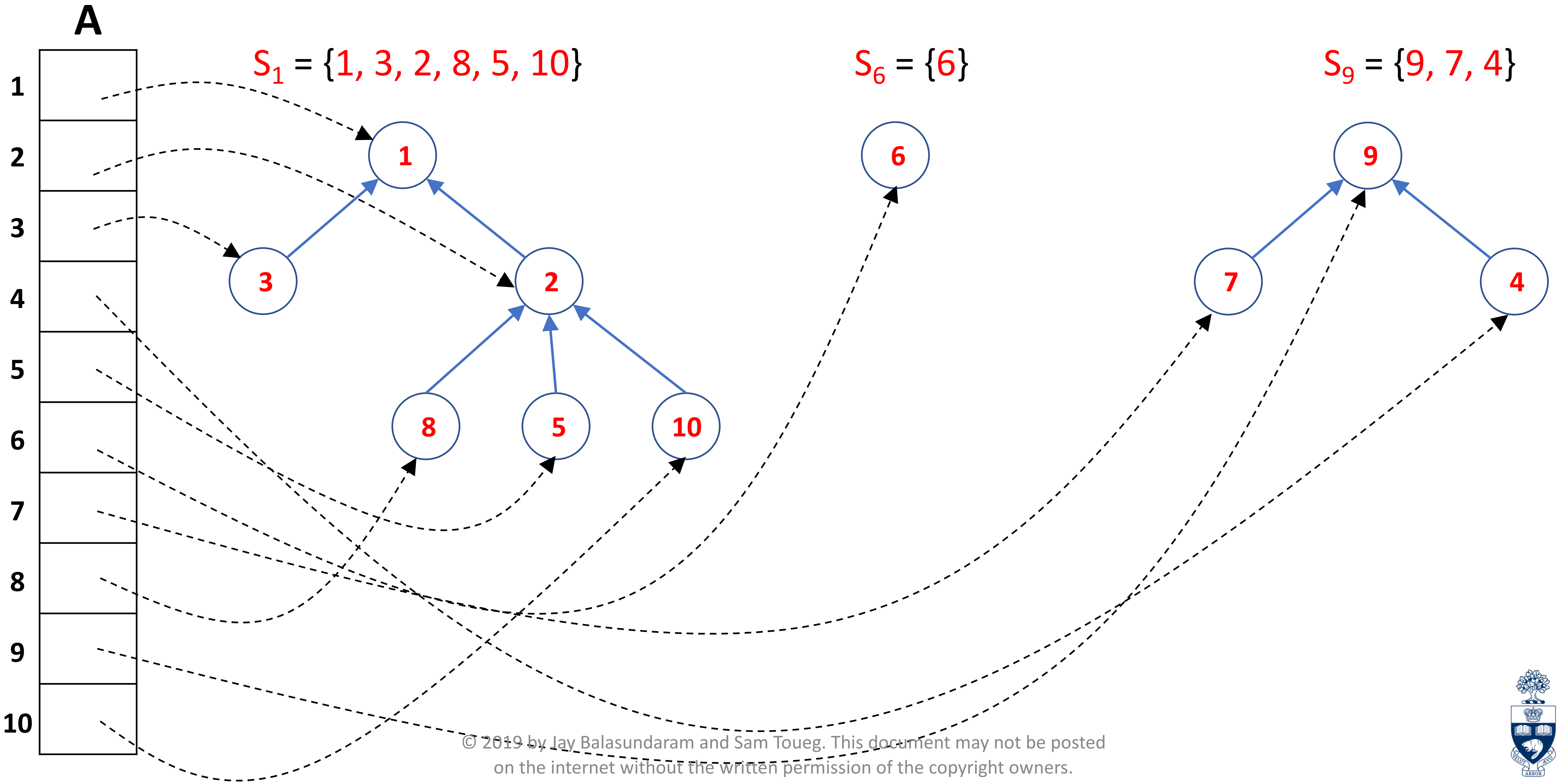
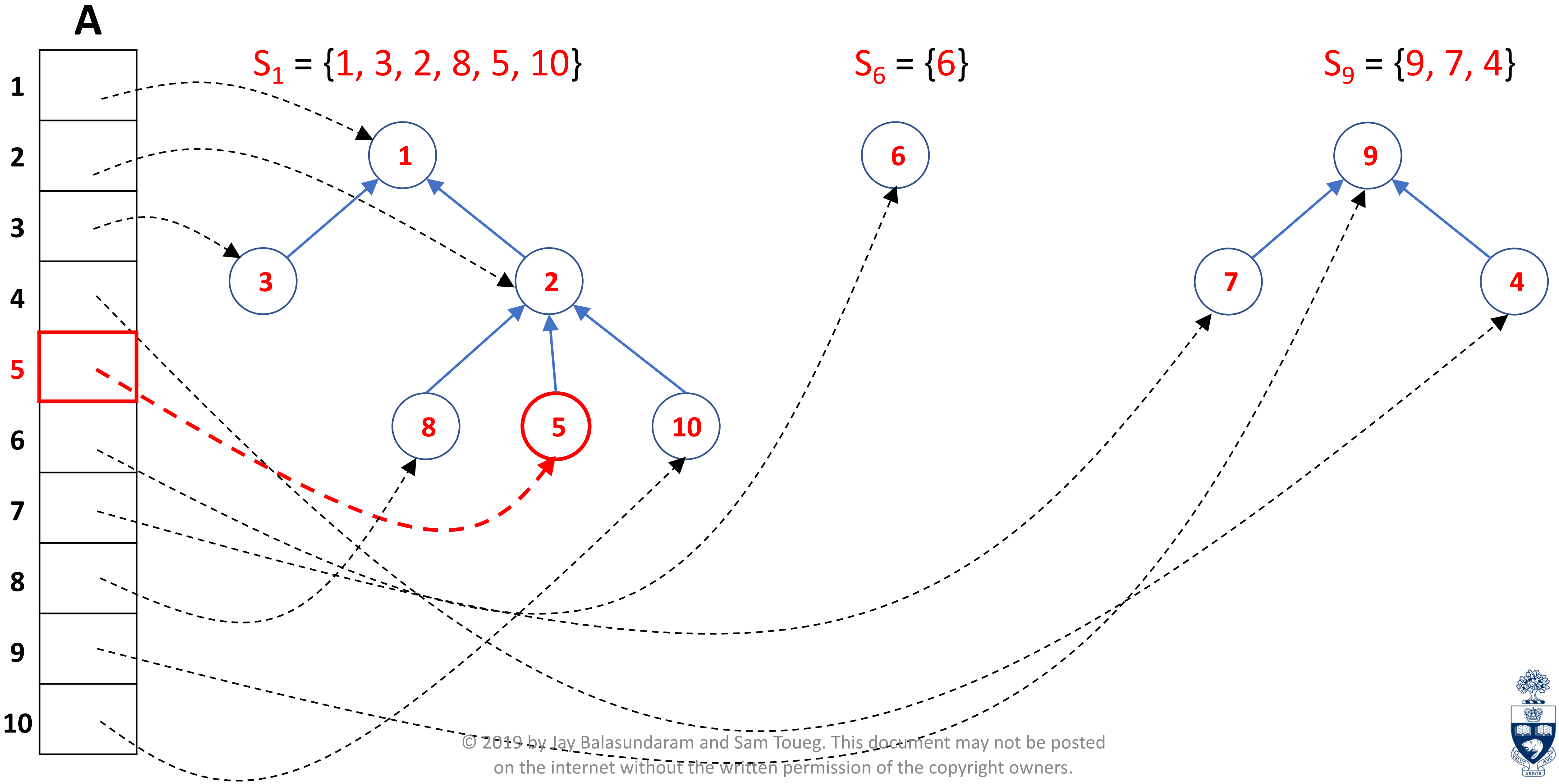
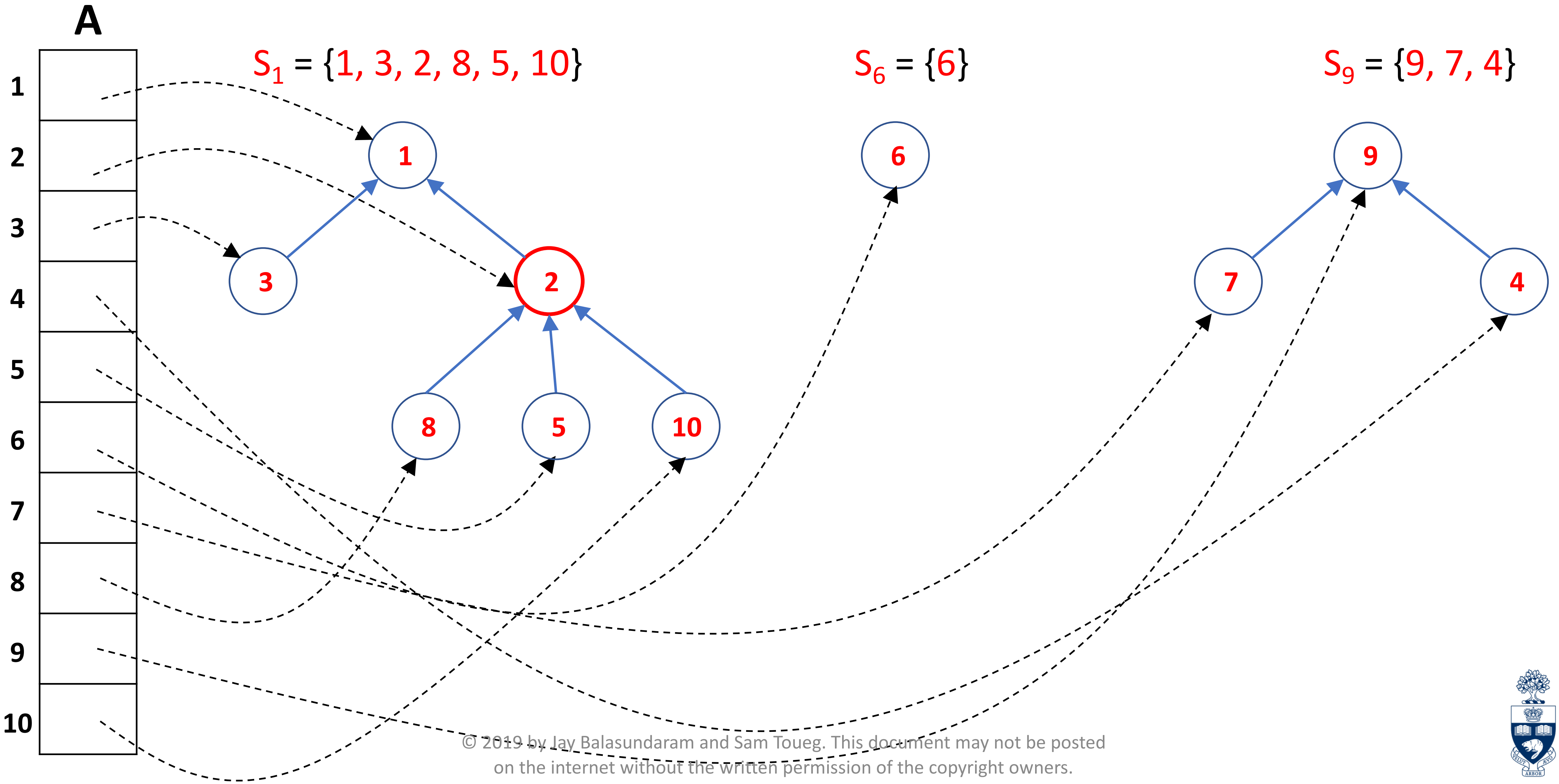


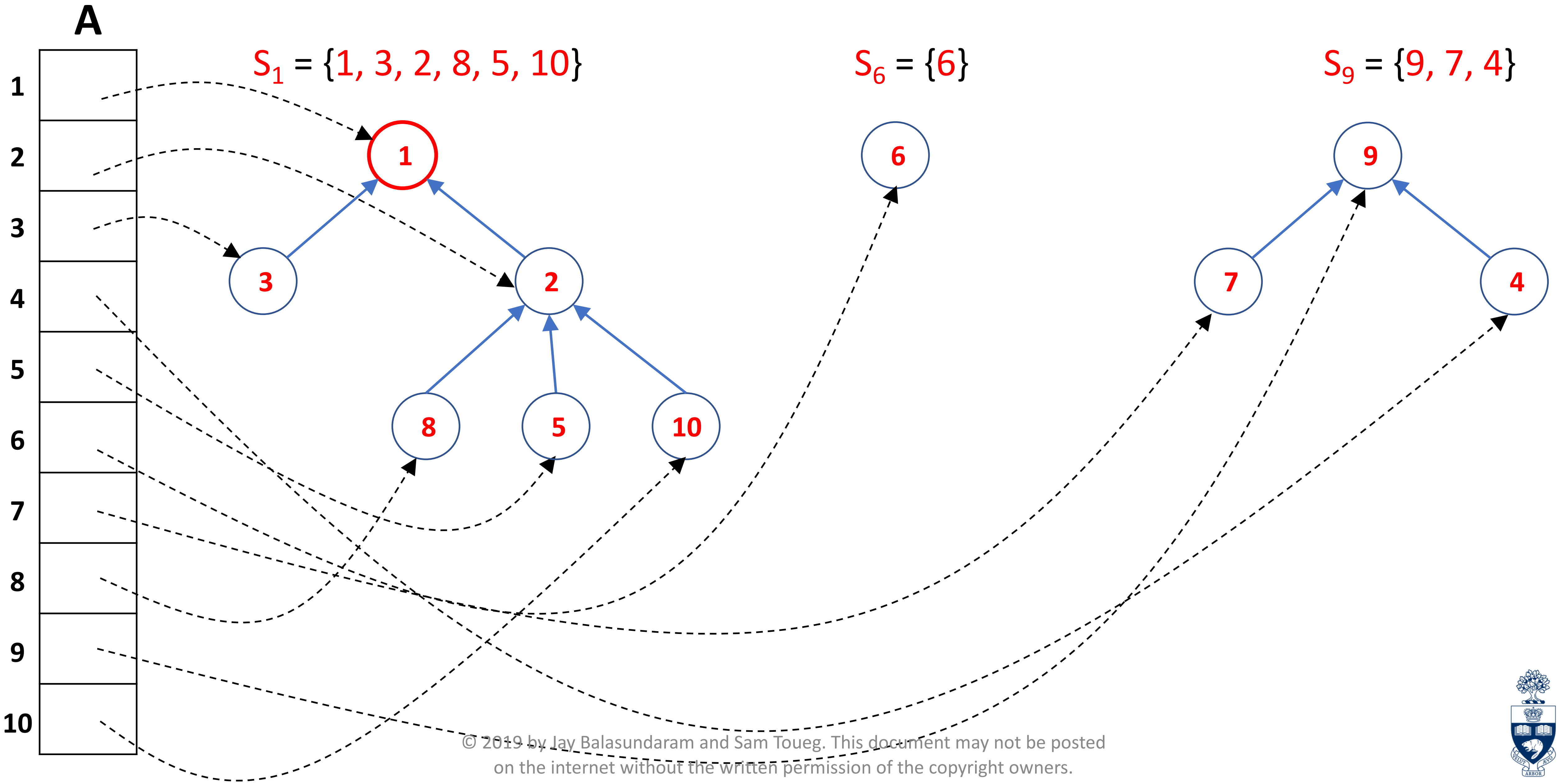
Find(5)



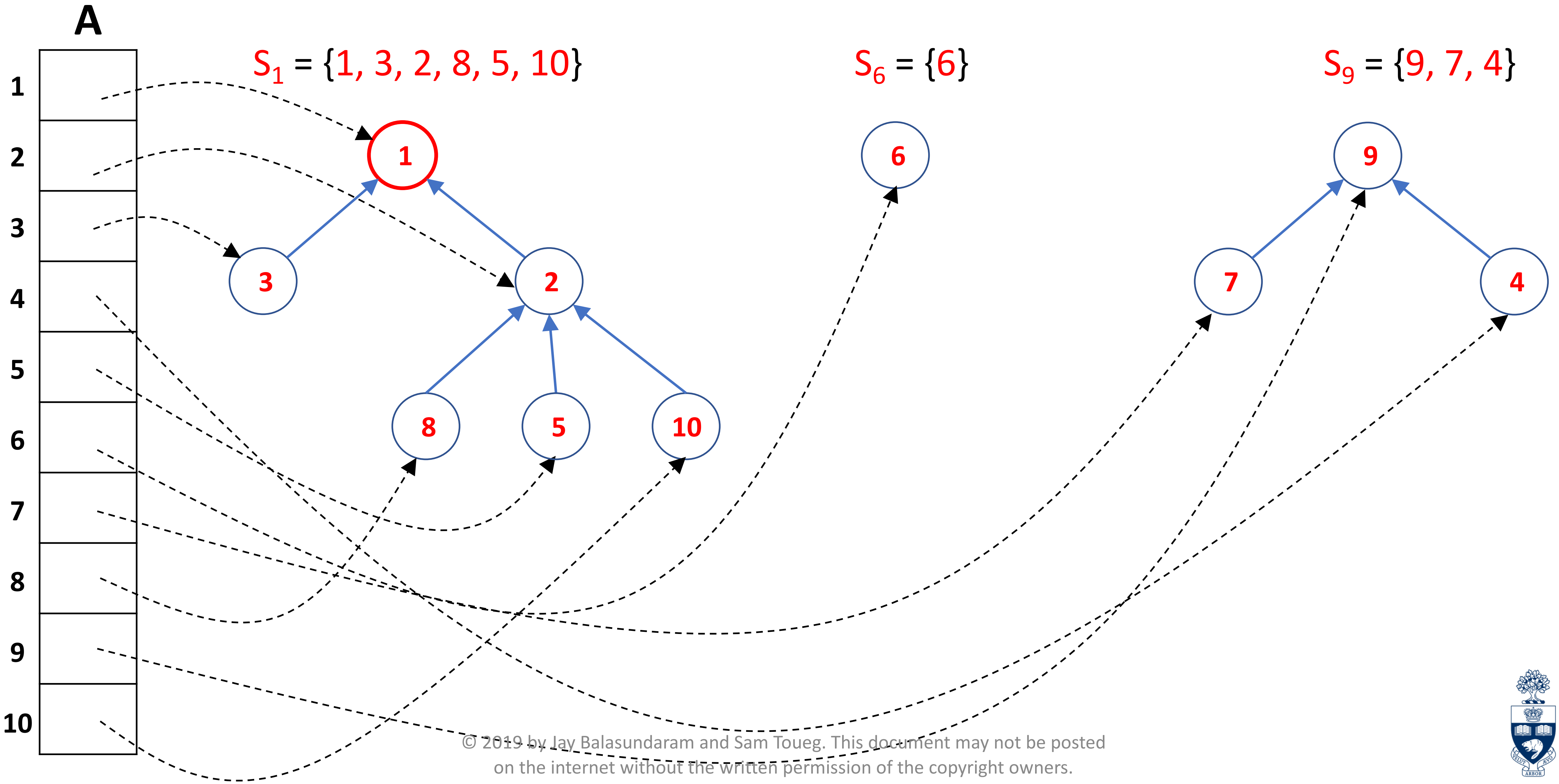
Find(5)



Find(5)



Find(5) : Return (ptr to) 1



Operations

- **Find**(x): Follow path from x up to root, return ptr to the root

  Cost is O(1 + length of the **Find** path)

- **Union**(S_x, S_y): Make root of S_x the child of root of S_y

  Cost is O(1)

Disjoint Forest: Time Complexity

Initially :

Union(1, 2) :

Union(2, 3) :

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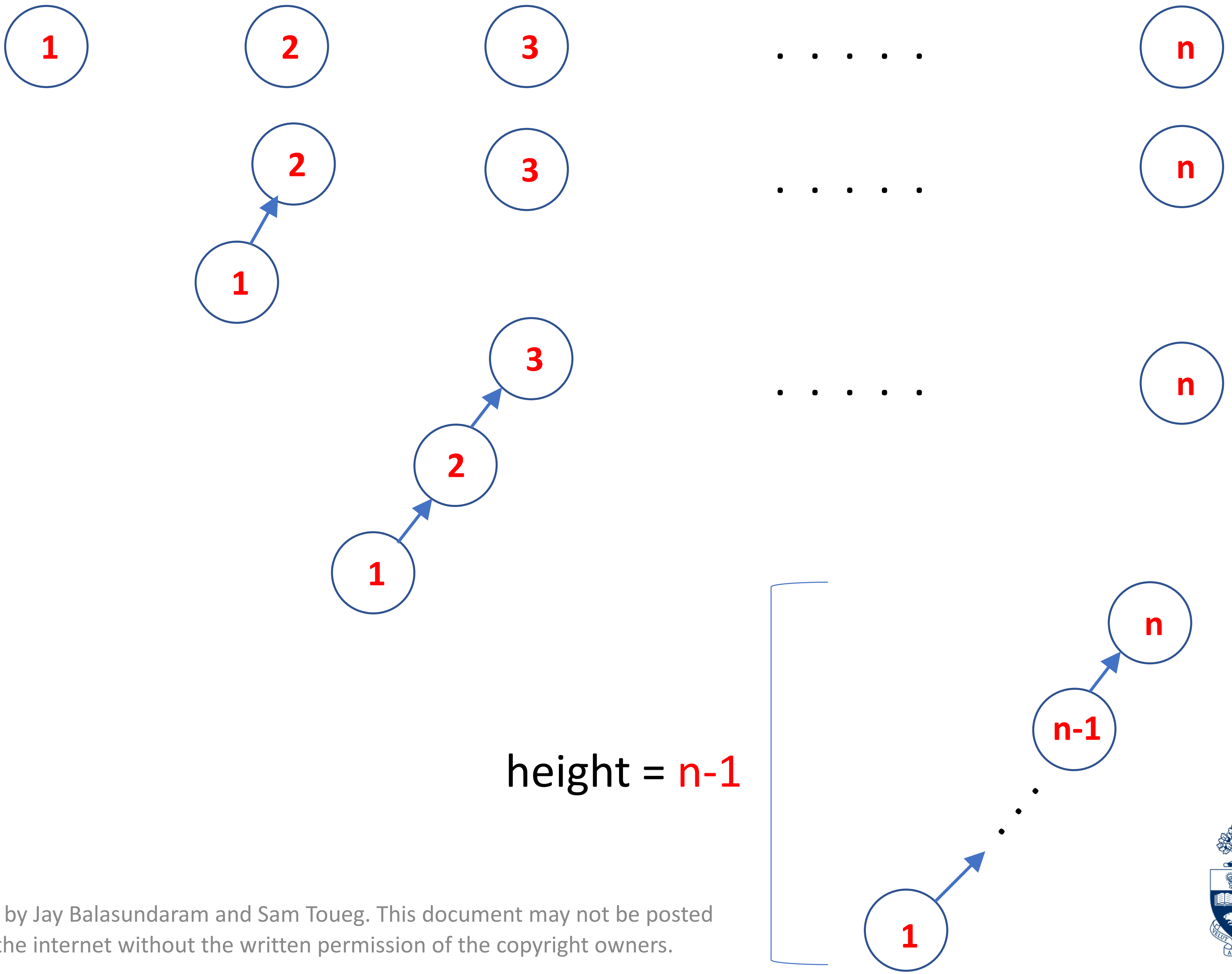
.
.

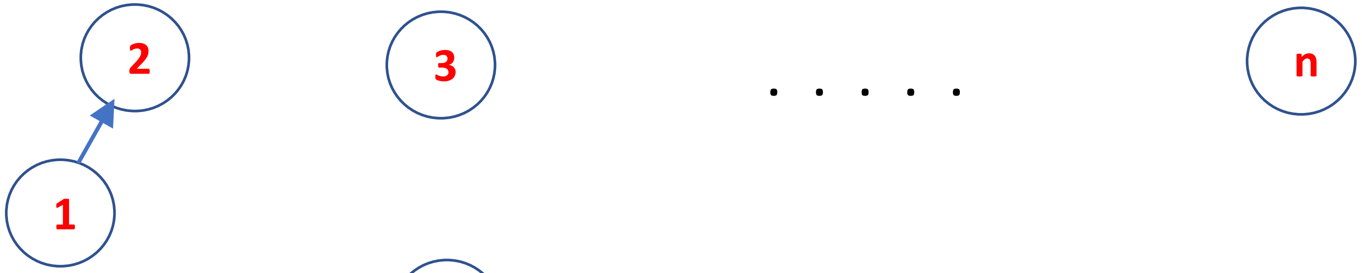
Union(n-1, n) :



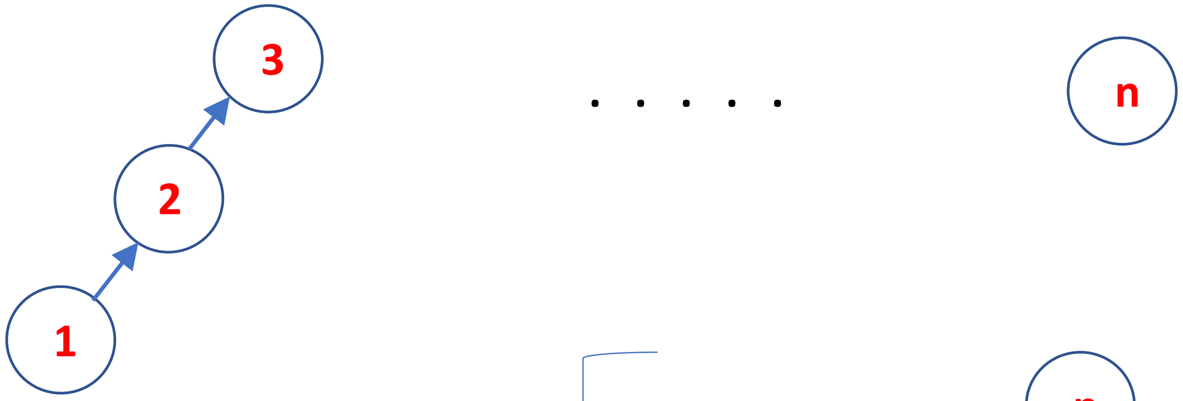
Disjoint Forest: Time Complexity

Initially :

Union(1, 2) :



Union(2, 3) :



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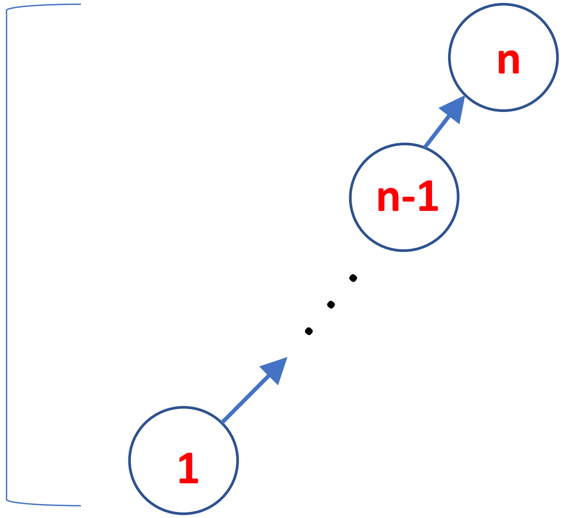
Union(n-1, n) :



height = n-1

Cost of **Find**(1) : Ω(n) !

$\Rightarrow$ worst–case cost of executing σ is Ω(m.n)

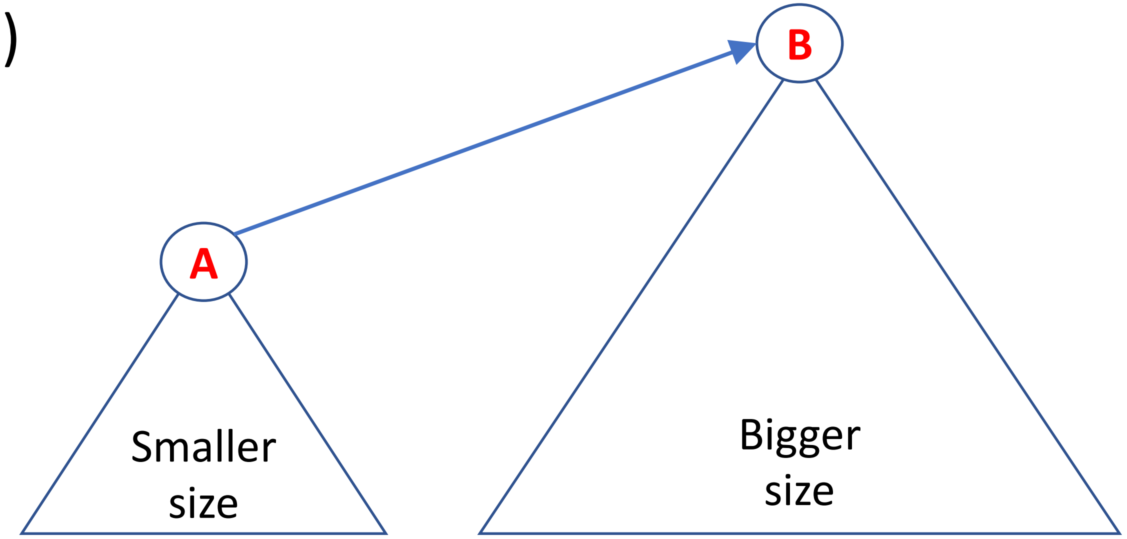
Disjoint Forest: Time Complexity

To reduce cost of executing σ, reduce the length of **Find** paths

$\Rightarrow$ reduce height of the trees formed during the execution of σ

Heuristic 1: **Weighted Union (WU)** by Size

Union(A, B)



size: # of nodes in the tree

WU rule (by size): Smaller size tree becomes the child of the bigger size tree

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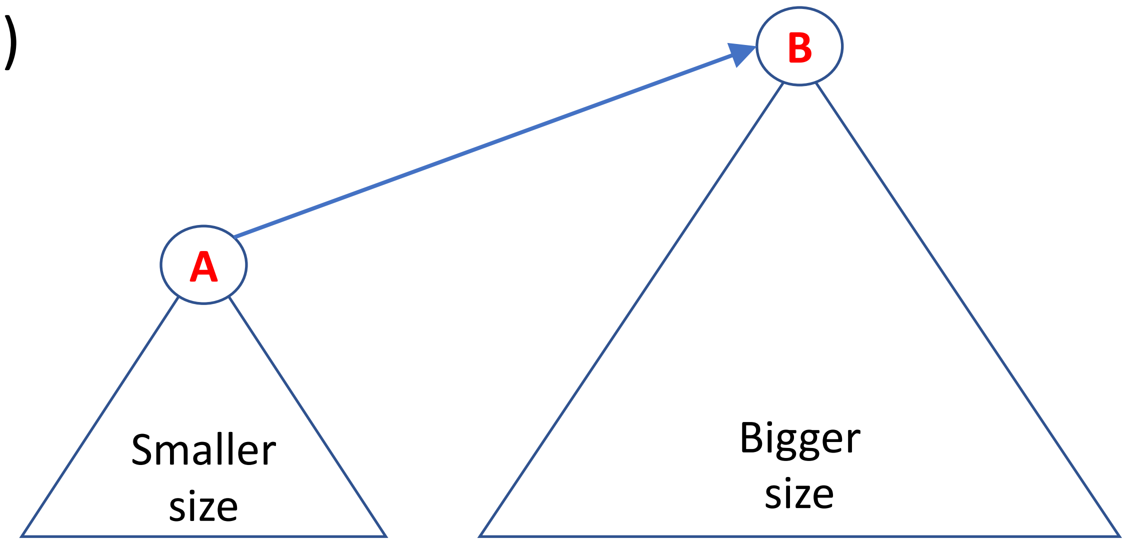

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With **WU**:

- Any tree T created during the execution of σ has height at most $\log_2 n$

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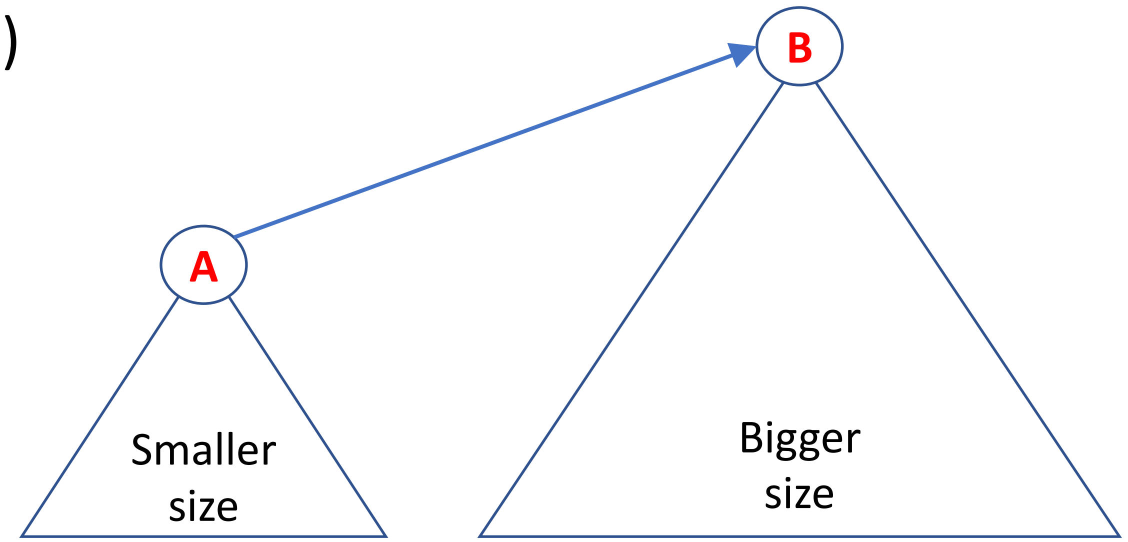

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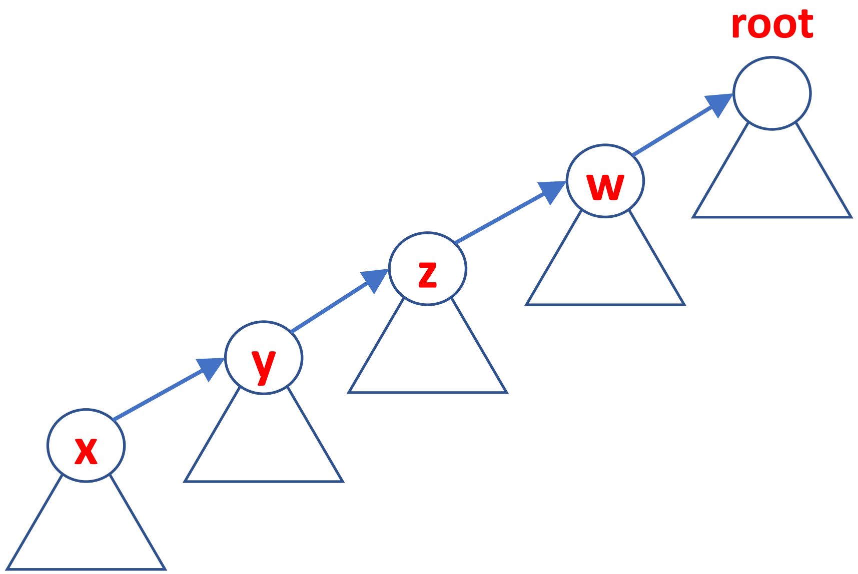
WU rule (by size): Smaller size tree becomes the child of the bigger size tree

With **WU**:

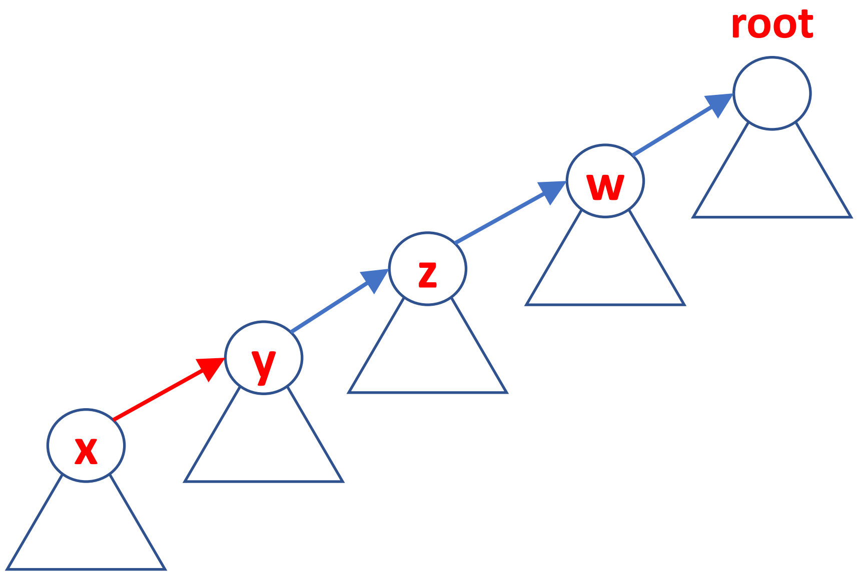
- Any tree T created during the execution of σ has height at most $\log_2 n$
- The worst-case cost of executing σ is O(m log n)

Heuristic 2: **P**ath **C**ompression (**PC**)



Find(x)

Heuristic 2: **Path Compression (PC)**



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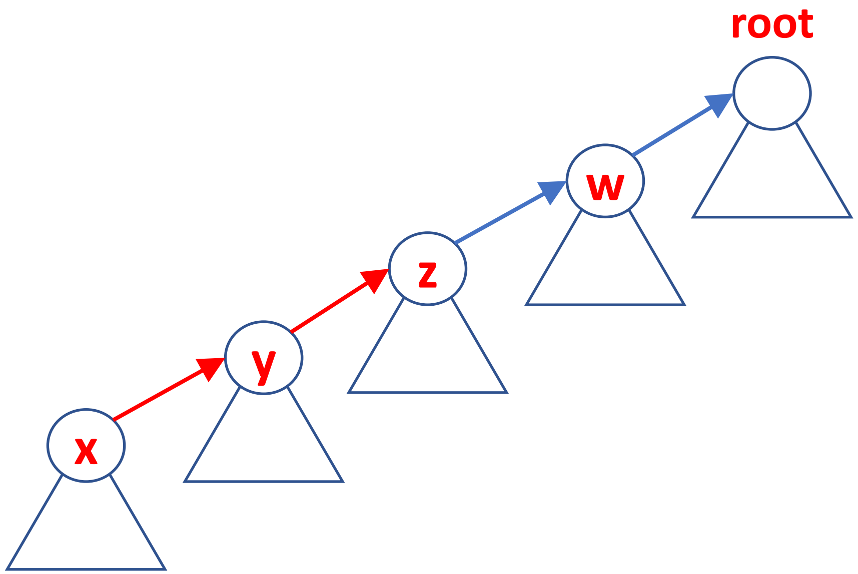

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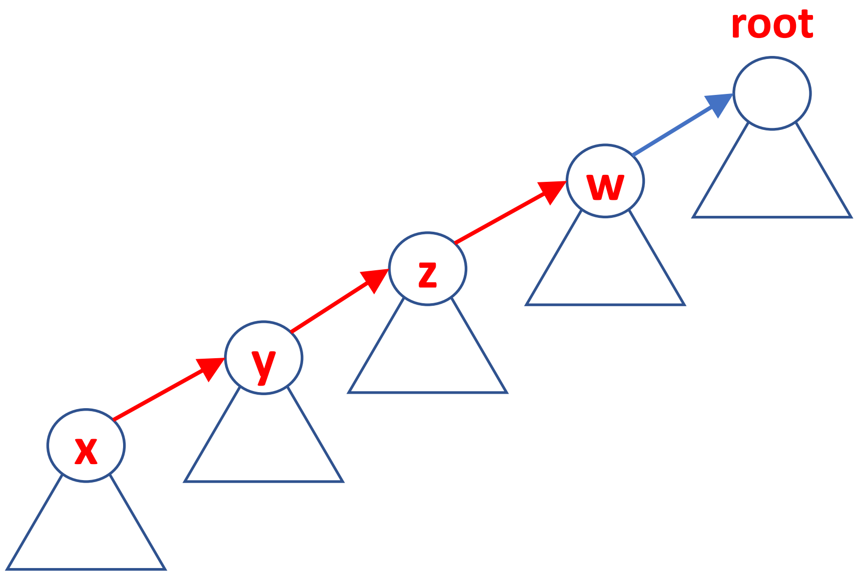

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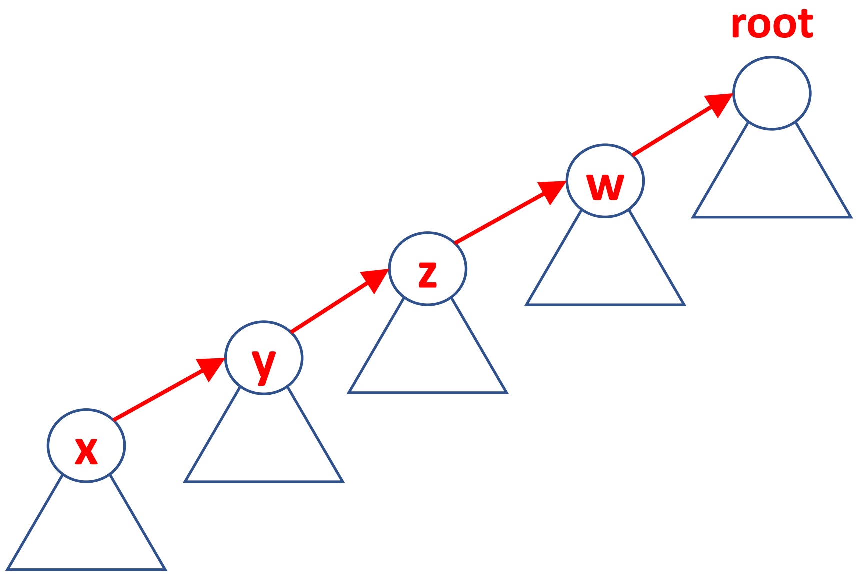

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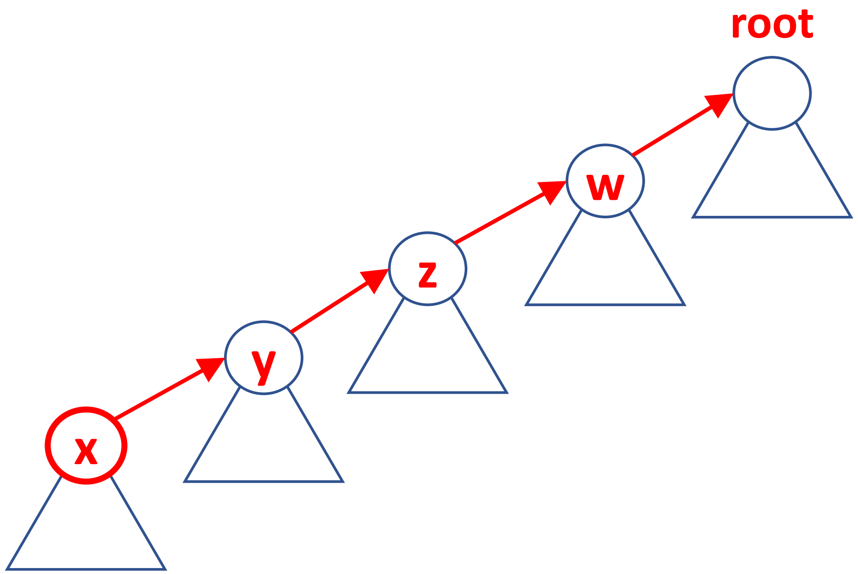

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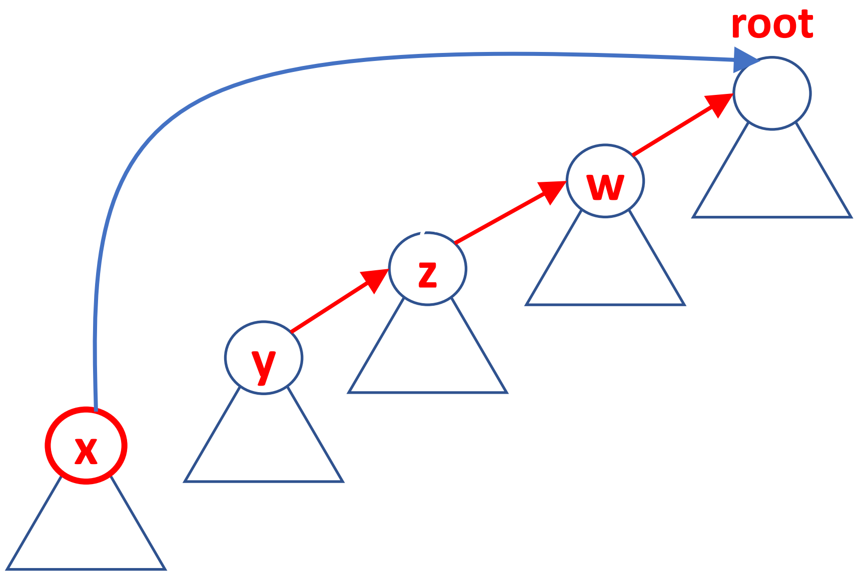

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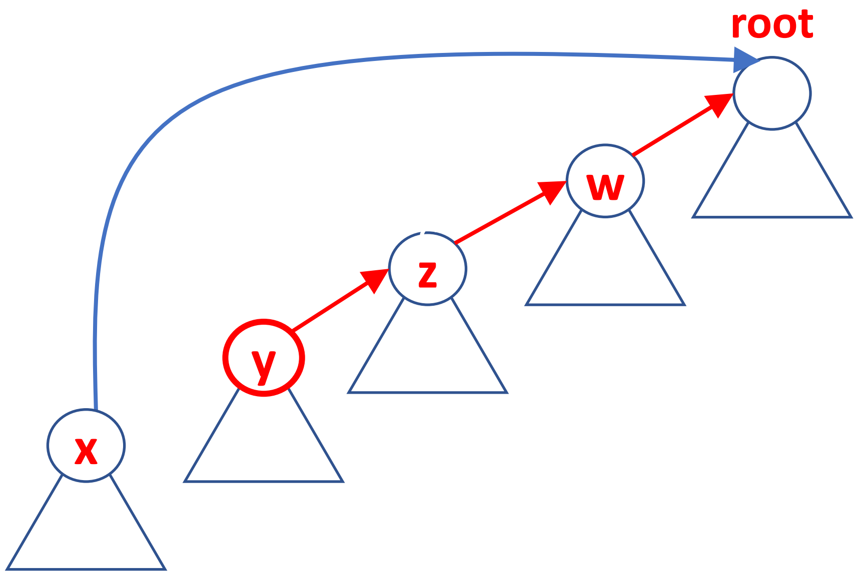

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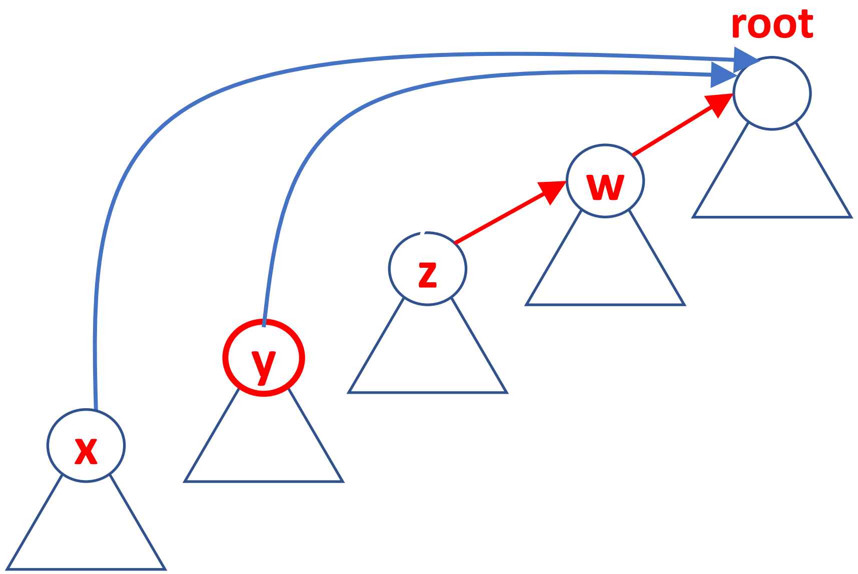

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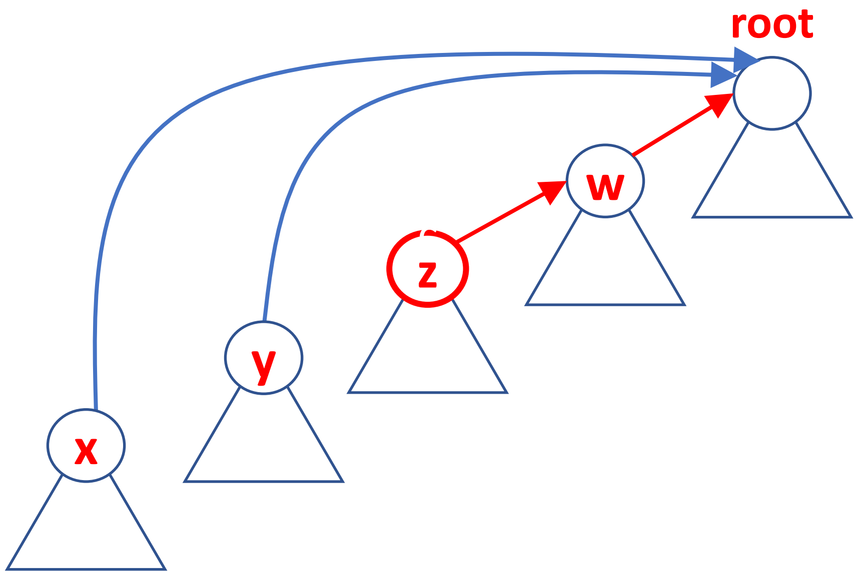

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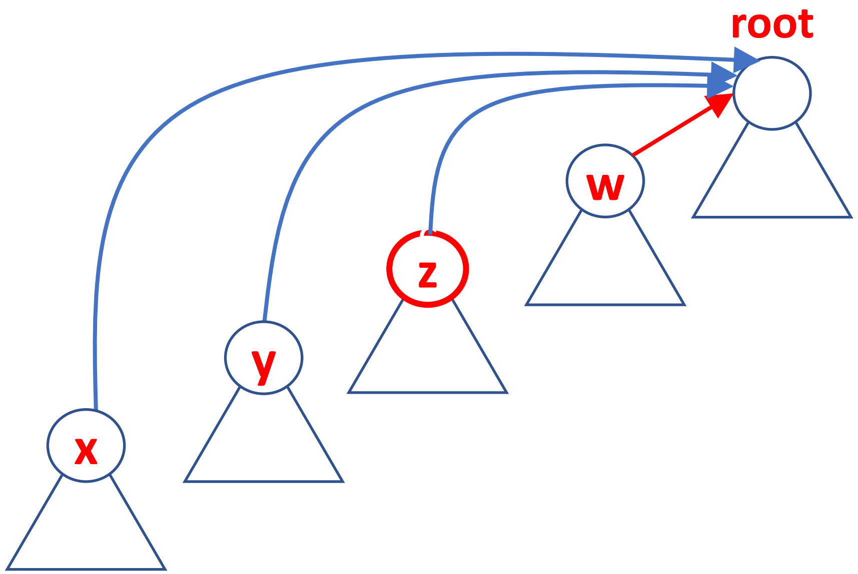

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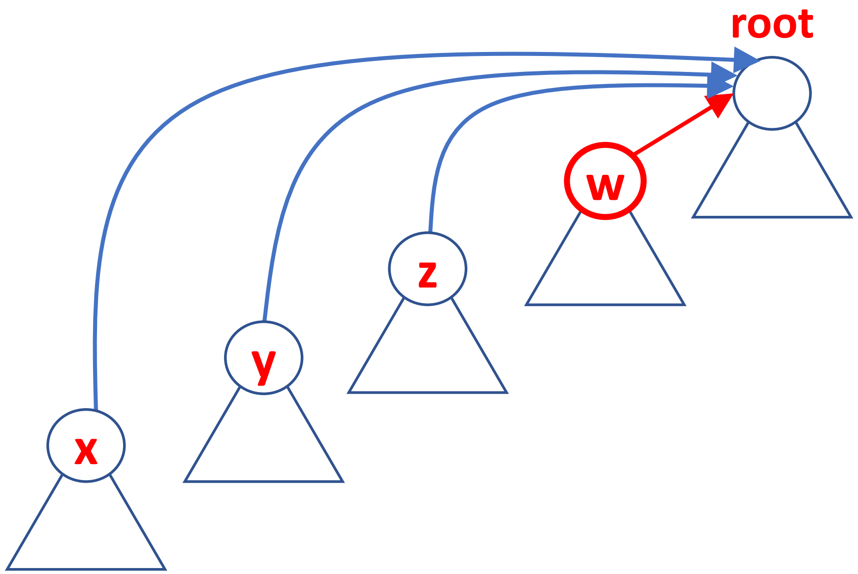

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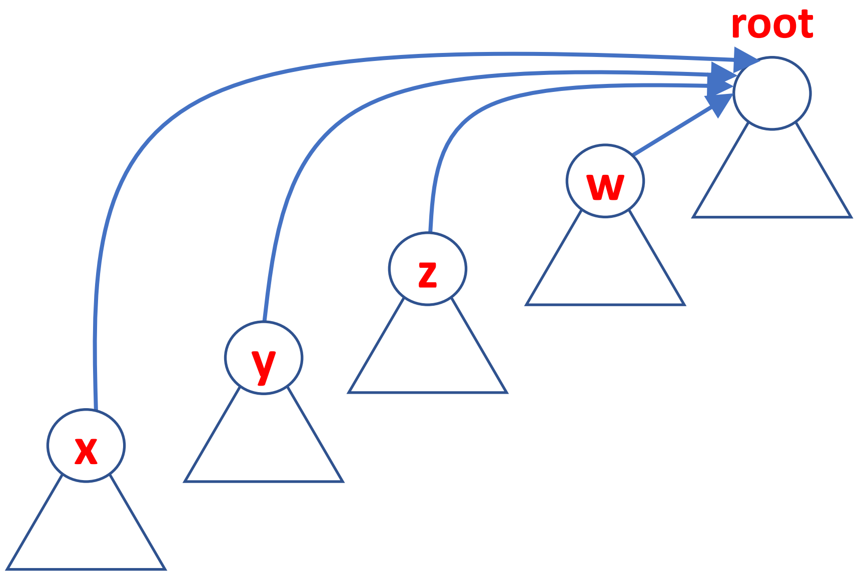

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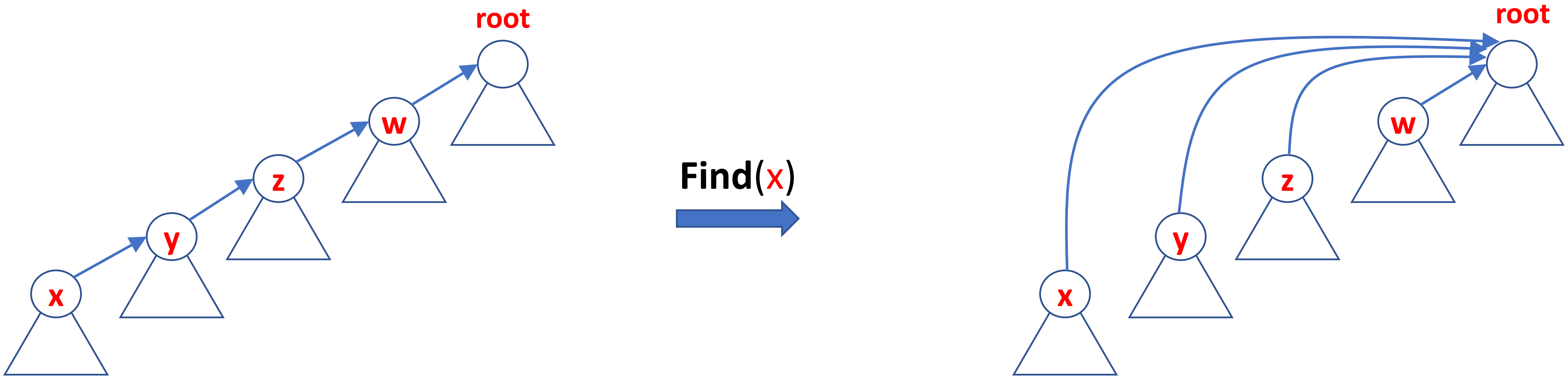
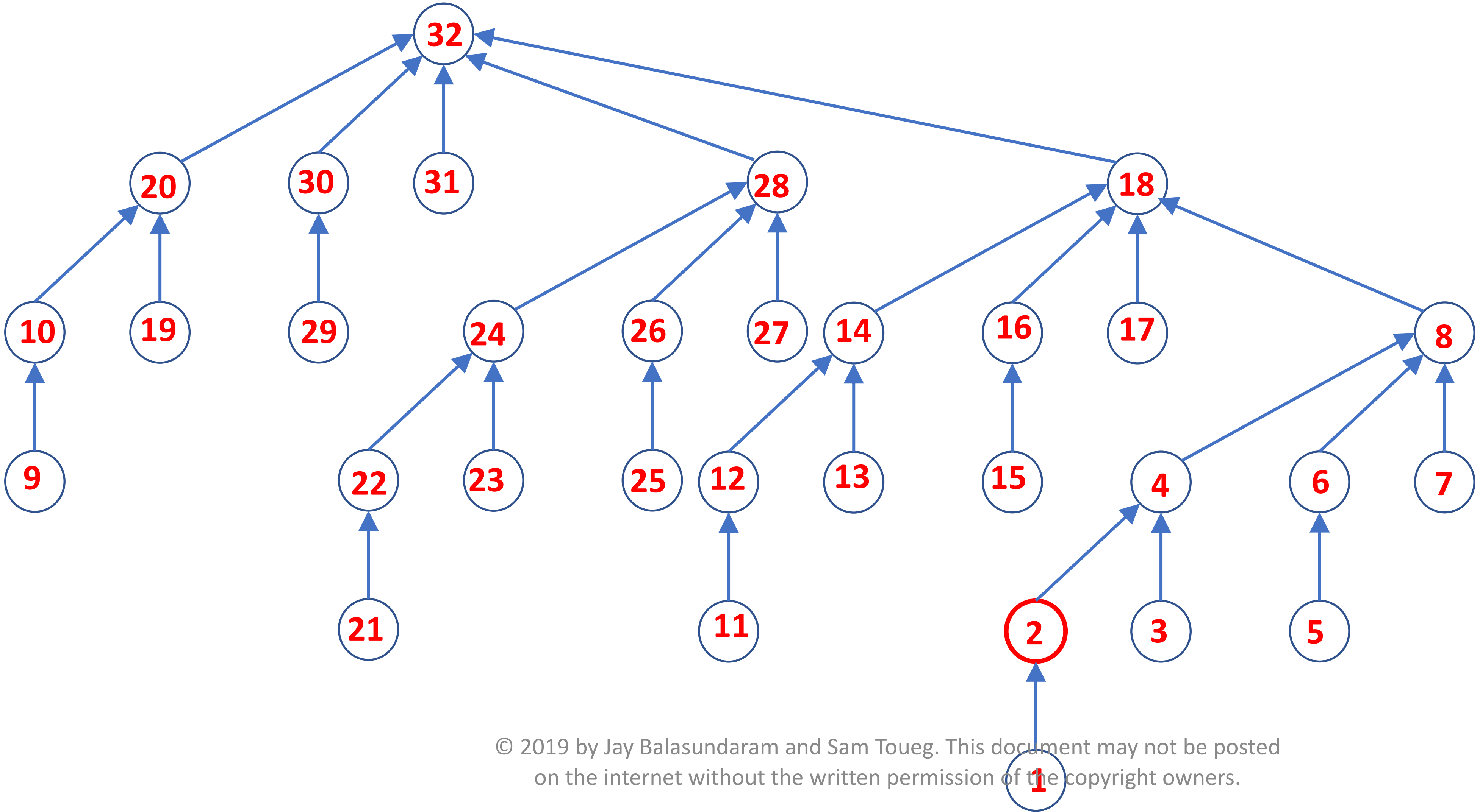


PC rule: In **Find**(x), make each vertex along the **Find** path a child of root

Example of Path Compression (PC)

Find(2)



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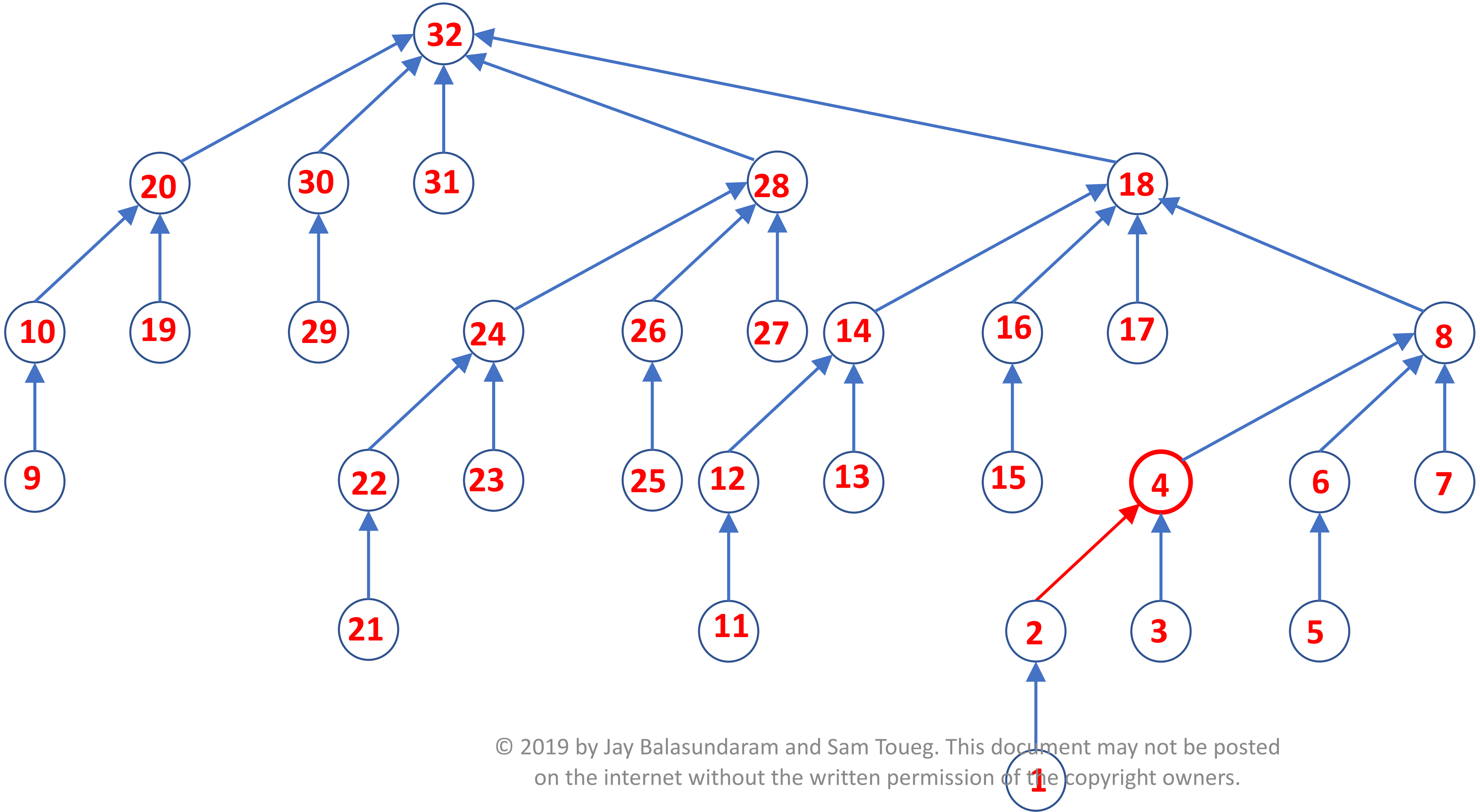

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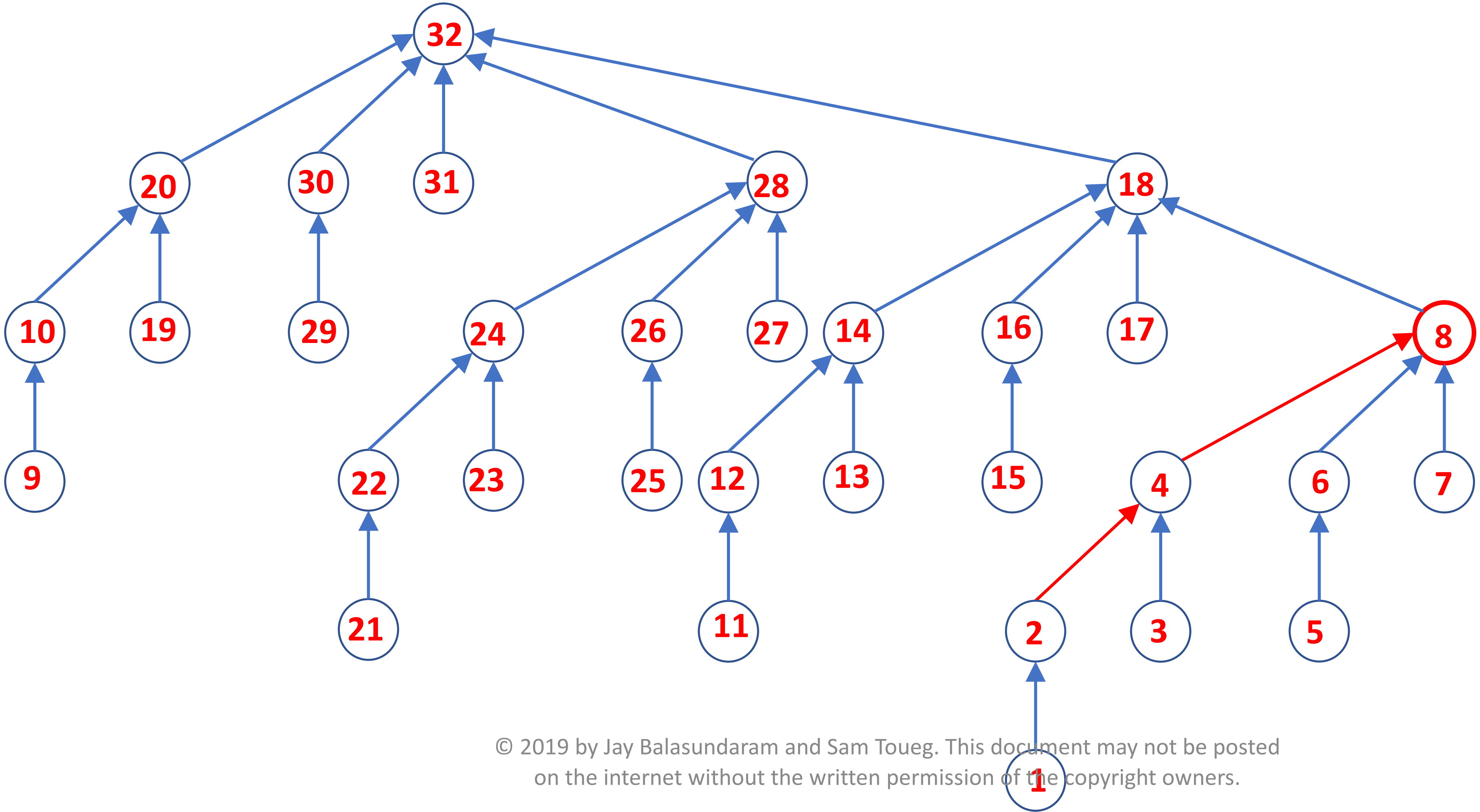

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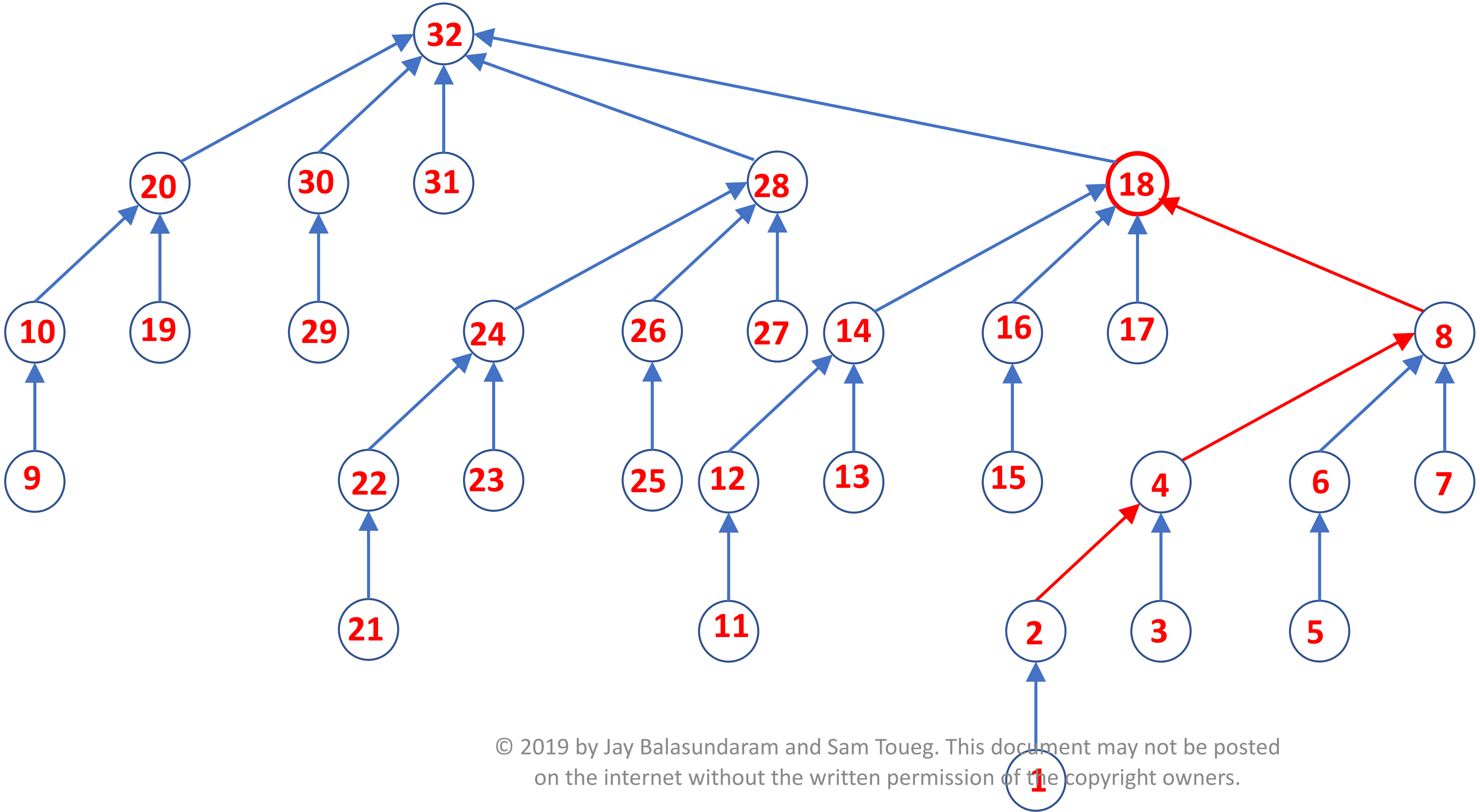

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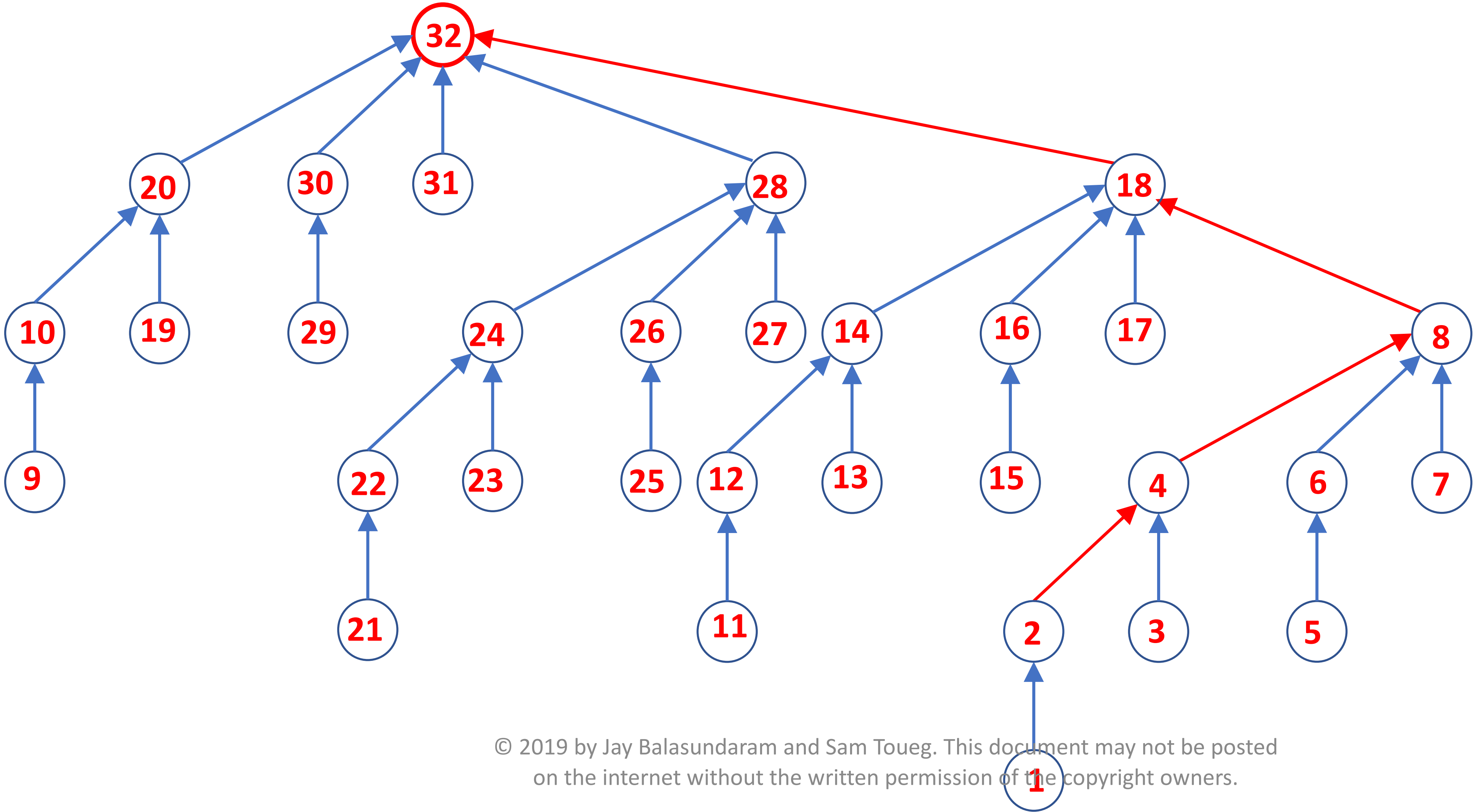

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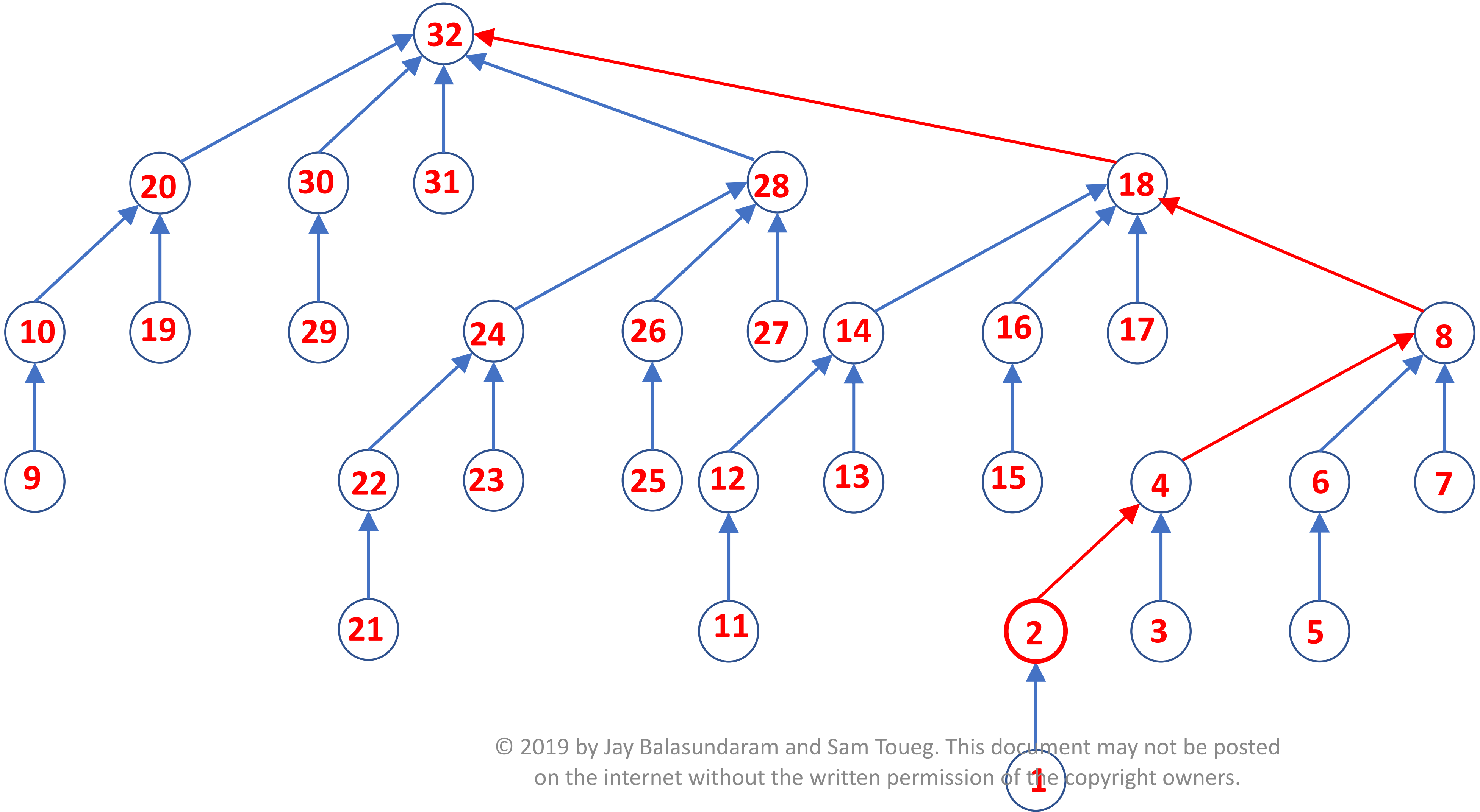

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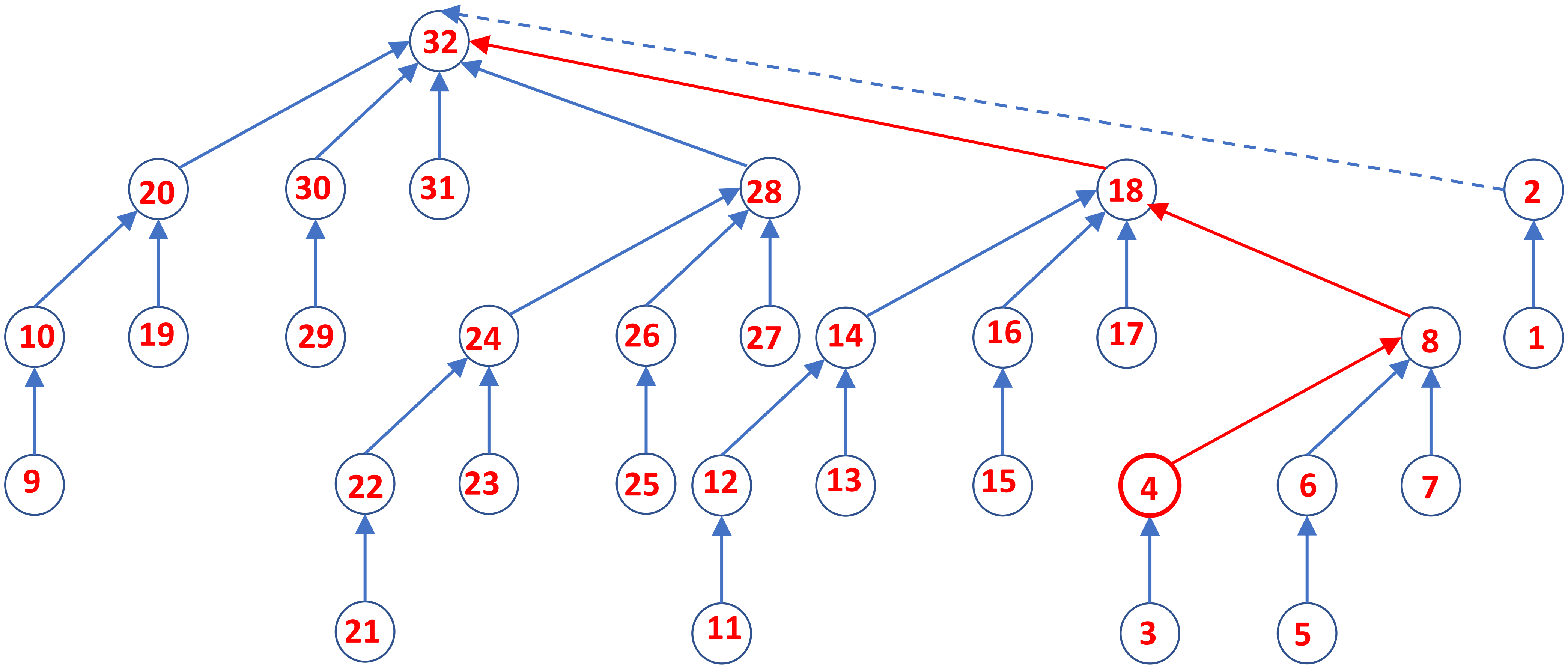

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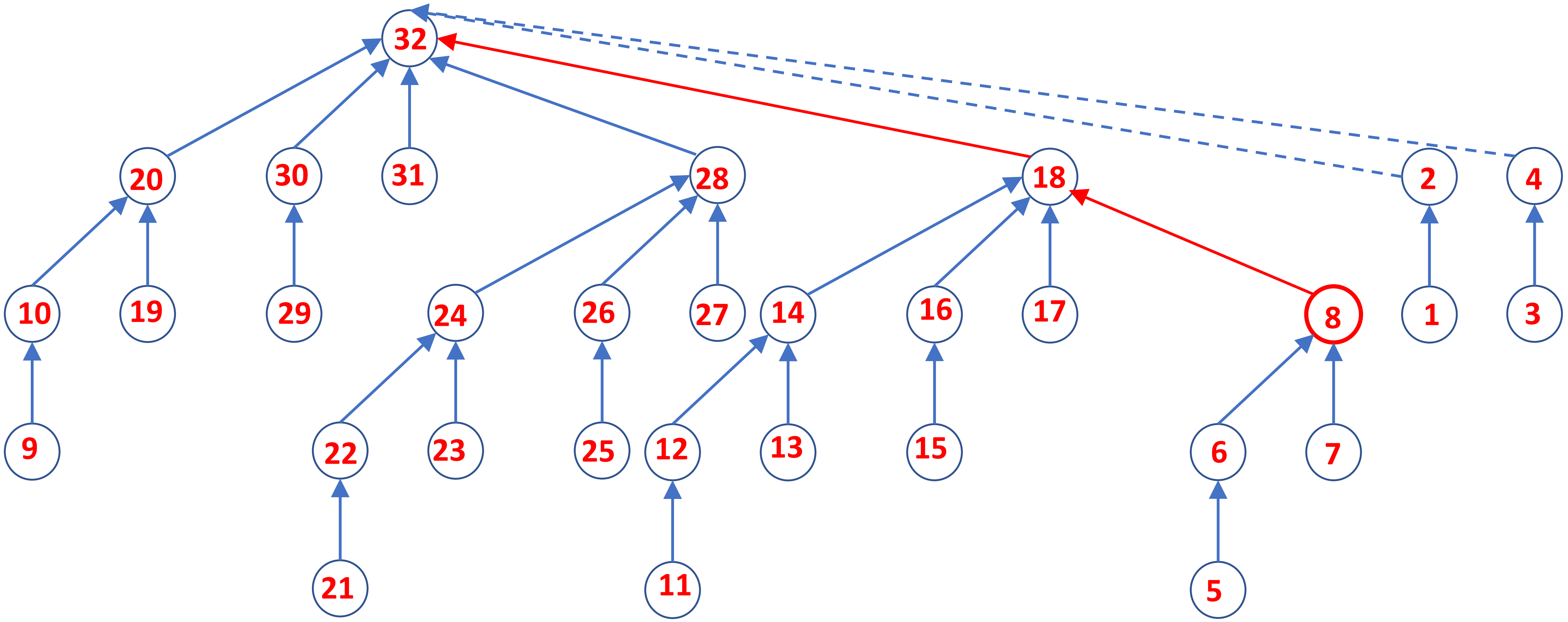

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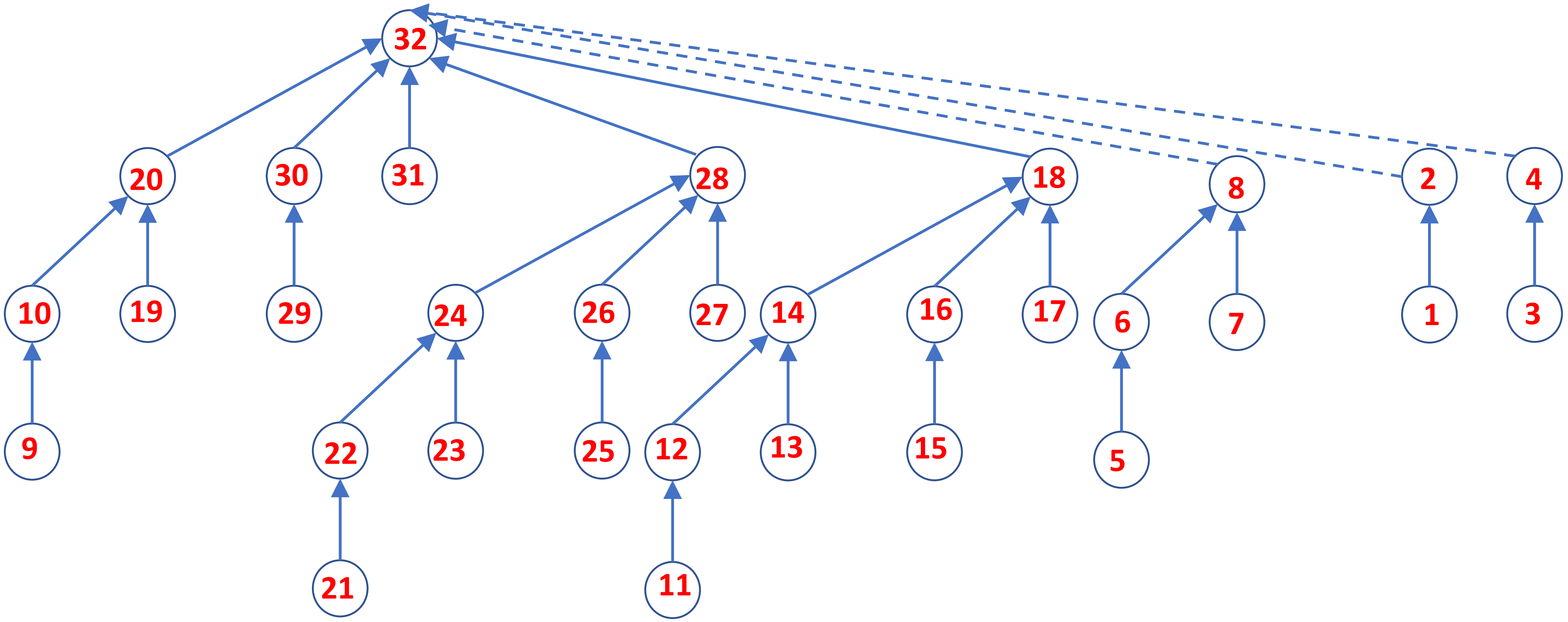

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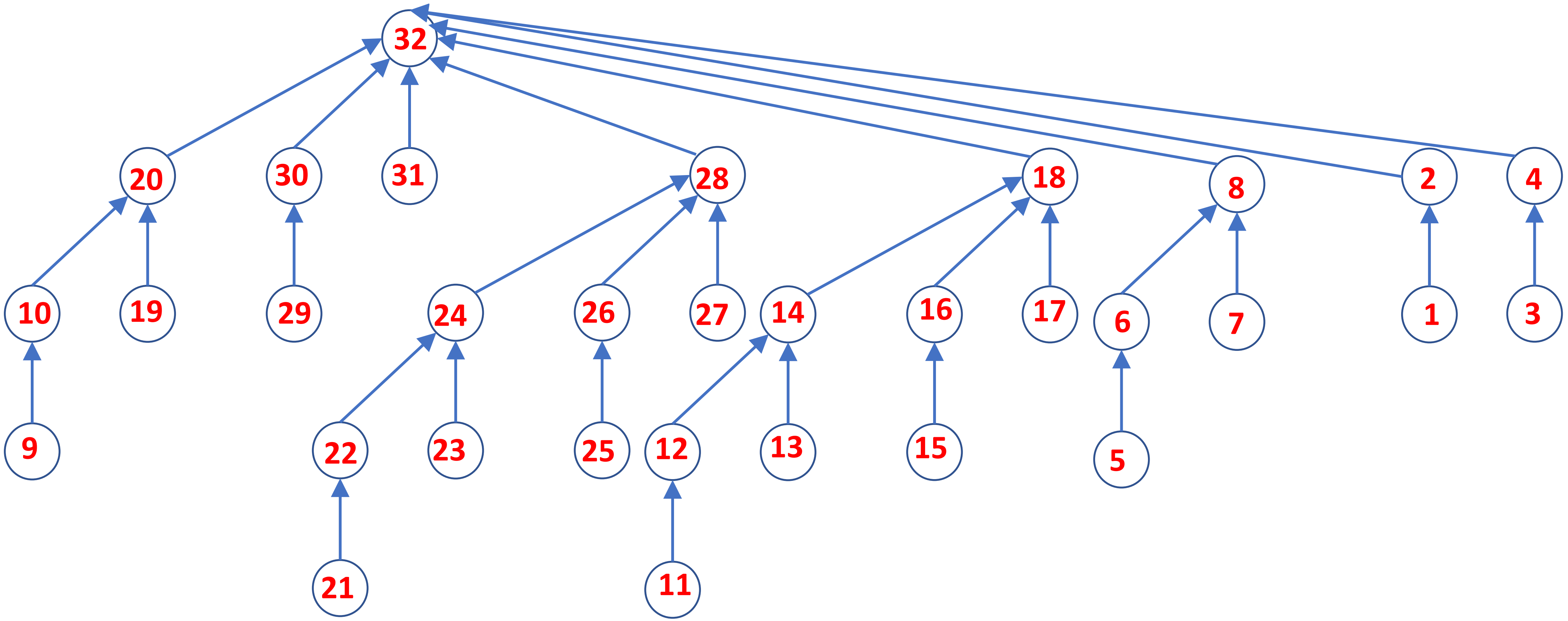
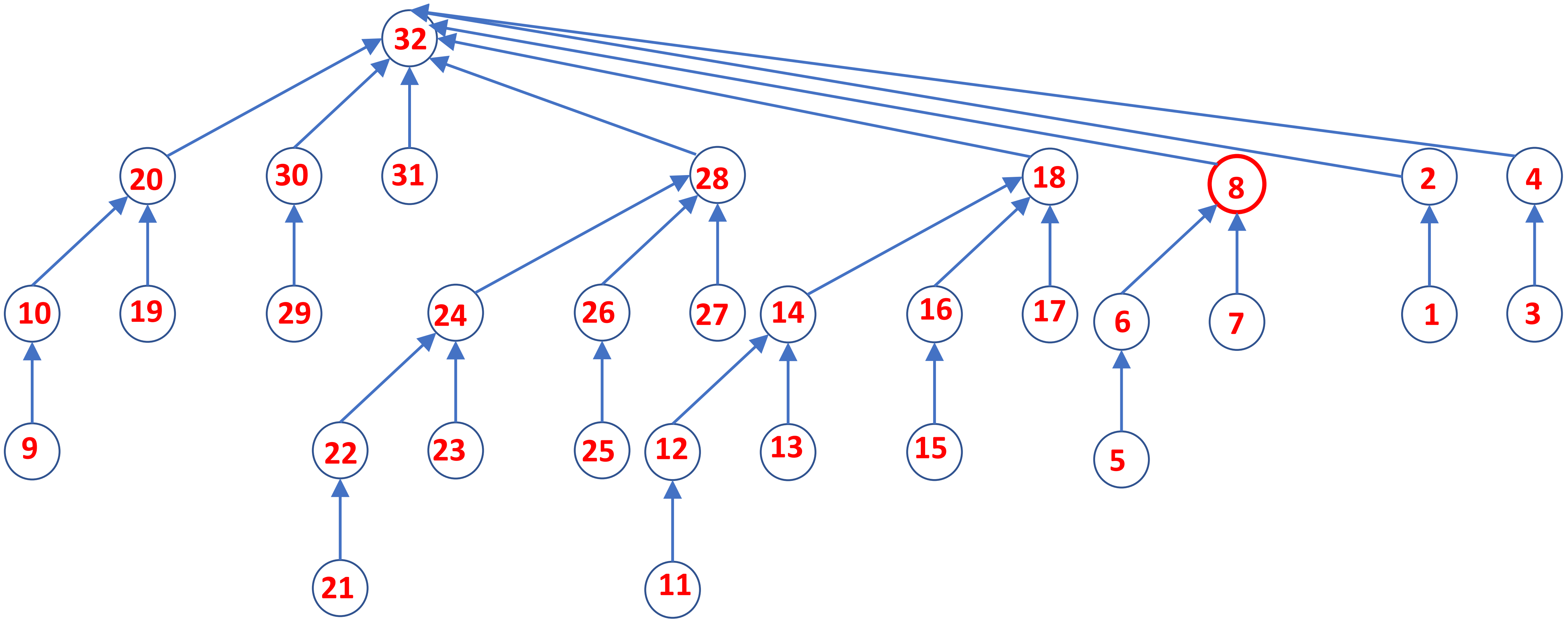


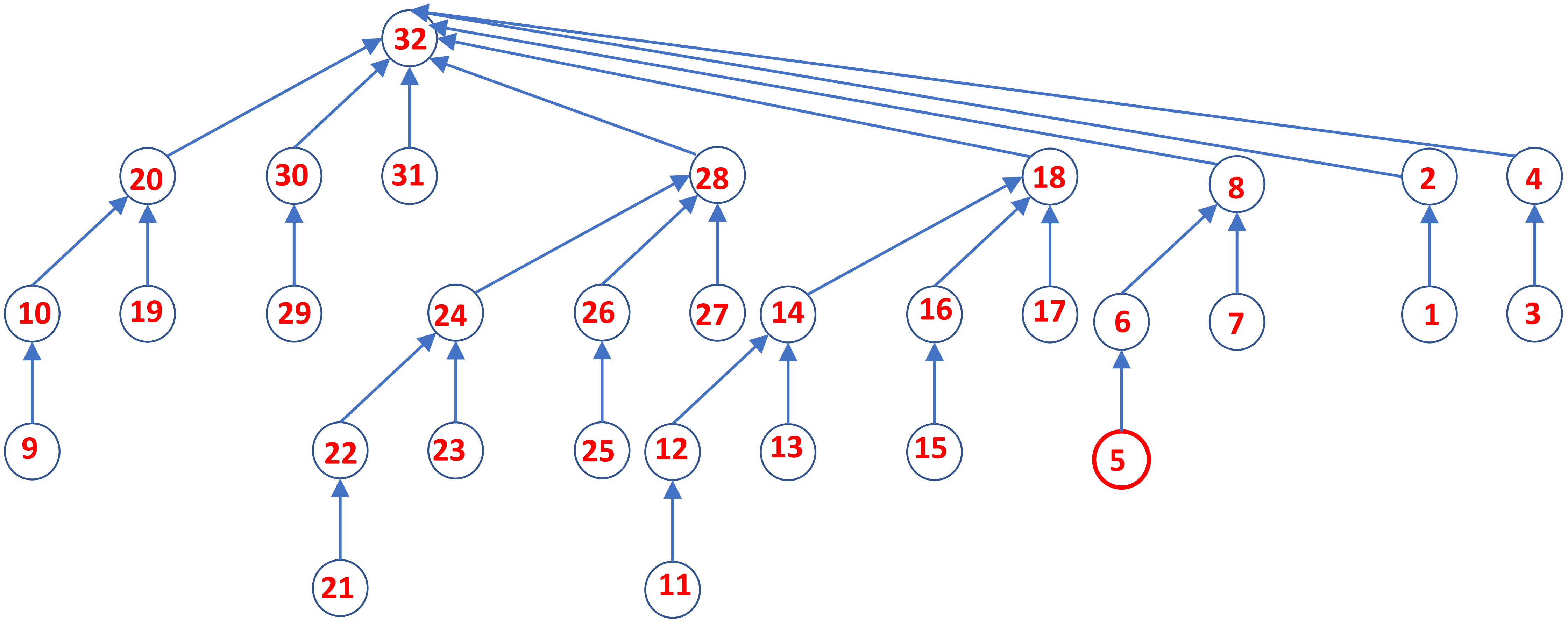
Example of Path Compression (PC)

Find(8)



Example of Path Compression (PC)

Find(5) ….



Heuristic 2: **Path Compression (PC)**

PC rule: In **Find**(x), make each vertex along the **Find** path a child of root

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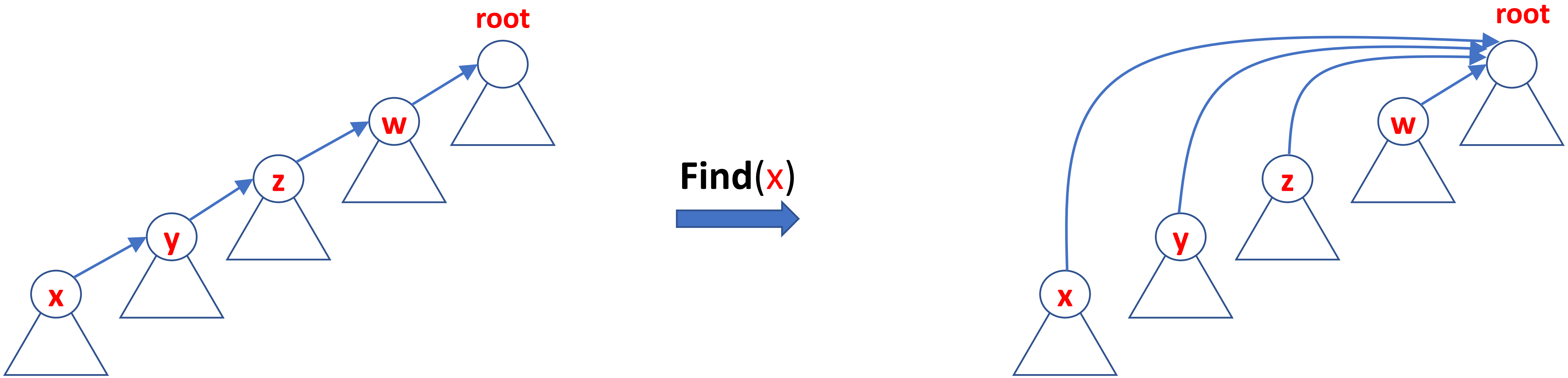

PC rule: In **Find**(x), make each vertex along the **Find** path a child of root

This increases the cost of **Find**(x), but makes several *future* **Find**s cheaper

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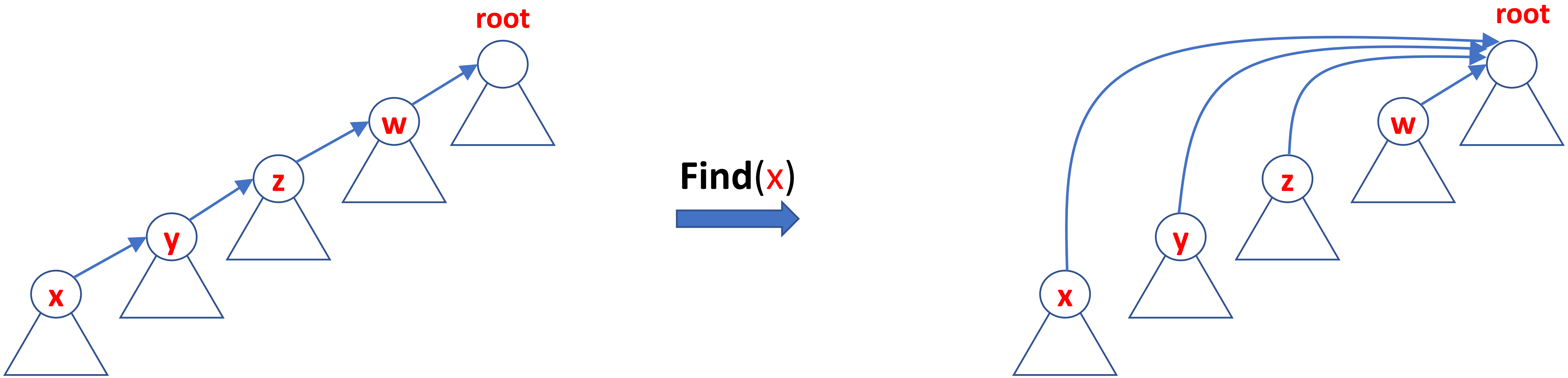

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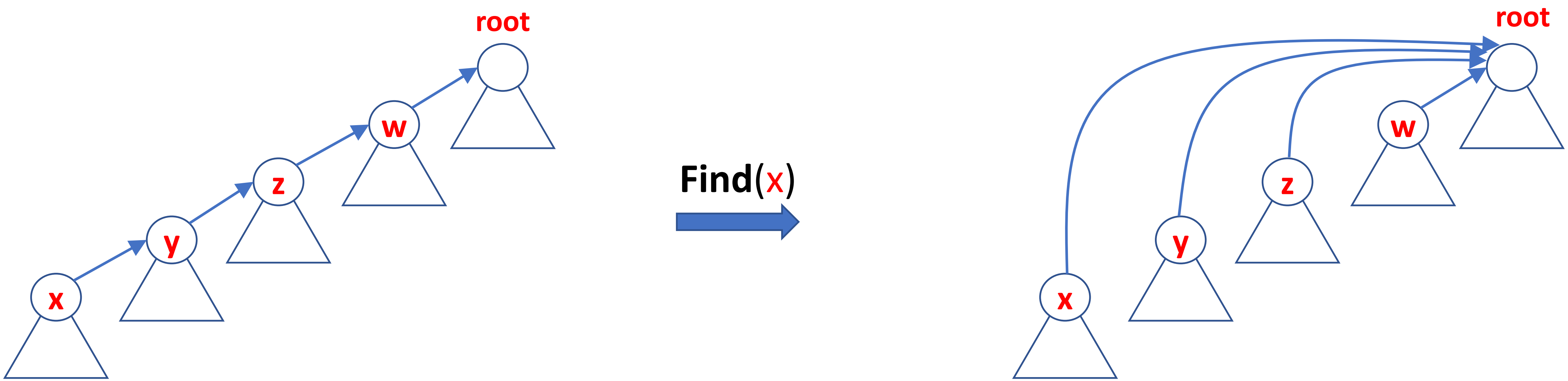

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Amortization

With **WU** and **PC** , Time Complexity of σ?

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A quick detour…

Definition: 2^{*n}

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$2^{*0} = 1$

$2^{*n+1} = 2^{2^{*n}}$, $n >= 0$

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$2^{*0} = 1$

$2^{*n+1} = 2^{2^{*n}}$, n >= 0

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Estimated # of atoms in observable universe $\approx 10^{80}$

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Definition: $\log^* n$

$\log^* n = \min\{k : 2^{*k} \geq n\}$

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n	1
log* n	0

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n	1	2
log* n	0	1

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n	1	2	3, 4
log* n	0	1	2

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log* n	0	1	2	3

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n	1	2	3, 4	5, 6, 7,…., 16	17, 18, 19, …., 65536	65537,….., 10^{19729}
log* n	0	1	2	3	4	5

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Definition: log*n grows very slowly with n !!

$\log^* n = \min\{k : 2^{*k} \geq n\}$

n	1	2	3, 4	5, 6, 7,...., 16	17, 18, 19, ..., 65536	65537,....., 10^{19729}	.....
log* n	0	1	2	3	4	5	

Time Complexity of σ, with **WU** and **PC**

σ: Sequence of n-1 **Unions** mixed with $m \geq n$ **Finds**

Theorem: With **WU** and **PC**, executing every such σ takes $O(m \log^* n)$ time

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That is, executing ***every*** such σ takes ***at most*** $m \log^* n$ time,
for large m, n, and within a constant factor

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However: Is there ***some*** σ that takes ***at least*** m log* n time ?

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However: Is there ***some*** σ that takes ***at least*** $m \log^* n$ time ?

Is the following claim true?

Claim: With **WU** and **PC**, executing every such σ takes $O(m)$ time

Analysis of Disjoint Forest: A Timeline

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Forest Implementation introduced

Bernard A. Galler
Michael J. Fischer

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O(m log* n) upper-bound for Forest Implementation

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1979

Ω(m α(m,n))
lower-bound
for *any*
Disjoint-Set data
structure that satisfies
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Robert E. Tarjan

Analysis of Disjoint Forest: A Timeline

1964	1973	1975	1979	1989
Forest Implementation introduced	$O(m \log^* n)$ upper-bound for Forest Implementation	$O(m\,\alpha(m,n))$ upper-bound for Forest Implementation	$\Omega(m\,\alpha(m,n))$ lower-bound for *any* Disjoint-Set data structure that satisfies some “technical assumptions”	$\Omega(m\,\alpha(m,n))$ lower-bound for *any* Disjoint-Set data structure
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