MEEN 651

Final Exam

Prof. Pilwon Hur

Take Home Exam

Issue: 12/4/2019

Due 12/10/2019 Midnight

Your Name: ________________________

An Aggie does not lie, cheat or steal or tolerate those who do.

By my signature below I pledge that my conduct on this exam is in every way consistent with the Aggie Code of Honor.

Signature: __________________________

Score

1-1	1-2	1-3	2-1	2-2	3-1	3-2	3-3	3-4	Total
/10	/10	/10	/10	/10	/10	/10	/10	/20	/100

1. You are given a linear vector space V defined on the field of *reals* with an inner product $\langle \cdot , \cdot \rangle$. You are further given n orthonormal (and naturally linearly independent) vectors $x_1, \dots x_n$ in V, and another vector $y \in V$ which is linearly independent of $\{x_1, \dots x_n\}$. Let

$$S \triangleq \mathrm{Span}\{x_1, \dots x_n\}$$

Given a vector v in S, we say that "$v \in S$ is an orthononal projection of y into S" if the vector $e \triangleq y - v$ has minimum norm. Such a vector v can be clearly expressed as

$$v = \sum_{i=1}^{n} \alpha_i x_i$$

for some scalars $\alpha_i, i = 1, \dots, n$ and the projection property says that the α_i's should be chosen in such a way that the function

$$f(\alpha_1, \dots, \alpha_n) \triangleq \|e\| = \left\| y - \sum_{i=1}^{n} \alpha_i x_i \right\|$$

is minimized.

1) Show that the unique choices for the α_i's are

$$\alpha_i = \langle y, x_i \rangle, \qquad i = 1, \dots, n$$

(Hint: Simply differentiate $f(\alpha_1, \dots, \alpha_n)$ with respect to α_i's and set the resulting expressions equal to zero) (10pt)

2) The result of 1) can easily be generalized to the case when the x_i's that generate S are not orthonormal, or even orthogonal, but simply linearly independent. In this case too each α_i will be unique, but it will be given by a different expression, which is again obtained by differentiation of $f(\alpha_1, \dots, \alpha_n)$. Carry out this procedure, and solve for α_i's when $V = (R^4, R) = (\text{vectors}, \text{field})$, $x_1 = (1,0,2,0)^T, x_2 = (0,1,0,-1)^T, x_3 = (1,0,2,1)^T$, and $y = (-2,0,10,0)^T$ with inner product on V being the standard one, that is, $\langle x, y \rangle = x^T y$, for $x, y \in R^4$. Also compute the minimum norm of the error vector e. (10pt)

3) Do the same as above when V is the space of continuous functions defined on the interval $[-1,1]$, with inner product

$$\langle x, y \rangle = \int_{-1}^{1} x(t) y(t) dt$$

$x_1 = 1, x_2 = t, x_3 = t^2$, and $y = t^3$ (10pt)

2. Consider the LTI system $\dot{x} = Ax + Bu,\ y = Cx$, where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix}$$

1) Determine an appropriate linear state feedback control law $u = -Kx + Gr$ $(G \in R)$ so that the closed-loop transfer function (from r to y) is equal to a given desired transfer function

$$H_m = \frac{1}{s^2 + 3s + 2}$$

Note: This is an example of model matching, that is compensating a given system so that it matches the input-output behavior of a desired model. In the present case, state feedback is used. However, output feedback is more common in model matching. (10pt)

2) Is the compensated system in i) controllable? Is it observable? Explain your answer. (10pt)

3. Given a scalar open-loop unstable linear system described by

$$\dot{x} = \frac{1}{4}x + u, \qquad x(0) = 2$$

Consider the optimal control problem of minimizing the performance index $J_0^{t_f}(u)$ where

$$J_t^{t_f}(u) = x(t_f)^4 + \int_t^{t_f} 2x(s)^4 + u(s)^4 ds, \quad 0 \le t \le t_f$$

Here, u is the control, x is the state, and $t_f < \infty$ is a fixed terminal time.
For any fixed $t \in [0, t_f]$, let $V(x,t)$ denote the minimum value of $J_t^{t_f}(u)$, with x denoting the initial state at time t. As mentioned in class, this function $V(x,t)$ is the optimum cost-to-go function associated with the optimal control problem.

1) Show that the optimum cost-to-go function is $V(x,t) = x^4$ for all $t \le t_f$ and all $t_f > 0$, and obtain a state feedback controller (that is a control, u, that depends on the state x) that minimizes $J_0^{t_f}(u)$. What is the minimum value of $J_0^{t_f}(u)$? (10pt)

2) What is the optimum control for the case when $t_f = \infty$? (10pt)

3) If we had used the Pontryagin's minimum principle to solve for the optimal control above, what would the corresponding co-state equation be? If $p = \frac{\partial}{\partial x}V(x,t)$, what would be $x^*(t), p^*(t), u^*(t)$? (10pt)

4) Solve the above problem numerically with $t_f = 2$. You can use both shooting method and direct collocation method. (10pt for each method)