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PHY422/820: Classical Mechanics

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Worksheet #5 (Sep 27 – Oct 1)

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1 Preparation

- Lemos, Section 1.6
- Goldstein, Sections 1.5

2 Symmetries and Effective Theories

2.1 Effective Theories

If we look back over the history of physics, we will note that again and again the existing theories of physical phenomena were revealed to be **effective theories**, i.e., limiting cases of more fundamental underlying theories. In fact, nature seems to be best grasped through a "tower" of such effective theories that will allow us to make progress despite not knowing what the ultimate underlying theory — perhaps a theory of everything — is going to be.

Classical mechanics is no different in this regard: It is an effective theory of Quantum Mechanics in the limit where energy can be treated as continuous instead of quantized, which we can formally express as $\frac{\hbar\omega}{E} \ll 1$ (or sometimes as $\hbar \to 0$). Furthermore, it is the limit of Special Relativity for $\frac{v}{c} \ll 1$, which is itself a limit of General Relativity for weak gravitational fields.

2.1.1 Example: Effective Theories of Gravity

The treatment of gravity in Introductory Physics is likely your earliest encounter with an effective theory. Gravity is first introduced either in force or potential form as

$$\vec{F} = m\vec{g}\,, \quad V = mgh\,, \tag{3}$$

where h is the elevation of an object with mass m over the surface of the Earth. Some time later, this is revealed to be a limit of the more general expression

$$\vec{F} = -G\frac{m_1 m_2}{r^2}\vec{e}_r\,, \quad V = -G\frac{m_1 m_2}{r}\,, \tag{4}$$

where $\vec{r}$ is the distance between the masses m_1 and m_2. For the case of an object near the surface of the Earth, we have $m_1 = m$, $m_2 = M_E$ is the mass of the Earth, and $r = R_E + h$ is the distance of object from the Earth's center of mass, so that

$$V = -G\frac{mM_E}{R_E + h} = -G\frac{mM_E}{R_E}\frac{1}{1 + h/R_E}$$

$$
\begin{aligned}
&= -G\frac{mM_E}{R_E}\sum_{n=0}^{\infty}\left(\frac{h}{R_E}\right)^n = -G\frac{mM_E}{R_E}\left(1-\frac{h}{R_E}+\mathcal{O}\left(\frac{h^2}{R_E^2}\right)\right) \\
&= -G\frac{mM_E}{R_E}+G\frac{mM_E}{R_E^2}h+\mathcal{O}\left(\frac{h^2}{R_E^2}\right) \\
&= V_0 + mgh + \mathcal{O}\left(\frac{h^2}{R_E^2}\right)\,, \qquad (5)
\end{aligned}
$$

where we have introduced $g \equiv G\frac{M_E}{R_E^2}$ and $V_0 = -mgR_E$ is the potential at ground level. Thus, the first form of gravity you learn is the leading-order effective theory of Newtonian gravity, Eq. (4). For sufficiently high elevation, we may need to take corrections in $\frac{h}{R_E}$ into account, and once $\frac{h}{R_E} \geq 1$, the geometric series expansion in Eq. (5) is no longer converging. We say that the effective theory **breaks down** at the distance **scale** R_E.

2.1.2 General Strategy for Effective Theories

The general strategy for setting up an effective theory consists of identifying a small expansion parameter that is usually associated with a **separation of scales**. In our example, an everyday elevation h is $\mathcal{O}(1\,\mathrm{m})$ to $\mathcal{O}(10\,\mathrm{m})$, which is small compared to the radius of the Earth, $R_E = 6.371 \times 10^6\,\mathrm{m}$. Moreover, we can neglect whatever the effects of corrections from a hypothetical quantum theory of gravity would be. Another example is nonrelativistic mechanics, where objects most with $v \ll c$. We can now perform an expansion in the small parameter and set up what is called a **power-counting scheme**, treating all observables through some order n in the ratio of the small and large scales. In this context, we frequently use the terms leading-order (LO), next-to-leading order (NLO), next-to-next-to-leading order (NNLO) etc.

In the example of gravity, we knew the underlying theory and used it to construct the effective theory for objects near the surface of the Earth. What can we do if the underlying theory is unknown? If we are able to identify a possible new scale — e.g., by noting where deviations form the effective theory behavior become noticeable — we can attempt to **define** the expansion parameter x as the ratio of our typical distances, momenta, etc. and the identified scale, and make a power-series **ansatz** for the theory:

$$
V = \sum_{k=0}^{\infty} c_k x^k\,. \qquad (1)
$$

The expansion coefficients[1] c_k are then constrained by the symmetries of the theory (see Sec. 2.2), and we can determine them at some given order of the expansion by fitting them to experimental data. The LO theory will fix c_1 and observables will have an error of order $\mathcal{O}(x^2)$ from the omitted terms in the expansion. At NLO, we fix c_1 and c_2, possibly with slightly different values than at LO, and the overall error of the theory will be $\mathcal{O}(x^3)$, and so on. The effective theory can then be used to make predictions for any observables that were **not** used to fit the c_k.

2.2 Deriving Lagrangians from Symmetries

We have seen that the laws of classical mechanics can be cast in the form of d'Alembert's principle or (with certain limitations) the form of the principle of least action. However, nothing in these

[1] Effective theories are frequently used as low-momentum or low-energy approximations to the underlying theory, hence the expansion parameter is a ratio of momentum or energy scales. In such cases, the expansion coefficients are referred to as the low-energy constants (LECs), of the effective (field) theory.

principles enforces a particular shape of the Lagrangian, aside perhaps from the idea that the action should be *minimal* in the latter case.

In our discussion of conservation laws, we have found the deep relationships between symmetries and invariances of the Lagrangian. Moreover, we have seen that we had to *impose* certain properties on the potentials appearing in the Lagrangian to produce a desired invariance — or, in other words, to make the potentials compatible with the fundamental symmetries of space. This is a very powerful idea that we want to explore in a bit more, and connect to the idea of effective theories laid out in the previous section.

Lagrangian of a Free Particle

Consider the Lagrangian governing the dynamics of a free particle, which should be a function of the particle's position, velocity, and possibly time,

$$L = L(\vec{r}, \dot{\vec{r}}, t)\,. \tag{2}$$

First, we make use of the **homogeneity of space**, which implies that the dynamics of a particle cannot depend on the choice of coordinate system that we use to describe its motion. In fact, his means that L cannot explicitly depend on $\vec{r}$ at all. Likewise, **time is homogeneous** if we do not have external forces, so the free particle's dynamics cannot depend explicitly on the time we make an observation either. Thus, we must have

$$L = L(\dot{\vec{r}})\,. \tag{3}$$

Next, we use the **isotropy of space**, which means that there is no preferred direction and the dynamics of the particle cannot depend on the orientation of the coordinate system we use to describe it. Thus, the Lagrangian can only be a function of the magnitude of the velocity vector,

$$\dot{\vec{r}} \cdot \dot{\vec{r}} = |\dot{\vec{r}}|^2 = v^2\,, \tag{4}$$

and we can write

$$L = L(v^2)\,. \tag{5}$$

Finally, we consider the principle of (Newtonian) relativity, which implies the invariance of the Lagrangian under Galiliean boosts, so that observers in different inertial systems will derive the same equations of motion for the particle. If the particle is observed from a coordinate system that is moving with a small relative velocity $\vec{u}$ with respect to ours, the Lagrangian will be

$$L' = L((\vec{v} + \vec{u})^2) = L(v^2) + 2\vec{u} \cdot \vec{v} \frac{\partial L}{\partial v^2} + O(\vec{u}^2) \tag{6}$$

Since the equations of motion for L' must be identical to those for L, the additional term must be a total time derivative. We can rewrite it as

$$2\left(\frac{d\vec{r}}{dt}\right) \cdot \vec{u} \frac{\partial L}{\partial v^2} = \frac{d}{dt}\left(2\vec{r} \cdot \vec{u} \frac{\partial L}{\partial v^2}\right) - 2\vec{r} \cdot \vec{u} \left(\frac{d}{dt}\frac{\partial L}{\partial v^2}\right)\,. \tag{7}$$

The first term on the right-hand side is our total time derivative, so the second term must vanish. The direction of $\vec{u}$ is arbitrary, so the scalar product $\vec{r} \cdot \vec{u}$ does not vanish general, and we must have

$$\frac{d}{dt}\frac{\partial L}{\partial v^2} = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial v^2} = \text{const.} \tag{8}$$

This actually means that the Lagrangian must be a scalar multiple of the square of the velocity, i.e.,

$$L(v^2) = Cv^2 \,, \tag{9}$$

and the boosted Lagrangian would be

$$L_0'(v^2) = C(v^2 + 2\vec{v}\cdot\vec{u} + u^2) = Cv^2 + C\frac{d}{dt}\left(2\vec{r}\cdot\vec{u} + u^2 t\right) \,. \tag{10}$$

The unknown constant C could now be determined by fitting observables like the momentum or the (kinetic) energy of the particle to experimental data.

Enter Relativity

If we extract C from experimental measurements of objects that move with low velocities, we will find that with very high accuracy $C = m/2$, as expected. As we try to test the universality of this determination for objects with increasing speed (or with increasing accuracy), we will notice meaningful deviations between C and $m/2$, which will force us to revisit the assumptions we made in our derivation of the free Lagrangian.

The homogenity of space and time and the isotropy of space are such fundamental properties that it is natural to question the final element we used in our derivation of L, which is the invariance under Galilean boosts. Giving up this requirement, we allow L to be a more general function of v^2; however, our experimental observations imply that in the low-velocity limit, we should have $L \approx \frac{1}{2}mv^2$. Next, we realize that low velocity implies the question *Low compared to what?* Applying the effective theory strategy, we would conclude that we should compare v to a velocity scale c which controls when we will start seeing deviations from the low-velocity limit, i.e., the LO effective theory, and we can make the ansatz

$$L(v^2) = \sum_{k=0}^{\infty} C_k \left(\frac{v}{c}\right)^{2k} \,, \tag{11}$$

where we have allowed for the presence of a constant term that has no impact on the dynamics of our particle.

From Eq. (11) and $L \approx \frac{1}{2}mv^2$ for $v/c \ll 1$, we can conclude that c is *finite*, and $C_1 \sim c^2$. If c were infinite, then we would have $L = C_0$, in contradiction to our experimental observations. Armed with this knowledge, we can fit the expansion coefficients at increasing orders of the effective theory to high-precision data, which allows us to pin down c with increasing accuracy and ultimately makes it possible to identify the underlying relativistic Lagrangian

$$L = \sqrt{(\vec{p}c)^2 + m^2c^4} \,. \tag{12}$$

The existence of the finite maximum speed c will then allow us to replace Galilean boosts by Lorentz boosts, and "restore" the ability to make coordinate transformations between arbitrary inertial frames.

Adding Interactions

In our discussion of the fundamental spacetime symmetries and the associated conservation laws we made note of the conditions we had to impose on the interaction terms in the Lagrangian to make the derivation of conserved quantities possible. We impose these conditions — as well as any additional conditions that might derive from additional, non-spacetime symmetries of our system, like electric charge conservation — on the general ansatz for an effective theory that is given in

Sec. (2.1.2). By doing so, we will usually restrict the form of the allowed interaction terms and expansion coefficients. For instance, we might have considered an expansion of the form

$$V \sim \sum_{klm} C_{klm} a^k r^l (\vec{a} \cdot \vec{r})^m \,, \quad \vec{a} = \text{const.} \tag{13}$$

but rotational invariance requires that we only allow terms with $m = 0$, because otherwise $\vec{a}$ would define a preferred direction.

3 Velocity-Dependent Potentials

When we derived the Lagrange equations from d'Alembert's principle, we first obtained them in the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j \,. \tag{14}$$

Assuming that the generalized forces can be written as the gradients of a potential $V(q)$,

$$Q_j = \frac{\partial V}{\partial q_j} \,, \tag{15}$$

we moved them to the left-hand side, using that $\frac{\partial V}{\partial \dot{q}_j} = 0$:

$$\frac{d}{dt}\frac{\partial (T-V)}{\partial \dot{q}_j} - \frac{\partial (T-V)}{\partial q_j} = 0 \,. \tag{16}$$

It is easy to see that the Lagrange equations (i.e., d'Alembert's principle) would also be satisfied for more general forces of the form

$$\boxed{Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_j} \,,} \tag{17}$$

where $U(q, \dot{q})$ is a velocity-dependent potential. The definition of the Lagrangian simply becomes

$$\boxed{L \equiv T - U \,,} \tag{18}$$

since $V(q)$ would be a special case of $U(q, \dot{q})$.

Example: Particle Moving in an Electromagnetic Field

Let us consider a particle with mass m and charge q that moves in an *external* electromagnetic field. We not impose any constraints, hence we can work in Cartesian coordinates:

$$(q_1, q_2, q_3) = (x_1, x_2, x_3) = (x, y, z) \,. \tag{19}$$

The Lagrangian of this particle is given by

$$L(\vec{r}, \dot{\vec{r}}, t) = \frac{1}{2} m \dot{\vec{r}}^2 - q\phi(\vec{r}, t) + q\vec{A}(\vec{r}, t) \cdot \dot{\vec{r}} \,, \tag{20}$$

with the explicitly velocity-dependent potential

$$U(\vec{r}, \dot{\vec{r}}) = q\phi - q\vec{A} \cdot \dot{\vec{x}} \,. \tag{21}$$

In this case, we have

$$\frac{\partial L}{\partial \dot{x}_i} = m\dot{x}_i + qA_i(\vec{r},t) \ , \tag{22}$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} = m\ddot{x}_i + q\left(\sum_{k=1}^{3}\frac{\partial A_i}{\partial x_k}\dot{x}_k + \frac{\partial A_i}{\partial t}\right) \ . \tag{23}$$

We also have

$$\frac{\partial L}{\partial x_i} = -q\frac{\partial \phi}{\partial x_i} + q\sum_{k=1}^{3}\frac{\partial A_k}{\partial x_i}\dot{x}_k \,. \tag{24}$$

Combining the derivatives and switching to a vectorial form, we obtain the Lagrange equation

$$m\ddot{\vec{r}} + q\left((\dot{\vec{r}}\cdot\vec{\nabla})\vec{A} + \frac{\partial \vec{A}}{\partial t}\right) = -q\vec{\nabla}\phi + q\vec{\nabla}(\vec{A}\cdot\dot{\vec{r}}) \,. \tag{25}$$

It is left as an exercise (see homework #4) to show that this equation reduces to the usual equation of motion under the influence of the **Lorentz force**,

$$m\ddot{\vec{r}} = q(\vec{E} + \dot{\vec{r}}\times\vec{B}) \,, \tag{26}$$

where the electric and magnetic fields are defined as

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial A}{\partial t} \tag{27}$$

and

$$\vec{B} = \vec{\nabla}\times\vec{A} \,. \tag{28}$$

Thus, the Lorentz force can indeed be derived from the velocity-dependent potential (21)

$$F_i = q(\vec{E} + \dot{\vec{r}}\times\vec{B})_i = -\frac{\partial U}{\partial x_i} + \frac{d}{dt}\frac{\partial U}{\partial \dot{x}_i} \,. \tag{29}$$

Let us have a quick look how work performed by or against the Lorentz force can be path independent and allow us to define a potential while the same is not true for other types of velocity-dependent forces like friction (cf. Sec. 4). For an arbitrary curve $\vec{r}(t)$, we obtain

$$W = \int_\gamma d\vec{r}\cdot\vec{F} = \int_{t_1}^{t_2} dt\,\dot{\vec{r}}\cdot\vec{F} = q\int_{t_1}^{t_2} dt\,\left(\dot{\vec{r}}\cdot\vec{E} + \underbrace{\dot{\vec{r}}\cdot\left(\dot{\vec{r}}\times\vec{B}\right)}_{=0}\right) = q\int_{t_1}^{t_2} dt\,\dot{\vec{r}}\cdot\vec{E} \,, \tag{30}$$

where we have used that $\dot{\vec{r}}$ is orthogonal to the cross product $\dot{\vec{r}}\times\vec{B}$ by construction. Thus, the velocity-dependent part of the Lorentz force doesn't actually contribute to the work.

To conclude our discussion, we compute the Jacobi integral $h(\vec{r},\dot{\vec{r}})$:

$$\begin{aligned} h(\vec{r},\dot{\vec{r}},t) &= \left(m\dot{\vec{r}} + q\vec{A}(\vec{r},t)\right)\cdot\dot{\vec{r}} - \frac{1}{2}m\dot{\vec{r}}^2 + q\left(\phi(\vec{r},t) - \dot{\vec{r}}\cdot\vec{A}(\vec{r},t)\right) \\ &= \frac{1}{2}m\dot{\vec{r}}^2 + q\phi(\vec{r},t) \,. \end{aligned} \tag{31}$$

Since the fields can be time dependent, h is generally *not conserved.* It does represent the total energy of the particle, but not the total energy of the complete system consisting of the particle and the fields. Energy can be transferred between the particles and the fields, and changes of the external fields $\phi(\vec{r},t), \vec{A}(\vec{r},t)$ *may* require some form of external work, e.g., for moving additional charges into the source region.

4 Dissipation

4.1 The Dissipation Function

In realistic mechanical systems, dissipative forces like dry or viscous friction will resist the *relative* motion of extended solids, surfaces, or fluid layers, causing a loss of mechanical energy. While the details of frictional mechanisms are usually microscopic in nature and beyond the scope of Classical Mechanics, a wide class of frictional phenomena can be modeled by forces of the form [1, 2]

$$\vec{F}^D = -\mu(v)\frac{\vec{v}}{v}\,, \tag{32}$$

pointing in the opposite direction of the relative velocity $\vec{v}$ between the moving object and the environment. Here, $\mu(v)$ is a positive function that could also depend on the coordinates. If the environment is static, we can identify

$$\vec{v} = \dot{\vec{r}}. \tag{33}$$

Since friction forces are nonconservative — i.e., the work done against these forces depends on the trajectory — they must be treated explicitly as generalized forces in the Lagrange equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = Q_j^D\,, \tag{34}$$

where we assumed that all *conservative* forces are included in the Lagrangian. The generalized dissipative forces are obtained from

$$Q_j^D = \sum_{i=1}^{N} \vec{F}_i^D \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_{i=1}^{N} \vec{F}_i^D \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} = -\sum_{i=1}^{N} \mu_i(v_i)\frac{\vec{v}_i}{v_i}\cdot\frac{\partial \vec{v}_i}{\partial \dot{q}_j}\,, \tag{35}$$

where we have used the "cancellation of dots",

$$\frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}\,. \tag{36}$$

Noticing that

$$\vec{v}_i \cdot \frac{\partial \vec{v}_i}{\partial \dot{q}_j} = \frac{1}{2}\frac{\partial}{\partial \dot{q}_j}\left(\vec{v}_i\cdot\vec{v}_i\right) = \frac{1}{2}\frac{\partial v_i^2}{\partial \dot{q}_j} = v_i\frac{\partial v_i}{\partial \dot{q}_j}\,, \tag{37}$$

we obtain

$$Q_j^D = -\sum_{i=1}^{N} \mu_i(v_i)\frac{\partial v_i}{\partial \dot{q}_j}\,. \tag{38}$$

We can rewrite this expression further: First, we note that

$$\mu_i(v_i) = \frac{\partial}{\partial v_i}\left(\int_0^{v_i} \mu_i(v')\,dv'\right)\,, \tag{39}$$

so the chain rule implies

$$\mu_i(v_i)\frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial v_i}{\partial \dot{q}_j}\mu_i(v_i) = \frac{\partial}{\partial \dot{q}_j}\left(\int_0^{v_i} \mu_i(v')\,dv'\right)\,, \tag{40}$$

i.e.,

$$Q_j^D = -\frac{\partial}{\partial \dot{q}_j}\sum_{i=1}^{N}\int_0^{v_i} \mu_i(v')\,dv'\,. \tag{41}$$

We see that instead of treating friction using the multi-component generalized forces Q_j^D, we can introduce a scalar **dissipation function** $D(q, \dot{q})$,

$$D = \sum_{i=1}^{N} \int_0^{v_i} \mu_i(v')\, dv' \,, \tag{42}$$

and write the Q_j^D as its derivatives with respect to $\dot{q}_j$. The Lagrange equations with the generalized friction forces now become

$$\frac{d}{d}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = -\frac{\partial D}{\partial \dot{q}_j} \,, \tag{43}$$

and we can apply them to systems with friction by specifying L and D.

Interpretation of the Dissipation Function

To understand the physical meaning of D, we consider the rate of change of the total energy $(T+V)$:

$$\frac{d}{dt}(T+V) = \sum_{j=1}^{s} \left(\frac{\partial T}{\partial q_j}\dot{q}_j + \frac{\partial T}{\partial \dot{q}_j}\ddot{q}_j \right) + \frac{dV}{dt} \,. \tag{44}$$

The second term can be rewritten with the usual trick,

$$\sum_{j=1}^{n} \frac{\partial T}{\partial \dot{q}_j}\ddot{q}_j = \frac{d}{dt}\left(\sum_{j=1}^{n} \frac{\partial T}{\partial \dot{q}_j}\dot{q}_j \right) - \sum_{j=1}^{n} \dot{q}_j \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} \,. \tag{45}$$

If we assume scleronomic constraints for simplicity, the kinetic energy T is a homogeneous function of degree 2 in the generalized velocities, and the parenthesis simply gives us $2T$ (see worksheet #4).

We can use this result along with the first version of the Lagrange equations (cf. worksheet #3) and (43) to rewrite Eq. (44):

$$\begin{aligned}
\frac{d}{dt}(T+V) &= \sum_{j=1}^{n} \left(\frac{\partial T}{\partial q_j} - \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} \right) \dot{q}_j + \frac{d}{dt}(2T) + \frac{dV}{dt} \\
&= \sum_{j=1}^{n} \left(\frac{\partial T}{\partial q_j} - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} \right) \dot{q}_j + \frac{d}{dt}(2T) + \frac{dV}{dt} \\
&= \sum_{j=1}^{n} \frac{\partial V}{\partial q_j}\dot{q}_j + \frac{d}{dt}(2T+V) + \sum_{j=1}^{n} \frac{\partial D}{\partial \dot{q}_j}\dot{q}_j \\
&= 2\frac{d}{dt}(T+V) + \sum_{j=1}^{n} \frac{\partial D}{\partial \dot{q}_j}\dot{q}_j \,,
\end{aligned} \tag{46}$$

and rearranging, we have

$$\frac{d}{dt}(T+V) = -\sum_{j=1}^{n} \frac{\partial D}{\partial \dot{q}_j}\dot{q}_j \,. \tag{47}$$

The work that a dynamical system of N particles must do against the frictional forces under an infinitesimal displacement along the generalized coordinates is

$$dW^D = \sum_{i=1}^{N} \vec{F}_i^D \cdot d\vec{r}_i = \sum_j Q_j^D dq_j = -\sum_j \frac{\partial D}{\partial \dot{q}_j} dq_j \tag{48}$$

and therefore

$$\frac{dW^D}{dt} = -\sum_j \frac{\partial D}{\partial \dot{q}_j} \dot{q}_j \,, \tag{49}$$

so we can also write

$$\boxed{\frac{d}{dt}(T+V) = \frac{d}{dt} W^D \,. \tag{50}}$$

Thus, the change in the systems total energy is equal to the work it does against dissipative forces. Since we restricted ourselves to scleronomic constraints above, the total energy is identical to the Jacobi integral $h(q, \dot{q})$, and we also have

$$\frac{d}{dt} h(q, \dot{q}) = -\sum_{j=1}^{n} \frac{\partial D}{\partial \dot{q}_j} \dot{q}_j \,. \tag{51}$$

4.2 Viscuous Friction and Rayleigh's Dissipation Function

Rayleigh considered the special case where the friction forces acting on a particle i are linear in the velocities, i.e., $\mu(v) = bv$. Assuming that the friction force is identical for all particles in the system, but possibly dependent on the direction in which they are moving, the force on particle i can be written as

$$\vec{F}_i = -\boldsymbol{B} \dot{\vec{r}}_i \,, \tag{52}$$

where $\boldsymbol{B}$ is a symmetric 3×3 matrix, or in components

$$F_r^{(i)} = -\sum_{s=1}^{3} B_{rs} \dot{x}_s^{(i)} \,, \quad r, s = 1, 2, 3 \,, \tag{53}$$

with $B_{rs} = B_{sr}$. Switching to generalized velocities and forces, we obtain

$$Q_j^D = \sum_{i=1}^{N} \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_{i=1}^{N} \sum_{k=1}^{n} \boldsymbol{B} \frac{\partial \vec{r}_i}{\partial q_k} \dot{q}_k \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_{k=1}^{n} \beta_{jk} \dot{q}_k \,, \tag{54}$$

where we have defined the symmetric dissipation matrix in generalized coordinates,

$$\beta_{jk} \equiv \sum_{i=1}^{N} \boldsymbol{B} \frac{\partial \vec{r}_i}{\partial q_j} \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \sum_{r,s=1}^{3} \frac{\partial x_s^{(i)}}{\partial q_j} B_{rs} \frac{\partial x_r^{(i)}}{\partial q_k} \tag{55}$$

We can use β_{jk} to write the quadratic dissipation function — also referred to as **Rayleigh's dissipation function** — in a compact form:

$$D = \frac{1}{2} \sum_{j,k=1}^{n} \beta_{jk} \dot{q}_j \dot{q}_k \,. \tag{56}$$

Since β_{jk} does not depend on the generalized velocities, we readily obtain Eq. (54) when we evaluate $Q_j^D = \frac{\partial D}{\partial \dot{q}_j}$.

We note that Rayleigh's dissipation function is a homogeneous function of degree 2, so the change in the total energy (and the Hamiltonian) is given by

$$\frac{dE}{dt} = \frac{d}{dt} h(q, \dot{q}) = -2D \tag{57}$$

(see Eqs. (47), (51)).

Example: Stokes's Law

As an example, we consider a sphere of mass m that is moving through a liquid at slow velocity, experiencing a drag force due to the fluid's **laminar flow** around its surface. The drag force is[2]

$$\vec{F}_D = -6\pi\eta R\vec{v} \equiv -\beta\vec{v}\,, \tag{58}$$

where η is the viscosity of the fluid and R the radius of the sphere. Considering the motion in one dimension, with z increasing in downward direction, the Lagrangian becomes

$$L = T - V = \frac{m}{2}\dot{z}^2 + mgz\ , \tag{59}$$

and the drag can be modeled by the dissipation function

$$D = \frac{1}{2}\beta v^2 = \frac{1}{2}\beta\dot{z}^2\,. \tag{60}$$

Thus, we obtain the Lagrange equation

$$m\ddot{z} - mg = -\beta\dot{z}\ . \tag{61}$$

We can write

$$\frac{d}{dt}\dot{z} = g - \frac{\beta}{m}\dot{z} \quad \Rightarrow \quad dt = -\frac{d\dot{z}}{\frac{\beta}{m}\dot{z} - g}\ , \tag{62}$$

and integrate:

$$t - t_0 = -\frac{m}{\beta}\ln\frac{\beta\dot{z} + mg}{\beta v_0 - mg}\,. \tag{63}$$

With the initial conditions $t_0 = 0, v_0 = 0$, we can exponential to obtain

$$\exp\left(-\frac{\beta}{m}t\right) = \frac{\beta\dot{z} - mg}{-mg} \tag{64}$$

$$\Leftrightarrow \quad \dot{z} = \frac{mg}{\beta}\left[1 - \exp\left(-\frac{\beta}{m}t\right)\right]\ . \tag{65}$$

Thus, we see that the velocity remains finite for large t, and approaches the *terminal velocity*

$$v_\infty = \frac{mg}{\beta}\,. \tag{66}$$

[2]The form of this drag force was first derived in 1851 by G. Stokes, who computed the friction between a viscous fluid and a solid sphere at the sphere's surface using fluid dynamics. Thus, such drag forces are referred to as Stokes's drag or Stokes's friction in the literature.

4.3 Drag Due to Turbulent Flow

If the flow of a fluid or gas around an object is **turbulent** instead of laminar, e.g., due to the object's high velocity, the drag force is no longer linear as in Stokes's law, but quadratic:

$$\vec{F}_D = -\frac{1}{2} C_D \rho A v^2 \frac{\vec{v}}{v} \equiv -\frac{1}{2} \beta v^2 \frac{\vec{v}}{v}\,, \tag{67}$$

where ρ is the density of the fluid, A the cross-section area of the object orthogonal to its direction of motion, v the relative velocity of object and fluid flow, and C_D the so-called drag coefficient. We will discuss the trajectory of a skydiver as a concrete example in the group exercises.

4.4 Coulomb or Dry Friction

The simple models for static and kinetic friction forces that are discussed in introductory mechanics classes are examples of **dry** or **Coulomb friction**. Coulomb modeled the frictional forces between dry surfaces with the ansatz

$$\vec{F}_{s,k} = -\mu_{s,k} N \frac{\vec{v}}{v}\,, \tag{68}$$

where μ_s and μ_k are the (constant) static and kinetic friction coefficients that are tabulated for a variety of materials and surface types, and N is the absolute value of the normal force pressing the surfaces together. We recall that static friction is only considered for masses *at rest*, to define a a critical force that is required to set objects in motion. Once the static friction force is overcome, the model switches from the static to the kinetic friction coefficient.

In the framework described in this section, we can obtain the kinetic friction force by setting

$$\mu(v) = \mu_k N = \text{const.} \tag{69}$$

and the dissipation function becomes

$$D = \mu_k N v\,. \tag{70}$$

In principle, we can also model the transition from static to kinetic friction with a steep but still smooth change in the friction function $\mu(v)$ at small velocities (see next example.)

Overall, Coulomb's model is a significant simplification of the underlying microscopic effects that nevertheless proves to be not only versatile, but adequate for many physical systems.

Example: Wooden Block on a Conveyor Belt

Consider a block of mass m that is sitting on a conveyor belt. At $t = 0$, a worker pushes is with the force $\vec{F} = (F_x, F_y)^T$. We describe the block's motion from the rest frame of the worker, assign coordinates x, y to it. The absolute value of the relative velocity between block and conveyor belt is given by

$$v_r = \sqrt{(\dot{x} - v_0)^2 + \dot{y}^2}\,, \tag{71}$$

where v_0 is the speed of the belt.

We use the dry friction force

$$\vec{F}_f = -\mu(v) \frac{\vec{v}}{v}\,, \tag{72}$$

The friction function for wood on belt rubber is given by [2]

$$\mu(v) = \left(\frac{\mu_0 - \mu_\infty}{1 + av} + \mu_\infty \right) N = \left(\frac{\mu_0 - \mu_\infty}{1 + av} + \mu_\infty \right) mg\,, \tag{73}$$

where μ_0, μ_∞, v_0 and a are positive constants, and we have plugged in the magnitude of the normal force, $|\vec{N}| = mg$. Note that

$$\mu(v) \underset{v\to\infty}{\longrightarrow} \mu_\infty N\,, \quad \mu(v) \underset{v\to 0}{\longrightarrow} \mu_0 N\,, \tag{74}$$

so μ_0 essentially corresponds to the coefficient of static friction in a Coulomb model, as discussed above.

According to Eq. (42), the dissipation function is now given by

$$D = mg\int_0^{v_r} \mu(u)\, du = mg\left(\frac{\mu_0-\mu_\infty}{a}\ln\left(1+a\sqrt{(\dot{x}-v_0)^2+\dot{y}^2}\right)+\mu_\infty\sqrt{(\dot{x}-v_0)^2+\dot{y}^2}\right)\,, \tag{75}$$

and its derivatives are

$$\begin{aligned}
\frac{\partial D}{\partial \dot{x}} &= mg\left(\frac{\mu_0-\mu_\infty}{a}\frac{1}{1+a\sqrt{(\dot{x}-v_0)^2+\dot{y}^2}}+\mu_\infty\right)\frac{\dot{x}-v_0}{\sqrt{(\dot{x}-v_0)^2+\dot{y}^2}}\,,\\
\frac{\partial D}{\partial \dot{y}} &= mg\left(\frac{\mu_0-\mu_\infty}{a}\frac{1}{1+a\sqrt{(\dot{x}-v_0)^2+\dot{y}^2}}+\mu_\infty\right)\frac{\dot{y}}{\sqrt{(\dot{x}-v_0)^2+\dot{y}^2}}\,.
\end{aligned} \tag{76}$$

Thus, the equations of motion for the block read

$$m\ddot{x} = F_x - mg\left(\frac{\mu_0-\mu_\infty}{a}\frac{1}{1+a\sqrt{(\dot{x}-v_0)^2+\dot{y}^2}}+\mu_\infty\right)\frac{\dot{x}-v_0}{\sqrt{(\dot{x}-v_0)^2+\dot{y}^2}}\,, \tag{77}$$

$$m\ddot{y} = F_y - mg\left(\frac{\mu_0-\mu_\infty}{a}\frac{1}{1+a\sqrt{(\dot{x}-v_0)^2+\dot{y}^2}}+\mu_\infty\right)\frac{\dot{y}}{\sqrt{(\dot{x}-v_0)^2+\dot{y}^2}}\,. \tag{78}$$

References

[1] A. I. Lurie, *Analytical Mechanics* (Springer, 2002).

[2] E. Minguzzi, European Journal of Physics **36**, 035014 (2015).

5 Group Exercises

Problem G12 – Particle in a Magnetic Field

[cf. Lemos, problem 2.18] A particle of mass m and electric charge q moves in a constant magnetic field $\vec{B} = B\vec{e}_z$. The Lagrangian is given by (cf. homework problem H10)

$$L = \frac{1}{2}m\dot{\vec{r}}^2 + q\dot{\vec{r}} \cdot \vec{A}(\vec{r})\,, \quad \vec{A} = \frac{1}{2}\vec{B} \times \vec{r}\,. \tag{79}$$

1. Show explicitly that the vector potential $\vec{A}$ produces the magnetic field $\vec{B}$.
2. Derive the Lagrange equations in Cartesian coordinates, and show that they are equivalent to

   $$\dot{\vec{v}} = -\vec{\omega} \times \vec{v}\,. \tag{80}$$

   Determine the frequency ω in terms of B.
3. Express the Lagrangian in cylindrical coordinates ρ, ϕ, z. Show that although ϕ is a cyclic coordinate, the angular momentum $l_z = m\rho^2\dot{\phi}$ is *not* conserved. Discuss.

Problem G13 – Aerodynamic Drag

A skydiver is falling under the influence of gravity while also being subject to aerodynamic drag of the form

$$\vec{F}_D = -\beta v^2 \vec{e}_v\,. \tag{81}$$

1. Identify $\vec{v}$ appropriately, and construct the dissipation function in terms of the skydiver's coordinates.
2. Derive the Lagrange equations.
3. Determine the skydiver's terminal velocity v_f from the dynamical equilibrium condition, and use replace β in the equation of motion. Solve the equations of motion and show that $v(t) \to v_f$ for $t \to \infty$.

   INTEGRALS:

   $$\int \frac{dz}{1-z^2} = \operatorname{artanh} z = \frac{1}{2}\log\frac{1+z}{1-h}\,, \quad |z| < 1 \tag{82}$$

   $$\int \tanh z\; dz = \log\cosh z \tag{83}$$