

PHY422/820: Classical Mechanics

FS 2021

Homework #12 (Due: Nov 29)

November 24, 2021

Problem H23 – Normal Modes and Generalized Forces

[10 Points] In our discussion of coupled oscillators, we only considered the forces resulting from a potential, e.g., a spring potential, or the Taylor expansion of a general potential around equilibrium. Let us now assume that each degree of freedom η_j is also subject to a generalized external force Q_j , e.g., friction or a driving term. We can write the equations of motion componentwise as

$$\sum_k T_{jk} \ddot{\eta}_k + \sum_k V_{jk} \eta_k = Q_j, \quad j = 1, \dots, N, \quad (1)$$

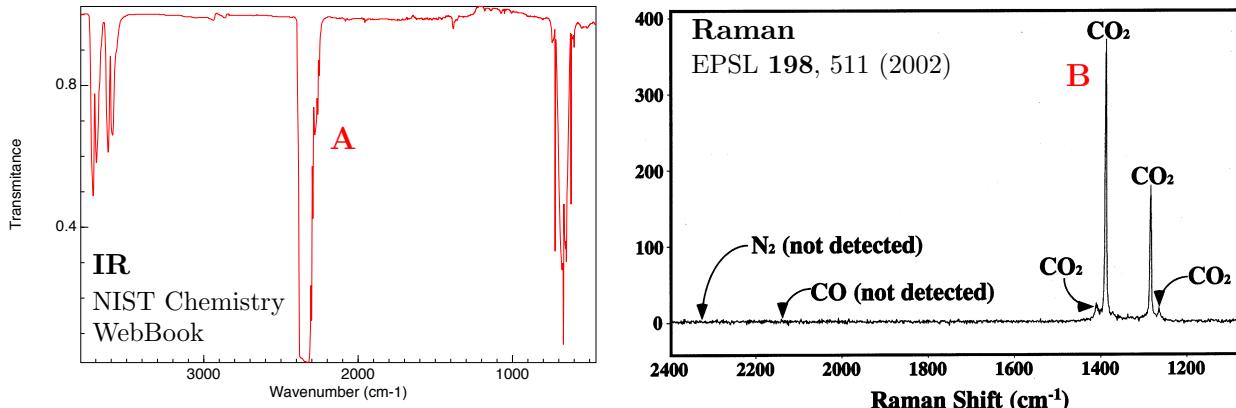
or vectorially as

$$\mathbf{T} \ddot{\vec{\eta}} + \mathbf{V} \vec{\eta} = \vec{Q}. \quad (2)$$

1. Show that the equations of motion in the normal mode basis are given by

$$\ddot{\zeta}_s + \omega_s^2 \zeta_s = \sum_k A_{sk} Q_k = (\mathbf{A}^T \vec{Q})_s, \quad (3)$$

where the indices s and k refer to the normal mode and original bases, respectively, and \mathbf{A} is the modal matrix built from the characteristic vectors.



2. Let us now consider CO₂ as a classical tri-atomic molecule. To perform molecular spectroscopy, samples are exposed to (nearly) monochromatic radiation to excite their molecular normal modes. The changing fields of the electromagnetic waves can be viewed as a periodic driving force with a (nearly) unique external frequency ω_{ext} .

The left figure shows an *infrared absorption spectrum*. Dips in the radiation transmission indicate wave numbers $k = 2\pi/\lambda = \omega/c$ at which the molecule is absorbing high amounts of the incoming radiation, i.e., a normal mode's resonance frequency.

The right figure shows a spectrum from *Raman scattering*, which is an approach that can excite normal modes that are not activated by simple IR irradiation.

Which normal modes of CO₂ do the highlighted peaks *A* ($k_A \approx 2350 \text{ cm}^{-1}$) and *B* ($k_B \approx 1388 \text{ cm}^{-1}$) correspond to? Are the relative positions of the peaks (roughly) consistent with expectations, based on the mass ratio of the C and O atoms?

Problem H24 – Hamiltonian of a Particle in an Electromagnetic Field

[10 Points] We recall from problem H10 that the Lagrangian of a particle of mass m and charge q in an electromagnetic field is given (in SI units) by

$$L(\vec{r}, \dot{\vec{r}}) = \frac{1}{2}m\dot{\vec{r}}^2 - q\left(\phi(\vec{r}, t) - \vec{r} \cdot \vec{A}(\vec{r}, t)\right). \quad (4)$$

The electric and magnetic fields are obtained from the scalar and vector potentials via

$$\vec{E}(\vec{r}, t) = -\nabla\phi(\vec{r}, t) - \frac{\partial}{\partial t}\vec{A}(\vec{r}, t), \quad \vec{B}(\vec{r}, t) = \nabla \times \vec{A}(\vec{r}, t). \quad (5)$$

1. Compute the canonical momentum \vec{p} . How is it related to the mechanical momentum $m\dot{\vec{r}}$ of the particle? Use \vec{p} to construct the Hamiltonian.
2. Derive Hamilton's equations and show that they yield the familiar Lorentz force

$$\vec{F}(\vec{r}, \dot{\vec{r}}) = q\vec{E}(\vec{r}, t) + q\dot{\vec{r}} \times \vec{B}(\vec{r}, t). \quad (6)$$

Problem H25 – Hamiltonians and Reference Frames

[10 points] We consider the Lagrangian of a mass falling in a gravitational field,

$$L(y, \dot{y}) = \frac{1}{2}m\dot{y}^2 - mgy. \quad (7)$$

1. Make a change of coordinates to an inertial frame that is moving with a constant velocity v_0 , $y \rightarrow y' = y + v_0 t$. Construct the Lagrangian L' in the new frame, and show that the Lagrange equations remain unchanged.
2. Construct the Hamiltonians in the original and transformed frames, $H(y, p, t)$ and $H'(y', p', t)$, and compute their time derivatives. Are the Hamiltonians conserved? Do they correspond to the total energy of the falling mass?

HINT: Eliminate \dot{p} and \dot{y} using the Hamilton equations.