5/8 KUPT #4

- $Y_1, Y_2, \cdots, Y_n \sim$ indep $Bin(m_i, p)$ $i = 1, 2, 3, \cdots, n$

find distribution of $\sum_{i=1}^{n} Y_i$

Y분포 주어지고 합의분포를 구하는문제

⇒ mgf 쓰면 쉽게 풀린다.

★ $M_X(t) = E[e^{tX}]$

R.V "X"의 mgf

$Y_i \sim$ indep $Bin(m_i, p)$

mgf는 확률분포의 지문!

ex) $Z \sim$ 분포모름

mgf $E(e^{tZ}) = [(1-p) + pe^t]^n$

↳ 만 알면 RV이 어떤 분포를 가지는지 알 수 있다

* 수리통계학에서는 MGF가 주어진다

ex) $X \sim Bin(n, p)$

$M_X(t) = [(1-p) + pe^t]^n$

증명 필요없이 가져다쓰면 됨.

$$M_{Y_i}(t) = E[e^{tY_i}] = [(1-p) + pe^t]^{m_i}$$

목표 : ΣY_i의 분포 ⇒ ΣY_i의 mgf

$$\boxed{M_{\Sigma Y_i}(t)} = E[e^{t\Sigma Y_i}] = E[e^{t(Y_1 + Y_2 + \cdots + Y_n)}]$$

$$= E[e^{tY_1} e^{tY_2} \cdots e^{tY_n}]$$

$E(AB) = E(A)E(B)$

A, B 독립일 때만!

$$= E[e^{tY_1}] \times E[e^{tY_2}] \times \cdots \times E[e^{tY_n}]$$

$$= [(1-p) + pe^t]^{m_1} [\quad]^{m_2} \cdots [\quad]^{m_2}$$

$$= [(1-p) + pe^t]^{m_1 + m_2 + \cdots + m_n}$$

$\Sigma Y_i \sim Bin(\Sigma m_i, p)$

$$E\left[e^{t\Sigma Y_i}\right] = \prod_{i=1}^{n} E(e^{tY_i})$$

$$= \prod_{i=1}^{n} \left[(1-p)+pe^t\right]^{m_i}$$

$$= \left[(1-p)+pe^t\right]^{\sum_{i=1}^{n} m_i}$$

$\therefore \Sigma Y_i \sim Bin(\Sigma m_i, p)$

< Sufficient Statistic >

☆ · Factorization Thm

SS of θ ?

$$L(\theta) = g(u(y_1, \cdots, y_n), \theta) \times h(y_1, \cdots, y_2)$$

ex) $Y_i \sim exp(\theta)$. SS for θ ?

$$f_Y(y) = \frac{1}{\theta} e^{-y/\theta}, \quad y > 0$$

pdf를 1부터 n까지 곱하기

$$L(\theta) = \prod_{i=1}^{n} \left(\frac{1}{\theta} e^{-y_i/\theta} \cdot I_{(0,\infty)}(y_i) \right)$$

$I_{(0,\infty)}(\min y_i)$로도 표현 가능

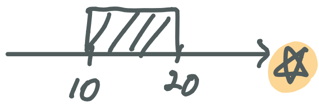
$$= \underbrace{\frac{1}{\theta^n} e^{-\frac{\Sigma y_i}{\theta}}}_{g(\Sigma y_i, \theta)} \times \underbrace{\prod_{i=1}^{n} I_{(0,\infty)}(y_i)}_{h(y_1, \cdots, y_n)}$$

y에 대한 식

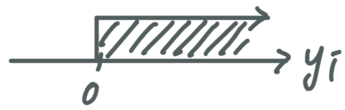
$\therefore$ by Factorization Thm, Σy_i is SS for θ.

- I (indicator function) $\Rightarrow$ 0 or 1

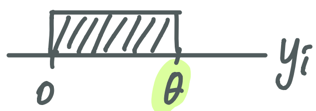
ex) $I_{(10,20)}(\star) = \begin{cases} 1 & , 10 < \star < 20 \\ 0 & , \text{o.w} \end{cases}$



- $\prod_{i=1}^{n} I_{(0,\infty)}(y_i) = I_{(0,\infty)}(\min(y_i))$



$\star$ • $\prod_{i=1}^{n} I_{(0,\theta)}(y_i)$



$y_{(1)} \; y_{(2)} \cdots y_{(n)}$

$= I_{(0,\infty)}(\min y_i) \times I_{(0,\theta)}(\max y_i) \Rightarrow \min y_i$ SS 아님

제일 작은애가 0보다 크고 　　　 제일 큰애가 θ보다 작다

$= I_{(0,\theta)}(\min y_i) \times I_{(0,\theta)}(\max y_i) \Rightarrow \min y_i$ SS 된다

$\therefore$ SS not unique

ex) $Y_i \sim \text{Uniform}(0,\theta)$

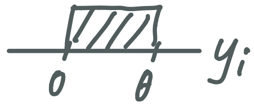
$f(y;\theta) = \frac{1}{\theta} \quad , 0 < y < \theta$

$= \frac{1}{\theta} \cdot I_{(0,\theta)}(y_i)$ 범위를 나타냄

SS for θ ?

$$L(\theta)=\prod_{i=1}^{n}\left(\frac{1}{\theta} I_{(0,\theta)}(y_i)\right)$$

$$=\frac{1}{\theta^n}\prod_{i=1}^{n} I_{(0,\theta)}(y_i)$$



$$=\frac{1}{\theta^n}\times I_{(0,\infty)}(\min y_i)\times I_{(0,\theta)}(\max y_i)$$

$$=\underbrace{\frac{1}{\theta^n} I_{(0,\theta)}(\max y_i)}_{g(\max(y_i),\,\theta)} \times \underbrace{I_{(0,\infty)}(\min y_i)}_{h(y_1,\cdots,y_n)}$$

$\therefore$ $\max(y_i)$ is SS for θ

또는

$$=\underbrace{\frac{1}{\theta^n} I_{(0,\theta)}(\min y_i)\cdot I_{(0,\theta)}(\max y_i)}_{g(\min(y_i),\,\max(y_i),\,\theta)} \times \underbrace{1}_{h(y_1,\cdots y_n)}$$

$\therefore$ $(\min y_i, \max y_i)$ is SS for θ

$\therefore$ SS not unique

〈Joint SS〉 시험나올확률↑↑

$f(y;\theta_1,\theta_2)$

$L(\theta)=\underbrace{\theta_1,\theta_2\text{ 에 depend}}_{g(\square,\square,\theta_1,\theta_2)} \times \underbrace{\theta_1,\theta_2\text{ depend X}}_{h(y_1,\cdots,y_n)}$

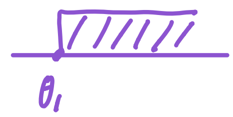
$\therefore$ $\square,\square$ is jointly SS for θ_1,θ_2

ex) $Y_1,\cdots,Y_n$

$$f(y;\theta_1,\theta_2)=\frac{1}{\theta_2}\times e^{-\frac{(y-\theta_1)}{\theta_2}}, \quad y>\theta_1$$

find Joint SS for θ_1, θ_2 → 지난 학기 중간고사 문제

$I_{(\theta_1, \infty)}(y_i)$



$$L(\theta) = \prod_{i=1}^{n} \left[\frac{1}{\theta^2} e^{-\frac{(y_i-\theta_1)}{\theta_2}} \cdot I_{(\theta_1,\infty)}(y_i) \right]$$

$$= \prod_{i=1}^{n} \left[\frac{1}{\theta_2} \times e^{\frac{\theta_1}{\theta_2}} \times e^{\frac{-y_i}{\theta_2}} \times I_{(\theta_1,\infty)}(y_i) \right]$$

$$= \left(\frac{1}{\theta^2} \times e^{\frac{\theta_1}{\theta_2}} \right)^n \times e^{\frac{-\sum y_i}{\theta_2}} \times \prod_{i=1}^{n} I_{(\theta_1,\infty)}(y_i)$$

$$= \underbrace{\left(\frac{1}{\theta^2} e^{\frac{\theta_1}{\theta_2}} \right)^n \times e^{\frac{-\sum y_i}{\theta_2}} \times I_{(\theta_1,\infty)}(\min y_i)}_{g(\sum Y_i,\ \min y_i,\ \theta_1, \theta_2)} \times \underbrace{1}_{h(y_1, \cdots, y_n)}$$

$\therefore \sum Y_i, \min y_i$ is joint SS for θ_1, θ_2

4-1 과제 문제 풀이

$f(y;\theta) = e^{-(y-\theta)}$, $y > \theta$ → 처음보는 분포
↓
① 변수변환
↓
아는 분포로 변환

(a) $W = \min(y_i)$

(b) $f_W(w) = n \cdot e^{-n(w-\theta)}$, $w > \theta$ → 처음보는 분포
↓
② 변수변환
↓
아는 분포로 변환

(c) $W - \theta$?

$U = W - \theta$

$f_U(u) = n \cdot e^{-un}$, $u > 0$ $\sim \exp\left(\frac{1}{n}\right)$

→ pivotal quantity

(d) U·E of θ based on $\bar{Y}, W$

(e) efficiency

$$\bar{Y} \to E(\bar{Y}) \qquad W \to E(W)$$

i) $E(\bar{Y}) = E(Y) = \int_\theta^\infty y \cdot e^{-(y-\theta)} dy$

$$= e^\theta \int_\theta^\infty y e^{-y} dy =$$

부분적분으로 풀기

$$Var(\bar{Y}) = \frac{Var(Y)}{n}$$

$$E(Y^2) = \int_\theta^\infty y^2 e^{-y} dy$$

ii) $E(W) = \int_\theta^\infty w f_w(w) dw$

$$f_W(w) = n \cdot e^{-n(w-\theta)}, \quad w > \theta$$

계산이 너무 복잡한 것 같다 → 우리가 이미 아는 분포 생각해보기!

① $f(y;\theta)$

② $f_W(w)$

③ $f_U(u) \sim exp(\frac{1}{n})$

$$U = W - \theta$$

$$E(W) = E(U+\theta) = E(U) + \theta = \frac{1}{n} + \theta$$

$$Var(W) = Var(U+\theta) = Var(U) = \frac{1}{n^2}$$

→ $f(y) = e^{-(y-\theta)}, \quad y > \theta$ shifted exp

지수분포 범위는 '0' 보다 크기

↓

지수분포나 감마분포로 만들어보자!

$X = Y - \theta$, $X > 0$

$F_X(x) = P(X \le x) = P(Y-\theta < x) = P(Y < x+\theta)$

$= F_Y(x+\theta)$

$f_X(x) = f_Y(x+\theta) \times 1$

$= e^{-x}$, $x > 0$

$X \sim exp(1)$

we want $E(\bar{Y}) = E(Y)$

$= E(X+\theta)$

$= E(X) + \theta$

$= 1 + \theta$

$Var(\bar{Y}-1) = Var(\bar{Y}) = \frac{Var(Y)}{n} = \frac{1}{n}$

$Var(Y) = Var(X+\theta) = Var(X) = 1$

* 예제

1. $f(y;\theta) = \frac{1}{2\theta^3} \times e^{3y} \times e^{-e^y/\theta}$, $-\infty < y < \infty$

(a) find pdf of $X = e^Y$ 변수변환 방향 알려줄것

$f_X(x) = \frac{1}{2\theta^3} x^2 e^{-\frac{1}{\theta}x}$, $x > 0$

$X \sim Gamma(3, \theta)$

기말 고난이도
: 변수변환 방향 없음
⇓
말도 안되게 복잡한 분포
무조건 변수변환

보통 $X = e^Y$
$\ln(Y+1)$

결과론적으로 생각해보기

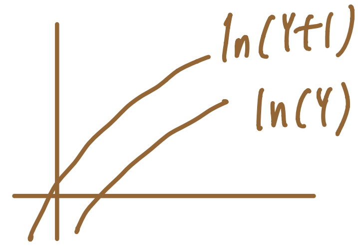
① 범위 $-\infty < y < \infty \Rightarrow y > 0$ ⟨ 지수 취하기 / 로그 취하기

$$X = e^Y \qquad e^{-\infty} < e^y < e^{\infty}$$

$$\downarrow \qquad\qquad \downarrow$$

$$F_X(x) \qquad e^y > 0$$

$X = \ln(Y+1)$



2. $f(y;\theta) = \dfrac{\theta}{(y+1)^{\theta+1}}, \quad y > 0, \quad \theta > 0$

저번학기 중간고사 문제

Show that $\boxed{\dfrac{n}{\sum_{i=1}^{n} \ln(Y_i+1)}}$ is consistent estimator of θ

$\xrightarrow{P} \theta$?

consistent

Tool 1 : WLLN $\frac{1}{n}\sum \square \to E[\square]$

Tool 2

$$= \frac{1}{\frac{1}{n}\sum_{i=1}^{n} \ln(Y_i+1)}$$

$\xrightarrow{P}$ by WLLN

$$E[\ln(Y+1)] = \frac{1}{\theta} \quad \text{by Tool 1}$$

$X = \ln(Y+1)$

$$P_X(x) = P(X \le x) = P[\ln(Y+1) < x]$$

$$= P(Y+1 < e^x)$$

$$= P(Y \le e^x - 1)$$

$$= F_Y(e^x - 1)$$

$$f_X(x) = f_Y(e^x - 1)\, e^x$$

$$= \theta e^{-\theta x}, \quad x > 0$$

$X \sim exp\left(\frac{1}{\theta}\right)$ 가 된다.