5/28 KUPT #6

<Beta distribution>

Beta (a.b)

$$f(y) = \begin{cases} \underbrace{\frac{1}{B(a.b)}}_{\text{상수처럼}} y^{a-1}(1-y)^{b-1} & , 0<y<1 \\ 0 & , ow \end{cases}$$

$$\Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x} dx$$

$$B(a.b) \begin{cases} \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \\ \int_0^1 x^{a-1}(1-x)^{b-1} dx \end{cases}$$

$$E(Y) = \frac{a}{a+b} \quad , Var(Y) = \frac{ab}{(a+b)^2(a+b+1)}$$

$$\hookrightarrow \int_0^1 y \cdot \frac{1}{B(a.b)} y^{a-1}(1-b)^{b-1} dy = \frac{1}{B(a.b)} \underbrace{\int_0^1 y^a (1-y)^{b-1} dy}_{B(a+1,b)}$$

$$= \frac{B(a+1,b)}{B(a,b)}$$

$$\Gamma(a) = (a-1)!$$

개념이 중요!!

☆ <MLE (Maximum Likelihood Estimator)>

ex) $Y_1, \cdots, Y_n \overset{iid}{\sim} Poisson(\lambda)$

① MLE of λ → parameter가 중요

② MLE of λ^{-1}

$$f(y|\lambda) = \frac{e^{-\lambda} \cdot \lambda^y}{y!}$$

① $L(\lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{y_i}}{y_i!}$

$$= \frac{e^{-\lambda n}\lambda^{\Sigma y_i}}{\prod_{i=1}^{n} y_i!}$$

$\ln L(\lambda) = -n\lambda + \Sigma y_i \ln\lambda - \ln\left(\prod_{i=1}^{n} y_i!\right)$

$\frac{\partial}{\partial\lambda}\ln L(\lambda) = -n + \frac{\Sigma y_i}{\lambda} \overset{set}{=} 0$

$\Rightarrow \hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} y_i = \bar{y}$

$\hat{\lambda}^{MLE} = \bar{y}$

$\Rightarrow \frac{\partial^2}{\partial\lambda^2}\ln L(\lambda) = \frac{-\Sigma y_i}{\lambda^2} < 0$

② MLE of $e^{-\lambda}$

i) set $\alpha = e^{-\lambda}$

MLE of α?

$\alpha = e^{-\lambda}$

$L(\lambda) = \frac{e^{-\lambda n} \cdot \lambda^{\Sigma y_i}}{\prod_{i=1}^{n} y_i!}$

$L(\alpha) = \frac{\alpha^n \cdot (-\ln\alpha)^{\Sigma y_i}}{\prod_{i=1}^{n} y_i!}$

$\ln L(\alpha)$

$\frac{\partial}{\partial\alpha}\ln L(\alpha) = \qquad \overset{set}{=} 0$

ii) by invariance property of MLE,

MLE of $e^{-\lambda}$ is $e^{-\bar{y}}$

☆☆ < Large Sample Properties of MLE >

$Y_1, \cdots, Y_n \overset{iid}{\sim} f(y|\theta)$

ⓐ $\hat{\theta}_n^{MLE} \xrightarrow{P} \theta$

asymptonic variance of MLE

ⓑ $\hat{\theta}_n^{MLE} \xrightarrow{D} \underbrace{N(\theta, \frac{1}{nI(\theta)})}_{\text{approximate distribution}}$ (CLT)

ⓒ $I(\theta) = E\left[-\frac{\partial^2}{\partial\theta^2} \ln f(Y|\theta)\right]$

☆☆ < $\hat{\theta}^{MLE}$ 이용해 CI of θ 구하기 >

⇒ CLT, Slutsky 이용

ex) $Y_1, \cdots, Y_n \sim exp(\theta)$

- $\hat{\theta}^{MLE} = \bar{y}$
- $I(\theta) = E\left[-\frac{\partial^2}{\partial\theta^2} \ln f(Y|\theta)\right] = \frac{1}{\theta^2}$
- $\hat{\theta}^{MLE} \xrightarrow{D} N(\theta, \frac{\theta^2}{n})$

· 95% CI of θ ?

① MLE로 $N(0,1)$ 만들기

$$\frac{\hat{\theta}^{MLE} - E(\hat{\theta}^{MLE})}{\sqrt{Var(\hat{\theta}^{MLE})}} \xrightarrow{D} N(0,1) \quad \text{by CLT}$$

$$\frac{\bar{Y}-\theta}{\frac{\theta}{\sqrt{n}}} \xrightarrow{D} N(0,1) \quad \text{by CLT}$$

② $P\left(-1.96 < \frac{\bar{Y}-\theta}{\theta/\sqrt{n}} < 1.96\right) = 95\%$

③ $\frac{\bar{Y}-\theta}{\frac{\hat{\theta}^{MLE}}{\sqrt{n}}} \xrightarrow{D} N(0,1)$ Yes 이면 새로운 pivotal quantity로 사용

⇒ Slutsky

$$\frac{\overbrace{\frac{\bar{Y}-\theta}{\theta/\sqrt{n}}}^{\rightsquigarrow N(0,1)}}{\underbrace{\hat{\theta}/\theta}_{\xrightarrow{P} 1}} = \frac{\bar{Y}-\theta}{\hat{\theta}/\sqrt{n}} \xrightarrow{D} N(0,1)$$

$$\hat{\theta}^{MLE} \xrightarrow{P} \theta$$

$$\frac{\hat{\theta}^{MLE}}{\theta} \xrightarrow{P} 1$$

④ $P\left(-1.96 < \frac{\bar{Y}-\theta}{\frac{\hat{\theta}^{MLE}}{\sqrt{n}}} < 1.96\right) = 0.95$

⟨L(θ)가 미분 불가능일 때⟩

ex) $Y_1, \ldots, Y_n \sim Unif(0, \theta)$

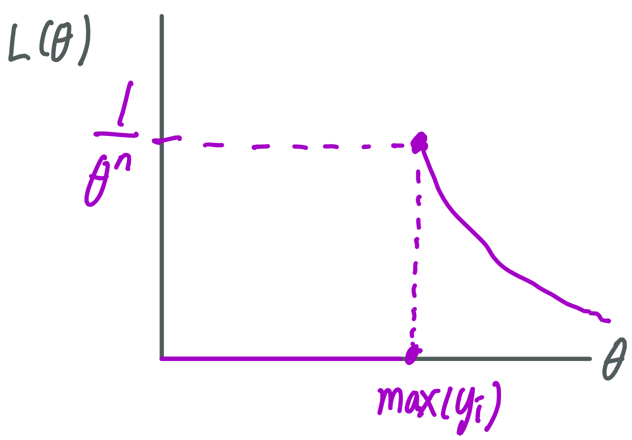
$$L(\theta) = \prod_{i=1}^{n} \frac{1}{\theta} I_{(0,\theta)}(y_i)$$

$$= \frac{1}{\theta^n} \cdot I_{(\max y_i, \infty)}(\theta)$$

$\frac{\partial}{\partial\theta} L(\theta)$ 불가능 Indicator가 θ에 dependent

↓

plot 그리기



∴ Likelihood $L(\theta)$를 maximize 하는 $\hat{\theta}^{MLE} = \max(y_i)$