5/15 KUPT

< MVUE (Minimum Variance U·E) >

- Is U·E unique? NO

~ HW 4.2
U·E of θ based on $\bar{Y}$ & W

$$\text{UE}\begin{cases} \hat{\theta}_1 \longrightarrow \theta \\ \hat{\theta}_2 \longrightarrow \theta \\ \hat{\theta}_3 \quad\quad \text{〃} \\ \vdots \quad\quad\quad \vdots \\ \hat{\theta}_n \quad\quad\quad \vdots \end{cases}$$

function of SS (group 1)

$$\text{UE}\begin{cases} \hat{\theta}_1 \longrightarrow \theta \\ \hat{\theta}_2 \longrightarrow \theta \\ \hat{\theta}_3 \quad\quad \text{〃} \\ \vdots \quad\quad\quad \vdots \\ \hat{\theta}_n \quad\quad\quad \vdots \end{cases}$$

not function of SS (group 2)

- Rao Blackwell thm : 문제 풀 때는 거의 안쓰임.

$Var(\hat{\theta}\ \text{group 1}) \leq Var(\hat{\theta}\ \text{group 2})$

then group 1 : $\hat{\theta}_1, \hat{\theta}_2$ 중에서는 $Var(\hat{\theta}_1) \lesseqgtr Var(\hat{\theta}_2)$

알 수 없음

$\Rightarrow$ unique MVUE 찾지 못함.

✡ Lemann - Scheffe Theorem

function of CSS ⇒ $\hat{\theta}_1$ is MVUE

무조건 하나

UE $\left\{ \begin{array}{ll} \hat{\theta}_1 & \longrightarrow \theta \\ \hat{\theta}_2 & \longrightarrow \theta \\ \hat{\theta}_3 & \quad '' \\ \vdots & \quad \vdots \\ \hat{\theta}_n & \end{array} \right.$

어떤 estimate가 $\underbrace{\text{UE}}_{\text{조건①}} + \underbrace{\text{function of CSS}}_{\text{조건②}} \Rightarrow \text{MVUE}$

✡ θ의 MVUE 구하기

① find CSS

✡② $E(CSS) = \square\theta$ (by 변수변환 등등)

U.E 만들기

③ 이항 $E[\square CSS] = \theta$

UE + function of CSS

④ By Lemann-Scheffe Thm, $\square CSS$ is MVUE of θ

꼭 암기하기!

⟨CSS⟩ complete SS

정의: W is SS of θ

$E[g(W)]=0$ 일때 $P(g(W)=0)=1$ 이면

항상 $g(W)=0$

W is CSS for θ

ex) $Y_1, \cdots, Y_n \sim Unif(0, \theta)$

$W = \max(y_i)$ is SS for θ, $f_W(w) = \frac{n \cdot w^{n-1}}{\theta^n}$, $0 < w < \theta$

by factorization thm

is $W = \max(y_i)$ CSS ?

Suppose $E[g(W)] = 0 \quad \forall \theta > 0$, is $P(g(W)=0) = 1$?

$$E[g(W)] = \int_0^\theta g(w) \cdot \frac{\cancel{n} w^{n-1}}{\cancel{\theta^n}} dw = 0$$

$$\Rightarrow \int_0^\theta g(w) \cdot w^{n-1} dw = 0$$

$$\Rightarrow \int_0^\theta g(w) w^{n-1} dw = 0$$

θ로 미분

$$\Rightarrow g(\theta) \cdot \cancel{\theta^{n-1}} = 0$$

$$\Rightarrow g(\theta) = 0 \quad \forall \theta > 0$$

$$\Rightarrow g(w) = 0 \quad \forall w > 0$$

CSS 인지 확인 (not 찾기)

$\therefore W = \max(y_i)$ is CSS of θ

Then, MVUE of θ ?

$$E[\max(y_i)] = \int_0^\theta w \cdot \frac{n w^{n-1}}{\theta^n} dw = \frac{n}{n+1} \theta$$

$$E\Big[\underbrace{\frac{n+1}{n} \max(y_i)}\Big] = \theta$$

① UE of θ, ② function of CSS

$\therefore$ By Lemann-Scheffe Thm, $\left(\frac{n+1}{n}\right) \max(y_i)$ is MVUE of θ

※ CSS 찾기

θ의 MVUE 구하기

① CSS 찾기 (REF thm)

② $E(CSS) = \square\theta$

③ 이항 $E(\square CSS) = \theta$

① UE of θ, ② function of CSS

④ By Lamann-Scheffe Thm, $\square CSS$ is MVUE of θ.

< REF Theorem > - Find CSS

$Y_1, \cdots, Y_n \sim f(y|\theta) = a(\theta)b(y)e^{c(\theta)d(y)}$

일 때

$U = \sum_{i=1}^{n} d(Y_i)$ is CSS of θ

REF (Regular Exponential Family) ⇒ $\square \sim$ pivotal quantity 조건 2개 필요. 풀이방법 유사

$f(y|\theta)$ is REF if

조건4개 암기!

① Same Support for all θ (y의 범위) ✓
& doesn't depend on θ

② $f(y|\theta) = a(\theta)b(y)\exp(c(\theta), d(y))$

③ $c(\theta)$ is 1 to 1 function

④ Ω is an interval (Parameter) ✓

ex)

$X \sim gamma(\alpha, \beta)$

support $x > 0$

$\alpha > 0$
$\beta > 0$

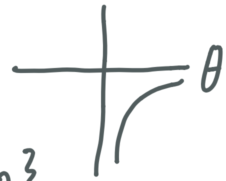
ex) $\exp(\theta)$ REF?

$f(y|\theta) = \frac{1}{\theta}e^{-y/\theta}$, $y > 0$

① support : $(0, \infty)$

② $f(y|\theta) = \underbrace{\frac{1}{\theta}}_{a(\theta)} e^{\underbrace{-\frac{1}{\theta}}_{c(\theta)} \times \underbrace{y}_{d(y)}} \cdot \underbrace{I_{(0,\infty)}(y)}_{b(y)}$

③ $c(\theta) = -\frac{1}{\theta}$



④ $\Omega : \{\theta : 0 < \theta < \infty\}$

$\Downarrow$

* CSS of θ? $d(y) = y$

By REF thm, $\sum_{i=1}^{n} y_i$ is CSS of θ.

* MVUE of θ?

$E\left[\sum_{i=1}^{n} Y_i\right] = nE(Y) = n\theta$

$E\left[\frac{1}{n}\sum_{i=1}^{n} Y_i\right] = \theta$

① UE of θ, ② function of CSS

$\therefore$ By Lemann-Scheffe Thm, $\bar{Y}$ is MVUE of θ

* MVUE of θ^2?

$\Rightarrow E[\boxed{}] = \theta^2$ 만들어야함

① UE of θ, ② function of CSS

$\Rightarrow E[\bar{y}^2] = Var(\bar{y}) + E(\bar{y})^2 = \frac{\theta^2}{n} + \theta^2 = \left(\frac{n+1}{n}\right)\theta^2$

*tip. 먼저 $\bar{y}^2$ 넣어보기

$\Rightarrow E\left(\frac{n}{n+1}\bar{y}^2\right) = \theta^2$

$\therefore$ By Lemann-Scheffe Thm, $\left(\frac{n}{n+1}\bar{y}^2\right)$ is MVUE of θ

과제 #5

1. $Y_1, \cdots, Y_n \sim$ iid Poisson (λ)

$$f(y|\lambda) = \frac{\lambda^y \cdot e^{-\lambda}}{y!}$$

ⓐ MVUE of λ ? → complete thm, REF

Find CSS by using REF

① Support : 0. 1. 2. $\cdots$ & doesn't depend on λ

$Y \sim$ Poisson (λ)

↓

$0.1.2.\cdots, \infty$

② $f(y) = a(\lambda) b(y) \exp[c(\lambda) d(y)]$

$a(\lambda) = e^{-\lambda}$, $b(y) = \frac{1}{y!}$

$\exp(\ln \lambda^y)$

$e^{y \ln \lambda}$, $y \to d(y)$, $\ln\lambda \to c(\theta)$

$\exp(\ln(x)) = x$

$e^{\log x} = x$ 이용

③ $\ln \lambda$: 1 to 1 function

④ $\Omega = \{\lambda : \lambda > 0\}$

By REF thm, CSS is $\sum_{i=1}^{n} y_i$

$$E\left[\sum_{i=1}^{n} Y_i\right] = n E(Y) = n\lambda$$

$$E\left[\frac{1}{n}\sum_{i=1}^{n} Y_i\right] = \lambda$$

① UE of θ, ② function of CSS

∴ By Lemann-Scheffe Thm, $\frac{1}{n}\sum_{i=1}^{n} Y_i = \bar{Y}$ is MVUE of θ

ⓑ MVUE of λ^2?

$$E[\bar{Y}^2] = Var(\bar{Y}) + E(\bar{Y})^2$$

$$= \frac{\lambda}{n} + \lambda^2$$

$E(\bar{Y})$로 생각하기!

$$\Rightarrow E\left(\bar{Y}^2 - \frac{\bar{Y}}{n}\right) = \lambda^2$$