4/17 KUPT

CLT (Central Limit theorem) 중심극한정리

$X_i \sim (\mu, \sigma^2)$

$$\frac{\sum_{i=1}^{n} X_i - E\left(\sum_{i=1}^{n} X_i\right)}{\sqrt{Var\left(\sum_{i=1}^{n} X_i\right)}} \xrightarrow{D} N(0,1)$$

$$\Leftrightarrow \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}} \xrightarrow{D} N(0,1)$$

$$\Leftrightarrow \bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right) = N\left(E(X), \frac{Var(X)}{n}\right)$$

$$\Leftrightarrow \boxed{\frac{\bar{X}-E(\bar{X})}{\sqrt{Var(\bar{X})}} \xrightarrow{D} N(0,1)}$$

외우기!

$E(X)=E(\bar{X})$

$$\Leftrightarrow \frac{\bar{X}-E(X)}{\sqrt{\frac{Var(X)}{n}}} \xrightarrow{D} N(0,1)$$

$Var(\bar{X}) = \frac{Var(X)}{n}$

ex) $E(X) = 3\theta$

$Var(X) = 3\theta^2$

$\sqrt{3n}(\bar{X}-3\theta)/3\theta \quad \sim ? \quad N(0,1)$

$$\frac{\bar{X}-E(\bar{X})}{\sqrt{Var(\bar{X})}} \overset{D}{\sim} N(0,1)$$

$E(\bar{X}) = E(X) = 3\theta$

$Var(\bar{X}) = \frac{Var(X)}{n} = \frac{3\theta^2}{n}$, $\sqrt{Var(\bar{X})} = \frac{\sqrt{3}\theta}{\sqrt{n}}$

$$\frac{\bar{X}-3\theta}{\frac{\sqrt{3}\theta}{\sqrt{n}}} \sim N(0,1) \Rightarrow \frac{\bar{X}-3\theta}{\frac{3\theta}{\sqrt{3n}}} \sim N(0,1)$$

T분포 : 특징. 생김새 알아두기.

$$T = \frac{Z}{\sqrt{W/V}} \sim t(V)$$

$Z \sim N(0,1)$

$W \sim \chi^2(V)$

$T \sim t(V)$ T는 $df = V$ 인 t분포를 따른다

$E(T) = 0$

F 분포

$W_1 \sim \chi^2(V_1) \perp\!\!\!\perp W_2 \sim \chi^2(V_2)$ 일 때

$$F = \frac{W_1/V_1}{W_2/V_2} \sim F(V_1, V_2)$$

우리가 아는 값

$$\frac{(n-1)s^2}{\sigma^2} \sim \chi^2(n-1)$$

ex) $Y_1, Y_2, \cdots, Y_n \sim N(\mu, \sigma^2)$ find 95% CI for μ.

target : $[\, c_1 < \mu < c_2 \,] = 95\%$

is $\underbrace{\dfrac{\bar{Y} - \mu}{\sigma/\sqrt{n}}}_{\sim N(0,1)}$ pivotal quantity? → NO!

μ, σ 둘다 모르기 때문

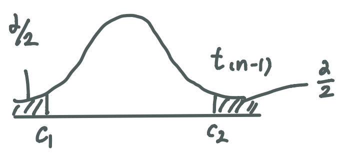
$$\frac{\dfrac{\bar{Y}-\mu}{\sigma/\sqrt{n}} \sim Z}{\sqrt{\dfrac{(n-1)s^2/\sigma^2 \sim \chi^2}{n-1 \ df}}} \sim t(n-1)$$

$\parallel$

$\dfrac{\bar{Y}-\mu}{s/\sqrt{n}}$ is pivotal quantity.

~ $t(n-1)$ 이므로

$$P\left[\, c_1 < \frac{\bar{Y}-\mu}{s/\sqrt{n}} < c_2 \,\right] = 95\%$$



$$P\left[\, -t_{\alpha/2, n-1} < \frac{\bar{Y}-\mu}{s/\sqrt{n}} < t_{\alpha/2, n-1} \,\right] = 95\%$$

$$P\left[\, \bar{Y} - t_{\alpha/2, n-1}\frac{s}{\sqrt{n}} < \mu < \bar{Y} + t_{\alpha/2, n-1}\frac{s}{\sqrt{n}} \,\right] = 95\%$$

예제1 : 변수변환. : 저번 학기 기출, 변수변환은 필수로 알아두기.

Q. $Y_1, Y_2 \cdots Y_n$ follow $f(y;\theta) = \frac{1}{2\theta^3} e^{3y} e^{-\frac{e^y}{\theta}}$, $-\infty < y < \infty$

(a) Find pdf of $X = e^Y$ and $E(X)$

$$F_X(x) = P(X \le x) = P(e^Y < x)$$
$$= P(Y < \ln x) = F_Y(\ln x)$$

$\downarrow$ 미분

$$f_X(x) = f_Y(\ln x) \cdot \frac{1}{x}$$
$$= \frac{1}{2\theta^3} \underbrace{e^{3\ln x}}_{e^{\ln x^3}} e^{-\frac{x}{\theta}} \cdot \frac{1}{x}$$
$$= \frac{1}{2\theta^3} x^3 e^{-\frac{1}{\theta}x} \cdot x^{-1} = \frac{1}{2\theta^3} x^{3-1} e^{-\frac{1}{\theta}x}$$

$-\infty < y < \infty$
$0 < x < \infty$ since $\ln x$

$$= \frac{1}{\Gamma(3)\theta^3} x^{3-1} e^{-\frac{1}{\theta}x} \sim Gamma(3, \theta) \Rightarrow 3\theta$$

예제2 : pivotal quantity 임을 보이기

Q. $U_i = \frac{2}{\theta} X_i$, $i = 1, 2, 3 \ldots n$

target parameter.

show that $\sum_{i=1}^n U_i$ is pivotal quantity for θ

$X \sim Gam(3, \theta)$

$U_i = \frac{2}{\theta}(X_i) \sim Gam(3, 2)$

$\sum_{i=1}^n U_i \sim Gamma(3n, 2) \sim \chi^2(6n)$

① $\sum_{i=1}^n U_i = \sum_{i=1}^n \frac{2}{\theta} X_i = \frac{2}{\theta} \sum_{i=1}^n X_i$

↳ only parameter 이다

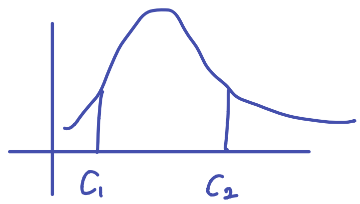
② $\sum_{i=1}^n U_i \sim \chi^2(6n)$

↳ doesn't depend on θ

예제3 : CI 구하기

Q. Find 95% CI for θ when $n=10$, $\sum_{i=1}^{n} X_i = 100$.

$$P\left[c_1 < \underbrace{\frac{2}{\theta}\sum_{i=1}^{n} X_i}_{\sim \chi^2(6n)} < c_2\right] = 95\%$$



$c_1 = \chi^2_{0.975,6n}$ $\quad$ $c_2 = \chi^2_{0.025,6n}$

$$P\left[\chi^2_{0.975,6n} < \frac{2}{\theta}\sum_{i=1}^{n} X_i < \chi^2_{0.025,6n}\right] = 95\%$$

$$P\left[\qquad < \theta < \qquad\right] = 95\%.$$

★ 문제에 많이 쓰이는 적분

$$\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} e^{-y}\, dy$$

gamma pdf

$$f(x;k,\theta) = x^{k-1}\frac{e^{-x/\theta}}{\theta^k\Gamma(k)}, \quad x>0$$

$$= x^{k-1} \times \frac{1}{\theta^k\Gamma(k)} e^{-\frac{x}{\theta}}$$

① $\int_0^\infty y^2 e^{-y}\, dy = \Gamma(3) = 2!$

$y^2 e^{-y} = y^{3-1}e^{-y}$

② $\int_0^\infty x^3 e^{-\frac{x}{2}}\, dx = \underbrace{\int_0^\infty \frac{1}{\Gamma(4)2^4} x^{4-1} e^{-\frac{x}{2}}\, dx}_{=1} \times \Gamma(4)2^4 = 3\times2\times1\times2^4 = 96$

$x^3 e^{-\frac{x}{2}} = x^{4-1}e^{-\frac{x}{2}}$

③ $\int_0^\infty y^6 e^{-y}\, dy = \Gamma(7) = 6!$

$y^{7-1}e^{-y}$