20.06.12 # KUPT 8

< N-P lemma >

$\Rightarrow$ MP RR, UMP RR

ex) $Y_1, Y_2, \cdots, Y_n \overset{iid}{\sim} exp(\theta)$

$H_0 : \theta = 1$ vs $H_a : \theta = 2$

simple　　simple → 구체적으로 parameter가 무엇인지 주어진 것

① $L(\theta)$ 구하기

$$L(\theta) = \prod_{i=1}^{n} f(y_i) = \frac{1}{\theta^n} e^{-\Sigma y_i/\theta}$$

② $\frac{L(\theta_0)}{L(\theta_a)} \le k$ 일 때 reject H_0 : N-P lemma

$$\frac{L(1)}{L(2)} = \frac{e^{-\Sigma y_i}}{\frac{1}{2^n} e^{-\Sigma y_i/2}} \le k \text{ 일 때 reject } H_0$$

y로 정리

$$\boxed{2^n \cdot e^{-\Sigma y_i/2}} \le k \text{ 일 때 reject } H_0$$

→ decreasing function of Σy_i

$\Leftrightarrow \underbrace{\{\Sigma y_i \ge c\}}_{RR}$ 일 때 reject H_0

*참고
Σy_i 분포구하기 어려울경우
$\{\bar{y} \ge c^*\}$ 로 풀기

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③ $P(RR \mid H_0) = \alpha$ 이용해서　c 구하기 = RR 구하기

$$P(\Sigma y_i \ge c \mid H_0) = \alpha$$

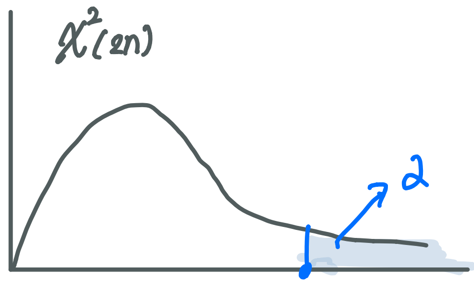
$Y_i \sim exp(\theta)$

$\Sigma Y_i \sim gamma(n, 1)$

$2\Sigma Y_i \sim gamma(n, 2) \sim \chi^2_{(2n)}$

$P(2\Sigma Y_i \geq 2c \mid H_0) = \alpha$

$P(2\Sigma Y_i \geq 2c \mid \theta = 1) = \alpha$



$2C$

$\parallel$

$\chi^2_{2n,\alpha}$

$C = \frac{1}{2}\chi^2_{2n,\alpha}$

$\therefore \{\Sigma Y_i \geq \frac{1}{2}\chi^2_{2n,\alpha}\}$ is MP RR of level α for testing

$H_0 : \theta = 1$ vs $H_a : \theta = 2$

④ UMP ?

$H_0 : \theta = 1$ vs $H_a : \theta_a > 1$

위의 RR이 이 testing의 RR과 같나?

$\Rightarrow \{\Sigma Y_i \geq \frac{1}{2}\chi^2_{2n,\alpha}\}$ doesn't depend on θ_a

$\therefore \{\Sigma Y_i \geq \frac{1}{2}\chi^2_{2n,\alpha}\}$ is UMP

⟨LRT⟩

two side testing

$H_0 : \theta = \theta_0 \qquad H_a : \theta \neq \theta_0$

이 θ가 L(θ)를 최대로 만들고 $\theta \subset \Omega_0$

$$\lambda = \frac{\max_{\theta \subset \Omega_0} L(\theta)}{\max_{\theta \subset \Omega} L(\theta)} \le K \text{ 일 때 Reject } H_0$$

이 θ가 L(θ)를 최대로 만든다

θ는 $\hat{\theta}^{MLE}$

EX) $Y_1, \cdots, Y_n \sim \text{Poisson}(\theta)$

$H_0 : \theta = \theta_0 \quad vs \quad H_a : \theta \neq \theta_a$

$\Omega = \{\theta : 0 < \theta < \infty\}, \quad \Omega_0 = \{\theta = \theta_0\}$

① $L(\theta)$, $\hat{\theta}^{MLE}$ 구하기

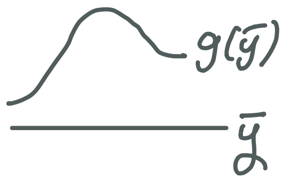
$\hat{\theta}^{MLE} = \bar{y}$

② $\lambda = \dfrac{L(\theta_0)}{L(\hat{\theta}^{MLE})} = e^{n(\bar{y}-\theta_0)} \left(\dfrac{\theta_0}{\bar{y}}\right)^{\sum y_i} \le K$ 일 때 reject H_0

$$\Leftrightarrow \underbrace{e^{n(\bar{y}-\theta_0)} \left(\frac{\theta_0}{\bar{y}}\right)^{n\bar{y}}}_{= g(\bar{y})} \le K \text{ 일 때 reject } H_0$$

$\Leftrightarrow g(\bar{y}) \le K$ 일 때 reject H_0

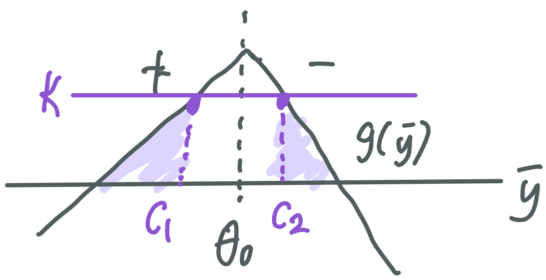
③ $g(\bar{y})$ 어떻게 생겼는지 보기



$$\ln g(\bar{y}) = n(\bar{y}-\theta_0) + n\bar{y}\ln\theta_0 - n\bar{y}(\ln\bar{y})$$

$$\frac{\partial}{\partial\bar{y}} \ln g(\bar{y}) = n + n\ln\theta_0 - n\ln\bar{y} - n$$

$$= n(\ln\theta_0 - \ln\bar{y})$$



④ $g(\bar{y}) \le K$ 일 때 reject H_0

$\Leftrightarrow \{\bar{y} \le C_1$ or $\bar{y} \ge C_2\}$ 일 때 reject H_0

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⑤ $P(RR|H_0) = \alpha$ 이용해서 RR 완성하기

$\{\bar{y} \le C_1$ or $\bar{y} \ge C_2\} \Leftrightarrow \{\sum y_i \le C_1^*$ or $\sum y_i \ge C_2^*\}$

$\sum Y_i \sim \text{Poisson}(n\theta)$

$$P\left(\sum^n y_i \le C_1^*\right) = \frac{\alpha}{2}, \quad P\left(\sum^n y_i \ge C_2^*\right) = \frac{\alpha}{2}$$

< LRT >

$Y_1, \dots, Y_n \quad exp(\theta)$

$H_0 : \theta = 3 \quad H_a : \theta \neq 3$

LRT ⇒ 계산복잡
실수주의
Y, C_1, C_2

UMP만 구하기 : NP lemma 문제

※ 변수변환
$\ln Y, e^Y,$

범위바꾸기 $Y > -1$
$Y+1 > 0$

1. $\hat{\theta}^{MLE}$

$$L(\theta) = \left(\frac{1}{\theta^n}\right) e^{-\Sigma y_i/\theta}$$

$$\ln L(\theta)$$

$$\frac{\partial}{\partial\theta} \ln L(\theta) \overset{set}{=} 0 \quad , \quad \frac{\partial^2}{\partial\theta^2} \ln L(\theta) < 0$$

$$\hat{\theta}^{MLE} = \frac{\Sigma Y_i}{n} = \bar{y}$$

2.

귀무가설

$$\lambda = \frac{L(3)}{L(\bar{y})} = \frac{\frac{1}{3^n} e^{-\Sigma Y_i/3}}{\frac{1}{\bar{y}^n} e^{-\Sigma Y_i/\bar{y}}}$$

MLE

$$= \left(\frac{\bar{y}}{3}\right)^n e^{-\overset{-n\bar{y}}{\Sigma y_i}\left(\frac{1}{3} - \frac{1}{\bar{y}}\right)} \le k$$

N-P lemma : increase/decrease 찾기

LRT : $g(x)$

$$g(\bar{y}) \to g(x) = \left(\frac{x}{3}\right)^n e^{-nx\left(\frac{x-3}{3x}\right)}$$

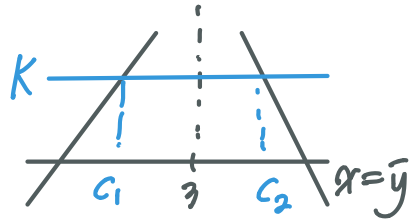
$$= \left(\frac{x}{3}\right)^n e^{-n\left(\frac{x-3}{3}\right)}$$

$g(x)$의 개형 실수주의

$$\ln(g(x)) = n\ln x - n\ln 3 - n\left(\frac{x-3}{3}\right)$$

$$\frac{\partial}{\partial x}\ln g(x) = \frac{n}{x} - \frac{n}{3} \overset{set}{=} 0$$

critical pt $x=3$



$g(x)\le k$ 일 때 reject H_0

$\Leftrightarrow \{\bar{y}\le c_1 \text{ or } \bar{y}\ge c_2\}$

$Y \sim exp(\theta)$

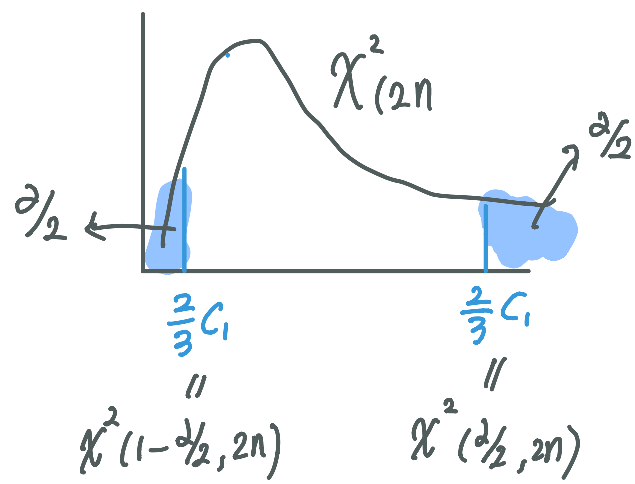
$\Leftrightarrow \{\sum y_i \le c_1^* \text{ or } \sum y_i \ge c_2^*\}$ 일 때 reject H_0

$\sum Y_i \sim gamma(n,\theta)$

$P(RR|H_0)=\alpha$ 이용 $\quad H_0:\theta=3$

$\frac{2}{3}\sum Y_i \sim gamma(n,2)\ \chi^2(2n)$

$$P\left(\frac{2}{3}\sum Y_i \le \frac{2}{3}c_1 \text{ or } \frac{2}{3}\sum Y_i \ge \frac{2}{3}c_2 \,\middle|\, H_0\right)=\alpha$$



$c_1 = \frac{3}{2}\chi^2(1-\alpha/2, 2n)$

$c_2 = \frac{3}{2}\chi^2(\alpha/2, 2n)$