Homework 4

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Equivalence of ERM and probabilistic approaches

Q1

The logistic loss function is

$$\ell_{\text{logistic}}(y, w) = \log\big(1 + \exp(-yw^T x)\big)$$

Then the objective funciton of ERM for logistic loss is given by

$$\hat{R}_n(w) = \frac{1}{n}\sum_{i=1}^{n} \log\big(1 + \exp(-y_i w^T x_i)\big)$$

Define the Bernoulli indicator as

$$y_i' = \begin{cases} 1 & y_i = 1 \\ 0 & y_i = -1 \end{cases}$$

Then we have the negative log-likelihood to be

$$\begin{aligned} NLL_D(w) &= -\sum_{i=1}^{n} y_i' \log \frac{1}{1 + \exp(-w^T x_i)} + (1 - y_i') \log(1 - \frac{1}{1 + \exp(-w^T x_i)} \\ &= \sum_{i=1}^{n} -y_i' \log \frac{1}{1 + \exp(-w^T x_i)} + (y_i' - 1) \log(1 - \frac{1}{1 + \exp(-w^T x_i)}) \end{aligned}$$

When $y_i = 1$, that is, $y_i' = 1$,

$$\log\big(1 + \exp(-y_i w^T x_i)\big) = -y_i' \log \frac{1}{1 + \exp(-w^T x_i)} = \log\big(1 + \exp(-w^T x_i)\big)$$

When $y_i = -1$, that is, $y_i' = 0$,

$$\log\big(1 + \exp(-y_i w^T x_i)\big) = (y_i' - 1) \log(1 - \frac{1}{1 + \exp(-w^T x_i)}) = \log\big(1 + \exp(w^T x_i)\big)$$

Therefore,

$$NLL_D(w) = n\hat{R}_n(w)$$

Thus, the two approaches are equivalent.

Linearly Separable Data

Q2

We predict y by

$$\hat{y} = \text{sign}(x^T w)$$

that is, for all $i = 1, 2, \ldots, n$, we predict y_i by

$$\hat{y}_i {x_i}^T w > 0$$

Therefore, the decision boundary of logistic regression is given by $\{x : x^T w = 0\}$

Q3

For $y = 1$, the likelihood function with respect to $c\hat{w}$ is given by

$$\begin{aligned} L(c\hat{w}) &= p(y = 1 \mid x; c\hat{w}) \\ &= 1/(1 + e^{-x^T c\hat{w}}) \\ &= (1 + e^{-x^T \hat{w} \cdot c})^{-1} \end{aligned}$$

Take derivative of $L(c\hat{w})$ with respect to c, we get

$$\begin{aligned} \frac{dL}{dc} &= -(1 + e^{-x^T \hat{w} c})^{-2} \cdot (-x^T w \cdot e^{-x^T wc}) \\ &= x^T w \cdot e^{-x^T wc} \cdot (1 + e^{-x^T \hat{w} c})^{-2} \end{aligned}$$

Since all examples are classified correctly,

$$x^T w = y = 1 > 0$$

And we also have

$$e^{-x^T wc} > 0, (1 + e^{-x^T \hat{w} c})^{-2} > 0$$

That is,

$$\frac{dL}{dc} > 0$$

Similarly, $\frac{dL}{dc} > 0$ also holds for $y = 0$

Therefore, as c increases, the likelihood of the data would always increase, which means that MLE is not well-defined in this case.

Regularized Logistic Regression

Q4

$$\begin{aligned} J_{\text{logistic}}(w) &= \hat{R}_n(w) + \lambda \|w\|^2 \\ &= \frac{1}{n} \sum_{i=1}^{n} \log\left(1 + \exp\left(-y^{(i)} w^T x^{(i)}\right)\right) + \lambda \|w\|^2. \end{aligned}$$

From notes 3.1.3, we know that $\exp\left(-y^{(i)} w^T x^{(i)}\right)$ is convex ,

$$\begin{aligned} \exp\left(-y^{(i)} w^T x^{(i)}\right) \text{ is convex } &\implies 1 + \exp\left(-y^{(i)} w^T x^{(i)}\right) \text{ is convex} \\ &\implies \log\left(1 + \exp\left(-y^{(i)} w^T x^{(i)}\right)\right) \text{ is convex} \\ &\implies \hat{R}_n(w) \text{ is convex} \end{aligned}$$

Also From notes 3.1.3, we know every norm is convex, so

$$\lambda\|w\|^2 \text{ is convex}$$

Thus, the objective function $J_{\text{logistic}}(w)$ is convex.

Q5

In [1]:

```
def f_objective(theta, X, y, l2_param=1):
    '''
    Args:
        theta: 1D numpy array of size num_features
        X: 2D numpy array of size (num_instances, num_features)
        y: 1D numpy array of size num_instances
        l2_param: regularization parameter

    Returns:
        objective: scalar value of objective function
    '''
    num_instances = X.shape[0]
    num_features = X.shape[1]
    sum_risk = 0
    for i in range(num_instances):
        sum_risk += np.logaddexp(0, -y[i]*np.dot(theta,X[i]))

    risk = sum_risk/num_instances
    reg = l2_param * np.dot(theta,theta)
    objective = risk + reg
    return objective
```

Q6

In [2]:

```
import numpy as np
from sklearn.preprocessing import StandardScaler
from scipy.optimize import minimize
from functools import partial
import matplotlib.pyplot as plt
```

In [3]:

```
# read data
X_train = np.genfromtxt('./logistic-code/X_train.txt', delimiter = ',')
X_val = np.genfromtxt('./logistic-code/X_val.txt', delimiter = ',')
y_train = np.genfromtxt('./logistic-code/y_train.txt', delimiter = ',')
y_val = np.genfromtxt('./logistic-code/y_val.txt', delimiter = ',')

# change label space from {0,1} to {-1,1}
y_train[y_train==0] = -1
y_val[y_val==0] = -1

# standardize data
scaler = StandardScaler()
X_train = scaler.fit_transform(X_train)
X_val = scaler.fit_transform(X_val)

# add bias term
bias_train = np.ones([X_train.shape[0], 1])
bias_val = np.ones([X_val.shape[0], 1])
X_train = np.hstack((X_train, bias_train))
X_val = np.hstack((X_val, bias_val))
```

In [4]:
```
def fit_logistic_reg(X, y, objective_function, l2_param=1):
    '''
    Args:
        X: 2D numpy array of size (num_instances, num_features)
        y: 1D numpy array of size num_instances
        objective_function: function returning the value of the objective
        l2_param: regularization parameter

    Returns:
        optimal_theta: 1D numpy array of size num_features
    '''
    num_features = X.shape[1]
    theta = np.ones(num_features)
    optimal_theta = minimize(objective_function, theta).x

    return optimal_theta
```

In [5]:
```
# train the model
X = X_train
y = y_train
objective_function = partial(f_objective, X=X, y=y, l2_param=1)
optimal_theta = fit_logistic_reg(X, y, objective_function, l2_param=1)
```

In [6]:
```
optimal_theta
```

Out[6]:
```
array([ 0.00095626, -0.00029962,  0.0030268 ,  0.10532762, -0.00358736,
       -0.0013585 , -0.00385288, -0.00079013, -0.00114407, -0.07178432,
        0.00654892, -0.00451097,  0.01124928, -0.00386437, -0.00271262,
        0.00150363, -0.00278399, -0.0091906 , -0.00682276, -0.01027486,
        0.00281868])
```

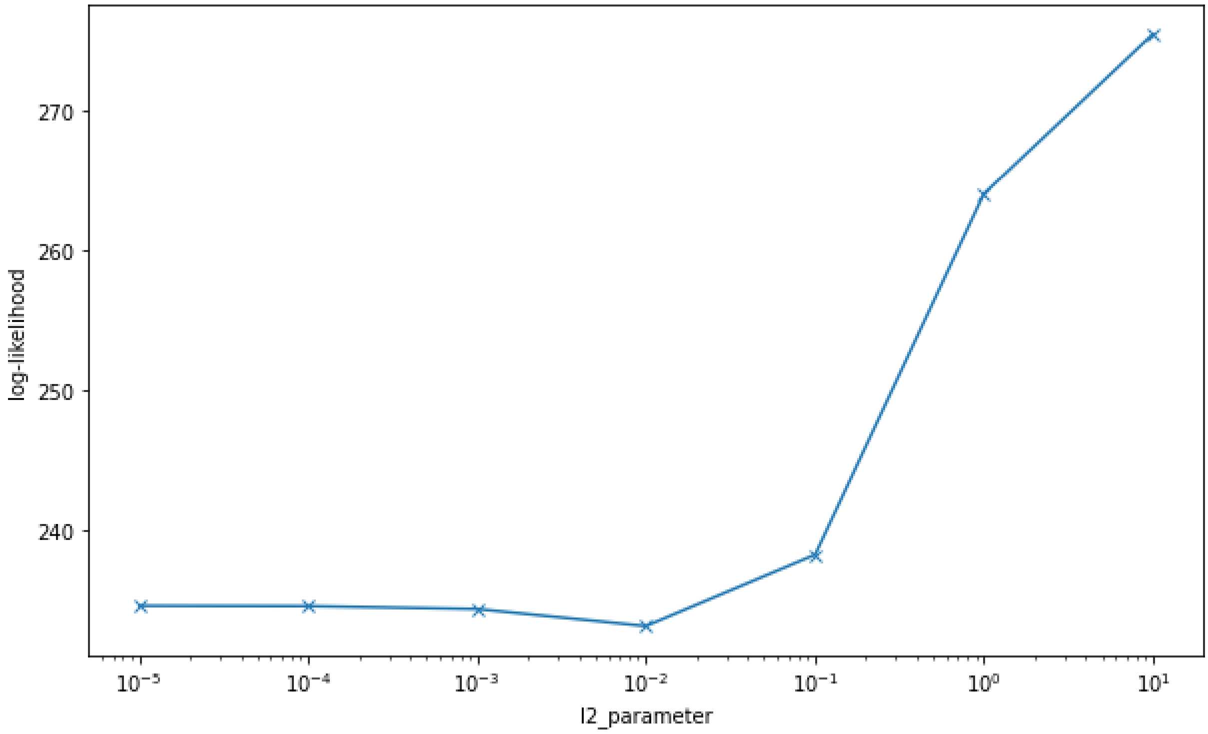
Q7

In [7]:
```
l2_param_set = [1e-5, 1e-4, 1e-3, 1e-2, 1e-1, 1, 10]
log_L_set = []

# train the model for each l2_param
for l2_param in l2_param_set:
    objective_function = partial(f_objective, X=X_train, y=y_train, l2_param=l2_param
    optimal_theta = fit_logistic_reg(X_train, y_train, objective_function, l2_param)

    # calculate the log loss on the validation set with the optimal theta
    log_L = 0
    for i in range(X_val.shape[0]):
        log_L += np.logaddexp(0, -y_val[i]*np.dot(optimal_theta,X_val[i]))
    log_L_set.append(log_L)
```

In [8]:
```
# Plot the log-likelihood for different values of the regularization parameter
plt.figure(figsize = (10,6))
plt.xscale('log')
plt.xlabel('l2_parameter')
plt.ylabel('log-likelihood')
plt.plot(l2_param_set, log_L_set, 'x-')
plt.show()
```



In [9]:

```
best_param = l2_param_set[np.argmin(log_L_set)]
print('The regularization parameter that minimizes the log-likelihood on the validatic
print('l2_param = ', best_param)
```

```
The regularization parameter that minimizes the log-likelihood on the validation data
is:
l2_param =  0.01
```

Bayesian Logistic Regression with Gaussian Priors

Q9

By Bayes rule, we can write the posterior distribution as

$$p(w|D) = \frac{p(D|w) \cdot p(w)}{p(D)}$$

Consider both sides as function of w, for fixed D, then

$$p(w|D) = c \cdot p(D|w) \cdot p(w), \text{ for some constant } c = \frac{1}{p(D)}$$

Also,

$$p(D|w) = L_D(w) = \exp(-NLL_D(w))$$

Therefore,

$$p(w|D) = c \cdot \exp(-NLL_D(w)) \cdot p(w), \text{ where constant } c = \frac{1}{p(D)}$$

Q10

No, the Gaussian $p(w)$ is not a conjugate prior.

From Q9, we know that $p(w|D) = c \cdot \exp(-NLL_D(w)) \cdot p(w)$, where the likelihood function $\exp(-NLL_D(w))$ is logistic and $p(w)$ is Gaussian.

Therefore, $\exp(-NLL_D(w)) \cdot p(w)$ will never be Gaussian, that is, the posterior $p(w|D)$ is not in the same family(Gaussian) as $p(w)$.

Q11

The MAP for w minimizes the negative log posterior of w and the regularized logistic function:

$$\begin{aligned} \hat{w}_{MAP} &= argmin(-\log p(w|D)) \\ &= argmin \frac{1}{n} \sum_{i=1}^{n} \log(1 + \exp(y_i w^T x_i)) + \lambda \|w\|^2 \end{aligned} \tag{1}$$

From Q10, we have that

$$\begin{aligned} -\log p(w|D)) &= -\log(c \cdot \exp(-NLL_D(w)) \cdot p(w)) \\ &= -\log c + NLL_D(w) - \log(p(w)) \end{aligned}$$

where c is a constant $NLL_D(w) = n \cdot \hat{R}_n(w)$

Given the piror $p(w)$ is Gaussian,

$$\log(p(w)) = -\frac{1}{2}\log(|2\pi\Sigma|) + \frac{1}{2} w^T \Sigma^{-1} w$$

Want to find Σ such that equation (1) holds.

Notice that the only term that match with $\frac{1}{2} w^T \Sigma^{-1} w$ is $\lambda \|w\|^2$, and since $NLL_D(w) = n \cdot \hat{R}_n(w)$, we need to have

$$\frac{1}{2} w^T \Sigma^{-1} w = n \cdot \lambda \|w\|^2$$

Solve for Σ, we get

$$\Sigma = \frac{1}{2n\lambda} I$$

Q12

To have the ERM equal to the MAP estimator, we need

$$\Sigma = \frac{1}{2n\lambda} I$$

Here we choose $\Sigma = I$, then

$$\lambda = \frac{1}{2n}$$

Q13

$$\begin{aligned}
p(x=H|\theta_1,\theta_2) &= p(x=H,z=H|\theta_1,\theta_2)+p(x=H,z=T|\theta_1,\theta_2) \\
&= p(x=H|z=H,\theta_2)\cdot p(z=H|\theta_1)+p(x=H|z=T,\theta_2)\cdot p(z=T|\theta_1) \\
&= \theta_2\cdot\theta_1+0\cdot(1-\theta_1) \\
&= \theta_1\theta_2
\end{aligned}$$

Q14

The likelihood of D_r as a function of θ_1 and θ_2:

$$p(D_r|\theta_1,\theta_2)=\binom{N_r}{n_h}\cdot(\theta_1\theta_2)^{nh}\cdot(1-\theta_1\theta_2)^{nt}$$

Q15

No, we only have a dataset with reported results where head is reported with probability $\theta_1\theta_2$ and tail is reported with $1-\theta_1\theta_2$. This means that we can only use MLE to estimate the parameter $\theta_1\theta_2$, but not θ_1 and θ_2 seperately.

Q16

The likelihood function on the two datasets is:

$$\begin{aligned}
L(\theta_1,\theta_2) &= p(D_r,D_c|\theta_1,\theta_2) \\
&= p(D_r|\theta_1,\theta_2)\cdot p(D_c|\theta_1) && (D_c \text{ only depends on } \theta_1) \\
&= \binom{N_r}{n_h}\cdot(\theta_1\theta_2)^{nh}\cdot(1-\theta_1\theta_2)^{nt}\cdot\binom{N_c}{c_h}\cdot\theta_1^{ch}\cdot(1-\theta_1)^{ct}
\end{aligned}$$

Then the log likelihood function is:

$$\log(L(\theta_1,\theta_2))=\log\binom{N_r}{n_h}+\log\binom{N_c}{c_h}+n_h\log(\theta_1\theta_2)+n_t\log(1-\theta_1\theta_2)+c_h\log\theta_1+c_t\log($$

We want to find θ_1 and θ_2 that maximize the log likelihood, so we take the partial derivatives for θ_1 and θ_2:

$$\begin{aligned}
\frac{\partial\log(L(\theta_1,\theta_2))}{\partial\theta_1} &= \frac{n_h}{\theta_1}-\frac{n_t\theta_2}{1-\theta_1\theta_2}+\frac{c_h}{\theta_1}-\frac{c_t}{1-\theta_1} \\
\frac{\partial\log(L(\theta_1,\theta_2))}{\partial\theta_2} &= \frac{n_h}{\theta_2}-\frac{n_t\theta_1}{1-\theta_1\theta_2}
\end{aligned}$$

To have both the partial derivatives equal to 0, we get

$$\begin{aligned}
\theta_1\theta_2 &= \frac{n_h}{n_h+n_t} \\
\theta_1 &= \frac{c_h}{c_h+c_t}
\end{aligned}$$

Further solve for θ_2, the MLE estimate for θ_1 and θ_2 is:

$$\begin{aligned}
\theta_1 &= \frac{c_h}{c_h+c_t} \\
\theta_2 &= \frac{n_h(c_h+c_t)}{c_h(n_h+n_t)}
\end{aligned}$$

Q17

The prior distribution for θ_1 is:

$$\theta_1 \sim Beta(h, t)$$

After obtaining the dataset D_c, the posterior distribution becomes:

$$\theta_1 \sim Beta(h + c_h, t + c_t)$$

Then the MAP estimate for θ_1 is the mode of the Beta distribution, which is

$$\theta_1 = \frac{h + c_h - 1}{h + c_h + t + c_t - 2}$$

From the previous question, we obtain that the MLE for $\theta_1\theta_2$ is

$$\theta_1\theta_2 = \frac{n_h}{n_h + n_t}$$

Solve for θ_2, the MAP estimate for θ_2 is:

$$\theta_2 = \frac{n_h(h + c_h + t + c_t - 2)}{(n_h + n_t)(h + c_h - 1)}$$