T2.1 using $\sum_{n=1}^{d}(a_n-\lambda b_n)^2 \geq 0$, $\forall a_n, b_n \in \mathbb{R}^d$, $\lambda \in \mathbb{R}$, prove: $(\sum_{n=1}^{d} a_n b_n)^2 \leq (\sum a_n^2)(\sum b_n^2)$

proof: 已知: $\sum_{n=1}^{d}(a_n^T a_n - 2\lambda \cdot a_n^T b_n + \lambda^2 \cdot b_n^T \cdot b_n) \geq 0$

$\Rightarrow \lambda^2(\sum_{n=1}^{d} b_n^T \cdot b_n) - 2\lambda \cdot (\sum_{n=1}^{d} a_n^T b_n) + \sum_{n=1}^{d} a_n^T \cdot a_n \geq 0$ 关于λ的2次方程无解/仅一解

$\Rightarrow [2\sum_{n=1}^{d}(a_n^T b_n)]^2 \leq 4(\sum_{n=1}^{d} b_n^T \cdot b_n) \cdot (\sum_{n=1}^{d} a_n^T a_n)$

$\Rightarrow (\sum_{n=1}^{d} a_n b_n)^2 \leq (\sum a_n^2)(\sum b_n^2)$

T2.2 prove $(X_1, d_1) \times (X_2, d_2) \times \cdots (X_n, d_n)$ is metric space, with metric Q_p

$Q_p(x,y) = Q_p((x_1, x_2, \cdots x_n), (y_1, y_2, \cdots y_n))$ $\quad x_i, y_i \in X_i$

$$= \begin{cases} \max_{j=1,\cdots n} d_j(x_j, y_j) & p = +\infty \\ (\sum_{j=1}^{n} d_j(x_j, y_j)^p)^{\frac{1}{p}}, & 1 \leq p < +\infty \end{cases}$$

proof: ① $p=+\infty$ $Q(x,y)=Q(y,x)$, $Q(x,y) \geq 0$, $Q(x,y)=0 \Leftrightarrow x=y$

$Q(x,y)+Q(x,z)-Q(y,z) = d_t(x_t, y_t) + d_l(x_l, z_l) - d_m(y_m, z_m)$. 即t,l,m处取max

$\geq d_m(x_m, y_t) + d_m(x_m, z_m) - d_m(y_m, z_m)$

≥ 0 since d_m is metric

② $p<+\infty$ 即证: $\underbrace{(\sum d_j(x_j, y_j)^p)^{\frac{1}{p}}}_{\text{LHS}} \leq \underbrace{(\sum d_i(x_i, y_i)^p)^{\frac{1}{p}}}_{记为v} + \underbrace{(\sum d_i(y_i, z_i)^p)^{\frac{1}{p}}}_{记为w} = \text{RHS}$

$\text{LHS} \leq (\sum d_i(x_i, z_i)^p + \sum d_i(y_i, z_i)^p)^{\frac{1}{p}}$ $\therefore \text{LHS}^p \leq \sum d_i(x_i, z_i)^p + \sum d_i(y_i, z_i) = v^p + w^p$

$\text{RHS} = v+w$, $\quad \text{RHS}^p = (v+w)^p$

$\therefore \text{RHS}^p - \text{LHS}^p \geq v^p + w^p + \underbrace{(C_p^1 v^{p-1} w + C_p^2 v^{p-2} w^2 + \cdots + C_p^1 v \cdot w^{p-1})}_{\geq 0} - (v^p + w^p) \geq 0$

原不等式得证 ✓

T2.3 证d is metric, then $\hat{d}(x,y) = \frac{d(x,y)}{1+d(x,y)}$ also

Proof: $\hat{d}(x,y) = \hat{d}(y,x)$, $\hat{d}(x,y) \geq 0$, $\hat{d}(x,y) = 0$ iff $x=y$ obv

WTS: $\frac{d(x,y)}{1+d(x,y)} + \frac{d(x,z)}{1+d(x,z)} \geq \frac{d(y,z)}{1+d(y,z)}$ 记为 $\frac{u}{1+u} + \frac{v}{1+v} \geq \frac{w}{1+w}$ ($u+v \geq w$ 已知)

$\Leftrightarrow u\overbrace{(1+w)(1+v)}^{1+wv+w+v} + v(1+u)(1+w) \geq w(1+u)(1+v)$

$\Leftrightarrow u + \underline{wvu} + \underline{uw} + uv + v + wvu + \underline{vw} + vu \geq w + \underline{uvw} + \underline{wu} + \underline{wv}$

$\Leftrightarrow (u+v-w) + 2uv + vuw \geq 0$ $\quad$ --(*)

$u+v-w \geq 0$ since d is metric (*) holds obv $\Rightarrow \hat{d}$ is metric

T2.4 S(K) is space of sequence $X=\{x_i\}_{i=1}^{+\infty}$, $x_i\in\mathbb{K}$; 证 $d(X,Y)=\sum_{n=1}^{+\infty}\frac{1}{2^n}\cdot\frac{|x_n-y_n|}{1+|x_n-y_n|}$ is metric

证: $X^n\to Y \Leftrightarrow x_i^n\to y_i$

proof: (1): 正定性, 对称性显然

WTS: $\sum_{n=1}^{+\infty}\frac{1}{2^n}\cdot\frac{|x_n-y_n|}{1+|x_n-y_n|}+\sum_{n=1}^{+\infty}\frac{1}{2^n}\frac{|x_n-z_n|}{1+|x_n-z_n|}\geqslant\sum_{n=1}^{+\infty}\frac{1}{2^n}\cdot\frac{|y_n-z_n|}{1+|y_n-z_n|}\quad\cdots ①$

先证: (*): $\frac{|x_n-y_n|}{1+|x_n-y_n|}+\frac{|x_n-z_n|}{1+|x_n-z_n|}\geqslant\frac{|y_n-z_n|}{1+|y_n-z_n|}$ ⇐ 由T2.3可知此式成立

(2.3中 $d(x,y)$ 由 $|x-y|$ 代入 即知)

"$\sum_{n=1}^{+\infty}\frac{1}{2^n}(*)$" 得原不等式①成立

∴ $d(X,Y)$ is metric in S(K)

(2): $X_1=\{x_1^1,x_2^1,x_3^1,\cdots x_i^1\cdots\}$ $\quad Y=\{y_1,y_2,y_3\cdots y_i\cdots\}$ $\quad d(X^n,Y)=\sum^{+\infty}\frac{1}{2^i}\cdot\frac{|x_i^n-y_i|}{1+|x_i^n-y_i|}$

$$\begin{array}{ccccc} X_2=\{x_1^2, & x_2^2, & x_3^2, & \cdots & x_i^2\cdots\} \\ \cdots\ \downarrow & \downarrow & & \cdots & \downarrow\ \cdots \\ y_1 & y_2 & & \cdots & y_i \end{array}$$

"⇐" 设 $x_1^n\to y_1$, $x_2^n\to y_2\cdots$

$\forall\varepsilon>0$, $\exists N_1$ s.t. $\forall n\geqslant N$ $|x_1^{N_1}-y_1|<\varepsilon$; similarly $N_2, N_3\cdots N_t$ (任取某一个 t)

then $\frac{|x_i^n-y_i|}{1+|x_i^n-y_i|}<\frac{\varepsilon}{1+\varepsilon}$ $i=1,2,\cdots t$, $\forall n\geqslant\max(N_1,N_2\cdots N_t)\overset{\text{denoted}}{=\!=}M$

$\therefore d(X^n,Y)\leqslant\sum_{i=1}^{t}\frac{1}{2^i}\times\frac{\varepsilon}{1+\varepsilon}+\sum_{t+1}^{+\infty}\frac{1}{2^i}\times 1$

$\leqslant\underline{(1-\frac{1}{2^t})\times\frac{\varepsilon}{1+\varepsilon}}+\frac{1}{2^{t+1}}=\underline{\underline{A}}+B$

let $\varepsilon\to 0$. $A\to 0$, then let $t\to 0$. $B\to 0$. ;

即 $\forall\sigma>0$, let $\varepsilon<\frac{1}{2}\sigma$, $\frac{1}{2^{t+1}}<\frac{1}{2}\sigma$, then. $d(X^n,Y)<\sigma$. for $\forall n\geqslant M$

∴ $X^n\to Y$ 成立

"⇒" $X^n\to Y$

$\forall\sigma>0$, $\exists M$ s.t. $\forall n\geqslant M$; $\sum_{i=1}^{+\infty}\frac{1}{2^i}\frac{|x_i^n-y_i|}{1+|x_i^n-y_i|}\leqslant\sigma$. -- ①

反证: 设 $\exists\varepsilon$ I s.t. $\forall N$. $\exists n\geqslant N$, $|x_i^n-y_i|\geqslant\varepsilon$

then let $\sigma<\frac{1}{2^I}\frac{\varepsilon}{1+\varepsilon}$. contradict with ①

综上. $X^n\to Y\Leftrightarrow x_1^n\to y_1$, $x_2^n\to y_2\cdots x_i^n\to y_i\cdots$

T2.15: $\{F_\alpha\}$ is closed, $\Rightarrow \bigcup_{n=1}^{n=N} F_n$ is closed; $\cap F_\alpha$ is closed

$\{F_n\}$ 有限并，任意交

proof ①: ~~x is limit pt of $\bigcup_{n=1}^{n=N} F_n$,~~

~~$\forall r>0$, $B_r(x) \cup (\bigcup_{n=1}^{N} F_n) \neq \phi$, 且有 x 之外的点~~ x 不定是 F_n 的 limit pt (反例说)

lem: $\{G_n\}$ open, $\bigcap_{n=1}^{n=N} G_n$ open

if $x \in \bigcap_{n=1}^{N} G_n$, $x \in G_n$, $n=1,2\cdots$

$\exists r_n$, $B_{r_n}(x) \subseteq G_n$ since G_n open

$\therefore B_r(x) \subseteq \bigcap_{n=1}^{N} G_n$, $r = \min\{r_1, r_2 \cdots r_N\}$ $\Rightarrow \bigcap_{n=1}^{N} G_n$ is open.

let $F_n^c = G_n$. G_n is open.

$\therefore \bigcap_{n=1}^{N} F_n^c$ is open by lemma

$\therefore (\bigcap_{n=1}^{N} F_n^c)^c = \bigcup_{n=1}^{N} F_n$ is closed

②: $\cap F_\alpha$ has limit point y, then: $\forall r>0$, $B_r(y) \setminus \{y\} \cap (\cap F_\alpha) \neq \phi$

$\therefore B_r(y) \setminus \{y\} \cap F_\alpha \neq \phi \quad \forall \alpha$

$\therefore y$ is limit point of F_α, $\forall \alpha$

F_α is closed $\therefore y \in F_\alpha \ \forall \alpha \Rightarrow y \in \cap F_\alpha$, $\cap F_\alpha$ is closed

T2.7. in metric space, every open set 可以看成 open balls' union

proof: $U \subseteq X$, $U = \bigcup_{x \in U} B_{r_x}(x)$. r_x 与 x 有关

$\left.\begin{array}{l} \forall x.\ \exists r_x \text{ s.t. } B_{r_x}(x) \subseteq U \text{ since } U \text{ open} \quad \therefore \bigcup_{x \in U} B_{r_x}(x) \subseteq U \\ \forall x \in U,\ x \in B_{r_x}(x) \subseteq \bigcup_{x \in U} B_{r_x}(x) \quad \therefore U \subseteq \bigcup_{x \in U} B_{r_x}(x) \end{array}\right\} \Rightarrow U = \bigcup_{x \in U} B_{r_x}(x)$ 证毕