Ch12

def12.1 F is filter on a set X if : F is a collection of subsets of X, $X \in F$, $\phi \notin F$

F closed under finite intersection. $A \in F, B \in F \Rightarrow A \cap B \in F$

F closed upward under "$\subseteq$": $A \in F, B \subseteq X, A \subseteq B \Rightarrow B \in F$

F = "large": A is large ($A \in F$), B is larger than A ($B \subseteq X, A \subseteq X$), then B is "larger" ($B \in F$)

I is ideal of a set X if : I is ···, $X \notin I$, $\phi \in I$

$\parallel$ "small"

I is closed under finite unions $A \in I, B \in I \Rightarrow A \cup B \in I$

I is closed downwards under "$\subseteq$", $A \in I, B \subseteq X, B \subseteq A \Rightarrow B \in I$

if F is a filter, then $I = \{X \setminus A : A \in F\}$ is the dual ideal of F; conversely, $F = \{X \setminus B : B \in I\}$ is ..

$A \subseteq X$ is I positive if $A \notin I$, being not small

def12.3 F is k-complete if F is closed under intersection of size less than k

i.e. $\forall \lambda < k$, $\{A_\alpha : \alpha < \lambda\} \subseteq F$, then $\bigcap_{\alpha < \lambda} A_\alpha = F$

filter F is maximal on X if $\nexists$ F' on X s.t. $F \subsetneq F'$

F is ultrafilter if : $\forall A \subseteq X$, either $A \in F$, or $(X - A) \in F$

$\Big\} \Rightarrow$ ultrafilter $\rightleftharpoons$ maximal Lem12.4/12.5

TH12.6 (Tarski ZFC) F is a filter on X, then F can be extended to a ultrafilter. $F' \supseteq F$

pf: suppose $\mathbb{F}$ is a chain of filters,

i.e. $F_1 \in \mathbb{F}, F_2 \in \mathbb{F}$, either $F_1 \subseteq F_2$, or $F_2 \subseteq F_1$

$\therefore \bigcup \mathbb{F} = \{x : \exists \text{ set } y \in \mathbb{F}, \text{ s.t. } x \in y\}$ is a filter. $\Rightarrow$ upper bounded

$\Rightarrow \exists$ maximal by Zorn's lem (验证一下max)

def12.7 filter F on X is principle if $\exists$ some $A \subseteq X$, s.t. $B \in F$ iff $A \subseteq B$

集中在A周围,由A生成

F is ultrafilter, then F is principle $\Rightarrow |A| = 1$

or F is free, $F = \{A \subseteq \mathbb{N} : \mathbb{N} - A \text{ is finite}\}$ (A cofinite) 无限集上无中心点 $\rightleftharpoons \bigcap F = \phi$

def12.9 F is a filter on ω, $\langle a_n : n \in \omega \rangle$ is a sequence of real numbers.

$\lim_F a_n = x$ iff $\forall \varepsilon > 0$, $\{n : |a_n - x| < \varepsilon\} \in F$

$\rightleftharpoons \forall \varepsilon > 0$, $\{n : |a_n - x| \geq \varepsilon\} \in I$