19.4 Show that a seminorm p on a vector space X satisfies

(i) $p(0) = 0$;

(ii) $|p(x) - p(y)| \le p(x - y)$;

(iii) $p(x) \ge 0$; and

(iv) $\{x : p(x) = 0\}$ is a subspace of X.

(Rudin, 1991)

T19.4: def: $\begin{cases} p(x+y) \le p(x)+p(y) \\ p(\lambda x) = |\lambda| p(x) \quad \forall \lambda \in \mathbb{F} \end{cases}$ let $\lambda = 0$ $p(0) = 0$. ··· (i)

$p(x-y) + p(y) \ge p(x)$

$\underbrace{p(x-y)}_{\text{let } \lambda = -1} + p(x) = p(y-x) + p(x) \ge p(y)$ $\Big\} \Rightarrow p(x-y) \ge |p(x) - p(y)|$ ··· (ii)

$\xRightarrow{\text{let } y=0}$ $p(x) \ge |p(x) - 0| \ge 0$ ··· (iii)

if $p(x) = 0$, $p(y) = 0$ $p(ax+by) \le p(ax) + p(by) = 0$ $\forall a, b \in \mathbb{F}$

$p(ax+by) \ge 0$ $\Big\} \Rightarrow ax+by \in \{x : p = 0\}$

$\Rightarrow$ subspace

19.6 Suppose that X is a real separable normed space, and W a closed linear subspace of X. Show that there exists a sequence of unit vectors $(z_j) \in X$ such that

$$z_{j+1} \notin W_j := \operatorname{Span} W \cup \{z_1, \ldots, z_j\},$$

and if we define

$$W_\infty = \operatorname{Span} W \cup \{z_j\}_{j=1}^{\infty},$$

then $\overline{W_\infty} = X$.

T19.6 $\{x_n\} \subseteq X$ dense. countable

let $\{y_n\}_{n=1}^{+\infty} = X \setminus W$, then $\{z_n\}_{n=1}^{+\infty} \subseteq \{y_n\}_{n=1}^{+\infty}$

从$\{y_n\}_{n=1}^{+\infty}$中，找到第i个元素不在 $\operatorname{span}(W \cup \{z_1, z_2 \cdots z_{i-1}\})$中，记为$z_i$，$i \ge 2$，$z_1$选取第一个不在$W$ span中的

$\Rightarrow$ WTS: $\{y_n\}_{n=1}^{+\infty} \subseteq \overline{\operatorname{span}(W \cup \{z_i\}_{n=1}^{+\infty})}$

if $y_n \notin \{z_i\}_{i=1}^{+\infty}$, then $y_n \in \operatorname{span}(W \cup \{z_1, z_2 \cdots z_k\})$ for some k --- 由上述选择方法知

$\therefore \{y_n\} \subseteq \overline{\operatorname{span}}$

$\Rightarrow$ WTS: $\forall x \in X$, $x \in \overline{\operatorname{span}}$

if $x \notin W$, $x = y_k$ some k thus in $\overline{\operatorname{span}}$

or $x = \lim_{n_k \to +\infty} y_{n_k}$

$\because y_{n_k} \in \overline{\operatorname{span}}$, $\overline{\operatorname{span}}$ is closed set $\Big\} \Rightarrow$ limit point $x \in \overline{\operatorname{span}}$

if $x \in W$, obv

$\therefore \{z_n\}_{n=1}^{+\infty}$ satisfy that $\overline{\operatorname{span}(W \cup \{z_n\}_{n=1}^{+\infty})} = X$

19.7 Suppose that X is a real separable normed space, and that W is a closed linear subspace of X. Use the results of Exercises 19.3 and 19.6, along with the 'extension to one more dimension' part of the proof of the Hahn–Banach Theorem given in Section 19.1 to show that any $\phi \in W^*$ has an extension to an $f \in X^*$ with $\|f\|_{X^*} = \|\phi\|_{W^*}$. (For a separable space this gives a proof of the Hahn–Banach Theorem that does not require Zorn's Lemma.) (Rynne and Youngson, 2008)

T19.7 lem1: $\phi \in Y^*$, ϕ is linear. $\phi(x) \le p(x)$ $\forall x \in Y$. p is sublinear or subnorm → 定义域等价于

$\forall y_0 \notin Y$, ϕ has linear extension f on $Y \cup \{y_0\}$, $f(x) \le p(x)$ $\forall x \in Y \cup \{y_0\}$, $\mathrm{span}(Y \cup \{y_0\})$

let $p(x) = \|\phi\|_{X^*} \|x\|$, X是全空间, $\Rightarrow \|\phi\|_{Y^*} = \|f\|_{(Y\cup\{y_0\})^*}$ ... Th19.1

lem2: $\phi \in Y^*$, ϕ is linear. $\phi(x) \le M \cdot \|x\|$ $\forall x \in Y$

$\exists$ unique linear extension $\hat{\phi}$ on $\bar{Y}$, $\hat{\phi}(x) \le M \cdot \|x\|$, $\forall x \in \bar{Y}$

pf: exend ϕ to ϕ_1 on $\mathrm{span}(W \cup \{z_1\})$, $\|\phi_1\| = \|\phi\|$..

similarly extend ϕ_i to ϕ_{i+1} on $\mathrm{span}(W \cup \{z_1, \dots z_i, z_{i+1}\})$, $\|\phi_{i+1}\| = \|\phi\|_{H_{i+1}}$

defined $f_{tp}(x) = \phi_n(x)$ if $x \in \mathrm{span}(W \cup \{z_1, z_2 \dots z_n\})$

this is well-defined, since: if $\mathrm{span}(W \cup \{z_1 \dots z_n\}) \subseteq \mathrm{span}(W \cup \{z_1, \dots z_m\})$ $n \le m$,

then $\phi_m|_{\mathrm{span}(W \cup \{z_1 \dots z_n\})} = \phi_n$

$\|\phi_n\| = \|\phi\|_m$ by lem1

$\Rightarrow$ f_{tp} is extension to $\mathrm{span}(W \cup \{z_1, z_2 \dots\})$, $\|f\| = \|\phi_n\| = \|\phi\|$

use lem2: f_{tp} canbe extended to $\hat{f}_{tp}$ on $\overline{\mathrm{span}(W \cup \{z_1, z_2 \dots\})}$, 证毕

2012

20.1 If H is a Hilbert space, given $x \in H$, find an explicit form for the functional $f \in H^*$ such that $\|f\|_{H^*} = 1$ and $f(x) = \|x\|$ (as in Lemma

T20.1 $\|f\|_{H^*} = 1$ $f(x) = \|x\|$

by Riesz's Representation, $f(x) = (x, z)$ $\forall x$, $\|f\| = \|z\| = 1$

$$(x, z) = \|x\| \Rightarrow z = \begin{cases} 0 & x = 0 \\ \frac{1x}{\|x\|} & x \ne 0 \end{cases}$$

即 $f(x) = \begin{cases} (x, \frac{x}{\|x\|}) & x \ne 0 \\ 0 & x = 0 \end{cases}$

20.2 Let X be a normed space, $\{e_j\}_{j=1}^n \in X$ a linearly independent set, and $\{a_j\}_{j=1}^n \in \mathbb{K}$. Show that there exists $f \in X^*$ such that

$$f(e_j) = a_j \qquad j = 1, \ldots, n.$$

T20.2 let $U = \operatorname{span}\{e_1, e_2, \ldots, e_n\}$. U is a subspace with basis $\{e_i\}_{i=1}^n$

$\phi: \sum_{i=1}^n \alpha_i e_i \mapsto \sum_{i=1}^n \alpha_i a_i$

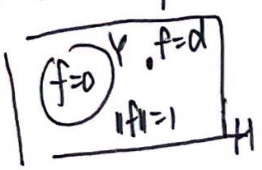
ϕ is bounded since U is finite-dimensional, $\phi \in U^*$

extend ϕ to f on X. then $\|f\|_{X^*} = \|\phi\|_{U^*}$, 即 $f \in X^*$ $f(e_i) = a_i$

没法直接在 X 中定义 f：“bounded”，“linear”
要定义其它 $X \setminus \{e_1, \ldots, e_n\}$ 才知道

20.4 Find an explicit form for the distance functional of Proposition 20.4 when X is a Hilbert space.

T20.4 distance functional. X is Hilbert, Y is closed. linear subspace



$\Downarrow$

$\Rightarrow \therefore X = Y \oplus Y^\perp$

$\Rightarrow \forall x_0 \in X \setminus Y,\ \exists y_0 \in Y$ s.t. $\|x_0 - y_0\| = \operatorname{dist}(x_0, Y) = d$

$f(x) = \langle x, v \rangle \quad v \in Y^\perp, \|v\| = 1, \langle x_0, v \rangle = d$

let $x_0 = u_0 + y_0 \quad u_0 \in Y^\perp, y_0 \in Y \quad \langle x_0, v \rangle = \langle u_0, v \rangle$

$\inf \|x_0 - y\| = \inf \|u_0 + y_0 - y\| = u_0 = d$ ☆ 忘记了，垂直分解 分量的几何意义是距离!

$\Rightarrow x_0 = u_0 + y_0 \quad y_0 \in Y, u_0 \in Y^\perp ; \|u_0\| = d$

$f(x) = \langle x, \frac{u_0}{\|u_0\|} \rangle \quad \begin{cases} f(y) = 0 \quad \forall y \in Y \\ f(x_0) = \langle u_0, \frac{u_0}{\|u_0\|} \rangle = d \\ \|f\| = 1 \end{cases}$

20.7 Deduce the existence of a support functional as a corollary of Proposition 20.4.

T20.7 Prop20.4 distance functional $\Rightarrow$ support functional

let Y closed linear subspace in X, $\underset{x_0 \in X \setminus Y}{\wedge}$ $\exists f \in X^*$. $\|f\| = 1$. $f(x_0) = d_{x_0}$, $f(Y) = 0$

let $Y = \operatorname{span}\{0\}$. then $f(0) = 0$

$\operatorname{dist}(x_0, Y) = \|x_0 - 0\| = \|x_0\| \quad \therefore f(x_0) = d_{x_0} = \|x_0\|$

20.8 Show that if $f \in X^*$ and $f \neq 0$ then

$$\operatorname{dist}(x, \operatorname{Ker}(f)) = \frac{|f(x)|}{\|f\|_{X^*}}.$$

T20.8 $f \in X^*, f \neq 0 \Rightarrow \operatorname{dist}(x, \ker f) = \dfrac{|f(x)|}{\|f\|_{X^*}}$

我的pf: $|f(x)| = |f(x) - f(y)| = |f(x-y)| \leq \|f\|_{X^*} \cdot \|x-y\|, \ \forall y \in \ker f$

$\therefore |f(x)| \leq \operatorname{dist}(x, \ker f) \cdot \|f\|_{X^*}$ ···①

$\forall \varepsilon > 0, \exists u \in X, \|u\| = 1, \ f(u) > \|f\|_{X^*} - \varepsilon$

let $\hat{y} = x - \frac{f(x)}{f(u)} \cdot u$ then $\hat{y} \in \ker f$

$\therefore \operatorname{dist}(x, \ker f) \leq \|x - \hat{y}\| = \dfrac{|f(x)|}{|f(u)|} < \dfrac{|f(x)|}{\|f\|_{X^*} - \varepsilon} \quad \forall \varepsilon > 0 \Rightarrow \operatorname{dist} \leq \dfrac{|f(x)|}{\|f\|_{X^*}}$ ···②

①+② $\Rightarrow \operatorname{dist} = \dfrac{|f(x)|}{\|f\|_{X^*}}$

20.9 Let X be a separable Banach space. Show that X is isometrically isomorphic to a subspace of ℓ^∞. [Hint: let (x_n) be a dense sequence in the unit sphere of X, let (ϕ_n) be support functionals at (x_n), and show that $T : X \to \ell^\infty$ defined by setting

$$Tx := (\phi_1(x), \phi_2(x), \ldots)$$

is a linear isometry.] (Heinonen, 2003)

T20.9 X separable. Banach $\Rightarrow$ isometric. Isomorphic to subspace of $l^{+\infty}$

pf: let $\mathbb{B}$ be unit sphere in X 的边界, $\mathbb{B} = \{x \in X : \|x\| = 1\}$

X sepa $\Rightarrow \mathbb{B}$ sepa

let $\{x_n\}_{n=1}^{+\infty}$ dense in $\mathbb{B}$.

let ϕ_n be support functionals at x_n, $\|\phi_n\|_{\{x_n\}^*} = 1$, $\phi_n(x_n) = \|x_n\| \ \forall n$

$\Rightarrow$ extend ϕ_n to $\hat{\phi}_n$ on X, $\|\hat{\phi}_n\| = 1 \ \hat{\phi}_n(x_n) = \|x_n\|$

let $T : X \to l^{+\infty}$, $Tx := (\phi_1(x), \phi_2(x), \cdots)$ WTS: $\|T\| = 1$

T is linear obv

$\|T\|_{l^{+\infty}} \leq \sup\limits_{\substack{\|x\|=1 \\ n=1,2\cdots}} \|\phi_n(x)\| = 1 \ \therefore$ bounded $\Big\} \Rightarrow T \in B(X, l^{+\infty})$, $\|T\| \leq 1$ ···①

$\|x\| = 1, \forall \varepsilon > 0$, then $x = x_n$ or $\exists$ some n. $\|x - x_n\| < \varepsilon$

$|\phi_n(x)| = |\phi_n(x_n) + \phi_n(x - x_n)| = |1 + \phi_n(x - x_n)| \geq 1 - \|\phi_n\| \cdot \|x - x_n\| > 1 - 1 \times \cancel{\Delta}\varepsilon$

let $\varepsilon \to 0$

$\therefore |\phi_n(x)| \geq 1 - \varepsilon$, $\forall x \in X, \|x\| = 1 \Rightarrow \|Tx\| = \sup\limits_{n=1,2\cdots} |\phi_n(x)| \geq 1$, $\|T\| \geq 1$ ···②

①.② $\Rightarrow$ T is isometric

问题：怎么说明 bijective？