Hilbert Spaces II

Dual Spaces and the Riesz Representation Theorem

If X is a normed space over $\mathbb{K}$, then a linear map from X into $\mathbb{K}$ is called a linear functional on X.

We denote by X^* the collection of all *bounded* linear functionals on X, i.e. $X^* = B(X, \mathbb{K})$; we equip X^* with the norm

$$\|f\|_{X^*} = \sup_{\|x\|=1} |f(x)| \qquad \text{for each } f \in X^*,$$

The space X^* is called the dual (space) of X.

Example 12.1 Take $X = \mathbb{R}^n$. Then if $\boldsymbol{e}^{(j)}$ is the jth coordinate vector, we have $\boldsymbol{x} = \sum_{j=1}^{n} x_j \boldsymbol{e}^{(j)}$, and so if $f : \mathbb{R}^n \to \mathbb{R}$ is linear, then

$$f(\boldsymbol{x}) = f\left(\sum_{j=1}^{n} x_j \boldsymbol{e}^{(j)}\right) = \sum_{j=1}^{n} x_j f(\boldsymbol{e}^{(j)});$$

if we write $\boldsymbol{y}$ for the element of $\mathbb{R}^n$ with $y_j = f(\boldsymbol{e}^{(j)})$, then we can write this as

$$f(\boldsymbol{x}) = \sum_{j=1}^{n} x_j y_j = (\boldsymbol{x}, \boldsymbol{y}). \tag{12.1}$$

So with any $f \in (\mathbb{R}^n)^*$ we can associate some $\boldsymbol{y} \in \mathbb{R}^n$ such that (12.1) holds; since

$$|f(\boldsymbol{x})| \le \|\boldsymbol{y}\|_{\ell^2} \|\boldsymbol{x}\|_{\ell^2} \qquad \text{and} \qquad |f(\boldsymbol{y})| = \|\boldsymbol{y}\|_{\ell^2}^2,$$

it follows that

$$\|f\|_{(\mathbb{R}^n)^*} = \|\boldsymbol{y}\|_{\ell^2}.$$

In this way $(\mathbb{R}^n)^* \equiv \mathbb{R}^n$.

Lemma 12.3 *If H is a Hilbert space over $\mathbb{K}$ and $y \in H$, then the map $f_y : H \to \mathbb{K}$ defined by setting*

$$f_y(x) = (x, y) \tag{12.2}$$

is an element of H^* with $\|f_y\|_{H^*} = \|y\|_H$.

Note that this shows in particular that $\|x\| = \max_{\|y\|=1} |(x, y)|$.

Theorem 12.4 (Riesz Representation Theorem) *If H is a Hilbert space, then for every $f \in H^*$ there exists a unique element $y \in H$ such that*

$$f(x) = (x, y) \qquad \text{for all} \qquad x \in H; \tag{12.3}$$

and $\|y\|_H = \|f\|_{H^*}$. In particular, the Riesz map $R : H \to H^*$ defined via (12.2) by setting $R(y) = f_y$ maps H onto H^*.

Note if H is real, then R is a bijective linear isometry and $H \equiv H^*$.

Proof Let $K = \operatorname{Ker} f$; since f is bounded this is a closed linear subspace of H (Lemma 11.12). We claim that $K^\perp$ is a one-dimensional linear subspace of H. Indeed, given $u, v \in K^\perp$ we have

$$f\Big(f(u)v - f(v)u\Big) = f(u)f(v) - f(v)f(u) = 0, \tag{12.4}$$

since f is linear. Since $u, v \in K^\perp$, it follows that $f(u)v - f(v)u \in K^\perp$, while (12.4) shows that $f(u)v - f(v)u \in K$. Since $K \cap K^\perp = \{0\}$, it follows that

$$f(u)v - f(v)u = 0,$$

and so u and v are linearly dependent.

Therefore we can choose $z \in K^\perp$ such that $\|z\| = 1$, and use Proposition 10.4 to decompose any $x \in H$ as

$$x = (x, z)z + w \qquad \text{with} \qquad w \in (K^\perp)^\perp = K,$$

where we have used Lemma 10.5 and the fact that K is closed to guarantee that $(K^\perp)^\perp = K$. Thus

$$f(x) = (x, z)f(z) = (x, \overline{f(z)}z),$$

and setting $y = \overline{f(z)}z$ we obtain (12.3).

To show that this choice of y is unique, suppose that

$$(x, y) = (x, \hat{y}) \qquad \text{for all} \qquad x \in H.$$

Then $(x, y - \hat{y}) = 0$ for all $x \in H$; taking $x = y - \hat{y}$ gives $\|y - \hat{y}\|^2 = 0$.

Finally, Lemma 12.3 shows that $\|y\|_H = \|f\|_{H^*}$. □

Let H be a Hilbert space over $\mathbb{R}$. A linear operator A : H → H is strictly positive definite if there exists $\beta > 0$ such that

$$(Au, u) \geq \beta \|u\|^2, \qquad \forall u \in H. \qquad (1)$$

Theorem (Inverse of a positive definite operator). Let H be a real Hilbert space and A : H → H be a strictly positive definite bonded linear operator so that (1) holds. Then, for every f ∈ H, ∃! u= $A^{-1} f \in H$ such that

$$Au = f \qquad (2)$$

The inverse operator A^{-1} satisfies $\|A^{-1}\| \leq \frac{1}{\beta}$.

Proof We have $\beta \|u\|^2 \leq (Au, u) \leq \|Au\| \|u\|$. Hence

$$\beta \|u\| \leq \|Au\| \qquad (3)$$

and so A is $1-1$. Let $(v_n = Au_n)$ be a sequence in Rang(A) such that $v_n \to v$. From $\|u_m - u_n\| \leq \frac{1}{\beta} \|Au_m - Au_n\|$, we know that (u_n) is Cauchy and so converges.

Let $u_n \to u$; then $Au_n \to Au$. Thus $v = Au$ which shows that Range(A) is closed.

We now claim that Range(A) = H. If not, since Range(A) is closed, we could find a nonzero vector $\omega \perp$ Range(A). This is a contradiction.

Thus A is bijective. It follows from (2) and (3) that

$$\|A^{-1} f\| \leq \frac{1}{\beta} \|f\|$$

and so $\|A^{-1}\| \leq \frac{1}{\beta}$.

Theorem (Lax-Milgram). *Let H be a Hilbert space over the reals and let $B : H \times H \mapsto \mathbb{R}$ be a continuous bilinear functional. This means that*

$$\begin{aligned} B[au + bu' \,,\ v] &= aB[u,v] + bB[u',v], \\ B[u\,,\ av + bv'] &= aB[u,v] + bB[u,v']\,, \\ |B[u,v]| &\leq C\,\|u\|\,\|v\|\,, \end{aligned}$$

for some constant C and all $u, u', v, v' \in H$, $a, b \in \mathbb{R}$. In addition, assume that B is strictly positive definite, i.e., there exists a constant $\beta > 0$ such that

$$B[u,u] \ \geq\ \beta\,\|u\|^2 \qquad \text{for all} \quad u \in H\,. \tag{4}$$

Then, for every $f \in H$, there exists a unique $u \in H$ such that

$$B[u,v] \ =\ (f,v) \qquad \text{for all} \quad v \in H\,. \tag{5}$$

Moreover,

$$\|u\| \ \leq\ \beta^{-1}\|f\|\,.$$

Proof. For every fixed $u \in H$ the map $v \mapsto B[u,v]$ is a continuous linear functional on H. By the Riesz representation theorem, there exists a unique vector, which we call $Au \in H$, such that

$$B[u,v] = (Au, v) \qquad \text{for all } \; v \in H\,.$$

We claim that A is a bounded, positive definite linear operator.

The linearity of A is easy to check. To prove that A is bounded we observe that, for every $u \in H$,

$$\|Au\| = \sup_{\|v\|=1} |(Au,v)| = \sup_{\|v\|=1} |B[u,v]| \leq C\,\|u\|\,.$$

Hence $\|A\| \leq C$.

Moreover,

$$(Au,u) = B[u,u] \geq \beta\,\|u\|^2\,,$$

proving that A is strictly positive definite.

We can apply the above theorem to conclude that the equation $Au = f$ has a unique solution $u = A^{-1}f$, satisfying $\|u\| \leq \beta^{-1}\|f\|$. By the definition of A, this provides a solution to (5).

A sequence (x_n) in a Hilbert space H converges weakly to x, if $(x_n - x, y) \to 0, \forall y \in H$.

Lemma *Let $(H,(\cdot,\cdot),\|\cdot\|)$ be a real Hilbert space and let $A : H \to H$ be a not necessarily linear operator satisfying*

(a) $(Au - Av, u - v) \geq c\|u - v\|^2$ for all $u, v \in H$ (strong monotonicity);

(b) $\|Au - Av\| \leq L\|u - v\|$ for all $u, v \in H$ (Lipschitz condition),

where c and L are given positive constants. Then for all $w \in H$ there exists a unique $u^* \in H$ such that $Au^* = w$, i.e., A is a bijection.

We only prove existence: First we note that $c \leq L$ by using (a) and (b) together with Cauchy–Schwarz. For a fixed $w \in H$, define $B : H \to H$ by

$$Bu = u - t(Au - w), \quad t > 0,\ u \in H\,.$$

Note that if there is a fixed point of B then it is u^* as desired. We wish to apply the Banach Contraction Principle in (H, d).

We have for all $u, v \in H$

$$\begin{aligned} d(Bu, Bv)^2 &= \|Bu - Bv\|^2 \\ &= \|u - v\|^2 - 2t(u - v, Au - Av) + t^2\|Au - Av\|^2 \\ &\leq \|u - v\|^2 - \underbrace{2tc\|u - v\|^2}_{\text{from (a)}} + \underbrace{t^2L^2\|u - v\|^2}_{\text{from (b)}} \\ &= \underbrace{\left(1 - 2tc + t^2L^2\right)}_{\text{call this } m}\|u - v\|^2 \\ &= m\|u - v\|^2 \end{aligned}$$

Obviously, $m \geq 0$. We choose t to minimize $m = m(t)$ and find that $t = \frac{c}{L^2}$. Thus the minimum value of m is

$$m = 1 - 2\frac{c^2}{L^2} + \frac{c^2}{L^2} = 1 - \frac{c^2}{L^2} \geq 0\,,$$

since $c \leq L$. If $c = L$, then $m = 0$, so B is constant, i.e., $Bu = w_0$, so that $w_0 = u - (c/L^2)(Au - w)$. In this case A is affine, namely

$$Au = \frac{L^2}{c}(u - w_0) + w\,,$$

so that $u^* = w_0$.
When $c < L$ then $0 < m < 1$ so that B is a contraction and hence by the Banach Contraction Principle, B has a unique fixed point u^*. □

Theorem (Nonlinear Lax–Milgram Theorem). *Let H be a real Hilbert space and consider two functionals $a : H \times H \to \mathbb{R}$ and $b : H \to \mathbb{R}$ satisfying*

1. *For all $u \in H$ the map $v \mapsto a(u,v)$ is linear and continuous on H (i.e., it belongs to H^*);*
2. $a(u, u-v) - a(v, u-v) \geq c\|u-v\|^2$ *for all $u, v \in H$ and some $c > 0$;*
3. $|a(u,w) - a(v,w)| \leq L\|u-v\| \cdot \|w\|$ *for all $u, v, w \in H$ and some $L > 0$;*
4. *b is a continuous linear functional (i.e., $b \in H^*$).*

Then there exists a unique $u \in H$ such that

$$(\sharp) \qquad a(u,v) = b(v) \quad \forall v \in H\,.$$

Proof. By the first assumption and the Riesz Representation Theorem for all $u \in H$ there exists a unique $z \in H$ such that $a(u,v) = (v,z)$ for all $v \in H$. So there exists an operator $A : H \to H$ defined by $Au := z$. We now rewrite the second condition

$$\begin{aligned} a(u,u-v) - a(v,u-v) &= (u-v, Au) - (u-v, Av) \\ &= (u-v, Au - Av) \\ &= (Au - Av, u-v) \\ &\geq c\|u-v\|^2 , \end{aligned}$$

for all $u, v \in H$, so A satisfies condition (a) of the previous lemma. From the third assumption we have for all $u, v, z \in H$

$$\begin{aligned} |a(u,z) - a(v,z)| &= |(z, Au) - (z, Av)| \\ &= |(z, Au - Av)| \\ &\leq L\|u-v\| \cdot \|z\| . \end{aligned}$$

Choosing $z = Au - Av$ we see that operator A also satisfies condition (b) of Lemma above.
On the other hand, by the fourth assumption and the Riesz Representation Theorem there exists a unique w such that $b(v) = (v,w)$ for all $v \in H$. Now (#) can be written as

$$\Big[(v, Au) = (v, w), \quad \forall v \in H\Big] \iff Au = w ,$$

so the conclusion of the theorem follows. $\square$

• **Theorem** Let H be a real Hilbert space. *Let $K \subset H$ be a nonempty closed convex set. Then for every $f \in H$ there exists a unique element $u \in K$ such that*

$$|f-u| = \min_{v\in K} |f-v| = \operatorname{dist}(f,K). \tag{2}$$

*Moreover, u is **characterized** by the property*

$$u \in K \ \ \textit{and} \ \ (f-u, v-u) \le 0 \quad \forall v \in K. \tag{3}$$

Notation. The above element u is called the *projection* of f onto K and is denoted by

$$\boxed{u = P_K f.}$$

Proposition *Let $K \subset H$ be a nonempty closed convex set. Then P_K does not increase distance, i.e.,*

$$|P_K f_1 - P_K f_2| \le |f_1 - f_2| \quad \forall f_1, f_2 \in H.$$

Proof. Set $u_1 = P_K f_1$ and $u_2 = P_K f_2$. We have

$$(f_1 - u_1, v - u_1) \le 0 \quad \forall v \in K \tag{6}$$

$$(f_2 - u_2, v - u_2) \le 0 \quad \forall v \in K. \tag{7}$$

Choosing $v = u_2$ in (6) and $v = u_1$ in (7) and adding the corresponding inequalities, we obtain

$$|u_1 - u_2|^2 \le (f_1 - f_2, u_1 - u_2).$$

It follows that $|u_1 - u_2| \le |f_1 - f_2|$.

Definition. A bilinear form $a : H \times H \to \mathbb{R}$ is said to be

(i) *continuous* if there is a constant C such that

$$|a(u, v)| \leq C|u|\,|v| \quad \forall u, v \in H;$$

(ii) *coercive* if there is a constant $\alpha > 0$ such that

$$a(v, v) \geq \alpha|v|^2 \quad \forall v \in H.$$

Theorem (Stampacchia). *Assume that $a(u, v)$ is a continuous coercive bilinear form on H. Let $K \subset H$ be a nonempty closed and convex subset. Then, given any $\varphi \in H^\star$, there exists a unique element $u \in K$ such that*

$$a(u, v - u) \geq \langle \varphi, v - u \rangle \quad \forall v \in K. \tag{10}$$

Moreover, if a is symmetric, then u is characterized by the property

$$\boxed{u \in K \quad \text{and} \quad \frac{1}{2}a(u, u) - \langle \varphi, u \rangle = \min_{v \in K} \left\{ \frac{1}{2}a(v, v) - \langle \varphi, v \rangle \right\}.} \tag{11}$$

Proof From the Riesz–Fréchet representation theorem we know that there exists a unique $f \in H$ such that

$$\langle \varphi, v \rangle = (f, v) \quad \forall v \in H.$$

On the other hand, if we *fix* $u \in H$, the map $v \mapsto a(u, v)$ is a continuous linear functional on H. Using once more the Riesz–Fréchet representation theorem we find a unique element in H, denoted by Au, such that $a(u, v) = (Au, v)\ \forall v \in H$. Clearly A is a linear operator from H into H satisfying

$$|Au| \le C|u| \quad \forall u \in H, \tag{12}$$

$$(Au, u) \ge \alpha |u|^2 \quad \forall u \in H. \tag{13}$$

Problem (10) amounts to finding some $u \in K$ such that

$$(Au, v - u) \ge (f, v - u) \quad \forall v \in K. \tag{14}$$

Let $\rho > 0$ be a constant (to be determined later). Note that (14) is equivalent to

$$(\rho f - \rho Au + u - u, v - u) \le 0 \quad \forall v \in K, \tag{15}$$

i.e.,

$$u = P_K(\rho f - \rho Au + u).$$

For every $v \in K$, set $Sv = P_K(\rho f - \rho Av + v)$. We claim that if $\rho > 0$ is properly chosen then S is a strict contraction. Indeed, since P_K does not increase distance , we have

$$|Sv_1 - Sv_2| \le |(v_1 - v_2) - \rho(Av_1 - Av_2)|$$

and thus

$$\begin{aligned}|Sv_1 - Sv_2|^2 &\le |v_1 - v_2|^2 - 2\rho(Av_1 - Av_2, v_1 - v_2) + \rho^2|Av_1 - Av_2|^2 \\ &\le |v_1 - v_2|^2(1 - 2\rho\alpha + \rho^2 C^2).\end{aligned}$$

Choosing $\rho > 0$ in such a way that $1-2\rho\alpha+\rho^2 C^2 < 1$ (i.e., $0 < \rho < 2\alpha/C^2$) we find that S has a unique fixed point.

Assume now that the form $a(u, v)$ is also *symmetric*. Then $a(u, v)$ defines a *new scalar product* on H; the corresponding norm $a(u, u)^{1/2}$ is equivalent to the original norm $|u|$. It follows that H is also a Hilbert space for this new scalar product. Using the Riesz–Fréchet theorem we may now represent the functional φ through the new scalar product, i.e., there exists some unique element $g \in H$ such that

$$\langle\varphi, v\rangle = a(g, v) \quad \forall v \in H.$$

Problem (10) amounts to finding some $u \in K$ such that

$$a(g - u, v - u) \le 0 \quad \forall v \in K. \tag{16}$$

u is the projection onto K of g for the new inner product a and is the unique element of K that achieves

$$\min_{v\in K} a(g - v, g - v)^{1/2}.$$

This amounts to minimizing on K the function

$$v \mapsto a(g-v, g-v) = a(v, v) - 2a(g, v) + a(g, g) = a(v, v) - 2\langle\varphi, v\rangle + a(g, g),$$

or equivalently the function

$$v \mapsto \frac{1}{2}a(v, v) - \langle\varphi, v\rangle.$$

The Hilbert Adjoint of a Linear Operator

Theorem 13.1 *Let H and K be Hilbert spaces and $T \in B(H, K)$. Then there exists a unique operator $T^* \in B(K, H)$, which we call the* (Hilbert) adjoint of *T, such that*

$$(Tx, y)_K = (x, T^*y)_H \tag{13.1}$$

for all $x \in H$, $y \in K$. Furthermore, $T^{**} := (T^*)^* = T$ and

$$\|T^*\|_{B(K,H)} = \|T\|_{B(H,K)}.$$

Proof Let $y \in K$ and consider $f : H \to \mathbb{K}$ defined by $f(x) := (Tx, y)_K$. Then clearly f is linear and

$$\begin{aligned} |f(x)| &= |(Tx, y)_K| \\ &\leq \|Tx\|_K \|y\|_K \\ &\leq \|T\|_{B(H,K)} \|x\|_H \|y\|_K. \end{aligned}$$

It follows that $f \in H^*$, and so by the Riesz Representation Theorem there exists a unique $z \in H$ such that

$$(Tx, y)_K = (x, z)_H \qquad \text{for all} \qquad x \in H.$$

We now define $T^* : K \to H$ by setting $T^*y = z$. By definition we have

$$(Tx, y)_K = (x, T^*y)_H \qquad \text{for all} \qquad x \in H, \; y \in K,$$

i.e. (13.1). However, it remains to show that $T^* \in B(K, H)$. First, T^* is linear since for all $\alpha, \beta \in \mathbb{K}$, $y_1, y_2 \in Y$,

$$
\begin{aligned}
(x, T^*(\alpha y_1 + \beta y_2))_H &= (Tx, \alpha y_1 + \beta y_2)_K \\
&= \overline{\alpha}(Tx, y_1)_K + \overline{\beta}(Tx, y_2)_K \\
&= \overline{\alpha}(x, T^* y_1)_H + \overline{\beta}(x, T^* y_2)_H \\
&= (x, \alpha T^* y_1 + \beta T^* y_2)_H,
\end{aligned}
$$

i.e. $T^*(\alpha y_1 + \beta y_2) = \alpha T^* y_1 + \beta T^* y_2$. To show that T^* is bounded, we can write

$$
\begin{aligned}
\|T^* y\|_H^2 &= (T^* y, T^* y)_H \\
&= (TT^* y, y)_K \\
&\le \|TT^* y\|_K \|y\|_K \\
&\le \|T\|_{B(H,K)} \|T^* y\|_H \|y\|_K.
\end{aligned}
$$

If $\|T^* y\|_H \neq 0$, then we can divide both sides by $\|T^* y\|_H$ to obtain

$$
\|T^* y\|_H \le \|T\|_{B(H,K)} \|y\|_K,
$$

while this final inequality is trivially true if $\|T^* y\|_H = 0$. Thus $T^* \in B(K, H)$ with $\|T^*\|_{B(K,H)} \le \|T\|_{B(H,K)}$.

We now show that $T^{**} := (T^*)^* = T$, from which can obtain equality of the norms of T and T^*. Indeed, if we have $T^{**} = T$, then it follows that

$$
\|T\|_{B(H,K)} = \|(T^*)^*\|_{B(H,K)} \le \|T^*\|_{B(K,H)},
$$

which combined with $\|T^*\|_{B(K,H)} \le \|T\|_{B(H,K)}$ shows that

$$
\|T^*\|_{B(K,H)} = \|T\|_{B(H,K)}.
$$

To prove that $T^{**} = T$, note that since $T^* \in B(K, H)$ it follows that $(T^*)^* \in B(H, K)$, and by definition for all $x \in K$, $y \in H$ we have

$$\begin{aligned}(x, (T^*)^* y)_K &= (T^* x, y)_H \\ &= \overline{(y, T^* x)_H} \\ &= \overline{(Ty, x)_K} \\ &= (x, Ty)_K,\end{aligned}$$

i.e. $(T^*)^* y = Ty$ for all $y \in H$, which is exactly $(T^*)^* = T$.

Finally, we show that the requirement that (13.1) holds defines T^* uniquely. Suppose that $T^*, \hat{T} : K \to H$ are such that

$$(x, T^* y)_H = (x, \hat{T} y)_H \qquad \text{for all} \quad x \in H, \;\; y \in K.$$

Then for each $y \in K$ we have

$$(x, (T^* - \hat{T}) y)_H = 0 \qquad \text{for every } x \in H;$$

this shows that $(T^* - \hat{T}) y = 0$ for each $y \in K$, i.e. that $\hat{T} = T^*$. $\square$

Lemma 13.2 *Let H, K, and J be Hilbert spaces, $R, S \in B(H, K)$, and $T \in B(K, J)$; then*

(a) $(\alpha R + \beta S)^* = \overline{\alpha} R^* + \overline{\beta} S^*$ *and*
(b) $(TR)^* = R^* T^*$.

Proof (a) For any $x \in H$, $y \in K$ we have

$$\begin{aligned}(x, (\alpha R + \beta S)^* y)_H &= ((\alpha R + \beta S)x, y)_K \\ &= \alpha (Rx, y)_K + \beta (Sx, y)_K \\ &= \alpha (x, R^* y)_H + \beta (x, S^* y)_H \\ &= (x, \overline{\alpha} R^* y + \overline{\beta} S^* y)_H = (x, (\overline{\alpha} R^* + \overline{\beta} S^*) y)_H;\end{aligned}$$

the uniqueness argument from Theorem 13.1 now guarantees that (a) holds.

(b) We have

$$(x, (TR)^* y)_H = (TRx, y)_J = (Rx, T^* y)_K = (x, R^* T^* y)_H,$$

and again we use the uniqueness argument from Theorem 13.1. □

Definition 13.3 If H is a Hilbert space and $T \in B(H)$, then T is *self-adjoint* if $T = T^*$.

Equivalently $T \in B(H)$ is self-adjoint if and only if it is *symmetric*, i.e.

$$(x, Ty) = (Tx, y) \qquad \text{for all} \qquad x, y \in H. \tag{13.2}$$

Example Let $H = K = \mathbb{K}^n$ with its standard inner product. Then any matrix $A = (a_{ij}) \in \mathbb{K}^{n \times n}$ defines a linear map T_A on $\mathbb{K}^n$ by mapping $\boldsymbol{x}$ to $A\boldsymbol{x}$, where

$$(A\boldsymbol{x})_i = \sum_{j=1}^{n} a_{ij} x_j .$$

Then we have

$$\begin{aligned}(T_A \boldsymbol{x}, \boldsymbol{y}) &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right) \overline{y_i} \\ &= \sum_{j=1}^{n} x_j \sum_{i=1}^{n} \overline{(\overline{a_{ij}}\, y_i)} = (\boldsymbol{x}, T_{A^*} \boldsymbol{y}),\end{aligned}$$

where A^* is the Hermitian conjugate of A, i.e. $A^* = \overline{A}^T$.

Definition If H is a Hilbert space and $T \in B(H)$, then T is *self-adjoint* if $T = T^*$.

Example Consider the right- and left- shift operators $\mathfrak{s}_r : \ell^2 \to \ell^2$ and $\mathfrak{s}_l : \ell^2 \to \ell^2$, given by

$$\mathfrak{s}_r(\boldsymbol{x}) = (0, x_1, x_2, \ldots) \qquad \text{and} \qquad \mathfrak{s}_l(\boldsymbol{x}) = (x_2, x_3, x_4, \ldots).$$

Both operators are linear with $\|\mathfrak{s}_r\| = \|\mathfrak{s}_l\| = 1$.

We have

$$(\mathfrak{s}_r\boldsymbol{x}, \boldsymbol{y}) = x_1y_2 + x_2y_3 + x_3y_4 + \cdots = (\boldsymbol{x}, \mathfrak{s}_r^*\boldsymbol{y});$$

so $\mathfrak{s}_r^*\boldsymbol{y} = (y_2, y_3, y_4, \ldots)$, i.e. $\mathfrak{s}_r^* = \mathfrak{s}_l$.

Similarly for the left shift $\mathfrak{s}_l\boldsymbol{x} = (x_2, x_3, x_4, \ldots)$ we have

$$(\mathfrak{s}_l\boldsymbol{x}, \boldsymbol{y}) = x_2y_1 + x_3y_2 + x_4y_3 + \cdots = (\boldsymbol{x}, \mathfrak{s}_l^*\boldsymbol{y});$$

so $\mathfrak{s}_l^*\boldsymbol{y} = (0, y_1, y_2, \ldots)$, i.e. $\mathfrak{s}_l^* = \mathfrak{s}_r$.

These maps are not self-adjoint, but we do have $\mathfrak{s}_l^{**} = \mathfrak{s}_l$ and $\mathfrak{s}_r^{**} = \mathfrak{s}_r$.