(2): λ is eigenvalue of T, then $T_{\mathbb{C}}$; λ is real eigenvalue of $T_{\mathbb{C}}$, then T

pf: $Tx=\lambda x \quad x\neq 0$.

$T_{\mathbb{C}}(x+ix) = Tx+iTx = \lambda x + i\lambda x = \lambda(x+ix)$ ∴ λ也是$T_{\mathbb{C}}$的e.v.

$T_{\mathbb{C}}(x+iy) = Tx+iTy = ux+iuy \quad u\in\mathbb{R}$

$Tx-ux+i(Ty-uy)=0 \Rightarrow Tx=ux,\ Ty=uy$ ($u\in\mathbb{R}$ 实部虚部=0) $\Rightarrow u$也是T的e.v.

T1(4) $D_a: l^2(\mathbb{C}) \to l^2(\mathbb{C})$, $(x_1,x_2,\dots) \mapsto (a_1x_1, a_2x_2, \dots)$, $a\in l^{\infty}(\mathbb{C})$

(1): $\sigma_p(D_a)=\{a_i\}_{i=1}^{+\infty}$ (2) $\sigma(D_a)=\overline{\sigma_p(D_a)}$ (closure)

pf: (1) $(a_1x_1, a_2x_2,\dots) = \lambda(x_1,x_2,\dots)$

$a_ix_i=\lambda x_i,\ i=1,2,\dots \quad x\neq 0$

$\Rightarrow \lambda=a_i,\ x_i=1,\ x_j=0$ if $i\neq j$ ∴ $\sigma_p(D_a)=\{a_i\}_{i=1}^{+\infty}$, (若$\lambda\notin\{a_i\}$…, $x=0$ 舍)

(2): $D_a-\lambda I: (x_1,x_2,\dots) \mapsto ((a_1-\lambda)x_1, (a_2-\lambda)x_2, \dots)$ not invertible

$= (y_1,y_2,\dots)$

$\Rightarrow$ $(a_i-\lambda)x_i$ 没法得出 y_i

$\{ (a_i-\lambda)x_i = (a_i-\lambda)x_i',\ x_i\neq x_i' \Rightarrow a_i=\lambda$

若 $\lambda\notin\overline{\sigma_p(D_a)}$, 即 $\exists\delta>0$ s.t. $|\lambda-a_i|\geq\delta\ \forall i$

$\|D_a-\lambda I\| = |\lambda-a_i| \geq \delta \Rightarrow$ injective

$\forall y_i, \exists x_i$ s.t. $(\lambda-a_i)x_i=y_i \Rightarrow$ surjective

∴ $\lambda\notin\overline{\sigma_p(D_a)}$, $\lambda\notin\sigma(D_a) \Rightarrow \sigma(D_a)\subseteq\overline{\sigma_p(D_a)}$

已知 onto, invertible $\Leftrightarrow \|D_a-\lambda I\|\geq\delta$

即 $\lambda\notin\sigma(D_a) \Rightarrow \lambda\notin\overline{\sigma_p(D_a)} \Rightarrow \sigma(D_a)\supseteq\overline{\sigma_p(D_a)}$ ∴ 取等号.

(3): compact subset of $\mathbb{C}$ 一定是 spectrum of operator of this form D_a

pf: $a\in l^{\infty}(\mathbb{C})$ ∴ $\max|a_i|<+\infty$

∴ $\overline{\sigma_p(D_a)}$ compact in $\mathbb{C}$; 反之 if $A\subseteq\mathbb{C}$. A cpt thus bounded, $\max\limits_{a\in A} a<+\infty$ 记为 a_M

Consider Disk $=\{r\in\mathbb{C}: |r|\leq a_m\}$

Let $R=\{r\in \text{Disk}: |r|\leq a_m, |r|\in\mathbb{Q}, \arg r\in[0,2\pi)\cap\mathbb{Q}\}$ 其中arg r指与实轴夹角

R countable, $\overline{R}=$ Disk

∴ $\overline{R\cap(\text{Disk}\cap A)} = A$ 证明略 ($\overline{\mathbb{Q}}=\mathbb{R}$ 所以这个也是对的)

而 $R\cap(\text{Disk}\cap A)\subseteq R$ countable, 可以写成 $(a_1,a_2,\dots,a_n,\dots)=a$ 证毕

T14.4: $X \subseteq \mathbb{C}$ Banach, $T\in B(X)$, 定义 spectral radius $r_\sigma(T)=\sup\{|\lambda|:\lambda\in\sigma(T)\}$

$$r_\sigma(T)\le \liminf_{n\to+\infty}\|T^n\|^{\frac{1}{n}}$$

pf: 反证假设. if $r_\sigma(T)>\liminf\|T^n\|^{\frac{1}{n}}$ （如果是 $<$, $\exists N$ s.t. $\forall n\ge N$, $\|T^n\|^{\frac{1}{n}}>r_\sigma(T)$）

$\forall N$, $\exists n\ge N$, $\|T^n\|^{\frac{1}{n}}<r_\sigma(T)$

订正: $Q(\sigma(T))=\sigma(Q(T))$

$\Rightarrow (\sigma(T))^n=\sigma(T^n)$, $\sigma(T)=(\sigma(T^n))^{\frac{1}{n}}$, $\forall n$

又∵ $\sigma(T)\subseteq\{\lambda\in\mathbb{C}:|\lambda|\le\|T\|\}$

$\Rightarrow \lambda\in\sigma(T)=(\sigma(T^n))^{\frac{1}{n}}$, $\lambda\le\|T^n\|^{\frac{1}{n}}$ $\forall n$

∴ $r_\sigma(T)\le\liminf\|T^n\|^{\frac{1}{n}}$

T14.5: $X=C([0,1])$, show: $\forall\lambda\neq0$ $\lambda\in\rho(T)$, $T\in B(X)$ $Tf(x)=\int_0^x f(s)ds$; $0\in\sigma(T)$ but not in $\sigma_p(T)$

pf: ~~$(T-\lambda I)f(x)=\int_0^x f(s)ds-\lambda f(x)$ 记为 $S\cdot f(x)$~~ 我本来想用 contract mapping

~~$|Sf-Sg|=|\int_0^x(f(s)-g(s))ds-\lambda(f(x)-g(x))|$~~ 但这个压缩条件不一定成立的!(不必要!)

T11.7 $k\in C([a,b]^2)$, $\|k\|_{+\infty}\le M$, $Tf(x)=\int_a^x k(x,y)f(y)dy$, T bdd + linear

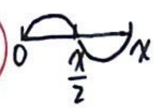
$|T^nf|\le M^n\|f\|_{+\infty}\frac{(x-a)^n}{n!}$ induction.

$f(x)=g(x)+\lambda\int_a^x k(x,y)f(y)dy$ 有唯一解

由T14.4 $r_\sigma(T)\le\liminf_{n\to+\infty}\|T^n\|^{\frac{1}{n}}$ $\xrightarrow[b=1\ a=0]{\text{在本题中}k=1}$ $r_\sigma(T)\le\lim_{n\to+\infty}\|f\|_{+\infty}\frac{(b-a)^n}{n!}\to0$,

∴ $\forall\lambda>0$, $\lambda\in\rho(T)$ --- (1)

$(T-0I)f(x)=\int_0^x f(s)ds$ not invertible T not surjective, $Tf\in C'([0,1])$ 而不可微函数没有原像!

since T not ~~inj~~. 若 $Tf=0$ 不是$Tf\ x=0$!  和  均可 ∴ $0\in\sigma(T)$

if $(T-0I)f=0$, then $(T-0I)f(x)=0$ $\forall x\in[0,1]$ $\Rightarrow f(x)=0$ in $[0,1]$

$(\int_0^x f(s)ds)'=f(s)=0$ 求导可知. ∴ $0\notin\sigma_p(T)$

T15.3: $T:l^2\to l^2$ $(x_1,x_2,\cdots)\longmapsto(x_1,\frac{x_2}{2},\frac{x_3}{3},\cdots)$ cpt

pf: $\{e_i\}_{i=1}^{+\infty}$ is orthonormal basis in H. $e_i=\{0,0,\cdots,\underset{\text{ith slot}}{1},0,\cdots\}$

$\sum_{i=1}^{+\infty}\|Te_i\|^2=\sum_{i=1}^{+\infty}(\frac{1}{i})^2<+\infty$

∴ T is Hilbert-Schmidt $\Rightarrow$ T is compact

T15.5 WTS: $\|T\|_{B(H)} \le \|T\|_{HS} < +\infty$ given

pf: $\|Tx\| = \|T(\sum_{n=1}^{+\infty} a_n e_n)\| = \|\sum_{n=1}^{+\infty} a_n Te_n\|$ since $T \in B(H)$, then cts, $\{e_i\}$为标准正交基

$\cancel{\le \sum_{n=1}^{+\infty} |a_n| \|Te_n\|}$ $\|x\|^2 = (x,x) = \sum_{n=1}^{+\infty} |a_n|^2$

$\|Tx\|^2 = (\sum_{n=1}^{+\infty} a_n Te_n)^2 \le (\sum_{n=1}^{+\infty} |a_n| \|Te_n\|)^2 \le (\sum_{n=1}^{+\infty} |a_n|^2)(\sum_{n=1}^{+\infty} \|Te_n\|^2) \le \|x\|^2 \|T\|_{HS}^2$

$\Rightarrow \|Tx\| \le \|x\|^2 \|T\|_{HS}^2$

let $\|x\|=1$ 则 $\|T\|_{B(H)} = \sup_{\|x\|=1} \|Tx\| \le \|T\|_{HS}^2$ 证毕

T15.7 $\{k_{ij}\}_{i,j=1}^{+\infty} \subseteq \mathbb{K}$, $\sum_{i,j} |k_{ij}|^2 < +\infty$, 证明 $S: l^2 \to l^2$, $(Sx)_i = \sum_{j=1}^{+\infty} k_{ij} x_j$ compact

pf: $\{e_i\}$为标准正交基 $e_i = (0,0,\cdots 1\ 0\cdots)$ $S(e_i) = (Se_i)_i = \sum_{j=1}^{+\infty} k_{ij} e_j$

$\therefore \|S\|_{HS} = \sum_{i=1}^{+\infty} \|Se_i\|^2 = \sum_{i=1}^{+\infty} \sum_{j=1}^{+\infty} \|k_{ij} e_j\|^2 \le \sum_{i=1}^{+\infty} \sum_{j=1}^{+\infty} \|k_{ij}\|^2 \underbrace{\|e_j\|^2}_{=1} < +\infty$

$\therefore S$ Hilbert-Schmidt $\Rightarrow$ cpt

T15.9 $T: l^2 \to l^2$, $(x_1, x_2, \cdots) \mapsto (0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \cdots)$ cpt. 且无 eigenvalue

pf: T15.3中证了 $(x_1, x_2 \cdots) \xrightarrow{T_1} (x_1, \frac{x_2}{2}, \cdots)$ cpt

$T_2: (x_1, x_2 \cdots) \mapsto (0, x_1, x_2, \cdots)$. $T_2 = S_r$ compact by Example 11.7

$\Rightarrow T = S_r \circ T_1 \therefore$ cpt

$Tx = \lambda x$ $\quad \lambda x_1 = 0$ $\quad$ 若 $\lambda = 0$. $\quad$ 若 $x_1 = 0$

$\lambda x_2 = x_1$ $\quad x_n = \lambda x_{n+1} = 0$ $n = 1, 2, \cdots$ $\quad x_{n+1} = \lambda^n x_1 = 0 \Rightarrow x = 0$

$\vdots$

$\therefore T$ 不存在 eigenvalue