L12:

Def: sublinear. $p: V \to \mathbb{R}$ $\begin{cases} p(x+y) \le p(x)+p(y) \\ p(\lambda x) = \lambda p(x) \quad \forall \lambda \in \mathbb{R}^+ \end{cases}$

$\downarrow$

$p(\lambda x) = |\lambda| \cdot p(x) \quad \forall \lambda \in \mathbb{R}$ or $\mathbb{C}$ $\Rightarrow$ subnorm

prop: $p(x-y) \geq |p(x)-p(y)|$ implies subnorm ≥ 0

Real Hahn-Banach TH (H-B) X is real, U is subspace $\le X$, $p: X \to \mathbb{R}$ is sublinear

$\Downarrow$

X complex, p is semi-norm ✓

$\phi: U \to \mathbb{R}$ is linear, $\phi \le p$ in U $\xRightarrow{\text{extend}}$ $\exists f: X \to \mathbb{R}$ linear $f|_U = \phi$. $f \le p$ in X

pf: ① $P := \{(G, g): U \le G \le X, g|_U = \phi, g \le p\}$, "$\le$" subspace

$(G_1, g_1) \le (G_2, g_2)$ if $G_1 \le G_2$, $g_2|_{G_1} = g_1$

P upper bounded $\Rightarrow$ $\max P = (Y, f)$

② if $Y < X$, $\exists z \in X \setminus Y$

$\exists F: \text{span}\{z\} \cup Y \to \mathbb{R}$ s.t. $F(u+\partial z) = f(u) + \partial F(z) \le p(u+\partial z) \quad \forall \partial \in \mathbb{R}$

$\rightleftharpoons \begin{cases} F(z) \le p(\frac{u}{\partial}+z) - f(\frac{u}{\partial}) & \partial > 0 \\ F(z) \geq (p(u+z) - f(u)) \cdot \frac{1}{\partial}. & \partial < 0 \end{cases}$

$\rightleftharpoons f(y_1) - p(y_1 - z) \le F(z) \le p(y_2+z) - f(y_2)$

(inf RHS > sup LHS)

③ $p(x) = \|\phi\|_{U^*} \|x\|$

$\Rightarrow \|f\| = \|\phi\|$ under this p (norm)

TH: V is complex, $f: V \to \mathbb{C}$ linear $\Rightarrow$ $\exists$ unique real linear $\phi: V \to \mathbb{R}$ s.t. $f(v) = \phi(v) - i\phi(iv) \quad \forall v \in V$

$\begin{cases} p \text{ is semi-norm} \quad |f(v)| \le p(v) \rightleftharpoons |\phi(v)| \le p(v) \\ f \in V^*, \text{ then } \phi \in V_{\mathbb{R}}^* \quad \text{i.e. } \|\phi\|_{V_{\mathbb{R}}^*} = \|f\|_{V^*} \end{cases}$ ...... ⊛

pf: ① $f(v) = \phi_1(v) + i\phi_2(v)$, $\phi_i: V \to \mathbb{R}$

$f(iv) = \phi_1(iv) + i\phi_2(iv)$

$= i\phi_1(v) - \phi_2(v) = if(v)$

$\Big\} \Rightarrow \phi_2(v) = -\phi_1(iv)$, $\phi_2(iv) = \phi_1(v)$ $\Rightarrow f = \phi(v) - i\phi(iv)$

② $f(v) = re^{i\theta}$ $\begin{cases} \phi(v) = r\cos\theta \\ -\phi(iv) = r\sin\theta \end{cases}$ $\Rightarrow |f(v)| = e^{-i\theta} \cdot f(v) = f(e^{-i\theta} v) = \phi(e^{-i\theta} v) - i\underbrace{\phi(i e^{-i\theta} v)}_{=0} \in \mathbb{R}$

$= \phi(e^{-i\theta} v)$

$\le \|\phi\| \cdot \|e^{-i\theta} v\| \cdot \|v\|$

$\Rightarrow \|f\| \le \|\phi\|$ then obv

TH': $\phi: V \to \mathbb{R}$ is real linear, $\Rightarrow f(v) = \phi(v) - i\phi(iv)$ is linear,

$f: V \to \mathbb{C}$ satisfy ⊛, 同样用②证 $|f(v)| \le p(v)$

TH: $\forall x_0 \in X, \exists f \in X^*$ s.t. $\|f\| = \|x_0\|$; $f(x_0) = \|x_0\|^2$

Pf: let $G = \mathbb{R} \cdot x_0 \subseteq X$, $g(t x_0 \in G) = t\|x_0\|^2$

extend $g \in G^*$ to $f \in X^*$

注意 $\mathbb{R} \cdot x_0$ 型的空间之后常用!

TH: $\forall x_0 \in X$, $\|x_0\| = \sup\limits_{\substack{f \in X^* \\ \|f\| \leq 1}} |f(x_0)| = \max\limits_{\substack{f \in X^* \\ \|f\| \leq 1}} |f(x_0)|$

Pf: $\sup |f(x_0)| \leq \underset{\leq 1}{\|f\|} \cdot \|x_0\| \leq \|x_0\|$

$\exists f \in X^*$ s.t. $\|f\| = \|x_0\|$, $f(x_0) = \|x_0\|^2 \Rightarrow$ let $g := \frac{1}{\|x_0\|} \cdot f$. g satisfy

取等

或: 用 X^{**}: $\|x_0\| = \|x_0^{**}\|$

$\sup |f(x_0)| = \sup |x_0^{**}(f)| = \|x_0^{**}\|$

$\Downarrow$

TH: $x_0 \neq 0 \in X$, $\exists f \in X^*$. $\|f\| = 1$ s.t. $f(x) = \|x\|$

我经常记不住上面内容! 很少遇到, 易忘!!

TH: X^* separate points

$x \neq y$ in X, $\Rightarrow \exists f \in X^*$ s.t. $f(x) \neq f(y)$

Pf: $x \neq y \Rightarrow \exists g \in X^*$, $\|g\| = 1$, $g(x-y) = \|x-y\| \neq 0$