The Spectrum of a Bounded Linear Operator

Let X be a complex Banach space and $T \in B(X)$. The *resolvent set of* T, $\rho(T)$, is

$$\rho(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is invertible}\}.$$

The *spectrum of* T, $\sigma(T)$, is the complement of $\rho(T)$,

$$\begin{aligned} \sigma(T) &= \mathbb{C} \setminus \rho(T) \\ &= \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible}\}. \end{aligned}$$

The point spectrum of T is the set

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : (T - \lambda I)x = 0 \text{ for some non-zero } x \in X\}.$$

If $\lambda \in \sigma_p(T)$, then λ is an *eigenvalue* of T, $E_\lambda := \operatorname{Ker}(T - \lambda I)$ is the *eigenspace* corresponding to λ, and any non-zero $x \in E_\lambda$ is one of the corresponding *eigenvectors* (if $x \in E_\lambda$, then $Tx = \lambda x$); the dimension of E_λ is the *multiplicity* of λ.

Any $\lambda \in \sigma_p(T)$ satisfies $|\lambda| \leq \|T\|$: if there exists $x \neq 0$ such that $Tx = \lambda x$, then

$$|\lambda|\|x\| = \|\lambda x\| = \|Tx\| \leq \|T\|\|x\|,$$

which shows that $|\lambda| \leq \|T\|$.

Lemma 14.2 *Suppose that $T \in B(X)$ and that $\{\lambda_j\}_{j=1}^n$ are distinct eigenvalues of T. Then any set $\{e_j\}_{j=1}^n$ of corresponding eigenvectors (i.e. $Te_j = \lambda_j e_j$) is linearly independent.*

Lemma *If $T \in B(X)$, then $\sigma(T)$ is a closed subset of*

$$\{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}. \tag{14.3}$$

Proof First we prove the inclusion

$$\sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\}.$$

Note that for any $\lambda \neq 0$, we can write

$$T - \lambda I = \lambda \left(\frac{1}{\lambda} T - I \right),$$

so if $I - \frac{1}{\lambda} T$ is invertible, $\lambda \notin \sigma(T)$. But for $|\lambda| > \|T\|$ we have

$$\left\| \frac{1}{\lambda} T \right\| \|I\| < 1,$$

and then Lemma 11.16 guarantees that $I - \frac{1}{\lambda} T$ is invertible, i.e. $\lambda \in \rho(T)$, and the result follows.

To show that the spectrum is closed we show that the resolvent set is open. If $\lambda \in \rho(T)$, then $T - \lambda I$ is invertible and Lemma 11.16 shows that $(T - \lambda I) - \delta I$ is invertible provided that

$$\|\delta \mathrm{I}\| \|(T - \lambda I)^{-1}\| < 1,$$

i.e. $T - (\lambda + \delta) I$ is invertible for all δ with $|\delta| < \|(T - \lambda I)^{-1}\|^{-1}$, and so $\rho(T)$ is open. $\square$

Example The right-shift operator $\mathfrak{s}_{\mathrm{r}}$ on ℓ^2 has no eigenvalues.

Proof Observe that $\mathfrak{s}_r \boldsymbol{x} = \lambda \boldsymbol{x}$ implies that

$$(0, x_1, x_2, \ldots) = \lambda(x_1, x_2, x_3, \ldots)$$

and so

$$\lambda x_1 = 0, \quad \lambda x_2 = x_1, \quad \lambda x_3 = x_2, \ldots.$$

If $\lambda \neq 0$, then this implies that $x_1 = 0$, and then $x_2 = x_3 = x_4 = \ldots = 0$, and so λ is not an eigenvalue. If $\lambda = 0$, then we also obtain $\boldsymbol{x} = 0$, and so there are no eigenvalues, i.e. $\sigma_{\mathrm{p}}(\mathfrak{s}_r) = \varnothing$. $\square$

Example For the left-shift operator $\mathfrak{s}_l$ on ℓ^2 every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ is an eigenvalue.

Proof Observe that $\lambda \in \mathbb{C}$ is an eigenvalue if $\mathfrak{s}_l \boldsymbol{x} = \lambda \boldsymbol{x}$, i.e. if

$$(x_2, x_3, x_4 \ldots) = \lambda(x_1, x_2, x_3, \ldots),$$

i.e. if

$$x_2 = \lambda x_1, \quad x_3 = \lambda x_2, \quad x_4 = \lambda x_3, \quad \cdots$$

Given $\lambda \neq 0$ this gives a candidate eigenvector

$$\boldsymbol{x} = (1, \lambda, \lambda^2, \lambda^3, \ldots),$$

which is an element of ℓ^2 (and so is an actual eigenvector) provided that

$$\sum_{n=1}^{\infty} |\lambda|^{2n} = \frac{1}{1 - |\lambda|^2} < \infty,$$

which is the case for any λ with $|\lambda| < 1$. It follows that

$$\{\lambda \in \mathbb{C} : \ |\lambda| < 1\} \subseteq \sigma_{\mathrm{p}}(\mathfrak{s}_l). \qquad \square$$

Lemma *If H is a Hilbert space and $T \in B(H)$, then*

$$\sigma(T^*) = \{\overline{\lambda} : \lambda \in \sigma(T)\}.$$

Proof If $\lambda \notin \sigma(T)$, then $T - \lambda I$ has a bounded inverse,

$$(T - \lambda I)(T - \lambda I)^{-1} = I = (T - \lambda I)^{-1}(T - \lambda I).$$

Taking adjoints we obtain

$$[(T - \lambda I)^{-1}]^*(T^* - \overline{\lambda} I) = I = (T^* - \overline{\lambda} I)[(T - \lambda I)^{-1}]^*,$$

and so $T^* - \overline{\lambda} I$ has a bounded inverse, i.e. $\overline{\lambda} \notin \sigma(T^*)$. Starting instead with T^* we deduce that $\lambda \notin \sigma(T^*) \Rightarrow \overline{\lambda} \notin \sigma(T)$, which completes the proof. $\square$

Since $s_r = s_l^*$, $\{\lambda \in \mathbb{C}: |\lambda| < 1\} \subset \sigma_p(s_l) \subset \sigma(s_l)$, we have from this Lemma that $\{\lambda \in \mathbb{C}: |\lambda| < 1\} \subset \sigma(s_r)$.

Example The spectrum of $\mathfrak{s}_l$ and of $\mathfrak{s}_r$ (as operators on ℓ^2) are both equal to the unit disc in the complex plane:

$$\sigma(\mathfrak{s}_l) = \sigma(\mathfrak{s}_r) = \{\lambda \in \mathbb{C} : \ |\lambda| \leq 1\}.$$

Proof We showed earlier that for the shift operators $\mathfrak{s}_r$ and $\mathfrak{s}_l$ on ℓ^2,

$$\sigma(\mathfrak{s}_l^*) = \sigma(\mathfrak{s}_r) \supseteq \{\lambda \in \mathbb{C} : \ |\lambda| < 1\}.$$

Since the spectrum is closed and $\|\mathfrak{s}_r\| = \|\mathfrak{s}_l\| = 1$, it follows

$$\sigma(\mathfrak{s}_l) = \sigma(\mathfrak{s}_r) = \{\lambda \in \mathbb{C} : \ |\lambda| \leq 1\}. \qquad \square$$

Spectral mapping theorem Let X be a complex Banach space and $T \in B(X)$. If $Q(t) = \sum_{k=0}^{n} a_k t^k$ is a polynomial, then

$$Q(\sigma_p(T)) = \sigma_p(Q(T)). \qquad (*)$$

$$Q(\sigma(T)) = \sigma(Q(T)). \qquad (**)$$

Proof We have $Q(\sigma_p(T)) \subset \sigma_p(Q(T))$. Assume now by contradiction that there exists $\mu \in \sigma_p(Q(T))$ such that $\mu \notin Q(\sigma_p(T))$. Write

$$Q(t) - \mu = a(t - t_1)(t - t_2)\cdots(t - t_n),$$

with $a \neq 0$. Then $t_i \notin \sigma_p(T)$ $\forall i$. In fact, if for some $i, t_i \in \sigma_p(T)$, then $\mu = Q(t_i)$. This contradicts our assumption. In addition, we have some $x \neq 0$ such that $Q(T)x = \mu x$. Since for each i, $T - t_i I$ is injective, we conclude that $x = 0$. Impossible. This proves (*).

In order to prove (**), let us show firstly that $Q(\sigma(T)) \subset \sigma(Q(T))$. Suppose, by contradiction, that there exists some $\mu \in Q(\sigma(T))$ such that $\mu \notin \sigma(Q(T))$. Then $\mu = Q(\lambda)$ with $\lambda \in \sigma(T)$, and $Q(T) - Q(\lambda)I = S$ is bijective. We may write

$$Q(t) - Q(\lambda) = (t - \lambda)W(t) \quad \forall t \in \mathbb{C},$$

and thus

$$(T - \lambda I)W(T) = W(T)(T - \lambda I) = S.$$

Hence $T - \lambda I$ is bijective and $\lambda \in \rho(T)$; impossible. Therefore, $Q(\sigma(T)) \subset \sigma(Q(T))$. Now suppose there is a $\mu \in \sigma(Q(T))$ such that $\mu \notin Q(\sigma(T))$. Write $Q(t) - \mu$ as above with $t_i \notin \sigma(T)$ $\forall i$. Then $Q(T) - \mu I$ can be written as product of bijective operators. Therefore $Q(T) - \mu I$ is bijective, i.e., $\mu \in \rho(Q(T))$. Impossible. This proves (**).

Compact Operators

Definition Let X and Y be normed spaces. A linear operator $T: X \to Y$ is *compact* if for any bounded sequence $(x_n) \in X$, the sequence $(Tx_n) \in Y$ has a convergent subsequence (whose limit lies in Y).

T is compact if $T\mathbb{B}_X$ is a precompact subset of Y

Note that a compact operator must be bounded, since otherwise there exists a sequence $(x_n) \in X$ with $\|x_n\| = 1$ but $Tx_n \to \infty$, and clearly (Tx_n) cannot have a convergent subsequence.

Example Take $T \in B(X, Y)$ with finite-dimensional range. Then T is compact, since any bounded sequence in a finite-dimensional space has a convergent subsequence.

Noting that if $T, S: X \to Y$ are both compact, then $T + S$ is also compact, and that λT is compact for any $\lambda \in \mathbb{K}$, we can define the space $K(X, Y)$ of all compact linear operators from X into Y, and this is then a vector space.

Definition. A linear operator $T : X \to Y$ is said to be of *fiite rank* if the range of T, is finite-dimensional.

Theorem *Suppose that X is a normed space and Y is a Banach space. If $(K_n)_{n=1}^{\infty}$ is a sequence of compact (linear) operators in $K(X,Y)$ that converges to some $K \in B(X,Y)$, i.e.*

$$\|K_n - K\|_{B(X,Y)} \to 0 \qquad as \qquad n \to \infty,$$

then $K \in K(X,Y)$. In particular, $K(X,Y)$ is complete.

Proof

Since Y is complete it suffices to check that for every ε > 0 there is a finite covering of $K(\mathbb{B}_X)$ with balls of radius ε. Fix an integer n such that $\|K_n - K\| < \frac{\varepsilon}{2}$. Since $K_n(\mathbb{B}_X)$ has compact closure, there is a finite covering of $K_n(\mathbb{B}_X)$ by balls of radius ε/2, say

$K_n(\mathbb{B}_X) \subset \bigcup_{i\in I} B\left(f_i, \frac{\varepsilon}{2}\right)$. It follows that $K(\mathbb{B}_X) \subset \bigcup_{i\in I} B(f_i, \varepsilon)$.

Definition 15.5 An operator $T \in B(H)$ is *Hilbert–Schmidt* if for some orthonormal basis $\{e_j\}_{j=1}^{\infty}$ of H

$$\|T\|_{\mathrm{HS}}^2 := \sum_{j=1}^{\infty} \|Te_j\|^2 < \infty.$$

Proposition 15.6 *Any Hilbert–Schmidt operator T is compact.*

Proof Choose some orthonormal basis $\{e_j\}_{j=1}^{\infty}$ for H, and observe that since T is linear and continuous we can write

$$Tu = T\left(\sum_{j=1}^{\infty}(u, e_j)e_j\right) = \sum_{j=1}^{\infty}(u, e_j)Te_j.$$

Now for each n let $T_n : H \to H$ be defined by setting

$$T_n u := \sum_{j=1}^{n}(u, e_j)Te_j.$$

This operator is clearly linear, and its range is finite-dimensional since it is the linear span of $\{Te_j\}_{j=1}^{n}$. It follows that T_n is a compact operator for each n.

Thus

$$\begin{aligned}
\|(T_n - T)u\| &= \left\| \sum_{j=n+1}^{\infty} (u, e_j)Te_j \right\| \\
&\le \sum_{j=n+1}^{\infty} |(u, e_j)|\|Te_j\| \\
&\le \left(\sum_{j=n+1}^{\infty} |(u, e_j)|^2\right)^{1/2} \left(\sum_{j=n+1}^{\infty} \|Te_j\|^2\right)^{1/2} \\
&\le \|u\| \left(\sum_{j=n+1}^{\infty} \|Te_j\|^2\right)^{1/2},
\end{aligned}$$

which shows that

$$\|T_n - T\|_{B(H)} \le \left(\sum_{j=n+1}^{\infty} \|Te_j\|^2\right)^{1/2}$$

$$\to 0, n \to \infty.$$

Theorem Let T be a compact linear operator on a Hilbert space E. There exists a sequence of finite rank operators $\{T_n\}_{n=1}^{\infty}$ such that $\|T_n - T\| \to 0$.

Proof

Let $B_E = \{x \in \text{E}: \|x\| \le 1\}$, set $K = \overline{T(B_E)}$. Given $\varepsilon > 0$ there is a finite covering of K with balls of radius ε, say $K \subset \bigcup_{i \in I} B(f_i, \varepsilon)$. Let G denote the vector space spanned by the f_i's and set $T_\varepsilon = P_G T$, so that T_ε is of finite rank. We claim that $\|T_\varepsilon - T\| < 2\varepsilon$. For every $x \in B_E$ there is some $i_0 \in I$ such that

(1) $$\|Tx - f_{i_0}\| < \varepsilon.$$

Thus

$$\|P_G Tx - P_G f_{i_0}\| < \varepsilon,$$

that is,

(2) $$\|P_G Tx - f_{i_0}\| < \varepsilon.$$

Combining (1) and (2), one obtains

$$\|P_G Tx - Tx\| < 2\varepsilon \quad \forall x \in B_E,$$

that is,

$$\|T_\varepsilon - T\| < 2\varepsilon.$$

Lemma *If H is a Hilbert space and $T \in K(H)$, then $T^* \in K(H)$.*

Proof Since T is compact and T* is bounded, it follows that TT* is compact. For any bounded sequence $(x_n) \in H$,
TT^*x_n has a convergent subsequence (which we relabel). Therefore

$$|(TT^*(x_n - x_m), x_n - x_m)| \leq \|TT^*(x_n - x_m)\| \|x_n - x_m\| \to 0$$

as $\min(m, n) \to \infty$. But the left-hand side of this expression is

$$|(T^*(x_n - x_m), T^*(x_n - x_m))| = \|T^*(x_n - x_m)\|^2,$$

which shows that (T^*x_n) is Cauchy and thus convergent, showing that T^* is compact. □

Theorem *Suppose that X is an infinite-dimensional Banach space and $T \in K(X)$. Then $0 \in \sigma(T)$.*

The Hilbert–Schmidt Theorem

If T is self-adjoint, then the *numerical range* of T, $V(T)$, is the set

$$V(T) := \{(Tx, x) : x \in H, \ \|x\| = 1\}. \tag{16.1}$$

Theorem 16.1 *Let H be a Hilbert space and $T \in B(H)$ a self-adjoint operator. Then $V(T) \subset \mathbb{R}$ and*

$$\|T\|_{B(H)} = \sup\{|\lambda| : \lambda \in V(T)\}. \tag{16.2}$$

Proof We have

$$(Tx, x) = (x, Tx) = \overline{(Tx, x)},$$

and so (Tx, x) is real for every $x \in H$.

To prove (16.2) we let $M = \sup\{|(Tx, x)| : x \in H, \ \|x\| = 1\}$. Clearly

$$|(Tx, x)| \leq \|Tx\|\|x\| \leq \|T\|\|x\|^2 = \|T\|$$

when $\|x\| = 1$, and so $M \leq \|T\|$.

Now observe that for any $u, v \in H$ we have

$$\begin{aligned}(T(u+v), u+v) - (T(u-v), u-v) &= 2[(Tu, v) + (Tv, u)] \\ &= 2[(Tu, v) + (v, Tu)] \\ &= 4\,\mathrm{Re}(Tu, v),\end{aligned}$$

using the fact that $(Tv, u) = (v, Tu) = \overline{(Tu, v)}$ since T is self-adjoint.

$$
\begin{aligned}
4\,\mathrm{Re}(Tu, v) &= (T(u+v), u+v) - (T(u-v), u-v) \\
&\leq M(\|u+v\|^2 + \|u-v\|^2) \\
&= 2M(\|u\|^2 + \|v\|^2)
\end{aligned}
$$

using the Parallelogram Law. If $Tu \neq 0$ choose

$$v = \frac{\|u\|}{\|Tu\|}\, Tu$$

to obtain, since $\|v\| = \|u\|$, that

$$4\|u\|\|Tu\| \leq 4M\|u\|^2,$$

i.e. $\|Tu\| \leq M\|u\|$ if $Tu \neq 0$. The same inequality is trivial if $Tu = 0$, and so it follows that $\|T\| \leq M$ and therefore we obtain $\|T\| = M$, as required. □

Corollary 16.2 *If $T \in B(H)$ is self-adjoint, then*

(i) *all of its eigenvalues are real, and*
(ii) *if $Tx_1 = \lambda_1 x_1$ and $Tx_2 = \lambda_2 x_2$ with $\lambda_1 \neq \lambda_2$, then $(x_1, x_2) = 0$.*

Theorem 16.3 *Let H be a Hilbert space and $T \in B(H)$ a compact self-adjoint operator. Then at least one of $\pm\|T\|$ is an eigenvalue of T, and so in particular*

$$\|T\| = \max\{|\lambda| : \lambda \in \sigma_p(T)\}. \tag{16.3}$$

Proof Note that $|\lambda| \leq \|T\|$, $\forall \lambda \in \sigma_p(T)$.

$$\|T\| = \sup_{\|x\|=1} |(Tx, x)|,$$

so there exists a sequence (x_n) of unit vectors in H such that

$$(Tx_n, x_n) \to \alpha, \tag{16.4}$$

where α is either $\|T\|$ or $-\|T\|$. Since T is compact, there is a subsequence x_{n_j} such that Tx_{n_j} is convergent to some $y \in H$. Relabel x_{n_j} as x_n again, so that $Tx_n \to y$ and (16.4) still holds.

Now consider

$$\begin{aligned}\|Tx_n - \alpha x_n\|^2 &= \|Tx_n\|^2 + \alpha^2 - 2\alpha(Tx_n, x_n)\\ &\leq 2\alpha^2 - 2\alpha(Tx_n, x_n);\end{aligned}$$

by our choice of x_n, the right-hand side tends to zero as $n \to \infty$. It follows, since $Tx_n \to y$, that

$$\alpha x_n \to y,$$

and since $\alpha \neq 0$ is fixed we have $x_n \to x := y/\alpha$; note that $\|x\| = 1$ since it is the limit of the x_n and $\|x_n\| = 1$ for every n. Since T is bounded, it is continuous, so therefore

$$Tx = \lim_{n\to\infty} Tx_n = y = \alpha x.$$

We have found $x \in H$ with $\|x\| = 1$ such that $Tx = \alpha x$, so $\alpha \in \sigma_p(T)$.

Lemma 16.5 *If $T \in B(H)$ and Y is a closed linear subspace of H such that $TY \subseteq Y$, then $T^*Y^\perp \subseteq Y^\perp$. In particular, if $T \in B(H)$ is self-adjoint and Y is a closed linear subspace of H, then*

$$TY \subseteq Y \qquad \Rightarrow \qquad TY^\perp \subseteq Y^\perp.$$

Proof Let $x \in Y^\perp$ and $y \in Y$. Then $Ty \in Y$ and so

$$0 = (Ty, x) = (y, T^*x) \qquad \text{for all } y \in Y,$$

i.e. $T^*x \in Y^\perp$. $\square$

X : a Banach space. A bounded operator $A\colon X \to X$ is called *invertible* if it maps X one-to-one onto X. By the Banach theorem the inverse mapping A^{-1} is automatically continuous.

Definition. *The spectrum $\sigma(A)$ of a bounded linear operator A on a complex Banach space X consists of all $\lambda \in \mathbb{C}$ such that the operator $A - \lambda I$ is not invertible.*

The complement of the spectrum is called the *resolvent set* of the operator A and denoted by $\varrho(A)$. The points of the resolvent set are called *regular* points. For every $\lambda \in \varrho(A)$ the operator

$$R_\lambda(A) := (A - \lambda I)^{-1}$$

is called the *resolvent* of A (one should bear in mind that sometimes the resolvent is defined as the inverse to $\lambda I - A$). For $\lambda, \mu \in \varrho(A)$ we have the *Hilbert identity*

$$R_\lambda(A) - R_\mu(A) = (\lambda - \mu) R_\mu(A) R_\lambda(A),$$

which is easily verified by multiplying both sides by $(A - \lambda I)$ from the right and then multiplying by $(A - \mu I)$ from the left.

By Banach's inverse mapping theorem a point λ belongs to the spectrum if and only if either $\mathrm{Ker}(A - \lambda I) \neq 0$ or $(A - \lambda I)(X) \neq X$, where

$$\mathrm{Ker}(A - \lambda I) := \{x\colon\ Ax - \lambda x = 0\}.$$

In the first case λ is an *eigenvalue*, i.e., $Av = \lambda v$ for some vector $v \neq 0$ (called an *eigenvector*). In the finite-dimensional space both cases can happen only simultaneously, but in infinite-dimensional spaces the situation is different.

Lemma (i) *Let $A \in \mathsf{B}(X)$. Then, whenever $|\lambda| > \|A\|$, we have $\lambda \in \varrho(A)$ and*

$$R_\lambda(A) = -\sum_{k=0}^{\infty} \frac{A^k}{\lambda^{1+k}},$$

where the series converges in the operator norm.

(ii) *For every point $\lambda_0 \in \varrho(A)$, whenever $|\lambda - \lambda_0| < \|R_{\lambda_0}(A)\|^{-1}$, we have $\lambda \in \varrho(A)$ and*

$$R_\lambda(A) = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k R_{\lambda_0}(A)^{k+1},$$

where the series converges in the operator norm.

PROOF. (i) We have $A - \lambda I = -\lambda I + A$, where $\|A\| < |\lambda| = 1/\|(\lambda I)^{-1}\|$. Convergence of the series of $-\lambda^{-1-k} A^k$ in the operator norm is obvious from the estimate $\|\lambda^{-k} A^k\| \leqslant |\lambda|^{-k} \|A\|^k$. It is straightforward to show that for its sum S_λ we have $S_\lambda (A - \lambda I) = (A - \lambda I) S_\lambda = I$.

(ii) Convergence of the series with respect to the norm is justified similarly. For its sum S_λ we have

$$\begin{aligned} S_\lambda(A - \lambda I) &= \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k R_{\lambda_0}(A)^{k+1} \big(A - \lambda_0 I - (\lambda - \lambda_0) I\big) \\ &= \sum_{k=0}^{\infty} \big[(\lambda - \lambda_0)^k R_{\lambda_0}(A)^k - (\lambda - \lambda_0)^{k+1} R_{\lambda_0}(A)^{k+1}\big] = I. \end{aligned}$$

Similarly, $(A - \lambda I) S_\lambda = I$. $\square$

Let X be a complex or real Banach space and K be a compact operator on X.

Lemma A *Let K be a compact operator on X.*
(i) The kernel of the operator $I - K$ is finite-dimensional.
(ii) The range of the operator $I - K$ is closed.

PROOF. (i) On the kernel of the operator $I - K$ the operator I equals K and hence is compact, which is only possible if this kernel is finite-dimensional.

(ii) Let $y_n = x_n - Kx_n \to y$. We show that $y \in (I - K)(X)$. Suppose first that $\sup_n \|x_n\| < \infty$. By the compactness of K we can extract from $\{Kx_n\}$ a convergent subsequence $\{Kx_{n_i}\}$. Since $x_{n_i} = y_{n_i} + Kx_{n_i}$, the sequence $\{x_{n_i}\}$ converges as well. Denoting its limit by x, we obtain $y = x - Kx$.

We now consider the case where the sequence $\{x_n\}$ is not bounded. Set $Z = \mathrm{Ker}(I - K)$ and

$$d_n = \inf\{\|x_n - z\| \colon z \in Z\}.$$

Since Z is finite-dimensional, there exist vectors $z_n \in Z$ with $\|x_n - z_n\| = d_n$. We show that the sequence $\{d_n\}$ is bounded. Suppose the contrary. We can assume that $d_n \to +\infty$. Set

$$v_n = (x_n - z_n)/\|x_n - z_n\|.$$

Since $(I - K)z_n = 0$ and $\sup_n \|y_n\| < \infty$, we have

$$\|v_n\| = 1, \; v_n - Kv_n = (I - K)x_n/\|x_n - z_n\| = y_n/d_n \to 0.$$

The sequence $\{Kv_n\}$ contains a convergent subsequence $\{Kv_{n_i}\}$. Then $\{v_{n_i}\}$ converges to some vector $v \in X$. Moreover,

$$v - Kv = \lim_{i\to\infty} (v_{n_i} - Kv_{n_i}) = 0,$$

i.e., $v \in Z$. However, this is impossible, since $\mathrm{dist}(v, Z) \geqslant 1$, because

$$\|v_n - z\| = \frac{1}{d_n}\|x_n - z_n - d_n z\| \geqslant \frac{d_n}{d_n} = 1 \quad \text{for all } z \in Z,\ n \in \mathbb{N}.$$

Thus, the sequence $\{d_n\}$ is bounded. Now everything reduces to the first case, since $(I - K)(x_n - z_n) = (I - K)x_n = y_n$. $\square$

Theorem *Let K be a compact operator on a complex or real infinite-dimensional Banach space X. Then the spectrum of K either coincides with the point 0 or has the form*

$$\sigma(K) = \{0\} \cup \{k_n\},$$

where all numbers k_n are eigenvalues of K of finite multiplicity, which means that $\dim \operatorname{Ker}(K - k_n I) < \infty$, and the collection $\{k_n\}$ is either finite or is a sequence converging to zero.

PROOF. By the noncompactness of I the operator K is not invertible and hence $0 \in \sigma(K)$. Let $\lambda \in \sigma(K)$ and $\lambda \neq 0$. We show that λ is an eigenvalue. Suppose the contrary. Passing to the operator $\lambda^{-1}K$, we can assume that $\lambda = 1$. By the lemma the subspace $X_1 = (K - I)(X)$ is closed in X. In addition, we have $X_1 \neq X$, since otherwise $K - I$ would be invertible. Set

$$X_n = (K - I)^n(X) = (K - I)(X_{n-1}), \quad n \geqslant 2.$$

It is clear that $X_{n+1} \subset X_n$, since $X_1 \subset X$, whence $X_2 \subset X_1$ and so on. By the lemma we obtain that all subspaces X_n are closed. They are all different by the injectivity of $K - I$, since if

$$(K - I)(X_n) = (K - I)(X_{n-1}),$$

then $X_n = X_{n-1}$, whence we obtain $X_n = \cdots = X_1 = X$.

According to Riesz Lemma, there exist vectors $x_n \in X_n$ such that $\|x_n\| = 1$ and $\operatorname{dist}(x_n, X_{n+1}) \geqslant 1/2$. If $n < m$, we have

$$Kx_n - Kx_m = x_n - x_m + (K - I)x_n - (K - I)x_m,$$

where

$$-x_m + (K - I)x_n - (K - I)x_m \in X_m + X_{n+1} + X_{m+1} \subset X_{n+1}.$$

Hence $\|Kx_n - Kx_m\| \geqslant 1/2$, i.e., $\{Kx_n\}$ contains no Cauchy subsequence contrary to the compactness of K. The obtained contradiction means that λ is an eigenvalue of K. By the lemma $\dim \operatorname{Ker}(K - \lambda I) < \infty$, i.e., λ has a finite multiplicity.

We now show that $\sigma(K)$ has no nonzero limit points. Suppose that $\lambda_n \to \lambda$, where λ_n are eigenvalues and $\lambda \neq 0$. We can assume that λ_n are distinct and $|\lambda_n| \geqslant \sigma > 0$. Let us take $x_n \neq 0$ with $Kx_n = \lambda_n x_n$. It is readily seen that the vectors x_n are linearly independent. Denote by X_n the linear span of $x_1, \ldots, x_n$. It is clear that $K(X_n) \subset X_n$. By Riesz Lemma there exist $y_n \in X_n$ with $\|y_n\| = 1$ and $\operatorname{dist}(y_n, X_{n-1}) \geqslant 1/2$, $n > 1$. We have

$$y_n = \alpha_n x_n + z_n, \quad z_n \in X_{n-1}.$$

Then for $n > m$ we have

$$\begin{aligned} Ky_n - Ky_m &= K(\alpha_n x_n) + Kz_n - Ky_m = \alpha_n \lambda_n x_n + Kz_n - Ky_m \\ &= \lambda_n (y_n - z_n + \lambda_n^{-1} K z_n - \lambda_n^{-1} K y_m), \end{aligned}$$

where $-z_n + \lambda_n^{-1} K z_n - \lambda_n^{-1} K y_m \in X_{n-1}$, because $z_n \in X_{n-1}$, $Kz_n \in X_{n-1}$, $Ky_m \in X_m \subset X_{n-1}$. Since $|\lambda_n| \geqslant \sigma$ and $\operatorname{dist}(y_n, X_{n-1}) \geqslant 1/2$, we have $\|Ky_n - Ky_m\| \geqslant \sigma/2$. Hence $\{Ky_n\}$ contains no Cauchy subsequence, which is a contradiction. $\square$

Theorem 16.6 *(Hilbert–Schmidt Theorem). Let H be a Hilbert space and $T \in B(H)$ a compact self-adjoint operator. Then there exists a finite or countably infinite orthonormal sequence (w_j) consisting of eigenvectors of T, with corresponding non-zero real eigenvalues (λ_j), such that for all $x \in H$*

$$Tx = \sum_j \lambda_j (x, w_j) w_j. \tag{16.5}$$

Proof By Theorem 16.3 there exists $w_1 \in H$ such that $Tw_1 = \pm\|T\|w_1$ and $\|w_1\| = 1$.

Consider the subspace of H perpendicular to w_1,

$$H_2 = w_1^{\perp}.$$

Since $H_2 \subset H$ is closed, it is a Hilbert space (Lemma 8.11). Then since T is self-adjoint, Lemma 16.5 shows that T leaves H_2 invariant. If we consider $T_2 = T|_{H_2}$, then we have $T_2 \in B(H_2, H_2)$ with T_2 compact; this operator is still self-adjoint, since for all $x, y \in H_2$

$$(x, T_2 y) = (x, Ty) = (Tx, y) = (T_2 x, y).$$

Now apply Theorem 16.3 to the operator T_2 on the Hilbert space H_2 find an eigenvalue $\lambda_2 = \pm\|T_2\|$ and an eigenvector $w_2 \in H_2$ with $\|w_2\| = 1$.

Now if we let $H_3 = \{w_1, w_2\}^{\perp}$, then H_3 is a closed subspace of H_2 and $T_3 = T|_{H_3}$ is compact and self-adjoint. We can once more apply Theorem 16.3 to find an eigenvalue $\lambda_3 = \pm\|T_3\|$ and a corresponding eigenvector $w_3 \in H_3$ with $\|w_3\| = 1$. We continue this process as long as $T_n \neq 0$.

If $T_n = 0$ for some n, then, for any given $x \in H$, if we set

$$y := x - \sum_{j=1}^{n-1} (x, w_j) w_j \in H_n,$$

we have

$$0 = T_n y = T y = T x - \sum_{j=1}^{n-1} (x, w_j) T w_j = T x - \sum_{j=1}^{n-1} \lambda_j (x, w_j) w_j,$$

which is (16.5).

If T_n is never zero, then, given $x \in H$, consider

$$y_n := x - \sum_{j=1}^{n-1} (x, w_j) w_j \in H_n$$

(for $n \geq 2$). We have

$$\|x\|^2 = \|y_n\|^2 + \sum_{j=1}^{n-1} |(x, w_j)|^2,$$

and so $\|y_n\| \leq \|x\|$. It follows, since $T_n = T|_{H_n}$, that

$$\left\| T x - \sum_{j=1}^{n-1} \lambda_j (x, w_j) w_j \right\| = \|T y_n\| \leq \|T_n\| \|y_n\| = |\lambda_n| \|x\|,$$

and since $|\lambda_n| \to 0$ as $n \to \infty$ (Theorem above) we obtain (16.5). $\square$