期中后

Perron's Method

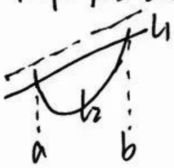
Def1: $v\in C^2$ is subharmonic if $-\Delta v\le 0$

Def2: $v\in C$ is subharmonic if: $\forall$ ball B, $\forall$ harmonic $w\in C^2(D)\cap C(\bar D)$, $w|_{\partial B}\ge v|_{\partial B}$, then:

$v\le w$ in B, 即任意调和函数 w，任意球 $w|_{\partial B}\ge v|_{\partial B}\Rightarrow v\le w$ in B

Rmk: if $d=1$, subharmonic $\Leftrightarrow$ convex, harmonic = straight line

Def2可以当成 convex 的等价定义，原因如下：



在 $[a,b]$ 中 $l_1\ge l_2$ at $x=a, x=b$, then $l_1\ge l_2$ in (a,b)

l_1 = harmonic, $x=a, x=b$ = ∂B, (a,b) = interior of B

Th1 (Harmonic Lifting) $u\in C(\bar U)$, u is subharmonic, B is a ball with $\bar B\subseteq U$, Let $\begin{cases}\Delta w=0, & B\\ w=u & U\setminus B\end{cases}$

then w is subharmonic in U, called: harmonic lifting

proof: Let v harmonic, $v|_{\partial B_1}\ge w|_{\partial B_1}$ for some $B_1\subseteq U$

in $U\setminus B$, $\left.\begin{array}{l} v|_{\partial B_1}\ge w|_{\partial B_1}=u|_{\partial B_1}\\ u \text{ is subharmonic}\end{array}\right\}\xrightarrow{\text{def}_2} v\ge \overset{=w}{u}$ in $B_1\setminus B$ ……①

in $B_1\cap B$, $\partial(B_1\cap B)\subseteq\partial B_1$ $\therefore \left.\begin{array}{l} v|_{\partial(B_1\cap B)}\ge w|_{\partial(B_1\cap B)}\\ v, w \text{ are hamonic in } B\end{array}\right\}\xrightarrow{\text{Maximum Principle}} v\ge w$ $B_1\cap B$ ……②

①+②: in $(B_1\cap B)\cup(B_1\setminus B)=B_1$ $v\ge w$

that is: $\forall$ ball $B_1\subseteq U$, $\forall$ possible harmonic $v|_{\partial B_1}\ge w|_{\partial B_1}$, then $v\ge w$ in B_1; satisfy def2

Rmk: 一个次调和函数 u 的一部分 和调和函数 $w|_B$，可以合成一个 $w|_U$ 为次调和函数

Th2: $\varphi\in C(\partial U)$, $\mathcal{S}_\varphi=\{u\in C(\bar U): u$ is subharmonic in U, $u\le\varphi$ on $\partial U\}$

define $u_\varphi(x):=\sup\{u(x): u\in\mathcal{S}_\varphi\}$. (point-wise supremum), then u_φ is harmonic in U

proof: Let $x_0\in U$, $B_r(x_0)\subseteq U$, it suffices to show "u_φ" is harmonic in $\forall B_r(x_0)$"

①: $u_\varphi(x)$ is the supremum $\therefore\exists v_n\in\mathcal{S}_\varphi$ s.t. $v_n(x_0)\nearrow u_\varphi(x_0)$ 递增逼近

Let $\tilde v_n$ be the harmonic lifting of v_n, 即 $\begin{cases}\Delta\tilde v_n=0 & B_r(x_0)\\ \tilde v_n=v_n & U\setminus B_r(x_0)\end{cases}$

then $v_n(x_0)=\tilde v_n(x_0)\le u_\varphi(x_0)$ in $U\setminus B_r(x_0)$

$\therefore \tilde v_n$ bounded, harmonic in $B_r(x_0)$ $\therefore D\tilde v$ bounded $\Rightarrow$ equ-cts

$\therefore \exists\{\tilde v_{n_k}\}_{k=1}^{\infty}$ s.t. $\tilde v_{n_k}\to v_*$ in $\overline{B_r(x_0)}$ uniformly

the limit v_* is also harmonic by mean property　　　cpt, 点有界, equcts $\Rightarrow$ uni 子列

$\therefore\exists v_*$ harmonic in $B_r(x_0)$ s.t. $v_*(x_0)\le u_\varphi(x_0)$ find v_n 使端点取等 $\therefore$ (4)'

②: 接下来要证明 $v_* = u_\varphi$ in the whole $B_r(x_0)$

Let $\tilde{x}_0 \in B_r(x_0)$, $\exists\, w_n \in S_\varphi$ s.t. $w_n(\tilde{x}_0) \uparrow u_\varphi(\tilde{x}_0)$

不妨设 $w_n \geq \tilde{v}_n$, 否则 $w_n := \max(w_n, \tilde{v}_n)$ point-wise ... (2)

$\tilde{w}_n$ is the harmonic lifting of w_n,

∴ $\exists\, n_k$ s.t. $\tilde{w}_{n_k} \xrightarrow{uni} w_*$ in $\overline{B_r(x_0)}$, w_* harmonic 证明同: $\tilde{v}_{n_k} \to v_*$

let $w_*(\tilde{x}_0) = u_\varphi(\tilde{x}_0)$ be the choice of w_n ... (3)

then $w_*(x) = \lim w_n(x) \geq \lim \tilde{v}_n(x) = v_*(x)$ for $\forall x \in B_r(x_0)$ ... (4)
$u_\varphi(x_0)$ if $x = x_0$

又∵ $w_*(x_0) \leq u_\varphi(x_0)$ since $w_* \in S_\varphi$,

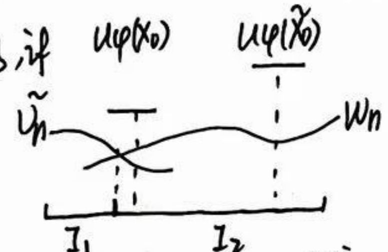
⇒ $w_*(x_0) = v_*(x_0)$

by maximum principle, $(w_* - v_*)$ harmonic in B. $w_* - v_*$ maximum $= 0$ at $x_0 \in B$ ⇒ $v_* - w_* \equiv 0$ ... (5)

问题: (1) uniform conv 缺少条件 eq-cts, 或如何证 $\{\tilde{v}_n\}$ eq-cts

✓(2) 为什么可以 let $w_n \geq \tilde{v}_n$ $\forall x \in B_r(x_0)$; 如果 let $u_\varphi(\tilde{x}_0) \geq u_\varphi(x_0)$, 只说明可能行

⇒ 肯定可以, 对



, $w_n := \begin{cases} \tilde{v}_n & I_1 \\ w_n & I_2 \end{cases}$ 即为 original w_n 的 harmonic lifting

∴ w_n 仍为 subharmonic

(3): $w_n(\tilde{x}_0) \nearrow u_\varphi(\tilde{x}_0)$, 为什么: $\tilde{w}_{n_k} \xrightarrow{uni} w_*$. $w_*(\tilde{x}_0) = u_\varphi(\tilde{x}_0)$ 是一定存在的.

即: 为什么 ∃: $w_*(\tilde{x}_0) = \lim_{n_k} \tilde{w}_{n_k}(\tilde{x}_0) \overset{?}{=} \lim_n w_n(\tilde{x}_0) = u_\varphi(\tilde{x}_0)$ 一定存在

(4) 这里为什么 $\forall x$ in B, $\{w_n\}$ 不一定 uni conv 呀?

(5) 为什么. MVP 和 MP 我觉得都没用　~~(6)~~ 为什么要 v_* 直接任选 $\tilde{v}_n$: 这样 (4) 中 $v_*(x) = u_\varphi(x_0)$ 不行 $(\tilde{v}_n \neq u_\varphi)|_{x_0}$

①+② ⇒ ∃ harmonic $v_*(x_0) = u_\varphi(x_0)$, $w_*(\tilde{x}_0) = u_\varphi(\tilde{x}_0)$, and $v_* = w_*$ in B

∴ 同样有 $v_*(x) = u_\varphi(x)$, $w_*(y) = u_\varphi(y)$ $\forall x, y$. v_*, w_* harmonic $v_* = w_*$ in B

∴ in $B_r(x_0)$, ∃ harmonic f. satisfy $f(x) = u_\varphi(x)$ $\forall x \in B_r(x_0)$

将 $B_r(x_0)$ extends to U, $f = u_\varphi$ in U, ∴ u_φ harmonic since f is harmonic

Th3: Assume $\exists\, W_{x_0}$ subharmonic s.t. $W_{x_0}(x_0)=0$, $W_{x_0}(x)<0$ for $\forall x\in\partial U\setminus\{x_0\}$ $\Leftarrow$ 边界条件 $x_0\in\partial U$

then: $\lim_{y\to x_0} u_\varphi(y)=\varphi(x_0)$

proof: $\varphi\in C(\partial U)$; $\forall\varepsilon>0\ \exists\delta>0$ s.t. $|\varphi(x)-\varphi(x_0)|<\varepsilon$, $\forall |x-x_0|<\delta$ $x\in\partial U$

且: $\exists k>0$ s.t. $\frac{k}{2}|\underset{W_{x_0}\text{记为}}{\underset{\shortparallel}{w(x)}}|\geq M:=\max_U\varphi$, $\forall x\in\partial U\setminus B_\delta(x_0)$ since $|W(x)|>0$, for $x\in\partial U\setminus B_\delta(x_0)$

$$①:\begin{cases}\text{in } \partial U\cap B_\delta(x_0): & \varphi(x_0)-\varepsilon+k\cdot w(x)\leq\varphi(x_0)-\varepsilon\leq\varphi(x)\\ \text{in } \partial U\setminus B_\delta(x): & \varphi(x_0)-\varepsilon+k\cdot w(x)\leq-\varepsilon-2M+\varphi(x_0)\leq\varphi(x)\end{cases}$$

又 $\because \varphi(x_0)-\varepsilon+k\cdot w(x)$ is subharmonic ($kw(x)+$常数)

$\therefore \varphi(x_0)-\varepsilon+k\cdot w(x)\in\mathcal{S}_\varphi$, thus $\because\leq u_\varphi(x_0)$

$$②\begin{cases}\text{in } \partial U\cap B_\delta(x_0) & \varphi(x_0)+\varepsilon-kw(x)\geq\varphi(x_0)+\varepsilon\geq\varphi(x)\geq v(x)\ \ \forall v\in\mathcal{S}_\varphi,\ \text{at } \forall x\in U\\ \text{in } \partial U\setminus B_\delta(x_0) & \varphi(x_0)+\varepsilon-kw(x)\geq 2M+\varphi(x_0)+\varepsilon\geq\varphi(x)\geq v(x)\end{cases}$$

$\therefore \varphi(x_0)+\varepsilon-k\cdot w(x)\geq\sup_{v\in\mathcal{S}_\varphi}v(x)=u_\varphi(x)$

$$①+② \Rightarrow \begin{cases}\varphi(x_0)-\varepsilon+kw\leq u_\varphi\\ \varphi(x_0)+\varepsilon-kw\geq u_\varphi\end{cases}\Rightarrow |\varphi(x_0)-u_\varphi(x)|\leq\varepsilon-kw(x),\ \forall x\in U$$

Let $\varepsilon\to0$. $x\to x_0$ $\lim \varepsilon-kw(x_0)=0=\lim_{x\to x_0}\varphi(x_0)-u_\varphi(x)$

Rmk: goal: $|\varphi(x_0)-u_\varphi(y)|\xrightarrow{y\to x_0}0$, 故而将这个绝对值用"$y\to x_0$时会$\to0$的某个式子"控制: $w(x)$, ε

Th4: $\forall\varphi\in C(\bar U)$. U satisfies the "exterior ball condition".

then: $\exists u\in C^{+\infty}(U)\cap C(\bar U)$, solving: $\begin{cases}\Delta u=0 & U\\ u=\varphi & \partial U\end{cases}$

proof: $G(x,y)=\Phi(x,y)-v(x)$ (不是构造只有一种)

$v:\begin{cases}\Delta v=0\\ v|_{\partial U}=\Phi(x,y)\end{cases}$, then: $\begin{cases}-\Delta G=\delta(x-y)\\ G(\cdot,y)|_{\partial U}=0\end{cases}$ 故而这个G是个符合条件的Green function

Φ不用管∂U满足梯度即可: $u(x)=\int_U\delta(x-y)u(y)dy=\int_U-\Delta G\cdot u(y)dy$ 即可

$-\Delta\Phi(x,y)=\delta(x)$