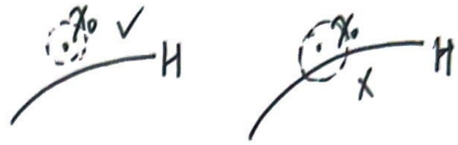
243

Def: $H = [f = \alpha]$ separates A, B if: $f(a) \le \alpha \ \forall a \in A$; $f(b) \ge \alpha, \ \forall b \in B$

Strictly separates A, B if: $\exists \varepsilon > 0$. $\begin{cases} f(a) \le \alpha - \varepsilon, \ \forall a \in A \\ f(b) \ge \alpha - \varepsilon, \ \forall b \in B \end{cases}$

Th: $H = [f = \alpha]$ is closed $\rightleftharpoons$ f is continuous



pf: H is closed $\Rightarrow$ $H^c = [f < \alpha] \cup [f > \alpha]$ open

$\Rightarrow x_0 \in H^c$, $f(x_0) < \alpha$; $\exists r > 0$ s.t. $B(x_0, r) \subseteq H^c$

if $\exists x \in B(x_0, r)$ s.t. $f(x) > \alpha$

let $x_t := (1-t)x_0 + tx$, $t \in [0,1]$ $\quad x_t \in B(x_0, r)$ since convexity

let $t = \dfrac{\alpha - f(x_0)}{f(x) - f(x_0)}$ $\quad \Rightarrow f(x_t) = \alpha \in H$

contradict!

$\Rightarrow B(x_0, r) \subseteq [f = \alpha]^c$ implies $B(x_0, r) \subseteq [f < \alpha]$; $f(x_0) < \alpha$

$\Rightarrow$ i.e. $[f < \alpha]$, similarly $[f > \alpha]$ both open

$\therefore \forall x_0 \in [f < \alpha]$, $\exists r > 0$, s.t. $f(x_0 + rz) \le \alpha \quad \forall z \in X, \|z\| = 1$

$\Rightarrow |f(z)| \le \dfrac{\alpha - f(x_0)}{r} < +\infty, \ \forall z \in X, \|z\| = 1$

$\Rightarrow f$ bounded, thus cts

rmk: $[f < \alpha] \cup [f > \alpha]$ open $\Rightarrow$ $[f < \alpha]$ open $\Rightarrow$ 开球常见所以得出 $f(z) < \cdots < +\infty \ \|z\| = 1$. $\rightleftharpoons$ bdd. = cts

Minkowski functional: X Banach, $C \subseteq X$ is open convex, $0 \in C$; $p(x) := \inf\{\lambda > 0 : \frac{1}{\lambda}x \in C\}$

$$\Rightarrow \begin{cases} (1)\ p \text{ sublinear} \\ (2)\ \exists M > 0 \text{ s.t. } 0 \le p(x) \le M \cdot \|x\| \\ (3)\ C = \{x \in X : p(x) < 1\} \end{cases}$$

core: ① C convex, $\therefore tx + (1-t)y \in C$ for $\forall x, y \in C$, $t \in [0,1]$

let $y = 0 \Rightarrow \forall x \in C$, implies $tx \in C$ $\forall t \in [0,1]$

② C open $\Rightarrow \forall x \in C$, $\exists \bar{t} > 1$ s.t. $\bar{t}x \in C$

①+② $\overset{(*)}{\Rightarrow} \begin{cases} \text{if } \lambda > p,\ \frac{1}{\lambda} < \frac{1}{p}\ \ \frac{1}{\lambda}x \in C \\ \text{if } \frac{1}{\lambda}x \in C,\ p \le \lambda \end{cases}$ 这里不取等号的方不进行

(1): WTS: $p(x+y) \le p(x) + p(y)$

let $p(x) = a$, $p(y) = b$, $p(x+y) \le a+b$

$\Leftarrow \forall \varepsilon > 0$, $p(x+y) \le a+b+\varepsilon$, $\Leftarrow \frac{1}{a+b+\varepsilon}(x+y) \in C$

pf: $\forall \sigma > 0$, $\frac{1}{a+\sigma}x \in C$, $\frac{1}{b+\sigma}y \in C$

$\Rightarrow \frac{a+\sigma}{a+b+2\sigma} \cdot \frac{1}{a+\sigma}x + \frac{b+\sigma}{a+b+2\sigma} \cdot \frac{1}{b+\sigma}y \in C$; i.e. $\frac{1}{a+b+2\sigma}(x+y) \in C$ $\forall \sigma > 0$

$\Rightarrow p(x+y) \le a+b+2\sigma$ $\forall \sigma > 0$

(2): ② $\Rightarrow \frac{1}{\lambda} > 1$ $\therefore \lambda < 1$, $p(x) < 1$ $\forall x \in C$

$\forall x \in C$, $\exists r > 0$ s.t. $B(x, r) \subseteq C$ $\Rightarrow p(x + r\frac{z}{\|z\|}) < 1$ $\forall z \in X$

$\Rightarrow p(z) \le \frac{1-p(x)}{r} \cdot \|z\| \le \frac{1}{r}\|z\|$, r可以从任意$x \in C$得出

(3): ② $\Rightarrow p(x) < 1$, $\forall x \in C$

$\Rightarrow C \subseteq \{x \in X : p(x) < 1\}$

x satisfies $p(x) < 1$ $\overset{(*)}{\Rightarrow} \exists\ p(x) < \lambda < 1$ s.t. $\frac{1}{\lambda}x \in C$

$\Rightarrow \lambda(\frac{1}{\lambda}x) + (1-\lambda) \cdot 0 = x \in C$; $C \supseteq \{x \in X : p(x) < 1\}$

TH: X Banach, $C \subseteq X$ open convex, $x_0 \in X \setminus C \Rightarrow \exists f \in X^*$ s.t. $f(x) < f(x_0)$ $\forall x \in C$

"$\geq$ is OK"

$\Downarrow$

可以分离C和C外任意一点

Pf: C is open convex $\Rightarrow$ define $p(x) = \inf\{\lambda : \frac{1}{\lambda} x \in C\}$ of C on X

let $G := \{t x_0 : t \in \mathbb{R}\} = \mathbb{R} x_0 \Rightarrow G$ is subspace $\subseteq X$; $G \cap C = \phi$

$$g : G \longrightarrow \mathbb{R}$$
$$x = t x_0 \longmapsto g(t x_0) = t$$

$\Rightarrow g(x \in G) = g(t x_0) = t \leq t \cdot p(x_0) = p(x)$; ie. $g(x) \leq p(x)$

$\Rightarrow$ extend $g \in G^*$ to $f \in X^*$

$f(x) \leq p(x) < 1 = f(x_0)$ $\forall x \in C$

rmk: 在$G := \mathbb{R} x_0$上 $g(t x_0) = t$, $g(x) \leq p(x)$ 延拓成 $f \in X^*$

Functional Separation TH: X is real Banach, $A, B \subseteq X$, non-empty, disjoint, convex

$$\Rightarrow \begin{cases} (1)\ A \text{ open}, \Rightarrow \exists f \in X^*, \gamma \in \mathbb{R}' \text{ s.t. } f(a) < \gamma \le f(b), \forall a \in A, b \in B \\ (2)\ A \text{ compact}, B \text{ closed} \Rightarrow \exists f \in X^*, \gamma \in \mathbb{R}', \varepsilon > 0 \text{ s.t. } f(a) \le \gamma - \varepsilon < \gamma + \varepsilon \le f(b)\ \forall a b \end{cases}$$

pf. (1): A, B convex $\Rightarrow A-B$ convex, 记 $A-B=C$

A open $\Rightarrow C$ open

$A\,B$ disjoint $\therefore 0 \notin C$

$\Rightarrow \exists f \in X^*$, s.t. $f(z) \le f(0)\ \forall z \in C$

$\therefore f(a-b) \le 0\ \forall a \in A, b \in B \Rightarrow f(a) \le f(b)\ \forall a \in A, b \in B$

$\Rightarrow \exists \gamma \in \mathbb{R}$ s.t. $\sup(f(A)) \le \gamma \le \inf(f(B))$

if $\gamma = \sup(f(A))$ $\begin{cases} \text{if } \gamma = f(a) \text{ some } a \in A \Rightarrow a \text{ is interior point} \Rightarrow \exists f(a + r\frac{z}{\|z\|}) > f(a) \\ \text{if } \gamma = \sup f(A), \text{ not obtained in } A \Rightarrow f(a) < \gamma\ \forall a\ ✓ \end{cases}$

$f(z) > 0 \Rightarrow r > 0$
$f(z) < 0 \Rightarrow r < 0$

Contradict!

$\therefore f(a) < \gamma \le f(b)\ \forall a \in A, b \in B$

rmk: $\overset{=C}{A-B}$ open convex, $0 \notin C \Rightarrow 0$和C可以分离 $\Rightarrow$ 再说明 $f(a) \ne \gamma$

书上: $C = w_0 + A - B$

$w_0 = a_0 - b_0 \notin C$

let $U = \mathrm{span}(w_0)$, $\phi: U \to \mathbb{R}$

$\phi(t w_0) = t$

$\Rightarrow \phi \le p$. extend ϕ to f

有用! (1) 证明时用了 $A-B$ open

(2): $C := A-B$ convex closed

$0 \notin C \Rightarrow \exists r$ s.t. $B(0,r) \subseteq C^c$

$\Rightarrow B(0,r), C$ are convex, disjoint; $B(0,r)$ is open

$\overset{(1)}{\Rightarrow} \exists f \in X^*$ s.t. $f(x) \ge f(c)$, $\forall x \in B(0,r), \forall c \in C$

$\Rightarrow f(a) - f(b) \le f(rz) = r f(z)$, $\forall z \in X, \|z\| = 1$

$\Rightarrow f(a) - f(b) \le -r\|f\|$, let $\varepsilon = \frac{r}{2} \cdot \|f\|$

$\Rightarrow f(a) + \varepsilon \le f(b) - \varepsilon$, let $\gamma \in [\sup LHS, \inf RHS]$

rmk: 即要说明 $f(a) - f(b) \le$ (严格小于0的数) $\Leftarrow \|f\| \ne 0$ since $d(A,B) > 0$

Distance Functional TH: X is normed space, Y is closed subspace $\subseteq X$, $\forall x_0 \in X \setminus Y$ $\quad \cdots (*)$

$\Rightarrow \exists f \in X^*$ s.t. $\|f\|_{X^*} = 1$, $f(Y) = 0$, $f(x_0) = d(x_0, Y) = \inf\limits_{y \in Y} \|x_0 - y\|$

Pf: let $U := \mathrm{span}(Y \cup \{x_0\})$, convex

def $\phi: U \to \mathbb{F}$, $\phi(y + t x_0) = t \cdot d$ $\Rightarrow \phi(Y) = 0$, $\phi(x_0) = d$

① $|\phi(y + t x_0)| = |t| \cdot d \le |t| \cdot \|\frac{1}{t} y + x_0\| = \|y + t x_0\| \Rightarrow \|\phi\| \le 1$

② $\exists \{y_n\} \subseteq Y$ s.t. $d \le \|x_0 - y_n\| \le d(1 + \frac{1}{n})$

$\therefore |\phi(y_n - x_0)| = d \ge \frac{n}{n+1} \|y_n - x_0\| \ \forall n \Rightarrow \|\phi\| \ge 1$

①, ② $\Rightarrow \|\phi\| = 1$ $\qquad \Rightarrow \phi \to f \in X^*$

rmk: Lem: in $(*)$, $d(x_0, Y)$ can be obtained; i.e. $\exists y \in Y$, $d(x_0, Y) = \|x_0 - y\|$

② $\Leftarrow$ $|\phi(y - x_0)| = d = \|y - x_0\|$

$\xRightarrow{+①}$ $\|\phi\| = 1$ can be obtained

思路：定义$U = \mathrm{span}(Y \cup \{x_0\})$ $\Rightarrow$ 凑出ϕ满足x_0, Y上的值 $\Rightarrow$ 证明$\|\phi\| = 1$

$$\Downarrow$$

TH: X^* separable $\Rightarrow$ X separable

Pf: X^* separable, $S_{X^*} = \{f \in X^* : \|f\| = 1\}$ infinite $\Rightarrow S_{X^*}$ separable

$\Rightarrow \exists \Phi = \{f_n\}_{n=1}^{\infty} \subseteq S_{X^*}$, s.t. $\overline{\Phi} = S_{X^*}$, ie. Φ dense

$\|f_n\| = 1 \ \forall n \Rightarrow \forall n, \varepsilon_n \in (0,1], \exists x_n \in X, \|x_n\| = 1$ s.t. $|f_n(x_n)| \ge 1 - \varepsilon_n$

$\Rightarrow$ let $M = \overline{\mathrm{span}\{x_n\}}$, M separable

if $M \ne X$, $\exists x_0 \in X \setminus M \Rightarrow \exists f \in X^*, \|f\| = 1$ s.t. $f(x_0) = d(x_0, M) > 0$, $f(M) = 0$

$\Rightarrow 1 - \varepsilon_n \le |f_n(x_n)| = |f_n(x_n) - f(x_n)| \le \|f_n - f\|$

$\Rightarrow f \notin \Phi$, $f \notin \overline{\Phi}$ $\Rightarrow$ contradict, since Φ dense!

rmk: S_{X^*}中Φ可数稠密，$\Rightarrow \|f_n\| = 1$，找出$\{x_n\}$，构造可分M

$\Rightarrow$ 反证 $M \ne X$，用Distance functional 找出$f \notin \Phi$，$f \notin \overline{\Phi}$，矛盾

$\|f - f_n\| \ge |\underbrace{f(x_n)}_{=0} - f_n(x_n)| \ge 1 - \varepsilon_n$

TH: $1<p<+\infty$. $(l^p)^* \cong l^q$ if $\frac{1}{p}+\frac{1}{q}=1$ $\longrightarrow$ Similarly, $(L^p)^* \cong L^q$, if $\frac{1}{p}+\frac{1}{q}=1$

use Holder's for integration

pf: define $L: l^p \to (l^q)^*$

$x \mapsto L(x): y \mapsto \sum_{i=1}^{+\infty} x_i y_i$

1. $\|L(x)(y)\| = |\sum_{i=1}^{+\infty} x_i y_i| \le \sum_{i=1}^{+\infty} |x_i y_i| \le \|x\|_{l^p} \cdot \|y\|_{l^q} \Rightarrow \|L(x)\| \le \|x\|_{l^p}$ ... ①

Holder's Equality $\Leftrightarrow (\frac{|x_i|}{\|x\|_p})^p = (\frac{|y_i|}{\|y\|_q})^q$ $\forall i$

$\Rightarrow$ let $|y_i| = (\frac{|x_i|}{\|x\|_p})^{\frac{p}{q}} \times \|y\|_q$, $y_i \cdot x_i \ge 0 \Rightarrow |L(x)(y)| = \|x\|_p \cdot \|y\|_q$

① $\Rightarrow \|L(x)\| = \|x\|_{l^p}$; $\|L\| = 1$

2. L is linear isometry $\Rightarrow$ L is injective

3.1 $\forall f \in (l^q)^*$, if $Lx = f$, then: $\sum_{i=1}^{+\infty} x_i y_i = f(y)$, $\forall y \in l^q$

let $e_1 = (1,0,0,\cdots)$, $e_2 = (0,1,0,\cdots)$, $e_3 = (0,0,1,0,\cdots)$, ...

$\Rightarrow x = (f(e_1), f(e_2), \cdots f(e_n), \cdots)$

rmk: l^q separable $\Rightarrow (l^p)^* \cdots \Rightarrow l^p \cdots$

$\Rightarrow (l^q)^*$

$\therefore 1<p<+\infty$ l^p sepa $\Leftrightarrow (l^p)^*$ sepa

3.2. construct $\{z_n\}_{n=1}^{+\infty}$ $z_{n,i} = \begin{cases} |x_i|^p \div x_i & , i \le n, |x_i| \ne 0 \\ 0 & , \text{otherwise} \end{cases}$

☆.

$\Rightarrow |f(z_n)| = |\sum_{i=1}^{+\infty} z_{n,i} x_i| = \sum_{i=1}^{+\infty} |x_i|^p$

$\Rightarrow \sum_{i=1}^{+\infty} |x_i|^p \le \|f\| \cdot \|z_n\| = \|f\| \cdot (\sum_{i=1}^{n} |x_i|^{q(p-1)})^{1/q} = \|f\| \cdot (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{q}}$

$\Rightarrow \|x_i\|_p \le \|f\|_{l^{q*}}$ $\xRightarrow{3.1 + 3.2}$ $\forall f \in (l^q)^*$, $\exists x \in l^p$ s.t. $L(x) = f$

$\Rightarrow$ L is surjective

rmk: $L: l^p \to (l^q)^*$

$x \mapsto L(x): y \mapsto \sum_{i=1}^{+\infty} x_i y_i$

这个超级常用！背下来！ $\Rightarrow$ Holder's取等条件，说明 $\|L\|=1$

$\Rightarrow$ isometry implies injective

$\Rightarrow \exists x = (f(e_1), f(e_2), \cdots)$ 也很常用！背下来！ $\Rightarrow$ 凑 $|f(z_n)| = |\sum_{i=1}^{+\infty} z_{n,i} \cdot x_i| = \sum_{i=1}^{+\infty} |x_i|^p$，说明 $\exists x \in l^p$

$\Rightarrow$ surjective

TH: Similarly $T_1: l^{+\infty} \to (l^1)^*$, $T_2: L^{+\infty} \to (L^1)^*$ are isometrically isomorphic

$\Rightarrow \begin{cases} l^{+\infty} \cong (l^1)^*, \\ l^1 \ncong (l^{+\infty})^* \text{ since } l^{+\infty} \text{ not separable, } l^p \text{ separable } \forall 1 \le p < +\infty \end{cases}$

$\Downarrow$

counter-example: $E = l^1$, E separable $\not\Rightarrow E^* = l^{+\infty}$ separable

Def: $T \in B(X,Y)$ $\quad T^*: Y^* \to X^*$

$y^* \mapsto T^*y^*: X \to \mathbb{F}$

$x \mapsto T^*y^*x = y^*(Tx)$ (Note: irrelevent to Hilbert adjoint "T^*")

TH: $\|T^*\| = \|T\|$, ie. $T^* \in B(Y^*, X^*)$

Pf: $|T^*y^*x| = |y^*(Tx)| \le \|y^*\| \cdot \|Tx\| \overset{\|x\|=1}{\Longrightarrow} \|T^*y^*\| \le \|y^*\| \cdot \|T\| \cdots$ ①

Support Functional: if $Tx \ne 0$, $\exists\, y^* \in Y^*$, $\|y^*\|=1$ s.t. $|y^*(Tx)| = \|Tx\|$

关键引理

$\Rightarrow |T^*y^*x| = |y^*(Tx)| = \|Tx\|$

let $\{x_n\} \subseteq X$, $\|x_n\|=1$, $\|Tx_n\| > \|T\| - \frac{1}{n}$ 代入 $\Rightarrow \|T^*\| \ge \|T\| \cdots$ ②

①+② $\Rightarrow \|T^*\| = \|T\|$

TH: E, F are Banach, $T \in K(E,F)$ ie. $T \in B(E,F)$, T compact $\Rightarrow T^* \in K(F^*, E^*)$

WTS: $\{f_n\} \subseteq F^*$ bounded $\Rightarrow \exists \{f_{n_k}\} \subseteq \{f_n\}$ s.t. $\{T^*f_{n_k}\}$ converges

$\Rightarrow$ ie. $\forall$ fixed x, $\{T^*f_{n_k}(x)\} = \{f_{n_k}(Tx)\}$ converges

$\Rightarrow$ suffices to show: $\{f_n(TB_x)\}$ compact

$\Rightarrow$ ------ : $\{f_n\}$ compact

Pf: let $\{f_n\} \subseteq F^*$, $\|f_n\|=1$ $\forall n$

$\forall \varepsilon > 0$ $\exists \delta$ s.t. $\|\tilde{y} - y\| < \delta \Rightarrow |f_n(\tilde{y}) - f_n(y)| \le \|f_n\| \cdot \|\tilde{y} - y\| \le \delta < \varepsilon \Rightarrow$ equ-cts $\cdots$ ①

if not totally-bdd, $\exists\, \varepsilon > 0$ s.t. $\exists$ infinite $\{g_n\} \subseteq \{f_n\}$ s.t. $\|g_i - g_n\| > \varepsilon$ $\forall i = 1, 2, \ldots n-1$

$\Rightarrow$ ie. ($\|g_n\| \ge \|g_i\| + \varepsilon$ or $\|g_n\| \le \|g_i\| - \varepsilon$), $\forall i$ contradict!

$\Rightarrow \{f_n\}$ totally bounded $\cdots$ ②

①+② $\Rightarrow$ Ascoli-Arzela's TH: $\{f_n\} \subseteq F^*$ $\|f_n\|=1$, $\{f_n\}_{n=1}^{+\infty}$ compact

$\Rightarrow \exists \{f_{n_k}\} \subseteq \{f_n\}$ s.t. f_{n_k} converges

$\Rightarrow \forall x \in X$, $f_{n_k}(Tx) \to t \in \mathbb{F}$; ie. $T^* \underline{f_{n_k}} \cdot x \to t \in \mathbb{F}$

$\Rightarrow \exists$, $T^* f_{n_k}$ converges, for $\forall \{f_n\} \subseteq F^*$, $\|f_n\|=1$

Rmk: 实际上问题可以化简为 "$\{f_n\} \subseteq F^*$, $\|f_n\|=1 \Rightarrow \{f_n\}$ compact" 这很好证的