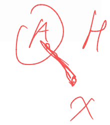
10.1 Show that the result on the existence of a unique closest point in Proposition 10.1 is equivalent to the following statement: if K is a non-empty closed convex subset of a Hilbert space that does not contain zero, then K contains a unique element with minimum norm.



Proposition 10.1 *Let A be a non-empty closed convex subset of a Hilbert space H and let $x \in H \setminus A$. Then there exists a unique $\hat{a} \in A$ such that*

$$\|x - \hat{a}\| = \operatorname{dist}(x, A) := \inf\{\|x - a\| : a \in A\}.$$

proof: • claim: A is closed convex $\rightleftarrows$ $x-A$ is closed convex. for $x \notin A$

"$\Rightarrow$" $\lambda(x-a)+(1-\lambda)(x-b) = x-\lambda a+(1-\lambda)b \in x-A$ since $\lambda a+(1-\lambda)b \in A$ for $\forall \lambda \in [0,1], a,b \in A$ $\Rightarrow$ convex

if t is limit point of $x-A$

$\forall \varepsilon>0$, $\exists\ x-a \in x-A \setminus \{t\}$, $d(x-a,t)<\varepsilon$

in normed space $d(x-a,t) = \|x-a-t\| = d(x-t,a)$ $\therefore x-t$ is limit point of A, $x-t \in A$

$\therefore x-A$ closed convex closed $\Leftarrow$ $t \in x-A$

"$\Leftarrow$" $\lambda(x-a)+(1-\lambda)(x-b) = x-(\lambda a+(1-\lambda)b) \in x-A$ $\forall \lambda \in [0,1], a,b \in A$

$\therefore \lambda a+(1-\lambda)b \in A \Rightarrow$ convex

similary, if t is limit point of A, $x-t$ is limit point of $x-A$

$\therefore x-t \in x-A$ then $t \in A$ $\Rightarrow$ closed

$\therefore A$ is closed convex

• if $x \in H \setminus A$, $\exists$ unique $\hat{a} \in A$ s.t. $\|x-\hat{a}\| = \operatorname{dist}(x,A) = \inf_{a \in A}\|x-a\|$

since $\inf_{a \in A}\|x-a\| = \inf_{v \in x-A}\|v\|$

$\left\{\begin{array}{l} \bar{a} \text{ unique} \Rightarrow \bar{v} = x-\bar{a} \text{ unique } \therefore \text{in } x-A\ \inf_{v \in x-A}\|v\| \text{ is unique, } \inf = \|x-\bar{a}\| \\ \bar{v} \text{ unique} \Rightarrow \bar{a} = x-\bar{v} \text{ unique. } \inf_{a \in A}\|x-a\| \text{ is unique. with } \inf = \|\bar{v}\| \end{array}\right.$ 证毕

uniformly 不保证 existence (T10.7)

10.3 A Banach space is called *strictly convex* if $x, y \in X$, $x \neq y$, with $\|x\| = \|y\| = 1$ implies that $\|x+y\| < 2$. Show that if X is strictly convex and U is a closed linear subspace of X, then given any $x \notin U$, any closed point to x in U (should one exist) is unique.

不一定存在吗?

proof: "唯一" 设 $\|x-\bar{y}\| = \|x-y\|$, $\bar{y} \neq y$.

$\|2x-(y+\bar{y})\|^2 + \|y-\bar{y}\|^2 = 2\|x-\bar{y}\|^2 + 2\|x-y\|^2 = 4\rho^2$

$\|2x-(y+\bar{y})\|^2 = 4\|x-y'\|^2 \geq 4\rho^2$

$\therefore \|y-\bar{y}\|^2 \leq 0 \Rightarrow y = \bar{y}$

(三角不等式只能说明"$\leq$", strict convex 才说明"$<$")

以前证过: 内积定义范数化 平行四边形法则 这里用不了的!

" ② $\|x-\bar{y}\| = \rho$, $\|x-y\| = \rho \Rightarrow \|2x-\bar{y}-y\| < 2\rho$ if $x-\bar{y} \neq x-y$ } strictly convex 用处

而 $\|2x-\bar{y}-y\| \geq 2\rho$ since $\inf\|x-u\| = \rho$

$\Rightarrow$ 只能 $x-\bar{y} = x-y$ $\therefore \bar{y} = y$

10.4 Show that a uniformly convex Banach space (see Exercise 8.10) is strictly convex.

A Banach space is called *uniformly convex* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x - y\| > \varepsilon, \ \|x\| = \|y\| = 1 \qquad \Rightarrow \qquad \left\| \frac{x+y}{2} \right\| < 1 - \delta.$$

Use the parallelogram identity to show that every Hilbert space is uniformly convex.

proof: if not, $\exists\, x \neq y$ $\|x\| = \|y\| = 1$, $\|x+y\| \geq 2$

$\|x-y\| > \varepsilon$ some ε. $\therefore \exists\, \delta > 0$ s.t. $\left\|\frac{x+y}{2}\right\| < 1-\delta \Rightarrow \|x+y\| < 2 - 2\delta < 2$ } contradict

$\therefore$ strictly convex

10.5 Show that if $2 \leq p < \infty$, then for any $\alpha, \beta \geq 0$

$$\alpha^p + \beta^p \leq (\alpha^2 + \beta^2)^{p/2},$$

and deduce, using the fact that $t \mapsto |t|^{p/2}$ is convex, that

$$\left| \frac{a+b}{2} \right|^p + \left| \frac{a-b}{2} \right|^p \leq \frac{1}{2}(|a|^p + |b|^p), \qquad a, b \in \mathbb{C}.$$

Hence obtain Clarkson's first inequality

$$\left\| \frac{f+g}{2} \right\|_{L^p}^p + \left\| \frac{f-g}{2} \right\|_{L^p}^p \leq \frac{1}{2}(\|f\|_{L^p}^p + \|g\|_{L^p}^p), \qquad f, g \in L^p. \tag{10.3}$$

?

(The same inequality also holds in ℓ^p, $2 \leq p < \infty$, by a similar argument.)

proof ①: $\alpha^p + \beta^p \leq (\alpha^2+\beta^2)^{p/2} \Leftarrow A^{p/2} + B^{p/2} \leq (A+B)^{p/2}$, $A = \alpha^2$ $B = \beta^2$

设 $f(x) = (A+x)^{p/2} - A^{\frac{p}{2}} - x^{p/2}$

$f'(x) = \frac{p}{2}(A+x)^{\frac{p}{2}-1} - \frac{p}{2}x^{\frac{p}{2}-1} = \frac{p}{2}\left((A+x)^{\frac{p}{2}-1} - x^{\frac{p}{2}-1}\right)$, $\frac{p}{2}-1 \geq 0$.

$\therefore f'(x) \geq 0$ $\forall x \geq 0$,

$\therefore \min f = f(0) = 0$ $\therefore f(x) \geq 0$ $\forall x$

$\Rightarrow (A+B)^{p/2} \geq A^{p/2} + B^{p/2} \Rightarrow \alpha^p + \beta^p \leq (\alpha^2+\beta^2)^{p/2}$

②: $\left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \leq \left|\frac{a^2+b^2}{2}\right|^{\frac{p}{2}}$, by ①

$\left|\frac{a^2+b^2}{2}\right|^{\frac{p}{2}} \leq \frac{1}{2}(a^2)^{\frac{p}{2}} + \frac{1}{2}(b^2)^{\frac{p}{2}} = \frac{1}{2}(|a|^p + |b|^p)$

} $\Rightarrow \left|\frac{a+b}{2}\right|^p + \left|\frac{a-b}{2}\right|^p \leq \frac{1}{2}(|a|^p + |b|^p)$

③ WTS: $\|\cdot\|^{\frac{p}{2}}: f \mapsto \|f\|^{\frac{p}{2}}$ convex

设 $\|\cdot\|$ 为 F_1, $(\cdot)^{\frac{p}{2}}$ 为 F_2; 由 norm 的定义知 $F_1: f \to \mathbb{K}$ convex

$p \geq 2$ $\therefore F_2: \mathbb{K} \to \mathbb{K}$ convex }

$\therefore \forall \lambda \in [0,1]$ $F_2 \circ F_1(\lambda f_1 + (1-\lambda) f_2) \leq F_2(\lambda F_1(f_1) + (1-\lambda) F_1(f_2)) \leq \lambda F_2 \circ F_1(f_1) + (1-\lambda) F_2 \circ F_1(f_2)$, $\forall f_1, f_2$

$\Rightarrow F_2 \circ F_1 = \|\cdot\|^{\frac{p}{2}}$ convex too, 即可以仿照 "$t \mapsto \|t\|^{\frac{p}{2}}$ convex 证②式" 的过程

$\therefore$ ②中的 $|\cdot|$ 可以换成 $\|\cdot\|_{L^p}$, a, b 用 f, g 代

10.6 Use Clarkson's first inequality to show that L^p is uniformly convex for all $2 \le p < \infty$.

Clarkson's second equality is valid for the range $1 < p \le 2$: for all $f, g \in L^p$ we have

$$\left\| \frac{f+g}{2} \right\|_{L^p}^q + \left\| \frac{f-g}{2} \right\|_{L^p}^q \le \left(\frac{1}{2} \|f\|_{L^p}^p + \frac{1}{2} \|g\|_{L^p}^p \right)^{1/(p-1)}$$

where p and q are conjugate; this is much less straightforward to prove than Clarkson's first inequality (Clarkson, 1936). Use this inequality to show that L^p is uniformly convex for $1 < p \le 2$. (The same arguments work in the ℓ^p spaces for $1 < p < \infty$.)

proof: if $\|f\|_{L^p} = \|g\|_{L^p} = 1$ then $\|\frac{f+g}{2}\|_{L^p}^q + \|\frac{f-g}{2}\|_{L^p}^q \le 1$

∴ if $\|f-g\|_{L^p} > \varepsilon$, then $\|\frac{f-g}{2}\|_{L^p}^q > (\frac{\varepsilon}{2})^q$

$\Rightarrow \|\frac{f+g}{2}\| \le \sqrt[q]{1-(\frac{\varepsilon}{2})^q}$ ∴ exist $\delta = 1 - \sqrt[q]{1-(\frac{\varepsilon}{2})^q}$, $\|\frac{f+g}{2}\|_{L^p} \le 1-\delta$

10.7 In this exercise we show that if X is a uniformly convex Banach space and K is a closed convex subset of X that does not contain 0, then K has a unique element of minimum norm. Let $(k_n) \in K$ be a sequence such that $k_n \to \inf_{k \in K} \|k\| =: d$

(i) Set $x_n = k_n / \|k_n\|$ and use the convexity of K to show that

$$\|\tfrac{1}{2}(x_n + x_m)\| \ge \frac{d}{2} \left(\frac{1}{\|k_n\|} + \frac{1}{\|k_m\|} \right);$$

(ii) deduce that $\|\frac{1}{2}(x_n + x_m)\| \to 1$ as $\max(n, m) \to \infty$;

(iii) use the uniform convexity of X to show that (x_n) is a Cauchy sequence; and

(iv) finally, use the fact that $k_n = \|k_n\| x_n$ to show that (k_n) is also Cauchy.

This result implies that if K is a closed convex subset of a uniformly convex Banach space and $x \notin K$, then there exists a unique closest point to x in K (just consider $K' = K - x$; see the solution of Exercise 10.1). (Lax, 2002)

(i): 设 $\inf_{k \in K} \|k\| = r$:

由inf定义, ∃ $\{k_n\}_{n=1}^{+\infty}$ s.t. $\|k_n\| < r + \frac{1}{n}$, that is $\|k_n\| \to r$

let: $x_n = \frac{k_n}{\|k_n\|}$, $x_m = \frac{k_m}{\|k_m\|}$, $\|x_n\| = \|x_m\| = 1$

∴ by convexity: $k_n \in K$, $k_m \in K$, then $\frac{\|k_m\| \cdot \|k_n\|}{\|k_m\| + \|k_n\|} \left(\frac{k_n}{\|k_n\|} + \frac{k_m}{\|k_m\|} \right) \in K$, 凑$k_n$和$k_m$的系数之和=1

where $\frac{\|k_m\|}{\|k_m\| + \|k_n\|} + \frac{\|k_n\|}{\|k_m\| + \|k_n\|} = 1$,

$\| \frac{\|k_m\| \cdot \|k_n\|}{\|k_m\| + \|k_n\|} (x_n + x_m) \| \ge r \Rightarrow \|x_n + x_m\| \ge r \left(\frac{1}{\|k_n\|} + \frac{1}{\|k_m\|} \right)$ 证毕.

(2): X is uniformly convex $\Rightarrow$ Strictly convex (T10.5)

$\|x_m\|=\|x_n\|=1$ $\therefore \|x_m+x_n\|<2$ since $x_m \neq x_n$

$\|x_m+x_n\| \geq r\left(\frac{1}{\|k_n\|}+\frac{1}{\|k_m\|}\right) \geq \frac{r}{r+\frac{1}{n}}+\frac{r}{r+\frac{1}{m}}=2-\left(\frac{1}{1+nr}+\frac{1}{1+mr}\right)$

$\Rightarrow \therefore$ as $\min(m,n)\to+\infty$, $\|x_m+x_n\|\to 2$ 为什么是$\max(m,n)\to+\infty$？我觉的要 $\min(m,n)$

(3): X is uniformly convex

$\therefore \forall \varepsilon>0$ $\|x_i-x_j\|>\varepsilon$, $\|x_i\|=\|x_j\|=1 \Rightarrow \exists \delta$, $\left\|\frac{x_i+x_j}{2}\right\|<1-\delta$ ... by def

as $m,n\to+\infty$, $\|x_m+x_n\| \geq 2-\left(\frac{1}{1+nr}+\frac{1}{1+mr}\right)$ ... by (1)

$\exists N$ s.t. $m,n\geq N$, $\frac{1}{1+nr}+\frac{1}{1+mr}<2\delta \Rightarrow \|x_m-x_n\|<\varepsilon$ --- 否则与def矛盾

$\Rightarrow \forall\varepsilon>0$, $\exists N$ s.t. $m,n\geq N$, then $\|x_m-x_n\|<\varepsilon$

$\therefore \{x_n\}_{n=1}^{+\infty}$ is Cauchy sequence in X

(4): $k_n=\|k_n\|\cdot x_n$

$\|k_n-k_m\|=\|\,\|k_n\|\cdot x_n-\|k_m\|\cdot x_m\|=\|(\|k_n\|-\|k_m\|)\cdot x_n+\|k_m\|(x_n-x_m)\|$

$\leq\|(\|k_n\|-\|k_m\|)\cdot x_n\|+\|k_m\|\cdot\|x_n-x_m\|$

$\leq \max\left(\frac{1}{m},\frac{1}{n}\right)\cdot\|x_n\|+\left(\frac{1}{m}+r\right)\cdot\varepsilon$, $\|x_n\|=1$

$\leq \frac{1}{N}+(r+1)\varepsilon$ when $m,n\geq N$, $\|x_n-x_m\|<\varepsilon$ N from (3)

$\leq (r+2)\varepsilon$ let $N^*=\max(N,\lceil\frac{1}{\varepsilon}\rceil)$, substitute N with N^*

$\therefore \forall\xi>0$, let $\varepsilon<\frac{1}{r+2}\xi$, then obtain N, s.t. $\|x_m-x_n\|<\varepsilon$ for $\forall m,n\geq N$.

then substitute this N with $N^*=\max(N,\lceil\frac{1}{\varepsilon}\rceil)$, $\lceil\frac{1}{\varepsilon}\rceil=\text{ceil}(\frac{1}{\varepsilon})$

$\Rightarrow \forall m,n\geq N^*$, $\|k_m-k_n\|\leq(r+2)\varepsilon<\xi$

$\therefore \{k_n\}_{n=1}^{+\infty}$ is Cauchy, and in complete set U, $k_n\to k_0\in U$, $\|k_0\|=r$