8.3 Show that if H and K are Hilbert spaces with inner products $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_K$, respectively, then $H \times K$ is a Hilbert space with inner product

$$((x,\xi),(y,\eta))_{H\times K} := (x,y)_H + (\xi,\eta)_K.$$

T8.3 $((x,\xi)(x,\xi))_{H\times K} = (x,x)_H + (\xi,\overset{=\xi}{y})_K \geq 0$, and equals to $0 \Leftrightarrow x=0, \xi=0$. ---①

$(a(x,\xi),(y,\eta))_{H\times K} = ((ax,a\xi),(y,\eta)) = (ax,y)_H + (a\xi,\eta)_K = a(x,y)_H + a(\xi,\eta)_K$

这个不是内积!就是H×K的元素! $= a((x,\xi),(y,\eta))_{H\times K}$ ---②

$\therefore a(x,\xi) = (ax, a\xi)$

$((x,\xi),(y,\eta)) + ((x',\xi'),(y,\eta)) = (x,y)+(x',y)+(\xi,\eta)+(\xi',\eta) = (x+x',y)+(\xi+\xi',y)$

$= ((x+x',\xi+\xi'),(y,\eta))$ ---③

$((x,\xi),(y,\eta)) = (x,y)+(\xi,\eta) = \overline{(y,x)}+\overline{(y,\xi)}$

$= \overline{(y,x)+(y,\xi)} = \overline{((y,y),(x,\xi))}$ ---④

①–④ $((x,\xi),(y,\eta))$ is well-defined inner product

8.4 Let $T : H \to K$ be a linear surjective mapping between two real Hilbert spaces. Use the polarisation identity (8.8) to show that

$$(Tx, Ty)_K = (x,y)_H \qquad \text{for every } x, y \in H \tag{8.12}$$

if and only if $\|Tx\|_K = \|x\|_H$ for every $x \in H$. (In this case we say that T is *unitary*.) (Young, 1988)

T8.4. WTS: $(Tx,Ty)_K = (x,y)_H \ \forall x,y \in H$, $\Leftrightarrow \|Tx\|_K = \|x\|_H \ \forall x$

"⇒": let $y=x$. $(Tx,Tx)=(x,x)$ $\therefore \|Tx\|=\|x\|$ ($T\neq 0$ since surjective)

"⇐": $(Tx,Ty) = \frac{1}{4}\left(\|Tx+Ty\|^2 - \|Tx-Ty\|^2\right) = \frac{1}{4}\left(\|x+y\|^2-\|x-y\|^2\right) = (x,y)$

if $\|Tx\|=\|x\|$ then $Tx+Ty = T(x+y)$, $\|Tx+Ty\| = \|x+y\|$

$Tx-Ty = T(x-y)$, $\|Tx-Ty\| = \|x-y\|$ since T is linear, surjective

$\therefore (Tx,Ty)_K = (x,y)_H$

8.7 If $\|\cdot\|$ is a norm on a vector space X induced by an inner product show that it satisfies Apollonius's identity,

$$\|z-x\|^2 + \|z-y\|^2 = \frac{1}{2}\|x-y\|^2 + 2\left\|z-\frac{1}{2}(x+y)\right\|^2$$

T8.7. RHS $= \frac{1}{2}\|x-y\|^2 - 2\|z+\frac{1}{2}(x+y)\|^2 + 2\|z+\frac{1}{2}(x+y)\|^2 + 2\|z-\frac{1}{2}(x-y)\|^2$

$= \frac{1}{2}\|x-y\|^2 - 2\|z\|^2 - \frac{1}{2}\|x+y\|^2 - 2(z,x+y) + 2\left(\|z+\frac{1}{2}(x+y)\|^2 + \|z-\frac{1}{2}(x-y)\|^2\right)$

$= \frac{1}{2}\|x-y\|^2 - \frac{1}{2}\|x+y\|^2 - 2\|z\|^2 - 2(z,x+y) + 4\|z\|^2 + \|x+y\|^2$

$= \frac{1}{2}\|x-y\|^2 + \frac{1}{2}\|x+y\|^2 + 2\|z\|^2 - 2(z,x+y)$

$= \|x\|^2 + \|y\|^2 + \|z\|^2 + \|z\|^2 - 2(z,x) - 2(z,y) = \|z-x\|^2 + \|z-y\|^2 = $ LHS

9.2 Show that $(x, y) = 0$ if and only if (i) $\|x + \alpha y\| \geq \|x\|$ for every $\alpha \in \mathbb{K}$ or (ii) $\|x + \alpha y\| = \|x - \alpha y\|$ for every $\alpha \in \mathbb{K}$. (Giles, 2000)

T9.2 WTS $(x,y)=0 \rightleftharpoons \|x+\alpha y\| \geq \|x\| \ \forall \alpha \in \mathbb{K}$ ... (2)

或 $\|x+\alpha y\| = \|x-\alpha y\| \ \forall \alpha \in \mathbb{K}$ ... (1)

没说 $\mathbb{K}$ 是 $\mathbb{R}$! $(x,y)+\overline{(x,y)}$

(1): $(x,y)=0 \rightleftharpoons 2\alpha(x,y) = 2\bar{\alpha}(y,x) = 0 \ \forall \alpha$. $\rightleftharpoons \|x+\alpha y\|^2 - \|x-\alpha y\|^2 = 2\bar{\alpha}(x,y) + 2\alpha(y,x) = 0 \ \forall \alpha$

$\bar{\alpha}, \alpha$

"$\Rightarrow$" obv, "$\Leftarrow$": $\|x+\alpha y\|^2 = \|x-\alpha y\|^2$. $\therefore \alpha(x,y)+\alpha(y,x) = -\alpha(x,y)-\alpha(y,x)$

$\therefore \alpha(x,y) = \alpha(y,x) = 0$.

(2): $(x,y)=0 \Rightarrow \|x+\alpha y\|^2 = \|x\|^2 + |\alpha|^2\|y\|^2 \geq \|x\|^2 \therefore \|x+\alpha y\| \geq \|x\|$

$\|x+\alpha y\|^2 - \|x\|^2 = \alpha(x,y) + \alpha(y,x) \geq 0 \ \forall \alpha$

if $(x,y) \neq 0$, say $(x,y) = \lambda$. $\alpha(x,y) + \alpha(y,x) = \alpha(\lambda + \bar{\lambda}) \Rightarrow \exists \alpha$ s.t. $\alpha(\lambda+\bar{\lambda}) < 0$. $\therefore (x,y) = 0$

9.3 Use Bessel's inequality to show that if $\{e_j\}_{j=1}^{\infty}$ is an orthonormal set in an inner-product space V, then for any $x \in V$

$$|\{j : |(x, e_j)| > M\}| \leq \frac{\|x\|^2}{M^2}.$$

T9.3. WTS: $\{e_i\}_{i=1}^{+\infty}$ orthonormal, $\Rightarrow |\{i : |(x,e_i)| > M\}| \leq \frac{\|x\|^2}{M^2}$

Bessel's: $\sum_{i=1}^{+\infty} |(x,e_i)|^2 \leq \|x\|^2$

if $|\{i : |(x,e_i)| > M\}| = n$, $\|x\|^2 \geq n \cdot M^2 + \sum_{\text{other } i}^{+\infty} |(x,e_i)|^2$

$\therefore n \leq \frac{\|x\|^2 - \sum_{\text{other}}^{+\infty} |(x,e_i)|^2}{M^2} \leq \frac{\|x\|^2}{M^2}$

9.5 Show that if $\{e_j\}_{j=1}^{\infty}$ is an orthonormal basis for H then

$$(u, v) = \sum_{j=1}^{\infty} (u, e_j)(e_j, v)$$

for every $u, v \in H$. (This is a more general version of Parseval's

T9.5. WTS: $(u,v) = \sum_{i=1}^{+\infty} (u,e_i)(e_i,v)$

$$(u,v) = \left(\sum_{i=1}^{+\infty} (u,e_i)e_i, \sum_{t=1}^{+\infty} (v,e_t)e_t\right)$$

$$= \lim_{I \to +\infty} \lim_{T \to +\infty} \left(\sum_{i=1}^{I} (u,e_i)e_i, \sum_{t=1}^{T} (v,e_t)e_t\right)$$

$$= \lim_{I \to +\infty} \lim_{T \to +\infty} \sum_{t=1}^{I} \left((u,e_i), (v,e_t)\right) \text{ where } t=i, \text{ otherwise } (e_i,e_t)=0.$$

$$= \lim_{I \to +\infty} \sum_{t=1}^{I} (u,e_t)(e_t,v)$$

$$= \sum_{t=1}^{+\infty} (u,e_t)(e_i,v) \text{ 证毕}$$

9.8 Proposition 9.14 shows that if $\{e_j\}_{j=1}^{\infty}$ is an orthonormal set, then it is a basis if its linear span is dense in H. This exercise gives an example to show that this is not true without the assumption that $\{e_j\}$ is orthonormal.

Let $(e_j)_{j=1}^{\infty}$ be an orthonormal sequence that forms a basis for a Hilbert space H. Set

$$f_n = \sum_{j=1}^{n} \frac{1}{j} e_j.$$

Show that the linear span of $\{f_j\}$ is dense in H, but that $\{f_j\}$ is not a basis for H. (Show that $x = \sum_{j=1}^{\infty} j^{-1} e_j$ cannot be written in terms of the $\{f_j\}$.)

T9.8 $E=\{e_i\}_{i=1}^{+\infty}$ orthonormal, if span(E) dense in H, $\Rightarrow$ E is basis

if E not orthonormal ... $\not\Rightarrow$

例: E orthonormal, $F=\{f_n\}_{n=1}^{+\infty}$, $f_n=\sum_{i=1}^{n}\frac{1}{i}e_i$ $\Rightarrow$ span(F) dense in H, but not basis

☆! 无穷项 不一定相等

(1) if $v \neq f_n$, $v \in H$, then $v = a_1e_1 + a_2e_2 + \cdots$

if $\bar{v} = \beta_1 f_1 + \cdots \beta_n f_n + \cdots = \sum_{i=1}^{+\infty}\beta_i e_1 + \frac{1}{2}\sum_{i=2}^{+\infty}\beta_i e_2 + \cdots \frac{1}{n}\sum_{i=n}^{+\infty}\beta_i e_n + \cdots$

want $\bar{v} \to v$, $\Rightarrow \beta_1 = a_1 - 2a_2$, $\beta_2 = 2a_2 - 3a_3$, $\cdots \beta_n = n\cdot a_n - (n+1)a_{n+1} \cdots$

~~$\sum \beta_i = a_1$, 由极限在内积上的传导知 $\bar{v} \to v$ as $n \to +\infty$~~

(2): $v = \sum_{i=1}^{+\infty}\frac{1}{i}e_i$ $\Rightarrow$ if $\bar{v} = v$, then $\beta_1 = 1 - 2\times\frac{1}{2} = 0$, $\cdots \beta_n = n\times\frac{1}{n} - (n+1)\frac{1}{n+1} = 0 \cdots$ 矛盾了!

$$\bar{v}_m = \sum_{i=1}^{+\infty}\beta_i e_1 + \cdots \frac{1}{m}\sum_{i=m}^{+\infty}\beta_i e_m$$

$\bar{v}_m \to v$ since $\beta_n = n\cdot a_n - (n+1)a_{n+1}$

$$\Rightarrow \|v - \bar{v}_m\| = \left\|\frac{1}{m+1}\sum_{i=m+1}^{+\infty}\beta_i e_{m+1} + \frac{1}{m+2}\sum_{i=m+2}^{+\infty}\beta_i e_{m+2} + \cdots\right\| \to 0$$

(由定理: $\{p_n\}$ converges iff

$\forall \varepsilon > 0, \exists M$, s.t. $\forall m > n \geq M$. $\sum_{i=n}^{m}|p_i| < \varepsilon$)