Review for final

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1 General concepts

- Multi-dimensional integration by parts or the divergence theorem.
- Classical solution (for heat equation, elliptic equation, parabolic equation, first-order equation).
- Boundary conditions: periodic, Dirichlet, Neumann, mixed.
- Well-posedness: existence, uniqueness and stability.
- Weak solutions.

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2 First-order PDEs and Hamilton–Jacobi equations

Reference: Evans 2.1, 3.2 - 3.4, 10.1 - 10.3

- Method of characteristics: characteristic ODE, Hamilton's ODE.
- Intersection of characteristics: examples of intersection and non-intersection, relation between the occurrence of intersection and the well-posedness of the equation.
- Important nonlinear equations: Burgers equation, Hamilton–Jacobi equation.
- representation formula: Hopf-Lax formula, variational problem, properties of the variational solutions such as principle of dynamic programming, Lipchitz continuity, etc.
- Entropy condition: conservation laws, Rankien-Hugoniot conditions.
- Viscosity solutions: definitions, vanishing viscosity limit

3 Heat equation

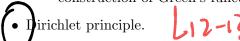
Refrence: Zhou

- Solution of the equation.
 - On the whole space.
 - * Find fundamental solutions via Fourier transform or scaling invariance.
 - * Solution to Cauchy problem via *convolution*, meaning of the initial condition, smoothness of the solutions

- * Extend to other domains via symmetry: half-space, periodic intervals.
- On bounded domains.
 - * Principle of superposition, separation of variables, the Sturm-Liouville/eigenvalue problem.
 - * In 1d: Fourier series, boundary conditions, Green's function.
- Inhomogeneous problem.
 - * Duhamel's principle.
 - * On the whole space: using fundamental solution.
 - * On bounded intervals: using Fourier series.
- Uniqueness and stability
 - Maximum principle, generalization to other parabolic operators, comparison principle.
 - Uniqueness via the maximum/comparison principle.
 - $-L^{\infty}$ -stability via the maximum/comparison principle.
 - Energy estimates and L^2 -stability.

4 Elliptic equation

- Solution theory.
 - Fundamental solutions, solutions to the Laplace's equation, form of the fundamental solutions in \mathbb{R}^d .
 - Green's function, solutions to the Poisson's equation. Poisson's kernel in half-space or balls, properties such as smoothness, symmetry, decay, etc.
 - Meaning of the boundary condition
- Harmonic functions.
 - Mean-value property.
 - Weak/strong maximum principle.
 - $-\mathcal{C}^{\infty}$ -smoothness of harmonic functions.
 - Derivative estimate, Liouville theorem.
 - Perron's method, sub-harmonic functions, local uniform convergence of harmonic functions, construction of Green's function.



- Variational problem, convexity and uniqueness of minimizers, first variation, Euler-Lagrange equation.
- Weak derivatives, Sobolev spaces $H^k(U)$, $H^k_0(U)$, weak solutions, weak convergence, weak compactness.
- Poincare's inequality.
- Free boundary condition, compatible condition.
- Stability.
 - Energy estimate, L^2 -stability.
 - Comparison principle and L^{∞} -stability.

5 Wave equation

- Solutions.
 - D'Alembert's formula, finite speed of propagation.
 - Wave equation on half-line with reflecting/absorbing boundary condition.
 - Method of spherical means, method of descent.
 - Duhamel's principle.
 - Separation of variables, resonance.
- Stability.
 - Energy method.
 - Domain of dependence.

6 Linear evolution equations

L17-18

Reference: Evans 7.1.1, 7.1.2, 9.2.1.

- Definition of the weak solutions: $\bar{u} \in L^2(0,T;H^1_0(U)), \bar{u}' \in L^2(0,T;H^{-1}(U)).$
- Energy estimates, uniqueness.
- Galerkin approximation, existence of weak solutions.
- Fixed point method.