L20

Th1: legendre transform and duality

$$\begin{cases} L: q \mapsto L(q) \text{ convex}, \ \lim_{|q|\to+\infty} \frac{L(q)}{|q|} = +\infty, \ q \in \mathbb{R}^n \\ H(p) = L^*(p) = \sup_q \{p\cdot q - L(q)\} \end{cases}$$

then (1): $p \mapsto H(p)$ is also convex, $\lim_{|p|\to+\infty} \frac{H(p)}{|p|} = +\infty$ — obv

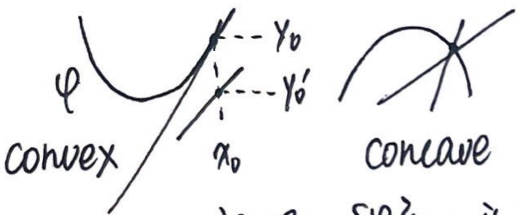
(2): $H = L^*$, then $L = H^*$

(1): Lem1:(凸函数的等价定义): φ is convex 若: $\varphi(x_0) \geq y_0$ some x_0, y_0; then $\exists$ line l:

s.t. $\varphi(x) \geq l(x)$ $\forall x$, with $l(x_0) = y_0$

$\Downarrow$

割线(切线)在外围



Lem2: $\{\varphi_a\}_{a\in A}$ is family of convex functions, then $\varphi^* = \sup_{a\in A} \varphi_a(x)$ is convex

proof: if $\varphi^*(x_0) \geq y_0$,

$\forall \varepsilon > 0$. $\varphi^*(x_0) > y_0 - \varepsilon$, $\exists\ \varphi_a(x_0) \geq y_0 - \varepsilon$

$\therefore \exists$ line $L(x_0) = y_0 - \varepsilon$, $L(x) \leq \varphi_a(x) \leq \varphi^*(x)$

$\Downarrow$

Q: How to let $\varepsilon \to 0$, and show such line exist?

(or show lem1 $\Leftarrow$ $\varphi(x_0) > y_0$)

proof (1): $\exists\ \{\varphi_q\}_{q\in A}$ $\varphi_q(p) = p\cdot q - L(q)$

$\partial_p^2 \varphi_q(p) = 0 \geq 0$, $\therefore \varphi_q$ convex $\forall q \in \mathbb{R}^n$,

then $H(p) = \sup\{\varphi_q\}$ convex by lem

proof (2): $H(p) = \sup_q \{p\cdot q - L(q)\}$ $\forall p, q \in \mathbb{R}^n$

$\therefore H(p) \geq p\cdot q - L(q) \Rightarrow L(q) \geq -H(p) + p\cdot q$ $\forall p \in \mathbb{R}^n$

$\Rightarrow L(q) \geq H^*(p)$ ... ①

$H^*(q) = \sup_p \{p\cdot q - H(p)\}$

$= \sup_p \{p\cdot q - \sup_r \{p\cdot r - L(r)\}\}$

sup$_p$和inf$_r$无关∴可以提取出

$= \sup_p \inf_r \{p(q-r) + L(r)\}$

$\geq \inf_r \{a(r-q) + L(r)\}$

: How to show "a" exist": L is convex. $\therefore$ 若 $L(x_0) \geq y_0$ for (x_0, y_0)

$\Rightarrow \begin{cases} \exists \text{ line } y(x) = a(x - x_0) + y_0 \\ \text{s.t. } a(x - x_0) + y_0 \leq L(x) \ \forall x \in \mathbb{R}^n \end{cases}$

$\Rightarrow$ let $L(x_0) = y_0$

$\therefore L(x) + a(x_0 - x) \geq y_0 = L(x_0)$; 对应到: $x_0 := q$, $x := r$

$\geq L(q)$ ... ②

①+② $\Rightarrow H^*(q) = L(q)$. ($H(p) = L^*(p)$ by definition)

variation method via dynamic program

if: $\partial_t u + H(\partial_x u, x, t) = 0$, $u(0,x) = g(x)$ ⋯①

then: $u(t,x) = \inf\{\int_0^t L(\dot{w}(s), w(s), s)ds + g(w(0))\}$ ⋯② , $x = w(s)$ $q = \dot{w}(s)$

$\begin{cases} H = L^* = \sup_q\{p\cdot q - L(q)\}, \ L \text{ convex}, \ \frac{L(q)}{|q|} \to +\infty \\ ① \rightleftharpoons ②, \end{cases}$ refer L_q & Th1 L20. "所选取的特征线,表现时间趋势"

∵ refer to Th5, L20 case $L(q,x,t) = L(q)$, $H(\partial_x u, x, t) = H(\partial_x u)$

dynamic program (物理意义: 物体会沿能量下降最少的轨迹运动)

Th2: Hopf-Lax formula

if: $L(q,x,t) = L(q)$ q为时间t,空间x以外的状态量

then: $u(t,x) = \min_y\{t\cdot L(\frac{x-y}{t}) + g(y)\}$, g linear growth , $x \in U$

Q: 不同的w效果不同呀??

w为什么可以任选?可以吗?

proof2: fix (t,x), $w(s) = z + \frac{s}{t}(x-y)$, $0 \le s \le t$ $y = w(0)$ 在之后会赋值

$J_t[w] = \int_0^t L(\dot{w}(s), w(s), s)ds = \int_0^t L(\dot{w}(s))ds$ (since $L(q,x,t) = L(q)$)

$= \int_0^t L(\frac{x-z}{t})ds = L(\frac{x-z}{t}) \times t$

∴ $u(t,x) \le L(\frac{x-z}{t})\cdot t + g(w(0))$ by def ⋯①

$L(\frac{x-w(0)}{t}) = L(\frac{1}{t}\int_0^t \dot{w}(s)ds) \le \frac{1}{t}\int_0^t L(\dot{w}(s))ds$ ⇐ "convex"; $f(x_{mean}) \le mean f$; ∵ Jensen

∴ $L(\frac{x-w(0)}{t})\cdot t + g(w(0)) \le \int_0^t L(\dot{w}(s))ds + g(w(0))$ $\forall t$

∴ $L(\frac{x-w(0)}{t})\cdot t + g(w(0)) \le \inf\{\int_0^t L ds + g\} = u(t,x)$ ⋯②

⇒ ①+② , let $w(0) = y \in \mathbb{R}^n$ 任何y均可,即计时开始点不重要

$\inf = \min$ since the domain of y is compact

Th3: $u(t,x) = \inf_y\{(t-s)\cdot L(\frac{x-y}{t-s}) + u(s,y)\}$ $t \ge s$ 在Th2中说明了$s=0$的情况 含义: A→B→C 能耗最少 A→B, B→C 均最少

proof3: $u(t,x) \le t\cdot L(\frac{x-z}{t}) + g(z)$ $\forall z \in \mathbb{R}^d$, $\forall y \in \mathbb{R}^d$

$\le (t-s)\cdot L(\frac{x-y}{t-s}) + s\cdot L(\frac{y-z}{s}) + g(z)$

⋯①

$L(\frac{x-z}{t}) \le \frac{t-s}{t}\cdot L(\frac{x-y}{t-s}) + \frac{s}{t}\cdot L(\frac{y-z}{s})$

since $(\frac{x-z}{t}) = \frac{t-s}{t}\cdot(\frac{x-y}{t-s}) + \frac{s}{t}(\frac{y-z}{s})$ ∵ Jensen

from Th2: $u(s,y) = \inf_z\{s\cdot L(\frac{y-z}{s}) + g(z)\}$

∴ $\forall \varepsilon > 0$, $\exists z \in \mathbb{R}^n$ st. $u(s,y) \le s\cdot L(\frac{y-z}{s}) + g(z) \le u(s,y) + \varepsilon$ ⋯②

⇒ ①+② let "z" in ① = "z" in ② ; y, s 也一样 (不过②中z固定, y, s 均 Arbitarary)

∴ $u(t,x) \le (t-s)\cdot L(\frac{x-y}{t-s}) + s\cdot L(\frac{y-z}{s}) + g(z) \le (t-s)\cdot L(\frac{x-y}{t-s}) + u(s,y) + \varepsilon$

∴ $u(t,x) \le \inf\{(t-s)\cdot L(\frac{x-y}{t-s}) + u(s,y)\}$, $\varepsilon \to 0$, ⋯③

$(t-s)\cdot L\left(\frac{x-y}{t-s}\right) + u(s,z) \leq (t-s) L\left(\frac{x-y}{t-s}\right) + s\cdot L\left(\frac{x-z}{s}\right) + g(z) \quad \forall z \in \mathbb{R}^n$ by Th

$= t\cdot L\left(\frac{x-z}{t}\right) + g(z)$

$\leq u(t,x) + \varepsilon$ let $y = \frac{s}{t}\cdot x + (1-\frac{s}{t})\cdot z$, Jenson取等 为什么? $\frac{x-y}{t-s} = \frac{x-z}{s} = \frac{x-z}{t}$

$\therefore u(t,x) \geq (t-s)\cdot L\left(\frac{x-y}{t-s}\right) + u(s,z) - \varepsilon$

$\therefore u(t,x) \geq \inf_y \{(t-s) L\left(\frac{x-y}{t-s}\right) + u(s,y)\}$, $\varepsilon \to 0$, ... ④

$\Rightarrow$ ③+④, $u(t,x) = \inf$

在 $L = L(q,x,t)$ (不仅 $L = L(q)$) 也成立, 这部分证明省略

Th4: $u(t,x)$ lipschitz, both in t, x; $u(0,x) = g(x)$, (g 也 lipschitz obv), 证系数可以是 $\mathrm{lip}(g)$ for x in u, g

$\Rightarrow |u(t,\tilde{x}) - u(t,x)| \leq \mathrm{lip}(g)\cdot |\tilde{x} - x|$

$\Rightarrow u(t,x) = \inf_y \{t\cdot L\left(\frac{x-y}{t}\right) + g(y)\} \geq g(x) + \min\{-\mathrm{lip}(g)(x-y) + t\cdot L\left(\frac{x-y}{t}\right)\}$

$\geq g(x) - t \times \max_y \{\mathrm{lip}(g)|z| - L(z)\}$ let $z = \frac{x-y}{t}$

$\geq g(x) - t\cdot H(w)$, $|w| = \mathrm{lip}(g)$, $H = L^*$

$\therefore |u(t,x) - g(x)| \leq C\cdot t$ for some C

Th5: u is differentiate at (t,x), then variation method holds!

proof: $u(x+hq, t+h) = \min_y \{h\cdot L\left(\frac{x+hq-y}{h}\right) + u(t,y)\} \leq h\cdot L\left(\frac{x+hq-x}{h}\right) + u(t,x)$, let $y := x$

$\therefore u(x+hq, t+h) - u(x,t) \leq h \times L(q) \quad \forall h \in \mathbb{R}^+, q \in \mathbb{R}^n$.

在 $h=0$ 时, 上式取等, $\therefore$ 在 $h \to 0^+$ (某个0的邻域内) $\frac{\partial LHS}{\partial h} \leq \frac{\partial RHS}{\partial h}$

$\therefore \partial_t u + \partial_x u \cdot q \leq L(q)$, $\forall q \in \mathbb{R}^n$

$\Rightarrow \partial_t u + H(\partial_x u) = \partial_t u + \sup_q \{q\cdot \partial_x u - L(q)\} \leq 0$ ... ①

设 $u(t,x) = \inf\{t\cdot L\left(\frac{x-z}{t}\right) + g(z)\}$ 在 z 取到, $u(t,x) = t L\left(\frac{x-z}{t}\right) + g(z)$.

$u(s,y) \leq s\cdot L\left(\frac{y-z}{s}\right) + g(z)$ obv

$\therefore u(t,x) - u(s,y) \geq t\cdot L\left(\frac{x-z}{t}\right) - s\cdot L\left(\frac{y-z}{s}\right)$

$= (t-s)\cdot L\left(\frac{x-z}{t}\right)$ let: $s = t-h$, $h>0$, $\frac{x-z}{t} = \frac{y-z}{s}$ (在 t, x, z 确定情况下, s, y 也定了)

即: $u(t,x) - u(t-h, (1-\frac{h}{t})x + \frac{h}{t}z) \geq h\cdot L\left(\frac{x-z}{t}\right)$

同样 $h=0$, LHS=RHS $\therefore h \to 0^+$, $\frac{\partial LHS}{\partial h} \geq \frac{\partial RHS}{\partial h}$

$\therefore \partial_t u + \frac{x-z}{t}\cdot \partial_x u \geq L\left(\frac{x-z}{t}\right)$

$\Rightarrow \partial_t u + H(\partial_x u) = \partial_t u + \sup_q \{ q \cdot \partial_x u - L(q) \} \geq 0 \quad \cdots ②$

☆ ①+②: when $u = \inf \{ \int_0^t L(\dot{w}(s)) + g(w(0)) \}$, u satisfy: $\partial_t u + H(\partial_x u) = 0$