L14.

Baire TH: (X,d) complete, $\begin{cases} U_i \text{ open}, \overline{U_i} = X \text{ (dense)} & \Rightarrow \bigcap_{i=1}^{+\infty} U_i \text{ dense} \\ F_i \text{ closed}, F_i^{\circ} = \phi & \Rightarrow \left(\bigcup_{i=1}^{+\infty} F_i\right)^{\circ} = \phi \end{cases}$

Uniform Boundness TH (X,d) complete, $\{T_\lambda\}_{\lambda\in\Lambda} \subseteq B(X,Y)$; $\forall x\in X$ (fixed) $\sup_{\lambda\in\Lambda} \|T_\lambda(x)\| < +\infty$

$\Rightarrow \sup_{\lambda\in\Lambda} \|T_\lambda\| = \sup_{\substack{\lambda\in\Lambda \\ \|x\|\le 1}} \|T_\lambda(x)\| < +\infty$

pf: let $X_n := \{x\in X : \sup_{\lambda\in\Lambda} \|T_\lambda(x)\| \le n\} \Rightarrow \bigcup_{n=1}^{+\infty} X_n = X$

X_n is closed

$\xrightarrow{\text{Baire TH}} X_N^{\circ} \neq \phi$ some N

need pf. not obv

$\Rightarrow$ some $x_0 \in X_N^{\circ}$, $\exists r>0$ s.t. $B(x_0,r) \subseteq X_n$

i.e. $\sup_{\lambda\in\Lambda} \|T_\lambda(x_0 + rz)\| \le n$, $\forall z\in X, \|z\|=1$

$\Rightarrow r \cdot \sup_{\substack{\lambda\in\Lambda \\ \|z\|=1}} \|T_\lambda(z)\| \le \sup_{\lambda\in\Lambda} \|T_\lambda(x_0)\| + n$

$\Rightarrow \sup_{\substack{\lambda\in\Lambda \\ \|z\|=1}} \|T_\lambda(z)\| \le \dfrac{\sup\|\cdot\| + n}{r} < +\infty$ 用 $B(x_0,r)$ 构造出 $T_\lambda(z)$ $\|z\|=1$

~~Open Mapping~~

Inverse Mapping TH X, Y Banach, $T\in B(X,Y)$ bijective $\Rightarrow T$ invertible

$\Leftrightarrow$ ie. $\exists T^{-1} \in B(Y,X)$, $T^{-1}\circ T = T\circ T^{-1} = I$

pf: T bijective

$\Rightarrow T^{-1}: Y \to X$,

$y \mapsto x$ s.t. $y = Tx$ linear. well-defined, $T^{-1}\circ T = T\circ T^{-1} = I$

T surj. $X\,Y$ Banach $\Rightarrow T$ maps open set to open set

$\Leftrightarrow$ preimage of open set of T^{-1} is open set

$\Rightarrow T^{-1}$ cts $\Leftrightarrow T^{-1}$ bounded

Open Mapping TH, X, Y Banach, $T \in B(X,Y)$ surjective $\Rightarrow$ T: open set $\subseteq X$ $\to$ open set $\subseteq Y$

Pf: $U \subseteq X$ is open

$\Rightarrow \forall x_0 \in U, \exists r > 0$ s.t. $B(x_0, r) \subseteq U$.

$\Rightarrow T(B(x_0,r)) = r \cdot T(B(0,1)) + \underset{\overset{\parallel}{y}}{Tx_0} \subseteq U$

X, Y Banach, $T \in B(X,Y)$ surj $\Rightarrow \exists c > 0$ s.t. $B_Y(0,c) \subseteq T(B_X(0,1))$ ← Lem(*)

$\Rightarrow B_Y(0, r\cdot c) + Tx_0 \subseteq U$

$\therefore \forall y \in T(U)$, let $y = Tx_0$, $x_0 \in U \Rightarrow \exists r \cdot c > 0$ s.t. $B_Y(Tx_0 = y, rc) \subseteq U$

$\Rightarrow T(U)$ open

Lem: X, Y Banach, $T \in B(X,Y)$ surjective $\Rightarrow \exists c > 0$ s.t. $B_Y(0,c) \subseteq T(B_X(0,1))$

1. $Y_n := \overline{T(B_X(0,n))}$

$\bigcup_{n=1}^{+\infty} Y_n = Y$. Y_n closed $\Rightarrow \exists Y_N^\circ \neq \phi$

$\Rightarrow$ i.e. $\exists y_0 \in Y_N$ s.t. $\exists r > 0$ s.t. $B_Y(y_0, r) \subseteq Y_N = \overline{T(B_X(0,N))}$

2. $y_0 \in Y_N \therefore -y_0 \in Y_N$

$\Rightarrow B_Y(y_0, r) + \{-y_0\} \subseteq Y_N + Y_N \Rightarrow B_Y(0,r) \subseteq \overline{T(B_X(0,2N))}$ 去掉closure

surj $\Rightarrow$ 3. $\forall u_0 \in B_Y(0,r)$

⊗ ...
$$\begin{cases} \exists x_1 \in B_X(0,2N) \text{ s.t. } \|Tx_1 - u_0\| < \frac{1}{2} r \Rightarrow Tx_1 - u_0 \in B_Y(0, \frac{r}{2}) \subseteq \overline{T(B_X(0,N))} \\ \exists x_2 \in B_X(0,N) \text{ s.t. } \|-Tx_2 - (Tx_1 - u_0)\| < \frac{1}{4} r \Rightarrow Tx_1 + Tx_2 - u_0 \in B_Y(0, \frac{r}{4}) \subseteq \overline{T(B_X(0,\frac{N}{2}))} \\ \dots \\ \exists x_k \in B_X(0, \frac{N}{2^{k-1}}) \text{ s.t. } \|u_0 - Tx_1 - Tx_2 - \dots - Tx_k\| < \frac{r}{2^k} \Rightarrow u_0 - T(\sum^k x_i) \in B_Y(0, \frac{r}{2^k}) \subseteq \overline{T \cdot B_X(\dots)} \end{cases}$$

let $x := \sum_{i=1}^{+\infty} x_i$, $u_0 = Tx \in T(B_X(0, 2N + N + \dots)) = T(B_X(0,4N))$

$\Rightarrow B_Y(0,r) \subseteq T(B_X(0,4N))$

⊗: $u_0 \in B_Y(0,r)$. WTS $u_0 \in T(B_X(0, zN))$ some $z \in \mathbb{Z}^+$

$$\Rightarrow \begin{cases} u_0 \in \partial(T(B_X(0,2N))) \Rightarrow \forall \varepsilon > 0, \exists y_1 \in T(B_X(0,2N)) \text{ s.t. } \|y_1 - u_0\| < \varepsilon, \text{ let } y_1 = Tx_1,\ \varepsilon = \frac{r}{2} \\ u_0 \in T(B_X(0,2N)) \quad \text{Done} \end{cases}$$

rmk: 1. $\overline{T(B_X(0,N))}$ has interior y_0

2. Move y_0 to $0 \in (\overline{TB_X})^\circ \Rightarrow 0$ is interior, too

3. extend to $T(B_X(0,2N))$

3 替代证明: Alternative pf: $B_Y(0,r)$ is open $\quad \left.\begin{matrix} A \subseteq B \\ A \text{ open} \end{matrix}\right\} \Rightarrow A \subseteq B^\circ$