Heat Equation (Homogeneous)

- Fourier transformation

  - 定义和常用性质:

$$f \in L^1(\mathbb{R}^d),\ \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi\cdot x} f(x)\,dx, \quad \text{here: } \xi, x \in \mathbb{R}^d$$

$$\check{f}(\xi) = \int_{\mathbb{R}^d} e^{2\pi i \xi\cdot x} f(x)\,dx$$

① linearity: $(\alpha f + \beta g)^\wedge = \alpha \hat{f} + \beta \hat{g}$, $\forall f, g \in L^1(\mathbb{R}^d)$, $\alpha, \beta \in \mathbb{C}$

② translation: $(f(x-k))^\wedge = e^{-2\pi i \xi\cdot k}\hat{f}(\xi)$; $\forall k \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} e^{-2\pi i \xi\cdot x} f(x-k)\,dx = \int_{\mathbb{R}^d} e^{-2\pi i \xi\cdot k}\, e^{-2\pi i \xi\cdot(x-k)} f(x-k)\,d(x-k) = \cdots$$

③ dialation: $(f(kx))^\wedge = \frac{1}{|k|}\hat{f}\left(\frac{\xi}{|k|}\right) \quad \forall k \in \mathbb{R}\setminus\{0\}$

$$\int_{\mathbb{R}} e^{-2\pi i \xi\cdot x} f(kx)\,dx = \int_{\mathbb{R}} e^{-2\pi i \frac{\xi}{|k|}\cdot |k|\cdot x} f(|k|\cdot x)\,d|k|x = \cdots$$

④: $(f(x))^\vee = \hat{f}(-\xi)$,

$f^{\vee\vee} = f(x)$

⑤: differentiation:

$d=1$: $(f')^\wedge = (2\pi i \xi)\hat{f}(\xi)$

$$\lim_{N\to\infty} \int_{-N}^{N} e^{-2\pi i \xi\cdot x} f'(x)\,dx \overset{\text{分部积分}}{=} e^{-2\pi i \xi\cdot x} f(x)\Big|_{-N}^{N} + (2\pi i \xi)\int_{-N}^{N} e^{-2\pi i \xi\cdot x} f(x)\,dx$$

$d>1$: $(\partial_{x_j} f)^\wedge = (2\pi i \xi_j)\hat{f}(\xi)$, $f \in C^{\alpha_1 \cdots \alpha_d}(\mathbb{R}^d)$

$$\left.\begin{aligned} &\text{记: } \alpha^d = (\alpha_1, \alpha_2, \cdots \alpha_d),\ |\alpha| = \textstyle\sum_{i=1}^{d} \alpha_i \\ &|x|^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d} \\ &D^\alpha f = \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d} f \\ &f = f(x_1, x_2, \cdots x_d) \end{aligned}\right\} \Rightarrow (D^\alpha f)^\wedge = (2\pi i)^{|\alpha|} \xi^{|\alpha|} \hat{f}(\xi)$$

⑥: convolution:

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x-y)\cdot g(y)\,dy = \int_{\mathbb{R}^d} g(x-y)\cdot f(y)\,dy, \quad f, g \in L^1(\mathbb{R}^d)$$

$$(f * g)^\wedge = f^\wedge \cdot g^\wedge$$

prop: $f,g\in L^1(\mathbb{R}^d)$, $\partial_{x_i} f\in C(\mathbb{R}^d)\cap L^1(\mathbb{R}^d)\cap L^{\infty}(\mathbb{R}^d)$

则: $\partial_{x_i}(g*f) = g*(\partial_{x_i} f) = f*(\partial_{x_i} g)$

prop: $f,g\in L^1(\mathbb{R}^d) \Rightarrow f*g = g*f \in L^1(\mathbb{R}^d)$;

$$\int_{\mathbb{R}^d} |f*g(x)|\,dx \le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)\cdot g(y)|\,dy\cdot dx$$

$$\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x-y)|\cdot|g(y)|\cdot dy\cdot dx$$

$$= \int_{\mathbb{R}^d} |g(y)|\cdot\left(\int_{\mathbb{R}^d}|f(x-y)|\,dx\right)dy$$

$$= \|g\|_{L^1}\cdot\|f\|_{L^1}$$

Theorem: $(f*g)^\wedge = \hat{f}(\xi)\cdot\hat{g}(\xi)$

pf: $$(f*g)^\wedge(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i\xi\cdot x}\left(\int_{\mathbb{R}^d} f(x-y)\cdot g(y)\,dy\right)dx$$

$$= \int_{\mathbb{R}^d} e^{-2\pi i\xi\cdot(x-y)}\times e^{-2\pi i\xi y}\times\left(\int_{\mathbb{R}^d} f(x-y)g(y)\,dy\right)d(x-y)$$

积分结果与y无关 可换序

$$= \underline{\int_{\mathbb{R}^d} e^{-2\pi i\xi\cdot(x-y)}\cdot f(x-y)\,d(x-y)}\cdot\int_{\mathbb{R}^d} e^{-2\pi i\xi y}\cdot g(y)\,dy$$

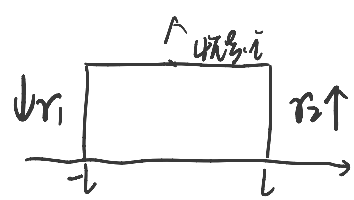
$$= \hat{f}(\xi)\,\hat{g}(\xi)$$

例: $\left(\frac{1}{\sqrt{4\pi}}e^{-\frac14 x^2}\right)^\wedge = e^{-4\pi^2\xi^2}$;

其中 $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac12(x-\mu)^2/\sigma^2} \sim N(0,2)$, $\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi}}e^{-\frac14 x^2}\,dx = 1$

$$(f)^\wedge = \int_{\mathbb{R}} e^{-2\pi i\xi x}\frac{1}{\sqrt{4\pi}}e^{-\frac14 x^2}\,dx$$

$$= \underline{\int_{\mathbb{R}}\frac{1}{\sqrt{4\pi}}\cdot e^{-\frac14(x+4\pi i\xi)^2}}\,e^{-4\pi^2\xi^2}\,dx$$



$\|$

设 $g(z) = \frac{1}{\sqrt{4\pi}}e^{-\frac14 z^2}$, $z\in\mathbb{C}$

$$\oint g\,dz = 0 = \int_{-L}^{L}\frac{1}{\sqrt{4\pi}}e^{-\frac14 x^2}\,dx + \int_{\gamma_2} + \int_{L}^{-L}\frac{1}{\sqrt{4\pi}}e^{-\frac14(x+4\pi i\xi)^2}\,dx$$

$\int_{\gamma_2} g\,dz = \int_0^{4\pi\xi}\frac{1}{\sqrt{4\pi}}e^{-\frac14(L+ti)^2}\,d(L+ti) \to 0$ as $L\to\infty$ 同理 $\int_{\gamma_1}$

example: $\begin{cases} \partial_t u = \Delta u \\ u(0,x) = \phi(x) \end{cases}$

选定 x 作为变量, 对 u 进行变换.

ODE: $\begin{cases} \partial_t \hat{u}(t,\xi) = -4\pi^2\xi^2 \hat{u}(t,\xi) \\ \hat{u}(0,\xi) = \hat{\phi}(\xi) \end{cases}$ 记 $\hat{u}(t,\xi) = g(t)$

$\rightleftarrows \begin{cases} g(t)' = -4\pi^2\xi^2 g(t) \\ g(0) = \hat{\phi}(\xi) \end{cases}$

$\Rightarrow \hat{u}(t,\xi) = g(t) = e^{-4\pi^2\xi^2 t} \times \hat{\phi}(\xi) = (h(t) * \phi(\xi))^\wedge$

$\| \quad \hat{h}(t)$: $h(t) = F^{-1}(e^{-4\pi^2\xi^2 t}) = \sqrt{\frac{1}{4\pi t}} \times e^{-x^2/4t}$

$\therefore u(t,\xi) = h(t) \cdot \phi(\xi) = \sqrt{\frac{1}{4\pi t}} \times e^{-x^2/4t} * \phi(\xi)$, $\forall \xi \in \mathbb{R}^d$

$u(t,x) = \sqrt{\frac{1}{4\pi t}} \times e^{-x^2/4t} * \phi(x)$

· verify $G * \phi(x)$

Th: HE on $\mathbb{R}^d$: $\begin{cases} \partial_t u = \Delta u \\ u(0,x) = \phi(x) \end{cases}$. $t>0$ $x \in \mathbb{R}^d$ $\phi \in C(\mathbb{R}^d) \cap L^{\infty}(\mathbb{R}^d)$

let $u(t,x) = (G_t * \phi)(x)$ from Fourier

then: ① $u \in C^{\infty}((0,+\infty) \times \mathbb{R}^d)$ (important!)

② $\lim_{(t,x)\to(0,x^0)} u(t,x) = \phi(x^0)$; 即 u 为经典解

$(G_t * \phi)(x)$ 可以说明 u 在 x 连续; 但这是 x 的卷积, t 的性质要另外证明

引入

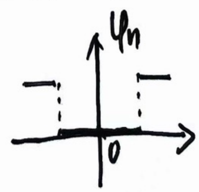
Dirac function: 所有值集中在0. 积分为1;

$\begin{cases} \delta(x) = 0 \ \forall x \neq 0. \ \int_{\mathbb{R}^d} \delta(x) dx = 1 \end{cases}$

then: $\Rightarrow \int_{\mathbb{R}^d} f(x)\delta(x) dx = f(0)$; $\forall f$ bounded, cts near 0

如: $y_n(x) = 2n \times 1(|x| < \frac{1}{n})$

$y_n(x) \to$ Dirac f, $n \to +\infty$



$$\left|\int_{\mathbb{R}^d} f(x)\delta(x)dx - f(0)\right| \quad ; \quad \text{let } \varepsilon > 0.$$

$$= \left|\int_{\mathbb{R}^d} f(x)\delta(x)dx - \int_{\mathbb{R}^d} f(0)\delta(x)dx\right| = \left|\int_{\mathbb{R}^d} (f(x)-f(0))\delta(x)dx\right|$$

$$= \left|\int_{|x|>\varepsilon} (f(x)-f(0))\delta(x)dx\right| + \left|\int_{|x|\le\varepsilon} (f(x)-f(0))\delta(x)dx\right|$$

$$\le 2\sup|f| \cdot \underbrace{\left|\int_{|x|>\varepsilon} \delta(x)dx\right|}_{=0} + \underbrace{|f(x)-f(0)|}_{\to 0} \cdot \left|\int_{|x|\le\varepsilon} \delta(x)dx\right|$$

Approximate identity: $\{K_n\} \to \delta(x)$

$$\begin{cases} ① \ K_n(x) = 0 \ \ \forall x \ne 0, \quad ② \int_{\mathbb{R}^d} K_n(x)dx = 1 \\ \lim_{n\to+\infty} \int_{|x|>\varepsilon} K_n(x)dx = 0, \quad \forall \varepsilon > 0 \end{cases}$$

$\int_{x>\varepsilon}$ 不一定是0

绝对值

$$\Rightarrow \lim_{n\to+\infty} \int_{\mathbb{R}^d} K_n(x)\cdot g(x)dx = g(0) = \int_{\mathbb{R}^d} g(x)\cdot\delta(x)dx \ ; \ \forall \text{ bdd } g. \text{ cts near } 0$$

ϕ **bdd**: $u = G_t * \phi(x)$, $G_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{1}{4}x^2/t}$

Let $G_n = \frac{1}{\sqrt{4\pi n}} e^{-\frac{1}{4}x^2\cdot n}$; $\{G_n\}$ 是 Approximate identity

$$\therefore \lim_{n\to+\infty} G_n * \phi = \lim_{n\to+\infty} \int \underset{\underset{f(y)}{\parallel}}{\phi(x-y)} \cdot G_n(y)dy = \underset{\underset{f(0)}{\parallel}}{\phi(x)}$$

ϕ **unbounded**:

1. 此处的收敛为绝对收敛
2. if φ not bounded $\Rightarrow$ HW1.5, $|\phi(x)| \le e^{Ax^2}$, $A > 0$; $t \in (0, (4A)^{-1})$

在 $|\phi| \le e^{Ax^2}$, $t \in (0, (4A)^{-1})$ 时, $G_t * \phi(x)$ 有定义, 即卷积收敛 …… (1)

fix an x. for any $\varepsilon > 0$, take an $\xi > 0$; s.t. $|\phi(x_1) - \phi(x_2)| < \varepsilon/2$ $\forall x_1, x_2 \in [x-\xi, x+\xi]$

$u(t,x) = \int_{\mathbb{R}^d} G_t(x-y)\cdot\phi(y)dy$; G_t 为 Approximate identity

$\Rightarrow$ 代入: $\left| u(t,x) - \int_{|x-y|\le\xi} G_t(x-y)\phi(x)dy \right|$

$$\le \underbrace{\left|\int_{|x-y|>\xi} G_t(x-y)\cdot\phi(y)dy\right|}_{①} + \underbrace{\left|\int_{|x-y|\le\xi} G_t(x-y)(\phi(y)-\phi(x))\cdot dy\right|}_{②}$$

(1) + by DCT:

$$\begin{cases} \lim_{t\to 0^+} ① = \left|\int_{|x-y|>\xi} \left(\lim_{t\to 0^+} G_t(x-y)\right)\phi(y)dy\right| \le |\phi(y)| \cdot \left|\int_{|x-y|>\xi} \underset{\underset{\delta(x-y)}{\parallel}}{\left(\lim_{t\to 0^+} G_t(x-y)\right)} dy\right| = 0 \\ ② \le |\phi(y) - \phi(x)| \cdot \left|\int_{|x-y|\le\xi} G_t(x-y)dy\right| < \varepsilon \times 1 = \varepsilon \end{cases}$$

即: fix x. $\forall \varepsilon>0$ $\exists \delta$ s.t.:

$$\left| u(t,x) - \int_{|x-y|\le\delta} G_t(x-y)\cdot\phi(x)\,dy \right| < \varepsilon, \text{ as } t\to 0^+$$

又∵ $t\to 0$ 时 $\int_{|x-y|\le\delta} G_t(x-y)\cdot\phi(x)\,dy = \phi(x)\cdot\int_{|z|\le\delta} G_t(z)\,dz = \phi(x)\times 1$

$$\Rightarrow \quad \lim_{t\to 0^+} u(t,x) = \phi(x)$$

Want: $u(t,x) - \phi(x) \to 0$

$\phi(x) = \phi(x)\times 1 \xrightarrow[t\to+\infty]{} \int_{\mathbb{R}} G_t(x)\,dx = \int_{|x|<\delta} G_t(x)\,dx$

$= \int_{|x-y|<\delta} G_t(x-y)\,dy$

这里用 $G_t(x-y)$, ϕ not bdd 时

用不了 Dirac 的 0 值结论