Metric Spaces

Definition 1 A metric space is a pair (X, d), where X is a set and $d : X \times X \to [0, \infty)$ for all $x, y, z \in X$ has the following properties:

- (Positivity) $d(x, y) = 0 \iff x = y$,
- (Symmetry) $d(x, y) = d(y, x)$,
- (Triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$.

A function $d : X \times X \to [0, \infty)$ that satisfies these axioms is called a distance function on X. A subset $U \subset X$ of a metric space (X, d) is called open if, for every $x \in U$, there exists a constant $\epsilon > 0$ such that the open ball

$$B(\epsilon, x) := \{y \in X | d(x, y) < \epsilon\}$$

(centered at x with radius ϵ) is contained in U.

A subset F of a metric space (X, d) is closed if its complement F^c is open.

Some basic facts about open sets

• Every ball $B(x,r)$ is open, for if $y \in B(x,r)$ and $d(x,y)=s$ then $B(y,r-s) \subset B(x,r)$.

• X and $\emptyset$ are both open and closed.

• The union of any family of open sets is open, and hence the intersection of any family of closed sets is closed.

• The intersection (resp. union) of any finite family of open (resp. closed) sets is open (resp. closed). Indeed, if $U_1, \cdots, U_n$ are open and $x \in \cap_{i=1}^n U_i$, for each j there exists $r_j > 0$ such that $B(x, r_j) \subset U_j$, and then $B(x,r) \subset \cap_{i=1}^n U_i$ where $r = \min(r_1, \cdots, r_n)$, so $\cap_{i=1}^n U_i$ is open.

Let $\mathcal{P}(X)$ be the collection of all subsets of X. Recall that a *topology* on X is a subfamily $\tau \subset \mathcal{P}(X)$ satisfying

- $\emptyset \in \tau$ and $X \in \tau$;
- if $U_i \in \tau (i = 1, ..., n)$, then $\cap_{i=1}^{n} U_i \in \tau$;
- if $U_\alpha (\alpha \in \mathcal{I})$ is an arbitrary collection in τ, then $\cup_{\alpha \in \mathcal{I}} U_\alpha \in \tau$.

The set of open subsets of (X, d) will be denoted by

$$U(X, d) := \{U \subset X | U \text{ is open}\}.$$

It follows from the definitions that the collection $U(X, d)$ in a metric space (X, d) satisfies the axioms of a topology and so (X, d) is a topological space.

We will use $\mathbb{F}$ to denote either $\mathbb{C}$ or $\mathbb{R}$.

Example 1. The set $\mathbb{R}$ of all real numbers endowed with the distance function $d(x,y) = |x-y|$, where $|x|$ is the absolute value of x, is a metric space.

Similarly, the set of all complex numbers $\mathbb{C}$ is a metric space with the distance function $d(z,w) = |z-w|$, where $|z|$ is the modulus of z in $\mathbb{C}$.

Example 2. Let X be a nonempty set. The function

$$d(x,y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x \neq y, \end{cases}$$

is a metric, called the discrete metric (also known as the trivial metric) on X. The space (X,d) is called the discrete metric space.

Example 3. Let $C[a,b] = \{x(t) : x(t) \text{ is continuous on } [a,b]\}$ and define

$$d_1(x,y) := \max_{a \le t \le b} |x(t) - y(t)|, \ d_2(x,y) := \int_a^b |x(t) - y(t)| dt.$$

Then d_1 and d_2 are metrics on $C[a,b]$.

Example 4. For any integer $n \ge 1$, the function $d : \mathbb{K}^k \times \mathbb{K}^n \to [0, \infty)$ defined by

$$d(x,y) = \left(\sum_{j=1}^{n} |x_j - y_j|^2 \right)^{1/2},$$

is a metric on the set $\mathbb{K}^n$, called the standard metric on $\mathbb{K}^n$.

Example 5. More generally, take $X = \mathbb{K}^n$ with any one of the metrics

$$d_{l^p}(x,y) = \begin{cases} \left(\sum_{j=1}^n |x_j - y_j|^p\right)^{1/p}, & 1 \le p < \infty, \\ \max_{j=1,\cdots,n} |x_j - y_j|, & p = \infty. \end{cases}$$

It is easy to see that d_{l^∞} is a metric. For the case $1 \le p < \infty$, we need only to use the

(Minkowski's inequality.) For arbitrary complex numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$ and a real number $p \ge 1$,

$$\left(\sum_{j=1}^n |x_j + y_j|^p\right)^{1/p} \le \left(\sum_{j=1}^n |x_j|^p\right)^{1/p} + \left(\sum_{j=1}^n |y_j|^p\right)^{1/p}.$$

Proof. We may assume that both real numbers

$$u = \left(\sum_{j=1}^n |x_j|^p\right)^{1/p} \text{ and } v = \left(\sum_{j=1}^n |y_j|^p\right)^{1/p}$$

are positive. By the triangle inequality, we have

$$|x_k + y_k|^p \le (|x_k| + |y_k|)^p = (u+v)^p \left(\frac{u}{u+v} \frac{|x_k|}{u} + \frac{v}{u+v} \frac{|y_k|}{v} \right)^p.$$

Since $\frac{u}{u+v} + \frac{v}{u+v} = 1$ and x^p is convex for $p \ge 1$, we have

$$\left(\frac{u}{u+v} \frac{|x_k|}{u} + \frac{v}{u+v} \frac{|y_k|}{v} \right)^p \le \frac{u}{u+v} \frac{|x_k|^p}{u^p} + \frac{v}{u+v} \frac{|y_k|^p}{v^p}.$$

Hence

$$|x_k + y_k|^p \le (u+v)^p \left(\frac{u}{u+v} \frac{|x_k|^p}{u^p} + \frac{v}{u+v} \frac{|y_k|^p}{v^p} \right).$$

By summing both sides of the above inequality, we obtain

$$\sum_{j=1}^{n} |x_j + y_j|^p \le (u+v)^p.$$

Example 6. If d is a metric on X and $A \subset X$, then $d|_{(A \times A)}$ is a metric on A.

Example 7. If (X_1, d_1) and (X_2, d_2) are metric spaces, the product metric d on $X_1 \times X_2$ is given by

$$d((x_1, x_2), (y_1, y_2)) = \max\{d(x_1, y_1), d(x_2, y_2)\}.$$

Other metrics are sometimes used on $X_1 \times X_2$, for instance,

$$d(x_1, y_1) + d(x_2, y_2) \text{ or } \sqrt{d(x_1, y_1)^2 + d(x_2, y_2)^2}.$$

Definition 2. A point x in a metric space X is said to be a limit of a sequence of points $(x_n) \subset X$ if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$, for all $n \geq N$.

If x is a limit of the sequence (x_n), we say that (x_n) converges to x and write $x_n \to x$.

If a sequence has a limit, it is called convergent. Otherwise, it is called divergent. Observe that $x_n \to x \Leftrightarrow d(x_n, x) \to 0$.

A subset Y of a metric space (X, d) is bounded if there exists $x \in X$ and $r > 0$, such that $Y \subset B(r, x)$. Otherwise, Y is unbounded.

Any convergent sequence is bounded, since if $x_n \to x$, then there exists $N \in \mathbb{N}$ such that $d(x_n, x) < 1$ for all $n \geq N$ and so

$$d(x_n, x) \leq \max\left(1, \max_{j=1,\cdots,N-1} d(x_j, x)\right), \ \forall n \in \mathbb{N}.$$

Definition 3. A point $x \in E$ is said to be an interior point of E if

$$\exists r > 0, \ s.t. \ B(x, r) \subset A.$$

The interior of E is the set of all its interior points and is denoted by E^o. A point x (not in E) is an exterior point of E when

$$\exists r > 0, \ s.t. B(x, r) \subset X \setminus E.$$

All other points are called boundary points of E.
The set of interior and boundary points of E is called the closure of E and denoted by $\overline{E} = E^o \cup \partial E$. Note that $\overline{E}$ is also the intersection of all closed sets containing E.

The set X is partitioned into three parts: its interior E^o, its exterior $(\overline{E})^c$, and its boundary ∂E.
We call E is dense in X if $\overline{E} = X$, and nowhere dense if $\overline{E}$ has empty interior.

Lemma 1. $x_n \to x \Leftrightarrow \forall$ open set U that contains x there exists an N such that $x_n \in U$ for every $n \geq N$.

Proof $\Rightarrow$: Given any open set U that contains x there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset U$, and so $\exists$ N such that $x_n \in B(x, \epsilon) \subset U$ for all $n \geq N$.
$\Leftarrow$: For any $\epsilon > 0$, since the set $B(x, \epsilon)$ is open and contains x, there exists an N such that $x_n \in B(x, \epsilon)$ for every $n \geq N$. Thus $x_n \to x$. $\square$

Lemma 2. A subset A of (X, d) is closed $\Leftrightarrow$ whenever $(x_n) \subset A$ with $x_n \to x$ it follows that $x \in A$.

Proof $\Rightarrow$: Let $(x_n) \subset A$ with $x_n \to x$. If $x \notin A$, i.e. $x \in A^c$, there would exist $B(x, r) \subset A^c$ since A^c is open. It follows that $B(x, r) \cap A = \emptyset$, which contradicts to $x_n \to x$.
$\Leftarrow$: Take $x \notin A$. If for any $\epsilon > 0$, $B(x, r) \cap A \neq \emptyset$, by taking $\epsilon = \frac{1}{n}$, we would get $(x_n) \subset A$ such that $d(x, x_n) < \frac{1}{n}$, which implies that $x_n \to x$ and so $x \in A$. This is a contradiction. Thus there is some $r > 0$ such that $B(x, r) \cap A = \emptyset$, that is $B(x, r) \subset A^c$. Consequently, A^c is open. $\square$

A set $V \subset X$ is said to be a neighborhood of a point $p \in X$ if there is an $r > 0$ such that $B(p, r) \subset V$. In particular, any open set D is a neighborhood of any $p \in D$.

A point p is an accumulation point (or limit point) of a set A if every open ball around it contains other points of A,

$$\forall \epsilon > 0, \ \exists q \neq p, q \in A \cap B(p, \epsilon).$$

Note that p is not necessarily an element of S. If $q \in S$ and q is not an accumulation point of S, then q is an isolated point of S.

By Lemma 2, S is closed $\Leftrightarrow$ S contains all its accumulation points.

The boundary of a set $S \subset (X, d)$ is also the set $\partial S = \overline{S} \cap \overline{S^c}$. We note that $p \in \partial S \Leftrightarrow B(p, \epsilon) \cap S \neq \emptyset$ and $B(p, \epsilon) \cap S^c \neq \emptyset$ for any $\epsilon > 0$.

Remark 1. p is a limit point of $S \Leftrightarrow V \cap S$ is an infinite set for every neighborhood V of p.

Remark 2. p is a limit point of $S \Leftrightarrow$ there exists a sequence with distinct points, $(p_n) \subset S$, such that $p_n \to p$.

Lemma 3

$$x \in \overline{A} \Leftrightarrow B(x, \epsilon) \cap A \neq \emptyset, \forall \epsilon > 0, \qquad (0.1)$$

that is, $x \notin \overline{A} \Leftrightarrow \exists \epsilon_0 > 0$ *such that* $B(x, \epsilon_0) \cap A = \emptyset$. *It follows that* $x \in \overline{A} \Leftrightarrow \exists$ *a sequence* $(x_n) \subset A$ *such that* $x_n \to x$.

Proof If $x \notin \overline{A}$, then there is some closed set K that contains A such that $x \notin K$. Since K is closed, $X \setminus K$ is open, and so $B(x, \epsilon_0) \cap K = \emptyset$ for some $\epsilon_0 > 0$, which shows that $B(x, \epsilon_0) \cap A = \emptyset$ since $K \supset A$. On the other hand, if there exists $\epsilon_0 > 0$ such that $B(x, \epsilon_0) \cap A = \emptyset$, then x is not contained in the closed set $X \setminus B(x, \epsilon_0)$, which contains A; so $x \notin \overline{A}$.

Finally, if $x \in \overline{A}$, then (0.1) implies that for any $n \in \mathbb{N}$ we have $B(x, 1/n) \cap A \neq \emptyset$, so $\exists x_n \in A$ such that $d(x_n, x) < 1/n$ and thus $x_n \to x$. Conversely, if $(x_n) \in A$ with $x_n \to x$, then $\forall \epsilon > 0, d(x_n, x) < \epsilon$ for n sufficiently large, and so $B(x, \epsilon) \cap A \neq \emptyset$ which implies by (0.1) that $x \in \overline{A}$. □

Lemma 4 $(\overline{A^c})^c = A^o$ *and so* $\overline{A^c} = X \Leftrightarrow A^o = \emptyset$.

Proof $x \in (\overline{A^c})^c \Leftrightarrow x \notin \overline{A^c} \Leftrightarrow \; B(x, \epsilon_0) \cap A^c = \emptyset$ for some $\epsilon_0 > 0$ $\Leftrightarrow B(x, \epsilon_0) \subset A$ for some $\epsilon_0 > 0 \Leftrightarrow x \in A^o$. $\square$

Recall that A is dense in X if and only if $\overline{A} = X$. It follows from Lemma 3 that A is dense in $X \Leftrightarrow \forall x \in X, \; \epsilon > 0, B(x, \epsilon) \cap A \neq \emptyset$, i.e. $\exists p \in A$ such that $d(p, x) < \epsilon$.

A sequence (x_n) in a metric space (X, d) is called a Cauchy sequence if, for every $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for any two integers $n, m \geq n_0$, we have $d(x_n, x_m) < \epsilon$. A metric space (X, d) is called complete if every Cauchy sequence in X converges.

For any $n \in \mathbb{N}$, $\mathbb{R}^n$, equipped with the Euclidean metric, is complete, because a Cauchy sequence in $\mathbb{R}^n$ is Cauchy in each coordinate.

Theorem $\forall$ metric space X, $\exists$ a complete metric space Y and a map $j : X \to Y$ s.t.
1) $d_Y(j(x), j(w)) = d_X(x, w), \ \forall x, w \in X$. 2) $\overline{j(X)} = Y$.
If Z is another complete metric space and $k : X \to Z$ is a map satisfying the 1) and 2), then $\exists$ a bijective map $f : Y \to Z$ s.t. $d_Z(f(y_1), f(y_2)) = d_Y(y_1, y_2)\ \forall y_1, y_2 \in Y$ and $f(j(x)) = k(x)\ \forall x \in X$.

The metric space Y is called the completion of X.

Let A be a non-empty set of the metric space (X, d). The diameter A is defined as $\mathrm{diam}(A) = \sup_{x,y\in A} d(x, y)$.

Theorem 1 (Baire) *Let (X, d) be complete and let $A_n \subset X, n \in \mathbb{N}$, be closed satisfying $A_n^o = \emptyset, \ \forall n \in \mathbb{N}$. Then,*

$$(\cup_{n=1}^{\infty} A_n)^o = \emptyset. \tag{0.2}$$

Proof $\overline{A_n^c} = X$. Thus A_n^c is dense in X and is also open $\forall n \in \mathbb{N}$. Let $D_n = A_n^c$; by Lemma 4, we need to show that $\overline{(\cup_{n=1}^{\infty} A_n)^c} = X$ or, that $M := \cap_{n=1}^{\infty} D_n$ is dense in X, i.e., for every open $U = B(x_0, r_0) \subset X, r_0 > 0$ we have $U \cap M \neq \emptyset$. Fix such an open U. Since D_1 is open and dense in X there exist $x_1 \in U \cap D_1$ and $r_1 > 0$ such that

$$\overline{B(x_1, r_1)} \subset U \cap D_1, \ 0 < r_1 < r_0/2.$$

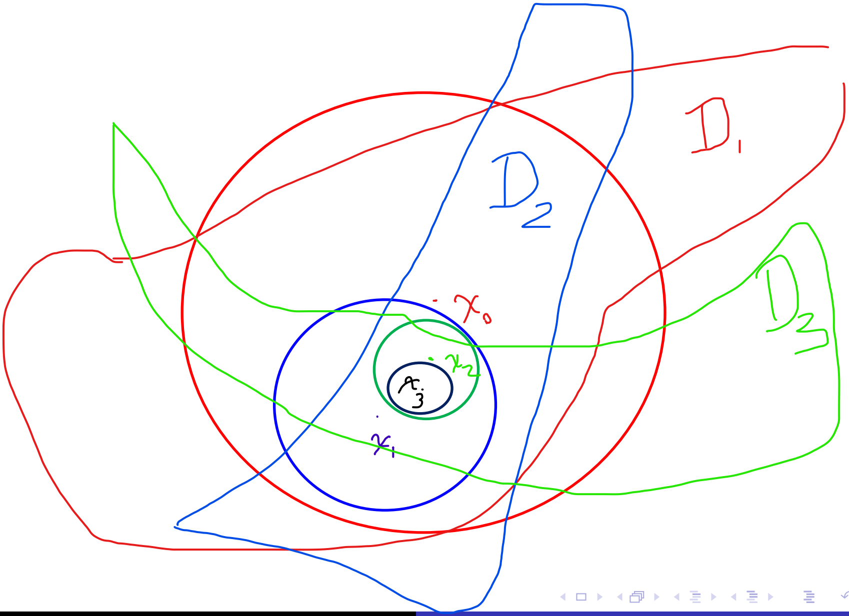
By induction one can find sequences (x_n) and (r_n) such that

$$\overline{B(x_{n+1}, r_{n+1})} \subset B(x_n, r_n) \cap D_{n+1}, \ 0 < r_{n+1} < r_n/2.$$

for $n = 0, 1, 2, \ldots$. Since for $l \geq k$,

$$d(x_k, x_l) \leq \sum_{i=k}^{l-1} d(x_i, x_{i+1}) \leq \frac{r_0}{2^{k-1}},$$

(x_n) is Cauchy. Let $y = \lim x_n$. Since $x_l \in B(x_k, r_k), \forall l \geq k$, $\Rightarrow y \in \overline{B(x_k, r_k)} \subset D_k, \ \forall k \in \mathbb{N}$ and so $y \in U \cap M$. $\square$



D_1
D_2
D_3
x_0
x_1
x_2
x_3

Definition. Let (X, d) be a metric space. The diameter of X is $\operatorname{diam}(X) = \sup_{x,y \in X} d(x, y)$.

Theorem 2. (Nested sets theorem). *Let $A_1 \supset A_2 \supset \cdots$ be a decreasing chain of non-empty closed subsets of a complete metric space (X, d) and let $\operatorname{diam}(A_n) \to 0$ as $n \to \infty$. Then $\cap_{n=1}^{\infty} A_n$ consists of exactly one point.*

Proof. Pick in each A_n a point a_n. If $N \in \mathbb{N}$ and $k, j > N$, then since $A_n \downarrow$, the points a_k and a_j belong to A_N. Thus, $d(a_j, a_k) \leq \operatorname{diam}(A_N) \to 0$ as $N \to \infty$, i.e., (a_n) is Cauchy. Let $a = \lim a_n$. For any N and any $k > N$, $a_k \in A_N$. Hence, $a = \lim a_k \in A_N$, i.e., $a \in \cap_{N=1}^{\infty} A_N$. Note that $\cap_{n=1}^{\infty} A_n \subset A_N$ for all N, and so

$$\operatorname{diam}\left(\cap_{n=1}^{\infty} A_n\right) \leq \operatorname{diam}(A_N) \to 0, \ N \to \infty.$$

But a set of diameter zero reduces to a single point. □

Contracted mapping theorem

Suppose (X, ρ) is a complete MS, $T : X \to X$ satisfies: $\exists \theta \in [0, 1)$ s.t.

$$\forall x \in X, \forall y \in X, \quad \rho(Tx, Ty) \leq \theta \rho(x, y). \quad (\Rightarrow T \text{ is conti.})$$

Then $\exists 1 \bar{x} \in X$ s.t. $T\bar{x} = \bar{x}$.

Proof. Step 1. Let $x_0 \in X$, $x_1 := Tx_0, \ldots, x_{n+1} := Tx_n, \ldots$.

$$\rho(x_n, x_{n+1}) \leq \theta \rho(x_n, x_{n-1}) \leq \theta^2 \rho(x_{n-1}, x_{n-2}) \leq \ldots \leq \theta^n \rho(x_0, Tx_0)$$

$\Rightarrow$ for any $n, p \in \mathbb{N}$,

$$\rho(x_n, x_{n+p}) \leq \rho(x_n, x_{n+1}) + \rho(x_{n+1}, x_{n+2}) + \ldots + \rho(x_{n+p-1}, x_{n+p})$$
$$\leq (\theta^n + \theta^{n+1} + \ldots + \theta^{n+p-1}) \rho(x_0, Tx_0) \leq \frac{\theta^n}{1 - \theta} \rho(x_0, Tx_0)$$

$\Rightarrow \{x_n\}_n$ is Cauchy. X is complete$\Rightarrow \exists \bar{x} \in X$ s.t. $\lim x_n = \bar{x}$.
$\Rightarrow T\bar{x} = \bar{x}$.

Step 2. Suppose $\exists \bar{x}, \tilde{x} \in X$ s.t. $\bar{x} = T\bar{x}$, $\tilde{x} = T\tilde{x}$. Then

$$\rho(T\bar{x}, T\tilde{x}) \leq \theta \rho(\bar{x}, \tilde{x}) = \theta \rho(T\bar{x}, T\tilde{x}) \Rightarrow \bar{x} = \tilde{x}$$

because $\theta < 1$.

Def. A metric space (X, d) is separable if it contains a countable dense subset, i.e., there exists a countable subset $B \subset X$ such that $\overline{B} = X$.

Separability means that elements of X can be approximated arbitrarily closely by some countable collection $\{x_1, x_2, \cdots, \}$:

$$\forall\ x \in X \text{ and } \epsilon > 0, \exists j \in \mathbb{N}\ s.t.\ d(x_j, x) < \epsilon.$$

Examples. $\mathbb{R}$ and $\mathbb{C}$ are separable, since $\mathbb{Q}$ and $\mathbb{Q} + i\mathbb{Q}$ are countable dense subsets of $\mathbb{R}$ and $\mathbb{C}$, respectively. Since separability of (X, d_X) and (Y, d_Y) implies separability of $X \times Y$ (with an appropriate metric), it follows that $\mathbb{R}^n$ and $\mathbb{C}^n$ are separable.

Lemma 5 If (X, d) separable and $Y \subset X$, then (Y, d) is also separable.

Proof. Let's construct a countable dense subset A of Y. Suppose that $\{x_1, x_2, \cdots, \}$ is dense in X. For each $k, n \in \mathbb{N}$, if $B(x_n, \frac{1}{k}) \cap Y \neq \emptyset$, then we choose one point from $B(x_n, \frac{1}{k}) \cap Y$ and add it to A. Constructed in this way A is (at most) a countable set. Given $y \in Y$ and $\epsilon > 0$, take $k \in \mathbb{N}$ such that $\frac{1}{k} < \frac{\epsilon}{2}$. Let x_n be such that $d(x_n, y) < \frac{1}{k}$; then $B(x_n, \frac{1}{k}) \cap Y \neq \emptyset$. It follows that there exists a $z \in A \cap B(x_n, \frac{1}{k}) \cap Y$ and so

$$d(y, z) \leq d(y, x_n) + d(x_n, z) < \frac{2}{k} < \epsilon.$$

Hence $\overline{A} = Y$, i.e., Y is separable.

Definition Let $(X, d), (Y, d_1)$ be metric spaces and $T : X \to Y$. For $x_0 \in X, T$ is continuous at x_0 iff $\forall \epsilon > 0$, $\exists \delta > 0$, such that $d(x, x_0) < \delta \Rightarrow d_1(Tx, Tx_0) < \epsilon$. T is continuous on D iff $\forall x \in D$, it is continuous at x. T is called uniformly continuous on D if δ can be the same for all $x_0 \in D$, i.e., δ is independent of $x_0 \in D$. T is a homeomorphism iff it is bijective, and T and T^{-1} are continuous.

Theorem 3. T is continuous at $x_0 \Leftrightarrow \forall \{x_n\} \subset X, x_n \to x_0 \Rightarrow Tx_n \to Tx_0$.

$\Rightarrow$ is obvious.
$\Leftarrow$: if T is not continuous at x_0, then $\exists \epsilon_0 > 0$, such that $\forall n \in \mathbb{N}, \exists x_n \in X, d(x_n, x_0) < \frac{1}{n}, d_1(Tx_n, Tx_0) \geq \epsilon_0$. Hence $x_n \to x_0$, $Tx_n \nrightarrow x_0$.

- T is continuous
$\Leftrightarrow \forall$open $G \subset Y, T^{-1}(G)$ is open in X.

Proof. $\Rightarrow$: Let $G \subset Y$ be open, $x_0 \in T^{-1}(G)$, then $T(x_0) \in G$, so $\exists\ B(Tx_0, \epsilon) \subset G$. By continuity, $\exists \delta > 0$ s.t. $T(B(x_0, \delta)) \subset B(Tx_0, \epsilon)$. Thus, $B(x_0, \delta) \subset T^{-1}(B(Tx_0, \epsilon)) \subset G^{-1}(G)$. Hence, $T^{-1}(G)$ is open.

$\Leftarrow$: Let $\{x_n\} \subset X, x_n \to x_0$. Fix $\epsilon > 0$. Since $T^{-1}(B(Tx_0, \epsilon))$ is open, $\exists \delta > 0$ s.t. $B(x_0, \delta) \subset T^{-1}(B(Tx_0, \epsilon))$. Thus, $T(B(x_0, \delta)) \subset B(Tx_0, \epsilon)$. From $x_n \to x_0$, $\exists N_0 \in \mathbb{N}$ s.t. $d(x_n, x_0) < \delta, \forall n \geq N_0$. Hence, $d(Tx_n, Tx_0) < \epsilon$, $\forall n \geq N_0$.

Def. $T: (X, d) \to (Y, d_1)$ is an *isometry* if it preserves distances, i.e.,

$$d_1(Tx, Ty) = d(x, y)\ \forall x, y \in X.$$

X is said to be *isometric* to Y if there is a bijective isometry from X onto Y.

Def. (X, d) is *compact* if every sequence in X has a convergent subsequence. A subset Y of X is called *compact* if it is a compact subspace of X.

Hence a subset Y of a metric space X is compact $\Leftrightarrow$ every sequence in Y has a subsequence that converges to a point in Y.

- A compact set in a metric space is closed and bounded.

Proof. Let $E(\subset (X,d))$ be compact and $x \in \overline{E}$, then $\exists (x_n) \subset E$ s.t. $x_n \to x$. Since E is compact, $x \in E$. Thus E is closed. If E is not bounded, then $\forall n \in \mathbb{N}, \exists x_n \in E$ s.t. $d(x_n, a) > n$, where $a \in X$ is a fixed point. Since every convergent sequence in X is bounded, (x_n) has no convergent subsequence. This contradicts the fact that E is compact.

Lemma. Let $\{U_i\}_{i \in J}$ be an open covering of a compact space X. Then $\exists\ r > 0$ s.t. $\forall x \in X$, $B(x, r)$ is contained in some U_i for some $i \in J$.

Proof. Assume $\forall n \in \mathbb{N}, \exists x_n \in X$ s.t. $B(x_n, \dfrac{1}{n}) \nsubseteq U_i, \forall i \in J$. From the compactness of X, $\exists$ $(x_{n_k}) \subset (x_n)$ that converges to some $x \in X$. Let $x \in U_{i_0}$. Since U_{i_0} is open, $\exists m$ s.t. $B(x, 1/m) \subset U_{i_0}$. From $x_{n_k} \to x, \exists n_w \geq 2m$ s.t. $x_{n_w} \in B(x, \dfrac{1}{2m})$. Thus

$$B\left(x_{n_w}, \frac{1}{n_w}\right) \subset B\left(x_{n_w}, \frac{1}{2m}\right) \subset B\left(x, \frac{1}{m}\right) \subset U_{i_0},$$

which contradicts the assumption that $B(x_n, 1/n) \nsubseteq U_i$ for all $n \in \mathbb{N}$ and $i \in J$.

Theorem (Borel–Lebesgue) (X, d) is compact ⇔ every open covering $\{U_i\}_{i \in J}$ of X contains a finite subcovering.

Proof. ⇒: Let $\{U_i\}_{i \in J}$ be an open covering of a compact X. $\exists r > 0$ s.t. for every $x \in X$, we have $B(x, r) \subset U_i$ for some $i \in J$. Let's prove that X can be covered by a finite number of $B(x, r)$.
If $B(x_1, r) = X$ for some $x_1 \in X$, we are done. Otherwise, choose $x_2 \in X \backslash B(x_1, r)$. If

$B(x_1, r) \cup B(x_2, r) = X,$ the proof is over. If by continuing this process we obtain X on some step, the proof is over. Otherwise, $\exists$ $(x_n) \subset X$ such that

$$x_{n+1} \notin B(x_1, r) \cup \cdots \cup B(x_n, r)$$

for every $n \in \mathbb{N}$. Since $d(x_n, x_m) \geq r, \forall m, n \in \mathbb{N},$ (x_n) has no convergent subsequence, a contradiction.

$\Leftarrow$: Let $(x_n) \subset X$ and assume that (x_n) has no convergent subsequence. Then $\forall x \in X, \exists$ an open ball $B(x, r_x)$ that contains no points of the sequence (x_n) except possibly x itself. $\{B(x, r_x)\}_{x \in X}$ is an open covering of X and thus contains a finite subcovering. Hence,

$$X = B(x_1, r_1) \cup \cdots \cup B(a_n, r_n),$$

for a finite set $A = \{a_1, \cdots, a_n\}$ in X. By the choice of $B(x, r_x),$ we have $x_k \in A$ for all $k \in \mathbb{N},$ which contradicts the assumption that (x_n) has no convergent subsequence.

• A subset of $\mathbb{K}^n$ (with the usual metric) is compact if and only if it is closed and bounded.

Theorem Suppose that K is a compact subset of (X, d_X) and that $f : (X, d_X) \to (Y, d_Y)$ is continuous. Then $f(K)$ is compact.

Proof Take $\{f(x_n)\} \subset f(K)$. There exists $\{x_{n_k}\}$ such that $x_{n_k} \to x_0 \in K$ since K is compact. Therefore, $f(x_{n_k}) \to f(x_0) \in f(K)$ by the continuity of f. $\square$

Proposition Let K be a compact subset of (X, d). Then any continuous function $f : K \to \mathbb{R}$ is bounded and attains its bounds.

Proof $f(K) \subset \mathbb{R}$ is compact and so is bounded and closed. Let $l = \sup\{f(x), x \in K\}$; then $l < \infty$. Take $\{f(x_n)\} \subset f(K)$ so that $f(x_n) \to l$; then $l \in f(K)$ since $f(K)$ is closed. Hence, there is a $z \in K$ such that $l = f(z)$. Similarly, there is a $y \in K$ such that $f(y) = \inf\{f(x), x \in K\}$.

$\square$

Lemma If $f : (X, d_X) \to (Y, d_Y)$ is continuous and X is compact, then f is uniformly continuous on X : $\forall \epsilon > 0\ \exists \delta > 0$ such that

$$d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon,\ \ x, y \in X. \tag{0.3}$$

Proof If f is not uniformly continuous then $\exists \epsilon > 0$ s.t. $\forall \delta > 0$, $\exists x, y \in X$ with $d_X(x,y) < \delta$ and $d_Y(f(x), f(y)) \geq \epsilon$. Taking $\delta = 1/n$, we can find $x_n, y_n \in X$ such that

$$d_X(x_n, y_n) < 1/n \text{ and } d_Y(f(x_n), f(y_n)) \geq \epsilon. \tag{0.4}$$

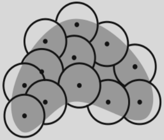
Since X is compact, $\exists \{x_{n_k}\} \subset \{x_n\}$ s.t. $x_{n_k} \to x$. It follows that $y_{n_k} \to x$ also. Since f is continuous at x, we can find $\delta > 0$ such that $d_X(z,x) < \delta$ ensures that $d_Y(f(z), f(x)) < \epsilon/2$. Thus for j sufficiently large we have $d_X(x_{n_j}, x) < \delta$, $d_X(y_{n_k}, x) < \delta$. Hence

$$d_Y(f((x_{n_j}), f((y_{n_j}) \leq d_Y(f((x_{n_j}), f(x)) + d_Y(f((y_{n_j}), f(x)) < \epsilon,$$

contradicting (0.4). $\square$

A subset $B \subseteq X$ is **totally bounded** when it can be covered by a finite number of ϵ-balls, however small their radii ϵ,

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N},\ \exists a_1, \ldots, a_N \in X, \quad B \subseteq \bigcup_{n=1}^{N} B_\epsilon(a_n).$$



- A totally bounded space X is separable.

Proof For each $n = 1, 2, \ldots$, consider finite covers of X by balls $B_{1/n}(a_{i,n})$ and let $A_n := \{a_{i,n}\}$ be the finite set of the centers, so $A := \bigcup_{n=1}^{\infty} A_n$ is countable. For any $\epsilon > 0$ and any point $x \in X$, let $n \geqslant 1/\epsilon$, then x is covered by some ball $B_{1/n}(a_{i,n})$, i.e., $d(x, a_{i,n}) < \epsilon$, thus $\bar{A} = X$.

A uniformly continuous function maps totally bounded sets to totally bounded sets.

Proof Let $f : X \to Y$ be a uniformly continuous function,

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x \in X, \quad f B_\delta(x) \subseteq B_\epsilon(f(x)).$$

Let A be a totally bounded subset of X, covered by a finite number of balls $A \subseteq \bigcup_{n=1}^N B_\delta(x_n)$. Then

$$f A \subseteq \bigcup_{n=1}^{N} f B_\delta(x_n) \subseteq \bigcup_{n=1}^{N} B_\epsilon(f(x_n)).$$

A subset A of a topological space is said to be *relatively compact* or *precompact*, if its closure is compact.

Every relatively compact subset of a metric space X is totally bounded; if the space X is complete then every totally bounded subset of X is relatively compact.

It is easy to show that *a subset M of a metric space is relatively compact if and only if every sequence (x_n) has a convergent subsequence; in this case the limit of the subsequence need not be in M.*

Proposition. *A set A in a metric space X is totally bounded precisely when every infinite sequence of its elements contains a Cauchy subsequence.*

PROOF. Let A be totally bounded and let $\{x_n\} \subset A$ be infinite. Let us cover A by finitely many balls of radius 1. At least one ball U_1 of this cover contains an infinite part of $\{x_n\}$. The set $A \cap U_1$ can be covered by finitely many balls of radius $1/2$. We can find among them a ball U_2 such that $U_1 \cap U_2$ contains an infinite part of $\{x_n\}$. Continuing by induction, for every n we obtain a ball U_n of radius $1/n$ with the property that $V_n := U_1 \cap \cdots \cap U_n$ contains infinitely many points of the original sequence. Now we can find pairwise distinct elements $x_{k_n} \in V_n$. Clearly, we have obtained a Cauchy sequence.

Conversely, suppose that A possesses the indicated property. Suppose that for some $\varepsilon > 0$ there is no finite ε-net in A. By induction we construct a sequence of points $a_n \in A$ with mutual distances at least ε: for a_1 we take an arbitrary element of A; if points $a_1, \ldots, a_n$ are already constructed, there exists a point a_{n+1} with the distances at least ε to all of them, since otherwise the sets $a_1, \ldots, a_n$ would form an ε-net. Such a sequence does not contain a Cauchy subsequence. $\square$