27.2 Use Theorem 27.12 to show that ℓ^1 is not reflexive.

Theorem 27.12 *Let X be a reflexive Banach space. Then any bounded sequence in X has a weakly convergent subsequence.*

T27.2 $\{x_n\} \subseteq l^1$ bounded $x_n = (0, 0, \cdots 1, 0 \cdots)$ n-th slot:=1

if l^1 reflexive $\Rightarrow$ $\exists \{x_{n_k}\} \subseteq \{x_n\}$ s.t. $x_{n_k} \rightharpoonup x_0$

consider $L_f: l^1 \to F$

$x \mapsto L_f(x) = \langle f, x \rangle$ $\quad L_f \in (l^1)^*$, $f = (-1, 1, -1, 1, \cdots) \in l^{+\infty}$

$\Rightarrow L_f(x_{n_k}) = (-1)^{n_k}$, $\{L_f(x_{n_k})\}$ never converges

contradict!

27.4 Recall (see Exercise 8.10) that a space X is uniformly convex if for every $\varepsilon > 0$ there exists $\delta > 0$ s.t.

$$\|x - y\| > \varepsilon, \; x, y \in \mathbb{B}_X \qquad \Rightarrow \qquad \left\| \frac{x+y}{2} \right\| < 1 - \delta,$$

where $\mathbb{B}_X$ is the closed unit ball in X. Show that if X is uniformly convex, then

$$x_n \rightharpoonup x \quad \text{and} \quad \|x_n\| \to \|x\| \quad \Rightarrow \quad x_n \to x.$$

(Set $y_n = x_n/\|x_n\|$, $y = x/\|x\|$, and show that $\|(y_n + y)/2\| \to 1$. Then argue by contradiction using the uniform convexity of X.)

T27.4 if $x_n \rightharpoonup x$, $\|x_n\| \to \|x\|$, but $x_n \not\to x$ i.e. $\|x_n - x\| \not\to 0$

pf: let $y_n = \frac{x_n}{\|x_n\|}$, $y = \frac{x}{\|x\|}$, $\|x_n\| \to \|x\|$ $\Rightarrow$: $x_n \to x \Leftarrow y_n \to y$ (WTS)

$x_n \rightharpoonup x \Rightarrow \forall f \in X^*, f(x_n) \to f(x)$, $\|x_n\| \to \|x\|$ $\Rightarrow \forall f \in X^*, f(y_n) \to f(y)$, $y_n \rightharpoonup y$

$\Rightarrow y_n + y \rightharpoonup 2y$

$\Rightarrow \|2y\| \le \liminf_{n \to +\infty} \|y_n + y\|$ (取f s.t. $f(y) = \|y\|$)

$\liminf_{n \to +\infty} \|y_n + y\| \le \|y_n\| + \|y\| = 2$

$\Rightarrow \left\|\frac{y_n + y}{2}\right\| \to 1$

$\forall \varepsilon > 0, \exists \delta$ s.t. $\left\|\frac{y_n+y}{2}\right\| > 1 - \delta$ implies $\|y_n - y\| \le \varepsilon$ --- uni-convexity

$\Rightarrow \forall \varepsilon > 0, \exists N$ s.t. $\forall n \ge N$, $\left\|\frac{y_n+y}{2}\right\| > 1 - \delta$. then $\|y_n - y\| < \varepsilon$

$\Rightarrow \|y_n - y\| \to 0$; $\|x_n - x\| \to 0$

27.9 Suppose that X is a real Banach space. A theorem due to James (1964) states that if X is not reflexive, then there exists $\theta \in (0,1)$ and sequences $(f_n) \in S_{X^*}$, $(x_n) \in S_X$, such that

$$f_n(x_j) \geq \theta,\ n \leq j, \qquad f_n(x_j) = 0,\ n > j.$$

Show that the sets $C_n := \overline{\operatorname{conv}\{x_n, x_{n+1}, x_{n+2}, \ldots\}}$ form a decreasing sequence of non-empty closed bounded convex sets in X that satisfies $\cap_j C_j = \varnothing$. (Show that if $x \in C_k$ for some k, then $f_n(x) \to 0$ as $n \to \infty$, but that if $x \in \cap_j C_j$, then $f_n(x) \geq \theta$ for every n.) (Megginson, 1998)

T27.9 X not reflexive $\Rightarrow \exists\, \theta \in (0,1)$, $\{f_n\} \subseteq S_{X^*}$ $\{x_n\} \subseteq S_X$ s.t. $\begin{cases} f_n(x_i) \geq \theta & \forall n \leq i \\ f_n(x_i) = 0 & \forall n > i. \end{cases}$

在上面的设定下 $C_n = \overline{\operatorname{conv}\{x_n, x_{n+1}, x_{n+2}, \ldots\}}$ $C_1 \supseteq C_2 \supseteq \cdots$

Show C_n closed bounded, convex, $\bigcap_{i=1}^{+\infty} C_i = \phi$

$\|\cdot\| \leq 1$, $\therefore$ closed, 这几个容易, 略.

Pf: if $x \in C_k$ ① $x = a_1 x_k + a_2 x_{k+1} + \cdots a_m x_{k+m-1} \in C_k^\circ$, $\sum^m a_i = 1$

$f_n(x) \to 0$ since $f_n(x) = 0$ for $\forall n > k+m-1$ $\downarrow$

② $x \in \partial C_n \setminus C_n$, $\forall \varepsilon\ \exists\, y \in C_k^\circ$, $y = a_1 x_k + a_2 x_{k+1} + \cdots a_m \cdot x_{k+m-1}$ s.t. $\|x - y\| < \varepsilon$

$f_n(x) = f_n(y) + f_n(x-y)$

$\leq f_n(y) + \|f_n\| \cdot \varepsilon$

let $\varepsilon \to 0$, $f_n(x) \to 0$ as $n > k+m-1$ $\Rightarrow$ $x \in C_k$ implies $f_n(x) \to 0$

if $x \in \bigcap_{i=1}^{+\infty} C_i$ for $\forall n$, $x \in C_k$ $k > n$

①' $x = a_1 x_k + \cdots a_m x_{k+m-1}$, $\sum_{i=1}^m a_i = 1$

$f_n(x) \geq (a_1 + a_2 + \cdots a_m)\theta = 1 \cdot \theta$

②' $x \in \partial C_k \setminus C_k$, as ②

$f_n(x) \geq f(x) - \|f_n\| \cdot \varepsilon \geq \theta - \|f_n\| \cdot \varepsilon$ $\forall \varepsilon > 0 \Rightarrow f_n(x) \geq \theta$

$\Rightarrow \bigcap_{i=1}^{+\infty} C_i = \phi$. since $\forall x \in C_k$. $x \notin \bigcap_{i=1}^{+\infty} C_i$

27.10 Let X be a Banach space. Show that if every bounded sequence in X has a weakly convergent subsequence, then whenever (C_n) is a decreasing sequence ($C_{n+1} \subseteq C_n$) of non-empty closed bounded convex sets in X, $\cap_n C_n \neq \varnothing$. Deduce, using the previous exercise, that X is reflexive if and only if its closed unit ball is weakly sequentially compact. [Hint: use Corollary 21.8.] (Megginson, 1998)

T2.7.10 X Banach, "$\{x_n\} \subseteq X$ bounded $\Rightarrow \exists \{x_{n_k}\} \subseteq \{x_n\}$ s.t. $x_{n_k} \rightharpoonup x_0$" ; (1) $\Rightarrow$ $C_1 \supseteq C_2 \supseteq C_3 \cdots$

$C_n \neq \phi$. closed. bdd. convex in X

then $\bigcap_{i=1}^{+\infty} C_i \neq \phi$

(2) X reflexive $\rightleftharpoons$ closed unit ball $\bar{S}_X$ weakly cpt

Pf: (1) ① if $C_k \neq C_{k+1}$ $\forall k$

若 $\bigcap_{i=1}^{+\infty} C_i = \phi$, let $x_k \in C_k \setminus C_{k+1}$, $x_{n_k} \rightharpoonup x_0$

for $\forall x_{n_k}$; $\exists C_t$ s.t. $x_{n_k} \notin C_t$

$\therefore \exists f \in X^*$, $\varepsilon > 0$ s.t. $f(y) < f(x_{n_k}) - \varepsilon$ $\forall y \in C_t$

$\Rightarrow \therefore \|x_{n_t} - x_{n_k}\| > \varepsilon$ $\forall n_t \geq t$ $x_{n_k} \not\rightharpoonup x_0$ contradict.

② 若 $\exists C_k = C_{k+1}$; let $x_k = x_l$. x_l 满足 $C_l \setminus C_{l+1} \neq \phi$ 最大的 l

若 $C_k = C_{k+1}$ $\forall k \geq$ some K $\Rightarrow \bigcap_{i=1}^{+\infty} C_i \neq \phi$ obv.

(2): "$\Rightarrow$" X reflexive, $\{x_n\} \subseteq X$ bounded $\Rightarrow x_{n_k} \rightharpoonup x_0$ $\therefore$ weakly cpt

"$\Leftarrow$" if not reflexive, by T2.7.9 we can show $\exists \{x_n\}$ bounded in $\bar{S}_X$, $\{f_n\} \subseteq S_{X^*}$ s.t. $\bigcap_{i=1}^{+\infty} C_i = \phi$

but (1) shows $\bigcap_{i=1}^{+\infty} C_i \neq \phi$

$\Rightarrow$ contradict!