easy

11.1 Show that

$$\|T\|_{B(X,Y)} = \sup_{\|x\|_X \le 1} \|Tx\|_Y \qquad \|T\|_{B(X,Y)} = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X}.$$

T11.1 by definition $\|T\|_{B(X,Y)} := \inf\{M : \|Tx\|_Y \le M \cdot \|x\|_X,\ x \in X\}$
$= \inf\{M : \|Tv\|_Y \le M,\ v \in X,\ \|v\|_X = 1\}$

(1) WTS: $\inf\{M : \|Tv\|_Y \le M,\ v \in X,\ \|v\|_X = 1\} = \sup_{\|x\|_X = 1} \|Tx\|_Y = \sup_{\|x\|_X \le 1} \|Tx\|_Y$

①: $\sup_{\|x\|_X = 1} \|Tx\|_Y \le \sup_{\|x\|_X \le 1} \|Tx\|_Y$ obv

if $\sup_{\|x\|_X = 1} \|Tx\|_Y < \sup_{\|x\|_X \le 1} \|Tx\|_Y$, then it implies $\sup_{\|x\|_X \le 1} \|Tx\|_Y = \|Tx_0\|_Y$ some $x_0 \in X$ $\|x_0\| < 1$

T is linear, $\therefore \|T(\frac{1}{\|x_0\|_X} x_0)\|_Y = \frac{1}{\|x_0\|_X} \cdot \|Tx_0\|_Y > \|Tx_0\|_Y$, contradict!

$\therefore \sup_{\|x\|_X = 1} \|Tx\|_Y = \sup_{\|x\|_X \le 1} \|Tx\|_Y$ holds

② if $N = \sup_{\|x\|_Y = 1} \|Tx\|_Y$, $N \ge \|Tx\|_Y$ $\forall x \in X,\ \|x\|_X = 1 \Rightarrow N \ge \|T\|_{B(X,Y)}$ ⋯ (*)

$\forall \varepsilon > 0, \exists x \in X, \|x\|_X = 1$ $N < \|Tx\|_Y + \varepsilon \le M + \varepsilon$ (arbitary M satisfies $\|Tv\|_Y \le M$ $\|v\|_X = 1$)

$\therefore N \le \inf(M + \varepsilon) = \|T\|_{B(X,Y)} + \varepsilon$ ⋯ $(*)'$

$(*) + (*)' \Rightarrow$ let $\varepsilon \to 0$, $N = \|T\|_{B(X,Y)} = \sup_{\|x\|_X = 1} \|Tx\|_Y = \sup_{\|x\|_X \le 1} \|Tx\|_Y$

(2): WTS $\|T\|_{B(X,Y)} = \sup_{x \ne 0} \frac{\|Tx\|_Y}{\|x\|_X}$

T is linear, $\therefore \frac{\|Tx\|_Y}{\|x\|_X} = \|T \frac{x}{\|x\|_X}\|_Y = \|Tv\|_Y$ $v = \frac{x}{\|x\|_X}$

arbitary $x \ne 0$ implies arbitary $v = \frac{x}{\|x\|_X}$

$\therefore \sup_{x \ne 0} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\|x\| = 1} \|Tx\|_Y$, then by (1) + ① $\Rightarrow$ (2) holds

easy

11.2 Let $X = C_b([0,\infty))$ with the supremum norm. Show that the map $T : X \to X$ defined by setting $[Tf](0) = f(0)$ and

$$[Tf](x) = \frac{1}{x} \int_0^x f(s)\,\mathrm{d}s$$

is linear and bounded with $\|T\|_{B(X)} = 1$.

T11.2 $Tf(x) = \frac{1}{x}\int_0^x f(s)ds$ is linear since $T(\alpha f + \beta g) = \frac{1}{x}\int_0^x (\alpha f + \beta g)(s) ds = \alpha \frac{1}{x}\int_0^x f + \beta \frac{1}{x}\int_0^x g$
$= \alpha Tf(x) + \beta Tg(x)$

$\|Tf(x)\| = \|\frac{1}{x}\int_0^x f(s)ds\| \le \|\frac{1}{x}\int_0^x 1 ds\| \cdot \|f\|_{\infty} = 1 \times \|f\|_{\infty}$

$\therefore \|T\| \le 1$ ⋯ ① (bounded)

$\|T\|_{B(X,X)} = \sup_{\|f\|_{\infty} = 1} \|Tf\| \ge \|T1\| = \|\frac{1}{x}\int_0^x 1 ds\| = 1$, let $f = 1$ const

$\therefore \|T\| \ge 1$ ⋯ ② ①+② $\Rightarrow$ T is linear, bounded, $\|T\|_{B(X)} = 1$

11.3 Show that $T \in L(X, Y)$ is bounded if and only if

$$\sum_{j=1}^{\infty} Tx_j = T\left(\sum_{j=1}^{\infty} x_j\right)$$

whenever the sum on the right-hand side converges. (Pryce, 1973)

T11.3: if T is bounded, then T is cts (since linear) 过题抄答案】

$\therefore T(\sum_{j=1}^{+\infty} x_j) = T(\lim_{n\to+\infty} \sum_{j=1}^{n} x_j) \overset{\text{cts用法!}}{=} \lim_{n\to+\infty}(T\sum_{j=1}^{n} x_j) = \lim_{n\to+\infty}(\sum_{j=1}^{n} Tx_j) = \sum_{j=1}^{+\infty} Tx_j$ ... (1)

if $T(\sum_{i=1}^{+\infty} x_i) = \sum_{i=1}^{+\infty}(Tx_i)$

$\forall z \in X$, construct $\{z_n\}_{n=0}^{+\infty}$ s.t. $z_n \to z$,

construct $\{x_k\}_{k=1}^{+\infty}$ s.t. $x_k = z_k - z_{k-1}$, then $\sum_{i=1}^{n} x_i = z_n$, $T(\sum_{i=1}^{n} x_i) = \sum_{i=1}^{n} Tx_i = Tz_n$

$\therefore T(\sum_{i=1}^{+\infty} x_i) = T(\lim_{n\to+\infty} \sum_{i=1}^{n} x_i) = T(\lim_{n\to+\infty} z_n) = Tz$

$\sum_{i=1}^{+\infty}(Tx_i) = \lim_{n\to+\infty} \sum_{i=1}^{n}(Tx_i) = \lim_{n\to+\infty} T(\sum_{i=1}^{n}(x_i)) = \lim_{n\to+\infty} Tz_n$ $\Rightarrow$ T cts. thus bounded at z

z is arbitary, $\therefore T$ is bounded in X ... (2)

(1)+(2): $\sum_{i=1}^{+\infty} Tx_i$ converges, then $T \in B(X,Y)$ bounded $\rightleftarrows T(\sum_{i=1}^{+\infty} x_i) = \sum_{i=1}^{+\infty}(Tx_i)$

easy 11.4 Suppose that $(T_n) \in B(X, Y)$ and $(S_n) \in B(Y, Z)$ are such that $T_n \to T$ and $S_n \to S$. Show that $S_nT_n \to ST$ in $B(X, Z)$.

T11.4 $\forall \varepsilon > 0$, $\exists N_T$ s.t. $\|T_n - T\| \le \varepsilon$ $\forall n \ge N_T$ ($\|T_n x - Tx\| < \varepsilon$, $\|x\| = 1$) $\therefore \|T_n\| \le \|T\| + \varepsilon$

$\forall \varepsilon > 0$, $\exists N_S$ s.t. $\|S_n - S\| \le \varepsilon$ $\forall n \ge N_S$

$\therefore n \ge \max(N_T, N_S)$, $\|S_nT_n - ST\| = \|S_nT_n - ST_n + ST_n - ST\|$

$\le \|S_n - S\| \cdot \|T_n\| + \|S\| \cdot \|T_n - T\|$

$\le \varepsilon(\|T\| + \varepsilon) + \|S\| \cdot \varepsilon$

$\therefore \forall y > 0$, let ε small enough s.t. $\varepsilon(\|T\| + \varepsilon) + \|S\| \cdot \varepsilon < y$, $\exists N = \max(N_T, N_S)$ s.t.

$\|S_nT_n - ST\| \le y$ $\forall n \ge N$.

11.5 Suppose that X is a Banach space and $T \in B(X)$ is such that

$$\sum_{j=1}^{\infty} \|T^n\|_{B(X)} < \infty.$$

谢条件是$\|T\|<1 \rightleftharpoons \sum_{n=1}^{+\infty}\|T\|^n<+\infty$

Show that

$$(I-T)^{-1} = I + T + T^2 + \cdots = \sum_{j=0}^{\infty} T^j.$$

(This is known as the Neumann series for $(I-T)^{-1}$.) In the case that $\|T\| < 1$ deduce that

$\|T\|<1$是为了说明这个可以展开

$$\|(I-T)^{-1}\| \le (1-\|T\|)^{-1}.$$

T11.5: (1): let $S_n = \sum_{i=0}^{n} T^i$

$(I-T)(\sum_{i=0}^{n} T^i) = \sum_{i=0}^{n} T^i - \sum_{i=1}^{n+1} T^i = \underbrace{\sum_{i=0}^{n} T^i - \sum_{i=1}^{n} T^i}_{I} - T^n = I - T^n$

$\therefore \|(I-T)(\sum_{i=0}^{n} T^i) - I\| = \|T^n\| \le$ ~~$\|I\|+\|T^n\|$~~ $\cdots$ ①

$\sum_{i=1}^{+\infty}\|T^n\| < +\infty \quad \therefore \forall \varepsilon>0 \ \exists N$ s.t. $\|T^n\|<\varepsilon \ \forall n \ge N \cdots$ ②

$\therefore \forall \varepsilon>0, \ \|(I-T)(\sum_{i=1}^{n} T^i) - I\| < \varepsilon \ \forall n \ge N, \quad \cdots$ by ①+②

$\therefore \lim_{n\to+\infty} \|(I-T)(\sum_{i=1}^{n} T^i) - I\| = \|\lim_{n\to+\infty}(I-T)(\sum_{i=1}^{n} T^i) - I\| = 0$

$\Rightarrow (I-T)\cdot\sum_{i=1}^{+\infty} T^i = I, \ (I-T)^{-1} = \sum_{i=1}^{+\infty} T^i, \quad (\sum_{i=1}^{+\infty} T^i$ converge since $\sum_{i=1}^{+\infty}\|T^i\| < +\infty)$

(2): $(1-\|T\|)^{-1} = 1 + \|T\| + \|T\|^2 + \cdots = \sum_{i=0}^{+\infty}\|T\|^i$

$\|(1-T)^{-1}\| = \|1+T+T^2+\cdots\| = \|\sum_{i=0}^{+\infty} T^i\| \le \sum_{i=0}^{+\infty}\|T^i\| \le \sum_{i=1}^{+\infty}\|T\|^i$

$\therefore (1-\|T\|)^{-1} \ge \|(1-T)^{-1}\|$

11.6 Use the result of the previous exercise to show that if X and Y are Banach spaces and $T \in B(X,Y)$ is invertible, then so is $T+S$ for any $S \in B(X,Y)$ with $\|S\|\|T^{-1}\| < 1$, and then

$$\|(T+S)^{-1}\| \le \frac{\|T^{-1}\|}{1-\|S\|\|T^{-1}\|}. \tag{11.15}$$

T11.6 (2)抄答案的

let $P = T^{-1}(T+S) = I + T^{-1}S$; $T+S, T$ invertible $\therefore P$ invertible

$\therefore \|P\|^{-1} \le \|P^{-1}\| = \|(I+T^{-1}S)^{-1}\| \le \frac{1}{1-\|T^{-1}S\|}$ $\cdots$ 由T11.5(2)知 $I^{-1}S$相当于11.5中的"$-T$"

$\therefore \|(T+S)^{-1}\| = \|P^{-1}T^{-1}\| \le \|P^{-1}\|\cdot\|T^{-1}\| \le \frac{\|T^{-1}\|}{1-\|T^{-1}S\|}$

11):

($N_r(p)$:= neighbor of radius r, centered at p)

Lem11.16 $(X,\|\cdot\|_X),(Y,\|\cdot\|_Y)$ are Banach spaces, $T\in B(X,Y)$ invertible, then:

$\forall S\in B(X,Y), \|S\|\cdot\|T^{-1}\|<1, \Rightarrow T+S\in B(X,Y)$

i.e. $S\in N_{\|T^{-1}\|^{-1}}(T)$ is invertible; the invertible set of operators 记 I is open

pf: • consider linear map $J: X\to X,\ x\mapsto T^{-1}(y-Sx)$ | 思路: 希望 $\forall y$, $\exists x_0$ s.t. $(T+S)(x_0)=y$

$\|J(x_1)-J(x_2)\| = \|T^{-1}S(x_1-x_2)\|$ | $\Leftrightarrow T^{-1}(y-Sx)=x$ 有解,

$\le \|T^{-1}\|\cdot\|S\|\cdot\|x_1-x_2\|,\ \|T^{-1}\|\cdot\|S\|<\alpha<1$ | 这么定义J是因为 $\|T^{-1}\|\cdot\|S\|<1$

$< \alpha\|x_1-x_2\|$

X is Banach, J is contracting $\Rightarrow \exists$ unique x_0 s.t. $Jx_0=x_0$ i.e. $Tx_0=y-Sx_0$ for given y

$\therefore \forall y\in Y, \exists$ unique x_0 s.t. $(T+S)(x_0)=y$ ... (*)

$\therefore$ T+S is onto

• $\|S\|\cdot\|T^{-1}\|<1 \quad \therefore \frac{1}{\|T^{-1}\|}-\|S\|=\beta>0$

$\|(T+S)(x)\|_Y \ge \|Tx\|-\|Sx\|$

$\ge\left(\frac{1}{\|T^{-1}\|}-\|S\|\right)\cdot\|x\| = \beta\|x\| \quad \therefore$ T+S is injective

$\Rightarrow$ T+S invertible

11.9 Suppose that X is a Banach space, Y a normed space, and take some $T \in B(X, Y)$. Show that if there exists $\alpha > 0$ such that $\|Tx\| \geq \alpha\|x\|$, then Range(T) is closed. (Rynne and Youngson, 2008)

T11.9. WTS: $\forall y_n=Tx_n,\ y_n\to y$. $\exists x\in X$ s.t. $Tx=y$

$\forall\varepsilon>0, \exists N$ s.t. $\|y_n-y\|<\varepsilon\ \forall n\ge N$.

$\Rightarrow \|y_n-y_m\|<\varepsilon\cdot 2\quad \forall m,n\ge N \quad \therefore \|Tx_n-Tx_m\|<2\varepsilon$ then

$\Rightarrow \|x_n-x_m\|<\frac{1}{\alpha}\varepsilon$ since $\|Tx\|\ge\alpha\|x\|$

$\therefore \{x_n\}$ is Cauchy $x_n\to x\in X$,

又∵ $\lim_{n\to\infty}T(x_n)=T(\lim_{n\to\infty}x_n)=Tx$ since $T\in B(X,Y)$ bounded thus continous

$\Rightarrow y=Tx\in \text{rang}(T)$

$\therefore$ rang(T) is closed