L1:

all defined in complex Banach X $\Rightarrow$ need "Banach" to let T^{-1} must in $B(Y,X)$

Def: resolvent, spectrum, point spectrum $\Rightarrow \rho(T), \sigma(T), \sigma_p(T)$ the inverse is "good"

TH: $T \in B(X)$, ① $\lambda_1, \lambda_2, \cdots \lambda_n$ distinct in $\sigma_p(T) \Rightarrow x_1, x_2, \cdots x_n$ independent

② $\dim \cdot \mathrm{Ker}(T-\lambda I) = \lambda$-multiplicity

otherwise

$\exists\, T^{-1}$ s.t. $T^{-1} \circ T = T \circ T^{-1} = I$; $\|T^{-1}\| = +\infty$

T in σ / ρ 都不适合!

TH: $T \in B(X)$ $\begin{cases} |\lambda| > \|T\| \Rightarrow \lambda \in \rho(T) \\ \sigma(T) \text{ closed}, \ \sigma(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|T\|\} \end{cases}$

TH: $T \in B(\text{Hilbert})$, $\sigma(T^*) = \{\bar{\lambda} : \lambda \in \sigma(T)\}$

Def: $T: X \to Y$ compact if $\forall \{x_n\} \subseteq X$ bounded, $\exists \{x_{n_k}\} \subseteq \{x_n\}$ s.t. $Tx_{n_k} \to y \in Y$

不一定是 $y = T(\cdot)$! not necessarily

Example: $S_l : (x_1, x_2, \cdots) \in l^2 \longmapsto \underset{\in l^2}{(x_2, x_3, \cdots)}$, $S_r : (x_1, x_2, \cdots) \in l^2 \longmapsto (0, x_1, x_2, \cdots) \in l^2$ compact

pf: l^p complete $1 \leq p \leq +\infty \Rightarrow$ converges $\Leftrightarrow$ Cauchy

TH: $(X, \|\cdot\|_X)$ normed, Y Banach ; $\{T_n\}_{n=1}^{+\infty}$ compact, $T_n \to T \in B(X,Y)$ ie. $\|T_n - T\| \to 0$

$\Rightarrow T \in K(X,Y)$, thus $K(X,Y)$ complete

pf: $\{x_n\} \subseteq X$ bounded, $T_1 x_{11}, T_1 x_{12}, T_1 x_{13}, \cdots \to y_1 \in Y$

$T_2 x_{21}, T_2 x_{22}, \cdots \to y_2 \in Y$

$\Rightarrow \{T(x_{nn})\}$ Cauchy thus conv 康托对角线

TH: $T \in B(X,Y)$, $\dim(\mathrm{Range}\,Y) < +\infty \Rightarrow T$ compact

$\Downarrow$

$\begin{cases} \cong \mathbb{R}^n, \ \{x_n\} \text{ bdd} \Rightarrow \exists \{x_{n_k}\} \text{ converges} \\ \text{Complete} \end{cases}$

$\Downarrow$

$\|Tx_n - Tx_m\| \leq \|T\| \cdot \varepsilon \to 0 \Rightarrow T$ compact

Spectral Mapping Th: X complex Banach, $T\in B(X)$, $Q(t)=\sum_{k=0}^{n} a_k t^k$ is polynomial

$\Rightarrow Q(\sigma_p(T))=\sigma_p(Q(T))$, $Q(\sigma(T))=\sigma(Q(T))$

(only prove "$\supseteq$" since "$\subseteq$" obv)

pf (1) $u\in\sigma_p(Q(T))$

$\Rightarrow \exists x\neq 0$ s.t. $(Q(T)-u)(x)=0$

$\Rightarrow (T-u_n)(T-u_{n-1})\cdots(T-u_1)(x)=0$

只要存在 u_k 为特征值! $1\le k\le n$

if $(T-u_1)(x)=0 \Rightarrow (u_1,x)$ is eigen-pair

if $(T-u_1)(x)\neq 0$, $(T-u_2)(T-u_1)(x)=0 \Rightarrow (u_2,(T-u_1)(x))$ is eigen-pair

$\cdots\cdots$

find k s.t. $(T-u_i)(T-u_{i-1})\cdots(T-u_1)(x)\neq 0\ \forall i\le k-1$; $(T-u_k)\cdots(T-u_1)(x)=0$

$\Rightarrow (u_k, \underbrace{(T-u_{k-1})\cdots(T-u_1)(x)}_{x_k})$ is eigen-pair

$Q(T)(x)=u\cdot(x)$

$\Rightarrow (Q(T)-u)(x)\cdot\big((T-u_{k-1})\cdots(T-u_1)\big)=(Q(T)-u)\big((T-u_{k-1})\cdots(T-u_1)\big)(x)=(Q(T)-u)(x_k)=0$

$\because Q(T)-u=(T-\cdot)(T-\cdot)\cdots(T-\cdot)$ form

$\Rightarrow Q(T)(x_k)=u\cdot x_k$, $u_k\neq 0$ $\cdots\cdots$ ①

(u_k,x_k) is eigen-pair $\Rightarrow Q(T)(x_k)=Q(u_k)\cdot(x_k)$ $\cdots\cdots$ ②

$\Big\}\Rightarrow Q(u_k)\cdot x_k=u\cdot x_k$

$\Rightarrow u=Q(u_k)$, i.e. $u\in Q(\sigma_p(T))$

rmk: $Q(T)x=u\cdot x \Rightarrow$ factorize to find (u_k,x_k), $Q(T)x_k=Q(u_k)\cdot x_k$

$\Rightarrow Q(T)x_k=u\cdot x_k$

$\Big\}\Rightarrow \checkmark$

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(2): $u\in\sigma(Q(T)) \Rightarrow Q(T)-uI$ not inv

$u\in Q(\sigma(T)) \Leftrightarrow u=Q(u_i)$, $u_i\in\sigma(T)$

$\Leftrightarrow Q(t)-u=a_n(t-t_1)(t-t_2)\cdots(t-t_n)$, $t_j=u_i$ some j

把多项式老实写出来!

$\Leftrightarrow$ 即 $(t-t_1)(t-t_2)\cdots(t-t_n)$ 中 $\exists t_i\in\sigma(T)$, $T-t_iI$ not inv

$\Leftrightarrow Q(T)-uI$ not inv

$\therefore$ if $u\notin Q(\sigma(T)) \Rightarrow Q(t)-u=a_n(t-t_1)(t-t_2)\cdots(t-t_n)$, $t_i\in P(T)$ $\forall i$

$\Leftrightarrow Q(T)-u$ inv

$\Rightarrow$ contradicto!

TH: $k \in C([a,b]\times[a,b])$, then $T: L^2(a,b) \to L^2(a,b)$, $[Tu](x) = \int_a^b k(x,y)\,u(y)\,dy$ is compact

Method1 : $k_n(x,y) = \sum_{i=1}^{n} f_i(y)\cdot g_i(x)$, $k_n \to k$

$[T_n u](x) = \sum_{i=1}^{n} \int_a^b g_i(y) u(y) dy \cdot f_i(x) = \sum_{i=1}^{n} c_i \cdot f_i(x)$

finite-dimensional range $\Rightarrow$ $T_n u$ is compact --- ①

$|[T_n u](x) - [Tu](x)| \le \int_a^b |k(x,y) - k_n(x,y)| \cdot |u(y)| \cdot dy$

$\le \left(\int_a^b \|k-k_n\|^p\right)^{\frac{1}{p}} \cdot \left(\int_a^b \|u\|^q\right)^{\frac{1}{q}}$, $\frac{1}{p}+\frac{1}{q}=1$, $1<p,q<+\infty$

$\Rightarrow T_n u \to Tu$ --- ②

①+② $\Rightarrow$ Tu is compact

Method2: Approximate by Rieman-sum

Method3: $\{Tu : u \in L^2(a,b)\}$ equ-cts + totally bounded $\xRightarrow{\text{Ascoli-Arzela's TH}}$ cpt!

Def: $T \in B(H)$ is Hilbert-Schmidt if $\exists$ orthonormal basis $\{e_n\}_{n=1}^{+\infty} \subseteq H$ s.t.

$$\|T\|_{HS}^2 := \sum_{i=1}^{+\infty} \|Te_i\|^2 < +\infty$$

note that: $\|T\|_{B(H)} = \sup_{\|x\|=1} \|Tx\| \le \|T\|_{HS}$

TH: $\|T\|_{HS} < +\infty \Rightarrow T$ is compact

pf: $T \in B(H)$, cts $\Rightarrow$ $Tx = T\left(\sum_{i=1}^{+\infty} (x,e_i) e_i\right) \overset{\text{cts}}{=} \sum_{i=1}^{+\infty} (x,e_i)\cdot Te_i$

let $T_n : H \to H$, $T_n x := \sum_{i=1}^{n} (x,e_i) Te_i$ $\Rightarrow$ T_n compact since $\dim \operatorname{Range} T_n = n$

$\|(T_n - T)u\| = \|\sum_{i=n+1}^{+\infty} (x,e_i)\cdot Te_i\| \le \sum_{i=n+1}^{+\infty} \|(x,e_i)\| \cdot \|Te_i\|$

$\le \underbrace{\left(\sum_{i=n+1}^{+\infty} \|(x,e_i)\|^2\right)^{\frac{1}{2}}}_{\to 0} \underbrace{\left(\sum_{i=n+1}^{+\infty} \|Te_i\|^2\right)^{\frac{1}{2}}}_{<+\infty} \to 0$

$T_n \to T$ $\therefore$ T compact

$<+\infty$ = 这里用到 T is B-H

TH: $\underbrace{T: H \to H}_{\text{project}}$ is compact linear $\Rightarrow \exists \{T_n\}_{n=1}^{\infty}$ of finite rank s.t. $T_n \to T$

H→H cpt算子可以用有限维算子逼近

pf: let $\mathbb{B} := \{x \in H : \|x\| \le 1\}$, $K := \overline{T(\mathbb{B})}$

T is compact $\rightleftharpoons$ $K = \overline{T(\mathbb{B})}$ is compact set

$\Rightarrow \forall \varepsilon > 0, \exists v_1, v_2 \cdots v_n \in H$ s.t. $K \subseteq \bigcup_{i=1}^{n} B(v_i, \varepsilon)$

$\Rightarrow \forall x \in \mathbb{B}, \exists v_i$ s.t. $\|Tx - v_i\| \le \varepsilon$ ······ ①

let $V_\varepsilon := \mathrm{span}(v_1, v_2 \cdots v_n)$, $T_\varepsilon := P_V(T)$ = project T onto V

$\Rightarrow \dim \mathrm{Range}(T_\varepsilon) \le n$, $\therefore T_\varepsilon$ compact ······ ②

$\|P_V Tx - \underbrace{P_V v_i}_{= v_i}\| \le \|Tx - v_i\|$ $\Rightarrow$ $\|T_\varepsilon - T\| \le \|T_\varepsilon - v_i\| + \|v_i - T\|$

$\le \|T - v_i\| + \|T - v_i\|$ ①

$\le 2\varepsilon$

$\Rightarrow T_\varepsilon \to T$ as $\varepsilon \to 0$ ②

$\Rightarrow T$ compact

rmk: 核心: 用 $T_\varepsilon = P_V T$ 构造有限维,

- $\|P_V a - P_V b\| \le \|a - b\|$
- def of compact set $\Rightarrow \forall x \in \mathbb{B}, Tx \in K \therefore \exists v_i$ s.t. $\|Tx - v_i\| \downarrow$
- T compact $\rightleftharpoons$ $\overline{T(\mathbb{B})}$ cpt, or $\overline{T(S)}$ cpt

TH: $T \in K(H)$, $\Rightarrow T^* \in K(H)$, defined by $(Tx, y) = (x, T^*y)$

pf: $\|T^*\| = \|T\|$, $\Rightarrow \{x_n\} \subseteq H$ bounded implies $\{T^* x_n\} \subseteq H^*$ bdd

$\Rightarrow \exists \{x_{n_k}\} \subseteq \{x_n\}$ s.t. $\|T(T^* x_{n_k}) - T(T^* x_{n_l})\| \to 0$

$\Rightarrow TT^*$ compact!

$\therefore \underbrace{\|(T \circ T^*(x_n - x_m), x_m - x_n)\|}_{= \|T^*(x_n - x_m)\|} \le \|T \circ T^*(x_n - x_m)\| \cdot \|x_m - x_n\| \Rightarrow T^*$ cpt

rmk: T cpt $\Rightarrow T \circ T^*$ cpt $\Rightarrow T^*$ cpt

TH: X is Banach, $T\in K(X,X)$ $\dim(X)=+\infty \Rightarrow 0\in\sigma(T)$, ie. T not-invertible

比 H 广

无穷维 Banach 空间中，紧算子不可逆

pf: 反证假设 Assume T invertible, $\Rightarrow T^{-1}\in B(X)$

Lem: $A: X\to Y$ compact, $B: Z\to X$ bounded $\Rightarrow A\circ B$ compact

$\therefore T\circ T^{-1} = I$ compact $\Leftrightarrow I(\overline{\mathbb{B}})$ compact, $\mathbb{B}:=\{x\in X: \|x\|=1\}$

$\Leftrightarrow X$ is finite-dimensional (week5.Lec01) TH5.5 in text-book

TH: $T\in B(H)$, $T=T^*$ $\Rightarrow$ (1) eigenvalues are real, ie. $\sigma_p(T)\subseteq\mathbb{R}^1$ (actually $\sigma_p(T)\subseteq V(T)\subseteq\mathbb{R}^1$)

(2) $\lambda_1,\lambda_2\in\sigma_p(T)$, $\lambda_1\neq\lambda_2 \Rightarrow (x_1,x_2)=0$

obv proof

Def: $T\in B(H)$, $T=T^*$, numerical range of T: $V(T)=\{(Tx,x): x\in H, \|x\|=1\}$

$\lambda\in\sigma_p(T)$, $\exists x\in H$, $\|x\|=1$ s.t. $Tx=\lambda x \Rightarrow \sigma_p(T)\subseteq V(T)$

Hilbert-Schimidt TH. $T\in B(H)$, $T=T^*$, then $\Rightarrow$ (1) $V(T)\subseteq\mathbb{R}$, obv pf

(2) $\|T\|_{B(H)}=\sup\{|\lambda|:\lambda\in V(T)\}=\sup|V(T)|$

proof(2): $\overset{\text{in } V(T),\, \|x\|=1}{|(Tx,x)|}\leq\|Tx\|\cdot\|x\|\leq\|T\| \Rightarrow M=\sup|V(T)|\leq\|T\|$ ... ①

$$4\cdot Re(Tu,v) = 2(Tu,v)+2\underset{2(Tv,u)}{\underset{\|}{(v,Tu)}} = (T(u+v),u+v)-(T(u-v),u-v)$$

$$=\left(T\cdot\frac{u+v}{\|u+v\|},\frac{u+v}{\|u+v\|}\right)\cdot\|u+v\|^2-\left(T\cdot\frac{u-v}{\|u-v\|},\frac{u-v}{\|u-v\|}\right)\cdot\|u-v\|^2$$

$$\leq M\cdot\|u+v\|^2+M\cdot\|u-v\|^2 = 2M(\|u\|^2+\|v\|^2)$$

let $v=\frac{\|u\|}{\|Tu\|}\cdot Tu \Rightarrow \|v\|=\|u\|$

$\therefore 4\cdot Re(Tu,v)=4\cdot\|Tu\|\cdot\|u\|\leq 2M(\|u\|^2+\|u\|^2) \Rightarrow M\geq\|T\|$ ...②

①+② $\Rightarrow M=\sup(|V(T)|)=\|T\|$

核心: parallelogram $4Re(Tu,v)=(T(u+v),u+v)-(T(u-v),u-v)$

$\| \; 2Re(Tu,v)+2Re(Tv,u)$

$\downarrow$

let $v:=\frac{\|u\|}{\|Tu\|}\cdot u$, $(Tu,v)\in\mathbb{R}$, $(Tu,v)=\|Tu\|\cdot\|u\|$

TH: $T\in B(H)$ $T=T^*$; T is compact $\Rightarrow$ at least one of $\pm\|T\|$ is eigenvalue

$\therefore \|T\| = \max\{|\lambda| : \lambda\in\sigma_p(T)\} = \max\{|\lambda| : \lambda\in V(T)\}$

pf: given $T\in B(H)$ $T=T^* \Rightarrow \|T\| = \sup|V(T)| = \sup\{|(Tx,x)| : x\in H, \|x\|=1\}$

$\Rightarrow \exists\{x_n\}\subseteq H, \|x_n\|=1\ \forall n$, s.t. $(Tx_n,x_n)\to a$, $a=\|T\|$ or $-\|T\|$ ...①

T compact, $\{x_n\}\subseteq H$ bounded $\Rightarrow \exists\{x_{n_k}\}\subseteq\{x_n\}$ s.t. $Tx_{n_k}\to y\in H$ ...②

$$\|Tx_n - a x_n\|^2 = \|Tx_n\|^2 + a^2 - a(Tx_n,x_n) - a(x_n,Tx_n)$$
$$\le a^2+a^2-2a(Tx_n,x_n) \to 0 \quad ①$$

$\therefore Tx_n \to a x_n$

$\Rightarrow \|a x_{n_k} - y\| \le \|a x_{n_k} - Tx_{n_k}\| + \|Tx_{n_k} - y\| \to 0$ i.e. $a x_{n_k}\to y$, $\lim_k x_{n_k}$ exists

$\Rightarrow (a, \frac{1}{a}\lim Tx_{n_k})$ is eigen-pair

rmk: 定义$\|\cdot\|$: $\exists\ (Tx_n,x_n)\to a$

$\Rightarrow \|Tx_n - a x_n\|^2 \to 0 \Rightarrow Tx_n \to a x_n$

$\Rightarrow Tx_{n_k}\to y$

$\Big\}\Rightarrow a x_{n_k}\to y. \Rightarrow (a, \frac{1}{a}\lim Tx_{n_k})$ ✓

TH: $T\in B(H)$, Y is closed subspace $\subseteq H$, (1). $TY\subseteq Y \Rightarrow T^*Y^\perp \subseteq Y^\perp$

(2). $TY=Y \Rightarrow T^*Y^\perp = Y^\perp$ (Y closed, $\therefore (Y^\perp)^\perp = Y$)

TH(1). $A\in B(X)$, $|\lambda|>\|A\| \Rightarrow \lambda\in\rho(A)$. $R_\lambda(A) := (A-\lambda I)^{-1} = -\sum_{k=0}^{+\infty}(\frac{A}{\lambda})^k\cdot\frac{1}{\lambda}$

$\Downarrow$

$-\frac{1}{\lambda}(I-\frac{A}{\lambda})^{-1}$

(2): $\forall\lambda\in\rho(A)$, $|\lambda-\lambda_0| < \|R_\lambda(A)\|^{-1} \Rightarrow \lambda_0\in\rho(A)$, $R_{\lambda_0}(A) := \sum_{k=0}^{+\infty}(\lambda_0-\lambda)^k\cdot(R_\lambda(A))^{k+1}$ 背出来!

$A-\lambda_0 I = \underset{\text{inv}}{\underbrace{A-\lambda I}} + (\lambda-\lambda_0)I = (\frac{1}{\underset{\neq 0}{\lambda-\lambda_0}}(A-\lambda I) + I)(\cdots)$

$\begin{cases} \frac{1}{|\lambda-\lambda_0|}\|A-\lambda I\| < \frac{1}{\|I^{-1}\|} \text{ i.e. } |\lambda-\lambda_0| > \|A-\lambda I\| \Rightarrow A-\lambda_0 I \text{ inv} \\ |\lambda_0-\lambda| < \|A-\lambda I\|^{-1} \Rightarrow A-\lambda_0 I \text{ inv} \ ✓ \end{cases}$

$\Rightarrow R_{\lambda_0}(A) := (A-\lambda_0 I)^{-1}$

$(A-\lambda_0 I)\cdot\sum_{k=0}^{+\infty}(\lambda_0-\lambda)^k(R_\lambda(A))^{k+1}$

$$= \sum_{k=0}^{+\infty}(\lambda_0-\lambda)^k(R_\lambda(A))^{k+1}(A-\lambda_0 I-(\lambda-\lambda_0)I)$$

$$= \sum_{k=0}^{+\infty}(\lambda_0-\lambda)^k(R_\lambda(A))^k - (\lambda_0-\lambda)^{k+1}(R_\lambda(A))^{k+1}$$

TH: X is Banach. $T \in B(X)$, T is compact on X

$\Rightarrow$ (1) $Ker(I-T)$ finite-dimensional

(2) $range(I-T)$ is closed

Pf: (1): if $ker(I-T) = span(e_1, e_2, \dots e_n, \dots)$ $\|e_i\|=1$ $\forall i$. $e_i \perp e_j$

$\{e_n\}$ bounded, $\exists \{e_{n_k}\} \subseteq \{e_n\}$ s.t. $T(e_{n_k}) \to y$ $\Rightarrow$ $e_{n_k} \to y$ since $(T-I)e_n = 0$

$\Rightarrow$ contradict!

(2): let $y_n = (I-T)(x_n) \in$ Range, $\{y_n\}_{n=1}^{+\infty} \to y$

① $\{x_n\} \subseteq X$ bounded, $\Rightarrow \exists \{x_{n_k}\} \subseteq \{x_n\}$ s.t. $Tx_{n_k} \to y_0$ $\Big\}$ $\Rightarrow \{(I-T)(x_{n_k})\} \to y$, $Tx_{n_k} \to y_0$

$\therefore x_{n_k} \to y + y_0 \in X$

$\Rightarrow y_n = (I-T)(x_n) \to y = (I-T)(y+y_0) \in$ Range

② $\{x_n\} \subseteq X$ unbounded,

let $Z = ker(I-T)$. $d_n := \inf\{\|x_n - z\| : z \in Z\} = d(x_n, Z)$ $\langle$ let $x_n \notin Z$ $\forall n$

(1): $dim(Z) < +\infty$. $\Rightarrow \exists z_n \in Z$ s.t. $d_n = \|x_n - z_n\|$

(*): if $\{d_n\}$ unbounded, ie. let $d_n \to +\infty$

let $v_n = \dfrac{x_n - z_n}{\|x_n - z_n\|}$, $\Rightarrow v_n \notin Ker$ since $x_n \notin Ker$ ... ⊛

$(I-T)(v_n) = \dfrac{(I-T)(x_n)}{\|x_n - z_n\|} \to \dfrac{y}{\|x_n - z_n\|} \to 0$

T compact. $\{v_n\}$ bounded $\therefore Tv_{n_k} \to y_0$ $\Big\}$ $\Rightarrow v_{n_k} \to y_0 \in Ker$ ...⊗

⊛ + ⊗ $\Rightarrow$ $Z = ker$ closed, $v_n \notin Z \Rightarrow v \notin Z$

contradict!

$\Rightarrow \{d_n\}$ bounded

$\Rightarrow$ use $\{x_n - z_n\} \subseteq X$ replace $\{x_n\} \subseteq X$ in ①, similarly $y \in$ Range

Range closed: $\begin{cases} \{x_n\} \text{ bdd} \\ \{x_n - z_n\} \text{ bdd}, z_n \in Ker \text{ 不影响结果} \end{cases}$

$\left.\begin{matrix} (I-T)(\ ) \to \triangle \\ T(\ ) \to \square \end{matrix}\right\} \Rightarrow (\ ) \to \triangle + \square$ ✓

Th: X is Banach, $\dim(X) = +\infty$, $T \in B(X)$ compact $\Rightarrow \sigma(T) = \{0\}$

separable $\Rightarrow$ 可数 k_n

or $\sigma(T) = \{0\} \cup \{k_n\}$ $k_n \in \sigma_p(T)$; $\{k_n\}$ finite or $k_n \to 0$

(1): 若 $\exists \lambda \in \sigma(T), \lambda \neq 0 \Rightarrow \lambda \in \sigma_p(T)$

{循环论证} W16用这个证的

Pf: 反证 Assume $\lambda \notin \sigma_p(T)$, ie $T - \lambda I$ is injective; $\lambda \in \sigma(T)$, ie. $T-\lambda I$ not invertible $\Big\} \not\Rightarrow$ $T - \lambda I$ not surject ... ref to Week16

$T - \lambda I = (\frac{1}{\lambda} T - I) \cdot \lambda$, $\frac{1}{\lambda} T$ compact + X Banach $\Rightarrow$ Range is closed subspace

let $X_1 := (T - \lambda I)(X)$, ∴ $X_n := (T-\lambda I)(X_{n-1}) \Rightarrow X_n$ is closed subspace $\subseteq X_{n-1}$

if $X_n = X_{n+1}$, $T - \lambda I$ injective $\Rightarrow X_{n-1} = X_n, \dots, X_1 = X$, contradict! ?Why

$\Rightarrow X_n \subsetneq X_{n-1}$, $n = 2, 3, \dots$

∴ $\exists \{x_n\}$ s.t. $\|x_n\| = 1\ \forall n$, $x_n \in X_n \setminus X_{n+1}$; $d(x_n, X_{n+1}) \geq \frac{1}{2}$

$$Tx_n - Tx_m = \lambda(x_n - x_m) + (T - \lambda I)(x_n - x_m)$$

$$= \underbrace{\lambda x_n}_{\in X_n \setminus X_{n+1}} - \underbrace{\lambda x_m + (T-\lambda I)(x_n) - (T - \lambda I)(x_m)}_{\in X_{n+1}}; \quad \text{if } m > n$$

$\Rightarrow \|Tx_n - Tx_m\| \geq d(x_n, X_{n+1}) \geq \frac{1}{2}$; T compact, $\{x_n\}$ bounded, $\exists Tx_{n_k} \to y$ $\Big\} \Rightarrow$ contradict $\Rightarrow \lambda \in \sigma_p(T)$

(2): if $\{k_n\}$ infinite, $k_n \to 0$

Pf: 反证 Assume $\exists \delta > 0$ s.t. $|k_n| > \delta\ \forall n$

let $\{k_n\}$ distinct, corresponding $\{x_n\}$ independent, let $\|x_i\| = 1\ \forall i$

let $X_n = \mathrm{span}(x_1, x_2, \dots x_n) \Rightarrow \exists v_n \in X_n \setminus X_{n-1}$, $\|v_n\| = 1$ s.t. $d(v_n, X_{n-1}) \geq \frac{1}{2}$

let $v_n = a_n x_n + z_{n-1}$, $z_{n-1} \in X_{n-1}$

$$T \cdot v_n - Tv_m = a_n \lambda_n x_n + T \cdot (z_{n-1} - z_{m-1})$$

$$= \lambda_n \Big(\underbrace{v_n}_{\in X_n} - \underbrace{z_n + \tfrac{1}{\lambda_n}(Tz_{n-1} - T \cdot z_{m-1})}_{\in X_{n-1}} \Big), \quad m < n$$

$\Rightarrow \|Tv_m - Tv_n\| \geq |\lambda| \cdot d(v_n, X_{n-1}) \geq \frac{1}{2} \times \delta$; T compact, $\{v_n\}$ bounded ... $\Big\} \Rightarrow$ contradict $\Rightarrow k_n \to 0$

rmk: $\begin{cases} X_n = (T - \lambda I)^n (X) \\ X_n = \mathrm{span}(v_1, \dots v_n) \end{cases} \Rightarrow \begin{matrix} x_n \in X_n \setminus X_{n+1} \\ v_n \in X_n \setminus X_{n-1} \end{matrix} \Rightarrow \begin{matrix} d(x_n, X_{n+1}) \geq \frac{1}{2} \\ d(v_n, X_{n-1}) \geq \frac{1}{2} \end{matrix} \xRightarrow{Tv_n - Tv_m \cdots} \begin{matrix} T\{x_{n_k}\} \text{ diverge} \\ T\{v_{n_k}\} \text{ diverge} \end{matrix} \Rightarrow$ 矛盾!

(2)' if add: $T = T^*$, $\Rightarrow \|Tx_n - Tx_m\|^2 = |\lambda_n|^2 + |\lambda_m|^2$ since $\langle x_m, x_n \rangle = 0$; (2) obv to prove then

Hilbert-Schimidt TH', H is Hilbert, $T\in B(H)$ is compact, self-adjoint, then

$\Rightarrow \begin{cases} (1)\ \exists \text{ finite or countably infinite orthonormal } \{e_i\},\ e_i \text{ is eigenvector} \\ (2)\ \text{corresponding } \lambda_i \neq 0,\ \forall x\in H,\ \Rightarrow Tx=\sum_{i=1}^{+\infty}\lambda_i\cdot(x,e_i)\cdot e_i \end{cases}$

pf: (1) $T\in B(H)$, cpt, $T^*=T \Rightarrow \|T\|=\max\{|\lambda|: \lambda\in\sigma_p(T)\}$, $-\|T\|$ or $\|T\|$ is eigenvalue

$\Rightarrow \exists e_1\in H,\ Te_1=\lambda_1 e_1,\quad \lambda_1 = \text{''},\ \|e_1\|=1$

let $H_1=e_1^\perp \Rightarrow$ we can prove (omitted) H_1 is Hilbert since closed

$T\in B(H)$, Y closed subspace $\Rightarrow TY=Y \rightleftarrows T^*(Y^\perp)=Y^\perp$

$\therefore$ let $Y:=\text{span}(e_1)$, then $T^*(H_1)=H_1$, $T\cdot H_1=H_1$

$\Rightarrow T|_{H_1}\in B(H_1)$, cpt, $T^*_{H_1}=T|_{H_1} \Rightarrow \exists e_2,\ \lambda_2=\|T|_{H_1}\|$ or $-\|T|_{H_1}\| \Rightarrow H_2=\{e_1,e_2\}^\perp$

$\Rightarrow T|_{H_2}\in B(H_2), \cdots$

$\therefore$ obtain $\{e_i\}_{i=1}^{+\infty}$ $\|e_i\|=1$, if $T|_{H_n}=0$ some n, then $e_n=e_{n+1}=\cdots=0$

又 $\because e_n\in H_n=\{e_1,\cdots e_{n-1}\}^\perp$, $(e_n,e_i)=0\ \forall i\neq n \Rightarrow$ orthonormal

实际上H中必存在一组 orthonormal basis ... Gram-Schimidt

(2): ① 若 $T_n=0$ some n, H has ortho $\{e_1,\cdots e_{n-1}\}$

$\forall x\in H$. let $y:=x-\sum_{i=1}^{n-1}(x,e_i)\cdot e_i \Rightarrow y\in H_n,\ T_n y=Ty=0$

$\therefore Tx=\sum_{i=1}^{n-1}\lambda_i(x,e_i)\cdot e_i$

② 若 $T_n\neq 0\ \forall n$; $\forall x\in H$. let $y_n:=x-\sum_{i=1}^{n-1}(x,e_i)\cdot e_i\in H_n$

$\|x\|^2=\|y_n+\sum_{i=1}^{n-1}(x,e_i)\cdot e_i\|^2=\|y_n\|^2+\sum_{i=1}^{n-1}|(x,e_i)|^2$, since $y_n\perp e_i\ \forall i$

$\Rightarrow \|Tx-\sum_{i=1}^{n-1}\lambda_i(x,e_i)\cdot e_i\|=\|Ty_n\|\leq \|\lambda_n\|\cdot\underbrace{\|y_n\|}_{\leq\|x\|}\to 0$

X Banach, dim=+∞, T cpt, $\lambda_n\to 0$ ... previous TH

rmk: H Hilbert, U closed subspace $\leq H \Rightarrow H=U\oplus U^\perp$

$\Rightarrow$? $\{e_n\}_{n=1}^{+\infty}$ are basis 对吗.我觉的是, 那①②自然成立?