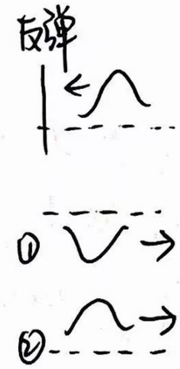
L15 仍考虑齐次式.

d=1: $u(t,x)=\frac{1}{2}(g(x+t)+g(x-t))+\frac{1}{2}\int_{x-t}^{x+t}h(y)dy$ 在L10中找出 full-space的解




half-line ① $u(t,0)=0$  $x>0$

odd extension $\bar{g}(x)=\begin{cases}g(x) & x>0\\ -g(-x) & x\le 0\end{cases}$, $\bar{h}(x)=\begin{cases}h(x) & x>0\\ -h(-x) & x\le 0.\end{cases}$

② $\partial_x u(t,0)=0$  $x>0$.

even extension $\bar{g}(x)=\begin{cases}g(x) & x>0\\ g(-x) & x\le 0\end{cases}=g(|x|)$, $\bar{h}(x)=h(|x|)$

① $u(t,0)=0$, then $\bar{u}(t,x)=\frac{1}{2}(\bar{g}(x+t)+\bar{g}(x-t))+\frac{1}{2}\int_{x-t}^{x+t}\bar{h}(y)dy$

$$=\begin{cases}\frac{1}{2}(g(x+t)+g(x-t))+\frac{1}{2}\int_{\cdots}h(y)dy=u(t,x), & x>t\\ \frac{1}{2}(g(x+t)-g(-x+t))+\frac{1}{2}\int_{t-x}^{x+t}h(y)dy & 0\le x<t\end{cases}$$

初值和非齐次项均为奇函数. Cauchy problem 的解也为…

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d>1: 齐次WE: $\begin{cases}\partial_t^2-\Delta u=0\\ u|_{t=0}=g(x),\ \partial_t u|_{t=0}=h(x)\end{cases}$

记 $\begin{cases}U(x;t,r)=U(t,r)=\fint_{\partial B_r(x)}u(t,y)dS(y), & \text{fix } x\in\mathbb{R}^d\\ G(x,t)=G(r)=\fint_{\partial B_r(x)}g(y)dy\\ H(x,t)=H(r)=\fint_{\partial B_r(x)}h(y)dy\end{cases}$

Th:(EPD) consider $U(t,r)$, $G(r)$, $H(r)$. fix $x\in\mathbb{R}^d$, then:

$$\begin{cases}U_{tt}=U_{rr}+\frac{d-1}{r}U_r\\ U|_{t=0}=G,\ U_t|_{t=0}=H\end{cases}$$

proof: ① $U(x,t)=\fint_{\partial B_r(x)}u(t,y)dS(y)=\frac{1}{|\partial B_r|}\int_{\partial B_1(0)}u(t,ry+x)dS(ry+x)$

$=\frac{1}{|\partial B_r|}\int_{\partial B_1(0)}u(t,ra+x)dS(ra+x)$ 变量换成 a: $ra+x=$ initial y

$=\frac{1}{|\partial B_1|}\int_{\partial B_1(0)}u(t,ra+x)dS(a)$

即要说明为什么: $dS(ra+x)=r^d\cdot dS(a)$ $\nRightarrow$ $\frac{dS(ra+x)}{da}=r^d\frac{dS(a)}{da}$, 代入 $ra+x=y$

LHS $=\frac{dS(ra+x)}{dy}\cdot\frac{dy}{da}=r^d\times\frac{dS(y)}{dy}$

☆ 我的疑问: $\frac{dS(y)}{dy}\ne\frac{dS(a)}{da}$, 如 $S(x)=x^2+x+1$  $\frac{dS(y)}{dy}=2x+1\big|_{x=y}\ne\frac{dS(a)}{da}=2x+1\big|_{x=a}$

或者 $S(y)=y^2+2y+1=(ra+x)^2+2(2a+x)+1=S(a)$ 这样满足 $\frac{dS(y)}{dy}=\frac{dS(a)}{da}$

②: $U_r(t,r) = \partial_r\left(\fint_{\partial B_1(0)} u(t, ra+x)\, dS(a)\right) = \fint_{\partial B_1(0)} D_2 u(t, ra+x) \times a\, dS(a)$ ; $D_2$表示对第2个slot求导

$$\overset{(1)}{=} \fint_{\partial B_1(0)} D_2 u(t, ra+x)\cdot \vec{n}\, dS(a)$$

$$\overset{(2)}{=} \frac{1}{|\partial B_1|}\int_{B_1(0)} \Delta u(t, ra+x)\, da \quad \text{代入 } a=\frac{y-x}{r}$$

$$= \frac{1}{|\partial B_1|}\int_{B_r(x)} \Delta u(t,y) \frac{1}{r^d}\, dy \overset{(3)}{=} \frac{r}{d} \fint_{B_r(x)} \Delta u(t,y)\, dy$$

Q: 问题: (1) $D_2u(t,ra+x)$ 我觉得是标量, 为什么 $= D_2u(t,ra+x)\cdot\vec{n}$

(2) $\int_{\partial U} F\cdot \vec{n}\, dS(y) = \int_U D_y F\, dy$. 我觉得(2) $= \frac{1}{|\partial B_1|}\int_{B_1(0)} D_a D_2 u(ra+x)\, da = \frac{1}{|\partial B_1|}\int_{B_1(0)} D_2^2 u(ra+x)\, da \times \underline{r^d}$.

(3) $\frac{|\partial B_1|}{|B_r|} = \left|\frac{\pi r^2}{\frac{4}{3}\pi r^3}\right|$ 似乎并不是 $\frac{r}{d}$ 的关系

③: $U_{rr}(t,r) = \partial_r\left(\frac{r}{d}\frac{1}{|B_r|}\times\int_{B_r(x)}\Delta u(y)\, dy\right)$

$$= \partial_r\left(\frac{r}{d}\cdot\frac{1}{\alpha(d)\cdot r^d}\times\int_{B_1(0)}\Delta u(ar+x)\cdot r^d\, da\right) = \partial_r\left(\frac{r}{d\cdot\alpha(d)}\cdot\int_{B_1(0)}\Delta u(ar+x)\, da\right)$$

$\vec{n}$要?

$$= \frac{1}{d\cdot\alpha(d)}\int_{B_1(0)}\Delta u(ar+x)\, da + \frac{r}{d\cdot\alpha(d)}\cdot\int_{B_1(0)} D_2(\Delta u(ar+x))\cdot\vec{n}\, da \quad = \int_{B_r(x)} D_2(\Delta u(y))\cdot\frac{\vec{n}}{r}\, dy\times\frac{1}{r^d}$$

$$= \frac{1}{d\cdot\alpha(d)}\int_{B_r(x)}\Delta u(y)\cdot\frac{1}{r^d}\, dy + \frac{r}{d\cdot\alpha(d)}\cdot\int_{\partial B_r(x)}\Delta u(y)\, dS(y)\cdot\frac{1}{r}\cdot\frac{1}{r^d}$$

$$= \frac{1}{d}\fint_{B_r(x)}\Delta u(y)\, dy + \frac{1}{d\cdot\alpha(d)\cdot r^d}\int_{\partial B_r(x)}\Delta u(y)\, dS(y)$$

Q: 老师写的是 $\left(\frac{1}{d}-1\right)\fint_{B_r} \times dy$ 由②(3)将 $|\partial B_r|$ 和 $|B_r|$ 看成 $\frac{r}{d}$. 这段就是 $\fint_{\partial B_r(x)}\Delta u(y)\, dS(y)$

$$\Rightarrow \begin{cases} U_r(t,r) = \frac{r}{d}\fint_{B_r(x)}\Delta u(t,y)\, dy \\ U_{rr}(t,r) = \left(\frac{1}{d}-1\right)\fint_{B_r(x)}\Delta u(t,y)\, dy + U_{tt} \end{cases} ;\quad U_{tt} = \fint_{\partial B_r(x)} u_{tt}(t,y)\, dy = \fint_{\partial B_r(x)}\Delta u(t,y)\, dy$$

$\therefore U_{tt} = U_{rr} - U_r\times\frac{d-1}{r}$ (euler-poisson-darboux)

$\Downarrow$

then: $r^2 U_{tt} = (r^2 U_r)_r$,

• let $\bar{U} = r\times U$, $\Rightarrow$ $\bar{U}_{tt} = \bar{U}_{rr}$ ⋯ ① 构造出了一维齐次WE; since $U, G, H$ 均为积分结果,是一维的

$$\begin{cases} \bar{U} = \fint_{\partial B_r(x)} u(t,y)\, dS(y)\times r, \\ \bar{G} = \fint_{\partial B_r(x)} g(y)\, dS(y)\times r = G\times r \\ \bar{H} = \fint_{\partial B_r(x)} h(y)\, dS(y)\times r = H\times r \end{cases}$$

$$\text{WE: } \begin{cases} \partial_t^2 u = \Delta_x u \\ u(0,x) = g(x) \\ \partial_t u(0,x) = h(x) \end{cases}$$

$\therefore$ let $t\to 0$, $\bar{U} = \bar{G}$, $\bar{U}_t = \bar{H}$ ⋯ ②

①+② $\Rightarrow \begin{cases} \partial_{tt}\bar{u} = \partial_{rr}\bar{u} \\ \bar{u} = \bar{G} \quad t=0 \\ \bar{u}_t(0,r) = \bar{H}(r) \end{cases}$ , $\bar{u}=0$ $r=0$ obv $\Rightarrow$ 在$r>0$时$\bar{u}|_{r=0}=0$ 故为 odd extension!

$\therefore \bar{u}(r,t) = \begin{cases} \frac{1}{2}(\bar{G}(r+t) - \bar{G}(t-r)) + \frac{1}{2}\int_{-r+t}^{r+t} \bar{H}(y)dy & r \le t \\ \frac{1}{2}(\bar{G}(r+t) + \bar{G}(r-t)) + \frac{1}{2}\int_{r-t}^{r+t} \bar{H}(y)dy & r \ge t, \end{cases}$ $\left(u = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy\right)$

- $u(t,x) = \lim_{r\to 0^+} \frac{\bar{u}}{r}$，由$\bar{u}$定义可知

$\bar{u}_r = \begin{cases} \frac{1}{2}(\bar{G}'(r+t) + \bar{G}'(t-r)) + \frac{1}{2}(\bar{H}(r+t) + \bar{H}(-r+t)) & r \le t \ \checkmark \\ \frac{1}{2}(\bar{G}'(r+t) + \bar{G}'(r-t)) + \frac{1}{2}(\bar{H}(r+t) - \bar{H}(r-t)) & r \ge t \end{cases}$ (因为 let $r \to 0^+$, $r \le t$)

$= \begin{cases} \bar{G}'(t) + H(t) & r \le t \ \checkmark \\ \frac{1}{2}(\bar{G}'(t) + \bar{G}'(-t)) + \frac{1}{2}(\bar{H}(t) - \bar{H}(-t)) & r \ge t. \end{cases}$

$\therefore u(t,x) = \bar{u}_r = \bar{G}'(t) + H(t)$

$= \int_{\partial B_t(x)} th(y) + g(y) + \nabla g(y) \times (y-x) \, dS(y) \Leftarrow$ Kirchhoff's formula for $d=3$