19.1 Let H be a Hilbert space and U a closed linear subspace of H. Use the Riesz Representation Theorem to show that any $\phi \in U^*$ has an extension to an element $f \in H^*$ such that $f(x) = \phi(x)$ for every $x \in U$ and $\|f\|_{H^*} = \|\phi\|_{U^*}$.

19.2 Show that the extension obtained in the previous exercise is unique.

19.3 Let X be a normed space and U a subspace of X that is not closed. If $\hat{\phi}: U \to \mathbb{K}$ is a linear map such that

$$|\hat{\phi}(x)| \leq M\|x\| \qquad \text{for every} \quad x \in U$$

show that $\hat{\phi}$ has a unique extension ϕ to $\overline{U}$ (the closure of U in X) that is linear and satisfies

$$|\phi(x)| \leq M\|x\|$$

for every $x \in \overline{U}$. (For any $x \in \overline{U}$ there exists a sequence $(x_n) \in U$ such that $x_n \to x$. Define

$$\phi(x) := \lim_{n\to\infty} \hat{\phi}(x_n).$$

Show that this is well defined and has the required properties.)

T19.1 Hilbert H, $U \subseteq H$ closed. linear. subspace

$\forall \phi \in U^*$, $\exists$ extension $f \in H^*$ s.t. $f(x) = \phi(x)$ $\forall x \in U$. $\|f\|_{H^*} = \|\phi\|_{U^*}$

Pf: $U \subseteq H$ U is closed $\therefore U$ is Hilbert

by Riesz Representation, $\overset{\text{bdd}}{\phi}: U \to \mathbb{R}^1$ can be equivalent to $\phi(u) = (u, u_0)$ $\forall u$, some u_0

$\{ \|\phi\|_{U^*} = \|u_0\|_U$ since $\|\phi(u)\| \leq \|u\| \cdot \|u_0\|$ implies $\|\phi\| \leq \|u_0\|$, let $u = u_0$ 取等

let $f(u) := (u, u_0)$ $\forall u \in H$. $\|f\|_{H^*} = \|u_0\| = \|\phi\|_{U^*}$, f satisfies the requirement

($\phi \in U^*$. U hilbert $\Rightarrow \phi$ is linear, cts & bdd necessarily)

T19.2: Show f in T19.1 is unique

Pf: 在T19.1中 u_0 is unique by Riesz Repre Th

if $\exists$ extension f_1, f_2, $f_1|_U = f_2|_U$ thus $f_1(u) = (u, u_1) = f_2(u) = (u, u_2)$ on U. $u_i \in U$

$(u, u_1 - u_2) = 0$ on U

let $u = u_1 - u_2$, then $\|u_1 - u_2\|^2 = 0$ $\therefore u_1 = u_2 \Rightarrow f_1 = f_2$ unique!

T19.3 $U \leq (X, \|\cdot\|_X)$ not closed, $\hat{\phi}: U \to \mathbb{K}$ is linear. $|\hat{\phi}(x)| \leq M \cdot \|x\|$ $\forall x \in U$

$\Rightarrow \hat{\phi}$ has unique extension ϕ to $\bar{U}$, $|\phi(x)| \leq M \cdot \|x\|$ $\forall x \in \bar{U}$

Pf: $\forall x \in \bar{U} \setminus U$, x is limit point of U

$\exists \{x_n\}_{n=1}^{+\infty} \subseteq U$ s.t. $x = \lim_{n\to+\infty} x_n$, define $\phi(x) := \lim_{n\to+\infty} \hat{\phi}(x_n)$; $\phi(x') = \hat{\phi}(x')$ $x' \in U$

if $x = \lim_{n\to+\infty} x_n$, $y \in \bar{U} \setminus U = \lim y_n$, $\alpha x + \beta y = \lim_{n\to+\infty}(\alpha x_n + \beta y_n)$, $\alpha x + \beta y \in \bar{U} \setminus U$ since U is subspace

$\phi(\alpha x + \beta y) = \lim_{n\to+\infty} \hat{\phi}(\alpha x_n + \beta y_n) = \alpha\phi(x) + \beta\phi(y)$ 前面证过 $\bar{U}$ 也一定是subspace (if U is)

$\therefore \phi$ is linear ... ①

$\|\phi(x)\| = \lim_{n\to+\infty} \|\hat{\phi}(x_n)\| \leq \lim_{n\to+\infty} M\|x_n\| = M \cdot \lim_{n\to+\infty} \|x_n\| = M\|x\|$ ... ②

$\therefore \phi$ is exetension of $\hat{\phi}$

if ϕ is not unique, i.e. ϕ_1, ϕ_2 are extension satisfy ①, ②

$\phi_1|_U = \phi_2|_U = \hat{\phi}$ $\hat{\phi}$ is linear, bounded $\Rightarrow$ continoues on U; ϕ is cts on $\bar{U}$ (linear, bdd)

$\forall x \in \bar{U} \setminus U$, $\exists \{x_n\}_{n=1}^{+\infty} \subseteq U$ s.t. $x_n \to x$ $\lim_{n\to+\infty} \phi(x_n) = \cancel{\lim} \phi(\lim_{n\to+\infty} x_n) = \phi(x)$ since ϕ is cts

the limit is unique $\Rightarrow \phi_1 = \phi_2 = \lim_{n\to+\infty} \phi(x_n)$ 由连续性所知ϕ唯一