L1 metric & general topology (L2也有)

def: metric space is a pair of (X,d); X is set, d is metric (distance function)
d satisfies ①. positivity ②. symmetry ③. triangle inequality

def: $P(X)$ is the collection of all subsets of X (幂集 2^X), $\tau \subseteq P(X)$ is topology if τ satisfies:
① $\phi \in \tau$, $X \in \tau$
② $U_\alpha \in \tau$ $\forall \alpha \in A$, then $\bigcup_{\alpha \in A} U_\alpha \in \tau$
③ $U_i \in \tau$, $i=1,2,3,\dots$ then $\bigcap_{n=1}^{+\infty} U_n \in \tau$
} "closed" under countable "$\cap$", any "$\cup$"

Example1: $X \neq \phi$, the function $d(x,y) = \begin{cases} 0 & x=y \\ 1 & x \neq y \end{cases}$ is a metric (discrete)

Ex2: $C[a,b] = \{x(t) : x(t)$ continuous on $[a,b]$ w.r.t $t\}$;
$d_1(x,y) = \sup_{t\in[a,b]} |x(t)-y(t)|$, $d_2(x,y) = \int_a^b |x(t)-y(t)|\,dt$ are metrics on $C[a,b]$

Ex3: $X = \mathbb{R}^n$, $d_p(x,y) = \begin{cases} \left(\sum_{i=1}^n |x_i-y_i|^p\right)^{\frac{1}{p}} & , 1 \le p < +\infty \\ \max_i |x_i - y_i| & , p = +\infty \end{cases}$ is metric on X $\forall p \ge 1$

lem: (Minkowski's) $\left(\sum_{i=1}^n |x_i+y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}} \stackrel{\text{denoted}}{=} u+v$

pf: $|x_i+y_i|^p \le (|x_i|+|y_i|)^p = (u+v)^p \cdot \left(\frac{1}{u+v}\right)^p \cdot \left(u \cdot \frac{|x_i|}{u} + v \cdot \frac{|y_i|}{v}\right)^p$
$= (u+v)^p \left(\frac{u}{u+v} \cdot \frac{|x_i|}{u} + \frac{v}{u+v} \cdot \frac{|y_i|}{v}\right)^p$, $\frac{u}{u+v} + \frac{v}{u+v} = 1$,
$f(z) = z^p$ convex
$\le (u+v)^p \cdot \left[\frac{u}{u+v} \cdot \left(\frac{|x_i|}{u}\right)^p + \frac{v}{u+v} \cdot \left(\frac{|y_i|}{v}\right)^p\right]$
$\Rightarrow \sum_{i=1}^n |x_i+y_i|^p \le (u+v)^p \times 1$; ✓

Q: $LHS^p \le \sum_{i=1}^n |x_i|^p + \sum_{i=1}^n |y_i|^p = u^p + v^p$, $RHS^p = (u+v)^p$
计 $p>1$, $u^p+v^p-(u+v)^p \le 0$ $\Rightarrow$ 证完, 没必要像上面pf一样 ✓

Ex4: $(X_1,d_1), (X_2,d_2)$ are metric spaces; the product metric d on $X_1 \times X_2$ can be
$d((y_1,y_2),(z_1,z_2)) = \max\{d_1(y_1,z_1), d_2(y_2,z_2)\}$
or $= d_1(y_1,z_1) + d_2(y_2,z_2)$
or $= \sqrt{d_1(y_1,z_1)^2 + d_2(y_2,z_2)^2}$; d 对 $A \times B \subseteq X_1 \times X_2$ 也适用, $d|_{A\times B}$ is metric ✓

def3: $\begin{cases} x \in E,\ x \text{ is interior point of } E \text{ if: } \exists r>0 \text{ s.t. } B_r(x) \subseteq E \Rightarrow x \in E^\circ \\ x \notin E,\ x \text{ is exterior point of } E \text{ if: } \exists r>0 \text{ s.t. } B_r(x) \subseteq E^c \Rightarrow x \in (E^c)^\circ \end{cases}$, and: $(E^c)^\circ \subseteq (E^\circ)^c$

all other points $x \in \partial E = \{$boundary points of $E\} = \partial(E^c)$
$\Leftrightarrow \forall r>0$, $B_r(x) \cap E \neq \phi$, $B_r(x) \cap E^c \neq \phi$

prop3.1: $\bar{E} = \partial E \cup E^\circ$ (obv) def4: E is open if: $\forall x \in E$, x is interior, $\Leftrightarrow E^\circ = E$

lem4.1: $x_n \to x \Leftrightarrow \forall$ open set U containing x, $\exists N \in \mathbb{N}^+$ s.t. $\forall n \geq N$, $x_n \in U$

即:任意小邻域中,有$\{x_N, x_{N+1}, x_{N+2}, \cdots\}$会集中. ; proof obv

def5: E is closed if: E^c is open

lem5.1 $\forall \{x_n\} \subseteq E$, $x_n \to x$ then $x \in E \Leftrightarrow E$ is closed set

proof5.1 "$\Rightarrow$" 设$F = E^c$

记: how to use lem!

let $y \in F$, $\{x_n\} \to y$; y and $\{x_n\}$ are arbitary

① if y is exterior pt of F, obv not!

希望证 y interior of F, 即排除 ∂F, $(F^c)^\circ$

② if $y \in \partial F$, $\forall r > 0$, $B_r(y) \cap F \neq \phi$, $B_r(y) \cap F^c \neq \phi$

$\Downarrow$

$B_r(y) \cap E \neq \phi$ then $y \in E$ or, $\exists N$ s.t. $\forall n \geq N$, $x_n \in E$

$\Rightarrow y \in E$, contradiction! ✓

$\therefore$ ①② both wrong, $y \in F^\circ$, F is open

"$\Leftarrow$" E is closed $\therefore$ F open

if $\exists \{x_n\} \subseteq E$, $x_n \to x$, $x \in F$;

$\exists r > 0$, $B_r(x) \subseteq F$

lem4.1. $\forall B_r(x)$, $\exists N$ s.t. $\forall n \geq N$, $x_n \in B_r(x)$ $\Rightarrow$ contradict with $\{x_n\} \subseteq E$ ✓

Rmk5.4 (equivalent definition:) E contains all limit point

def6: $p \in A$, p is limit point (accumulation) of A if: $\forall \varepsilon > 0$, $(B_\varepsilon(p) \setminus \{p\}) \cap A \neq \phi$

p is isolated point of A, otherwise

prop6.1: $p \in A$, p is limit point, $\Leftrightarrow \forall$ neighborhood of p, $V \cap A$ is infinite set

$\Leftrightarrow \{p_n\} \subseteq A$, $\{p_n\}$ distinct s.t. $p_n \to p$

prop6.2 in $\mathbb{R}^n$, A is discrete set,

if A infinite, bounded $\Rightarrow \exists$ limit point, contradiction! in $\mathbb{R}^n$ 离散点集中 bdd 和 infinite 不同时

def7: the closure $\bar{E} = E^\circ \cup \partial E$;

Th7.1 $\bar{E}$ is closed

proof7.1: $\bar{E} = E^\circ \cup \partial E = E \cup \partial E$ obv

$\forall \{x_n\} \subseteq \bar{E}$, $x_n \to x$, WTS: $x \in \bar{E} = E \cup \partial E$

if $x \notin E$, $x \in B_r(x) \cap E^c$ for $\forall r > 0$, $\cdots$ ①

consider $B_r(x) \cap E$: $\forall r > 0$, $\exists N$ s.t. $\forall n \geq N$, $x_n \in B_r(x)$ $\cdots$ from lem 4.1

if $\exists x_n \in E$, $n \geq N$, $B_r(x) \cap E \neq \phi$ $\cdots$ ②

if $\forall n \geq N$, $x_n \notin E$, then $x_n \in \partial E$; $\Rightarrow x_n$ is limit point of E, $\forall n \geq N$

take any $x_m \in B_r(x)$, $R < r - d(x_m, x)$

then $B_R(x_m) \subseteq B_r(x)$,

$B_R(x_m) \setminus \{x_m\} \cap E \neq \phi$ $\Big\} \Rightarrow B_r(x) \cap E \neq \phi$ ... ②''

①+② $\Rightarrow$: $x \in \partial E$

$\therefore$ if $x \notin E$, then $x \in \partial E$, $\Rightarrow x \in E \cup \partial E = \bar{E}$, $\bar{E}$ is closed

lem 7.2 $\bar{E} = E \cup E'$, where $E' = \{\text{limit pt of } E\}$

(given $\bar{E} = E^\circ \cup \partial E = E \cup \partial E$)

因为$x \notin E$

proof 7.2: $x \in E \cup \partial E$, $\begin{cases} x \in E, \Rightarrow x \in E \cup E' \\ x \notin E, x \in \partial E, \text{ then } \forall r > 0\ B_r(x) \cap E \neq \phi,\ B_r(x) \setminus \{x\} \cap E \neq \phi \Rightarrow x \in E' \end{cases}$

$\therefore x \in E \cup E'$, that is, $E \cup \partial E \subseteq E \cup E'$ ...①

$y \in E \cup E'$ $\begin{cases} y \in E, \Rightarrow y \in E \cup \partial E \\ y \notin E, y \in E', \text{ then } \forall r > 0,\ B_r(y) \cap E \neq \phi,\ B_r(y) \cap E^c \supseteq \{y\} \Rightarrow y \in \partial E \end{cases}$

$\therefore y \in E \cup \partial E$, that is, $E \cup \partial E \supseteq E \cup E'$ ...②

①+② $\Rightarrow E \cup \partial E = E \cup E'$

coro 7.2: E is closed, $\rightleftharpoons$ $\bar{E} = E$

coro 7.2': similarly, E' is closed, $E' = (\bar{E})' = (\bar{E'})$ $\qquad$ E和$\bar{E}$ limit pt 一样

coro 7.2'': $x \in E' \rightleftharpoons \forall r > 0,\ B_r(x) \cap E \neq \phi$ ($x \in E$ or $x \in E'$)

例: Ex5: $E' \neq (E')'$, if: $E = \{0, 1, \frac{1}{2}, \dots \frac{1}{n}, \dots\}$, $E' = \{0\}$, $(E')' = \phi$

Ex6: $E^\circ \neq (\bar{E})^\circ$ if: $E = \mathbb{Q}$, $E^\circ = \phi$ $(\bar{E})^\circ = (\mathbb{R})^\circ = \mathbb{R}$

$\mathbb{Q}$ not open, not closed

Ex7: $\bar{E} \neq (\bar{E^\circ})$ if: $E = \bigcirc$ (circle), $\bar{E} = E$, $(\bar{E^\circ}) = \bar{\phi} = \phi$

Th 7.3: $E \subseteq F$, F is closed set $\Rightarrow \bar{E} \subseteq F$

proof 7.3: F is closed, $\therefore F = \bar{F} = F \cup F'$ ... by coro 7.2 $\quad$ 这步关键!,直接证$x \in E' \Rightarrow x \in F$烦一点

$x \in E'$, then $\forall r > 0$, $B_r(x) \setminus \{x\} \cap E \neq \phi$

$\therefore B_r(x) \setminus \{x\} \cap F \supseteq B_r(x) \setminus \{x\} \cap E$, $x \in F'$ $\Rightarrow E \subseteq F$ implies $E' \subseteq F'$

$E' \subseteq F'$, $E \subseteq F \Rightarrow (\bar{E} = E' \cup E) \subseteq (\bar{F} = F' \cup F)$

coro 7.3: $E \subseteq F$, E is open set $\Rightarrow E \subseteq F^\circ$ (proved by complement)

Th7.4: $\bar{E}$ is the minimal closed set containing E. $\left(\Leftrightarrow \bar{E} = \bigcap_{\alpha\in A} E_\alpha,\ \ A = \{\alpha : E_\alpha \text{ is closed, } E_\alpha \supseteq E\}\right)$
$\underset{F}{\parallel}$

proof 7.4: $F \subseteq \bar{E}$ by def ... ($\bar{E}$ closed by Th7.2)
$E \subseteq F$, F is closed $\Rightarrow \bar{E} \subseteq F$ ... by Th7.3
$\Rightarrow \bar{E} = F$

Th7.5: $\overline{(E^c)} = \overline{(\bar{E})^c} = (A^o)^c$

def8 $\{x_n\}$ is Cauchy 当: $\forall \varepsilon > 0\ \exists N$ s.t. $\forall m,n \geq N$, $d(x_m, x_n) < \varepsilon$
(X,d) is complete if $\forall$ Cachy sequence converges in X.

prop 8.1 $\{x_n\}$ is Cauchy, then $\{x_n\}$ converges $\Leftrightarrow \exists$ subsequnce $\{x_{n_k}\}$, $x_{n_k} \to x_0$

Exa 8.2 $x_n = (1+\frac{1}{n})^n$, $x_n \to e$
$\{x_n\}$ is Cauchy in Q, but limit $e \notin Q$, $\therefore Q$ not complete

Exa 8.3 Q: $\{x_n\}$无 limit pt, 且 $\{x_n\}$ Cauchy 的例子?

def9. $\bar{A} = X$, then A is dense in X
$(\bar{A})^o = \phi$, $\bar{A}$ has empty interior, then A is nowhere dense

Lem9.1 $(\overline{A^c})^c = A^o$ or, $\overline{A^c} = (A^o)^c$

Th9.2: A is dense in X, $\Leftrightarrow \forall x \in X$, $r > 0$; $B_r(x) \cap A \neq \phi$

等价定义

proof 9.2: $\bar{A} = X$, Let $A^c = E$,
$\Rightarrow \overline{E^c} = X = (E^o)^c$, $\therefore E^o = (A^c)^o = \phi$ ... ☆! (补集无内点)

"$\Rightarrow$" 反证: 若 $\exists B_r(x) \cap A = \phi$, some $x \in X$, $r > 0$,
$B_r(x) \subseteq A^c$, $(A^c)^o \neq \phi$ ... contradict with ☆"

"$\Leftarrow$" obv, $x \in X \Rightarrow x \in A$, or $x \in A'$
$\therefore x \in A \cup A' = \bar{A}$,

Th9.3: $\{G_n\}$ is dense open subset of $\mathbb{R}^n$, then $\bigcap_{n=1}^{\infty} G_n$ is dense

proof 9.3: $\forall x_1, r_1$, $B_{r_1}(x_1) \cap G_1 \neq \phi$
$B_1(x_1)$, G_1 open, $\therefore B_{r_1}(x_1) \cap G_1$ open,

$\Rightarrow \exists x_2, r_2,\ B_{r_2}(x_2) \subseteq B_{r_1}(x_1) \cap G_1$

$B_{r_2}(x_2) \cap G_2 \neq \phi$

$\Rightarrow \exists x_n, r_n,\ B_{r_n}(x_n) \subseteq B_{r_{n-1}}(x_{n-1}) \cap G_{n-1}$; and $B_{r_n}(x_n) \cap G_n \neq \phi$

$\Rightarrow$ let $r_{n+1} \leq r_n$, $n=1,2,3,\cdots$ then $x_n \to x_0$ since $\{x_n\}$ Cauchy in $\mathbb{R}^n$

$\therefore B_{r_1}(x_1) \cap \left(\bigcap_{n=1}^{+\infty} G_n\right) \ni x_0$, x_1, r_1 arbitary $\Rightarrow \bigcap_{n=1}^{+\infty} G_n$ dense

Th9.4: (Baire) (X,d) complete, $A_n \subseteq X$, A_n is closed and $A_n^\circ = \phi \Rightarrow \left(\bigcup_{n=1}^{+\infty} A_n\right)^\circ = \phi$

proof9.4: $B_n = A_n^c$ is open

$(B_n^c)^\circ = \phi \Rightarrow B_n$ dense in X

$\therefore \bigcap_{n=1}^{+\infty} B_n = \bigcap_{n=1}^{+\infty} A_n^c = \left(\bigcup_{n=1}^{+\infty} A_n\right)^c$ is dense $\cdots$ by Th9.3

$\therefore \left[\left[\left(\bigcup_{n=1}^{+\infty} A_n\right)^c\right]^c\right]^\circ = \left(\bigcup_{n=1}^{+\infty} A_n\right)^\circ = \phi$

Rmk: A_n closed, $\left(\bigcup_{n=1}^{+\infty} A_n\right)^\circ \neq \phi \Rightarrow$ at least $\exists A_n$, s.t. $A_n^\circ \neq \phi$, 并集有内点,则必有某个A_n有内点
$\hat{X}$ complete