Additional Exercises in Functional Analysis (week 3)

1. Let (X, d) be a metric space. Let A, B be compact subsets of X such that $A \cap B = \emptyset$. Show that

$$d(A, B) = \inf_{x \in A, y \in B} d(x, y) > 0.$$

proof1: define $f_a : B \to \mathbb{R}^1$, s.t. $f_a(b) = d(a,b)$; define $F : A \to \mathbb{R}^1$, $F(a) = \inf_{b\in B} f_a(b)$

then $d(A,B) = \inf_{a\in A} \inf_{b\in B} f_a(b) = \inf_{a\in A} F(a)$

$|f_a(b_1) - f_a(b_2)| = |d(a,b_1) - d(a,b_2)| \le d(b_1,b_2)$, $f_a(b)$ is lipschitz cts thus uniformly cts on B, $\forall a\in A$

设 $F(a_1) = \inf_{b\in B} f_{a_1}(b) = \overset{\le f_{a_1}(b_2)}{f_{a_1}(b_1)}$, $F(a_2) = f_{a_2}(b_2) \le f_{a_2}(b_1)$

$\therefore F(a_1) - F(a_2) = d(a_1,b_1) - d(a_2,b_2) \le d(a_1,b_2) - d(a_2,b_2) \le d(a_1,a_2)$; similarly $F(a_2) - F(a_1)$

$\therefore |F(a_1) - F(a_2)| \le d(a_1,a_2)$ $\Rightarrow$ F is cts on A

反证假设(*): if $d(A,B) = \inf_{a\in A} F(a) = 0$ } $\Rightarrow$ $\inf_{a\in A} F(a) = \min_{a\in A} F(a)$, $\exists a_0 \in A$, $F(a_0) = 0$

$\Rightarrow \inf_{b\in B} f_{a_0}(b) = 0$

$f_{a_0}(b)$ cts on compact set B } $\Rightarrow \exists b_0 \in B$. $f_{a_0}(b_0) = d(a_0,b_0) = 0$

最后有订正,看一下!

$\Rightarrow a_0 = b_0 \in A\cap B$, contradict! (*)不成立, $d(A,B) > 0$

2. Let A, B be two subsets of a metric space (X, d) with

$$d(A, B) = \inf_{x \in A, y \in B} d(x, y) > 0.$$

Show that there are two open subsets U_1 and U_2 of X such that $U_1 \supset A$, $U_2 \supset B$ and $U_1 \cap U_2 = \emptyset$.

proof2: step1: to show $\bar{U}_1 = \bar{A}$, where U_1 is the minimal open set s.t. $U_1 \supseteq A$

$\bar{U}_1 = U_1 \cup U_1'$, $\bar{A} = A \cup A'$, obv $\bar{U}_1 \supseteq \bar{A}$

① 设 $\exists x \in U_1'$, $x \notin \bar{A}$, 且 $x \notin U_1$

$\Rightarrow \begin{cases} \forall r>0,\ B_r(x)\setminus\{x\} \cap U_1 \ne \emptyset, \text{ then } \bar{B}_r(x)\setminus\{x\}\cap U_1 \ne \emptyset,\ \bar{B}_r(x)\cap U_1 \ne \emptyset \\ \exists r_0 > 0,\ B_{r_0}(x)\setminus\{x\}\cap A = \emptyset \text{ then let } r_0' = \frac{r_0}{2},\ \bar{B}_{r_0'}(x)\setminus\{x\}\cap A = \emptyset \text{ too.} \end{cases}$

设 $x \notin A$, $\bar{B}_{r_0'}(x) \cap A = \emptyset$

$x \notin U_1$, $x \in (\bar{B}_{r_0'}(x_0))^c$ $\therefore$ $U_1 \cap (\bar{B}_{r_0'}(x_0))^c$ 比 U_1 严格更大

$\therefore U_1 \cap (\bar{B}_{r_0'}(x))^c \supseteq A$, $U_1 \cap (\bar{B}_{r_0'}(x))^c$ is smaller open set than U_1 $\Rightarrow$ contradict!

$\therefore$ "$x \notin A$" not holds. $x \in A$

即 if $x \in U_1$, $x \in A'$ or $x \in A$, $\therefore U_1 \subseteq \overline{A}$

② if $\exists\, x \in U_1$, $x \notin A$, that is $x \in U_1 \setminus A$

if $x \notin A'$, $\exists\, r > 0$ s.t. $B_r(x) \cap A = \phi$, then $\exists\, r'$ s.t. $\overline{B_{r'}(x)} \cap A = \phi$ }

$\Rightarrow U_1 \setminus \overline{B_{r'}(x)}$ is smaller open set than U_1, contradict !

$\therefore$ "$x \notin A'$" not holds. $x \in A'$

例: $A =$  A'不含; U_1'含.

即: if $x \in U_1$, $x \in A$ or $x \in A$, $\therefore U_1 \subseteq \overline{A}$

①+② $\Rightarrow U_1$ is the minimal open set s.t. $U_1 \supseteq A$; than $\overline{U_1} = \overline{A}$ ($U_1' = A'$ 不行的), $\overline{U_2} = \overline{B}$

Step2: $\inf\limits_{a \in A, b \in B} d(a,b) > 0$, to show $\overline{A} \cap \overline{B} = \phi$

if $x \in \overline{A} \cap \overline{B}$, $x \in A \cap B$ then let $a = b = x$, 矛盾, ... ①

$\therefore x \notin A$ or $x \notin B$. ②: 设 $x \notin A$, $x \in A' \setminus A$; $x \notin B$ $x \in B' \setminus B$.

$\forall r > 0$, $B_r(x) \setminus \{x\} \cap A \neq \phi$; $\forall r' > 0$, $B_{r'}(x) \setminus \{x\} \cap B \neq \phi$

设 $a_r \in B_r(x) \setminus \{x\} \cap A$, $b_{r'} \in B_{r'}(x) \setminus \{x\} \cap B$, then $d(a_r, b_{r'}) \leq d(a_r, x) + d(x, b_{r'}) < r + r'$

since r, r' approximates 0, $\therefore d(a_r, b_{r'}) \to 0$. 矛盾

③: 不妨设 $x \notin A$ $x \in A' \setminus A$; $x \in B$ ($\Leftrightarrow x \notin B$, $x \in B' \setminus B$; $x \in A$)

then let $b = x$, $\forall r > 0$, $\exists\, a_r \in A \setminus \{x\}$. $a_r \in B_r(x) \cap A$ then $d(a_r, b) < r \to 0$ 矛盾

①+②+③ $\Rightarrow \overline{A} \cap \overline{B} = \phi$

Step1 + Step2: $\overline{U_1} = \overline{A}$, $\overline{U_2} = \overline{B}$; $\overline{A} \cap \overline{B} = \phi \Rightarrow \overline{U_1} \cap \overline{U_2} = \phi \Rightarrow U_1 \cap U_2 = \phi$ 证毕

3. Let (X, d) be a metric space and Y be a compact subset of X. Assume that a map $T : Y \to Y$ satisfies $d(Tx, Ty) < d(x, y)$, $\forall x, y \in Y, x \neq y$. Show that T has a unique fixed point. (Hint: use that the function $g(x) = d(x, T(x))$ attains its minimum.)

proof3: 要证明: $\exists\, x$ s.t. $Tx = x$ ($\Leftrightarrow d(Tx, x) = 0$) $\Rightarrow$ 证 $g(x) = d(Tx, x)$ cts, $\therefore \inf = \min$

• 设 $g(x) := d(Tx, x)$, $g(x)$ is uniformly cts. Since:

$\because d(Tx, Ty) < d(x,y)$ $\therefore g(x) - g(y) < d(Tx, x) - d(Ty, y) \leq d(Tx, Ty) + d(Ty, x) - d(Ty, y) < 2d(x, y)$

- Let $x_0 \in Y$, $Tx_0 = x_1, \cdots Tx_n = x_{n+1}$ $\therefore d(Tx_n, x_n) = d(x_{n+1}, x_n) < d(x_n, x_{n-1})$ 记 $d(x_n, x_{n+1}) = g_n > 0$.
then construct $\{g_n\}_{n=1}^{+\infty}$, $g_1 > g_2 > \cdots g_n > \cdots$

经典写法

- $g: Y \to Y$ g cts and Y cpt $\Rightarrow g(Y)$ cpt

$\Rightarrow$ $\{g_n\}_{n=1}^{+\infty}$ converges to g_0, $g_0 < g_n \ \forall n$ obv
$g(Y)$ cpt, g cts $\therefore$ g_0 可以取到, 设 $g_0 = d(Tx_m, x_m)$ some m $\}$

$\Rightarrow$ if $g_0 > 0$, then $d(Tx_{m+1}, x_{m+1}) < g_0$ 与 $g_0 < g_n$ 矛盾 $\}$

$\therefore g_0 = 0$, $\exists$ some m s.t. $Tx_m = x_m$ ...... (existence)

if $Tx = x$, $Ty = y$ $x \neq y$, then $d(Tx, Ty) = d(x, y)$; $d(Tx, Ty) < d(x, y)$ since $x \neq y$ $\Rightarrow$ 矛盾 ...(unique)

3.1 Show that if d is a metric on a vector space X derived from a norm $\|\cdot\|$, i.e. $d(x, y) = \|x - y\|$, then d is translation invariant and homogeneous, i.e.

$$d(x + z, y + z) = d(x, y) \quad \text{and} \quad d(\alpha x, \alpha y) = \alpha d(x, y).$$

Deduce that the metric on $\mathfrak{s}$ in Exercise 2.4 does not come from a norm.

proof 3.1 即说明 $d(X, Y) = \sum_{n=1}^{+\infty} \frac{1}{2^n} \cdot \frac{x_n - y_n}{1 + |x_n - y_n|}$, $X = (x_1, x_2, \cdots)$ $Y = (y_1, y_2, \cdots)$ $x_i, y_i \in$ space K

不满足 $d(X+Z, Y+Z) = d(X, Y)$ 或 $d(\partial X, \partial Y) = \partial d(X, Y)$ $\forall \partial, X, Y$

$d(\partial X, \partial Y) = \sum_{n=1}^{+\infty} \frac{1}{2^n} \cdot \frac{\partial (x_n - y_n)}{1 + \partial |x_n - y_n|} < \sum_{n=1}^{+\infty} \frac{1}{2^n} \cdot \frac{\partial (x_n - y_n)}{1 + |x_n - y_n|} = \partial d(X, Y)$. when $\partial > 1$, $X \neq Y$

$d(\partial X, \partial Y) > \partial d(X, Y)$ when $\partial < 1$ $X \neq Y$

$\therefore d(\partial X, \partial Y) \neq \partial d(X, Y)$ if $X \neq Y$ 证毕

3.2 If A and B are subsets of a vector space, then we can define

$$A + B := \{a + b : a \in A, \ b \in B\}.$$

Show that if A and B are both convex, then so is $A + B$.

proof 3.2. $a_1, a_2 \in A$. $\forall \lambda_1 \in [0,1]$, $\lambda_1 a_1 + (1-\lambda_1) a_2 \in A$
$b_1, b_2 \in B$, $\forall \lambda_2 \in [0,1]$, $\lambda_2 b_1 + (1-\lambda_2) b_2 \in B$
$a_j + b_j \in A + B$, $\lambda \in [0,1]$ $\lambda(a_1 + b_1) + (1-\lambda)(a_2 + b_2) = (\lambda a_1 + (1-\lambda) a_2) + (\lambda b_1 + (1-\lambda) b_2) \overset{\text{denote}}{=} a + b$
let $\lambda_1 = \lambda_2 = \lambda$, then $a \in A$, $b \in B$, $a + b \in A + B$
$\therefore A + B$ is convex too.

3.3 If C is a closed subset of a vector space X show that C is convex if and only if $a, b \in C$ implies that $(a+b)/2 \in C$.

proof 3.3 "$\Rightarrow$" C convex, $a, b \in C$, $\lambda a + (1-\lambda) b \in C$ $\forall \lambda \in [0,1]$

let $\lambda = \frac{1}{2}$, then $\frac{a+b}{2} \in C$

"$\Leftarrow$" $a, b \in C$, then $\frac{1}{2}a + \frac{1}{2}b \in C$

$\Rightarrow \frac{1}{2}(\frac{1}{2}a+\frac{1}{2}b) + \frac{1}{2}a = \frac{3}{4}a + \frac{1}{4}b \in C$, 同理 $(\frac{1}{2})^n b + (1-(\frac{1}{2})^n) a \in C$, $(\frac{1}{2})^n a + (1-(\frac{1}{2})^n) b \in C$

$\frac{1}{2}(\frac{3}{4}a + \frac{1}{4}b) + \frac{1}{2}b \in C$ ...

$a_i, b_i \geq 0$ ✓

由上可知 $(a_n (\frac{1}{2})^n + a_{n-1}(\frac{1}{2})^{n-1} + \cdots a_1 \frac{1}{2}) a + (b_n (\frac{1}{2})^n + \cdots b_1 \frac{1}{2}) \cdot b \in C$, $\sum_{i=1}^{n} a_i (\frac{1}{2})^i + b_i (\frac{1}{2})^i = 1$

$\therefore \forall \lambda \in [0,1]$, $\exists$, $a_n (\frac{1}{2})^n + \cdots a_1 \frac{1}{2} = \lambda$ $a_i \geq 0$ $\forall i$ 这么写 λ 只能是 $\lambda \in \mathbb{Q}$!

这由 λ 的2进制展开即可求出, then $\sum b_i (\frac{1}{2})^i$ also fixed

$\therefore \lambda a + (1-\lambda) b \in C$, C is convex if $\lambda \notin \mathbb{Q}$, limit point, $\Rightarrow$ closed set!

3.4 Show that if $f : [a, b] \to \mathbb{R}$ is C^2 on (a, b) and C^1 on $[a, b]$, then f is convex on $[a, b]$ if $f''(x) \geq 0$. Deduce that $f(x) = e^x$ and $f(x) = |x|^p$, $1 \leq p < \infty$, are convex functions on $\mathbb{R}$.

proof 3.4: (1) $f(x) = e^x$ $f''(x) = e^x \geq 0$;

$$f(x) = |x|^p = \begin{cases} x^p & x \geq 0 \\ (-x)^p & x < 0 \end{cases} \qquad \begin{matrix} f''(x) = p(p-1)x^{p-2} & x > 0 \\ f''(x) = -p(-p+1)(-x)^{p-2} & x < 0 \end{matrix}$$

$f''(0) = \frac{x^p - (-x)^p}{x - (-x)} = 0$ as $x \to 0$, $\Rightarrow f''(x) \geq 0$.

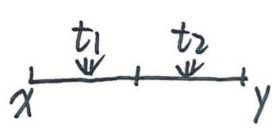
(2): $f''(x) \geq 0$ WTS: $f(\lambda x + (1-\lambda) y) \leq \lambda \cdot f(x) + (1-\lambda) \cdot f(y)$ $\forall \lambda \in [0,1]$, $x, y \in [a,b]$ ... (*)

$\Leftrightarrow \lambda (f(\lambda x + (1-\lambda)y) - f(x)) \leq (1-\lambda)(f(y) - f(\lambda x + (1-\lambda) y))$

$\Leftrightarrow \cancel{\lambda} \frac{f(\lambda x + (1-\lambda)y) - f(x)}{(1-\lambda)y - (1-\lambda)x} \times \cancel{(1-\lambda)} \leq \cancel{(1-\lambda)} \cdot \frac{f(y) - f(\lambda x + (1-\lambda) y)}{\lambda y - \lambda x} \times \cancel{\lambda}$, 不妨设 $y > x$

LHS $= f'(t_1)$ $t_1 \in (x, \lambda x + (1-\lambda) y)$

RHS $= f'(t_2)$ $t_2 \in (\lambda x + (1-\lambda) y, y)$



since $f''(x) \geq 0$, $t_2 \geq t_1$ $\therefore f'(t_2) \geq f'(t_1)$ $\Rightarrow$ RHS $\geq$ LHS

即(2)式成立,证毕

3.5 Show that if $m = \max_{j=1,\dots,n} |x_j|$, then for any $p \in [1, \infty)$,

$$m^p \leq \sum_{j=1}^{n} |x_j|^p \leq nm^p$$

有限维才行

and deduce that for any $\boldsymbol{x} \in \mathbb{K}^n$, $\|\boldsymbol{x}\|_{\ell^p} \to \|\boldsymbol{x}\|_{\ell^\infty}$ as $p \to \infty$.

proof 3.5 $m = \max_j |x_j| = |x_t|$ some $t \in \{1, 2, \cdots n\}$

$\therefore m^p = |x_t|^p \leq \sum_{j=1}^{n} |x_j|^p \leq \sum_{j=1}^{n} |x_t|^p = n \cdot m^p$, obv

$\Rightarrow \|x\|_{\ell^p} = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} \in [(m^p)^{\frac{1}{p}}, (n \times m^p)^{\frac{1}{p}}] = [m, n^{\frac{1}{p}} \times m]$

as $p \to +\infty$, $n^{\frac{1}{p}} \to 1$ $\forall n \in \mathbb{N}^+$

$\therefore \|x\|_{\ell^p} \to m = \|x\|_{\ell^{+\infty}}$ as $p \to +\infty$

~~3.8~~ Show that if U is an open subspace of a normed space X then $U = X$. (Naylor and Sell, 1982)

proof 3.8 U is subspace $\therefore$ $0 \in U$

$\left\{ \begin{array}{l} \forall \lambda \in \mathbb{F}, v \in U, \lambda v \in U \\ v, u \in U, \ v + u \in U \end{array} \right.$ $\Rightarrow$ let $\lambda = \frac{k}{\|v\|}$ if $\|v\| \neq 0$, $\lambda \cdot v \in U$ too

U is open $\therefore \exists\, r > 0$ s.t. $B_r(0) = \{v : \|v\| < r\} \subseteq U$

$\therefore \lambda v$ with norm $\|\frac{k}{\|v\|} \cdot v\| = |k| \in U$, $\forall |k| \in [r, +\infty)$

k取遍$[r, +\infty)$, 则可知 $\{v : \|v\| \geq r\} \subseteq U$

已知: $B_r(0) = \{v : \|v\| < r\} \subseteq U$ $\Rightarrow$ $\{v : \forall \|v\|\} = U$ $\therefore U = X$ 证毕

1. $d(A,B)>0$, $\Rightarrow \exists$ open $U_1 \supseteq A$, $U_2 \supseteq B$, s.t. $U_1 \cap U_2 = \phi$

lem: $A \cap \bar{B} = \phi$. $B \cap \bar{A} = \phi$ $\Rightarrow \exists$ open set $U_1 \supseteq A$, $U_2 \supseteq B$, s.t. $U_1 \cap U_2 = \phi$

if $A \neq \phi$. or $B \neq \phi$ ✓

if $A \neq \phi$. and $B \neq \phi$ $\quad f: x \longmapsto d(x,A) = \inf\{d(x,a): a \in A\}$

$g: x \longmapsto d(x,B) = \inf\{d(x,b): b \in B\}$

f, g cts. well-defined.

let: $U_1 = \{x: d(x,A) < d(x,B)\}$

$U_1 \supseteq A$ since: $A \cap \bar{B} = \bar{B} \cap A = \phi \Rightarrow d(a,B) > 0$ $\forall a \in A$. (但 $d(A,B)$ 可以 0)

let: $U_2 = \{x: d(x,A) > d(x,B)\}$

$U_2 \supseteq A$ similarly

$\Rightarrow U_1 \cap U_2 \neq \phi$. $U_1 \supseteq A$, $U_2 \supseteq B$ 证毕!

2. C is convex $\rightleftarrows$ $a, b \in C$, implies $\frac{a+b}{2} \in C$

"closed" 要闭的.