Ch6 The cumulative hierarchy

we define the cumulative hierarchy V of sets by transfinite iterating the powerset operation

def 6.1 for each ordinal α, define V_α as follows:

$$\begin{cases} V_0=\emptyset,\ V_{\alpha+1}=P(V_\alpha) \text{ for } \forall\alpha, & V_{\alpha+1}\text{由}V_\alpha\text{的全体子集组成} \\ V_\lambda=\bigcup_{\beta<\lambda}V_\beta & \text{若 } \lambda \text{ is a limit ordinal} \end{cases}$$

let the cumulative hierarchy V be the class $\bigcup_{\alpha\in ORD}V_\alpha=\{x:\exists\alpha \text{ s.t. } x\in V_\alpha\}$

Prop 6.2 (1). V_α is transitive, $\forall\alpha$

(2). $\alpha\le\beta$ implies $V_\alpha\subseteq V_\beta$, $\forall\alpha,\beta$

(3). $\alpha\in V_{\alpha+1}\setminus V_\alpha$, $\forall\alpha$

(4). $V_\alpha\in V_{\alpha+1}\setminus V_\alpha$, $\forall\alpha$

pf (1): by transfinite induction, $\emptyset$ is transitive (base case); assume V_α transitive

① 若 $a\in V_{\alpha+1}=P(V_\alpha)$ 子集的集 $\therefore a\subseteq V_\alpha$, ie. $\forall x\in a,\ x\in V_\alpha$

V_α transitive, $\therefore \forall x\in a,\ \forall y\in x$, then $y\in V_\alpha$

$\therefore \forall x\in a,\ x\subseteq V_\alpha$, ie. $x\in P(V_\alpha)=V_{\alpha+1}$

$\Rightarrow a\in V_{\alpha+1}$, then $\forall x\in a,\ x\in P(V_\alpha)=V_{\alpha+1}$; ie $V_{\alpha+1}$ transitive

② 若 λ is limit ordinal; assume $\forall\beta<\lambda$, V_β transitive

$\forall a\in V_\lambda=\bigcup_{\beta<\lambda}V_\beta$, then $\exists\beta<\lambda$ s.t. $a\in V_\beta$, V_β trans

$\forall b\in a$, $b\in V_\beta$ by ①, $\therefore b\in V_\lambda$

$\Rightarrow a\in V_\lambda$, then $\forall b\in a,\ b\in V_\lambda$; ie. V_α transitive.

注意: 1. $V_{\alpha+1}=P(V_\alpha)$, $\therefore a\in V_{\alpha+1}\Leftrightarrow a$是$V_\alpha$的子集, $a\subseteq V_\alpha$; 注意"$\subseteq$""$\in$","$P(\cdot)$"之间转换

2. 结论: ordinal $x<\lambda+1\Leftrightarrow x\le\lambda$

it implies: $x<\lambda+1\Leftrightarrow x+1\le\lambda+1 \Rightarrow y<\lambda\Leftrightarrow y+1\le\lambda$, 含极限(易证)

3. 在②中, $a\in V_\beta=P(V_{\beta'})$ where $\beta'+1=\beta$, assume $V_{\beta'}$ transitive

$a\subseteq V_{\beta'}$, $\therefore \forall x\in a,\ x\in V_{\beta'}$

$\forall y\in x,\ y\in V_{\beta'}$ $\therefore x\subseteq V_{\beta'}$ ie. $x\in P(V_{\beta'})=V_{\beta'+1}=V_\beta$

$\Rightarrow a\in V_\beta$ implies $\forall x\in a,\ x\in V_\beta\subseteq V_\lambda$

$a\in V_\lambda=\bigcup_{\beta<\lambda}V_\beta$ implies $\forall x\in a,\ x\in V_\lambda=\bigcup_{\beta<\lambda}V_\beta$ 这是②的详细步骤,我在上面没写,直接用了①的结论

pf(2): transfinite induction on β

① base case, $\beta=0$; $\alpha\le 0$ implies $\alpha=0$, $V_0\subseteq V_0$

if (2) holds for β, i.e. $\forall \alpha\le\beta$, $V_\alpha\subseteq V_\beta$

for $\beta+1$, $\alpha\le\beta+1$ implies $\alpha\le\beta$, or $\alpha>\beta$, $\alpha\le\beta+1$

since $\alpha>\beta$ is equivalent to $\alpha\ge\beta+1$, $\therefore \alpha=\beta+1$ $V_\alpha\subseteq V_{\beta+1}$

$\therefore V_\alpha\subseteq V_{\beta+1}$ for $\forall\alpha\le\beta+1$

② if λ is a limit ordinal, $V_\lambda=\bigcup_{\beta<\lambda}V_\beta$

if $\alpha<\lambda$, $V_\alpha\subseteq\bigcup_{\beta<\lambda}V_\beta=V_\lambda$ obviously

①+② => induct on successor ordinal β, and separately prove the limit ordinal λ, (2) holds for ORD

pf(3) ① $V_{\alpha+1}=P(V_\alpha)$ $\therefore \alpha\in V_{\alpha+1}\Leftrightarrow\alpha\subseteq V_\alpha$

transfinite induction on $\forall\beta<\alpha$

for $\forall\beta\in\alpha=\{\gamma:\gamma<\alpha\}$, $\beta<\alpha$ by definition of ordinal

by (2), $\beta<\alpha$ implies $V_\beta\subseteq V_\alpha$

$\beta\in V_{\beta+1}$, $\beta\subseteq V_\beta$ by assumption $\Big\}$ => $\beta\subseteq V_\alpha$ $\forall\beta\in\alpha$ => $\alpha\subseteq V_\alpha$

② first we want to show $V_\alpha\cap\text{Ord}=\alpha=\{\gamma:\gamma<\alpha\}$

transfinite induction on α

base case, $\alpha=0$, $V_0\cap\text{Ord}=\emptyset\cap\text{Ord}=0$

if $V_\beta\cap\text{Ord}=\beta$ holds for $\forall\beta<\alpha$

$V_{\beta+1}\cap\text{Ord}=P(V_\beta)\cap\text{Ord}$

ordinals bigger than (including)

by (3)①, $\beta\in V_{\beta+1}$; $P(V_\beta)\cap\text{Ord}$ will not introduce "$\beta+1$" in it ×

$\therefore \max(V_{\beta+1}\cap\text{Ord})<\beta+1$, which implies $\max(V_{\beta+1}\cap\text{Ord})\le\beta$

$\beta\in V_{\beta+1}\cap\text{Ord}$ $\Big\}$ => $\max(V_{\beta+1}\cap\text{Ord})=\beta$

$\therefore V_{\beta+1}\cap\text{Ord}=\{\gamma:\gamma\le\beta\}=\{\gamma:\gamma<\beta+1\}=\beta+1$

=> $\alpha\in V_\alpha$ for successor ordinal, limit case is trivial

更正: $V_{\beta+1}\cap\text{ORD}=\{\gamma\in\text{ORD}:\gamma\in V_{\beta+1}\}=\{\gamma\in\text{ORD}:\gamma\subseteq V_\beta\}$

assume $V_\beta\cap\text{ORD}=\beta$. $\therefore$ $\gamma\in\text{ORD}$, $\gamma\subseteq V_\beta\Leftrightarrow\gamma\subseteq\beta$ $\Big\}$ => $V_{\beta+1}\cap\text{ORD}=\{\gamma\in\text{ORD}:\gamma\subseteq\beta\}$

Lem: ordinal γ,δ satisfy $\gamma\subseteq\delta\Leftrightarrow\gamma\le\delta$ ($\alpha\ne\beta$ are ordinals, $\alpha\subseteq\beta\Leftrightarrow\alpha\in\beta$, i.e. $\alpha<\beta$)

$\therefore V_{\beta+1}\cap\text{ORD}=\{\gamma\in\text{ORD}:\gamma\subseteq\beta\}=\{\gamma\in\text{ORD}:\gamma<\beta+1\}=\beta+1$

$\gamma\le\beta\Leftrightarrow\gamma<\beta+1$

lem: $\forall \alpha, \forall x$, 若 $x \in V_\alpha$, then $\exists \beta < \alpha$ s.t. $x \subseteq V_\beta$. 每层元素会落在更早一层的子集中

pf: · base case $\alpha = 0$

· 若 $\alpha = \gamma + 1$ for some $\gamma \in \mathrm{ORD}$, then

若 $x \in V_{\gamma+1} = P(V_\gamma)$, then $x \subseteq V_\gamma$.

let $\beta = \gamma$, $\beta < \alpha$, then we conclude that: $x \in V_\alpha$ implies $x \subseteq V_\beta$ for some $\beta < \alpha$

· 若 α is limit ordinal,

若 $x \in V_\alpha = \bigcup_{\beta < \alpha} V_\beta$, then $\exists \beta < \alpha$ s.t. $x \in V_\beta$

$\therefore \exists \bar{\beta} < \beta < \alpha$, s.t. $x \subseteq V_{\bar{\beta}}$ by the previous case

pf(4): $V_{\alpha+1} = P(V_\alpha)$ $\therefore V_\alpha \in V_{\alpha+1}$ obv

$$\left.\begin{array}{l} \text{若 } V_\alpha \in V_\alpha, \text{ then } \exists \beta < \alpha \text{ s.t. } V_\alpha \subseteq V_\beta \text{ by lemma.} \\ \text{by (2) } \beta \leq \alpha \text{ implies } V_\beta \subseteq V_\alpha \end{array}\right\} \Rightarrow V_\alpha = V_\beta \text{ for this } \beta < \alpha$$

by (3) $\alpha \in V_{\alpha+1} = P(V_\alpha)$, then $\alpha \subseteq V_\alpha$

$$\left.\begin{array}{l} \therefore \alpha \subseteq V_\alpha = V_\beta,\ \alpha \in V_{\beta+1}. \\ \beta < \alpha \text{ implies } \beta + 1 \leq \alpha.\ \text{by (2) } V_{\beta+1} \subseteq V_\alpha \end{array}\right\} \Rightarrow \alpha \in V_\alpha \text{ contradict with (3) } \alpha \in V_{\alpha+1} \setminus V_\alpha$$

我觉得这个超级绕!

def 6.3 if $x \in V$, then $\mathrm{rank}(x)$ is the least ordinal α s.t. $x \subseteq V_\alpha$

compared to def 5.10 $\mathrm{rank}_p(x) = \begin{cases} \inf\{\alpha : x \in X_\alpha\} \\ \infty \end{cases}$

def 6.3 operates on the whole universe "V-class", with the Foundation Axiom,

在Foundation下处处有定义，永远是序数

$\mathrm{rank}(x) = \bigcup \{\mathrm{rank}(y) + 1 : y \in x\}$

def 6.5 $\forall$ set x, let $x_0 = x$, $x_{n+1} = x_n \cup (\bigcup x_n)$

then define the transitive closure of x to be the set $TC(x) = \bigcup_{n \in \omega} x_n = x \cup (\bigcup x) \cup (\bigcup\bigcup x) \cup (\bigcup\bigcup\bigcup x) \cup \cdots$

$\Rightarrow TC(x) = \{y : \exists n < \omega, \exists (a_0, a_1, \cdots a_n) \text{ s.t. } a_0 = y, a_n = x, a_i \in a_{i+1}\}$

from x, all elements that can be reached by "$\in$" within n steps are in $TC(x)$ 有限∈链可以到达的元素

since: $y \in TC(x) \rightleftharpoons y \in x$ or $y \in \bigcup x$ or $y \in \bigcup\bigcup x$ or $\cdots$

$\rightleftharpoons \quad n=1 \qquad n=2 \qquad n=3 \qquad \cdots$

Prop 1: $TC(x)$ is transitive

$z \in TC(x)$, $y \in z$ implies $y \in TC(x)$ $\forall y, z$

obv by $TC(x) = \{y : \exists n < \omega, \exists (a_0, a_1, \cdots a_0) \text{ s.t. } a_0 = y, a_n = x, a_i \in a_{i+1}\}$

Prop 2: $TC(x)$ is the smallest transitive set containing x

if $T(x)$ is transitive set containing x, $y \in TC(x)$

then any finite $\in$-chain from y to x is in T, since by definition $x \in T$, $z \in x$ implies $z \in T$

$\therefore y \in T(x)$, thus $TC(x) \subseteq T(x)$

Prop 3: (def 6.3) $\mathrm{rank}(x) = \min\{\alpha : x \subseteq V_\alpha\} = \min\{\alpha : TC(x) \subseteq V_\alpha\}$

① if $x \subseteq V_\alpha$, V_α transitive

for $\forall z \in \bigcup x$, $\exists y \in x$ s.t. $z \in y$;

~~$x \subseteq V_\alpha \rightleftharpoons x \in V_{\alpha+1}$, then $y \in x$ implies $y \in V_{\alpha+1}$, $\rightleftharpoons y \subseteq V_\alpha$~~ $x \subseteq V_\alpha$ implies $\forall y \in x$, $y \in V_\alpha$

这是⊆的定义啊，竟然也忘记了！

$\therefore \forall z \in \bigcup x$, $z \in V_\alpha$ by transitivity $\Rightarrow \bigcup x \subseteq V_\alpha$

$\therefore \bigcup(\bigcup x) \subseteq V_\alpha$, $\bigcup\bigcup\bigcup x \subseteq V_\alpha \cdots \Rightarrow TC(x) \subseteq V_\alpha$

② if $TC(x) \subseteq V_\alpha$, $x \subseteq TC(x) \Rightarrow x \subseteq V_\alpha$

①+② $\Rightarrow TC(x) \subseteq V_\alpha \rightleftharpoons x \subseteq V_\alpha$

prop 6.7 Assume ZF-Foundation, then the axiom of foundation is equivalent to $\forall x\,(x\in V)$

假设ZF中其它公理成立，则基础公理 $\Leftrightarrow \forall x\,(x\in V)$

pf: (1) if Foundation Axiom holds

claim: $\forall y\in TC(x)$, $\exists\,\alpha\in ORD$ s.t. $y\in V_\alpha$

if not, let $A=\{y\in TC(x): \forall\alpha\in ORD,\ y\notin V_\alpha\}$, $A\neq\emptyset$

by Foundation Axiom, $\exists$ $\in$-minimal of A, denoted y^*

$\forall z\in y^*,\ z\notin A$; $\forall z\in A,\ z\notin y^*$ $\Rightarrow$ $y^*\cap A=\emptyset$ 极小元的等价表达

$\therefore \forall z\in y^*,\ \exists\,\alpha_z\in ORD$ s.t. $z\in V_{\alpha_z}$

let $\beta=\sup\{\alpha_z+1: z\in y^*\}$
$\begin{cases} \alpha_z+1\in ORD,\ \therefore \overset{\beta+1\,\cdots}{\beta}\in ORD,\ \beta\geq\alpha_z+1\ \forall z \\ \forall z\in y^*,\ z\in V_{\alpha_z+1}\subseteq V_\beta,\ \therefore y^*\subseteq V_\beta, \text{ i.e. } y^*\in V_{\beta+1} \end{cases}$

$\Rightarrow y^*\in A,\ y^*\in V_{\beta+1}$ contradict!

by Replacement Axiom, $\{\text{rank}(y)+1: y\in TC(x)\}$ is a set

$\sup\{\text{rank}(y)+1: y\in TC(x)\}=\xi\in ORD$

$\forall y\in TC(x)$, $y\subseteq V_{\text{rank}(y)}$ by definition of rank, $\therefore y\in V_{\text{rank}(y)+1}\subseteq V_\xi$, i.e. $y\in V_\xi\ \forall y$

$\therefore \forall y\in TC(x),\ y\in V_\xi$, i.e. $TC(x)\subseteq V_\xi$

$x\subseteq TC(x)$ $\therefore x\subseteq V_\xi$, i.e. $x\in V_{\xi+1}$ (或证明 $x\subseteq V_\alpha \Leftrightarrow TC(x)\subseteq V_\alpha$)

x is arbitrary, $\therefore \forall x,\ \exists\,\xi+1\in ORD$ s.t. $x\in V_{\xi+1}\subseteq V$

(2): if $\forall x.\,(x\in V)$

let $X\neq\emptyset$, $\forall y\in X$, $\exists\,\alpha_y\in ORD$ s.t. $\alpha_y=\min\{\gamma: y\subseteq V_\gamma\}=\text{rank}(y)$ 这个想法非常常用! 序数最小 $\Rightarrow$ 元素极小

by Replacement Axiom, $\{\alpha_y: y\in X\}$ is a set contained in ORD

$\alpha=\min\{\alpha_y: y\in X\}$, $\text{rank}(y_0)=\alpha$ for some $y_0\in X$

if y_0 not $\in$-minimal in X, $\exists\, z\in X\cap y_0$

$\text{rank}(y_0)=\sup\{\text{rank}(u)+1: u\in y\}>\text{rank}(z)$ contradict! 这个式子也记住 $\text{rank}(y)=\sup\{\text{rank}(u)+1: u\in y$

$\Rightarrow \exists$ $\in$-minimum $y_0\in X$, Foundation Axiom holds