

Abstract Algebra

: Lecture 10

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Today we talk about group actions.

Example 1. Recall: Let $G = \text{GL}(V)$, this is a linear group. And we have a set V , also a vector space.

1. Each $g \in G$ is a bijection from V to V ;
2. $\forall v \in V$ and $1 \in G$ we have $v^1 = v$;
3. $\forall v \in V$ and $g, h \in G$ we have $v^{gh} = (v^g)^h$.

We called G acts on V .

Definition 2. Let G be a group and Ω be a set. We say that G acts on Ω if

- (1). each element of G is a bijection from Ω to Ω ;
- (2). $\forall \omega \in \Omega, \omega^1 = \omega$;
- (3). $\forall \omega \in \Omega$ and $g, h \in G$ we have $\omega^{gh} = (\omega^g)^h$.

We denote this action by $G \curvearrowright \Omega$.

Example 3. $\text{Sym}(\Omega)$ and $\text{Alt}(\Omega)$ acts on Ω naturally.

Example 4. Let G be a group, for each element $g \in G$, we define action $g : x \mapsto xg$ for all $x \in G$. We denote this map by \hat{g} , called right multiplication. This is a group action. We denote the group of this action by \hat{G} .

Example 5. Let G be a group, for each element $g \in G$, we define action $g : x \mapsto g^{-1}xg$ for all $x \in G$. We denote this map by \tilde{g} , called conjugation. This is a group action. We denote the group of this action by \tilde{G} .

Example 6. Let G be a group, for each element $g \in G$, we define action $g : x \mapsto gx$ for all $x \in G$. We denote this map by \check{g} , called left multiplication. This is a group action. We denote the group of this action by \check{G} .

Exercise 7. If we right group action in another way, i.e. $(gh)\omega = g(h\omega)$, check example 4, 5, 6.

Proposition 8. 1. $\hat{g}, \check{g}, \tilde{g} \in \text{Sym}(G)$.

2. $\hat{G}, \check{G}, \tilde{G} \leq \text{Sym}(G)$.

3. $\forall g, h \in G, \hat{g}\check{h} = \check{h}\hat{g}$.
4. $\forall g \in G, \hat{g}\check{g} = \check{g}$;
5. $\hat{G} \cap \check{G} = Z(\hat{G}) = Z(\check{G}), \hat{G} \cap \check{G} = \check{G} \cap \hat{G} = \{1\}$;
6. $\langle \hat{G}, \check{G} \rangle = \hat{G}\check{G} = \hat{G} \circ \check{G} = \hat{G} : \check{G} = \check{G} : \hat{G}$;

Definition 9. For two groups G and H , assume there exist $C \lesssim Z(G)$ and $C \lesssim Z(H)$. s.t. $C \neq \{1\}$. Let $Z_1 \leq Z(G)$ and $Z_2 \leq Z(H)$. s.t. $Z_1 \simeq Z_2 \simeq C$. Let ϕ be an isomorphism from Z_1 to Z_2 . Let $X = (G \times H) / \langle (x, x^\phi) | x \in Z_1 \rangle \simeq (G \times H) / C$. This group X is called a central product of G and H , denoted by $G \circ H$.

Definition 10. Let $H, K < G$ s.t. $H \triangleleft G$ and $H \cap K = \{1\}$. Then $\langle H, K \rangle = HK = H \rtimes K = H : K$ called a semi-direct product of H and K .

G acts on Ω also can be said as an action of G on Ω or group action of G on Ω .

Definition 11. Let G act on Ω . Then G partitions Ω into orbits, where an orbit is $\Delta = \omega^G = \{\omega^g | g \in G\}$, where $\omega \in \Omega$. So $\Omega = \bigsqcup_{\omega \in \Omega} \omega^G$.

Example 12. \tilde{G} acts on G naturally, i.e. $x^{\tilde{g}} = g^{-1}xg$. $G = x_0^G \sqcup x_1^G \sqcup \dots \sqcup x_t^G$, where $x_0 = e$, $x_i^G = \{g^{-1}x_i g | g \in G\}$ called a conjugacy class of x_i , denoted by $C(x_i)$. $C(x_i) = \{g^{-1}x_i g | g \in G\}$ is a orbit of the action of \tilde{G} on G .

Definition 13. For G acting on Ω , $G_\omega = \{g \in G | \omega^g = \omega\}$ called the stabilizer of ω in G . It's easy to check that G_ω is a subgroup of G .

Theorem 14. (Orbit-Stabilizer Theorem) For G acting on Ω , For $\omega \in \Omega$, $|G| = |\omega^G| \cdot |G_\omega|$.

证明. Let $\Delta = \omega^G = \{\delta_1, \delta_2, \dots, \delta_m\}$ write $\delta = \delta_1$. Let g_i s.t. $\delta^{g_i} = \delta_i$ for $1 \leq i \leq m$.

Claim: For any element $x \in G$, $\delta^x = \delta_i \Leftrightarrow x \in G_\delta g_i$. Which is due to $\delta^x = \delta_i = \delta^{g_i} \Leftrightarrow \delta^{xg_i^{-1}} = \delta \Leftrightarrow xg_i^{-1} \in G_\delta \Leftrightarrow x \in G_\delta g_i$.

Recall $G = G_\delta \sqcup G_\delta g_2 \sqcup \dots \sqcup G_\delta g_m$, we have $|G| = |G_\delta| + |G_\delta g_2| + \dots + |G_\delta g_m|$, i.e. $|G| = |\Delta| |G_\delta|$. \square

Oberserve: consider conjugate action of G on itself, $|x^G| = 1$ iff $x \in Z(G)$. So by Orbit-Stabilizer Theorem, $|G| = |Z(G)| + |C(g_1)| + \dots + |C(g_r)|$ and $|g_i^G| \mid |G|$ due to $|G| = |C(g_i)| \cdot |C_G(g_i)|$.

Theorem 15. (Sylow's 1st Theorem) Let G be a finite group, $|G| = p^e m$ s.t. p is a prime and $(p, m) = 1$, a subgroup H of G s.t. $|H| = p^e$ exists. And H is called a Sylow p -subgroup of G , denoted by $H \in \text{Syl}_p(G)$.

Recall: If G is abelian, then let $H = \{g \in G | |g| = p^e\}$. Then H is a subgroup of G and $|H| = p^e$. In other words H is a Sylow p -subgroup of G .

证明. Write $|G| = |Z(G)| + |C(g_1)| + \dots + |C(g_r)|$. If $p \nmid |Z(G)|$ then $Z(G)$ has a Sylow p -subgroup N and $N \triangleleft G$. Then $\bar{G} = G/N$ has order $|\bar{G}| < |G|$. If $|N| = p^e$ then N is a Sylow p -subgroup of G . If $|N| < p^e$ then $|\bar{G}| < p^e m$. By induction, \bar{G} has a Sylow p -subgroup \bar{N} , the preimage of \bar{N} is a Sylow p -subgroup of G .

Now suppose $p \nmid |Z(G)|$. Then $p \nmid |C(g_i)|$ for some i . By Orbit-Stabilizer Theorem, $|G| = |C(g_i)| \cdot |C_G(g_i)|$. Then $p^e \mid |C_G(g_i)|$ and by induction $C_G(g_i)$ has a Sylow p -subgroup N , which is also a Sylow p -subgroup of G . \square

Theorem 16. (*Cauchy*) If $p \mid |G|$ then G has a subgroup of order p .

Lemma 17. If G is a p -group i.e. $|G| = p^n$ then G has a non-trivial center.