CH5 Transfinite induction & recursion.

- transfinite induction: $\sim$ ordinary induction to prove statements about all natural numbers
- trans... recursion: $\sim$ ordinary recursion to define functions on the ω

TH5.1 C is a class of a class of ordinals s.t.

$$\begin{cases} 0\in C \cdots ① \\ \forall \alpha\in C, \text{ implies } \alpha+1\in C \cdots ② \\ \lambda \text{ is nonzero limit ordinal}, (\forall \alpha<\lambda, \alpha\in C) \text{ implies } \lambda\in C \cdots ③ \end{cases}$$

then $C=ORD$

pf: let $\bar\lambda$ be the least ordinal in $ORD-C$

$\bar\lambda\notin C$, ② $\Rightarrow \nexists\, \alpha\in C$, s.t. $\alpha+1=\bar\lambda$

$\bar\lambda$ is the least in $ORD-C$ $\therefore \nexists\, \alpha\in ORD-C$ s.t. $\alpha+1=\bar\lambda$, otherwise $\alpha<\bar\lambda$

$\Rightarrow \nexists\, \alpha\in ORD$, s.t. $\alpha+1=\bar\lambda$

$\therefore \bar\lambda$ is limit ordinal

$\bar\lambda$ is the least in $ORD-C$ $\therefore \forall\alpha<\bar\lambda$, $\alpha\in C$

$\Rightarrow$ if $\bar\lambda\neq 0$, ③, $\bar\lambda\in C$ contradict with $\bar\lambda\in ORD-C$

$\Rightarrow \therefore \bar\lambda=0$, contradict with $0\in C$

$\Rightarrow ORD-C=\emptyset$, $C=ORD$

rmk: it shows transfinite recursion can extend through all ordinals

TH5.2 G is a class function, then $\exists$ unique class function F s.t. $\forall\alpha\in ORD$

$F(\alpha)=G(F\restriction\alpha)=G(F\restriction\{\beta:\beta<\alpha\})$ 用小序数上的值域 $F\restriction\alpha$ 定义紧接着的 $F(\alpha)$

pf: step1 if $f(\alpha)=G(f\restriction\alpha)$, $f'(\alpha)=G(f'\restriction\alpha)$

let α be the least ordinal s.t. $f(\alpha)\neq f'(\alpha)$, $\alpha\in dom(f)\cap dom(f')$

then $f'\restriction\alpha=f\restriction\alpha$

$\therefore G(f'\restriction\alpha)=G(f\restriction\alpha)=f'(\alpha)=f(\alpha)$ contradict

$\therefore f\equiv f'$ in $dom(f)\cap dom(f')$

step2: def $F:=\{(\beta,y): \exists f \text{ s.t. } f(\alpha)=G(f\restriction\alpha) \text{ for } \forall\alpha\in dom f\subseteq ORD;\ \beta\in dom f, f(\beta)=y\}$

if $\exists f, f'$ satisfy $f(\alpha)=G(f\restriction\alpha)$, $f'(\alpha)=G(f'\restriction\alpha)$

then by step1, $\beta\in dom f\cap dom f'$, $f(\beta)=f'(\beta)$, $\therefore F$ is well-defined function

if $dom F\neq ORD$, let $\bar\beta$ be the least in $ORD-dom F$,

$\therefore \forall\gamma<\bar\beta$, $\exists f_\gamma$ s.t. $G(f_\gamma\restriction\gamma)=f_\gamma(\gamma)$, ie, $\gamma\in dom f$, and $\forall z<\gamma$, $z\in dom f$

let $\bar f(\gamma)=f_\gamma(\gamma)$ for $\forall\gamma<\bar\beta$, then let $\bar f(\bar\beta):=G(\bar f\restriction\bar\beta)$

add $(\bar\beta, \bar f(\bar\beta))$ to $dom F$, contradiction

Step2: 书上写的方式:

if $domF \neq ORD$, let $\bar\beta$ be the least in ORD-domF ; then $\forall \gamma < \bar\beta$, $\gamma \in domF$

(*) $\exists$ some f s.t. $dom f = \{\gamma: \gamma < \bar\beta\} = \bar\beta$; $f(\alpha) = G(f \restriction \alpha)$ $\forall \alpha \in dom f$

then $f \cup (\bar\beta, G(f \restriction \bar\beta))$ satisfies $f = G(f \restriction \cdot)$ as well

$\Rightarrow$ we can add $\bar\beta$ to domF if $\bar\beta$ initially not in domF

$\therefore domF = ORD$

注意 (*): Why such f exists?

$\forall \gamma < \bar\beta$, $\gamma \in domF$, $\therefore \exists f_\gamma$ s.t. $f_\gamma(\gamma) = G(f_\gamma \restriction \gamma)$, $\gamma \in dom f_\gamma$; and $\forall \gamma' < \gamma$, $\gamma' \in dom f_\gamma$

let $f(\gamma) = f_\gamma(\gamma)$

if $\gamma_1 < \gamma_2$, $\gamma_1 \in dom f_{\gamma_2}$, but by uniqueness, $f_{\gamma_2}(\gamma_1) = f_{\gamma_1}(\gamma_1) \Rightarrow f$ well-defined

Q: $f[\alpha] = \{f(x): x \in \alpha\} = f[\{\beta: \beta < \alpha\}]$ "image set"

$f(\alpha)$ is a single value determed by $f \restriction \alpha$. (f restricted to α)

What's the difference between $f[\alpha]$. $f(\alpha)$?

这对吗,确认一下例子部分

$f = \{(0,1), (1,2), (2,3), (3,5)\}$, $\alpha = 2$

$f \restriction \alpha = \{(0,1), (1,2)\}$ $f(\alpha) = 3$; $3 = G(\{(0,1), (1,2)\})$

$f[\alpha] = f[\{\beta: \beta < \alpha\}] = \{1, 2\}$

important lemma for CH5:

lem4.8 "<" is a well-founded ordering of any class (set) of ordinals

Pf: if $A \subseteq ORD$, $A \neq \phi$

α is an ordinal. $\alpha \in A$. $\begin{cases} \text{if } \alpha \text{ is the least ordinal in } A, \ A \cap \alpha = A \cap \{\beta: \beta < \alpha\} = \phi \\ \text{if } \alpha \text{ isn't the least} \cdots, \text{ then } A \cap \alpha \neq \phi \end{cases}$

$A \cap \alpha \subseteq \alpha$ $\therefore A \cap \alpha$ is an ordinal

$\therefore \exists$ least element of $A \cap \alpha$, (of A as well)

P23 why $w+1 \neq 1+w$. how to understand this?

def 5.10. R is a relation on a set X. define set X_α, $\alpha \in ORD$ by transfinite recursion:

$$\begin{cases} X_0 = \{x \in X : x \text{ is } R\text{-minimal in } X\} \\ X_\alpha = \left(\bigcup_{\beta<\alpha} X_\beta\right) \cup \{x \in X : x \text{ is } R\text{-minimal in } X \setminus \bigcup_{\beta<\alpha} X_\beta\} \end{cases} \Rightarrow X = \bigcup_{\alpha \in ORD} X_\alpha$$

define $rank_R : X \to ORD \cup \{\infty\}$ ① : if $x \notin X_\alpha$ $\forall \alpha \in ORD$, then $rank_R(x) = \infty$, but I think $X = \bigcup_{\alpha \in ORD} X_\alpha$

$rank_R(x) = \inf\{\alpha : x \in X_\alpha\}$, 若 $\exists\, \alpha \in ORD$, s.t. $x \in X_\alpha$; otherwise let $rank_R(x) = +\infty$

注意: here ∞ is just a formal symbol, interpret ∞ as being larger than any ordinal, $\infty < \infty$,

若 $C \subseteq ORD$. $C = \phi$, def $\inf(C) = \infty$

Rmk: rank在此处表示层级, 距离最底层(最小元)有多远。X_0: 所有没有前驱的元素

X_α新加入: 移除所有更低层($\beta<\alpha$)之后, 取剩下新的最小元; 这些加入的元素比之前的元素层级更高

Exa: $R = $ "$\in$", X is any set

Pf: $X_0 = \{x : x \text{ is the } R\text{-min in } X\} = \{x : \nexists\, y \in X, \ yRx\}$

$\therefore x$ is empty in X, $X_0 = \{\phi\}$

$X_1 = X_0 \cup \{x : x \text{ is the } R\text{-min in } X - X_0\} = X_0 \cup \{x : \nexists\, y \in X - X_0, \text{ s.t. } y \in x\}$

$= X_0 \cup \{x : \forall y \in x, \ y \notin X - X_0\}$, i.e. $y \in X_0$ or $y \notin X$

$\Rightarrow x \cap X \subseteq X_0$, ... ①

(注意下 $x \in X$ 是前提, 但 $x \in X \nRightarrow x \subseteq X$, 这里是 $x \cap X \subseteq X_0$ 而不是 $x \subseteq X_0$ 哦

actually ① can be extended to $x \cap X \subseteq \bigcup_{\beta<\alpha} X_\beta$ ... ①'

$\therefore \forall y \in x \cap X$, $rank(y) < \alpha$ by def, $rank_R(y) \neq \infty$

$\therefore$ we can define $rank_R(x) = \sup\{rank_R(y) + 1 : y \in x \cap X\}$, $rank_R(x) \neq \infty$

$\Rightarrow x \cap X \neq \phi$ ... ②

otherwise, the equivalent def in Lem 5.11 fails

①+② $\Rightarrow$ $x \in X - \bigcup_{\beta<\alpha} X_\beta$, i.e. x is the "$\in$-min" in $X - \bigcup_{\beta<\alpha} X_\beta$ $\rightleftharpoons \begin{cases} x \cap X \subseteq \bigcup_{\beta<\alpha} X_\beta, \ x \cap X \neq \phi \\ x \in X, \ x \notin \bigcup_{\beta<\alpha} X_\beta \end{cases}$

总结一下: ①: $x \in X_\alpha - \bigcup_{\beta<\alpha} X_\beta$. $\rightleftharpoons$ $x \subseteq \bigcup_{\beta<\alpha} X_\beta$, $x \notin \bigcup_{\beta<\alpha} X_\beta$; $x \subseteq X$

① $x \in X_\alpha - \bigcup_{\beta<\alpha} X_\beta \rightleftharpoons x \subseteq \bigcup_{\beta<\alpha} X_\beta$, $x \notin \bigcup_{\beta<\alpha} X_\beta$ (分别对应 $\in$-minimal, 和 $x \in X_\alpha - \bigcup X_\beta$ 定义

② $\{\phi, a, b\} \cap \{\phi, c, d\} = \{\phi\}$ not ϕ 类似的交并不要写错了!

lem5.1 (1) $\forall a \in X$, $rank_R(a) = \sup\{rank_R(b)+1 : bRa\}$ 这里只证了没有∞的情况!

(2) if aRb, then $rank_R(a) < rank_R(b)$

Pf: let $\alpha = rank_R(a)$, then a is the R-minimal in $X - \bigcup_{\beta<\alpha} X_\beta$

- $\forall bRa$, a is R-minimal in $X-\bigcup_{\beta<\alpha} X_\beta$, then $b \notin X - \bigcup_{\beta<\alpha} X_\beta$

$\therefore b \in \bigcup_{\beta<\alpha} X_\beta$, $rank_R(b) < rank_R(a)$ ... (2) is proved

if $\alpha > \sup\{rank_R(b)+1 : bRa\}$ $\therefore rank_R(a) \geq rank_R(b)+1$

$\Rightarrow rank_R(a) \geq \sup\{rank_R(b)+1 : bRa\}$

- if $\alpha > \sup\{rank_R(b)+1 : bRa\} \overset{\text{denote}}{=} \gamma$

then $\forall bRa$, $rank_R(b)+1 \leq \gamma < \alpha$, $rank_R(b) < \gamma$

for b, $\gamma_b := rank_R(b) = \inf\{\alpha : b \in X_\alpha\}$

$\therefore \begin{cases} \forall \gamma_b' < \gamma_b,\ b \notin X_{\gamma_b'} \\ b \in X_{\gamma_b} \end{cases}$

因为 $a\in X-\bigcup X_\beta$ 且比 a 小的 b 均不在 $X-\bigcup X_\beta$. $\therefore a$ 是 R-min

$\because \gamma_b < \gamma$ $\forall bRa$ $\therefore \{b : bRa\} \subseteq \bigcup_{\beta<\gamma} X_\beta$

similarly $\begin{cases} a \in X_\alpha \\ \forall \beta<\alpha,\ a \notin X_\beta \Rightarrow a \notin X_\gamma \text{ since } \gamma<\alpha \end{cases}$

$\Rightarrow a$ is the "R minimal" in $X - \bigcup_{\beta<\gamma} X_\beta$ --- (*)

$\Rightarrow rank_R(a) \leq \gamma$ Contradict!

注意: ① $rank_R(b) = \inf\{\alpha : b \in X_\alpha\} = \min\{\alpha : b \in X_\alpha\}$ --- by Lem4.8

② $rank_R(a) = \alpha$, means: $\begin{cases} \forall \beta<\alpha.\ a \notin X_\beta \\ a \in X_\alpha = (\bigcup_{\beta<\alpha} X_\beta) \cup \{x : x \text{ is } R\text{-minimal in } X - \bigcup_{\beta<\alpha} X_\beta\} \end{cases}$

*: 确认一下结论 GPT说对的.

Th: a is the R-minimal element of $X - \bigcup_{\beta<\alpha} X_\beta \Leftrightarrow rank_R(a) = \alpha$ (no "∞" here)

Pf: "$\Rightarrow$" $rank_R(a) = \inf\{\gamma : a \in X_\gamma\}$

$a \notin \bigcup_{\beta<\alpha} X_\beta$, $\therefore$ if $a \in X_\gamma$, then $\gamma \geq \alpha \Rightarrow rank_R(a) \geq \alpha$

$a \in X - \bigcup_{\beta<\alpha} X_\beta$ $\therefore a \in X_\alpha$; $\therefore rank_R(a) \leq \alpha$

$\Rightarrow$ RHS

"$\Leftarrow$" if $\inf\{\gamma : a \in X_\gamma\} = \alpha$; here inf = min by lem4.8

$\therefore a \in X_\alpha$, $a \notin X_\beta$ for $\forall \beta<\alpha$

$a \in X_\alpha - \bigcup_{\beta<\alpha} X_\beta$. by def of X_α, $\therefore a$ is the R-minimal of .. (LHS)

we should suppose R is well-founded i.e. every non-empty subset has a least element

$\Rightarrow X - \bigcup_{\beta<\alpha} X_\beta \neq \emptyset$. otherwise $rank_R(a) = \infty$ for some a

只说明有极小"基"，不说明别的！(如传递性…)

lem5.12 a relation R on a set X is well-founded iff & only iff $rank_R(a) \in ORD$, $\forall a \in X$

Pf: 1' iff R well-founded 这个证明不好!

let $Z_a = \{b \in X : bRa\}$ iff $Z_a = \Phi$ then $rank_R(a) = 0$, iff $Z_a \neq \Phi$, then $rank_R(a) = \sup\{rank_R(b)+1 : b \in Z_a\}$

and for $\forall b \in Z_a$, we can find a "Z_b" … then for $\forall c \in Z_b$, find "Z_c" …

since R is well-founded, $\exists$ min-element in X, ⇒没有无限下降链!

$\therefore$ the process of finding the set of predecessor can't be infinite

iff $\forall b \in Z_a$, $rank_R(b)$ is ordinal, then $rank_R(a)$ is ordinal

every $rank_R(b)$ is ordinal since there's no infinite descending chain

$a \to Z_a$

$b \to Z_b$ $\quad b' \to Z_{b'}$ -- $\quad b'' \to Z_{b''}$

$c \to Z_c$ -- $\quad c' \to Z_{c'}$ -- $\quad c'' \to Z_{c''}$ …

… $R b R a$

finite length chain

$\Rightarrow$ $rank_R(a)$ is ordinal

1. iff $\exists a \in X$, $rank_R(a) \notin ORD$

let $Y \subseteq X$, $Y := \{b \in X : rank_R(b) = \infty\} \neq \Phi$

(b_0 may not unique)

R-well founded, $\therefore \exists b_0$ is the mini- in Y; and $\forall c$ s.t. cRb, $c \notin Y$, ie. $rank_R(c) \in ORD$

$\therefore b_0$ satisfy $rank_R(b_0) = \sup\{rank_R(c)+1 : cRb_0\} \in ORD$

contradict!

$\therefore Y = \Phi$, ie. R is well-founded then $\forall a \in X$, $rank_R(a) \in ORD$

2. $\forall a \in X$, $rank_R(a) \in ORD$

$\forall Y \subseteq X$, $Y \neq \Phi$, $\exists$ corresponding set $Z := \{rank_R(y) : y \in Y\}$, Z is a set

$Z \subseteq ORD$, $\therefore \exists$ least element $rank(y_0)$ in Z, then y_0 is R-minimal

$\therefore \forall Y \subseteq X$, $Y \neq \Phi$, $\exists$ minimal element

$\Rightarrow$ R is well-founded on X

即证任意非空子集有极小元，非空 → 秩集在ORD中非空 → ORD必有least → ORD中的least对应Y中mini