L19: Fixed point Method:

$$\begin{cases} \partial_t u = -\mathcal{L}u + f(u) \\ u|_{t=0} = g \end{cases} \quad . \ \mathcal{L} = -\sum_{i,j} (a_{ij}\partial_{x_i})\partial_{x_j}$$

Banach fixed point th:

$X$ is Banach (complete + normed linear space), $A: X \to X$ nolinear,

if $\exists \gamma < 1$ s.t. $\|A[u] - A[v]\| \le \gamma \|u - v\|$, $\forall u, v \in X$

$\Rightarrow$ A的fixed point存在且唯一, $Au_0 = u_0$

proof: let $v = Au \in X$

$\|A[u] - A^2[u]\| \le \gamma \|u - A[u]\|$

$$\|A^n[u] - A^{m+n}[u]\| \le \|A^n[u] - A^{n+1}[u]\| + \|A^{n+1}[u] - A^{n+2}[u]\| + \cdots + \|A^{m+n-1}[u] - A^{m+n}[u]\|$$

$$\le \gamma^n \|u - A[u]\| + \gamma^{n+1} \|u - A[u]\| + \cdots + \gamma^{m+n-1} \|u - A[u]\|$$

$$\le \frac{1}{1-\gamma} \gamma^n \cdot \|u - A[u]\|$$

$\therefore \{A^n[u]\}_{n=1}^{+\infty}$ is Cauchy sequence, $\forall u \in X$.

$\therefore \exists w \in X$ s.t. $A^n[u] \to w$, $A^{n+1}[u] \to A[w] = w$; 这个Cauchy列的收敛点即是fixed point

Example: ODE: $\begin{cases} \dot{x}(t) = f(t, x) \\ x(0) = a \end{cases}$, $f$ bounded, Lipchitz cts in Banach space $X$

let $X = C[0,T] \cap \{x(0) = a\}$, X是函数空间, x可理解为X中的函数

let $A[x](t) = \int_0^t f(s, x(s))\,ds$ ; $A[x](t)' = f(t, x(t))$

then: $|A[x](t) - A[\tilde{x}](t)| \le \int_0^t |f(s, x(s)) - f(s, \tilde{x}(s))|\,ds$

$$\le LT \times \sup_{s \in [0,T]} |x(s) - \tilde{x}(s)|$$

$\therefore \|A[x](t) - A[\tilde{x}](t)\| \le LT \cdot \|x - \tilde{x}\|$, let $T < \frac{1}{L}$

在X中, $A^n[x](t) \to y$ , $A^{n+1}[x](t) \to A[y](t) = y$

Q: 在X中, $A[x](t)$ 就是ODE解, 为什么要去求 $A^{n+1}[x](t) \to y$

{ let $T < \frac{1}{L}$, restrict X, 求出的是局部解吧, y在$\mathbb{R}^n \supseteq X$, 并不一定远离w ?

Th: $\begin{cases} u_t = -\mathcal{L}u + f(u) \quad f: \mathbb{R} \to \mathbb{R} \text{ lipschitz cts}, |f(z)| \le C(1 + |z|) \\ u(t=0) = g \\ u|_{\partial U} = 0 \end{cases}$ , then:

(1): $\bar{u} \in L_t^2 H_0^1$, $\bar{u}' \in L_t^2 H^{-1}$ (即$\partial_t u$), $\Rightarrow \bar{u} \in C_t L^2$; (2): $\frac{d}{dt}\|\bar{u}\|_{L^2}^2 = 2\langle \bar{u}, \bar{u}' \rangle$, $\max_{0 \le t \le T} \|\bar{u}\|_{L^2} \le C(\|\bar{u}\|_{L^2 H_0^1}^2 + \|\bar{u}'\|_{L^2 H^{-1}}^2)$

recall: $C_0^k(U)=\{v\in C^k(U): \lim_{x\to\partial U}|v(x)|=0\}$

$C_c^k(U)=\{v\in C^k(U): \exists$ cpt $K\subseteq U$, s.t. $u=0$ in $K^c\}$ 即 support is cpt

$H^k(U)=\{u\in L^1_{loc}(U): D^\alpha u\in L^2(U)$ for $\forall|\alpha|\le k\}$

$H_0^k(U)=C_c^{+\infty}$ closure under $\|\cdot\|_{H^k(U)}$

$\|u\|_{H^k(U)}=\sum_{|\alpha|\le k}\|D^\alpha u\|_{L^2(U)}$

$\|f\|_{H^1(U)}=\int_U |f(x)|+|\nabla f(x)|dx$ , $L_{loc}$: 在每个cpt子集上可积 $\int_K|f|d\mu<+\infty$ $\forall$ cpt $K\subseteq X$

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proof: Let $X=C([0,T],L^2(U))$

1: 证 $W=A[u]$ is weak solution of: $\begin{cases}\partial_t w=-\mathcal{L}w+f(u)\\ w|_{t=0}=g\end{cases}$ ($A=\mathcal{L}+\partial_t$ 吗?) 即: $\int_U\phi\mathcal{L}(u)dx=\int_U\phi f dx$ $\forall$ test $\phi$ given $\mathcal{L}(u)=f$

$\int_0^T\int_U f^2(u(t,x))\le\int_0^T\int_U c(1+u^2)\le C(1+\|u\|^2_{L^2_tL^2_x})\le C(1+\|u\|^2_{L^2_t\cdot H^1_{0x}})<+\infty$

$\therefore f(u(t,x))\in L^2_tL^2_x$ $w\in X$ exist

2. T small enough, $u\to w=A[u]$ is contradiction

设: $w_1=A[u]$, $w_2=A[\tilde u]$ 均为解

$\partial_t u+\mathcal{L}u=f(u)$

$\Rightarrow\int\partial_t u\cdot v+\int\mathcal{L}u\cdot v=\int f(u)\cdot v$

consider weak solution: 以 $v$ 作为测试函数. $\forall v\in H_0^1$, $0\le t\le T$ A和B是什么?

$\begin{cases}(w_1,v)+B[w_1,v]=(f(u),v)\\(w_2,v)+B[w_2,v]=(f(\tilde u),v)\end{cases}$ ; ~~$B[w,v]=\int_U Dw\,Dv+(\sum b_i\partial_{x_i}u)v$, ...~~

$(w_1-w_2,v)+B[w_1-w_2,v]=(f(u)-f(\tilde u),v)$

? $\therefore\frac{d}{dt}\|w_1-w_2\|^2_{L^2}+\theta\|\tilde w_1-\tilde w_2\|^2_{H_0^1}\le L\cdot\|u-\tilde u\|_{L^2}\cdot\|\tilde w_1-\tilde w_2\|_{L^2}\le\frac{L}{2\varepsilon}\|u-\tilde u\|^2_{L^2}+\frac{L\varepsilon}{2}\|w_1-w_2\|^2_{L^2}$

$\le\frac{L}{2\varepsilon}\|u-\tilde u\|^2_{L^2}+\frac{L\varepsilon C}{2}\|w_1-w_2\|^2_{H_0^1}$

L: lipschitz constant. C: Poincaire const

choose $\varepsilon$: $\frac{\varepsilon LC}{2}\le\theta$

then: $\frac{d}{dt}\|w_1-w_2\|^2_{L^2}\le\frac{L}{2\varepsilon}\|u-\tilde u\|^2_{L^2}$

$\therefore\|w_1-w_2\|^2_{L^2}\le\int_0^T\frac{L}{2\varepsilon}\|\cdots\|^2_{L^2}dt\le\frac{LT}{2\varepsilon}\|u-\tilde u\|^2_X$; 在T很小 $T<\frac{4\varepsilon}{L}$ 时 $\frac{LT}{2\varepsilon}<\frac{1}{2}$ 满足 fixed point

即 $\|A[u]-A[\tilde u]\|^2_{L^2}\le\gamma\|u-\tilde u\|^2_X$

$\|A[u]-A[\tilde u]\|_{L^2}\le\gamma\|u-\tilde u\|_X$, $\gamma<1$

$\therefore\exists$ fixed point, $u_0$. 即为方程解

3. Uniqueness: (Gronwall)

$\|u-\tilde u\|^2_{L^2}\le\int_0^T\frac{L}{2\varepsilon}\|u-\tilde u\|^2_{L^2}dt$, 设 $\exists u_0'$ 也为解 $u_0'-u$ 代入 $\to 0$.

Halmilton-Jacobi equation: $\partial_t u(t,x) + H(\partial_x u, t, x) = 0$ -- (*)

H: halmilton function 系统总能量, u: 作用量, x,t 坐标、时间

换元 (method of char): $x = \eta(s)$, $t := s$

$$\begin{cases} z(s) = u(t,x) = u(s,\eta(s)) \\ p(s) = \partial_x u(t,x) = \partial_x u(s,\eta(s)) \end{cases} \Rightarrow \partial_t z(s) + H(p(s), s, \eta(s)) = 0$$ 只剩 s 变量

$\dot p(s) = \partial_x\partial_t u(s,\eta(s)) + \partial_x\partial_x u(s,\eta(s))\cdot\dot\eta(s)$ 此处 $\partial_t$, $\partial_x$ 表示对第i个 slot 求导

$\overset{or}{=} \partial_t\partial_x u(t,x) = \partial_x(-H(\partial_x u, t, x)) = -\partial_x H(p(s), s, \eta(s))$

$\dot\eta(s) = \partial_p H(p(s), s, \eta(s))$

since: 由 $p(s)$ 的不同表示方法知 $\partial_x\partial_t u + \partial_x^2 u\cdot\dot\eta(s) + \partial_x H = 0$

对(*)整体求x的偏导: $\partial_x\partial_t u + \partial_p H\cdot\partial_x^2 u + \partial_x H = 0$ $\Bigg\} \Rightarrow \dot\eta(s) = \partial_p H$

$\dot z(s) = \partial_t u + \partial_x u(s,\eta(s))\cdot\dot\eta(s)$

$= p(s)\cdot\dot\eta(s) - H(p,s,\eta(s)) = p(s)\cdot\partial_p H(p,s,\eta(s)) - H(p,s,\eta(s))$

$$\Rightarrow \text{Hamilton ODE: } \begin{cases} \partial_t z(s) + H(p(s),s,\eta(s)) = 0 \\ \dot\eta(s) = \partial_p H(p(s),s,\eta(s)) \\ \dot p(s) = -\partial_x H(p(s),s,\eta(s)) \end{cases}$$

若 $\partial_t u = 0$. energy not depends on time: conservative (物理意义)

$\dot H(s) = \partial_p H\cdot\dot p(s) + \partial_x H\cdot\dot\eta(s)$

$= \partial_p H\cdot(-\partial_x H) + \partial_x H\cdot\partial_p H = 0$

注意: 以 $\dot p(s) = \partial_t\partial_x u(s,\eta(s)) + \partial_x^2 u(s,\eta(s))\cdot\dot\eta(s)$

$= \partial_t\partial_x u(t,x) = \partial_x(-H(\partial_x u,t,x)) = -\partial_x H(p(s),s,\eta(s))$ 为例:

$p(s) = \partial_x u(t,x)$, $s=t$, $\therefore \dot p(s) = \partial_t(\partial_x u(t,x))$

区别: 在 $u(t,x)$ 这个表达中 t 和 x 是没有关系的, 换元的时候 let $x=\eta(s)$, $t=s$, 和原来的式子没关系

$x=\eta(t)$ 不行

Q: 为什么不能 $x=\eta(s)$, $t=s$ $\Rightarrow$ $\dot\eta(s) = \partial_t x$

$p(s) = \partial_x u(t,x)$, $s=t$, $\Rightarrow$ $\dot p(s) = \partial_t\partial_x u(t,x)$, 用了这个, 两者有什么区别?

both right, but meaningless if "dx/dt"

**Calculus of variation**: 接下来L均表示legrangian,

let $L(q,x,t)$ 表示 legrangian function; $q\ x\ t$ 是形式变量, $q=\dot{w}(s), x=w(s), t=s$ 代入本问题

$I[w]=\int_0^t L(\dot{w}(s), w(s), s)\,ds$ ; $q=\dot{w}(s), x=w(s), t=s$ ⋯①

$\mathcal{A}=\{w\in C^2[0,t]: w(0)=y, w(t)=x\}$

考虑变分问题: $\inf\limits_{w} I[w]$

let $i(\varepsilon)=I[w+\varepsilon v]$, $v\in C_0^2[0,t]$, $w\in\mathcal{A}$

$i'(\varepsilon)=\frac{d}{d\varepsilon}\int_0^t L(\dot{w}+\varepsilon\dot{v}, w+\varepsilon v, s)\,ds=\int_0^t (\partial_q L)\dot{v}+(\partial_x L)\cdot v\,ds=0$

$=\int_0^t(\partial_q L)dv+\int(\partial_x L)\cdot v\,ds=(\partial_q L)v\Big|_0^t+\int_0^t -\partial_s(\partial_q L)v+(\partial_x L)\cdot v\,ds$

$\therefore -\partial_s(\partial_q L)+\partial_x L=0$ since $v|_{\partial C_0^2[0,t]}=0$, ↖ 对 $\forall v\in C^2$ 要满足 $i'(\varepsilon)=0$, 对 $v\in C_0^2$ 当然也要

⇓

[Euler-lagrange equation] under ①. ⇒ legendre transform 的原因

例: $L(q,x,t)=L(q,x)=\frac{1}{2}mq^2-\phi(x)$ ; 求 $\overset{I[w]=\int L}{L}$ 的极值, 对应的 $q,x$ ⇒ $q,x$ 不是定值, 是一个轨迹, $q$ 与 $x$ 相关

$\partial_x L=-\phi'(x)$, $\partial_q L=mq$

换元: $q=\dot{w}(s)$, $x=w(s)$, then $\partial_x L=-\phi'(w(s))$, $\partial_q L=m\cdot\dot{w}(s)$

代入 Euler-lagrange: $-\partial_s(m\cdot\dot{w}(s))+[-\phi'(w(s))]=0$

$\therefore m\cdot\ddot{w}(s)=-\phi'(w(s))$

关于 $s$ 的 ODE, 在 $\phi$ 已知时, $w(s)$ 可以求, 代回 $q=\cdots, x=\cdots$ 可知 $q,x$ 的趋势轨迹

Q: $t$ 和 $s$ 没有关系. consider $L(q,x)$, $q=\dot{w}(s)$ $x=w(s)$ some $w,s$ ⇒ characteristic line