4.15.

TH: $(E,\|\cdot\|_1),(E,\|\cdot\|_2)$ complete, $\exists C>0$ s.t. $\|x\|_1 \le C\|x\|_2\ \forall x \Rightarrow \|\cdot\|_1 \simeq \|\cdot\|_2$

$(\Rightarrow \|x\|_2 \le C\|x\|_2)$

pf: $I_{1,2}: (E,\|\cdot\|_1) \to (E,\|\cdot\|_2)$

$x \mapsto x$

$\begin{cases} \text{surjective} \\ x|_{(E,\|\cdot\|_2)} = y|_{(E,\|\cdot\|_2)} \Rightarrow \|x-y\|_1 \le \|x-y\|_2 \Rightarrow x|_{(E,\|\cdot\|_2)} = y|_{(E,\|\cdot\|_1)} \text{ injective} \end{cases}$

只适用于 $E\to E$，如果 surjective 成立可以考虑 $E \leftrightarrow F$

Closed Graph TH, $(X,\|\cdot\|_1),(Y,\|\cdot\|_2)$ Banach, linear $T: X\to Y$; $G := \{(x,Tx): x\in X\} \subseteq (X\times Y, \|\cdot\| = \|x\|_1 + \|y\|_2)$

G is closed in $(X\times Y, \|\cdot\|) \Leftrightarrow T$ bounded

pf: $\left.\begin{array}{l} x_n \to x_0 \\ Tx_n \to y_0 = Tx_0 \text{ since } G \text{ closed} \end{array}\right\} \Rightarrow \underset{(\text{in } \|\cdot\|_1)}{x_n \to x_0}, Tx_n \to y_0$ then $y_0 = Tx_0$. (WTS)

我们需要的 $x_n \to x_0$, then $Tx_n \to y_0$, $y_0 = Tx_0$；即 $x_n \to x_0$ 应该推出 $Tx_n \to y_0$

$(X,\|\cdot\|_3): \|x\|_3 = \|x\|_1 + \|Tx\|_2$, $\|\cdot\|_3 \simeq \|\cdot\|_1$

$\therefore x_n \to x_0$ in $(X,\|\cdot\|_1) \Rightarrow x_n \to x_0$ in $(X,\|\cdot\|_3)$, $\|x_n - x_m\|_1 + \|Tx_n - Tx_m\|_2 \to 0$

$\Rightarrow Tx_n \to y_0$ in $(Y,\|\cdot\|_2)$

Def: $J: X \to X^{**}$

$: x \mapsto x^{**} \overset{记}{=} J(x): X^* \to \mathbb{F}$

$: f \mapsto f(x) = x^{**}(f)$

①. $|J(x)\cdot f| = |f(x)| \le \|f\|\cdot\|x\| \Rightarrow \|Jx\| \le \|x\|$

distance functional: X normed, $Y \le X$ closed, $x_0 \in X\setminus Y$, $\exists f\in X^*$, $f(Y)=0$ $\|f\|=1$, $f(x_0) = d(x_0,Y)$

$\Rightarrow f(x) = \|x\|$, $\|f\|=1$ exist

$\Rightarrow \|J(x)\| = \|x\|$,

② $J(X)$ is closed subspace in X^{**}

X^{**} 中不只有 x^{**}，可能还有别的；J injective. not necessarily surj

TH: H is reflexive

Rmk: Double Riesz Prepresentative TH

$\forall f \in H^*$, $\exists x_0$, x_0 unique s.t. $f(y) = (y, x_0)$ $\forall y \in H$

$R: H \to H^*$

$x_0 \mapsto R(x_0): H \to \mathbb{F}$

$y \mapsto R(x_0)(y) = \overline{(x_0, y)}$ bijective

WTS: $\forall F \in H^{**}$ $\exists x$ (must be unique) s.t. $x^{**} = F$

$\iff f(x) = F(f)$ $\forall f \in H^*$ ... apply 1st. Riesz Rep to f

$R(y) = f$ bijective $\iff (x, y) = F \circ R(y)$ $\forall y \in H$

$\iff (y, x) = \overline{F \circ R(y)}$ $\forall y \in H$ ... apply 2nd Riesz Rep to $\overline{F \circ R}$

$\overline{F \circ R}: H \to \mathbb{F} = (\cdot, x_0)$ some $x_0 \in H$, x_0 unique $\Rightarrow$ let $x := x_0$, satisfy $f(x) = F(f)$ $\forall f \in H^*$

这里也能映证 J must be inj

TH: X Banach. X reflexive $\iff$ X^* reflexive

need proof!

"$\Rightarrow$" WTS: $\forall F \in (X^*)^{**}$, $\exists y \in X^*$ s.t. $y^{**} = F$

$\iff f(y) = F(f)$ $\forall f \in (X^*)^* = X^{**}$ ... ①

Given: $\forall f \in X^{**}$, $\exists x \in X$ s.t. $x^{**} = f$

$\iff T(x) = f(T)$ $\forall T \in X^*$ (including y) ... ②

$\iff x^{**}(y) = F(x^{**})$

define $g: X \to \mathbb{F}$

$x \mapsto f(g) = F(f)$, where $f = x^{**}$, well-defined

$y := g$ satisfy!

注意此处 J bij $\Rightarrow f \iff x^{**}$

"$\Leftarrow$" Given ... ①

WTS ... ② if J not surj, $\exists f \in X^{**}$ s.t. ($\nexists x \in X$, s.t. $x^{**} = f$)

$J(X)$ closed subspace, $f \notin J(X)$

$\Rightarrow \exists \Phi \in (X^{**})^* = X^{***}$ s.t. $\Phi|_{J(X)} \equiv 0$, $\|\Phi\| = 1$, $\Phi(f) = dist(f, J(X)) > 0$

$\Phi \iff y$ in ① $f(y) = \underbrace{\Phi(f)}_{\neq 0} = y^{**}(f) = \underbrace{f^{**}(y^{**})}_{f^{**} \in X^{***}} = \underbrace{\Phi(y^{**})}_{=0}$

TH: X Banach, Y closed subspace of X $\Rightarrow$ Y reflexive

pf: WTS: $\forall F \in Y^{**}$, $\exists y \in Y$ s.t. $y^{**} = F$

we know that $\exists x \in X$ s.t. $x^{**} = F$, x is unique since J injective $\Big\} \Rightarrow$ WTS: $x = y$

$F(f) = f(x)\ \forall f \in X^{**}$

if $x \notin Y$, $\exists g \in X$ s.t. $g|_Y \equiv 0$ $\|g\| = 1$, $g(x) = dist(x, Y) > 0$.

$$\left.\begin{aligned} &F(g) = F(g|_Y) = 0 \text{ since } F \text{ is linear} \\ &F(g) = x^{**}(g) = g(x) > 0 \end{aligned}\right\} \Rightarrow \forall F, x^{**}, \text{ if } x \notin Y, \text{ there's a contradiction}$$

$\Rightarrow \forall F \in Y^{**}$, $x \in Y$

TH: $X \cong Y$, X reflexive $\Leftrightarrow$ Y reflexive

pf: $\exists T: (X, \|\cdot\|_1) \to (Y, \|\cdot\|_2)$

$x \mapsto Tx$ s.t. $C_1\|x\|_1 \le \|Tx\|_2 \le C_2\|x\|_2$, $C_1, C_2 > 0$

$\forall F \in X^{**}$, $\exists x_0^{**} = F$, i.e. $f(x_0) = F(f)\ \forall f \in X^*$

WTS: $\forall \Phi \in Y^{**}$, $\exists y^{**} = \Phi$, i.e. $\bar{f}(y) = \Phi(\bar{f})\ \forall \bar{f} \in Y^*$

consider $T^*: Y^* \to X^*$

$\bar{f} \mapsto T^*\bar{f}: X \to \mathbb{F}$

$x \mapsto T^*(\bar{f})(x) = \bar{f} \circ T(x)$; $(T^{-1})^*(f)(y) = f \circ T^{-1}(y)$

$\Rightarrow \bar{f} := f \circ T^{-1}$

$\Rightarrow \bar{f}(y) = \Phi(\bar{f}) \Leftrightarrow f \circ T^{-1}(y) = f(x) = \Phi(f \circ T^{-1})$

$\Phi(\cdot)T^{-1}: X^* \to \mathbb{F}$

$\triangle \mapsto \Phi(\underbrace{\triangle)T^{-1}}_{Y \to \mathbb{F}} \in \mathbb{F}$

$\Rightarrow \Phi(\cdot)T^{-1}: \cdot \mapsto \mathbb{F} \in X^{**}$, $\exists x_0 \in X$, $x_0^{**} = \Phi(\cdot)F^{-1}$

$\Rightarrow T^{-1}y = x_0$

实际上所有内容都是双射对应的

Def: $x_n \rightharpoonup x$: $f(x_n) \to f(x)\ \forall f \in X^*$

Counter-Example: in H, $f(\cdot) = (\cdot, y)$, $\{e_i\}_{i=1}^{\infty}$ orthonormal basis

$\sum_{i=1}^{+\infty} |(e_i, y)|^2 \le \|y\|^2$ --- Bessel's, here "=" holds

$\Rightarrow |(e_i, y)| \to 0$ i.e. $f(e_i) \to 0$

$e_i \rightharpoonup 0$ but $\|e_i\| = 1$

TH: $T: X \to Y$ is compact linear, $x_n \rightharpoonup x$ in X $\Rightarrow Tx_n \to Tx$ in Y

Pf: $x_n \rightharpoonup x \Rightarrow \{x_n\}$ bounded ··· ①

$\Rightarrow \exists \{x_{n_k}\} \subseteq \{x_n\}$ s.t. $Tx_{n_k} \to y \in Y$

$\forall f \in Y^*$, $f \circ T \in X^* \Rightarrow f\circ T(x_n) \to f \circ T(x)$

$\Rightarrow T(x_n) \rightharpoonup T(x)$ in Y, $T(x_{n_k}) \rightharpoonup T(x)$ in Y

$\Big\} \Rightarrow y = Tx$; $Tx_{n_k} \to Tx$

if $\exists \varepsilon$ s.t. $\forall N$, $\exists n > N$ s.t. $\|Tx_n - Tx\| > \varepsilon$

let $\{y_n\} \subseteq \{x_n\}$ satisfy $\|Ty_n - Tx\| > \varepsilon$ $\forall n$

let $x_n := y_n$ in ① $\Rightarrow \exists\, y_{n_k} \to Tx$

$\Big\} \Rightarrow$ contradict.

这个不反证说不明白

Def: $\{f_n\}_{n=1}^{+\infty} \subseteq X^*$, $f_n \overset{*}{\rightharpoonup} f$ if $f_n(x) \to f(x)$ $\forall x \in X$

$\Rightarrow$ "$x^{**}(f_n) \to x^{**}(f)$ $\forall x^{**} \in J(X)$" contained in "$F(f_n) \to F(f)$ $\forall F \in X^{**}$"

$\Vert$

$f_n \rightharpoonup f$

TH: ① : $f_n \rightharpoonup f \Rightarrow f_n \overset{*}{\rightharpoonup} f$

② : $f_n \overset{*}{\rightharpoonup} f \overset{\text{reflexive } X}{\Longrightarrow} f_n \rightharpoonup f$

$\Big\}$ obv from Def ; $\dim(X) < +\infty$ also implies ②

③
1. $\dim X < +\infty \Rightarrow f_n \rightharpoonup f$ implies $f_n \to f$; ···
2. $f_n \overset{*}{\rightharpoonup} f \Rightarrow \{\|f_n\|\}$ bounded ··· Uni-Boundness TH
3. $f_n \overset{*}{\rightharpoonup} f \Rightarrow \|f\|_{X^*} \le \liminf\limits_{n\to+\infty} \|f_n\|_{X^*}$

$\Big\}$ $f_n \rightharpoonup f$ or. $f_n \overset{*}{\rightharpoonup} f$ both holds!

Pf 3: $\forall \varepsilon > 0$, $\exists x \in X$ s.t. $\|x\| = 1$, $|f(x)| > \|f\| - \varepsilon$ $\Rightarrow f(x) < 0$, then let $f := -f$

$\Rightarrow \|f\| - \varepsilon < f(x) = \lim\limits_{n\to+\infty} f_n(x) = \liminf\limits_{n\to+\infty} f_n(x) \le \liminf \|f_n\| \cdot \|x\|$

这里没什么别的引理，用了 $\|f\| = \sup|f(x)|$ 的定义

TH: $\{f_n\} \subseteq X^*$ bounded ($\sup\|\cdot\| < M$), $f_n(a) \to f(a)$ $\forall a \in A$. A dense in X $\Rightarrow \lim\limits_{n\to+\infty} f_n(x)$ exist $\forall x \in X$

↓

有界函数列, conv in dense set $\Rightarrow$ conv in X

TH: X separable, $\{f_n\} \subseteq X^*$ bounded $\Rightarrow \exists \{f_{n_k}\} \subseteq \{f_n\}$ s.t. $\lim\limits_{n_k\to+\infty} f_{n_k}(x)$ exist $\forall x \in X$, i.e. $f_{n_k} \overset{*}{\rightharpoonup} f$ in X^*

康托对角线 $\Rightarrow f_{n_k}$ point-wise conv

$\Rightarrow f_{n_k} \overset{*}{\rightharpoonup} f$ in X^*