L22

例1: $u_t + u\cdot u_x = 0 \quad t\geq 0$

$$u(x,0)=\phi(x)=\begin{cases}1 & x<0\\ 1-x & 0<x<1\\ 0 & x>1\end{cases}$$

解: characteristic line: $\dot{x}(t)=u$, $\dot{u}(t)=0$, ...(*):

$$x(t)=ut+x_0$$

Let $y(t)=u(t,x)=u(t,x(t))$, $\dot{x}(t)=u$

$\dot{y}(t)=\partial_t u+\partial_x u\cdot\frac{dx}{dt}=\partial_t u+u\cdot\partial_x u=0$, 即特征线上 $u(t,x)$ 只与 x 有关

$\therefore u(t,x)=u(t,ut+x_0)=u(0,x_0)=u(0,x-ut)$

$$=\phi(x-ut)=\begin{cases}1 & x<ut\\ 1-x+ut & 0<x-ut<1\\ 0 & x-ut>1\end{cases}$$

(*): 疑问: $\begin{cases}\dot{x}(t)=u.\ \dot{u}(t)=0 \text{ exist? 为什么可以表示所有的解 } u\text{?}\\ u(t,x)=u(t,ut+x_0)\text{, 两个 } u \text{ 不一样, 为何例的 } u \text{ 要满足 } \dot{u}(t)=0\text{, 为什么可以如 } u=1-x+ut\end{cases}$

$\Rightarrow u=\frac{1-x}{1-t}$

① recall L2: $u(t,x)=\begin{cases}1 & x<t\\ (1-x)/(1-t) & t<x<1\\ 0 & x>1\end{cases}$ if $t\leq 1$, (no intersection)

② $t>1$, $u(t,x)=\frac{x-1}{t-1}$ $x\in(1,t)$ intersect

lem: [Link, The Rankine-Hugoniot jump condition]:

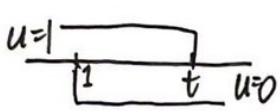
if u is a weak solution of $\begin{cases}u_t+\partial_x(f(u))=0 & x\in\mathbb{R}^n, t>0\\ u(x,0)=\phi(x)\end{cases}$; u discontinuous at $x=\xi(t)$,

but u smooth on either side of $x=\xi(t)$, then: $\frac{f(u^-)-f(u^+)}{u^--u^+}=\xi'(t)$

smooth = 连续高阶导

本题中 $\partial_t u+u\cdot\partial_x u=u_t+\partial_x\left(\frac{u^2}{2}\right)=0$, $f(u)=\frac{1}{2}u^2$

$u(t,x)=\begin{cases}1 & x<t\\ \frac{x-1}{t-1} & 1<x<t\\ 0 & x>1\end{cases}$

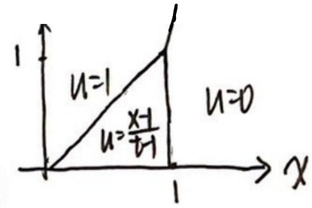


intersections happen $1<x<t$, $\therefore u^+=0,\ u^-=1$ $\quad \xi'(t)=\frac{\frac{1}{2}\cdot1^2-\frac{1}{2}0^2}{1-0}=\frac{1}{2}$

冲突起点 $x=\xi(t)$ contains $(1,1)$ $\quad\therefore x=\xi(t)=\frac{1}{2}(t+1)$

$$\Rightarrow u(x,t)=\begin{cases}1 & x<\frac{t+1}{2}\\ 0 & x>\frac{t+1}{2}\end{cases}$$

①+② ⇒: $u(x,t)=\begin{cases}1 & x<t\\ \frac{x-1}{t-1} & t<x<1\\ 0 & x>1\end{cases}$, if $t\le 1$, $u(x,t)=\begin{cases}1 & x<\frac{t+1}{2}\\ 0 & x>\frac{t+1}{2}\end{cases}$ if $t>1$



例2: $\begin{cases}u_t+u\cdot u_x=0 & t>0.\\ u(x,0)=\phi(x)=\begin{cases}0 & x<0\\ 1 & x>0\end{cases}\end{cases}$ (recall L2, 例1)

解: $\dot{x}(t)=u$, $\dot{u}(t)=0$, $x=u\cdot t+x_0$

let $y(t)=u(x(t),t)$

$y(t)'=\partial_x u\cdot\dot{x}(t)+\partial_t u=0$

$\therefore u(t,x)=u(t,ut+x_0)=u(0,x_0)=u(0,x-ut)=\begin{cases}0 & x<ut\\ 1 & x>ut\end{cases}=\begin{cases}0 & x<0\\ 1 & x>t\end{cases}$

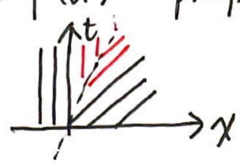
here we have no intersection, but "?"

How to find a "physical" solution

entrophy condition: $\begin{cases}u_t+\partial_x(f(u))=0\\ u(x,0)=\phi(x)\end{cases}$

$\Rightarrow f'(u^-)>\sigma=\xi'(t)>f'(u^+)$ "physical"

例2中, case1: $u(x,t)=\begin{cases}0 & x<\frac{1}{2}t\\ 1 & x>\frac{1}{2}t\end{cases}$



在 $x=\frac{1}{2}t$ 处断开需要检验

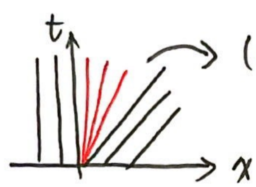
$u^-=0$, $u^+=1$,

$f(u)=\frac{1}{2}u^2$ $\therefore f'(u^-)=0$, $f'(u^+)=1$

$\sigma=\xi'(t)=\frac{f(u^-)-f(u^+)}{u^--u^+}=\frac{1}{2}$

⇒ not satisfy: $0>\frac{1}{2}>1$ fail!

case2: $u(x,t)=\begin{cases}0 & x<0\\ \frac{x}{t} & 0\le x\le t\\ 1 & x>1\end{cases}$



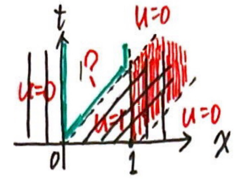
→ (called rarefaction wave)

it's continueous, ($x=0$, $x=t$) success ✓

例3: $u_t + u\cdot u_x = 0 \quad t\geq 0$

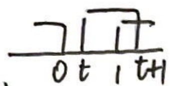
$u(x,0)=\phi(x)=\begin{cases}1 & 0<x<1\\ 0 & \text{otherwise}\end{cases}$

解: $u(t,x)=u(0,x-ut)=\begin{cases}1 & 0<x-\overset{1}{u}t<1\\ 0 & x-ut\le 0 \quad x-ut\geq 1\end{cases}$



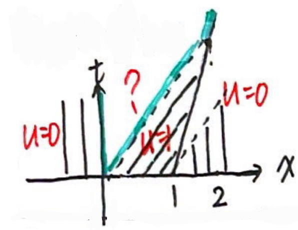
$x=\begin{cases} t+x_0 & t<x<t+1 \\ x_0 & x\le 0,\ x\geq 1 \ (\text{不是 } x\le t,\ x\geq t+1!)\end{cases}$

Step1: intersection



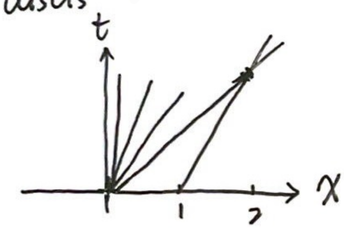
$\begin{cases}\xi'(t)=\dfrac{f(u^-)-f(u^+)}{u^- - u^+}=\dfrac{1}{2}; & f=\frac{1}{2}u^2,\ u^-=1,\ u^+=0. \\ x=\xi(t) \text{ contains } (x=1,t=0) & \end{cases}$

⇒ 冲突起点



$\Rightarrow x=\frac{1}{2}t+1$

Step2: discts

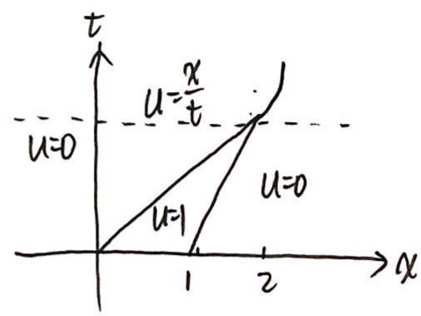


$u=\frac{x}{t}$ (记住! 对应 $\phi(x)=\frac{x}{\varepsilon}$, $|x|\le\varepsilon$)

Step3: intersection from step 2

$u^-=\frac{x}{t},\ u^+=0 \quad f=\frac{1}{2}u^2$

$\begin{cases}\because \dot{\xi}=\xi'(t)=\dfrac{x}{2t} \\ x=\xi(t) \text{ contains } (2,2)\end{cases} \Rightarrow x(t)=\sqrt{2t}$



$\Rightarrow$ Step 1+2+3: $u(t,x)=\begin{cases}0 & x<0\\ \frac{x}{t} & 0<x<t\\ 1 & t<x<1+\frac{t}{2}\\ 0 & x>1+\frac{t}{2}\end{cases}$ $(t\le 2)$; $u(t,x)=\begin{cases}0 & x<0\\ \frac{x}{t} & 0<x<\sqrt{2t}\\ 0 & \sqrt{2t}<x\end{cases}$ (if $t>2$)