L14

Thm: L^∞ Stability: $u\in C^2(U)\cap C(\bar U)$, $c\geq 0$.

$$\begin{cases} -\Delta u + cu = f(x) & U \\ u|_{\partial U} = g & \partial U \end{cases} \Rightarrow \max_{\bar U}|u| \leq C(\sup_{\bar U}|f| + \sup_{\bar U}|g|)$$

Lemma: (weak maximum principle) $u\in C^2(U)\cap C(\bar U)$, $\mathcal{L} = -\Delta + c$

$\mathcal{L}u \leq 0$ in U,

then: $\max_{\bar U} u \leq \max_{\partial U} u^+$, $u^+ = \max(u,0)$ $u<0|_{\partial U}$ $\max_{\bar U} u\leq 0$ 可能 $\max_{\bar U} u \geq u|_{\partial U}$?

(comparison principle) $u\in C^2(U)\cap C(\bar U)$, $\mathcal{L} = -\Delta + c$

$$\begin{cases} \mathcal{L}u \leq \mathcal{L}v & U \\ u\leq v & \partial U \end{cases} \Rightarrow \max_{\bar U}(u-v) \leq \max_{\partial U} u^+ = 0,\ 即\ u\leq v \text{ in } \bar U$$

proof1: 设 $x_0 = \arg\max_{\bar U} u(x)$, $\begin{cases} \text{if } u(x_0)\leq 0, \text{ or } x_0\in\partial U, 证毕 \\ \text{if } u(x_0)>0,\ x_0\in U; \end{cases}$

① if $\mathcal{L}u<0$ in U,

$\mathrm{Hess}\, u(x_0) \leq 0$ (负定) $\therefore -\Delta u(x_0) = -\mathrm{tr}(\mathrm{Hess}\, u(x_0)) \geq 0$.

$c\geq 0$ $\therefore cu(x_0)\geq 0$.

$\therefore \mathcal{L}u = -\Delta u + cu|_{x_0} \geq 0$ 矛盾, 和热方程的 Stability 一样. HE 中 $\mathcal{L}u = \Delta u - \partial_t u$. 分为 $\mathcal{L}u<0$ 和 $\leq$

② if $\mathcal{L}u\leq 0$. Let $u_\varepsilon(x) = u(x) + \varepsilon(e^{x_1} - l)$, $l > \sup_{\bar U} e^x$

$\mathcal{L}u_\varepsilon = \mathcal{L}u - \varepsilon e^{x_1} + c(x)\cdot\varepsilon(e^x - l) < 0$. 同①

$\therefore \max_{\bar U} u_\varepsilon \leq \max_{\partial U} u_\varepsilon^+$

Let $\varepsilon\to 0$, then $\max_{\bar U} u \leq \max_{\partial U} u_\varepsilon^+$

这证明了 $\mathcal{L} = -\Delta + c$ 时的 weak MP. 之前证过 $\mathcal{L} = -\Delta$ 的 weak MP; comparison principle 由此易知

proof 1': 设 $w = G + F(M - \lambda e^x)$ $G = \sup|g|$, $F = \sup|f|$

$\mathcal{L}w = c(G + F(M - \lambda e^x)) + F\cdot\lambda e^x$,

say $x\in[-L, L]$. Let $\lambda e^{-L} \geq 1$, $M \geq \lambda e^L$

$$\begin{cases} \mathcal{L}(u+w) \geq 0 \\ u+w|_{\partial U} \geq 0 \end{cases} \Rightarrow u+w\geq 0.\quad 同样的\ -u+w\geq 0.$$

$\therefore \max_{\bar U} u \leq C(G+F) = C(\sup|f| + \sup|g|)$ some const C 所以是要求 U bounded 吗? $x\in[-L,L]$

Wave Equation: $\begin{cases} \Delta u = \partial_{tt} u & (\text{即 } \partial_x^2 u = \partial_t^2 u) \\ -\Delta u + \partial_{tt} u = \cdots \end{cases}$

泛函: $\mathcal{L}[u] = \int_{U_{t_0}} \frac{1}{2}|\partial_t u|^2 - \frac{1}{2}|\partial_x u|^2$

$i(\varepsilon) = \mathcal{L}[u + \varepsilon v]$

$i'(\varepsilon) = \int \partial_t u \cdot \partial_t v + \varepsilon|\partial_t v|^2 - Dv \cdot Du + \varepsilon|Dv|^2$

$0 = i'(0) = \int (-\partial_{tt} u + \Delta u) v = 0 \quad \forall v \in C_0^{\infty}(U_{t_0}) \Rightarrow -\partial_{tt} u + \Delta u = 0$

boundary: $u(0,x) = g,\ u_t(0,x) = h$ 一维WE的解

解法一:(示波)

$U = \mathbb{R}^d$, full-space, 用 fourier 解 $\partial_{tt} u = \Delta u$.

$\partial_{tt} \hat{u}(t,\xi) = -4\pi^2|\xi|^2 \hat{u}(t,\xi) \quad \forall t$

$(f*g)^\wedge = \hat{f} \cdot \hat{g}$

$\therefore (\hat{f} \cdot \hat{g})^\vee = f * g$

解ODE: $\hat{u}(t,\xi) = \hat{g}(\xi) \times \cos(2\pi|\xi|t) + \sin(2\pi|\xi|t) \cdot \frac{\hat{h}(\xi)}{2\pi|\xi|}$

$\therefore u(t,x) = \left(\hat{g}(\xi) \times \cos(2\pi|\xi|t)\right)^\vee + \left(\sin(2\pi|\xi|t) \cdot \frac{\hat{h}(\xi)}{2\pi|\xi|}\right)^\vee$

$= \cos(2\pi|\xi|t)^\vee * g(x) + \left(\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\right)^\vee * h(x) \quad \cdots ①$

其中: $\begin{cases} \cos(2\pi|\xi|t)^\vee = \int e^{2\pi i \xi \cdot x} \cos(2\pi \xi t)\, d\xi = \frac{1}{2}\int e^{2\pi i|\xi|(x+t)} + e^{2\pi i|\xi|(x-t)}\, d\xi \\ \qquad = \frac{1}{2}(\delta(x+t) + \delta(x-t)) \\ \frac{\sin(2\pi|\xi|t)}{2\pi|\xi|} = \frac{\sin(2\pi\xi t)}{2\pi\xi} = \int_0^t \cos(2\pi\xi x)\, dx \end{cases}$

$1 \leftrightarrow \delta$

$\therefore \left(\frac{\sin(2\pi|\xi|t)}{2\pi|\xi|}\right)^\vee = \int_0^t (\cos 2\pi\xi s)^\vee ds = \int_0^t \frac{1}{2}(\delta(x-s) + \delta(x+s))\, ds$

$= \frac{1}{2}(1_{x \in (-t,0)} + 1_{x \in (0,t)})$ ← δ 的积分区间含0.则为1

$\therefore$ 代回①中: $u = \frac{1}{2}(\delta(x+t) + \delta(x-t)) * g(x) + \frac{1}{2}(1_{x \in (-t,0)} + 1_{x \in (0,t)}) * h(x)$

$= \frac{1}{2}(g(x-t) + g(x+t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy$

显然: $\lim_{(t,x) \to (0,x)} u(t,x) = g(x)$; $\lim_{(t,x) \to (0,x)} u_t(t,x) = h(x)$; 即在 ∂U 处满足 cts $\Rightarrow$ classic

解法二:(示波)

$\partial_{tt} - \partial_{xx} = (\partial_t - \partial_x)(\partial_t + \partial_x)$

设 $u(t,x) = F(x+t) + G(x-t)$, some F, G

$\begin{cases} u(0,x) = g \\ u_t(0,x) = h \end{cases} \Rightarrow \begin{cases} F(x) + G(x) = g(x) \\ F'(x) - G'(x) = h(x) \end{cases}$, then $F'(x) = \frac{1}{2}g'(x) + \frac{1}{2}h(x)$

用 $\int_0^x$ 这种符号, 显然在 $\mathbb{R}^1$

$\Rightarrow \begin{cases} F(x) = \frac{1}{2}g(x) + \frac{1}{2}\int_0^x h(y)\, dy \\ G(x) = \frac{1}{2}g(x) - \frac{1}{2}\int_0^x h(y)\, dy \end{cases}$ 这样和解法一求出来一样的