L7:

Ω bounded, $T>0$. parabolic interior $\Omega_T = (0,T] \times \Omega$ 默认单个Ω为x的区域

parabolic boundary $\Gamma_T = (\{0\}\times\Omega) \cup ([0,T]\times\partial\Omega) = \bar{\Omega}_T \setminus \Omega_T$

Th1 (weak) maximum principle: $u \in C^{1,2}(\Omega_T) \cap C(\bar{\Omega}_T)$

$$u_t - \Delta u = F(x,t) \le 0 \text{ in } \Omega_T$$

then: $\max\limits_{\bar{\Omega}_T} u = \max\limits_{\Gamma_T} u$ $\Rightarrow$ 即内部$F\le 0$, 则闭包中u最大值在边界取到

不含T (见纸质 Lecture Note)

Th1 (coro): classical solution $u \in C^{1,2}(\Omega_T) \cap C(\bar{\Omega}_T)$, u is unique

$$\text{HE}: \begin{cases} \partial_t u = \Delta u & \text{in } \Omega_T \\ u|_{\partial\Omega} = g(t) & \text{on } \partial\Omega \\ u|_{t=0} = \varphi \end{cases}$$

在Th1基础上: 设 u_1, u_2 be solution of HE, let $v = u_1 - u_2$

$u_1, u_2, v \in C^{1,2}(\Omega_T) \cap C(\bar{\Omega}_T)$

$$\begin{cases} \partial_t v = \Delta v \text{ in } \Omega_T \text{ ; } v=-v\text{代入也成立} \\ v|_{\partial\Omega} = 0 \end{cases}$$

$u|_{t=0} = \varphi$ 没用

by "MP": $\begin{cases} \max_{\bar{\Omega}_T} v \le \max_{\partial\Omega} v = 0 \\ \max_{\bar{\Omega}_T} (-v) \le \max_{\partial\Omega} (-v) = 0 \end{cases}$ $\Rightarrow$ $v \equiv 0$ in $\bar{\Omega}_T$, $u_1 = u_2$

对于Th1的证明:

① 设 $(t^*, x^*) \in \Omega_T$, s.t. $u(t^*, x^*) = \max_{\bar{\Omega}_T} u(t,x)$

in Ω_T: $\begin{cases} u(t^*,x^*) \ge u(t, x^*) \ \forall t, \ \therefore \partial_t u(t^*,x^*) \ge 0 \\ u(t^*,x^*) \ge u(t^*, x), \ \forall x, \ \therefore \partial_x^2 u(t^*,x^*) \le 0 \end{cases}$ then $\begin{cases} \partial_t u - \partial_x^2 u \ge 0 \\ u_t - \Delta u = F(x,t) \le 0 \text{ given} \end{cases}$

② then: let $u_\varepsilon = u(t,x) - \varepsilon t$, $\forall \varepsilon > 0$; 上面只解决$F<0$, 在$F\le 0$时: $F=0$ here

$$\partial_t u_\varepsilon - \Delta u_\varepsilon = \partial_t u - \Delta u - \varepsilon < 0 \text{ in } \Omega_T \text{ since: } \partial_t u - \Delta u \le 0$$

by "MP": $\max_{\bar{\Omega}_T} u_\varepsilon \le \max_{\partial\Omega_T} u_\varepsilon$, $\forall \varepsilon > 0$ ($F<0$情况在上面证过了, 已满足MP)

let $\varepsilon \to 0$ then $\max_{\bar{\Omega}_T} u \le \max_{\partial\Omega_T} u$

①+②: $F<0$ 和 $F=0$ 时: $\max_{\bar{\Omega}_T} u = \max_{\partial\Omega_T} u$; 证毕

补充前半部分为什么 $\partial_x^2 u \le 0$,

Hessian 负定, 临界点 (x,y) 为极大值, 负定(不)严格 $\Rightarrow$ (x,y) (不)严格极大

一个··

不定 (x,y) 啥也不是

by: $f(z_0+h) = f(z_0) + \nabla f(z_0) \cdot h + \frac{1}{2} h^T Hf(z_0) h + o(\|h\|^2)$, $h \to 0$. ∇f代表一阶导, Hf代表Hessian

此处 $\text{Hess}\, u(t^*, x^*) = \nabla^2 u(t^*, x^*) \le 0$

$\Downarrow$

$\Delta^2 u(t^*, x^*) \le 0$ (我理解为 $H \le 0$, $\therefore$ 特征值均≤ 0 $\sum$特征值 $= \frac{\partial^2 u}{\partial x^2} + \underset{\text{HE}=0}{\frac{\partial^2 u}{\partial t^2}} = \Delta^2 u$)

HE中所有二阶不讨论t, 只考虑x

Th3: $u\in C^{1,2}((0,+\infty)\times\mathbb{R}^d)\cap C([0,+\infty)\times\mathbb{R}^d)$

$$\begin{cases}\partial_t u=\Delta u, & t>0\\ u|_{t=0}=0\end{cases}$$

and $|u(t,x)|\le C\cdot e^{Ax^2}$ for some $A,C>0$, then $u\equiv 0$. $\Rightarrow$ u在x上的增速若能被 $C\times e^{Ax^2}$ 控制 L^∞

proof3: 要使用MP条件,要有确定边界值,

let $\Omega_{T,L}=(0,T]\times B_L(0)$,

let $T<\frac{1}{4A}$ $\therefore \exists\varepsilon>0$, $T+\varepsilon<\frac{1}{4A}$ ③中用来逼近

HW1中有 $\phi(x)=u(t,x)$, $|\phi|\le Ce^{Ax^2}$ 条件,实际上仅在此条件下, $t<\frac{1}{4A}$, $G_t*\phi$ 才有意义;再大不一定)

设 $v(t,x)=u(t,x)-\frac{C'}{(T+\varepsilon-t)^{d/2}}\times e^{|x|^2/4(T+\varepsilon-t)}$, $C'>0$

then $\partial_t v-\Delta v=0$

"MP": $\max_{\bar{\Omega}_{T,L}} v=\max_{\partial\Omega_{T,L}} v(t,x)$

t的边界 $t=T$ $t=0$, x边界 $|x|=L$:
$$\begin{cases}① & v(0,x)\le u(0,x) \text{ by definition } \therefore v(0,x)\le 0\\ ② & \forall C'>0,\ |x|=L,\ v(t,x)\le C\times e^{AL^2}-\frac{C'}{(T+\varepsilon)^{d/2}}\times e^{L^2/4(T+\varepsilon)}\\ & \le C\times e^{AL^2}-C'A\times e^{(A+\delta)L^2},\ \underset{>0}{\delta}=\frac{1}{4(T+\varepsilon)}-A\\ & \le 0 \text{ if } L\to+\infty\\ ③ & v(T,x)\le 0 \text{ let } \varepsilon\to 0\end{cases}$$

$\Rightarrow \max_{\bar{\Omega}_{T,L}} v\le 0$

$u(t,x)\le\frac{C'}{(T+\varepsilon-t)^{d/2}}\times e^{|x|^2/4(T+\varepsilon-t)}$ in $\Omega_{T,L}$, $T<\frac{1}{4A}$, $L\to+\infty$, $\underline{\forall C'}$;

let $C'\to 0$, then $u(t,x)\le 0$ in $\Omega_{T,L}$, 同理 $-u(t,x)\le 0$ in $\Omega_{T,L}$

$\therefore u(t,x)\equiv 0$

Th4: Stability: Let $u_i\in(C^{1,2}(U_T)\cap C(\bar{U}_T))$, $U_T=(0,T]\times U$ 详见TWP35

$$\begin{cases}\partial_t u_i=\Delta u_i+f_i & U_T\\ u_i|_{\partial U}=g_i\\ u_i|_{t=0}=\varphi_i, & f_i g_i \varphi_i \text{ cts}\end{cases}$$

then: $\max_{\bar{U}_T}|u_1-u_2|\le T\cdot\|f_1-f_2\|_{L^\infty}+\|g_1-g_2\|_{L^\infty}+\|\varphi_1-\varphi_2\|_{L^\infty}$;

即 the map from data to solution $(f,g,\varphi)\to u$ is table (cts) if f,g,φ cts

proof4: 设 $v(t,x)=u_1(t,x)-u_2(t,x)$

$w(t,x)=t\times\|f_1-f_2\|_{L^\infty}+\|g_1-g_2\|_{L^\infty}+\|\varphi_1-\varphi_2\|_{L^\infty}$ 所以这里就有界

then:
$$\begin{cases}\partial_t w-\Delta w\ge\partial_t v-\Delta v & \text{in } U_T\\ w\ge v & \text{on } \partial U\\ w|_{t=0}\ge v|_{t=0}\end{cases}$$

这么设原因在于:刚好满足"then"式

所以通过"then"式回推; ∂U说明w中应有 $\|g_1-g_2\|_{L^\infty}$; U_T说明w中应有 $t\times\|f_1-f_2\|_{L^\infty}$;

$$\text{简单解释：}\begin{cases} \text{in } U_T: \partial_t W-\Delta W=\|f_1-f_2\|_{L^\infty}-0,\ W\text{无关于}x;\ \partial_t v-\Delta v=f_1-f_2 \Rightarrow \|f_1-f_2\|_{L^\infty}\geq f_1-f_2 \\ \partial U: v=u_1-u_2=g_1-g_2\leq \|g_1-g_2\|_{L^\infty}\leq W \\ t=0: W|_{t=0}=\|g_1-g_2\|_{L^\infty}+\|\varphi_1-\varphi_2\|_{L^\infty}\geq \varphi_1-\varphi_2=v \end{cases}$$

$$\Rightarrow \text{即}\begin{cases} (\partial_t-\Delta)(-W+v)\leq 0 & \text{in } U_T \\ -W+v\leq 0 & \text{on } \partial U \\ (-W+v)|_{t=0}\leq 0 \end{cases}$$

$\therefore$ 由MP可知：$\max_{\bar U_T}(-W+v)\leq \max_{\partial U_T}(-W+v)$ in $\bar U_T$

$y=t\cdot\|f_1-f_2\|_{L^\infty}+\|g_1-g_2\|_{L^\infty}+\|\varphi_1-\varphi_2\|_{L^\infty}-(u_1-u_2)\geq y|_{\partial U_T},\ \forall t\in(0,T],\ \forall x$

$\therefore T\cdot\|f_1-f_2\|_{L^\infty}+\|g_1-g_2\|_{L^\infty}+\|\varphi_1-\varphi_2\|_{L^\infty}-\max_{\bar U_T}|u_1-u_2|\geq y|_{\partial U_T}\geq y|_{t=0}\geq 0$

$\Rightarrow \max_{\bar U_T}|u_1-u_2|\leq T\cdot\|f_1-f_2\|_{L^\infty}+\|g_1-g_2\|_{L^\infty}+\|\varphi_1-\varphi_2\|_{L^\infty}$

Th5：if $u\in C^{1,2}((0,+\infty)\times\mathbb{R}^d)\cap C([0,T]\times\mathbb{R}^d)$ $\to$ 即 $\bar U_T$

Th3后续

$$\begin{cases} \partial_t u-\Delta u=0 \\ u_{t=0}=\varphi,\ |u(t,x)|\leq C\cdot e^{A|x|^2} \end{cases}$$ ；"u"有这样的增速受控制，$\max_{\partial U_T}$可以进一步化简 $= \max_{\mathbb{R}^d}\varphi$

then：$\sup_{\bar U_T}|u|=\sup_{\mathbb{R}^d}|\varphi|$

设：$U_{T,L}=[0,T]\times B_L(0)$；$T<\frac{1}{4A}$，$\exists\varepsilon>0$ $T+\varepsilon<\frac{1}{4A}$

$$v(t,x)=u(t,x)-\frac{C'}{(T+\varepsilon-t)^{d/2}}\times e^{|x|^2/4(T+\varepsilon-t)}$$

then：$\partial_t v-\Delta v=0$（在proof3中也有上述过程）

$\therefore \sup_{U_{T,L}} v\leq \sup_{\partial U_{T,L}} v=\max(\sup_{\mathbb{R}^d}\varphi,\ \sup_{[0,T]\times\partial B_L(0)} v)$ …… ①

if：$|x|=L$ $\quad v(t,x)=u-\frac{C'}{(T+\varepsilon-t)^{d/2}}\times e^{|x|^2/4(T+\varepsilon-t)}$

$$\leq C\times e^{A|x|^2}-AC'\times e^{(A+\sigma)|x|^2}$$

$$=C\times e^{AL^2}-AC'\times e^{(A+\sigma)L^2}\leq 0 \text{ if } L\to+\infty$$，（具体可以看proof3）

…… ②

①+②：$\mathbb{R}^d$中 $\sup_{[0,T]\times\partial B_L(0)} v\leq\sup_{\mathbb{R}^d}\varphi$

$\therefore \sup_{\bar U_T}|u|=\sup_{\mathbb{R}^d}|\varphi|$

Thm $u\in C^{1,2}(U_T)\cap C^{0,1}(\bar U_T)$, $U=(0,l)$

$$\begin{cases}\partial_t u-\Delta u\ge 0 & \text{in } U_T\\ u|_{t=0}\ge 0 & \text{in } U\\ \frac{\partial u}{\partial n}+\beta u\big|_{\partial U}\ge 0 & t\in[0,T]\end{cases}$$

第一次出现"大于0"，比较定理中"x" bounded!

then $u\ge 0$ on $\bar U_T$ $\beta(t)\ge 0$

→ 流出热与热的关系

proof: ① $\frac{\partial u}{\partial n}+\beta u>0$ on ∂U

流出的热在当前法向方向上

$\partial_t u-\Delta u\ge 0$ in U_T ⇒ $\min_{\bar U_T}u=\min_{\partial U_T}u$

①' if $(t^*,x^*)\in[0,T]\times\partial U$ is the minimizer, then $\frac{\partial u}{\partial n}\le 0$ on ∂U

(*) ⇒ ∴ $\frac{\partial u}{\partial n}+\beta u\big|_{(t^*,x^*)}\le 0$ if $u(t^*,x^*)<0$

即若最小值在 $[0,T]\times\partial U$ 取，$\min_{\bar U_T}u=\min_{[0,T]\times\partial U}u=u(t^*,x^*)\ge 0$

①'' if $(t^*,x^*)\in\{0\}\times U$,

$\min_{\bar U_T}u=u(0,x^*)\ge 0$ by initial condition

② $\frac{\partial u}{\partial n}+\beta u\ge 0$ on ∂U ("=0"情况) ← (***)

let $w(t,x)=2t+(x-\frac{1}{2}l)^2$,

then: $\begin{cases}\partial_t w-\Delta w\ge 0 & \text{in } U_T\\ w|_{t=0}\ge 0 & \text{in } U\\ \frac{\partial w}{\partial n}+\beta w\big|_{\partial U}\ge c & \text{for some } c>0\end{cases}$ (**)

和 $\begin{cases}\partial_t u_\varepsilon-\Delta u_\varepsilon\ge 0\\ u_\varepsilon|_{t=0}\ge 0\end{cases}$

加l, 让其>0

⇒ 设 $u_\varepsilon=u(t,x)+\varepsilon\times w(t,x)$; u_ε 满足①的条件: $\frac{\partial u_\varepsilon}{\partial n}+\beta u_\varepsilon\big|_{\partial U}>0$, $\forall\varepsilon>0$

∴ $\min_{\bar U_T}u_\varepsilon\ge 0$

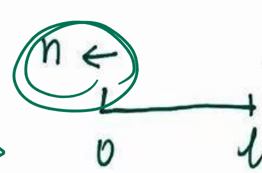
$\min_{\bar U_T}u\ge -\varepsilon\max_{\bar U_T}|2t+(x-\frac{1}{2}l)^2|$ $\forall\varepsilon>0$

let $\varepsilon\to 0$; obtain: $\min_{\bar U_T}u\ge 0$

解释 (*) 为什么不能直接 $\left.\begin{array}{l}\frac{\partial u}{\partial n}\le 0 \text{ on } \partial U\\ \frac{\partial u}{\partial n}+\beta u\big|_{\partial U}\ge 0\end{array}\right\}$ ⇒ $u(t^*,x^*)$ 证毕，为什么要分①② ⇒ $\beta(t)=0$

(**) 为什么 $\frac{\partial w}{\partial n}+\beta w\big|_{\partial U}\ge c>0$, $\frac{\partial w}{\partial n}$是什么 $=\frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial n}$ 吗？没法求吧？ u即x

(***) "$x-\frac{l}{2}$"目的呢, $\partial_t w-\Delta w\ge 0$ ⇒ 但取 $w=2t+x^2$ 会咋样



→ n 法向量边界指

在0处 $\frac{\partial w}{\partial n}=-w_x$