22.6 Suppose that $\boldsymbol{x}$ is a sequence (in $\mathbb{K}$) with the property that

$$\sum_{j=1}^{\infty} x_j y_j$$

converges for every $\boldsymbol{y} \in \ell^p$, $1 < p < \infty$. Show that $\boldsymbol{x} \in \ell^q$, where q is conjugate to p. (Meise and Vogt, 1997)

T22.6 $\sum_{i=1}^{+\infty} x_i y_i$ converges $\forall y \in \ell^p$ $1<p<+\infty \Rightarrow x \in \ell^q$. $\frac{1}{q}+\frac{1}{p}=1$

proof: Young's $\begin{cases} ab \le \frac{a^p}{p} + \frac{b^q}{q},\ \frac{1}{p}+\frac{1}{q}=1,\ 1<p,q<+\infty & \cdots ① \\ a^{\frac{1}{p}} \cdot b^{\frac{1}{q}} \le a\cdot\frac{1}{p} + b\cdot\frac{1}{q} & \cdots \text{由convexity of log(·)得出的原始式} \cdots ② \end{cases}$

$\Rightarrow \frac{|x_i|}{\|x\|_q} \cdot \frac{|y_i|}{\|y\|_p} \le \frac{1}{q} \cdot \frac{|x_i|^q}{\sum^{+\infty} |x_i|^q} + \frac{1}{p} \cdot \frac{|y_i|^p}{\sum^{+\infty} |y_i|^p}$

$\Rightarrow \sum_{i=1}^{+\infty} \frac{|x_i \cdot y_i|}{\|x\|_q \cdot \|y\|_p} \le \frac{1}{q} \sum^{+\infty}(\cdots) + \frac{1}{p} \sum^{+\infty}(\cdots) = 1$

$\Rightarrow$ Holder's $\sum_{i=1}^{+\infty} |x_i y_i| \le \|x\|_q \cdot \|y\|_p$

Young's equality $\Leftrightarrow$ ②中 $a=b$ $\Leftrightarrow$ ①中 $a^p = b^q$

$\Rightarrow$ Holder's equality $\Leftrightarrow \left(\frac{|x_i|}{\|x\|_q}\right)^q = \left(\frac{|y_i|}{\|y\|_p}\right)^p$ $\forall i$. ie: $\frac{|x_i|^q}{\sum^{+\infty} |x_i|^q} = \frac{|y_i|^p}{\sum^{+\infty} |y_i|^p}$ $\forall i$.

$\therefore$ we can let $\|y\|^p = 1$, $|y_i| = \left(\frac{|x_i|^q}{\sum_{i=1}^{+\infty} |x_i|^q}\right)^{\frac{1}{p}}$. y_i 符号取决于 x_i ie: $y_i = \begin{cases} |y_i| & \text{if } x_i = |x_i| \\ -|y_i| & \text{otherwise} \end{cases}$

$\Rightarrow \sum_{i=1}^{+\infty} x_i y_i = \sum_{i=1}^{+\infty} |x_i y_i| = \|x\|_q \cdot \|y\|_p$

$\Rightarrow \|x\|_q = \sum_{i=1}^{+\infty} |x_i y_i| \div \|y\|_p < +\infty$

$\Rightarrow x \in \ell^q$, $\frac{1}{q}+\frac{1}{p}=1$

23.1 Show that if (α_n) is a sequence of strictly positive real numbers such that $\sum_{n=1}^{\infty} \alpha_n < \infty$, then there is a sequence (y_n) with $y_n \to \infty$ as $n \to \infty$ such that $\sum_{n=1}^{\infty} \alpha_n y_n < \infty$. [Hint: consider the map $T\colon \ell^\infty \to \ell^1$ defined by setting

$$(T\boldsymbol{x})_j = \alpha_j x_j;$$

~~T23.1~~ $\{a_n > 0\ \forall n\}$, $\sum_{n=1}^{+\infty} a_n < +\infty \Rightarrow \exists \{y_n\}$ s.t. $y_n \to +\infty$, but $\sum_{n=1}^{+\infty} a_n \cdot y_n < +\infty$

proof: let $y_n = \frac{1}{a_n^t}$. $\sum_{n=1}^{+\infty} a_n \cdot y_n = \sum_{n=1}^{+\infty} a_n^{(1-t)}$, $t>0$

$\sum_{n=1}^{+\infty} a_n < +\infty \Rightarrow a_n \to 0$. $a_n^t \to 0\ \forall t>0$. $\Rightarrow y_n \to +\infty$

so WTS: $\exists t>0$ s.t. $\sum_{n=1}^{+\infty} a_n^{1-t} < +\infty$ // $\frac{a_n^{1-t}\, t^n \ln a_n^n}{n!}$

$a_n^{1-t} - a_n = \frac{a_n^{1-t} \ln a_n \cdot (-1)}{1!} \cdot (-t) + \frac{a_n^{1-t} (\ln a_n)^2 \cdot (-1)^2}{2!} \cdot (-t)^2 + \cdots \frac{a_n^{1-t} (\ln a_n)^n (-1)^n}{n!} (-t)^n + \cdots$

$= a_n^{1-t} \left(t \ln a_n + t^2 (\ln a_n)^2 + \cdots t^k \cdot (\ln a_n)^k + \cdots \right)$

$= a_n^{1-t} \cdot \frac{(t \ln a_n)^{k+1} - t \ln a_n}{t \ln a_n - 1}$ $k \to +\infty$ 没法收敛了… t可能不存在的!

$T: \ell^{+\infty} \subseteq X \to \ell^1$

$y = (y_1, y_2, \cdots) \mapsto a \cdot y = (a_1 y_1, a_2 y_2 \cdots)$, where $\sum_{i=1}^{+\infty} a_i < +\infty$, i.e. $a = (a_1, a_2, \cdots) \in \ell^1$ $\Rightarrow$ T bounded ①

WTS: $\exists\, y$ s.t. $y_n \to +\infty$, but $\sum_{i=1}^{+\infty} a_i y_i < +\infty$ i.e. $\exists\, y \notin \ell^{+\infty}$, $Ty \in \ell^1$ 翻译条件和问题!

Assume $\nexists\, y$, $\Rightarrow$ $y \notin \ell^{+\infty}$ implies $Ty \notin \ell^1$, $\forall y$

这是为什么?

$\Rightarrow$ if T not surjective, $\exists\, z \in \ell^1$ s.t. $z \notin T(\ell^{+\infty})$, i.e. $z \in T(X \setminus \ell^{+\infty})$ contradict

$\therefore T$ surjective

obv T injective, since $a \in \ell^1$ $\Big\}$ $\Rightarrow$ T bijective ②

①+② $\Rightarrow$ T has bounded linear inverse T^{-1}

$\Rightarrow$ $\ell^{+\infty}$ isomorphic to ℓ^1

$\ell^{+\infty}$ not separable, ℓ^1 sepa $\Big\}$ $\Rightarrow$ contradict!

23.3 Use the Closed Graph Theorem to show that if H is a Hilbert space and $T: H \to H$ is a linear operator that satisfies

$$(Tx, y) = (x, Ty) \qquad \text{for every} \qquad x, y \in H$$

then T is bounded. (This is the Hellinger–Toeplitz Theorem.)

T23.3 $T: H \to H$, $(Tx, y) = (x, Ty)$ $\Rightarrow$ T bounded

$\Leftarrow$ $x_n \to x$, $Tx_n \to z$; $z = Tx$ ·· by Hint

$\Rightarrow$ 即证 $(z, y) = (Tx, y)$ $\forall y \in H$

$(Tx_n, y) \to (z, y)$

$(x_n, Ty) \to (x, Ty) = (Tx, y)$ $\Big\}$ $\Rightarrow$ $(Tx_n, y) = (x_n, Ty)$ $\therefore (z, y) = (Tx, y)$ 证毕

rmk: 条件 $(Tx, y) = (x, Ty)$ 可以让 (Tx_n, y) 从不同角度取极限，可以体现 $\begin{cases} Tx_n \to z \\ x_n \to x_0 \end{cases}$

23.5 Use the Closed Graph Theorem to show that if X is a real Banach space and $T: X \to X^*$ is a linear map such that

$$(Tx)(x) \geq 0 \qquad \text{for every } x \in X,$$

then T is bounded. (Brezis, 2011)

T23.5 $Tx(x) \geq 0$ $\Rightarrow$ T bounded

$(Tx_n - Ty)(x_n - y) \geq 0$ $\forall y \in X$

$n \to +\infty$ $(f - Ty)(x - y) \geq 0$ $\forall y \in X$

let $y=x+tz$ $\Rightarrow$ $0 \le (f-Tx-tTz)(tz) = t(f(z)-Tx(z)) - t^2 \cdot Tz(z)$ $\forall t, \forall z \in X$

$\Rightarrow \begin{cases} 0 \le t \cdot Tz(z) \le (f-Tx)(z) & \text{let } t \ge 0,\ \forall z \in X \\ 0 \ge t \cdot Tz(z) \ge (f-Tx)(z) & \text{let } t < 0,\ \forall z \in X \end{cases}$

$\Rightarrow$ $0 \equiv (f-Tx)(z)$ $\forall z \in X$

$\Rightarrow$ $Tx = f$; ie. $x_n \to x$; then $Tx_n \to f = Tx$

26.1 Suppose that X and Y are Banach spaces, and that $T_X : X \to Y^*$ and $T_Y : Y \to X^*$ are both isometric isomorphisms (so that $X^* \equiv Y$ and $Y^* \equiv X$). Show that if

$$[T_X x](y) = [T_Y y](x) \qquad \text{for all} \qquad x \in X,\ y \in Y,$$

then X is reflexive. (The proof, generalising the argument we used to prove reflexivity of the ℓ^p and L^p spaces, gives some indication why $X^{**} \equiv X$ alone is not sufficient for X to be reflexive.)

T26.1 X,Y Banach, $T_X : X \to Y^*$ $T_Y : Y \to X^*$ isometric isomorphism ie. $X \equiv Y^*$, $Y \equiv X^*$

if $[T_X x](y) = [T_Y y](x)$ $\forall x \in X, y \in Y$ $\Rightarrow$ X reflexive.

proof: WTS: $J(X) = X^{**}$ ie. $\forall F \in X^{**}$, $\exists x \in X$ s.t. $J(x) = x^{**} = F$

$\Leftrightarrow$ $F(f) = x^{**}(f) = f(x)$ $\forall f \in X^*$ ...①

$T_Y : Y \to X^*$ isometric iso $\Rightarrow$ $\forall f \in X^*$, $\underset{\text{unique}}{\exists y \in Y}$ s.t. $T_Y y = f$ } $\Rightarrow$ ① $\Leftrightarrow$ $F(T_Y y) = T_Y y(x)$ $\forall y \in Y$

$\Leftrightarrow$ $F(T_Y y) \overset{②}{=} [T_Y y](x) \overset{③}{=} [T_X x](y)$ $\forall y \in Y$

$\Leftrightarrow$ WTS. $\exists g \in Y^*$ s.t. $F(T_Y y) = g(y)$ $\forall y \in Y$) ④

$\therefore$ we can define $g : Y \to \mathbb{F}$, $y \mapsto F(T_Y y)$

T_X isomorphic $\therefore$ $x = (T_X)^{-1} g$

rmk: ②: $F(T_Y y) = [T_Y y](x)$ $\forall y$ 这不一定存在的; 所以 ③用到了 $[T_X x](y) = [T_Y y](x)$

④用到了 T_X isomorphic

$X \equiv X^{**}$ not sufficient for reflexitivity of X ← 只有③ ($\Rightarrow X \equiv X^{**}$) 没有④

同构当然不够 ....

和26.3一个题型

26.2 Suppose that U is a subset of a Banach space X. Show that U is bounded if and only if for every $f \in X^*$ the set

$$f(U) = \{f(u) : u \in U\} \qquad \text{is bounded in } \mathbb{R}.$$

(Use the Principle of Uniform Boundedness on an appropriately chosen set of elements of X^{**}.)

T26.2 X Banach. U is bounded ⇔ $f \in X^*$. $f(U)$ bounded $\forall f$

Pf: U bdd $\sup_{u\in U}\|u\| \le M \Rightarrow \sup_{v\in f(U)}\|v\| \le \|f\|\cdot M$ bounded obv

recall: X complete $\{T_\lambda\}_{\lambda\in\Lambda} \subseteq B(X,Y)$, $\forall x\ \sup_{\lambda\in\Lambda}\|T_\lambda(x)\| < +\infty \Rightarrow \sup_{\lambda\in\Lambda}\|T_\lambda\| < +\infty$ ⋯ TH

☆ U bounded ⇔ U^{**} bounded

U^{**} 相当于 T_λ, $f\in X^*$ 相当于 TH 中的 $x\in X$

和U范数相关 ⇔ U^{**}范数

$\forall f\in X^*\ f(U) = \{f(u): u\in U\} = \{u^{**}(f): u\in U\}$ bounded

$\Rightarrow \forall f \in X^*\ \sup_{u^{**}\in U^{**}} \|u^{**}(f)\| < +\infty$

$\Rightarrow \sup_{u^{**}\in U^{**}} \|u^{**}\| < +\infty$, denoted $\sup \le M < +\infty$

$\Rightarrow \|u^{**}\| = \|u\|\ \forall u\in U \quad \therefore \sup_{u\in U}\|u\| \le M$

证毕!

和26.2一起复习 ✓

26.3 If $T: X \to Y$ is a linear map between Banach spaces and $\phi \circ T$ is bounded for every $\phi \in Y^*$ show that T is bounded. (Prove the contrapositive.)

if $T: X\to Y$ ubd $\overset{\text{WTS}}{\Rightarrow} \{\phi\circ T\}_{\phi\in Y^*}$ ubd, for some $\phi\in Y^*$

$\forall n\in \mathbb{N}^+, \exists x_n\in X, \|x_n\|=1$ s.t. $\|Tx_n\| > n \overset{\text{WTS}}{\Rightarrow} \{\phi\circ T(x_i)\}_{x_i\in\{x_n\}}$ is ubd for some $\phi\in Y^*$

Pf: $\phi: Tx_n \mapsto \phi(Tx_n) \in \mathbb{F}$, $\{(Tx_n)^{**}\} \subseteq Y^{**} = (Y^*)^*$, Y^* complete ⇒ 满足用 Uni-Bound Principle 条件

if $\{\phi\circ T(x_i)\}_{x_i\in\{x_n\}}$ bounded for $\forall \phi\in Y^* \Rightarrow \sup_{x_i\in\{x_n\}}\{(Tx_i)^{**}(\phi)\}$ bounded $\forall\phi$

$\Rightarrow \sup_{x_i\in\{x_n\},\ \phi\in Y^*}\{(Tx_i)^{**}(\phi)\}$ bounded

$\Rightarrow \sup_{x_i\in\{x_n\}}\{\|Tx_i\|_{**}\}$ bounded, let $\|\phi\|=1$ contradict!

☆元素 Tx_i 和 函数 $(Tx_i)^{**}$ 的转化

⇓

为了用Uniform-Bound-Principle