Banach Spaces

• **Theorem** (Baire) *Let X be a complete metric space and let $(X_n)_{n\geq 1}$ be a sequence of closed subsets in X. Assume that*

$$\operatorname{Int} X_n = \emptyset \quad \textit{for every } n \geq 1.$$

Then

$$\operatorname{Int}\left(\bigcup_{n=1}^{\infty} X_n\right) = \emptyset.$$

Remark The Baire category theorem is often used in the following form. Let X be a nonempty complete metric space. Let $(X_n)_{n\geq 1}$ be a sequence of closed subsets such that

$$\bigcup_{n=1}^{\infty} X_n = X.$$

Then there exists some n_0 such that $\operatorname{Int} X_{n_0} \neq \emptyset$.

• **Theorem** **(Banach–Steinhaus, uniform boundedness principle).** *Let E and F be two Banach spaces and let $(T_i)_{i\in I}$ be a family (not necessarily countable) of continuous linear operators from E into F. Assume that*

$$\sup_{i\in I} \|T_i x\| < \infty \quad \forall x \in E. \tag{1}$$

Then

$$\sup_{i\in I} \|T_i\|_{\mathscr{L}(E,F)} < \infty. \tag{2}$$

In other words, there exists a constant c such that

$$\|T_i x\| \le c\|x\| \quad \forall x \in E, \quad \forall i \in I.$$

Proof. For every $n \ge 1$, let

$$X_n = \{x \in E; \quad \forall i \in I, \ \|T_i x\| \le n\},$$

so that X_n is closed, and by (1) we have

$$\bigcup_{n=1}^{\infty} X_n = E.$$

It follows from the Baire category theorem that $\operatorname{Int}(X_{n_0}) \ne \emptyset$ for some $n_0 \ge 1$. Pick $x_0 \in E$ and $r > 0$ such that $B(x_0, r) \subset X_{n_0}$. We have

$$\|T_i(x_0 + rz)\| \le n_0 \quad \forall i \in I, \quad \forall z \in B(0,1).$$

This leads to

$$r\|T_i\|_{\mathcal{L}(E,F)} \le n_0 + \|T_i x_0\|,$$

which implies (2).

Remark. From the proof, we need only to assume that E is a Banach space.

Corollary *Suppose that X is a Banach space, Y a normed space, and that $T_n \in B(X, Y)$. Suppose that*

$$Tx := \lim_{n\to\infty} T_n x$$

exists for every $x \in X$. Then $T \in B(X, Y)$.

Proof The operator T is linear, since if $x, y \in X$ and $\alpha, \beta \in \mathbb{K}$, then

$$\begin{aligned} T(\alpha x + \beta y) = \lim_{n\to\infty} T_n(\alpha x + \beta y) &= \lim_{n\to\infty} \alpha T_n x + \beta T_n y \\ &= \alpha \lim_{n\to\infty} T_n x + \beta \lim_{n\to\infty} T_n y \\ &= \alpha T x + \beta T y. \end{aligned}$$

To show that T is bounded, observe that since

$$\lim_{n\to\infty} \|T_n x\|_Y$$

exists it follows that for every $x \in X$ the sequence $(T_n x)_{n=1}^{\infty}$ is bounded. The Principle of Uniform Boundedness now shows that $\|T_n\|_{B(X,Y)} \leq M$ for every $n \in \mathbb{N}$. It follows that

$$\|Tx\|_Y = \lim_{n\to\infty} \|T_n x\|_Y \leq M\|x\|_X$$

and so T is bounded. $\square$

• **Theorem 2.6 (open mapping theorem).** *Let E and F be two Banach spaces and let T be a continuous linear operator from E into F that is* **surjective** *(= onto). Then there exists a constant $c > 0$ such that*

$$T(B_E(0,1)) \supset B_F(0,c). \tag{7}$$

Remark 5. Property (7) implies that the image under T of any open set in E is an open set in F (which justifies the name given to this theorem!). Indeed, let us suppose U is open in E and let us prove that $T(U)$ is open. Fix any point $y_0 \in T(U)$, so that $y_0 = Tx_0$ for some $x_0 \in U$. Let $r > 0$ be such that $B(x_0, r) \subset U$, i.e., $x_0 + B(0,r) \subset U$. It follows that

$$y_0 + T(B(0,r)) \subset T(U).$$

Using (7) we obtain

$$T(B(0,r)) \supset B(0,rc)$$

and therefore

$$B(y_0, rc) \subset T(U).$$

• **Corollary 2.7.** *Let E and F be two Banach spaces and let T be a continuous linear operator from E into F that is* **bijective**, *i.e., injective (= one-to-one) and surjective. Then T^{-1} is also continuous (from F into E).*

Proof of Corollary 2.7. Property (7) and the assumption that T is injective imply that if $x \in E$ is chosen so that $\|Tx\| < c$, then $\|x\| < 1$. By homogeneity, we find that

$$\|x\| \le \frac{1}{c}\|Tx\| \quad \forall x \in E$$

and therefore T^{-1} is continuous.

Corollary 2.8. *Let E be a vector space provided with two norms, $\| \ \|_1$ and $\| \ \|_2$. Assume that E is a Banach space for* **both** *norms and that there exists a constant $C \geq 0$ such that*

$$\|x\|_2 \leq C\|x\|_1 \quad \forall x \in E.$$

Then the two norms are **equivalent**, *i.e., there is a constant $c > 0$ such that*

$$\|x\|_1 \leq c\|x\|_2 \quad \forall x \in E.$$

Proof of Corollary 2.8. Apply Corollary 2.7 with

$$E = (E, \| \ \|_1), \ F = (E, \| \ \|_2), \text{ and } T = I.$$

Proof of Theorem 2.6. We split the argument into two steps:

Step 1. Assume that T is a linear surjective operator from E onto F. Then there exists a constant $c > 0$ such that

$$\overline{T(B(0,1))} \supset B(0,2c). \tag{8}$$

Proof. Set $X_n = n\overline{T(B(0,1))}$. Since T is surjective, we have $\bigcup_{n=1}^{\infty} X_n = F$, and by the Baire category theorem there exists some n_0 such that $\operatorname{Int}(X_{n_0}) \neq \emptyset$. It follows that

$$\operatorname{Int}\left[\overline{T(B(0,1))}\right] \neq \emptyset.$$

Pick $c > 0$ and $y_0 \in F$ such that

$$B(y_0, 4c) \subset \overline{T(B(0,1))}. \tag{9}$$

In particular, $y_0 \in \overline{T(B(0,1))}$, and by symmetry,

$$-y_0 \in \overline{T(B(0,1))}. \tag{10}$$

Adding (9) and (10) leads to

$$B(0,4c) \subset \overline{T(B(0,1))} + \overline{T(B(0,1))}.$$

On the other hand, since $\overline{T(B(0,1))}$ is convex, we have

$$\overline{T(B(0,1))} + \overline{T(B(0,1))} = 2\overline{T(B(0,1))},$$

and (8) follows.

Step 2. Assume T is a continuous linear operator from E into F that satisfies (8). Then we have

$$T(B(0,1)) \supset B(0,c). \tag{11}$$

Proof. Choose any $y \in F$ with $\|y\| < c$. The aim is to find some $x \in E$ such that

$$\|x\| < 1 \quad \text{and} \quad Tx = y.$$

By (8) we know that

$$\forall \varepsilon > 0 \quad \exists z \in E \text{ with } \|z\| < \frac{1}{2} \text{ and } \|y - Tz\| < \varepsilon. \tag{12}$$

Choosing $\varepsilon = c/2$, we find some $z_1 \in E$ such that

$$\|z_1\| < \frac{1}{2} \quad \text{and} \quad \|y - Tz_1\| < \frac{c}{2}.$$

By the same construction applied to $y - Tz_1$ (instead of y) with $\varepsilon = c/4$ we find some $z_2 \in E$ such that

$$\|z_2\| < \frac{1}{4} \text{ and } \|(y - Tz_1) - Tz_2\| < \frac{c}{4}.$$

Proceeding similarly, by induction we obtain a sequence (z_n) such that

$$\|z_n\| < \frac{1}{2^n} \quad \text{and} \quad \|y - T(z_1 + z_2 + \cdots + z_n)\| < \frac{c}{2^n} \quad \forall n.$$

It follows that the sequence $x_n = z_1 + z_2 + \cdots + z_n$ is a Cauchy sequence. Let $x_n \to x$ with, clearly, $\|x\| < 1$ and $y = Tx$ (since T is continuous).

• **Theorem 2.9 (closed graph theorem).** *Let E and F be two Banach spaces. Let T be a linear operator from E into F. Assume that the graph of T, $G(T)$, is closed in $E \times F$. Then T is continuous.*

Remark 6. The converse is obviously true, since the graph of any continuous map (linear or not) is closed.

Proof of Theorem 2.9. Consider, on E, the two norms

$$\|x\|_1 = \|x\|_E + \|Tx\|_F \quad \text{and} \quad \|x\|_2 = \|x\|_E$$

(the norm $\| \ \|_1$ is called the *graph norm*).

It is easy to check, using the assumption that $G(T)$ is closed, that E is a Banach space for the norm $\| \ \|_1$. On the other hand, E is also a Banach space for the norm $\| \ \|_2$ and $\| \ \|_2 \leq \| \ \|_1$. It follows from Corollary 2.8 that the two norms are equivalent and thus there exists a constant $c > 0$ such that $\|x\|_1 \leq c\|x\|_2$. We conclude that $\|Tx\|_F \leq c\|x\|_E$.

Let X be a normed space and X^{**} be the set of bounded linear functionals from

X^* to $\mathbb{K}$, i.e. $X^{**} := B(X^*; \mathbb{K})$.

Lemma *For any normed space X we can isometrically map X onto a subspace of X^{**} via the canonical linear mapping $x \mapsto x^{**}$, where x^{**} is the element of X^{**} defined by setting*

$$x^{**}(f) = f(x) \qquad \textit{for each} \quad f \in X^*.$$

We denote this mapping by $J : X \to X^{**}$.

Proof We have to show that for any $x \in X$, x^{**} defines a linear functional on X^* (i.e. an element of X^{**}) with the same norm as x. Given $x \in X$ we set

$$x^{**}(f) := f(x) \qquad \text{for every } f \in X^*.$$

Then, since

$$|x^{**}(f)| = |f(x)| \leq \|f\|_{X^*} \|x\|_X,$$

it certainly follows that $x^{**} \in X^{**}$ and that $\|x^{**}\|_{X^{**}} \leq \|x\|_X$.

Let $f \in X^*$ be such that

$\|f\| = 1$ and $f(x) = \|x\|$, then we have

$$|x^{**}(f)| = |f(x)| = \|x\|_X = \|x\|_X \|f\|_{X^*}$$

(since $\|f\|_{X^*} = 1$) and it follows that $\|x^{**}\|_{X^{**}} \geq \|x\|_X$, which yields the required equality of norms. $\square$

Lemma *If X is a Banach space, then $J(X)$ is a closed subspace of X^{**}.*

Proof If $(F_n) \in J(X)$ with $F_n \to F$ in X^{**}, then (F_n) must be Cauchy in X^{**}. Since there exist $x_n \in X$ such that $F_n = x_n^{**}$ and the map J is a linear isometry, we have

$$\|x_n - x_m\|_X = \|F_n - F_m\|_{X^{**}},$$

so (x_n) is Cauchy in X. It follows that there exists $x \in X$ such that $x_n \to x$ in X, and so

$$\|F_n - x^{**}\|_{X^{**}} = \|x_n - x\|_X \to 0 \qquad \text{as} \qquad n \to \infty.$$

By uniqueness of limits it follows that $F = x^{**}$, so $J(X)$ is closed. □

Definition A Banach space X is *reflexive* if $J : X \to X^{**}$ is onto, i.e. if every $F \in X^{**}$ can be written as x^{**} for some $x \in X$.

Proposition *All Hilbert spaces are reflexive.*

Theorem *Let X be a Banach space. Then X is reflexive if and only if X^* is reflexive.*

Proof Suppose first that X is reflexive; we want to show that X^* is reflexive, i.e. that for any $\Phi \in (X^*)^{**}$ we can find an $f \in X^*$ such that $f^{**} = \Phi$, i.e. such that

$$\Phi(F) = F(f) \qquad \text{for every } F \in X^{**}.$$

This actually tells us what f should be. Since any $F \in X^{**}$ can be written as x^{**} for some $x \in X$, we require

$$\Phi(x^{**}) = x^{**}(f) \qquad \text{for every } x \in X.$$

But since, by definition, $x^{**}(f) = f(x)$, this says that we must have

$$f(x) = \Phi(x^{**}) \qquad \text{for every } x \in X,$$

and we now use this as the definition of f. We just have to check that f really is an element of X^*, i.e. is a bounded linear map from X into $\mathbb{K}$. But this follows immediately, as it is the composition of J, a bounded linear map from X into X^{**}, with Φ, which is a bounded linear map from X^{**} into $\mathbb{K}$.

For the converse, suppose that X^* is reflexive but X is not, i.e. there is an element $F \in X^{**}$ such that $F \neq x^{**}$ for any $x \in X$. Then the set

$$J(X) = \{x^{**} : x \in X\}$$

is a proper closed linear subspace of X^{**} and so there is some non-zero $\Phi \in (X^{**})^*$ such that $\Phi = 0$ on $J(X)$, i.e.

$$\Phi(x^{**}) = 0 \qquad \text{for all } x \in X.$$

Since $(X^{**})^* = (X^*)^{**}$ and X^* is reflexive, we know that $\Phi = f^{**}$ for some $f \in X^*$, and so if $x \in X$, we have

$$f(x) = x^{**}(f) = f^{**}(x^{**}) = \Phi(x^{**}) = 0.$$

But this means that $f = 0$, which in turn implies that $\Phi = 0$, a contradiction.

Lemma *Any closed subspace Y of a reflexive Banach space X is reflexive.*

Proof Take $f \in X^*$ and let f_Y denote the restriction of f to Y, so that $f_Y \in Y^*$. Because of the Hahn–Banach Theorem, any element of Y^* can be obtained as such a restriction.

To show that Y is reflexive we need to show that for any $\Psi \in Y^{**}$ there exists a $y \in Y$ such that

$$\Psi(f_Y) = y^{**}(f_Y) \qquad \text{for every} \quad f \in X^*.$$

First define an element $\hat{\Psi}\colon X^* \to \mathbb{R}$ by setting

$$\hat{\Psi}(f) = \Psi(f_Y),$$

and then

$$|\hat{\Psi}(f)| \leq \|\Psi\|\|f_Y\| \leq \|\Psi\|\|f\| \qquad \text{for any} \qquad f \in X^*,$$

so $\hat{\Psi} \in X^{**}$. Now we can use the fact that X is reflexive to find an $x \in X$ such that

$$\hat{\Psi} = x^{**}.$$

We only need now show that $x \in Y$.

Suppose that $x \notin Y$. Then the distance functional provides an $f \in X^*$ such that $f(x) \neq 0$ and $f(y) = 0$ for every $y \in Y$, i.e. such that $f_Y = 0$. Then

$$f(x) = x^{**}(f) = \hat{\Psi}(f) = \Psi(f_Y) = 0,$$

a contradiction. $\square$

Definition 27.1 We say that a sequence $(x_n) \in X$ *converges weakly* to $x \in X$, and write $x_n \rightharpoonup x$, if

$$f(x_n) \to f(x) \qquad \text{for all} \qquad f \in X^*.$$

Note that in a Hilbert space, where every linear functional is of the form $x \mapsto (x, y)$ for some $y \in H$, $x_n \rightharpoonup x$ if

$$(x_n, y) \to (x, y) \qquad \text{for all} \qquad y \in H.$$

This observation allows us to provide an example of a sequence that converges weakly but does not converge strongly. Pick any countable orthonormal sequence $(e_j)_{j=1}^{\infty}$ in H; then for any $y \in H$ Bessel's inequality

$$\sum_{j=1}^{\infty} |(y, e_j)|^2 \leq \|y\|^2$$

shows that the sum converges; it follows that $(y, e_j) \to 0$ as $j \to \infty$ for any $y \in H$, and hence that $e_j \rightharpoonup 0$.

Lemma 27.2 *Weak convergence has the following properties.*

(i) *Strong convergence implies weak convergence;*

(ii) *in a finite-dimensional normed space weak convergence and strong convergence are equivalent;*

(iii) *weak limits are unique;*

(iv) *weakly convergent sequences are bounded; and*

(v) *if $x_n \rightharpoonup x$, then*

$$\|x\| \leq \liminf_{n\to\infty} \|x_n\|. \tag{27.1}$$

Proof (i) If $x_n \to x$, then for any $f \in X^*$

$$|f(x_n) - f(x)| \leq \|f\|_{X^*}\|x_n - x\|_X \to 0 \qquad \text{as} \qquad n \to \infty,$$

so $f(x_n) \to f(x)$, and hence $x_n \rightharpoonup x$.

(ii) Due to part (i) we need only show that if V is a finite-dimensional normed space, then weak convergence in V implies strong convergence in V. If $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is a basis for V, then for each $i = 1, \ldots, n$ the map

$$\boldsymbol{x} = \sum_{j=1}^{n} x_j \boldsymbol{e}_j \mapsto x_i$$

is an element of V^*, so if $\boldsymbol{x}^{(k)} \rightharpoonup \boldsymbol{x}$ it follows that $x_j^{(k)} \to x_j$ for each $j = 1, \ldots, n$, and so

$$\boldsymbol{x}^{(k)} = \sum_{j=1}^{n} x_j^{(k)} \boldsymbol{e}_j \to \sum_{j=1}^{n} x_j \boldsymbol{e}_j = \boldsymbol{x}.$$

(iii) Suppose that $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$. Then for any $f \in X^*$,

$$f(x) = \lim_{n\to\infty} f(x_n) = f(y),$$

so x=y.

(iv) Since $f(x_n)$ converges, it follows that $f(x_n)$ is a bounded sequence (in $\mathbb{K}$) for every $f \in X^*$. If we consider the sequence $(x_n^{**}) \in X^{**}$, then, since

$$x_n^{**}(f) = f(x_n),$$

it follows that $(x_n^{**}(f))_n$ is bounded in $\mathbb{K}$ for every $f \in X^*$. We can now use the Principle of Uniform Boundedness to deduce that (x_n^{**}) is bounded in X^{**}. Since $\|x^{**}\|_{X^{**}} = \|x\|_X$, it follows that (x_n) is bounded in X.

(v) Choose $f \in X^*$ with $\|f\|_{X^*} = 1$ such that $f(x) = \|x\|$, then

$$\|x\| = f(x) = \lim_{n\to\infty} f(x_n),$$

so

$$\|x\| \leq \liminf_{n\to\infty} |f(x_n)| \leq \liminf_{n\to\infty} \|f\|_{X^*}\|x_n\|_X;$$

the result follows since $\|f\|_{X^*} = 1$. $\square$

Lemma 27.3 *Let H be a Hilbert space. If $(x_n) \in H$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $x_n \to x$.*

Proof Observe that

$$\|x - x_n\|^2 = (x - x_n, x - x_n) = \|x\|^2 - (x, x_n) - (x_n, x) + \|x_n\|^2.$$

Since $x_n \rightharpoonup x$, we have $(x_n, x) \to (x, x) = \|x\|^2$ and $\|x_n\|^2 \to \|x\|^2$ by assumption; so $\|x - x_n\|^2 \to 0$ as $n \to \infty$. □

Lemma 27.4 *Suppose that $T : X \to Y$ is a compact linear operator. If $(x_n) \in X$ with $x_n \rightharpoonup x$ in X, then $Tx_n \to Tx$ in Y.*

Proof We first show that $Tx_n \rightharpoonup Tx$ in Y; indeed, if $f \in Y^*$, then $f \circ T$ is an element of X^*, so that $x_n \rightharpoonup x$ implies that

$$f(Tx_n) \to f(Tx).$$

Now, suppose that $Tx_n \not\to Tx$; then there is an $\varepsilon > 0$ and a subsequence $(x_{n_j})_j$ such that

$$\|Tx_{n_j} - Tx\| > \varepsilon \qquad \text{for every } j. \tag{27.2}$$

Since x_{n_j} converges weakly, it is a bounded sequence in X (by part (iv) of Lemma 27.2); since T is compact it follows that (Tx_{n_j}) has a subsequence $(Tx_{n'_j})_j$ that converges to some $z \in Y$. Since strong convergence implies weak convergence (Lemma 27.2 (i)), we also have $Tx_{n'_j} \rightharpoonup z$; but weak limits are unique (part (iii) of Lemma 27.2) and we already know that $Tx_{n'_j} \rightharpoonup Tx$ (since $x_{n'_j}$ is a subsequence of x_n and we know that $Tx_n \rightharpoonup Tx$), so we must have $z = Tx$ and

$$\lim_{j\to\infty} \|Tx_{n'_j} - Tx\| \to 0 \qquad \text{as } j \to \infty.$$

Since $x_{n'_j}$ is a subsequence of x_{n_j} the preceding equation contradicts (27.2), and therefore $Tx_n \to Tx$ as claimed. □

Definition 27.8 If $(f_n)_{n=1}^{\infty} \in X^*$, then f_n *converges weakly-$*$ ('weakly star')* to f if

$$f_n(x) \to f(x) \qquad \text{for all} \qquad x \in X;$$

we write $f_n \overset{*}{\rightharpoonup} f$.

Note that weak-$*$ convergence is a very natural way to define convergence of sequences in X^*: it is the equivalent of pointwise convergence for continuous functions.

Lemma 27.9 *Weak-$*$ convergence has the following properties.*

(i) *Strong convergence in X^* implies weak-$*$ convergence in X^*;*
(ii) *weak-$*$ limits are unique;*
(iii) *weakly-$*$ convergent sequences are bounded;* (Assume X is Banach)
(iv) *if $f_n \overset{*}{\rightharpoonup} f$, then*

$$\|f\|_{X^*} \leq \liminf_{n\to\infty} \|f_n\|_{X^*};$$

(v) *weak convergence in X^* implies weak-$*$ convergence in X^*;*
(vi) *if X is reflexive, then weak-$*$ convergence in X^* implies weak convergence in X^*.*

Proof (i) If $f_n \to f$ in X^*, i.e. $\|f_n - f\|_{X^*} \to 0$, then for any $x \in X$ we have

$$|f_n(x) - f(x)| = |(f_n - f)(x)| \leq \|f_n - f\|_{X^*}\|x\|_X \to 0$$

and so $f_n \overset{*}{\rightharpoonup} f$.

(ii) If $f_n \overset{*}{\rightharpoonup} f$ and $f_n \overset{*}{\rightharpoonup} g$, then

$$f(x) = \lim_{n\to\infty} f_n(x) = g(x) \qquad \text{for every } x \in X,$$

so $f = g$.

(iii) We have $f_n \in B(X;\mathbb{K})$ for each n, and if $f_n(x) \to f(x)$ for every $x \in X$, then

$$\sup_n |f_n(x)| < \infty \qquad \text{for every } x \in X,$$

so it follows from the Principle of Uniform Boundedness that $\sup_n \|f_n\|_{X^*} < \infty$.

(iv) Given any $\varepsilon > 0$ we can find an $x \in X$ with $\|x\| = 1$ such that $f(x) > \|f\|_{X^*} - \varepsilon$; then

$$\|f\|_{X^*} - \varepsilon < f(x) = \lim_{n\to\infty} f_n(x) \leq \liminf_{n\to\infty} \|f_n\|_{X^*}\|x\| = \liminf_{n\to\infty} \|f_n\|_{X^*},$$

which yields the result since $\varepsilon > 0$ is arbitrary.

(v) $f_n \rightharpoonup f$ in X^* means that for every $F \in X^{**}$ we have

$$F(f_n) \to F(f).$$

Given any element $x \in X$ we can consider the corresponding $x^{**} \in X^{**}$. Since $f_n \rightharpoonup f$ in X^*, we have

$$f_n(x) = x^{**}(f_n) \to x^{**}f = f(x),$$

and so $f_n \overset{*}{\rightharpoonup} f$.

(vi) When X is reflexive any $F \in X^{**}$ is of the form x^{**} for some $x \in X$. So if $f_n \overset{*}{\rightharpoonup} f$ in X^* we have

$$F(f_n) = x^{**}(f_n) = f_n(x) \to f(x) = x^{**}(f) = F(f),$$

using the weak-$*$ convergence of f_n to f to take the limit. So $f_n \rightharpoonup f$ in X^*.

Lemma 27.10 *Suppose that* (f_n) *is a bounded sequence in* X^*, *so that* $\|f_n\|_{X^*} \leq M$ *for some* $M > 0$, *and suppose that* $f_n(a)$ *converges as* $n \to \infty$ *for every* $a \in A$, *where* A *is a dense subset of* X. *Then* $\lim_{n\to\infty} f_n(x)$ *exists for every* $x \in X$, *and the map* $f : X \to \mathbb{R}$ *defined by setting*

$$f(x) = \lim_{n\to\infty} f_n(x), \qquad \textit{for each} \qquad x \in X$$

is an element of X^* *with* $\|f\|_{X^*} \leq M$.

Proof We first prove that if $f_n(a)$ converges for every $a \in A$, then $f_n(x)$ converges for every $x \in X$. Given $\varepsilon > 0$ and $x \in X$, first choose a such that

$$\|x - a\|_X \leq \varepsilon/3M.$$

Now, using the fact that $f_n(a)$ converges as $n \to \infty$, choose n_0 sufficiently large that $|f_n(a) - f_m(a)| < \varepsilon/3$ for all $n, m \geq n_0$. Then for all $n, m \geq n_0$ we have

$$\begin{aligned}
&|f_n(x) - f_m(x)| \\
&\qquad \leq |f_n(x) - f_n(a)| + |f_n(a) - f_m(a)| + |f_m(a) - f_m(x)| \\
&\qquad \leq \|f_n\|_{X^*}\|x - a\| + \frac{\varepsilon}{3} + \|f_m\|_{X^*}\|a - x\| \\
&\qquad \leq \varepsilon.
\end{aligned}$$

It follows that $(f_n(x))$ is Cauchy and hence converges.

We now define $f : X \to \mathbb{R}$ by setting

$$f(x) := \lim_{n\to\infty} f_n(x).$$

Then f is linear since

$$f(x + \lambda y) = \lim_{n\to\infty} f_n(x + \lambda y) = \lim_{n\to\infty} f_n(x) + \lambda f_n(y) = f(x) + \lambda f(y)$$

and f is bounded since

$$|f(x)| = \lim_{n\to\infty} |f_n(x)| \leq M\|x\|. \qquad \square$$

Theorem 27.11 *Suppose that X is separable. Then any bounded sequence in X^* has a weakly-$*$ convergent subsequence.*

Proof Let $\{x_k\}$ be a countable dense subset of X, and (f_j) a sequence in X^* such that $\|f_j\|_{X^*} \le M$. As in the proof of Theorem 15.3 we will use a diagonal argument to find a subsequence of the (f_j) (which we relabel) such that $f_j(x_k)$ converges for every k.

Since $|f_n(x_1)| \le M\|x_1\|$, we can use the Bolzano–Weierstrass Theorem to find a subsequence $f_{n_{1,i}}$ such that $f_{n_{1,i}}(x_1)$ converges. Now, since $|f_{n_{1,i}}(x_2)| \le M\|x_2\|$ we can find a subsequence $f_{n_{2,i}}$ of $f_{n_{1,i}}$ such that $f_{n_{2,i}}(x_2)$ converges; $f_{n_{2,i}}(x_1)$ will still converge since it is a subsequence of $f_{n_{1,i}}(x_1)$ which we have already made converge. We continue in this way to find successive subsequences $f_{n_{m,i}}$ such that

$$f_{n_{m,i}}(x_k) \qquad \text{converges as } i \to \infty \text{ for every } k = 1, \ldots, m.$$

By taking the diagonal subsequence $f_m^* := f_{n_{m,m}}$ (as in the proof of the Arzelà–Ascoli Theorem) we can ensure that $f_m^*(x_k)$ converges for every $k \in \mathbb{N}$.

The proof concludes using Lemma 27.10. □

Theorem 27.12 *Let X be a reflexive Banach space. Then any bounded sequence in X has a weakly convergent subsequence.*

Proof Take a bounded sequence $(x_n) \in X$ and let

$$Y := \operatorname{clin}\{x_1, x_2, \ldots\}.$$

Then, Y is separable. Since $Y \subseteq X$ and X is reflexive, so is Y (Lemma 26.10). Therefore $Y^{**} \equiv Y$, which implies that Y^{**} is separable Lemma 20.5 implies that Y^* is separable.

Now, x_n^{**} is a bounded sequence in Y^{**}, so using Theorem 27.11 there is a subsequence x_{n_k} such that $x_{n_k}^{**}$ is weakly-$*$ convergent in Y^{**} to some limit $\Phi \in Y^{**}$. Since Y is reflexive, $\Phi = x^{**}$ for some $x \in Y \subseteq X$.

Now for any $f \in X^*$ we have $f_Y := f|_Y \in Y^*$, so

$$\begin{aligned}\lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} f_Y(x_{n_k}) &= \lim_{k\to\infty} x_{n_k}^{**}(f_Y) \\ &= x^{**}(f_Y) = f_Y(x) = f(x),\end{aligned}$$

i.e. $x_{n_k} \rightharpoonup x$. $\square$

Lemma 27.13 *Let X be a reflexive Banach space, and $T : X \to X$ a compact linear operator. Suppose that (x_n) is a sequence in X such that there exist c_1, c_2 with $0 < c_1 \le c_2$ so that $c_1 \le \|x_n\| \le c_2$ and*

$$\|Tx_n - x_n\| \to 0 \tag{27.9}$$

as $n \to \infty$. Then there exists a non-zero $x \in X$ such that $Tx = x$.

Proof Since (x_n) is a bounded sequence in a reflexive Banach space, by Theorem 27.12 it has a weakly convergent subsequence, $x_{n_j} \rightharpoonup x$. Since T is compact, it follows from Lemma 27.4 that $Tx_{n_j} \to Tx$ strongly in X. Since

$$\lim_{j\to\infty} Tx_{n_j} - x_{n_j} = 0,$$

it follows that $x_{n_j} \to Tx$. Since strong convergence implies weak convergence, we have $x_{n_j} \rightharpoonup Tx$, and since weak limits are unique and we already have $x_{n_j} \rightharpoonup x$ it follows that $x = Tx$.

To ensure that $x \neq 0$, note that since Tx_{n_j} converges strongly to $Tx = x$, it follows from (27.9) that x_{n_j} also converges strongly to x, and so $\|x\| \ge c_1$. $\square$

Lemma 27.14 *Suppose that X is reflexive and that K is closed convex subset of X. Then for any $x \in X \setminus K$ there exists at least one $k \in K$ such that*

$$\|x - k\| = \text{dist}(x, K) = \inf_{y \in K} \|x - y\|.$$

Proof Let (y_n) be a sequence in K such that $\|x - y_n\| \to \text{dist}(x, K)$. Then (y_n) is a bounded sequence in X, so has a subsequence y_{n_k} that converges weakly to some $k \in X$. Since K is closed and convex, it is also weakly closed (Theorem 27.7), and so $k \in K$. Since $y_{n_k} \rightharpoonup k$, $x - y_{n_k} \rightharpoonup x - k$, and so we have

$$\|x - k\| \leq \liminf_{k \to \infty} \|x - y_{n_k}\| = \text{dist}(x, K)$$

using (27.1). Since $\|x - k\| \geq \text{dist}(x, K)$, we have $\|x - k\| = \text{dist}(x, K)$ as required. $\square$

Banach space X possesses the *Banach–Saks property* if every norm bounded sequence $\{x_n\}$ in X contains a subsequence $\{x_{n_k}\}$ such that the sequence of arithmetic means

$$\frac{x_{n_1} + \cdots + x_{n_k}}{k}$$

converges in norm.

Example. Hilbert spaces possess the Banach–Saks property.

PROOF. Passing to a subsequence we can assume that $\{x_n\}$ converges weakly to some vector x. In addition, we can assume that $x = 0$. Set $n_1 = 1$. Since $(x_{n_1}, x_n) \to 0$, there exists a number $n_2 > n_1$ with $|(x_{n_1}, x_{n_2})| \leqslant 1$. If numbers $n_1 < n_2 < \cdots < n_k$ are already chosen, we find a number $n_{k+1} > n_k$ such that

$$|(x_{n_j}, x_{n_{k+1}})| \leqslant k^{-1}, \quad j = 1, \ldots, k.$$

This is obviously possible by weak convergence of $\{x_n\}$ to zero. We observe that $\sup_n \|x_n\| = M < \infty$. Hence

$$\frac{\|x_{n_1} + \cdots + x_{n_k}\|^2}{k^2} \leqslant \frac{kM^2 + 2 \cdot 1 + \cdots + 2(k-1)(k-1)^{-1}}{k^2} \leqslant \frac{M^2 + 2}{k},$$

which shows norm convergence of the arithmetic means. $\square$

Theorem. All uniformly convex Banach spaces possess the Banach–Saks property. Any space with the Banach–Saks property is reflexive.

Theorem. (THE FREDHOLM ALTERNATIVE) *Let K be a compact operator on a complex or real Banach space X. Then*

$$\mathrm{Ker}(K - I) = 0 \iff (K - I)(X) = X,$$

i.e., either the equation

$$Kx - x = y$$

is uniquely solvable for all $y \in X$ or for some vector $y \in X$ it has no solutions and then the homogeneous equation

$$Kx - x = 0$$

has nonzero solutions.

PROOF. If $\operatorname{Ker}(K-I)=0$, then we have $1 \notin \sigma(K)$. Hence $(K-I)(X)=X$. Conversely, suppose that

$$(K-I)(X)=X, \quad \text{but} \quad \operatorname{Ker}(K-I)\neq 0.$$

As we know, the operator K^* on X^* is also compact We observe that $\operatorname{Ker}(K^*-I)=0$. Indeed, if $f \in X^*$ and $(K^*-I)f=0$, then

$$f\big((K-I)x\big)=(K^*-I)f(x)=0 \quad \text{for all } x \in X.$$

Since $(K-I)(X)=X$, we have $f=0$. It follows that the operator K^*-I is invertible. We now take a nonzero element $a \in \operatorname{Ker}(K-I)$. By the Hahn–Banach theorem there is a functional $f \in X^*$ with $f(a)=1$. Let $g=(K^*-I)^{-1}f$. Then $(K^*-I)g(a)=f(a)=1$. On the other hand, $(K^*-I)g(a)=g\big((K-I)a\big)=0$, which is a contradiction.