Abstract Algebra
: Lecture 23

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1 Review

1.1 Group

As for groups, we mainly focus on structures and actions.
Structures: Let G be a group. Recall the concepts of normal subgroups, factor groups, subgroups.
For subgroups we know Sylow subgroups, and the Sylow theorems.
Composition factors: C_p or non-abelian simple group (A_n, $n \geqslant 5$).

Exercise 1. *Write the composition series of S_4 and S_3 .*

Actions: Conjugation action, coset action

1.2 Ring

Subrings, ideals, quotient rings. Chinese Remainder Theorem.
Prime ideal: $ab \in I$ implies $a \in I$ or $b \in I$.
Fractional field of integral domain.
Factorizations of elements in a ring.

Example 2. $R = \mathbb{Z}[\sqrt{-5}]$ *is an ID but not UFD.*

UFD, PID, ED.

1.3 Field

CharF=0 or p.
Extensions. We mainly focus on polynomials and algebraic extensions.
Degree of extension. Construction by straightedge and compass.
For $f(x) \in F[x]$ there exist an extension E of F s.t. $f(x)$ has root in E.
If CharF=0, each finite extension of F is simple.

Let $F \subset L \subset E$ where E is a splitting extension of F with CharF=0. Then $\mathrm{Gal}(E/L) \lhd \mathrm{Gal}(E/F)$ if and only if L is a splitting extension over F.

Example 3. *Let $f(x) = x^m - 2 \in \mathbb{Q}[X]$. Find $\mathrm{Gal}(f)$.*
Consider $F = \mathbb{Q} \subset L = \mathbb{Q}(\omega) \subset E$. $\mathrm{Gal}(f) = \mathrm{E/L}$. $E = L(\alpha)$, $\alpha = 2^{\frac{1}{m}}$. $\mathrm{Gal}(E/L) = Z_m$. $\mathrm{Gal}(E/L) \lhd \mathrm{Gal}(E/F)$.
L is a splitting extension of $f(x) = x^m - 1$ over F. $\mathrm{Gal}(L/F)$ is a permutation group on the set P_m of the m roots of $x^m - 1$, dividing P_m into orbits. An orbit consist of primitive roots of $x^d - 1$ where $d \mid m$. Actually $\mathrm{Gal}(L/F) \simeq \mathrm{Aut}(Z_m)$. $\mathrm{Gal}(f) \simeq \mathrm{Hol}(Z_m)$.

Example 4. *$|G| = 24$, how many different G?*

1. *Abelian: $Z_3 \times Z_8$, $Z_3 \times Z_4 \times Z_2$, $Z_3 \times (Z_2^3)$*
2. *Non-abelian: Sylow 2-subgroup, order 8. 5 different types. $8, 4 \times 2, 2 \times 2 \times 2$, D_8, Q_8.*

   nilpotent: $3 \times D_8$, $3 \times Q_8$.

   non-nilpotent: $P_2 \lhd G$ we have $G = P_2 \rtimes Z_3$, $Q_8 \rtimes 3$ and $A_4 \times Z_2$.

   $P_3 \lhd G$ we have $G = Z_3 \rtimes 8$. $(3 \rtimes 4) \times 2$, $(3 \rtimes 2) \times 2 \times 2$. $3 \rtimes D_8$ (two), $3 \rtimes Q_8$.

   $< \rho, \tau \mid |\rho| = 4, |\tau| = 2, \rho^\tau = \rho^{-1} >$. (1) $x^\rho = x^{-1}$, $x^\tau = x$. (2) $x^\rho = x$, $x = x^{-1}$.