lem1: $\forall a,b,p,q>0,\ \frac{1}{p}+\frac{1}{q}=1 \Rightarrow a^{\frac{1}{p}}\cdot b^{\frac{1}{q}} \le \frac{a}{p}+\frac{b}{q}$

proof: $f(x)=\ln x$ convex

$\therefore \sum_{i=1}^{n}\lambda_i=1,\ \lambda_i\ge 0.$ then $\lambda_1\ln x_1+\lambda_2\ln x_2+\cdots\lambda_n\cdot\ln x_n \le \ln(\lambda_1 x_1+\cdots\lambda_n x_n)$, for $\forall x_i$ s.t. $\sum_{i=1}^{n}\lambda_i\cdot x_i>0$

$\therefore \frac{1}{p}\cdot\ln a+\frac{1}{q}\cdot\ln b \le \ln(\frac{a}{p}+\frac{b}{q})$

$f(x)$ increasing with x, $\therefore\ a^{\frac{1}{p}}\cdot b^{\frac{1}{q}} \le \frac{a}{p}+\frac{b}{q} \Leftrightarrow a\times b \le \frac{1}{p}\cdot a^p+\frac{1}{q}b^q$, by letting $a:=a^p,\ b:=b^q$

Th1: Holder inequality: $\int|g(x)\cdot f(x)|\,du \le (\int|g(x)|^p du)^{\frac{1}{p}}(\int|f(x)|^q du)^{\frac{1}{q}}$

proof1: let $v(x)=\frac{f(x)}{\|f(x)\|_q}$, $u(x)=\frac{g(x)}{\|g(x)\|_p}$

then $\int v(x)\cdot u(x)\,du \le \int(\frac{1}{p}\cdot u(x)^p+\frac{1}{q}\cdot v(x)^q)\,du$

$=\int\frac{1}{p}u(x)^p du+\int\frac{1}{q}\cdot v(x)^q du=\frac{1}{p}\int(\frac{g(x)}{\|g\|_p})^p du+\frac{1}{q}\cdot\int(\frac{f(x)}{\|f\|_q})^q du=1$

$\Rightarrow \int\frac{f(x)}{\|f\|_q}\cdot\frac{g(x)}{\|g\|_p}du \le 1,$

prop2: if $\|\cdot\|_1, \|\cdot\|_2$ equivalent in X, then

(1): $\forall\{x_n\}\subseteq X,\ \|x_n-x\|_1\to 0 \Leftrightarrow \|x_n-x\|_2\to 0$

(2): $\{x_n\}$ is Cauchy in $(X,\|\cdot\|_1)$, $\Leftrightarrow$ $\{x_n\}$ is Cauchy in $(X,\|\cdot\|_2)$

(3): $(X,\|\cdot\|_1)$ complete $\Leftrightarrow (X,\|\cdot\|_2)$ complete

proof2: (omitted obv)

use $c_1\|x-y\|_2 \le \|x-y\|_1 \le c_2\cdot\|x-y\|_2\quad \forall x,y\in X$, some $c_1,c_2>0$

Th3: $\dim X=n,\ n<+\infty \Leftrightarrow$ any two norm $\|\cdot\|_1, \|\cdot\|_2$ are equivalent

proof3: "$\Rightarrow$" consider: $\|\cdot\|_E$ euclidean norm, $\|\cdot\|$ arbitrary norm

$\forall x\in X,\ x=\sum_{i=1}^{n}x_i e_i$, $\{e_i\}$ is othonormal basis of X, 注意:基和范数取法无关的.

$\|\sum_{i=1}^{n}x_i e_i\| \le \sum_{i=1}^{n}\|e_i\|\cdot\|x_i\| \le (\sum|x_i|^2)^{\frac{1}{2}}(\sum\|e_i\|^2)^{\frac{1}{2}}=\sqrt{n}\cdot\|x\|_E \Rightarrow \|x\|\le\sqrt{n}\cdot\|x\|_E$ --- ①

let $S=\{\sum_{i=1}^{n}y_i e_i : \sum_{i=1}^{n}|y_i|^2=1\}\subseteq(X,\|\cdot\|)$

let $f:\mathbb{F}^n\to(X,\|\cdot\|)$, s.t. $f:(y_1,y_2,\cdots y_n)\longmapsto\sum_{i=1}^{n}y_i e_i$

$\|\sum_{i=1}^{n}y_i e_i-\sum_{i=1}^{n}y_i' e_i\| \le \sqrt{n}\cdot\|y-y'\|_E$, $\therefore f$ is cts under $\|\cdot\|$.

$\mathrm{Dom} f=\{y:\sum_{i=1}^{n}|y_i|^2=1\}$ compact, $\therefore S=\mathrm{Range} f$ compact ....(*)

现在不能用closed+bdd ⇒ cpt, 这在$\|\cdot\|_E$成立, 我们还没证$\|\cdot\|_E\sim\|\cdot\|$

let $g(x)=\|x\|$, g cts

$\therefore g(x)$ attain $\min g, \max g$ in compact set S, denote $\min g=\|u_0\|$, $\|u_0\|\le\|u\|$ $\forall \sum_{i=1}^{n}|u_i|^2=1$

$\|\frac{w}{\|w\|_E}\|_E=1$ obv, $\therefore \forall w\in X$, $w\in S$

$\therefore \|\frac{w}{\|w\|_E}\|\geq\|u_0\| \Rightarrow \|w\|\geq\|w\|_E\cdot\|u_0\|$ --- ②

①+②: $\left.\begin{array}{ll}\|x\|\le\sqrt{n}\cdot\|x\|_E & \forall x\\ \|w\|\geq\|u_0\|\cdot\|w\|_E & \forall w\end{array}\right\}\Rightarrow \|\cdot\|\sim\|\cdot\|_E$ arbitrary norm, $\therefore \|\cdot\|_1\sim\|\cdot\|_2$

"$\Leftarrow$" $\dim X=+\infty$, X has hamel basis $\{e_\lambda\}_{\lambda\in\Lambda}$; $\{a_\lambda\}_{\lambda\in\Lambda}$ is any unbounded sequence

let $\|\cdot\|_1=\sum_\lambda|x_\lambda|$, $\|\cdot\|_2=\sum_\lambda a_\lambda|x_\lambda|$

if $\{x_\lambda\}_{\lambda\in\Lambda}$ unbdd, $\|\cdot\|_2\div\|\cdot\|_1$ unbdd

$\therefore \nexists C_1, C_2$ s.t. $C_1\|x\|_1\le\|x\|_2\le C_2\cdot\|x\|_1$,

Rmk: 不一定可数维. dim 与 order of basis (bijective or "=") 没证过.
没关系但注意一下

Th4: If X is a normed space, the following are equivalent:

(i) X is separable (i.e. X contains a countable dense subset);

(ii) The unit sphere in X, $S_X:=\{x\in X:\|x\|=1\}$, is separable;

(iii) X contains a countable set $\{x_j\}_{j=1}^\infty$ s.t. $\overline{\operatorname{Span}\{x_j\}_{j=1}^\infty}=X$.

proof4: (1) $\Rightarrow$ (2) obv since subset (not space) of sepa set is also sepa

(2) $\Rightarrow$ (3) let $E=\{x_1, x_2, \cdots x_n, \cdots\}$ dense in S_X;

$\forall y\in S_X, r>0$; $B_r(y)\cap E\neq\phi$

let $y=\frac{x}{\|x\|}$, $\forall r>0$, $\exists x_i\in E$ s.t. $\|x_i-\frac{x}{\|x\|}\|<r$ $\quad \varepsilon=r\cdot\|x\|$ is OK

$\Rightarrow \forall x\in X, \forall r>0$, $\exists x_i\cdot\|x\|\in X$ s.t. $\|x_i\cdot\|x\|-x\|<r\cdot\|x\|$, here $\|x\|$ is const!

$x_k\cdot\|x\|\in \operatorname{span}\{x_i\}$, $B_\varepsilon(x)\cap\operatorname{span}\{x_i\}\neq\phi$ $\forall x\in X$

$\therefore \overline{\operatorname{span}\{x_i\}}=X$, (2)和(3)的$\{x_i\}$是一样的.

(3) $\Rightarrow$ (1) if $\overline{\operatorname{span}\{x_i\}}=X$, $\{x_i\cdot\|x\|\}_{x_i\in E, x\in X}$ is dense

use $\{x_i q\}_{x_i\in E, q\in\mathbb{Q}}$ alternate $\{x_i\cdot\|x\|\}_{x_i\in E, x\in E}$, $\{x_i q\}$ is dense, countable

prop: If $(X, \|\cdot\|)$ is a Banach space and Y is a linear subspace of X then $(Y, \|\cdot\|)$ is a Banach space if and only Y is closed.

Th5: $1 \le p < +\infty$, space $l^p(\mathbb{K})$ is separable, $l^{+\infty}(\mathbb{K})$ not; $\mathbb{K}$ is the number field

proof5: 1: $1 \le p < +\infty$, $l^p(\mathbb{K}) = \{(x_1, x_2 \cdots x_n, \cdots) : \sum_{i=1}^{+\infty} |x_i|^p < +\infty\}$

let $e_i = (0, 0, \cdots 0, 1, 0, \cdots)$, with i-th slot 1, other slots 0

$\overline{\text{span}\{e_n\}} = X$ obv $\therefore \{e_n\}_{n=1}^{+\infty}$ dense countable.

2. let $S := \{x \in l^{+\infty}(\mathbb{K}) : x_i = 0 \text{ or } 1 \text{ arbitrarily } \forall i\}$

if S countable, $S = \{y_1, y_2, \cdots y_n \cdots\}$

construct y_0 s.t. i-th slot of $y_0 = 1 -$ i-th slot of y_i, $y \notin S$ contradict!

$\Rightarrow$ WTS: $\forall$ set E, E dense in $l^{+\infty}(\mathbb{K})$, $\exists$ injective $S \to E$, then $|E| \ge |S|$

if E dense: $\begin{cases} \forall x \in S \subseteq l^{+\infty}(\mathbb{K}), \exists x' \in E \text{ s.t. } \|x' - x\| < \frac{1}{3} \quad \cdots (*) \\ \forall y \in S \subseteq l^{+\infty}(\mathbb{K}), \exists y' \in E, \text{ s.t. } \|y' - y\| < \frac{1}{3} \end{cases}$

define $T: S \to E$, $T: x \mapsto x'$ by $(*)$. x' is arbitrary picked

for: $x \ne y$, $\|x' - y'\| \ge \underbrace{\|x - y\|}_{=1} - \|x - x'\| - \|y - y'\| > 1 - \frac{1}{3} - \frac{1}{3}$ $\therefore x' \ne y'$

1 since $x, y \in S$, $\|x\|_{+\infty} = \max |x_i|$

Th6: $\forall 1 \le p \le +\infty$, $l^p(\mathbb{K})$ complete with l_p standard norm

Easy!

proof6: let $1 \le p < +\infty$, consider $\{x^k\}_{k=1}^{+\infty}$, $x^k = (x_1^k, x_2^k, \cdots x_n^k, \cdots)$

$\{x^k\}$ Cauchy, then $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall m, n \ge N$, $\|x^m - x^n\|_{l^p} = \left(\sum_{i=1}^{+\infty} |x_i^m - x_i^n|^p\right)^{\frac{1}{p}} < \varepsilon$

$\therefore \{x_i^k\}_{k=1}^{+\infty}$ Cauchy for $\forall i$, $x_i^k \in \mathbb{K}$

$\mathbb{K}$ complete $\therefore x_i^k \xrightarrow{k \to +\infty} a_i$ for $\forall i$ $\Rightarrow \{x^k\}$ converges too.

let $p = +\infty$, ...

$\forall \varepsilon > 0$, $\exists N$ s.t. $\forall m, n \ge N$, $\|x^m - x^n\|_{l^{+\infty}} = \max |x_i^m - x_i^n| < \varepsilon$ .. then same with $p < +\infty$