L8 Thm(能量模估计): u范数^2 + Du范数^{2}的积分 $\le C$(初值范数^{2}的积分 + 边值范数^2)

$$\begin{cases} \mathcal{L}u = \partial_t u - \Delta u = f &, \text{in } U_T \\ u|_{t=0} = \varphi(x) &, \text{in } U=(0,l),\ t\in(0,+\infty)\text{时间} \\ u|_{x=0} = u|_{x=l} = 0 &, \text{on } \partial U \end{cases}$$

设 u is classical solution, then: $\sup\limits_{0\le t\le T}\int_0^l u^2(x,t)\,dx + 2\int_0^T\int_0^l Du^2(x,t)\,dx\,dt \le C\left(\int_0^l \varphi^2(x)\,dx + \int_0^T\int_0^l f^2(x,t)\,dx\,dt\right)$

$$\Leftrightarrow \sup_{0\le t\le T}\|u(t,\cdot)\|^2_{L^2(U)} + 2\int_0^T \|Du(t,\cdot)\|^2_{L^2(U)}\,dt \le C\left(\|\varphi\|^2_{L^2(U)} + \int_0^T \|f(t,\cdot)\|^2_{L^2(U)}\,dt\right)$$

全空间积分 = $L^2(U)$空间范数^2, $\int_U \cdots dx = \|\cdots\|^2_{L^2(U)}$

given: $\int u\cdot(\partial_t u - \Delta u)\,dx = \int uf\,dx$

$$\text{LHS:}\begin{cases} \int_U u\cdot\partial_t u(t,x)\,dx = \int_U \frac{1}{2}\partial_t(u^2)\,dx = \frac{1}{2}\frac{d}{dt}\int u^2(t,x)\,dx \\ \int_U u(-\Delta u)\,dx = \int_U |Du|^2 dx - \int_{\partial U}\frac{\partial u}{\partial n}\cdot u\,dS = \int_U |Du|^2 dx, \quad \text{since: } \partial U\text{上 } u|_{x=0} = u|_{x=l} = 0 \end{cases}$$

RHS: $\int_U uf\,dx \le \frac{1}{2}\int_U u^2 dx + \frac{1}{2}\int_U f^2 dx$

代入即: $\frac{1}{2}\frac{d}{dt}\int_U u^2(t,x)\,dx + \int_U |Du|^2 dx \le \frac{1}{2}\int_U u^2 dx + \frac{1}{2}\int_U f^2 dx$

插入:

$$\frac{1}{2}\frac{d}{dt}\int_U\int_0^t u^2(s,x)\,ds\,dx = \frac{1}{2}\int_U u^2(t,x)\,dx - \frac{1}{2}\int_U u^2(0,x)\,dx = \frac{1}{2}\int_U u^2(t,x)\,dx - \frac{1}{2}\|\varphi(\cdot)\|^2_{L^2(U)}$$

$\times$ [$\Rightarrow \|u(t,\cdot)\|^2_{L^2(U)} + 2\|Du\|^2_{L^2(U)} \le \|\varphi(\cdot)\|^2_{L^2(U)} + \|u^2(t,\cdot)\|^2_{L^2} + \|f^2(t,\cdot)\|^2_{L^2}$

: "$\frac{d}{dt}$"是针对"t"这个变量, 而是所代表的"first shoo"; (不然可以随便写了)

$$\therefore\ \frac{1}{2}\frac{d}{dt}\int_0^T\int_U u^2(t,x)\,dx\,dt + \int_0^T\int_U |Du|^2 dx\,dt \le \frac{1}{2}\int_0^T\int_U u^2(t,x)\,dx\,dt + \frac{1}{2}\int_0^T\int_U f^2(t,x)\,dx\,dt \quad \cdots ①$$

$$\frac{1}{2}\int_U u^2(t,x)\,dx\Big|_{t=0}^{t=T} = \frac{1}{2}\int_U u^2(T,x)\,dx - \frac{1}{2}\int_U u^2(0,x)\,dx = \frac{1}{2}\|u(T,\cdot)\|^2_{L^2(U)} - \frac{1}{2}\|\varphi(\cdot)\|^2_{L^2(U)}$$

$$\Rightarrow \left(\frac{1}{2}\|u(T,\cdot)\|^2_{L^2(U)} - \frac{1}{2}\|\varphi(\cdot)\|^2_{L^2(U)}\right) + \underbrace{\int_0^T \|Du(t,\cdot)\|^2_{L^2(U)}dt}_{(*)} \le \frac{1}{2}\int_0^T \|u(t,\cdot)\|^2_{L^2(U)}\,dt + \frac{1}{2}\int_0^T \|f(t,\cdot)\|^2_{L^2(U)}\,dt \quad \cdots ②$$

① $\Rightarrow$:

$$\text{here:}\begin{cases} G(t) = \frac{1}{2}\int_0^T \|u(t,\cdot)\|^2_{L^2(U)}\,dt \\ F(t) = \frac{1}{2}\int_0^T \|f(t,\cdot)\|^2_{L^2(U)}\,dt - \int_0^T \|Du\|^2_{L^2(U)}\,dt \end{cases}$$

Lem: Gronwall's inequality:

$$\begin{cases} G'(t) \le G(t) + F(t) \quad \forall t \\ G(0) = 0 \end{cases}$$

then $G(t) \le (e^t - 1)F(t)$ $\forall t$ 微分中学过 $F=0$ 情况

$\therefore \int_0^T \|u(t,\cdot)\|^2 dt \le (e^t-1)\left(\int_0^T \|f(t,\cdot)\|^2 dt - 2\int_0^T \|Du\|^2 dt\right)$

这么写问题在于没有φ, 我们尝试用②寻找 $G, F(t)$.

② $\Rightarrow$ 把T当成变量: $G(T) = \frac{1}{2}\int_0^T \|u(t,\cdot)\|^2_{L^2(U)}\,dt$, $G(0)=0$

$$\begin{cases} G'(T) = \frac{1}{2}\|u(T,\cdot)\|^2_{L^2(U)} \\ F(T) = \frac{1}{2}\int_0^T \|f(t,\cdot)\|^2_{L^2(U)}\,dt + \frac{1}{2}\|\varphi(\cdot)\|^2_{L^2(U)} \end{cases}$$

"$(*) \le 0$"代入②即可

$\frac{1}{2}\frac{d}{dt}\int_U u^2(t,x)dx + \int_U |\nabla u|^2 dx \le \frac{1}{2}\int_U u^2 dx + \frac{1}{2}\int_U f^2 dx$;…… LHS和RHS的简单变形

$\Downarrow$

对于first slot求导而不是t这个变量，"∂_t"改成"∂_1"更适合

$\frac{1}{2}\partial_1\int_U u^2(t,x)dx + \int_U |\nabla u|^2 dx \le \frac{1}{2}\int_U u^2 dx + \frac{1}{2}\int_U f^2 dx$; actually holds for $\forall t_0 \in [0,T]$

$\int_0^{t_0}\left(\frac{1}{2}\partial_1\int_U u^2(t,x)dx\right)dt + \int_0^{t_0}\int_U |\nabla u|^2 dx\,dt \le \frac{1}{2}\int_0^{t_0}\int_U u^2(t,x)dx\,dt + \frac{1}{2}\int_0^{t_0}\int_U f^2(t,x)dx\,dt$,

$\because$ here "t_0" is the variable, and "t" only a mediator，最后形式为 $u($ t_0函数, $x)$；∂_1作用于 t_0

$\therefore \int_0^{t_0}\left(\frac{1}{2}\partial_1\int_U u^2(t,x)dx\right)dt = \frac{1}{2}\int_U u^2(t,x)dx\Big|_{t=0}^{t=t_0}$

$= \frac{1}{2}\int_U u^2(t_0,x)dx - \frac{1}{2}\int_U u^2(0,x)dx = \frac{1}{2}\int_U u^2(t_0,x)dx - \frac{1}{2}\|\varphi(\cdot)\|^2_{L^2(U)}$

$\Rightarrow \left(\frac{1}{2}\int_U u^2(t_0,x)dx - \frac{1}{2}\|\varphi(\cdot)\|^2_{L^2(U)}\right) + \int_0^{t_0}\underbrace{\|\nabla u(t,\cdot)\|^2_{L^2(U)}}_{\ge 0}dt \le \frac{1}{2}\int_0^{t_0}\int_U u^2(t,x)dx\,dt + \frac{1}{2}\int_0^{t_0}\|f(t,\cdot)\|^2 dt$

$\therefore \underbrace{\frac{1}{2}\int_U u^2(t_0,x)dx}_{G'(t_0)} \le \underbrace{\frac{1}{2}\int_0^{t_0}\int_U u^2(t,x)dx\,dt}_{G(t_0)} + \underbrace{\frac{1}{2}\|\varphi(\cdot)\|^2_{L^2(U)} + \frac{1}{2}\int_0^{t_0}\|f(t,\cdot)\|^2_{L^2(U)}dt}_{F(t_0)}$

Gronwall's inequality: $\frac{1}{2}\int_0^{t_0}\|u^2(t,\cdot)\|^2_{L^2(U)}dt \le (e^{t_0}-1)\frac{1}{2}\left(\|\varphi(\cdot)\|^2 + \int_0^{t_0}\|f(t,\cdot)\|^2 dt\right)$ $\forall t_0 \in (0,T)$

let $C = \max(e^{t_0}-1) = e^T-1$ 即可.

常用梯度公式：① $\int_\Omega \nabla\cdot F\,dx = \int_{\partial\Omega} F\cdot\vec{n}\,dS$

② $\int_\Omega \nabla(u\nabla u)dx = \int_{\partial\Omega}(u\nabla u)\cdot\vec{n}\,dS$

$= \int_{\partial\Omega} u\cdot(\nabla u\cdot\vec{n})dS = \int_{\partial\Omega} u\cdot\frac{\partial u}{\partial\vec{n}}dS$ 不怎么用，略

③ $\nabla(u\nabla u) = \nabla u\cdot\nabla u + u\cdot\nabla\cdot\nabla u = |\nabla u|^2 + u\cdot\Delta u$

since $\nabla\cdot\nabla\cdot u = \nabla(\nabla u)$ 算子从右到左，向量不满足结合律

逆方向也要会推！

$\int_\Omega \nabla(u\nabla u)dx$

$= \int_\Omega |\nabla u|^2 dx + \int_\Omega u\cdot\Delta u\,dx$ 这个常用

④ $\int_\Omega \nabla u\,\nabla v\,dx = \int_\Omega \nabla v\cdot\nabla u\,dx$

$= \int_{\partial\Omega} u\nabla v\cdot\vec{n}\,dS = \int_{\partial\Omega} u\cdot\frac{\partial v}{\partial\vec{n}}dS = \int_{\partial\Omega} v\cdot\frac{\partial u}{\partial\vec{n}}dS$

③→⑤ $\int_\Omega \nabla(v\nabla u)dx = \int_\Omega \nabla u\cdot\nabla v\,dx + \int_\Omega v\cdot\nabla^2 u\,dx$

其中 $\nabla^2 u = \Delta u$

$\therefore \int_\Omega \nabla u\cdot\nabla v\,dx$

$= \int_\Omega \nabla(v\nabla u)dx - \int_\Omega v\nabla^2 u\,dx = \int_\Omega \nabla(u\nabla v)dx - \int_\Omega u\nabla^2 v\,dx$ (以上主要集中于P13 eigen"必考项!)

Thm Backward HE 周P132

$u_1,u_2 \in C^{1,2}((0,T]\times U)\cap C^1(\bar{U}_T)$; u_1,u_2 solve $\begin{cases} \partial_t u - \Delta u = 0, & \text{in } U_T \\ u|_{t=T} = \varphi(x), & \text{on } \partial U \quad (\text{usually, HE: } u|_{t=0}=\varphi(x)) \\ u|_{\partial U} = g(t), & t\in[0,T] \end{cases}$

then $u_1 \equiv u_2$ in $\bar{U}_T$, 即: Backward HE (从末状态 $t=T$ 出发) 解唯一

Thm1: 解不稳定, $\|u(t,\cdot)\| \to +\infty$

设 φ solve: $\begin{cases} -\Delta\varphi = \lambda\varphi & U, \\ \varphi|_{\partial U} = 0 \end{cases}$

let $u_\lambda(t,x) = e^{(T-t)\lambda}\cdot\varphi(x)$, then u_λ solves: $\begin{cases} \partial_t u = \Delta u, & U_T \\ u_\lambda|_{t=T} = \varphi(x), & U \cdots ① \end{cases}$

$\therefore \|u_\lambda(t,x)\|_{L^2(U)} = e^{(T-t)\lambda}\|\varphi(x)\|_{L^2(U)}$

given $\forall t\in(0,T]$, fix t, $\|u_\lambda\|_{L^2(U)} \to +\infty$ as $\lambda\to+\infty$

$\therefore$ Backward HE 解 unstable; $u_{\lambda_1}-u_{\lambda_2}$ 可以很大 "$\lambda\to+\infty$"在eigen章节讲过, 证明不要求

(In HE: 是找不到这样的 $u_\lambda(t,x)$ 的, 因为①若成立 T 改成 0, $\lambda\to+\infty$ $\|u_\lambda\|\to 0$ 了)

Thm2: Backward HE 解也唯一

(In HE, unique和stable均用到 Maximum principle; 而MP中 ∂U 不含 T, $\therefore$ 反向问题用不了)

设: $\begin{cases} w = u_1 - u_2 \\ e(t) = \int_U w^2(t,x)\,dx = \|w(t,\cdot)\|^2_{L^2(U)}, \quad e(T)=0 \text{ since: } u_1|_{t=T} = u_2|_{t=T} \end{cases}$

then: $\begin{cases} \partial_t w = \Delta w, & U_T \\ w|_{t=T} = 0, & \partial U \\ w|_{\partial U} = 0, & t\in[0,T] \end{cases}$

$e(t) \Rightarrow \begin{cases} e'(t) = \int_U \frac{dw^2}{dt}dx = \int_U 2w\cdot\frac{dw}{dt}dx \overset{\partial_t w = \Delta w}{=} \int_U 2w\cdot\Delta w\,dx \cdots ① \\ e''(t) = \underbrace{\int_U 2w_t\cdot\Delta w\,dx + \int_U 2w\cdot\Delta w_t\cdot dx} \end{cases}$

由梯度公式换算:

why?

$$e''(t) \overset{②}{=} -2\int_U \frac{d|Dw|^2}{dt}dx$$

$$= -4\int_U Dw\cdot Dw_t\,dx$$

$$= -4\int_U D(w\cdot Dw_t)dx + 4\int_U w\cdot D^2 w_t\,dx \quad \text{这样不好}$$

$$= -4\int_U Dw_t\,Dw\,dx = -4\int_U D(w_t\cdot Dw)dx + 4\int_U w_t\cdot D^2 w\,dx$$

$$= -4\int_{\partial U} w_t\cdot \underbrace{Dw\cdot\vec{n}}_{0}\,ds + 4\int_U w_t\,\Delta w\,dx$$

$$= 4\int_U |\Delta w|^2 dx$$

换个公式用:

$$e'(t) = \int_U 2w\,\Delta w\,dx$$

$$= 2\int_U D(wDw)dx - 2\int_U |Dw|^2 dx$$

$$= 2\int_{\partial U}(\overset{0}{w}Dw)\cdot\vec{n}\,ds - 2\int_U |Dw|^2 dx$$

$$= -2\int_U |Dw|^2 dx \cdots ②$$

$\therefore |e'(t)|^2 \overset{①}{=} 4\left|\int_U w\Delta w\,dx\right|^2 \le 4\cdot\int_U w^2dx \times \int_U |\Delta w|^2 dx = e(t)\cdot e''(t) \quad \forall t\in(0,T) \quad e(t)$ convex

$e(T)=0$, $e'(t)\le 0\ \forall t$ t在凸函数左段 log e(t) convex why $e\equiv 0$?