4.7 Show that if $f \in C(\mathbb{R};\mathbb{R})$ and $x \in C([0,T];\mathbb{R})$, then

$$\dot{x} = f(x), \qquad \text{with} \qquad x(0) = x_0, \qquad \text{for all } t \in [0,T] \tag{4.10}$$

if and only if

$$x(t) = x_0 + \int_0^t f(x(s))\,\mathrm{d}s \qquad \text{for all } t \in [0,T]. \tag{4.11}$$

proof 4.7 "$\Leftarrow$" if $x(t)=x_0+\int_0^t f(x(s))ds$

$x(0)=x_0$. $x'(t)=f(x(t))=f(x)$

"$\longrightarrow$" if $x(0)=x_0$, $\dot{x}=f(x)$, $\therefore x'(t)=f(x(t))$

订:

$x(t)-x(0)=\int_0^t x'(s)ds=\int_0^t f(x(s))ds$ $\therefore x(t)=x_0+\int_0^t f(x(s))ds$

小打即可

4.8 Suppose that $f:\mathbb{R}\to\mathbb{R}$ is Lipschitz continuous, i.e. satisfies

①$T\in(0,\frac{1}{2L})$, 已证出 x 存在, unique. 记为 x_1

then let. $x_2(0)=x_1(\frac{1}{2L})$ → 即"x_0"

$$|f(x)-f(y)| \le L|x-y|, \qquad x,y\in\mathbb{R},$$

订: for $LT>1$

②$T\in[\frac{1}{2L},\frac{1}{2L}\times 2)$

for some $L>0$. Use the Contraction Mapping Theorem in the space $C([0,T];\mathbb{R})$ on the mapping

用分段函数

$(\mathcal{J}x_2)(t)=x_1(\frac{1}{2L})+\int_{\frac{1}{2L}}^t f()ds$

定义范数!

$LT<1$

$$(\mathcal{J}x)(t) = x_0 + \int_0^t f(x(s))\,\mathrm{d}s$$

to show that (4.10) has a unique solution on any time interval $[0,T]$ with

proof 4.8 上题中的 $x(t)$ 仅有唯一解. if $|f(x)-f(y)|\le L|x-y|$, $L\cdot T<1$ $t\in[0,T]$

contraction mapping Th:

$T: X\to X$, $d(Tx,Ty)\le\theta d(x,y)$ for $\theta\in(0,1)$, $\Rightarrow$ $Tx=x$ has only one solution $x_0\in X$

需要证明 $J(X(t))=X(t)$ 只有一解: ↘ 本题中要 $d(x_1,x_2)$, 不是函数; "d"要先定义norm, 再用norm定义d

$J(X_1(t))=x_0+\int_0^t f(X_1(s))ds$

$J(X_2(t))=x_0+\int_0^t f(X_2(s))ds$

$|J(x_1(t))-J(x_2(t))|=|\int_0^t f(x_1(s))-f(x_2(s))ds|\le\int_0^t|f(x_1(s))-f(x_2(s))|ds\le\int_0^t L\cdot|x_1(s)-x_2(s)|ds$

let $\|x\|=\sup\limits_{s\in[0,T]}|x(s)|$

then $|J(x_1(t))-J(x_2(t))|\le\int_0^t L\|x_1-x_2\|ds\le T\cdot L\cdot\|x_1-x_2\|$ $\forall t\in[0,T]$

$\therefore$ $\|J(x_1)-J(x_2)\|\le T\cdot L\|x_1-x_2\|$

引入合理的norm和"d"

$\therefore Jx=x$ 仅有1解. 即: $x(t)=x_0+\int_0^t f(x(s))ds$ 仅有一解

5.4 Show that if Y is a finite-dimensional subspace of a normed space X and $x \in X \setminus Y$, then there exists $y \in Y$ with $\|x - y\| = \operatorname{dist}(x, Y)$.

proof 5.4: $x \in X\setminus Y$, $\exists y \in Y$ s.t. $\|x-y\| = dist(x,Y) = \inf_y \|x-y\|$

设 $d_x(y) = \|x-y\|$ x fixed

$|d_x(y_1) - d_x(y_2)| = |\|x-y_1\| - \|x-y_2\|| \le \|y_2-y_1\|$ $\therefore$ $d_x(y)$ is cts on y ⋯①

Y is finite-dimensional subspace

$\therefore$ Y的basis 可以由"$Av=0$"的v得到, $A_{m\times n}$ for $m,n<+\infty$.

A is linear thus continues; $\{0\}$ is closed

$\therefore$ $A^{-1}(0) = span(v) = Y$ is closed ⋯②

$dist(x,Y) \neq 0$, otherwise x is limit point of Y thus $x\in Y$

设 $d_x(y) = d_0$. ~~设$\exists$ $d_x(y_0) = d_0$. $y_0 \in Y$~~

$\forall \varepsilon = \frac{1}{2^i}$, $\exists y_i \in Y$ s.t. $d_0 < d_x(y_i) < d_0 + \frac{1}{2^i}$; $i=1,2,3,\cdots$

$d(y_1,y_n) \le d(y_1,x) + d(y_n,x) = d_x(y_1) + d_x(y_n) < 2d_0 + \frac{1}{2^1} + \frac{1}{2^n} < 2d_0 + 1$

$\therefore$ $\{y_1, y_2, \cdots y_n\}$ is bounded thus $\exists$ limit point s.t. $y_{n_k} \to y$, $y \in Y$ since Y closed.

$\forall \varepsilon > 0$, $\exists N$ s.t. $d(y, y_{n_k}) < \varepsilon$ $\forall k \ge N$

$\therefore$ $d_x(y) \le d(y, y_{n_k}) + d_x(y_{n_k}) < \varepsilon + d_0 + \frac{1}{2^{n_k}}$ $\forall k \ge N$ ($n_k \uparrow$ as $k \uparrow$)

$\therefore$ $d_x(y) \le \varepsilon + d_0$ as we let $k \to +\infty$, $n_k \to +\infty$

$\therefore$ $d_x(y) \le d_0$. as we let $\varepsilon \to 0$.

即 $d_x(y) = d_0$ ⋯③ 证毕!

(实际上, 连续函数 $f(x)$ 在 closed domain X 上必能取到 inf / sup)

距离总结笔记

5.6 Suppose that Y is a proper finite-dimensional subspace of a normed space X. Show that for any $y \in Y$ and $r > 0$ there exists $x \in X$ such that $\|x - y\| = \text{dist}(x, Y) = r$. (Megginson, 1998)

Recall: Riesz lemma: 结论可以改成 $\|x\|=1$, $\|x-y\| \geq \frac{1}{r}$ $\forall y \in Y, r>1$ (r可以去逼近1)

$\Rightarrow \exists x \in X, |\frac{x}{\|x\|} - y| \geq \frac{1}{r}$ $\forall y \in Y, r>1$

$\Rightarrow \exists x \in X, |x - y| \geq \frac{1}{r}\|x\|$, $\because$ since $\{\|x\|y : y \in Y\} = Y$

① 证明 $\{z : \|z - y_0\| = r\} = \{\frac{v}{\|v\|} \cdot r + y_0 : v \in X\}$

$\frac{v}{\|v\|} \cdot r + y_0 \in \{z : \|z - y_0\| = r\}$ $\therefore$ RHS $\subseteq$ LHS

if $z \in$ LHS, $z = \frac{w}{\|w\|} \cdot r$ some w $\therefore z \in$ RHS $\Rightarrow$ LHS $\subseteq$ RHS

ii: if $\dim X = +\infty$

②: $\exists$ some $v \in X$ s.t. $\|\frac{v}{\|v\|} \cdot r + y_0 - y\| \geq r$ $\forall$ other $y \in Y$

$\Leftrightarrow \|\frac{v}{\|v\|} + \frac{1}{r}y_0 - \frac{1}{r}y\| \geq 1$ let $w = \frac{v}{\|v\|}$. $\|w\| = 1$ 可以取遍 X 中全体 $\|\cdot\| = 1$ 的向量

用Riesz's 逼近

☆ 由 Riesz lem $\exists w_q \in X$ $\|w_q\| = 1$. $\|w_q - z\| \geq \frac{1}{q}$, $q > 1$, $\forall z \in Y$,

let $q = 1 + (\frac{1}{10})^n$, 记 $w_q = w_n$ then as $n \to +\infty$. $w_n \to w_0$.

$\|w_0\| = 1$, $w_0 \in X$ since $\{w \in X : \|w\| = 1\}$ is closed

$\Rightarrow$ with this w_0; $\|w_0 \cdot r + y_0 - y\| \geq r$ $\forall$ other $y \in Y$

$\therefore w_0 r + y_0$ satisfies: $\text{dist}(y, X) = d(w_0 r + y_0, y_0) = r$

5.7 Use the result of the previous exercise to show that no infinite-dimensional Banach space can have a countable Hamel basis. (Megginson, 1998) [Hint: given a Hamel basis $\{e_j\}_{j=1}^{\infty}$ for X, let $X_n = \text{Span}(e_1, \ldots, e_n)$. Now find a sequence (y_n) with $y_n \in X_n$ such that

$$\|y_n - y_{n-1}\| = \text{dist}(y_n, X_{n-1}) = 3^{-n}$$

and show that (y_n) is Cauchy in X but its limit cannot lie in any of the X_n.] For another, simpler, proof using the Baire Category Theorem, see Exercise 22.1.

proof 5.7: 证明 ∄ infinite dimensional Banach space has countable Hamel basis

~~(可数基 ⇔ separable in HW4, T3.16)~~ 有限维 ⇒ 有限 basis；无限维 + Banach ⇒ 不可数 ba

反证：∃ hamel basis $\{e_1, e_2, \cdots e_n, \cdots\}$

let $X_n = \mathrm{span}(e_1, e_2, \cdots e_n)$; $X = \mathrm{span}(e_1, \cdots e_n, \cdots)$

由 T5.6：$\exists\, y_n \in X_n$, $y_n \notin X_{n-1}$ s.t. $\|y_n - y_{n-1}\| = \mathrm{dist}(y_n, X_{n-1}) = (\frac{1}{3})^n$

$\therefore \{y_n\}$ is Cauchy，由于 X infinite-dimensional，∴ 基无限个，y_n 无限个

$\therefore y_n \to y_0 \in X$ since X is Banach

but $y_0 \notin X_n\ \forall n$ as $n \to +\infty$，contradict!

6.4 Prove Dini's Theorem: suppose that $(f_n) \in C([a, b]; \mathbb{R})$ is an increasing sequence ($f_{n+1}(x) \geq f_n(x)$) that converges pointwise to some $f \in C([a, b]; \mathbb{R})$. Show that f_n converges uniformly to f. (For each $\varepsilon > 0$ consider the sets

$$E_n := \{x \in [a, b] :\ f(x) - f_n(x) < \varepsilon\}$$

and use the compactness of $[a, b]$.)

let. $E_n = \{x \in [a,b] : f(x) - f_n(x) < \varepsilon\}$

$f_n \uparrow$, $\therefore E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n \subseteq \cdots$

f_n converges pointwise $\Rightarrow \forall x \in [a,b].\ x \in E_i$ some i

$\therefore \bigcup_{n=1}^{+\infty} E_n = [a,b]$

$[a,b]$ compact $\Rightarrow \exists$ finite $E_{n_1}, E_{n_2} \cdots E_{n_t}$ s.t. $\bigcup_{i=1}^{t} E_{n_i} \supseteq [a,b]$ ①

需要补充(*)

$\therefore$ for ε, $\exists N = \max(n_1, n_2, \cdots n_t)$ s.t. $f(x) - f_n(x) < \varepsilon\ \ \forall n \geq N,\ \forall x \in [a,b]$

$\Rightarrow f_n \to f$ uniformly

(*): $f_n \uparrow$ $\therefore x \in E_N \Rightarrow x \in E_n\ \forall n \geq N$;

$\therefore$ let $N = \max(n_1, n_2, \cdots n_t)$. $x \in E_{n_i} \Rightarrow x \in E_N\ \forall x$ $\therefore E_N = [a,b]$

(或直接 $E_i \subseteq E_j$ 对 $i \leq j$，∴将①式改成 $E_{n_t} = [a,b]$)

6.11证：totally bdd + equcts ⇒ $\{f_n\}$ cpt spaces

$\therefore \exists \{f_{n_k}\}$ s.t. f_{n_k} converges (函数本身的 converge)

$\therefore \forall \varepsilon > 0, \exists K$ s.t. $\|f_{n_k} - f\| < \varepsilon\ \ \forall k \in K$,

? 6.11 Use the Arzelà–Ascoli Theorem repeatedly to show that if a sequence $(f_n) \in C_b(\mathbb{R})$ is bounded and equicontinuous on $\mathbb{R}$, then it has a subsequence that converges uniformly on all compact subsets of $\mathbb{R}$.

对

Proof 6.11: (f_n) bdd, equi-cts on $\mathbb{R}'$ $\Rightarrow$ totally bounded

$\therefore \{f_n\}$ compact on $C_b(\mathbb{R})$ $\Rightarrow$ $f_{n_k} \to f$ uniformly

$f_{11}\ f_{12}\ f_{13} \cdots$ conv in $[-1,1]$

$\exists \{f_{2n}\} \subseteq \{f_{1n}\}$ s.t. $f_{21}, f_{22}, f_{23} \cdots$ conv in $[-2,2]$

$\cdots$

对角线法取出 $f_{11}, f_{22}, \cdots f_{nn}, \cdots$

或 $\therefore \exists \{f_{n_k}\}$ s.t. f_{n_k} converges pointwise on $\mathbb{R}$

converges pointwise + equi-cts + compact set $\Rightarrow$ f_{n_k} converge uniformly 证毕!

$\uparrow$

第五周第2节课内容

? 6.12 Suppose that $f \in C_b(\mathbb{R})$. Show that for every $\delta > 0$ the function

$$f_\delta(x) := \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} f(y)\, dy$$

is Lipschitz with $\|f_\delta\|_\infty \le \|f\|_\infty$. Show furthermore that f_δ converges uniformly to f on every bounded interval.

Proof 6.12: (1): $\because \|f_\delta\| \le \frac{1}{2\delta}\int_{-\delta+x}^{\delta+x} \|f\|\, dy = \frac{1}{2\delta} \cdot 2\delta \times \|f\| = \|f\|$

$$|f_\delta(x) - f_\delta(z)| = \left|\frac{1}{2\delta}\int_{-\delta+x}^{\delta+x} f(y)dy - \frac{1}{2\delta}\int_{-\delta+z}^{\delta+z} f(y)dy\right|$$

$$= \left|\frac{1}{2\delta}\int_{\delta+z}^{\delta+x} f(y)dy - \frac{1}{2\delta}\int_{-\delta+z}^{-\delta+x} f(y)dy\right|$$

$$\le \left|\frac{1}{2\delta}\int_{\delta+z}^{\delta+x} \|f\|dy\right| + \left|\frac{1}{2\delta}\int_{z-\delta}^{x-\delta} \|f\|dy\right| = \frac{1}{\delta}\|f\| \cdot |x-z| \Rightarrow \text{Lipschitz cts}$$

(2): WTS: in bounded $I \subseteq \mathbb{R}'$, $f_\delta \xrightarrow{uni} f$ as $\delta \to 0$.

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta_0$ s.t. $\left|\frac{1}{2\delta}\int_{x-\delta}^{x+\delta} (f(y) - f(x))dy\right| < \varepsilon$. $\forall \delta < \delta_0$

$\because$ Lipschitz cts. $\therefore \forall \varepsilon > 0, \exists \sigma$ s.t. $|f(x) - f(y)| < \varepsilon$. $\forall |x-y| < \sigma$.

$\therefore$ let $\sigma > \delta$, $\frac{1}{2\delta}\int_{x-\delta}^{x+\delta} |f(y) - f(x)| dy < \frac{1}{2\delta}\int_{x-\delta}^{x+\delta} \varepsilon\, dy = \varepsilon$ 与x无关 $\Rightarrow$ uni converge

(Q: bounded 有什么用? 不相关啊)

误

6.13 Suppose that $f \in C_b(\mathbb{R})$ with $\|f\|_\infty \le M$, and that $(f_n) \in C_b(\mathbb{R})$ is a sequence with $\|f_n\|_\infty \le M$ such that $f_n \to f$ uniformly on every bounded subinterval in $\mathbb{R}$. Use the Arzelà–Ascoli Theorem to show that if there exist $(x_n) \in C([0, T])$ such that

$$x_n(t) = x_0 + \int_0^t f_n(x_n(s))\, ds \qquad \text{for all } t \in [0, T] \qquad (6.9)$$

for each n then there exists $x \in C([0, T])$ such that

$$x(t) = x_0 + \int_0^t f(x(s))\, ds \qquad \text{for all } t \in [0, T]. \qquad (6.10)$$

proof 6.13: $\{f_n\}, f \subseteq C_b(\mathbb{R})$, $\|f_n\|, \|f\| \leq M$; $f_n \to f$ uniformly in $\forall$ bounded $I \subseteq \mathbb{R}$

$\exists \{x_n\} \subseteq C[0,T]$, $x_n(t) = x_0 + \int_0^t f_n(x_n(s))\,ds$

WTS: $\exists x \in C[0,T]$, $x(t) = x_0 + \int_0^t f(x(s))\,ds$

$f_n \to f$ uniformly, $\forall \varepsilon > 0$, $\exists N_1$ s.t. $\|f_n - f\| < \varepsilon$, $\forall n \geq N$ ... (*)

$\therefore \int_0^t |f_n(x_n(s)) - f(x_n(s))|\,ds < T \cdot \varepsilon \Rightarrow \int_0^t f_n(x_n(s)) \to \int_0^t f(x_n(s))$ uniformly ... ①

$x_n(t)$ bounded since $\|f_n\| \leq M$.

$\forall n$: $x_n(t) - x_n(k) = \int_k^t f(x(s))\,ds \leq M|t-k| \Rightarrow \{x_n\}$ equi-cts $\Big\} \Rightarrow \{x_n\}$ compact

$\therefore x_{n_k}(t) \to x(t)$ uniformly ②, for δ in (*)'', $\exists N_2$. $n_k \geq N_2 \Rightarrow \|x_{n_k} - x\| < \delta$ ... (*)'

$\therefore \left|\int_0^t f_{n_k}(x_{n_k}(s)) - f(x(s))\,ds\right| \leq \left|\int_0^t f_{n_k}(x_{n_k}(s)) - f(x_{n_k}(s))\,ds\right| + \left|\int_0^t f(x_{n_k}(s)) - f(x(s))\,ds\right|$

$\leq t \cdot \|f_{n_k} - f\| + t \cdot \varepsilon$ let $n_k \geq \max(N_1, N_2)$ in (*)', (*)

补充: f cts, $\therefore \forall \varepsilon > 0$, $\exists \delta$ s.t. $|x_{n_k} - x| < \delta$, then $|f(x_{n_k}) - f(x)| < \varepsilon$ (*)''

$\therefore \int_0^t f_{n_k}(x_{n_k}(s))\,ds \to \int_0^t f(x(s))\,ds$ uniformly

即: $\exists x_{n_k} \to x$, $x(t) = x_0 + \int_0^t f(x(s))\,ds$