L9: Poisson's equation: $-\Delta u = f$

fundamental solution: $-\Delta u = \delta(x)$, $x \in \mathbb{R}^d$

green function: $\begin{cases} -\Delta u = \delta(y-x) & x \in U, y \in U \subseteq \mathbb{R}^d \\ u|_{\partial U} = 0 \end{cases}$ 两者区别在于 $\delta(y-x)$, $\delta(x)$；边值条件

· 相比 fundamental solution 是一个解, green function 是求解的中间变量 $\begin{cases} -\Delta u = 0 \\ u|_{\partial\Omega} = g \end{cases}$

G satisfy green function: $u(y) = \int_U \delta(x-y) u(x)\, dx$

$= \int_U (-\Delta G) u(x)\, dx$

$= \int_U (-\underbrace{\Delta u}_{0}) G\, dx - \int_{\partial U} \frac{\partial G}{\partial n} u\, ds + \int_{\partial U} \frac{\partial u}{\partial n} \overset{0}{G}\, ds \quad \cdots ①$

$= \int_{\partial U} (-\frac{\partial G}{\partial n}) g\, ds + \int_U f G\, dx$ （和ch2中Duhamel类比）

$-\Delta G = \delta(y-x)$，把y当成const. x是G的变量, g的变量

$\forall$ fixed y: $u(y) = \int_{\partial U} (-\frac{\partial G}{\partial n}) g\, ds$，用变量x.s 过渡到 const y 的值

· $ds = d((\nabla_x u)^2 + x^2)^{\frac{1}{2}}$

u单变量 $u(x)$, G也是 $\therefore u(y) = \int_{\partial U} (-\frac{\partial G}{\partial n}) g(x) \sqrt{(\nabla_x u)^2 + x^2}'\, dx$

即x是Green function的中间值

· 对于 Φ satisfy fundamental solution: $-\Delta \Phi(x) = \delta(x)$

$\Phi(x) = \frac{1}{d(d-2)\cdot \alpha_d} \cdot |x|^{-(d-2)}$, $d \geq 3$, $\alpha_d = \frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}$

↓

Φ 和 G 都可以求出 $-\Delta u = f$ 的通解，

$u(x) = \int \Phi(x-y) f(y)\, dy$ 易知

And:

①: $\int_{B_r} (\Delta\Phi) \cdot 1\, dx = \int_{B_r} \Phi (\Delta 1)\, dx + \int_{\partial B_r} \frac{\partial \Phi}{\partial n} \cdot 1 - \int_{\partial B_r} \frac{\partial 1}{\partial n} \cdot \Phi\, ds = \int_{\partial B_r} \frac{\partial \Phi}{\partial n}\, ds$

$\therefore \int_{B_r} (\Delta\Phi) \cdot 1\, dx = \int_{\partial B_r} \frac{\partial \Phi}{\partial n}\, ds = -1$

②: $\int_{B_r} \Phi\, dx \to 0$. $\int_{\partial B_r} \Phi\, dx \to 0$ as $r \to 0^+$

· 如何求出 fundamental Φ,（symmetry）

L7写了一种没懂

want: $\Phi(x) = \Phi(|x|)$ s.t. $\Delta\Phi = 0$；在L4 hw1中证过 $\Phi(x) = 0 \Rightarrow \Phi(Ox) = 0$，"O"为 rotation

设 $\varphi(x) = u(r) = u(|x|)$

• 如何求 fundamental Φ (symmetry)

↳ 还有一种

HW4(1)中已证 $\Phi(x)=0 \Rightarrow \Phi(Ax)=0$, A is rotation.

设 $u_r(x)=u(r)=u(|x|)$; $\Delta u_r=0$

$$\begin{cases} \partial_{x_i} u_r(x) = u'(r)\cdot\frac{\partial r}{\partial x_i} \\ \partial_{x_i}^2 u_r(x) = \left(u''(r)\cdot\frac{\partial r}{\partial x_i}\right)\cdot\frac{\partial r}{\partial x_i} + u'(r)\cdot\partial_{x_i}\left(\frac{x_i}{r}\right) \end{cases}$$

代入: $\partial_{x_i} r = \partial_{x_i}\sqrt{x_1^2+x_2^2+\cdots x_d^2} = \frac{2x_i}{2\sqrt{\sum x_i^2}} = \frac{x_i}{r}$

$$= u''(r)\cdot\left(\frac{x_i}{r}\right)^2 + u'(r)\cdot\left(\frac{1}{r} - \frac{x_i}{r^2}\cdot\frac{x_i}{r}\right)$$

注意: $\partial_{x_i}\left(\frac{\partial r}{\partial x_i}\right) = \partial_{x_i}\left(\frac{x_i}{r}\right)$ 这时 r 仍然与 x_i 相关! $\ne \frac{1}{r}$

Summing over i:

$$\begin{cases} \partial_x u_r(x) = u'(r)\cdot\partial_x r \\ \Delta u_r = \partial_x^2 u_r(x) = u''(r)\frac{\sum x_i^2}{r^2} + u'(r)\left(\frac{d}{r} - \frac{\sum x_i^2}{r^3}\right) + \sum_{i\ne j}\left(u''(r)\frac{x_i x_j}{r^2} - u'(r)\frac{x_i x_j}{r^3}\right) \times 2 \end{cases}$$?

$\Delta u = \sum_{i=1}^{n} \partial_{x_i}^2 u$ 见P37 ?

Since: $\partial_x u_r(x) = \sum_{i=1}^{d} \partial_{x_i} u_r(x)$.

$$\partial_x^2 u_r(x) = \left(\sum_{i=1}^{d} \partial_{x_i} u_r(x)\right)' = \sum_{j=1}^{d} \partial_{x_j}\left(\sum_{i=1}^{d} \partial_{x_i} u_r(x)\right)$$

计 $i\ne j$. $\partial_{x_i}\partial_{x_j} u(r) = \partial_{x_j}\left(u'(r)\cdot\frac{x_i}{r}\right)$

$$= u''(r)\cdot\frac{\partial r}{\partial x_j}\cdot\frac{x_i}{r} + u'(r)\left(-\frac{x_i}{r^2}\right)\cdot\frac{\partial r}{\partial x_j}$$

$$= u''(r)\frac{x_i x_j}{r^2} - u'(r)\frac{x_i x_j}{r^3}$$

$\Rightarrow \Delta u = u''(r) + \frac{d-1}{r}u'(r) = 0$

解ODE: $\underbrace{(r u''(r) + u'(r))}_{\parallel} + (d-1)u'(r) = 0$

$\therefore$ $ru' + (d-2)u = \text{const}$ if $d \geq 2$

then: $u(r) = a\log r + b$ $d=2$

代入: $\int_{\partial B_r} \frac{\partial\Phi}{\partial n} ds = -1 \Rightarrow a\int_{\partial B_r} \frac{1}{r} dx = -1$? $\therefore c = -\frac{1}{2\pi}$

$$\Phi(x) = \begin{cases} -\frac{1}{2}|x| & d=1 \\ -\frac{1}{2\pi}\log|x| & d=2 \\ \frac{1}{d(d-2)\alpha_d |x|^{d-2}} & d\geq 3 \end{cases}$$

$d=3$时物理意义为在原点的单位正电荷引起的在全空间 $\mathbb{R}^3$ 上的电势分布

· $\Phi(x)$ 的物理意义

$$\Phi(x)=\begin{cases}-\frac{1}{2}|x| & d=1\\ -\frac{1}{2\pi}\log|x| & d=2\\ \frac{1}{d(d-2)\alpha_d}\cdot|x|^{2-d} & d\geq 3,\ \alpha_d=\frac{\pi^{d/2}}{\Gamma(\frac{d}{2}+1)}\ d\text{维 unit-ball 的体积}\end{cases}$$

① $d=3$. $\Phi(x)=\frac{1}{4\pi}\frac{1}{r}$

$$\vec{E}=\frac{q}{4\pi\varepsilon_0 x^2}\cdot\frac{\vec{x}}{|x|}$$

$$\phi(r)=-\frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{|x|}$$

② $d=2$. Coulomb's Law should be $\vec{E}=\frac{q}{2\pi\varepsilon_0 r}\cdot\frac{\vec{r}}{|r|}$. Since fundamental solution is now $\frac{-1}{2\pi}\log|x|$

基本解在广义函数下满足 $\begin{cases}-\Delta\Phi(x)=\delta(x)\\ \text{here}: \Phi(x)=\Phi(|x|),\ \Delta\Phi(x)=0\ \forall x\neq 0\ \text{位势方程的特殊基本条件}\end{cases}$

在含点源⊕的任意体积内 $\int_{Vx}-\Delta\Phi(x)dx=-1=\iiint_V \nabla\cdot\nabla\Phi\, dV$

$$=\iint_S \Phi_r\, ds=4\pi x^2\Phi_r(x)$$

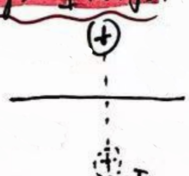
$$\therefore \Phi_r(x)=\frac{1}{4\pi x^2},\ \Phi(x)=\frac{1}{4\pi x}$$

· 格林函数 G

$$\begin{cases}-\Delta G(x,y)=\delta(x-y) & x\in U\\ G(\cdot;y)|_{\partial U}=0\end{cases}$$

case(1): $U=\mathbb{R}_+^d=\{(x_1,x_2,\cdots x_d): x_i>0\}$ half-space

$G(x,y)=\Phi(x,y)-\Phi(x,\bar{y})$, $\bar{y}=(-y_1,y_2,\cdots y_d)$

如同2维平面  ，求电势用镜像法

check: ①. $-\Delta G(x,y)=-\Delta_x\Phi(x,y)+\Delta_x\Phi(x,\bar{y})$

$$=-\Delta_x\Phi(x,y)$$

$$=\delta(x-y)$$

②. $G(x_0,y)=\Phi(|x_0-y|)-\Phi(|x_0-\bar{y}|)=0$ where $x_0=(0,x_2,\cdots x_d)$ 即满足 $G|_{\partial U}=0$

问题: $\begin{cases}G(x,y)=G(x-y)\text{吗}\\ -\Delta\Phi(x,\bar{y})=0\ \text{为什么}\\ \because\partial U\ \text{不只有}\ x_1=0\ \text{情况 if}\ x_0=(x_1,0,x_3,\cdots)\ \text{怎么办}\end{cases}$

因为 $\Delta\Phi(x,z)=\delta(x,z)=\begin{cases}0 & x\neq z\\ +\infty & x=z\end{cases}$

after symmetry reflection

$\bar{y}\neq x$ forever, since $\bar{y}\notin$ Domain

$\therefore \Delta\Phi(x,\bar{y})=0$

solve $\begin{cases} -\Delta u = 0, & U \\ u = g, & \partial U \end{cases}$

☆: 已知: $u(y) = \int_U \delta(x-y)\cdot u(x)\,dx = \int_U (-\Delta G)\cdot u(x)\,dx$

$$= \int_U (-\Delta u) G\,dx - \int_{\partial U} \frac{\partial G}{\partial n} u\,ds + \int_{\partial U} \frac{\partial u}{\partial n} G\,ds$$

$$= -\int_{\partial U} \left(\frac{\partial G}{\partial n}\right) g\,ds$$

这一步是恒成立的 case(1)和(2)都是

here $\int_{\partial U} \left(\frac{\partial G}{\partial n}\right) g\,ds = \int_{\partial U} \left(-\frac{\partial G}{\partial n}\right)(z;x)\, g(z)\, \overset{S(z)}{\underline{\underline{dz}}}, \quad z=(z_1,z_2)$

$$= \int_{-\infty \atop \forall z}^{+\infty} 2\partial_{z_1}\Phi(z;x)\Big|_{z_1=0} g(z)\,dz$$

?: 为什么从ds变成dz? z哪来, $\vec{n}$怎么找

为什么∂U是$z_1=0$. 前面也有这个疑问

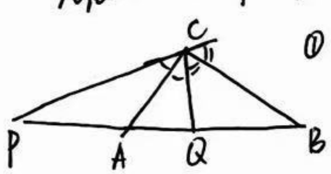
例: $d=2$ $\Phi(z;x) = -\frac{1}{4\pi}\log\left((z_1-x_1)^2+(z_2-x_2)^2\right)$

$$\partial_{z_1}\Phi(z;x)\Big| = -\frac{1}{2\pi}\cdot\frac{z_1-x_1}{(z_1-x_1)^2+(z_2-x_2)^2}\Big|_{z_1=0} = \frac{1}{2\pi}\cdot\frac{x_1}{x_1^2+(z_2-x_2)^2}$$

$$\therefore u(x_1,x_2) = \frac{1}{2}\int_{-\infty \atop \forall y}^{+\infty} \underbrace{\frac{x_1}{x_1^2+(y-x_2)^2}}_{\text{Poisson kernel}} g(y)\,dy$$

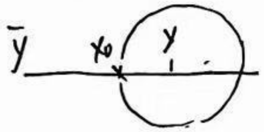
case(2): $U = B_R(0)$

考虑 circle of Apollonis



① $\frac{CP}{CQ} = \frac{AP}{AQ} = \frac{PB}{QB}$, $\forall C$, C形成的圆以$\frac{AB}{2}$为半径, mid(A,B)为圆心

② $\left|\frac{x-y}{x-\bar y}\right| = $ const., x轨迹也形成圆



$$= \left|\frac{x_0-y}{x_0-\bar y}\right|$$

所以性质 (

$$= \frac{R-|y|}{R^2/|y|-R} = \frac{|y|}{R}$$

、用①比较清楚

$\Rightarrow$ let $G(x,y) = \Phi(x-y) - \Phi\left(\frac{|y|}{R}(x-\bar y)\right)$ since $|x-y| = |x-\bar y|\cdot\frac{|y|}{R}$

☆ $\begin{cases} \frac{\partial\Phi}{\partial x_i}(x-y) = -\frac{1}{d\alpha_d}\frac{x_i-y_i}{|x-y|^d} \\ \frac{\partial\Phi}{\partial x_i}\left(\frac{|y|}{R}(x-\bar y)\right) = -\frac{1}{d\alpha_d}\frac{\frac{|y|^2}{R^2}(x_i-\bar y_i)}{\left|\frac{|y|}{R}(x-\bar y)\right|^d}, \quad \bar y = y\cdot\frac{R^2}{|y|^2},\ \bar y_i = y_i\cdot\frac{R^2}{|y|^2} \end{cases}$

summing over i: $\frac{\partial G}{\partial n}(x,y) = \sum_{i=1}^{d}\frac{x_i}{R}\left[\frac{\partial\Phi}{\partial x_i}(x-y) - \frac{\partial\Phi}{\partial x_i}\left(\frac{|y|}{R^2}(x_i-y_i)\right)\right]$

$$= -\frac{1}{d\alpha_d}\cdot\frac{1}{|x-y|^d}\times\frac{R^2-|y|^2}{R}$$

$\frac{\partial G}{\partial n} = \frac{\partial\Phi}{\partial n}(x-y) - \frac{\partial\Phi}{\partial n}\left(\frac{|y|}{R}(x-\bar y)\right)$, $\frac{\partial\Phi}{\partial n} = \frac{\partial\Phi}{\partial x_i}\cdot\frac{\partial x_i}{\partial R}$ ($\vec n = \vec R$) 所以应有 $\frac{\partial x_i}{\partial R} = \frac{x_i}{R}$?

接上面:

$$\frac{\partial G}{\partial n}(x,y)=\frac{\partial\Phi}{\partial n}(x-y)-\frac{\partial\Phi}{\partial n}\left(\frac{|y|}{R}(x-\bar{y})\right)$$

$$=\sum_{i=1}^{d}\frac{\partial\Phi}{\partial x_i}\cdot\frac{\partial x_i}{\partial n}(x-y)-\sum_{i=1}^{d}\frac{\partial\Phi}{\partial x_i}\cdot\frac{\partial x_i}{\partial n}\left(\frac{|y|}{R}(x-\bar{y})\right)$$

$$=\sum_{i=1}^{d}\frac{x_i}{R}\left[\frac{\partial\Phi}{\partial x_i}(x-y)-\frac{\partial\Phi}{\partial x_i}\left(\frac{|y|}{R}(x-\bar{y})\right)\right]$$

$\frac{\partial x_i}{\partial n}$

记 $R=\sqrt{x_1^2+\dots+x_d^2}$，$\frac{\partial R}{\partial x_i}=\frac{2x_i}{2\sqrt{\dots}}=\frac{x_i}{R}$ 反了?

$\vec{n}$: 方向怎么表示;

也不是这个表达式吗

$d\geq 3$ $\Phi(x)=\frac{1}{d(d-2)\alpha_d}|x|^{2-d}$

$$\Phi(x-y)=\frac{1}{d(d-2)\alpha_d}\cdot\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+\dots(x_d-y_d)^2}^{\,2-d}$$

$$\frac{\partial\Phi}{\partial x_i}(x-y)=\frac{1}{d(d-2)\alpha_d}\cdot\frac{2-d}{2}\left(\sum(x_i-y_i)^2\right)^{-\frac{d}{2}}\cdot 2(x_i-y_i)$$

$$=-\frac{1}{d\alpha_d}\times\frac{x_i-y_i}{|x-y|^d}$$

$$\frac{\partial\Phi}{\partial x_i}\left(\frac{|y|}{R}(x-\bar{y})\right)=\frac{1}{d(d-2)\alpha_d}\cdot\frac{2-d}{2}\left(\frac{|y|}{R}\right)^{2-d}\times\sum(x_i-\bar{y}_i)^{-\frac{d}{2}}\cdot 2(x_i-\bar{y}_i)$$

$$=-\frac{1}{d\alpha_d}\times\left(\frac{|y|}{R}\right)^{2-d}\times\frac{x_i-y_i}{|x-y|^d}$$

代入 $\frac{\partial\Phi}{\partial x_i}$ 得，$\frac{\partial G}{\partial n}(x,y)=-\frac{1}{d\alpha_d}\frac{1}{|x-y|^d}\cdot\frac{R^2-|y|^2}{R}$

$$\therefore u(y)=\int_{\partial U}\left(-\frac{\partial G}{\partial n}\right)g\,dS$$

$$=\int_{\partial B_r}\frac{R^2-|y|^2}{R}\cdot\frac{1}{d\cdot\alpha_d}\times\frac{1}{|x-y|^d}g(x)\,dS(x)\quad.\quad y\in B_R\quad S\text{用}x\text{表示}$$

此处 Poisson kernel 为 $k(x,y)=\frac{R^2-|y|^2}{R\cdot d\cdot\alpha_d}\cdot\frac{1}{|x-y|^d}$. for B_R

$$\Uparrow$$

$$\int k(x,y)\,dx=1$$