CH9 Cardinality in ZF

等势类中最小的序数

def9.4 an ordinal α is an cardinal if $|\alpha|\neq|\beta|$ for $\forall \beta<\alpha$

By transfinite induction, let ω_α be the α-th infinite ordinal which is a cardinal,

$\Rightarrow$ i.e. ω_α is the least ordinal whose cardinality is greater than ω_β for $\forall \beta<\alpha$, let $\aleph_\alpha$ denote the cardinality

$\omega_0=\omega$, ω_1 is the least uncountable ordinal

AC: any set X, $\exists$ ordering $<_X$ s.t. $(X,<_X)$ can be well-ordering

TH9.6 (ZF, Hartog's TH): $\forall$ set X, $\exists$ an ordinal which can't be injected into X

任意集合都有一个"太大"的序数，无法注入X

AC+CH: ordertype $>\omega_1$

or otp $=\omega_0$. (infinite)

pf: let $K=\{\alpha: \exists$ injection from α to $X\}$

$=\{\alpha: \exists$ wellordering of a subset of X of ordertype $\alpha\}$ … claim $\Leftarrow$ by ①+②

① if $\exists f:\alpha\to X$. f is injection

let $A=\text{range}(f)\subseteq X$, define $(A,<)$: $a<b \Leftrightarrow f^{-1}(a)\in f^{-1}(b)$,

$\left.\begin{array}{l}(\alpha,\in) \text{ is wellordering by the definition of ordinal,}\\ (A,<)\cong(\alpha,\in) \text{ is an order iso}\end{array}\right\} \Rightarrow (A,<)$ is wellordering

from the isomorphism, $\text{otp}(A,<)=\alpha$

② if $\exists A\subseteq X$, $(A,<)$ is wellordering, $\text{otp}(A,<)=\alpha$

$\therefore \exists$ order isomorphism $h:\alpha\to(A,<)$

$\therefore i\circ h:\alpha\to X$ is an injection, where i is the inclusion map

$\therefore K$ is a set by Replacement Axiom

if $\alpha\in K$, $\beta<\alpha$. $\exists$ injection $f:\alpha\to X$

the restriction $f|_\beta:\beta\to X$ is injection as well. $\therefore \beta\in K$, K is transitive

K is $\in$-well ordered since any $\alpha\in K$, $\exists$ least element by definition of ordinal

$\therefore K$ is an ordinal, $K\notin K$ 所有序数满足正规性 $\alpha\notin\alpha$，因为"$\in$"在ORD上是严格偏序

rmk: Replacement Axiom

F is class function. for $\forall$ set A, $\{F(a): a\in A\}$ is a set 要有"函数形式"的定义

$\therefore \{\alpha: \exists$ inj from α to $X\}$是不够的

wellordering on A is a subset of $X\times X$, $K=\{\text{otp}(A,<): (A,<)\in P(X\times X)\}$ is a set!

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ordinal α 可以 injected into X，说明 the cardinality $|\alpha|\le|X|$

$\Leftrightarrow \exists f:\alpha=\{\gamma:\gamma<\alpha\}\to X$, $f(\gamma_1)\neq f(\gamma_2)$ if $\gamma_1\neq\gamma_2$

$\therefore$ Hartog's TH 说明 $\forall$ set X, $\exists\,\alpha$ s.t. $|\alpha|>|X|$

def: $h(X) :=$ the least ordinal that can't be injected to X, called the Hartog's number of X

$\forall \alpha < h(X)$, α can be injected into X; $\therefore |\alpha| \le |X|$

$|h(X)| > |X| \ge |\alpha|$ for $\forall \alpha < h(X)$ $\Rightarrow$ $h(X)$ is a cardinality by def

由TH9.6知: $\forall$ ordinals α, $\exists$ ordinal λ of greater cardinality than α,

λ could be $h(\alpha) = \{\beta : |\beta| < |\alpha|\}$, ie. $|\{\beta : |\beta| < |\alpha|\}| > |\alpha|$

def 9.7 κ is a cardinal, then κ^+ denote the least ordinal of cardinality greater than κ

$\kappa^+ = h(\kappa)$ by def

表示为 $X \dot\cup Y = \{0\} \times X \cup \{1\} \times Y$

补充复习(最开始讲的) def 9.9 $|X| + |Y| = |X \dot\cup Y|$, $|X| \cdot |Y| = |X \times Y|$, $|Y|^{|X|} = |\{f : f : X \to Y\}|$

TH9.8 $|X| < |P(X)|$

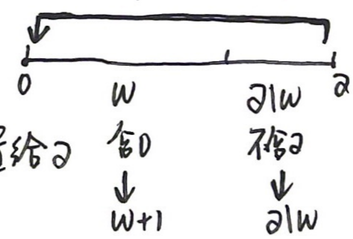
$i: X \to P(X)$, $i(x) = \{x\}$ is injection $\therefore |X| \le |P(X)|$

$f: X \to P(X)$ let $D = \{x \in X : x \notin f(x)\} \subseteq P(X)$, $D \ne f(a)$ for $\forall a \in X$ $\therefore |X| < |P(X)|$

Exe 9.11 for $\forall$ infinite ordinal α, $|\alpha| = |\alpha+1|$

$g: \alpha+1 \to \alpha$
$$\begin{cases} g(\alpha) = 0 & \\ g(n) = n+1 & n \in \omega \\ g(\xi) = \xi & \xi \in \alpha \setminus \omega \end{cases}$$

since $\alpha+1 = \{x : x < \alpha+1\} = \{x : x \le \alpha\}$



ω性质: 平移不变性, 所以可以移出一个位置给α

g inj. $|\alpha| \ge |\alpha+1|$

$|\alpha| \le |\alpha+1|$ $\Rightarrow$ $|\alpha| = |\alpha+1|$

prop: $\exists$ a strictly increasing continuous class function $F: ORD \to CARD$ s.t.
$F(0)=0$, $F(\alpha+1) = (F(\alpha))^+$, range$(F)$ = CARD)

即F把一切序数按次序"编号"为第α个基数，通常记 $\aleph_\alpha := F(\alpha)$

pf: def $F(0)=0$,

$F(\alpha+1) = (F(\alpha))^+$, successive cardinality exists by Hartog's Th

$F(\lambda) = \sup_{\beta<\lambda} F(\beta)$ if λ is limit ordinal

$\sup_{\beta<\lambda} F(\beta) = \bigcup_{\beta<\lambda} F(\beta)$ by property of ordinals，这里要证一下 $\sup F(\beta)$ is cardinal $\Rightarrow$ F well-defined ✓

if $\alpha<\gamma$, $\gamma \geq \alpha+1$ induct on γ. base case $\gamma=\alpha+1$; assume $\forall \alpha<\beta<\gamma$, $F(\alpha)<F(\beta)$

$F(\alpha) < F(\alpha)^+ = F(\alpha+1)$. $\therefore F(\alpha)<F(\gamma)$

if γ is not limit ordinal, $\exists \beta \in ORD$ s.t. $\beta+1=\gamma$ $\beta \geq \alpha$. $F(\beta)<F(\gamma)$ by previous step.
$F(\alpha)<F(\beta)$ by induction $\Rightarrow F(\alpha)<F(\gamma)$

if γ is limit ordinal, $F(\alpha) \leq \sup_{\beta<\lambda} F(\beta) = F(\lambda)$

$\therefore$ F is strictly increasing ✓

let $S = \{\kappa \in CARD : \exists \alpha \text{ s.t. } F(\alpha)=\kappa\} \subseteq CARD$, WTS: $S = CARD$

base case $0=F(0)\in S$

if $\kappa \in S$, then $\kappa = F(\alpha)$ some $\alpha \in ORD$; $\therefore \kappa^+ = F(\alpha+1) \in S$

if λ is limit cardinality, $\forall \mu<\lambda$, $\mu \in S$ ie. $F(\alpha_\mu)=\mu$, then $\alpha^* = \sup_{\mu<\lambda} \alpha_\mu$ is an cardinal

$\therefore \lambda \in S$

即transfinite induction在CARD也一样的，区分limit cardinal ($\lambda \neq \kappa^+$ $\forall \kappa \in CARD$) 和successive ($\lambda = \kappa^+$ some κ)

$\therefore S = \text{range}(F) = CARD$ ✓

记 $\Rightarrow \aleph_{\alpha+1} = (\aleph_\alpha)^+$, $\aleph_\lambda = \sup_{\beta<\lambda} \aleph_\beta$, $\aleph_0 = \omega$

注意：定义即保证了连续性 $F(\lambda) = \sup_{\beta<\lambda} F(\beta)$ 说明在每个极限点没有跳跃

所有可能的"大小" $0, 1, 2, \cdots, \aleph_0, \aleph_1, \aleph_2, \cdots$ 这条刻度线"不回头"(严格增)，"不断开"(极限连续)

$\alpha=0, \alpha=1, \alpha=2, \dots$

def: X is a set, λ is a cardinal, $[X]^\lambda = \{A \subseteq X : |A| = \lambda\} \subseteq P(X)$

X中势为λ的子集集合

Lem9.12 (ZF) 设 α is an infinite ordinal, then $|\alpha\times\alpha|=|\alpha|$

pf: induct on α

base case $\alpha=\omega$. $|\omega\times\omega|=|\omega|$ obv

设 $\forall\beta<\alpha$. $|\beta\times\beta|=|\beta|$ 设 α is not cardinal, then $\exists\beta<\alpha$ s.t $|\beta|=|\alpha|$ $\therefore|\alpha\times\alpha|=|\alpha|$

设 α is cardinal, define well-ordering $(\alpha\times\alpha,<)$ by:

$$(\beta,\gamma)<(\bar\beta,\bar\gamma)\Leftrightarrow\begin{cases}\max\{\beta,\gamma\}<\max\{\bar\beta,\bar\gamma\}\\ \max\{\beta,\gamma\}=\max\{\bar\beta,\bar\gamma\},\ \beta<\bar\beta\\ \max\{\beta,\gamma\}=\max\{\bar\beta,\bar\gamma\},\ \beta=\bar\beta,\ \gamma<\bar\gamma\end{cases}$$

this is the dictionary order of the tuple $(\max\{\beta,\gamma\},\beta,\gamma)$, well-ordered obv

let $x:=(\beta,\gamma)\in\alpha\times\alpha$. $\lambda:=\max\{\beta,\gamma\}+1<\alpha$

let $I_x:=\{y\in\alpha\times\alpha: y<x\}\subseteq\lambda\times\lambda$

$\therefore|I_x|\le|\lambda\times\lambda|=|\lambda|<|\alpha|=\alpha$, $\therefore otp(I_x)<\alpha$ $otp(I_x)\in\alpha$ 这是为了说明 f 良定义

$\therefore$ we can def $f:\alpha\times\alpha\to\alpha$, $f(x)=otp(I_x)\in\alpha$

f injective, $|\alpha\times\alpha|\le|\alpha|$

also def $g:\alpha\to\alpha\times\alpha$ $g(x)=(x,0)$ is injective $\therefore|\alpha\times\alpha|\ge|\alpha|$ $\Rightarrow|\alpha\times\alpha|=|\alpha|$

注意: 1. $|I_x|<|\alpha|=\alpha$, then $otp(I_x)<\alpha$ $\Rightarrow$ for any ordinal δ; $|\delta|<|\alpha|$, then $\delta<\alpha$

设 $otp(I_x)\ge\alpha$, $otp(I_x)=\{\gamma:\gamma<otp(I_x)\}\supseteq\alpha$

$\therefore\exists$ injection from α to $otp(I_x)$, ie. $|\alpha|\le|otp(I_x)|$ contradict

the converse is true

2. On any well-ordering $(W,<)$, $f:W\to ORD$, $f(w)=\underbrace{otp(\{u\in W: u<w\})}_{=otp(I_w)}$ is strictly increasing, thus injection

复习一下 lem2.4 Jech's. well-ordering $(W,<)$, $f:W\to W$ increasing, then $f(x)\ge x$ $\forall x\in W$

证明略

3. $x:=(\beta,\gamma)\in\alpha\times\alpha$, then $\lambda:=\max\{\beta,\gamma\}+1<\alpha$ elaborate on "$\lambda<\alpha$" instead of "$\lambda\le\alpha$"

claim: any infinite cardinal is limit

设 α is infinite cardinal, $\alpha=\beta+1$

Exe9.1 $|\alpha|=|\beta+1|=|\beta|$ contradict!

$\therefore(\beta,\gamma)\in\alpha\times\alpha$ implies $\max\{\beta,\gamma\}<\alpha$, thus $\lambda=\max\{\beta,\gamma\}+1<\alpha$

思路: 设 α is cardinal, 定义三元组字典序, $x:=(\beta,\gamma)\in\alpha\times\alpha$ $|I_x|\le|\lambda\times\lambda|=|\lambda|<|\alpha|=\alpha$ $otp(I_x)<\alpha$ 即 $otp(I_x)\in\alpha$

$f:\alpha\times\alpha\to\alpha$ $f(x)=otp(I_x)$ injective $|\alpha\times\alpha|\le|\alpha|$

实际这个是核心, 前面只是为了说明 $x\in\alpha\times\alpha$, $otp(I_x)\in\alpha$ (通过 $|I_x|\le|\alpha|$

Th: any infinite ordinal α, $|\alpha| = |[\alpha]^2| = |\{A \subseteq \alpha : |A| = 2\}|$

pf: $a \neq b \neq c$ in α, def $f: \alpha \to [\alpha]^2$

$$f(x) = \begin{cases} \{a, x\} & x \neq a, b \\ \{a, b\} & x = a \\ \{b, c\} & x = b \end{cases}$$, $f(x) \in [\alpha]^2$ obv, f injective $\therefore |\alpha| \le |[\alpha]^2|$

induction on α, to show $|[\alpha]^2| \le |\alpha|$

base case ω, $f: [\omega]^2 \to \omega$. $f(\{m, n\} \in [\omega]^2) = 2^m \cdot 3^n$ injective, $\therefore |[\omega]^2| \le |\omega|$

设 $\forall \beta < \alpha$, $|\beta| = |[\beta]^2|$, 若 α is not a cardinal, then $\exists \beta < \alpha$ s.t $|\beta| = |\alpha|$

$\therefore \exists g: \alpha \to \beta$ is bijection, then $\exists G: [\alpha]^2 \to [\beta]^2$ is bijection

$G(\{x, y\} \in [\alpha]^2) = \{g(x), g(y)\} \in [\beta]^2$

$\therefore |[\beta]^2| = |[\alpha]^2| = |\beta| = |\alpha|$ ...(1)

若 α is a cardinal, then α is a limit ordinal

设 $A \in [\alpha]^2$, $A = \{\xi, \eta\}$: $\xi < \alpha$, $\eta < \alpha$

let $\lambda = \max\{\xi, \eta\} < \alpha$, then $A \in [\lambda+1]^2$, $\therefore [\alpha]^2 \subseteq \bigcup_{\lambda < \alpha} [\lambda+1]^2$ ... ①

设 $A \in [\lambda+1]^2$, $\lambda < \alpha$ implies $\lambda+1 < \alpha$ since α is limit

$\therefore A \in [\alpha]^2$, ie. $[\alpha]^2 \supseteq \bigcup_{\lambda < \alpha} [\lambda+1]^2$ ...②

① + ② $\Rightarrow [\alpha]^2 = \bigcup_{\lambda < \alpha} [\lambda+1]^2$

$\therefore |[\alpha]^2| = |\bigcup_{\lambda < \alpha} [\lambda+1]^2| = \sup_{\lambda < \alpha} |[\lambda+1]^2| = \sup_{\lambda < \alpha} |\lambda+1| = \sup_{\lambda < \alpha} |\lambda| = |\alpha|$ ...(2)

(1) + (2) $\Rightarrow |\alpha| = |[\alpha]^2|$

Corollary: $|\alpha| = |\alpha \times \alpha|$

写成 $|\{0,1\} \times [\alpha]^2|$ 而非 $|[\alpha]^2|$ 是因为和 $|\alpha \times \alpha|$ 好对应

pf: $|\alpha| = |\{0,1\} \times \alpha| = |\{0,1\} \times [\alpha]^2| = |\alpha \times \alpha|$

$\{0,1\} \times [\alpha]^2 = \{(0, \{x, y\}) : x < \alpha, y < \alpha\} \cup \{(1, \{x, y\}) : x < \alpha, y < \alpha\}$, $x \neq y$

$\alpha \times \alpha = \{(x, y) : x < \alpha, y < \alpha, x \neq y\} \cup \{(x, x) : x < \alpha\}$

$|\{(x, y) : x < \alpha, y < \alpha, x \neq y\}| = |\{(0, \{x, y\}) : \cdots\} \cup \{(1, \{x, y\}) : \cdots\}|$ (x, y) 的顺序可以用 0, 1 表示

$\therefore |\alpha \times \alpha| = |\{0,1\} \times [\alpha]^2| + |\alpha|$

$= |\{0,1\} \times [\alpha]^2| + |[\alpha]^2|$

$= |\{0,1\} \times [\alpha]^2|$

注意: $\{0,1\} \times [\alpha]^2 = \{(0, \{x, y\}) : \cdots\} \cup \{(1, \{x, y\}) : \cdots\} \overset{记}{=} A \cup B$ $A \cap B = \phi$

$|A| = |B|$ obv, $|A \cup B| = |A| + |B| = |A|$ 无限基数相等, 则相加也不变

Exe9.13: $|X|, |Y|$ infinite, then $|X|+|Y| \le |X|\cdot|Y|$

pf: since X, Y infinite, $\exists$ injection $a: \omega \to X$, $b: \omega \to Y$

def $X \dot\cup Y = (X \times \{0\}) \cup (Y \times \{1\})$

def $f: X \dot\cup Y \to X \times Y$
$$\begin{cases} f(x,0) = (x, b(0)) & x \in X - \{a(n) : n \in \omega\} \\ f(y,1) = (a(0), y) & y \in Y - \{b(n) : n \in \omega\} \\ f(x,0) = (x, b(1)) & x \in \{a(n) : n \in \omega\} \\ f(y,1) = (a(1), y) & y \in \{b(n) : n \in \omega\} \end{cases}$$

f is injective, $\therefore |X|+|Y| \le |X|\cdot|Y|$ where we use Axiom of Choice here?

Corollary 9.14: $|\aleph_\alpha| + |\aleph_\beta| = |\aleph_\alpha|\cdot|\aleph_\beta| = |\aleph_{\max(\alpha,\beta)}|$，其中 $\aleph_\gamma$ is the γ-th infinite cardinal

pf: if $\alpha \le \beta$

$$|\aleph_\beta| \le \underbrace{|\aleph_\alpha| + |\aleph_\beta| \le |\aleph_\alpha|\cdot|\aleph_\beta|}_{\text{Exe9.13}} \le \underbrace{|\aleph_\beta|\cdot|\aleph_\beta| = |\aleph_\beta|}_{\text{infinite ordinal } \alpha \text{ satisfy } |\alpha| = |[\alpha]^2| = |\alpha\times\alpha| = |\alpha|\cdot|\alpha|}$$

$\therefore |\aleph_\alpha| + |\aleph_\beta| = |\aleph_\alpha|\cdot|\aleph_\beta| = |\aleph_{\max(\alpha,\beta)}|$

Exe9.15: α is a limit ordinal, then $\omega_\alpha = \sup_{\beta<\alpha} \omega_\beta$

by def: ω_α is the least ordinal whose cardinality is greater than ω_β, for $\forall \beta < \alpha$

$\quad\quad = \alpha$-th infinite ordinal that is a cardinal

pf: $|\omega_\alpha| > |\omega_\beta|$ $\forall \alpha > \beta$ $\therefore \omega_\alpha > \omega_\beta$ $\forall \alpha > \beta$ ($|\omega_\alpha| = \aleph_\alpha$)

$\therefore \omega_\alpha \ge \sup_{\beta<\alpha} \omega_\beta$ $\cdots$ ①

$\sup_{\beta<\alpha}\omega_\beta$ is a cardinal since if ordinal $\gamma < \sup_{\beta<\alpha}\omega_\beta$, then $\exists \beta < \alpha$ s.t. $\gamma < \omega_\beta$

$\therefore |\gamma| < |\omega_\beta| \le |\sup_{\beta<\alpha}\omega_\beta|$ $\cdots$ ③

$\sup_{\beta<\alpha}\omega_\beta > \omega_\beta$ for $\forall \beta < \alpha$

by def of ω_α, then $\omega_\alpha \le \sup_{\beta<\alpha}\omega_\beta$ $\cdots$ ②

①+② $\Rightarrow$ $\omega_\alpha = \sup_{\beta<\alpha}\omega_\beta$. ③ $\Rightarrow$ verify $\sup_{\beta<\alpha}\omega_\beta$ is a cardinal

Th9.15 (ZF) for $\forall$ infinite sets X, and Y, $|X|+|Y| = |X|\times|Y|$ then the axiom of choice is true