L8 Thm: $\forall$ finite Abel group is direct product of cyclic groups of prime-power order

$|G| = p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n}$ 1.中的H

1. $G = G_1 \times G_2 \times \cdots G_n$, $|G_i| = p_i^{r_i}$ 素数幂直积分解（每次拎一个 $p_i^{r_i}$ 出来找H）

2. $G_i = \langle a_{i1} \rangle \times \langle a_{i2} \rangle \times \cdots \langle a_{it_i} \rangle$ some t_i 循环群分解

1. 直积分解: $|G| = p^n \times m$, p prime, p^n, m coprime

$\Rightarrow G = H \times K$, $H = \{x \in G : x^{p^n} = e\}$, ~~$K = \{y, y^2, \cdots y^m = e\}$~~ $K = \{y \in G : y^m = e\}$ $= \langle y \rangle$, $|H| = p^n$, $|K| = m$

对于 Abel group G, $H \le G$, $K \le G$, $HK \times G \Leftrightarrow H \cap K = \{e\}$, $HK = G$

1.1 交集: 设 $x \in H \cap K$, $x^{p^n} = e$, $x^m = e$ since $x \in K$ $|\langle x \rangle| = o(g)$ 是 m 的因数

$o(g) \mid p^n$, $\gcd(p^n, m) = o(g)$ $\therefore o(g) = 1$ since p, m coprime

$\Updownarrow$ $H \cap K = \{e\}$

1.2 积: $\phi: HK \to G$

$(h, k) \to hk = g \in G$

- $\phi(h_1k_1) = h_1k_1$ $\phi(h_2k_2) = h_2k_2$ $\phi(h_1k_1h_2k_2) = h_1k_1h_2k_2 = \phi(h_1k_1) \cdot \phi(h_2k_2)$ homo

$g = h_1k_1 = h_2k_2$; $h_1 = h_2$, $k_1 = k_2$ since $H \cap K = \{e\}$. inj

- p, m coprime, $\therefore \exists z_1, z_2 \in \mathbb{Z}$ s.t. $z_1p^n + z_2m = 1$

$\forall g \in G$; $g = g^{z_1p^n + z_2m} = g^{z_1p^n} \times g^{z_2m}$ 常用

$\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad g_1 \quad\quad g_2$, $g_2 \in H$ obv

$(g_1)^m = e$, 但这种方法不说明 $g_1 \in K$ 把K换成 non-cyclic $\Rightarrow HK = G$

1.3. 阶数: $|HK| = \dfrac{|H| \cdot |K|}{|H \cap K|}$

$\therefore |H| \cdot |K| = p^n \times m$ since $p \nmid |K|$, $|H| = p^n$, $|K| = m$

实际上除了 Sylow 之后可不用 Abel

1.3.1: 引理: G is finite Abel, p prime $p \mid |G|$; 则 $\exists g \in G$, $o(g) = p$

suppose $|G| < n$, holds

① 任何 group 必有素数阶元素 $\Rightarrow$ $|G| = n$: arbitrarily find $x \in G$, $o(x) = q \ne p$, q prime

Since: $o(g) = xy$, $\gcd(x, y) = 1$ $\quad$ $\because \forall$ subgroup in Abel group is normal, $\langle x \rangle$ also

then $o(g^x) = y$, $o(g^y) = x$ $\quad$ $\Rightarrow \therefore \exists$ well-defined $G/\langle x \rangle$, $G/\langle x \rangle$ is also Abel

反面也用了这个

$|G/\langle x \rangle| = |G| \div |\langle x \rangle| = n \div q \in \mathbb{Z}^+$; $p \mid |G/\langle x \rangle|$ since p, q 互素

$|G/\langle x \rangle| < n$ thus $\exists g\langle x \rangle \in G/\langle x \rangle$, $|g\langle x \rangle| = p$

$(g\langle x \rangle)^p = g^p\langle x \rangle = e \in G/\langle x \rangle$ 即 $p\langle x \rangle$

下面

$(y\langle x\rangle)^p = y^p\langle x\rangle = \langle x\rangle$

∴ $y^p \in \langle x\rangle$, $o(\langle x\rangle)=q$ given

∴ $(y^p)^q = e$ 且q即为$o(y^p)$ ⟹ 如果 $o\langle x\rangle = q \in$ Prime；$\langle x\rangle \cong \mathbb{Z}_q$（或用拉格朗日）

完毕　　∴ $\langle x\rangle$中所有元素 $y\in\langle x\rangle$; $o\langle y\rangle = q$ $\forall y$ ①'

≈反面①

将m继续分成素数幂积，即证完 part I

then: 要将 $|H| \neq p^n$ 分解成 $\langle a\rangle\times K \to \langle a_1\rangle\langle a_2\rangle K' \to \langle\cdots$

2. G finite Abel group, |G| is prime-power order, $a\in G$为G中阶最大的元素,

⟹ $|G| = \langle a\rangle\times K$.

induct on $|G|$:, if $|G|<p^n$ prop holds

$|G|=p^n$: suppose $o(a)=p^m$. $m<n$ (if $m=n$, finished)

2.1 其余最小阶为p: find $b\in G\setminus\langle a\rangle$, with smallest order,

$o(b^p) = o(b)\div p < o(b)$ ∴ $b^p\in\langle a\rangle$

∴ $b^p = a^i$ some i

$b^{p^m} = (a^i)^{p^{m-1}} = e$ ∴ $o(a^i)<p^m$. a^i not the generator

$\langle a^i\rangle \neq \langle a\rangle$ ∴ $\gcd(p^m, i)\neq 1$ 即i与p^m有公因数

∴ let $i=pj$, $b^p=a^i=a^{pj}$

∴ $(ba^{-j})^p = e$, $o(ba^{-j})=p$, since $b\notin\langle a\rangle$ $b\neq a^j$

since $ba^{-j}\notin\langle a\rangle$, $o(b)$ should be p 不然与最小阶矛盾

最大阶　p阶

2.2 $\langle a\rangle\cap\langle b\rangle = \{e\}$

2.2.1 CAAP80: $\forall\, o(x)=n$, $\langle a^k\rangle = \langle a^{\gcd(n,k)}\rangle$, $|\langle a^k\rangle| = |\langle a^{\gcd(n,k)}\rangle| = n\div\gcd(n,k)$

⇒ $\langle a\rangle = \langle a^i\rangle \Leftrightarrow \gcd(n,i)=1 \Leftrightarrow$ 阶数相等　　HW写过！

⇒ $\langle a^i\rangle=\langle a^j\rangle \Leftrightarrow \gcd(i,n)=\gcd(j,n)\Leftrightarrow$ 阶数相等

∴ 设∃ $b^t = a^l \in\langle a\rangle\cap\langle b\rangle$　$\langle b^t\rangle = \langle b\rangle$ since $p\in$ Prime then $b\in\langle b\rangle\subseteq\langle a\rangle$ 矛盾

2.2.1' p为素数

∴ $\mathbb{Z}_p\cong\langle b\rangle$ $\mathbb{Z}_p$中每个元素为generator; $\langle b\rangle$中元素也是　≈①+①' 素数阶循环群

2.3 寻找商群（常规阶数induction的推导）

$|a\langle b\rangle| = p^m$, $a\langle b\rangle$在$G/\langle b\rangle$中阶最大

∴ $a\langle b\rangle\times H/\langle b\rangle \cong G/\langle b\rangle$；取 $K=\{k\in K: k\langle b\rangle\in H/\langle b\rangle\}$　$\langle a\rangle K = G$ ✓

Def: G为finite group, $x,y\in G$. $[x,y]=x^{-1}y^{-1}xy$: "commutator" of x,y

$H=\langle [x,y]: x,y\in G\rangle$ 为G的commutator group 记为G'

Prop1: $G'\lhd G$ 易证

例1. let $G=\langle a,b\rangle=D_{2n}$ $o(a)=n$, $o(b)=2$, $a^b=a^{-1}$; 求G', G/G'

($a^b = b^{-1}ab$ 共轭)

(1): $G'=\langle x^{-1}y^{-1}xy: x,y\in\langle a,b\rangle\rangle$

$x=a^ib^j$, $y=a^mb^k$; $k,j=0,1$, $m,i=1,2,\cdots n-1$

$[xy]=b^{-j}a^{-i}b^{-k}a^{-m}a^ib^ja^mb^k$

$=(b^{-1}a^{-i}b)\times(b^{-1}a^{-m+i}b^{-2})\cdot(b^{-1}a^mb)=a^i\cdot a^{m-i}\cdot a^m=e$ if $k=j=1$或0 or $k\neq j$

$=(b^{-j}a^{-i}b^j)(b^{-j+k}a^{-m+i}b^{j+k})(b^{-k}a^mb^k)$ →

Prop2: G/G' Abel

设$\bar{x}=xG'$, $\bar{y}=yG'$; $\bar{G}=G/G'$

由定义: $[x,y]\in G'$ ∴ $e\in\bar{G}=G'=[xy]G'$

∴ $[x,y]G'=x^{-1}y^{-1}xyG'=x^{-1}G'\cdot y^{-1}G'\cdot xG'yG'=(xG')^{-1}\cdot(yG')^{-1}\cdot xG'\cdot yG'$

$\Rightarrow \overline{x^{-1}y^{-1}xy}=\overline{x^{-1}}\cdot\overline{y^{-1}}\cdot\bar{x}\cdot\bar{y}=\bar{x}^{-1}\cdot\bar{y}^{-1}\cdot\bar{x}\cdot\bar{y}$

又∵ $[x,y]G'=e\in\bar{G}=G'$

即 $\overline{x^{-1}y^{-1}xy}=e=\bar{x}^{-1}\cdot\bar{y}^{-1}\cdot\bar{x}\cdot\bar{y}$ $\Rightarrow \bar{x}\bar{y}=\bar{y}\bar{x}$ $G/G'=\bar{G}$ Abel

Thm: $\forall H\lhd G$; G/H Abel $\Leftrightarrow G'\leq H$ $\Rightarrow$ G' is smallest subgroup s.t. G/G' Abel

$\Leftarrow$ $G'\leq H$ ∴ $G'\lhd H$, ∃ H/G'; 且 H/G' Abel since $H/G'\leq G/G'$;

∴ $G/H\cong(G/G')/(H/G')$

右式Abel, then 左式Abel

prop4用

$\Leftarrow$ $H\lhd G$, $K\lhd G$, $H\leq K$

$G/K\cong(G/H)/(K/H)$

$\Rightarrow$ G/H Abel

∴ $\bar{x},\bar{y}\in\bar{G}=G/H$ $\bar{x}\bar{y}=\bar{y}\cdot\bar{x}$

∴ $[\bar{x},\bar{y}]=e$, 即 $(xH)^{-1}(yH)^{-1}xH\cdot yH=G/H$, $x^{-1}y^{-1}xy\in H$

∴ $G'\leq H$

No.

Date

最终必有 $G^{(n)}=1$，或 $G^{(n)}=G^{(n+1)}$

不一定simple

如 $S_n \rhd A_n \rhd A_n' \cdots n\geq 5$

A_n simple $\therefore A_n=A_n'$

但S_n不simple

Prop3: $G \rhd G' \rhd G'' \cdots G^{(n)} \cdots$

since G finite, $G^{(n)}=G^{(n+1)}=\cdots$ for some n

例：G为non-Abel simple group 则 $G=G'$

Def3: $G=G' \Rightarrow G$ is perfect group

Def4: finite G is solvable: $1=G^{(n)}$ some n; $G^{(n)}=G^{(n+1)}$ 为单群定义 for G

Def5: $x, g\in G$, $x^g=g^{-1}xg$

函群solvable $x \to x^g=g^{-1}xg$ 是内自同构. induced by g

Prop4: G is solvable $\Leftrightarrow \exists\, G=G_0 \rhd G_1 \rhd G_2 \cdots \rhd G_n=1$ $\to$ WTS $G^{(n)}=1$ suffices to show $G^{(n)}\leq G_n$

其中 $G_{i+1}\lhd G_i$, G_i/G_{i+1} Abel

一定

"$\Leftarrow$" $G'\leq G_1$ since G/G_1 Abel (Th2)

Induction by $G^{(i)}\leq G_i$

then "$i+1$": G_i/G_{i+1} Abel; $\therefore G_i'\leq G_{i+1}$

$G^{(i+1)}=(G^{(i)})'\leq(G_i)'\leq G_{i+1}' \Rightarrow G^{(i)}\leq G_i\ \forall i.$

then $G_n=1$ implies $G^{(n)}=1$

Prop5: $\mathrm{Inn}(G)\cong G/Z(G)$

$\psi: G\to \mathrm{Auto}(G)$

$g \to \sigma_g: G\to G$

$h\to g^{-1}hg$

$\mathrm{Im}\psi=\{\sigma_g: h\to g^{-1}hg, g\in G\}$

$\mathrm{Im}\psi \lhd \mathrm{Aut}(G)$

$\ker\psi = Z(G)$

$\therefore$ 同态定理有 $\mathrm{Inn}(G)\cong G/Z(G)$

PS. sub的e和环/群的e不一定一样 如 $\begin{bmatrix}1&\\&1\end{bmatrix}$ 是 $\{\begin{bmatrix}a&b\\c&d\end{bmatrix}\}$ ring 的e

$\begin{bmatrix}1&\\&0\end{bmatrix}$ 是 $\{\begin{bmatrix}a&b\\c&0\end{bmatrix}\}$ subring 的e

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