3.9 Show that if U is a linear subspace of a normed space X then $\overline{U}$ is a closed linear subspace of X.

proof 3.9. U is subspace $\therefore \forall u,v \in U.\ a,b \in \mathbb{F},\ au+bv \in U.$

$\bar{U} = U \cup U' = U^{\circ} \cup U'$;

arbitrary $u,v \in \bar{U}$, ① if $u,v \in U' \setminus U$, then $\forall r>0$, $\exists\, x \in B_r(u)$, $y \in B_r(v)$

since $\forall a,b \in \mathbb{F}$, $ax \in U$, $by \in U$, $\|ax-au\| < |a| \cdot r$, $\|by-bv\| < |b| \cdot r$

$\therefore$ for $au+bv$, $\forall r>0$, $\exists\, ax+by \in B_R(au+bv)$, where $R = (|a|+|b|) r$

if $ax+by \neq au+bv$, $au+bv$ is limit point of U; or $au+bv = ax+by \in U$

$\therefore au+bv \in \bar{U}$ $\forall a,b \in \mathbb{F}$, $u,v \in U' \setminus U$

②: if $u \in U' \setminus U$, $v \in U$ (or $u \in U$, $v \in U' \setminus U$), $\forall r>0$, $\exists\, x \in B_r(u)$

$\forall a \in \mathbb{F}$, $\|ax-au\| < |a| \cdot r$

$\therefore$ similar to ①, $au \in U' \setminus U$, or $au = ax \in U$

if $au \in U' \setminus U$, $\forall b \in \mathbb{F}$, $\|ax+bv-(au+bv)\| < |a| \cdot r$, then $au+bv \in U' \subseteq \bar{U}$

if $au \in U$, $\forall b \in \mathbb{F}$, $au+bv \in U \subseteq \bar{U}$

$\therefore au+bv \in \bar{U}$, $\forall a,b \in \mathbb{F}$, $u \in U' \setminus U$, $v \in U$,

③ $u,v \in U$, $\forall a,b \in \mathbb{F}$. $au+bv \in U$ since U is subspace

①+②+③: $\bar{U}$ is closed under linear combination, $0 \in U \subseteq \bar{U}$ obv
$\bar{U}$ is closed subset
$\Big\} \Rightarrow \bar{U}$ is closed subspace

3.11 Show that if (f_n), $f \in C([0,1])$ and $\|f_n - f\|_\infty \to 0$ (uniform convergence) then $\|f_n - f\|_{L^p} \to 0$ (convergence in L^p) and $f_n(x) \to f(x)$ for every $x \in [0,1]$ ('pointwise convergence'). (For related results see Exercises 7.1–7.3.)

proof 3.11 (1): $\|f_n - f\|_{\infty} \to 0$

$\forall \varepsilon>0$, $\exists N$ s.t. $\|f_n - f\|_{\infty} = \max|f_n(x)-f(x)| < \varepsilon$. $\forall n \geq N$.

$\therefore \|f_n - f\|_{L^p} = \left(\int_0^1 |f_n(x)-f(x)|^p dx\right)^{\frac{1}{p}} < \left(\int_0^1 \varepsilon^p dx\right)^{\frac{1}{p}} = \varepsilon$ $\forall n \geq N$, $\Rightarrow \|f_n - f\|_{L^p} \to 0$,

(2): from (1), let $p=1$

then $\forall \varepsilon>0$, $\exists N$ s.t. $|f_n(x)-f(x)| < \varepsilon$ $\forall x \in [0,1]$

$\therefore f_n \to f$ uniformly convergent, thus pointwise convergent

3.12 Show that $c_0(\mathbb{K})$ is separable.

proof 3.12 $C_0(\mathbb{K})$: subspace of $l^{+\infty}(\mathbb{K})$; $C_0(\mathbb{K}) = \{x \in l^{+\infty}(\mathbb{K}) : x_i \to 0 ; as\ i \to +\infty\}$

$\forall \varepsilon > 0,\ x \in C_0(\mathbb{K}),\ \exists N,\ |x_i| < \varepsilon\ \forall i \geq N+1 \cdots (*)$

let: $Y_1 = \{(q_1, 0, 0, \cdots) : q_1 \in Q\}$

$Y_2 = \{(q_1, q_2, 0, 0, \cdots) : q_i \in Q\}$ ...

$Y_n = \{(q_1, q_2, \cdots q_n, 0, 0, \cdots) : q_i \in Q\}$ ..

$x \in C_0(\mathbb{K}) = (a_1, a_2, a_3, a_4 \cdots \overset{\to 0}{a_n}, \cdots)$

let $\begin{cases} |q_i - a_i| < \varepsilon. & \forall n \leq N, \\ |a_i| < \varepsilon. & \forall n \geq N \end{cases}$

Y_i countable, $\therefore \bigcup_{i=1}^{+\infty} Y_i$ countable, denote $E = \bigcup_{i=1}^{+\infty} Y_i$

$\therefore$ for x in $(*)$, consider $y \in Y_N$, $y_i = 0\ \forall i > N$ $\therefore |y_i - x_i| < \varepsilon\ \forall i \geq N+1 \cdots ①$

for $x_1, x_2, \cdots x_n, \cdots ②$

consider decimalism expansion $x_i = \sum_{j=1}^{+\infty} a_j (\frac{1}{10})^j,\ a_j \in \mathbb{N}^+$

$\forall \varepsilon > 0,\ \exists J \in \mathbb{N}^+$ s.t. $|a_{j+1}| (\frac{1}{10})^{j+1} < 9 \times (\frac{1}{10})^{j+1} < \varepsilon\ \ \forall j \geq J$

let $y_i = \sum_{j=1}^{J} a_j (\frac{1}{10})^j$, a_j decided by x_i, then $|y_i - x_i| < |a_{J+1}| \cdot (\frac{1}{10})^{J+1} < \varepsilon \cdots ②$ ✓

①+②. $E = \bigcup_{n=1}^{+\infty} Y_n$ countable

$\forall x \in C_0(\mathbb{K}),\ \forall \varepsilon > 0,\ \exists y \in E$ s.t. $\|x - y\|_{+\infty} = \max |x_i - y_i| < \varepsilon$ $\therefore E$ dense

$\Rightarrow C_0(\mathbb{K})$ is separable subspace

3.13 If $(X, \|\cdot\|_X) \simeq (Y, \|\cdot\|_Y)$ show that X is separable if and only if Y is separable.

proof 3.13: 证 X separable, let $E = \{x_1, x_2, \cdots x_n, \cdots\}$, $\overline{E} = X$ exists,

$\therefore \forall x \in X,\ \forall r > 0,\ \exists x_i \in E$ s.t. $x_i \in B_r(x)$

设 $T : X \to Y$ bijective, linear; $\exists c_1, c_2 > 0$ s.t. $c_1 \|x\|_X \leq \|Tx\|_Y \leq c_2 \|x\|_X\ \ \forall x \in X$

$\|x_i - x\|_X < r$ $\therefore \|Tx_i - Tx\| \leq c_2 \cdot r$

let $F = \{Tx_1, Tx_2, \cdots Tx_n \cdots\}$. countable

$\therefore \forall y \in T(X),\ \forall c_2 r = R > 0,\ \exists Tx_i \in F \subseteq T(X)$, s.t. $Tx_i \in B_R(y)$. $\therefore F$ dense in $Y \Rightarrow Y$ separable

similary, 证 Y separable, consider $T^{-1} : Y \to X$, $\Rightarrow X$ separable 和上面一样的 c_1, c_2 换 $c_1 c_2$

即可.

Definition 8 $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are isomorphic, write $X \simeq Y$, if $\exists$ a bijective linear map $T : X \to Y$ and $c_1 > 0, c_2 > 0$ such that

$$c_1\|x\|_X \le \|T(x)\|_Y \le c_2\|x\|_X \ \forall x \in X; \tag{1.14}$$

and are *isometric* if in addition T preserves the norm, i.e.,

$$\|T(x)\|_Y = \|x\|_X \ \forall x \in X,$$

and in this case, T is called an isometry from X to Y.

Metric space: sepa ⇔ 可数基

3.16 Show that a normed space $(X, \|\cdot\|)$ is separable if and only if

topo: 可数基 ⇒ sepa.

$$X = \overline{\bigcup_{j=1}^{\infty} X_j},$$

where the $\{X_j\}$ are finite-dimensional subspaces of X. (Zeidler, 1995)

"⇐" if $X = \overline{\bigcup_{i=1}^{\infty} X_i}$ than $\forall x \in X, \forall r>0, \exists a_i \in \mathbb{F}, x_i \in X_i$ s.t. $\|x - \sum_{i=1}^{\infty} a_i \cdot x_i\| < r \cdots (*)$

笔误

X_j finite dimensional ∴ has finite bases vector. ; (默认已知基数量 $= \dim(X_i)$ 已知)

let base of X_j be $\{x_{j1}, x_{j2}, \cdots x_{jd_j}\}$ $E = \{q_{11}x_{12}, q_{12}x_{12}, \cdots q_{jt}x_{jt} \cdots\}$ $\forall q_{jt} \in \mathbb{Q}$

then $\{x_{11}, x_{12}, \cdots x_{1d_1}; x_{21}, x_{22}, \cdots x_{2d_1}, \cdots x_{j1}, x_{j2}, \cdots x_{jd_j}, \cdots\}$ countable, denoted E.

$\forall x \in X, \forall r$. $\sum_{i=1}^{\infty} a_i \cdot x_i$ in $(*)$ $= \sum_{j=1}^{\infty} a_j \sum_{t=1}^{d_j} x_{jt} * b_t = \sum_{j=1}^{\infty}\sum_{t=1}^{d_j} a_j b_t \cdot x_{jt} \in E$,

∴ this vector in E (denoted v), $\|x - v\| < r$ ⇒ E dense ∴ X separable

"⇒" X has dense countable set $E = \{x_1, x_2, \cdots x_n, \cdots\}$ we can let $\|x_i\| = 1$ here (不相关)

$\forall x \in X, \forall r>0, \exists a_i \in \mathbb{F}$ s.t. $\|x - \sum_{i=1}^{\infty} a_i \cdot x_i\| < r \cdots (*)'$

let $V_n = \{q_1x_1 + q_2x_2 + \cdots q_nx_n : q_i \in \mathbb{Q}\}$. V_n is subspace obv

$\overline{\bigcup_{n=1}^{\infty} V_n} = X$ since ∵ $\sum a_ix_i$ in $(*)'$ can be approximated by $\sum q_ix_i$

let $|q_i - a_i| < (\frac{1}{2})^{i+M}$ ∵ 有理数集 dense in $\mathbb{R}$ ∴ 可以有 q_i 做到

∴ $\|x - \sum a_ix_i\| > \|x - \sum q_ix_i\| - \sum|q_i - a_i| \cdot \|x_i\| = \|x - \sum q_ix_i\| - (\frac{1}{2})^M$

∴ $\|x - \sum q_ix_i\| < (\frac{1}{2})^M + r$ for r in $(*)$ (m 可以 $\to \infty$) ; 即 $(*)'$ 仍满足

综上. X separable ⇔ $X = \overline{\bigcup_{i=1}^{\infty} X_i}$