# Normed Linear Spaces

Definition Let $X$ be a vector space over the field $\mathbb{K}$. A norm on $X$ is function on $\|\cdot\| : X \to [0, \infty)$, with the properties
i) $\|\lambda x\| = |\lambda|\|x\|$;
ii) $\|x + y\| \leq \|x\| + \|y\|$;
iii) $\|x\| = 0 \Leftrightarrow x = 0$.
for all $\lambda \in \mathbb{K}$ and $x, y \in X$. The pair $(X, \|\cdot\|)$ is said to be a normed space.

A function $p : X \to [0, \infty)$ is said to be a seminorm on $X$ if
(a) $p(\lambda x) = |\lambda| p(x)$,
(b) $p(x + y) \leq p(x) + p(y)$.

Every normed space is also a metric space with respect to the induced metric $d(x, y) := \|x - y\|$.

Convergence in a normed space is defined by the metric $d$, that is, $x_n \to x$ in $(X, \|\cdot\|)$ if $\|x_n - x\| \to 0$ as $n \to \infty$.

A Banach space is a normed space that is a complete w.r.t. the induced metric.

**Definition 1.1** A subset $K$ of a vector space $V$ is convex if whenever $x, y \in K$ the line segment joining $x$ and $y$, lies in $K$, i.e, for every $\lambda \in [0,1]$ we have $\lambda x + (1-\lambda)y \in K$.

**Lemma 1.1** In any normed space $(X, \|\cdot\|)$,
$\mathbb{B}_X := \{x \in X, \|x\| \leq 1\}$ and $B_X := \{x \in X, \|x\| < 1\}$ are convex.

**Lemma 1.2** Suppose that $N : X \to [0, \infty)$ satisfies
(i) $N(x) = 0$ if and only if $x = 0$;
(ii) $N(\lambda x) = |\lambda| N(x)$ for every $\lambda \in \mathbb{K}, x \in X$
and, in addition, that the set $B := \{x : N(x) \leq 1\}$ is convex.
Then $N$ satisfies

$$N(x+y) \leq N(x) + N(y) \tag{1.1}$$

and so $N$ defines a norm on $X$.

Proof We can assume that $N(x) > 0, N(y) > 0$. In this case $x/N(x) \in B$ and $y/N(y) \in B$ so using the convexity of $B$ we have

$$\frac{N(x)}{N(x)+N(y)}\frac{x}{N(x)} + \frac{N(y)}{N(x)+N(y)}\frac{y}{N(y)} \in B.$$

Thus

$$\frac{x+y}{N(x)+N(y)} \in B,$$

which implies that

$$1 \geq N\left(\frac{x+y}{N(x)+N(y)}\right) = \frac{N(x+y)}{N(x)+N(y)} \Rightarrow N(x+y) \leq N(x)+N(y).$$

$\square$

$f : [a,b] \to \mathbb{R}$, is convex if whenever $x, y \in [a,b]$ we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \ \forall \lambda \in (0,1). \tag{1.2}$$

If $f \in C^2(a,b) \cap C([a,b])$ and $f''(x) > 0 \ \forall x \in (a,b)$, then $f$ is convex.

Note that $s \to |s|^p$ is convex for all $1 \leq p < \infty$ and that $s \to e^s$ is convex.

Examples i) $(\mathbb{K}^n, \|\cdot\|)$ with

$$\|x\| = \|x\|_{l^2} = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \quad x \in \mathbb{K}^n. \tag{1.3}$$

ii) $(\mathbb{K}^n, \|\cdot\|_{l^p}),\ 1 \leq p < \infty$, where

$$\|x\|_{l^p} = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \quad x \in \mathbb{K}^n. \tag{1.4}$$

iii) $(\mathbb{K}^n, \|\cdot\|_{l^\infty})$ with

$$\|x\|_{l^\infty} = \max_{j=1,\cdots,n} |x_j|. \tag{1.5}$$

Lemma 1.3 (Minkowski's inequality in $\mathbb{K}^n$ ) For all $1 \le p \le \infty$, if $x, y \in \mathbb{K}^n$, then

$$\|x+y\|_{l^p} \le \|x\|_{l^p} + \|y\|_{l^p}. \tag{1.6}$$

Proof. The case $p = \infty$ is obvious. Suppose that $1 \le p < \infty$. By Lemma 1.2, we need only to show that

$$B = \{x \in \mathbb{K}^n : \|x\|_{l^p} \le 1\} = \{x \in \mathbb{K}^n : \|x\|_{l^p}^p \le 1\}$$

is convex. Note that $\forall 1 \le p < \infty$, $t \to |t|^p$ is convex. If $x, y \in \mathbb{K}^n$, $\lambda \in (0,1)$, then

$$\|\lambda x + (1-\lambda)y\|_{l^p}^p = \sum_{i=1}^n |\lambda x_i + (1-\lambda)y_i|^p$$

$$\le \sum_{i=1}^n (\lambda|x_i| + (1-\lambda)|y_i|)^p \le \sum_{i=1}^n (\lambda|x_i|^p + (1-\lambda)|y_i|^p) \le 1,$$

and so $\lambda x + (1-\lambda)y \in B$ and $B$ is convex. $\square$

Lemma 1.4 Let $\dim(V) < \infty$ and $\{e_j\}_{j=1}^n$ be a basis of $V$. Then

$$\|x\|_E = \left( \sum_{j=1}^{n} |\alpha_j|^2 \right)^{1/2}, \text{ for } x = \sum_{j=1}^{n} \alpha_j e_j \tag{1.7}$$

is a norm on $V$.

Proof. We check the triangle inequality. Let $x = \sum_{j=1}^n \alpha_j e_j, y = \sum_{j=1}^n \beta_j e_j \in V$; then

$$\|x+y\|_E = \|\alpha + \beta\|_{l^2} \leq \|\alpha\|_{l^2} + \|\beta\|_{l^2} = \|x\|_E + \|y\|_E,$$

where $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$. $\square$

Example iv) The sequence space $l^p(\mathbb{K}), 1 \leq p < \infty$ , consists of all $\mathbb{K}$- valued sequences $x = (x_j)_{j=1}^{\infty}$ such that the $l^p$ norm is finite, where

$$\|x\|_{l^p} = \left( \sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} ; \tag{1.8}$$

and $l^{\infty}(\mathbb{K})$ is the space of bounded sequences equipped with the norm

$$\|x\|_{\infty} = \sup_{j \in \mathbb{N}} |x_j|. \tag{1.9}$$

**Lemma 1.5** (Minkowski's inequality in $l^p(\mathbb{K})$ ) For all $1 \leq p \leq \infty$, if $x, y \in l^p(\mathbb{K})$, then $x + y \in l^p(\mathbb{K})$ and

$$\|x + y\|_{l^p} \leq \|x\|_{l^p} + \|y\|_{l^p}. \tag{1.10}$$

*Proof.* Let $1 \leq p \leq \infty$. We use (1.6) to get

$$\begin{aligned}\left(\sum_{j=1}^{n} |x_j + y_j|^p\right)^{1/p} &\leq \left(\sum_{j=1}^{n} |x_j|^p\right)^{1/p} + \left(\sum_{j=1}^{n} |y_j|^p\right)^{1/p} \\ &\leq \|x\|_{l^p} + \|y\|_{l^p}.\end{aligned}$$

We can take the limit as $n \to \infty$ to deduce (1.10). $\square$

Example v) On the Space C([0,1]), the collection of continuous functions on [0, 1], we define

$$\|f\| = \max_{x\in[0,1]} |f(x)|.$$

$(C([0,1]), \|\cdot\|)$ is a normed space.

Example vi) If $X$ is a metric space consider the space $C_b(X;\mathbb{K})$ of all bounded continuous functions from $X$ into $\mathbb{K}$. Then

$$\|x\|_\infty := \sup_{x\in X} |f(x)|$$

defines a norm on $C_b(X;\mathbb{K})$. If X is compact, then $C_b(X;\mathbb{K}) = C(X;\mathbb{K})$, and this norm is the same as the maximum norm

$$\|x\|_\infty := \max_{x\in X} |f(x)|.$$

Example vii) On the Space C([0,1]) for any $1 \leq p < \infty$,

$$\|f\|_p := \left( \int_0^1 |f(x)|^p dx \right)^{1/p}$$

defines a norm.

- If $x, y \in X$, then

$$|\|x\| - \|y\|| \leq \|x - y\|;$$

in particular, the map $x \to \|x\|$ is continuous from $(X, \| \cdot \|)$ into $\mathbb{R}$.

**Definition** Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space $X$ are equivalent - we write $\|\cdot\|_1 \sim \|\cdot\|_2$ -if there exist constants $0 < c_1 \leq c_2$ such that

$$c_1\|x\|_1 \leq \|x\|_2 \leq c_2\|x\|_1 \quad \text{for all } x \in X. \tag{1.11}$$

- The above notion of 'equivalence' is an equivalence relation.
- Two equivalent norms on $X$ induce the same topology on $X$.
- If $\|\cdot\|_1 \sim \|\cdot\|_2$ then

$$\|x_n - x\|_1 \to 0 \Leftrightarrow \|x_n - x\|_2 \to 0.$$

- If $(x_n), (y_n) \subset X$ with $x_n \to x, y_n \to y$ and if $(a_n) \subset \mathbb{K}^n$ with $a_n \to a$, then $\|x_n\| \to \|x\|;\ x_n + y_n \to x + y,$ and $a_n x_n \to ax.$
- If $1 \leq p \leq q \leq \infty$, then $l^p(\mathbb{K}) \subset l^q(\mathbb{K})$ and

$$\|x\|_{l^q} \leq \|x\|_{l^p},\ \forall x \in l^p(\mathbb{K}). \tag{1.12}$$

- Let $1 \leq p < q$. For any $C > 0$, we do not have

$$\|x\|_{l^p} \leq C\|x\|_{l^q}, \text{ for every } x \in l^1(\mathbb{K}). \tag{1.13}$$

In fact, define $x^{(n)} \in l^1$ by

$$x_j^{(n)} = \begin{cases} j^{-1/p}, & j = 1, \cdots, n \\ 0 & j = n+1, \cdots . \end{cases}$$

Then $\|x^{(n)}\|_{l^q} < \sum_{j=1}^{\infty} j^{-q/p} < \infty \ \ \forall n$ but $\|x^{(n)}\|_{l^p} = \sum_{j=1}^{n} j^{-1}$ is unbounded, so (1.13) cannot hold for any $C > 0$.

**Definition** $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are isomorphic, write $X \simeq Y$, if $\exists$ a bijective linear map $T : X \to Y$ and $c_1 > 0, c_2 > 0$ such that

$$c_1\|x\|_X \leq \|T(x)\|_Y \leq c_2\|x\|_X \;\forall x \in X; \tag{1.14}$$

and are *isometric* if in addition $T$ preserves the norm, i.e.,

$$\|T(x)\|_Y = \|x\|_X \;\forall x \in X,$$

and in this case, $T$ is called an isometry from $X$ to $Y$.

**Theorem 1.1** If $\dim(X) = n < \infty$, then any two norms on $X$ are equivalent.

**Proof.** Let $\{u_j\}_{j=1}^n$ be a base of $X$ and define

$$\left\|\sum_{j=1}^n x_j u_j\right\|_E = \left(\sum_{j=1}^n |x_j|^2\right)^{1/2}.$$

$\|\cdot\|_E$ is a norm on $X$. It suffices to show that any norm $\|\cdot\|$ on $X$ is equivalent to $\|\cdot\|_E$. We have

$$\left\|\sum_{j=1}^{n} x_j u_j\right\| \leq \beta \left\|\sum_{j=1}^{n} x_j u_j\right\|_E, \tag{1.15}$$

where $\beta = \left(\sum_{j=1}^{n} \|u_j\|^2\right)^{1/2}$. Note that
$f : (\mathbb{K}^n, \|\cdot\|) \to (X, \|\cdot\|_E)$, $f(x_1, \cdots, x_n) = \sum_{j=1}^{n} x_j u_j$ is an isometry. Thus,
$B := \{u \in X : \|u\|_E = 1\} = \{\sum_{j=1}^{n} x_j u_j : \sum_{j=1}^{n} |x_j|^2 = 1\}$ is a compact subset of $(X, \|\cdot\|_E)$. By (1.14), $\|\cdot\|$ is continuous on $(X, \|\cdot\|_E)$. Hence $\exists v_0 \in B$ such that $\|v_0\| \leq \|u\|$, $\forall u \in B$ and so $\|v_0\|\|w\|_E \leq \|w\|$, $\forall w \in X$. $\square$

• If $S$ is a nonempty subset of a linear space $X$, we denote by Span$(S)$ the collection of all finite linear combinations of elements of S, i.e.,

$$\mathrm{Span} S = \left\{ \sum_{i=1}^{n} x_i u_i, \ x_i \in \mathbb{K}, n \in \mathbb{N}, u_i \in S, i = 1, ..., n \right\}.$$

• $\mathrm{Span}(S)$ is a linear subspace of $X$, called the linear subspace spanned by $S$ and call $\mathrm{clin}(S) := \overline{\mathrm{Span(S)}}$ the closed linear span of $S$. If $X = \mathrm{Span}(S)$ we say that $S$ spans $X$ .

• A set $E \subset X$ is linearly independent if any finite collection of elements of $E$ is linearly independent.

• A Hamel basis for a vector space $X$ is any linearly independent spanning set.

• Every vector space has a Hamel basis.

**Lemma** If $X$ is a normed space, the following are equivalent:
(i) $X$ is separable (i.e. $X$ contains a countable dense subset);
(ii) The unit sphere in $X$, $S_X := \{x \in X : \|x\| = 1\}$, is separable;
(iii) $X$ contains a countable set $\{x_j\}_{j=1}^{\infty}$ s.t. $\overline{\mathrm{Span}\{x_j\}_{j=1}^{\infty}} = X$.

**Proof.** ii)$\Rightarrow$ iii): Let $\{x_j\}_{j=1}^{\infty}$ be a dense subset of $S_X$. If $0 \neq x \in X$ and $\epsilon > 0$, then from $x/\|x\| \in S_X$, $\exists x_k$ s.t.

$$\|x_k - x/\|x\|\| < \epsilon/\|x\|, \ i.e., \|x - x_k\|x\|\| < \epsilon,$$

and since $x_k\|x\| \in \overline{\mathrm{Span}\{x_j\}_{j=1}^{\infty}}$ this gives (iii).
iii)$\Rightarrow$ i): The collection of finite linear combinations of the $\{x_j\}$ with rational coefficients is countable. This countable collection is dense in $X$. □

- If $(X, \|\cdot\|)$ is a Banach space and $Y$ is a linear subspace of $X$ then $(Y, \|\cdot\|)$ is a Banach space if and only $Y$ is closed.

Lemma For $1 \leq p < \infty$ the space $l^p(\mathbb{K})$ is separable, but $l^\infty(\mathbb{K})$ is not separable.

Proof If $1 \leq p < \infty$, then the linear span of $\{e^{(j)}\}_{j=1}^\infty$, where $e_i^{(j)} = \delta_{ij}$ is dense in $l^p(\mathbb{K})$: given any $x \in l^p(\mathbb{K})$ and any $\epsilon > 0$ $\exists N$ s.t.

$$\sum_{j=n+1}^{\infty} |x_j|^p < \epsilon \quad \text{for every } n \geq N$$

and so

$$\left\| x - \sum_{j=1}^{n} x_j e^{(j)} \right\|_{l^p} = \left( \sum_{j=n+1}^{\infty} |x_j|^p \right)^{1/p} < \epsilon.$$

In the space $l^\infty(\mathbb{K})$, consider the uncountable set

$$S := \{x \in l^\infty : x_j = 0 \text{ or } 1 \text{ for each } j \in \mathbb{N}\}.$$

Any two distinct elements $x$ and $y$ in $S$ satisfy

$$\|x - y\|_{l^\infty} = 1$$

since they must differ by $1$ in at least one term. Any dense set $A$ of $l^\infty(\mathbb{K})$ must contain an uncountable number of elements: since $A$ is dense, for every $x \in S$ there must be some $x' \in A$ such that $\|x' - x\|_{l^\infty} < 1/3$. But if $x, y$ are distinct elements of $S$, then $x' \neq y'$ since

$$\|x' - y'\|_{l^\infty} > 1/3$$

by triangular inequality. Thus $A$ contains an uncountable number of elements.

Definition A normed space $(X, \|\cdot\|)$ is complete if every Cauchy sequence in $X$ converges in $X$ (to a limit that lies in $X$). A Banach space is a complete normed space.

Steps to show completeness:

(i) use the definition of what it means for a sequence to be Cauchy to identify a possible limit;

(ii) show that the original sequence converges to this 'possible limit' in the appropriate norm;

(iii) check that the 'limit' lies in the correct space.

**Theorem** For each $1 \le p \le \infty$ the space $l^p(\mathbb{K})$ is complete w.r.t. its standard norm.

**Proof** Let $1 \le p < \infty$ and $x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \cdots) \subset l^p(\mathbb{K})$ be a Cauchy sequence. Then $\forall \epsilon > 0, \exists N_\epsilon$ s.t.

$$\|x^{(n)} - x^{(m)}\|_{l^p}^p = \sum_{j=1}^{\infty} |x_j^{(n)} - x_j^{(m)}| < \epsilon^p \ \forall n, m \ge N_\epsilon. \quad (1.16)$$

Thus $(x_j^{(n)})_{n=1}^{\infty}$ is Cauchy in $\mathbb{K}$ for every $j$ and so for each $j \in \mathbb{N}$

$$x_j^{(n)} \to a_j \ \text{ as } \ n \to \infty$$

for some $a_j \in \mathbb{K}$. Set $a = (a_1, a_2, \cdots)$; then (1.16) implies that

$$\|x^{(n)} - a\|_{l^p}^p = \sum_{j=1}^{\infty} |x_j^{(n)} - a_j| < \epsilon^p \ \forall n \ge N_\epsilon,$$

and so $x^{(n)} - a \in l^p$ provided that $n \ge N_\epsilon$. This implies that $a \in l^p$ and $x^{(n)} \to a$ in $l^p(\mathbb{K})$.

**Theorem** Let X be a metric space and let $\mathbb{F}_b(X : \mathbb{K})$ be the collection of all functions $f : X \to \mathbb{K}$ that are bounded, i.e. $\sup_{x \in X} |f(x)| < \infty$. Then $\mathbb{F}_b(X : \mathbb{K})$ is complete with the supremum norm

$$\|f\|_\infty := \sup_{x \in X} |f(x)|.$$

**Remark**

$$\|f_n - f\|_\infty \to 0 \Leftrightarrow f_n \to f \text{ uniformly}.$$

If $(f_n)$ is a Cauchy sequence in $\mathbb{F}_b(X : \mathbb{K})$, then given any $\epsilon > 0$, $\exists N$ s.t.

$$\|f_n - f_m\|_\infty = \sup_{x \in X} |f_n(x) - f_m(x)| < \epsilon \ \ \forall m, n \geq N. \quad (1.17)$$

Thus for each $x \in X$, $(f_n(x))$ is a Cauchy sequence in $\mathbb{K}$ and so we can set

$$f(x) = \lim_{k \to \infty} f_k(x).$$

From (1.17) we have

$$|f_n(x) - f_m(x)| < \epsilon \ \ \forall m, n \geq N,$$

where $N$ does not depend on $x$. Letting $m \to \infty$ we obtain

$$|f_n(x) - f(x)| < \epsilon \ \ \forall n \geq N,$$

where $N$ does not depend on $x$ again. It follows that

$$\sup_{x \in X} |f_n(x) - f(x)| < \epsilon \ \ \forall n \geq N \tag{1.18}$$

and so $f_n \to f$ uniformly on $X$. Since $f_N$ is bounded and we have from (1.18) that $\|f_N - f\|_\infty < \epsilon$. Thus $f$ is bounded. $\square$

**Corollary** If $(X, d)$ is any metric space, then $C_b(X : \mathbb{K})$, the space of all bounded continuous functions from $X$ to $\mathbb{K}$, is complete when equipped with the supremum norm $\|f\|_\infty := \sup_{x \in X} |f(x)|$.

**Proof** Note that $C_b(X : \mathbb{K})$ is a linear subspace of $\mathbb{F}_b(X : \mathbb{K})$. It then suffices to show that $C_b(X : \mathbb{K})$ is a closed subspace. Let $(f_k) \in C_b(X : \mathbb{K})$ and $\|f_k - f\|_\infty \to 0$. Fix $x \in X$. Given $\epsilon > 0$, $\exists N$ such that $\|f_n - f\|_\infty < \epsilon/3$ for every $n \geq N$. Since $f_N \in C_b(X : \mathbb{K})$, $\exists \delta > 0$ s.t.
$d(x, y) < \delta \Rightarrow |f_N(y) - f_N(x)| < \epsilon/3$. Thus if $d(x, y) < \delta$, then

$$\begin{aligned} |f(y) - f(x)| &\leq |f(y) - f_N(x)| + |f_N(y) - f_N(x)| \\ &\quad + |f_N(x) - f(x)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon, \end{aligned}$$

which shows that $f$ is continuous at $x$. Since this holds for any $x \in X$, we conclude that $f \in C_b(X : \mathbb{K})$. $\square$

**Theorem** The space $C([a,b])$ of all continuously differentiable functions on $[a,b]$ is complete with the $C^1$ norm
$\|f\|_{C^1} = \|f\|_\infty + \|f'\|_\infty$.

**Proof** Let $(f_n) \in C([a,b])$ be Cauchy in $C^1$ norm. $\forall \epsilon > 0, \exists N$ s.t.

$$\|f_n - f_m\|_\infty + \|f_n' - f_m'\|_\infty < \epsilon \quad n, m \geq N. \tag{1.19}$$

Thus $(f_n)$ and $(f_n')$ are both Cauchy sequences in $C([a,b])$ with the supremum norm and so $\exists f, g \in C([a,b])$ s.t. such that $\|f_n - f\|_\infty \to 0, \|f_n' - g\|_\infty \to 0$. **Claim**: $g = f'$. In fact, from

$$\left| \int_a^x f_n'(t)dt - \int_a^x g(t)dt \right| \leq \int_a^x |f_n'(t) - g(t)|dt \leq (b-a)\|f_n' - g\|_\infty,$$

we have by taking $n \to \infty$ in $f_n(x) = f_n(a) + \int_a^x f_n'(t)dt$ that $f(x) = f(a) + \int_a^x g(t)dt$ and so $f' = g$. Hence $f \in C'([a,b])$.
Taking $m \to \infty$ in (1.19), we have
$\|f_n - f\|_\infty + \|f_n' - f'\|_\infty < \epsilon \;\; \forall n \geq N$ which shows that $f_n \to f$ in $C^1([a,b])$. $\square$

The space $C^k([a,b])$ of all $k$ times continuously differentiable functions on $[a,b]$ with the $C^k$ norm

$$\|f\|_{C^k} = \sum_{j=0}^{k} \|f^{(j)}\|_\infty, \text{ where } f^{(j)} = \frac{d^j f}{dx^j}.$$

is complete.

If $\{x_n\}$ is a sequence in a normed linear space $V$, the series $\sum_{n=1}^{\infty} x_n$ is said to converge to $x$ if $\sum_{n=1}^{N} x_n \to x$ as $N \to \infty$, and it is called absolutely convergent if $\sum_{n=1}^{\infty} ||x_n|| < \infty$

**Lemma** A normed linear space $V$ is complete iff every absolutely convergent series in $V$ converges.

**Proof.** If $V$ is complete and $\sum_{n=1}^{\infty} ||x_n|| < \infty$, let $S_N = \sum_{n=1}^{N} x_n$. Then for $N > M$, we have

$$||S_N - S_M|| \leq \sum_{n=M+1}^{N} ||x_n|| \to 0, \ as \ M, N \to \infty.$$

so the sequence $\{S_N\}$ is Cauchy and hence convergent.
Conversely, let $\{x_n\}$ be a Cauchy sequence. We can choose $n_1 < n_2 < \cdots$ such that $||x_n - x_m|| < 2^{-j}$ for $m, n \geq n_j$. Let $y_1 = x_{n_1}, \ y_j = x_{n_j} - x_{n_{j-1}}$ for $j > 1$. Then $\sum_{j=1}^{k} y_j = x_{n_k}$, and

$$\sum_{j=1}^{\infty} ||y_j|| \leq ||y_1|| + \sum_{j=1}^{\infty} \frac{1}{2^j} < \infty,$$

so $\lim_{k\to\infty} x_{n_k} = \sum_{j=1}^{\infty} y_j$ exists. But since $\{x_n\}$ is Cauchy, we know that $\{x_n\}$ converges to the same limit as $\{x_{n_k}\}$. $\square$

- If $(X, \|\cdot\|_X) \simeq (Y, \|\cdot\|_Y)$, then $(X, \|\cdot\|_X)$ is complete iff $(Y, \|\cdot\|_Y)$ is complete.

Proof Let $T : X \to Y$ be an isomorphism between $X$ and $Y$ with

$$c_1\|x\|_X \leq \|Tx\|_Y \leq c_2\|x\|_X, \ \forall x \in X$$

for some $c_1 > 0, c_2 > 0$. Assume that $Y$ is complete. Let $(x_n)$ be a Cauchy sequence in $X$; then $(Tx_n)$ is Cauchy in $Y$ and so $Tx_n \to y$ for some $y \in Y$. Let $x = T^{-1}y$; then

$$\|x_n - x\|_X \leq \frac{1}{c_1}\|Tx_n - y\|_Y \to 0.$$

Hence, $X$ is complete. $\square$

- If $X$ is a normed space and $\|\cdot\|_1)$ and $\|\cdot\|_2)$ are two equivalent norms on $X$, then $(X, \|\cdot\|_1)$ is complete $\Leftrightarrow$ $(X, \|\cdot\|_2)$ is complete.

• If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are complete, then $X \times Y$ is complete for the norm $\|(x, y)\|_1 := \|x\|_X + \|y\|_Y$ and also for the equivalent norm $\|(x, y)\|_2 := \left(\|x\|_X^2 + \|y\|_Y^2\right)^{1/2}$.

Proof If $(x_n, y_n)$ is a Cauchy sequence in $X \times Y$ using the norm $\|\cdot\|_1$, then from

$$\|(x_n, y_n) - (x_m, y_m)\|_1 = \|x_n - x_m\|_X + \|y_n - y_m\|_y$$

we know that $(x_n)$ is Cauchy in $(X, \|\cdot\|_X)$ and $(y_n)$ is Cauchy in $(Y, \|\cdot\|_Y)$. Thus $\exists x \in X, y \in Y$ s.t. $x_n \to x$ in $(X, \|\cdot\|_X)$, $y_n \to y$ in $(Y, \|\cdot\|_Y)$. Since

$$\|(x_n, y_n) - (x, y)\|_1 = \|x_n - x\|_X + \|y_n - y\|,$$

it follows that $(x_n, y_n) \to (x, y)$ in $(X \times Y, \|\cdot\|_1)$, and so this space is complete. $\square$

**Riesz's Lemma** Let $(X, \|\cdot\|)$ be a normed space and Y a proper closed subspace of $X$. Then $\exists x \in X$ with $\|x\| = 1$ such that $\|x - y\| > 1/2$ for every $y \in Y$.

**Proof** Let $x_0 \in X \setminus Y$ and set $d = d(x_0, Y) := \inf_{y \in Y} \|x_0 - y\|$. If $d = d(x_0, Y) = 0$, then $\exists (y_n) \in Y$ s.t. $\|y_n - y_0\| \to 0$, i.e. $y_n \to x_0$ and then since $Y$ is closed, we would have $x_0 \in Y$. Thus, $d > 0$.

Choose $y_0 \in Y$ s.t. $d \leq \|x_0 - y_0\| \leq 2d$ and set $x = \frac{x_0 - y_0}{\|x_0 - y_0\|}$; then $\|x\| = 1$, and for any $y \in Y$ we have

$$\begin{aligned}
\|x - y\| &= \left\| \frac{x_0 - y_0}{\|x_0 - y_0\|} - y \right\| \\
&= \frac{1}{\|x_0 - y_0\|} \|x_0 - [y_0 + y\|x_0 - y_0\|]\| \\
&\geq \frac{d}{\|x_0 - y_0\|} \geq \frac{d}{2d} = \frac{1}{2}.
\end{aligned}$$

$\square$

• If $(X, \|\cdot\|)$ is a Banach space and $Y$ is a linear subspace of $X$, then $(X, \|\cdot\|)$ is a Banach space if and only if $Y$ is closed.

• A finite-dimensional subspace of a normed linear space $X$ is complete and so is closed.

Theorem A normed space $X$ is finite-dimensional iff its closed unit ball is compact.

Proof Let $\dim X = \infty$. Take any $x_1 \in X$ with $\|x_1\| = 1$. Then $\mathrm{span}\{x_1\}$ is a proper closed linear subspace of $X$, so $\exists x_2 \in X$ with $\|x_2\| = 1$ and $\|x_2 - x_1\| \geq 1/2$. Now $\mathrm{Span}\{x_1, x_2\}$ is a proper closed linear subspace of $X$ so $\exists x_3 \in X$ with $\|x_3\| = 1$ and $\|x_3 - x_2\| \geq 1/2$ and $\|x_3 - x_1\| \geq 1/2$.
One can continue inductively to obtain a sequence $(x_n)$ with $\|x_n\| = 1$ and $\|x_i - x_j\| \geq 1/2$ whenever $i \neq j$. No subsequence of the $(x_n)$ can be Cauchy, so no subsequence can converge, from which it follows that the closed unit ball in $X$ is not compact. $\square$

- A set is precompact if its closure is compact.

Lemma A subset $A$ of a complete normed space $(X, \|\cdot\|)$ is precompact iff any sequence in $A$ has a Cauchy subsequence.

Proof

$\Rightarrow$: Let $A$ be precompact and that $(x_n) \in A$. Since $(x_n) \in \overline{A}$ and $\overline{A}$ is compact, $(x_n)$ has a convergent subsequence, and any convergent sequence is Cauchy.

$\Leftarrow$: Suppose that any sequence in $A$ has a Cauchy subsequence. Take a sequence $(y_n) \in \overline{A}$; then $\exists$ $(x_n) \in A$ such that $\|x_n - y_n\| < 1/n$. Let $x_{n_k}$ be a Cauchy subsequence of $(x_n)$; then $(y_{n_k})$ is Cauchy too, and so converges to a limit $y$, which is contained in $\overline{A}$ since this set is closed. This shows that $\overline{A}$ is compact. $\square$

Arzelá-Ascoli Theorem If $X$ is a compact metric space then $A \subset C(X : \mathbb{R})$ is precompact iff it is bounded ($\exists R > 0$ s.t. $\|f\|_\infty \leq R \; \forall f \in A$) and equicontinuous, i.e. $\forall \epsilon > 0 \;\; \exists \delta > 0$ s.t.

$$d(x, y) < \delta \Rightarrow |f(x) - f(y)| < \epsilon \text{ for every } f \in A, x, y \in X.$$

Proof $\Leftarrow$ : Take a countable set $(x_n)$ as below:

- (Exercise 2.14) In any compact metric space $\exists$ a countable subset $(x_n)_{n=1}^\infty$ s.t. $\forall \epsilon > 0 \; \exists M(\epsilon)$ such that for every $x \in X$ we have

$$d(x_j, x) < \epsilon \text{ for some } 1 < j < M(\epsilon).$$

Let $(f_j) \subset A$. We can use a 'diagonal argument' to find a subsequence (which we relabel) such that $f_j(x_k)$ converges for every $k$. The idea is to repeatedly extract subsequences to ensure that $f_j(x_k)$ converges for more and more of the $(x_k)$.

$(f_j)$ is bounded $\Rightarrow f_j(x_1)$ is bounded in $\mathbb{K} \Rightarrow \exists (f_{1,j}) \subset (f_j)$ s.t. $f_{1,j}(x_1)$ converges. Since $(f_{1,j}(x_2))$ is bounded in $\mathbb{K}$, $\exists (f_{2,j}) \subset (f_{1,j})$ s.t. $f_{2,j}(x_2)$ converges. Since $(f_{2,j}(x_1)) \subset (f_{1,j}(x_1))$, $(f_{2,j}(x_1))$ still converges.

We continue, extracting subsequences of subsequences, so that

$$f_{n,j}(x_i) \text{ converges for all } i = 1, ..., n.$$

Consider the sequence $(f_{m,m})_{m=1}^{\infty}$. This is a subsequence of $(f_j)$, and a subsequence of $(f_{n,j})$ once $m \geq n$. We have $(f_{m,m}(x_k))_{m=k}^{\infty} \subset (f_{k,j}(x_k))_{j=1}^{\infty}$. Thus

$$f_{m,m}(x_k) \text{ converges for all } k \in \mathbb{N}.$$

Set $g_m = f_{m,m}$. Let's show that $(g_m)$ is Cauchy in the supremum norm.

Given $\epsilon > 0$, since $(g_n) \subset (f_j)$ is equicontinuous $\exists \delta > 0$ s.t.

$$d(x,y) < \delta \Rightarrow |g_n(x) - g_n(y)| < \epsilon/3 \ \ \forall \ n, \ x, y.$$

$\exists M(\delta)$ s.t. $\forall x \in X$, $\exists \ x_i$ with $1 \le i \le M$ s.t. $d(x, x_i) < \delta$. Since $g_n(x_i)$ converges for every $i$, $\exists N$ s.t. if $n, m \ge N$, then

$$|g_n(x_i) - g_m(x_i)| < \epsilon/3, \ 1 \le i \le M$$

Thus, if $n, m > N$, we have for any $x$, by taking $i \in \{1, \cdots, M\}$ with $d(x, x_i) < \delta$ that

$$\begin{aligned} |g_n(x) - g_m(x)| \ \ &\le \ \ |g_n(x) - g_n(x_i)| + |g_n(x_i) - g_m(x_i)| \\ &\quad + |g_m(x_i) - g_m(x)| < \epsilon, \end{aligned}$$

which shows that if $n, m \ge N$, then $\|g_n - g_m\|_\infty \le \epsilon$ and so $(g_n)$ is Cauchy.

$\Rightarrow$: The boundedness of $A$ follows from the fact that $\overline{A}$ is compact.

Since $A$ is precompact, it is totally bounded. Thus, $\forall\ \epsilon > 0$, $\exists\{f_1, ..., f_n\}$ s.t. $\forall f \in A$, we have

$$\|f - f_i\|_\infty < \epsilon/3 \ \text{ for some } i \in \{1, ..., n\}.$$

Since the $f_j$ are all continuous functions on the compact set $X$ they are uniformly continuous, so $\exists \delta > 0$ s.t. $\forall i = 1, \cdots, n$

$$d(x, y) < \delta \Rightarrow |f_i(x) - f_i(y)| < \epsilon/3.$$

For any $f \in A$ choose $j$ such that $\|f - f_j\|_\infty < \epsilon/3$; then whenever $d(x, y) < \delta$ we have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| \\ &\leq \|f - f_j\|_\infty + |f_j(x) - f_j(y)| + \|f_j - f\|_\infty < \epsilon, \end{aligned}$$

so $A$ is equicontinuous. $\square$