

# Abstract Algebra

## : Lecture 9

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**Theorem 1.**  $A_n$  is a simple group when  $n \geq 5$ .

**Remark 2.** Let  $G = \text{Sym}(\Omega) = S_n$ , if  $|\Omega| = n$ .

- each element of  $G$  is a product of transpositions.
- each even permutation of  $\Omega$  is a product of some 3-cycles.

证明. Let  $N \triangleleft A_n$ , s.t.  $N \neq \{e\}$ . We aim to prove  $N = A_n$ .

Claim 1:  $N$  contains a 3-cycle.

Let  $e \neq g \in N$  s.t.  $|\text{Fix}(g)| \geq |\text{Fix}(x)|$  for all  $x \in A_n$ . In which  $\text{Fix}(x) = \{\omega \in \Omega \mid \omega^x = \omega, x \in S_n\}$  called the set of fixed points of  $x$ .

**Remark 3.** Since  $g$  is a even permutation,  $|\text{Fix}(g)| \leq n - 3$ .

If  $|\text{Fix}(g)| = n - 3$  then  $g$  is a 3-cycle as claimed.

To complete the proof of the claim, we assume that  $|\text{Fix}(g)| < n - 3$ . Then relabelling if necessary,

we may write  $g = \begin{cases} (12345 \dots) \dots \\ (1234)(5 \dots) \dots \\ (123)(45 \dots) \dots \\ (12)(34)(5 \dots) \dots \end{cases} \quad \text{or } g = (12)(34) \text{ which means that we write } g \text{ into a product}$

of disjoint cycles and put the longest cycle in the first place, and relabelling if necessary.

If  $\text{Fix}(g) \cap \{1, 2, 3, 4, 5\} = \emptyset$  and so  $\text{Fix}(g) \subseteq \{6, 7, \dots, n\}$ . In particular,  $1^g = 2$ .

Let  $\sigma = (345)$  and  $h = [\sigma, g]$ . Then  $h \in N$  and  $1^\sigma = 2^{g^{-1}} = 1$ . So  $1 \in \text{Fix}(h)$  and  $h$  fixes each point in  $\text{Fix}(g)$ . Thus  $|\text{Fix}(h)| \geq |\text{Fix}(g)| + 1$ , and  $h = 1$  by our assumption. Then  $[\sigma, g] = 1 \Rightarrow \sigma g = g \sigma$  it is obviously wrong, just need to consider the result of conjugacy action of  $\sigma$ .

This is a contradiction, so  $g$  is not a product of two disjoint cycles. It is just a 3-cycle as we claimed. □

**Remark 4.** If  $g = (12)(34)$  then  $h = (354)$  also has more fixed points.

**Lemma 5.** Let  $g = (ijk \dots)$ ,  $\sigma \in S_n$ ,  $g^\sigma = (i^\sigma j^\sigma k^\sigma \dots)$ .

**Remark 6.** 1.  $|\text{Fix}(g)| = n - 3 \Leftrightarrow g$  is a 3-cycle.

2. As  $g$  is even, if  $g$  is not a 3-cycle, then either  $g$  moves at least 5 points or  $g$  is a product of two disjoint transpositions.

Claim 2:  $N$  contains all 3-cycles.

**Remark 7.** each 3-cycle is conjugate to  $(123)$  by even permutations. Let  $(ijk)$  be an arbitrary 3-cycle. Then there exists  $x \in S_n$  such that  $(ijk)^x = (123)$ .

1. If  $\{i, j, k\} \cap \{1, 2, 3\} = \emptyset$ , then let  $x = (1i)(2k)(3j)(23)$ . Then  $(ijk)^x = (123)$ .

2. Assume  $|\{i, j, k\} \cap \{1, 2, 3\}| = 1$ . WLOG,  $i = 1$  and  $\{j, k\} \cap \{2, 3\} = \emptyset$ . Let  $x = (2j)(3k)$ . Then  $(ijk)^x = (123)$ .

3. Assume  $|\{i, j, k\} \cap \{1, 2, 3\}| = 2$ . WLOG,  $i = 1$ ,  $j = 2$  and  $k = 4$ . Let  $x = (k35)$ .

That means all 3-cycles are in  $N$  since  $N \triangleleft A_n$ , and each element of  $A_n$  is a product of 3-cycles so  $N = A_n$ .

**Remark 8.** If  $G$  is simple then for all  $e \neq g \in G$ , the normal closure  $g^G = G$  where  $g^G = \langle g^x | x \in G \rangle$ .

Permutation Group

**Definition 9.** For a set  $\Omega$ , each subgroup of  $\text{Sym}(\Omega)$  is called a permutation group on  $\Omega$ .

**Theorem 10.** (Cayley) Every finite group is isomorphic to a permutation group.

证明. Let  $G$  be a group,  $G = \{g_1, g_2, \dots, g_n\} = \Omega$ . For each element  $x \in G$  define a permutation on  $\Omega$  by  $\hat{x} : g_i \mapsto g_i x$ . Then  $\hat{x} \in \text{Sym}(\Omega)$  (the proof of  $\hat{x}$  is a bijection is easy to check).

Let  $\hat{G} = \{\hat{x} | x \in G\}$  then  $\hat{G}$  is a group which is isomorphic to  $G$  since  $\hat{x}\hat{y} = \widehat{xy}$ . Also  $\hat{G} \leq \text{Sym}(\Omega)$   $\square$

For any  $g_i, g_j \in G$  there exists  $\hat{x} \in \hat{G}$  s.t.  $g_i^{\hat{x}} = g_j$ . So  $\hat{G}$  is transitive on  $\Omega$ . And such  $\hat{x}$  is unique. So  $\hat{G}$  is regular on  $\Omega$  which means only the identity permutation fixes some point of  $\Omega$ .