CH10 Cofinality: measure how hard an ordinal is to approach from below

def10.1 α is an ordinal, $C \subseteq \alpha$; C is cofinal in α if $\forall \beta \in \alpha, \exists \gamma \in C$ s.t. $\gamma \geq \beta$

$\Downarrow$

ie. $\forall \gamma \in C, \gamma \in \alpha, \gamma < \alpha$

① 若 $\alpha+1$ is successive, then $\{\alpha\}$ is cofinal in $\alpha+1$ since:

$\alpha \in \alpha+1 \therefore \{\alpha\} \subseteq \alpha+1$; $\forall \beta \in \alpha+1$ $\beta \leq \alpha$, $\exists \gamma \in \{\alpha\}$ s.t $\overset{\gamma}{\alpha} \geq \beta$

(cofinal set not unique, but must contain α)

② 若 α is limit ordinal, $C \subseteq \alpha$ is cofinal $\Leftrightarrow \sup C = \alpha$ 这条对① 其实也是对的!

(not unique, $\alpha = \omega$, $C_1 = \{1, 2, 3 \cdots\}$ $C_2 = \{2, 4, 6, \cdots\}$)

def10.2 define $cf(\alpha)$: the least ordinal β s.t. $\exists f: \beta \to \alpha$ s.t. range(f) is cofinal in α

$\Rightarrow cf(\alpha) = \min\{\underset{}{\overset{\in ORD}{\beta}} : \exists f: \beta \to \alpha$, s.t. range($\beta$) cofinal in $\alpha\}$

$= \min\{|A| : A \subseteq \alpha, \sup A = \alpha\}$

证明略, 实际上可以加强成 Lem10.4

Exe10.3 若 α is limit ordinal, then $cf(\alpha) \geq \omega$ since $cf(\alpha) \geq cf(\omega) = \omega$

$\Rightarrow$ (1) $cf(\omega^{\omega}) = \omega$, since $\{\omega^n : n \in \omega\}$ cofinal in ω^{ω}

(2) $cf(\omega_\omega) = \omega$, since $\{\omega_n : n \in \omega\}$ cofinal in ω_ω

(3) $cf(\omega_1) = \omega_1$

复习定义: ω_ω: the least ordinal whose cardinality is greater than ω_β for $\forall \beta < \omega$

the ω-th infinite ordinal that is a cardinal

Lem10.4 α is a limit ordinal, then $cf(\alpha) = \min\{\beta \in ORD : \exists$ nondecreasing $f: \beta \to \alpha$ s.t. range(f) cofinal in $\alpha\}$

pf: $f: cf(\alpha) = \beta \to \alpha$, range($f$) is cofinal in α

then let $\bar{f}(\gamma) = \sup\{f(\delta) : \delta \leq \gamma\}$, then $\bar{f}: \beta \to \alpha$ non-decreasing, $\bar{f}(\beta)$ cofinal in α

Exe10.5. $\forall$ ordinal α, $cf(\alpha) =$ the shortest length λ of a strictly increasing sequence cofinal in α

$\parallel$

用 $\min\{|A| : A \subseteq \alpha, \sup A = \alpha\}$

$|A| = \lambda$ since $\exists$ bijection from A to the sequence

Ex10.6 α, β are limit ordinals, $\exists$ non-decreasing functions $f: \alpha \to \beta$, $g: \beta \to \alpha$ s.t. range(f) is cofinal in β, range(g) is cofinal in α; then $cf(\alpha) = cf(\beta)$

pf: by definition of $cf(\alpha)$, $cf(\beta)$

$\therefore \exists$ non-decreasing h, k s.t. $h: cf(\alpha) \to \alpha$ $k: cf(\beta) \to \beta$; $h(cf(\alpha))$ cofinal in α
$k(cf(\beta))$ cofinal in β

then $f \circ h: cf(\alpha) \to \beta$ is non-decreasing.

$\boxed{核心}$ 把$f: \alpha \to \beta$复合成 $f \circ h: cf(\alpha) \to \beta$
cofinal $\circ$ cofinal $\overset{还}{\Rightarrow}$ cofinal

$\sup(f \circ h(cf(\alpha))) = \sup f(\sup h(cf(\alpha))) = \sup f(\alpha) = \beta$

$\therefore$ rang$(f \circ h)$ cofinal in β

$\therefore cf(\beta) \leq$ dom$(f \circ h) = cf(\alpha)$ by claim (*).

similarly $cf(\beta) \geq cf(\alpha)$

$\Rightarrow cf(\beta) = cf(\alpha)$

claim(*) if α is limit ordinal, $\exists$ non-decreasing function $h: \lambda \to \alpha$, range$(h) = h(\lambda)$ cofinal in α

then $cf(\alpha) \leq \lambda$

pf: see $\bar{h}: \bar{\lambda} \to \alpha$ strictly increasing $\bar{h} = h|_{\bar{\lambda}}$, $\bar{\lambda} \subseteq \lambda$

rang$(\bar{h}) = \bar{h}(\bar{\lambda})$ cofinal in α as well

$cf(\alpha) =$ shortest length of $\uparrow$ sequence cofinal in α ("sequence" $\rightleftarrows$ range(function))

$\leq |\bar{h}(\bar{\lambda})| \leq |h(\lambda)| = |\lambda|$

因为单射

Lem10.7 $\forall$ ordinals α, $cf(cf(\alpha)) = cf(\alpha)$

pf: $f: \beta \to \alpha$, f non-de, rang$(f) = f(\beta)$ cofinal
$g: \gamma \to \beta$, g ---, rang$(g) = g(\gamma)$ ----

$\Rightarrow g \circ f: \gamma \to \alpha$ non-de, cofinal

by (*) claim: let $\beta := cf(\alpha)$, $\gamma := cf(cf(\alpha))$

$\therefore cf(\alpha) \leq \gamma = cf(cf(\alpha))$ ... ①

$cf(\gamma) = \min\{|A| : A \subseteq \gamma, \sup A = \gamma\} \Rightarrow \gamma \geq cf(\gamma)$ ... ②

①+②: $\therefore cf(\alpha) = cf(cf(\alpha))$, $\exists f: cf(\alpha) \to \alpha \ldots$, $\bar{g}: \lambda \to cf(\alpha) \ldots$

$f \circ \bar{g}: \lambda \to \alpha \ldots$

$\therefore$ by (*) claim. $cf(\alpha) \leq \lambda$

lem10.8: α not a cardinal, then $cf(\alpha) < \alpha$

pf: $\exists \beta < \alpha$ s.t $|\beta| = |\alpha|$, ie. $\exists$ bijection $f: \beta \to \alpha$

$f(\beta) = \alpha$, thus range(f) cofinal in α

$\therefore |cf(\alpha)| = \min\{\gamma : \exists f: \gamma \to \alpha$ s.t. $f(\gamma)$ cof in $\alpha\} \le \beta < \alpha$

lem10.9 $\forall \alpha \in ORD$, $cf(\alpha)$ is a cardinal

pf: if not, then $cf(cf(\alpha)) < cf(\alpha)$

contradict to lem 10.7

Def10.10. a cardinal k is regular if $cf(k) = k$; otherwise $cf(k) < k$, then k is singular.

($cf(k) \le k$ 必然的)

TH10.11 (ZFC). infinite successor cardinal k^+ is regular; ie $cf(k^+) = k^+$

pf: $cf(k^+) = \min\{|C| : C \subseteq k^+, C$ cofinal in $k^+\}$

$\therefore cf(k^+) = k^+ \Leftrightarrow \nexists C$ cofinal in k^+ s.t $|C| < k^+$

But we assume exist such C

$|C| < k^+$, $\therefore \forall \alpha \in C$, $|\alpha| < k^+$, $|\alpha| \le k$

by AC, we can find an $f_\alpha: \alpha \to k$ for $\forall \alpha \in C$, f_α injective

def: $F: \bigcup C \to C \times k$

$F(\beta) = (\alpha(\beta), f_{\alpha(\beta)}(\beta))$, $\alpha(\beta)$ is $\min\{\alpha : \beta \in \alpha, \alpha \in C\}$ 记住这个证明构造!

$\Rightarrow F$ injective $\quad \bigcup C = \sup C = k^+ \quad |C \times k| \le |k \times k| = |k|$

which implies $|\bigcup C| \le |C \times k|$ ... contradict!

Eg: 1. $\aleph_\omega$ is the first infinite singular cardinal

since $cf(\aleph_n) = \aleph_n$ $\forall n \in \omega$, ($\aleph_n$ is successor cardinal, then by TH10.11)

2. construct sequence $\alpha_0 = \omega$, $\alpha_{n+1} = \aleph_{\alpha_n}$

let $\alpha = \sup_n \alpha_n$, $\aleph_\alpha = \aleph_{\sup_n \alpha_n} \ge \aleph_{\alpha_n} = \alpha_{n+1}$ $\forall n$ $\therefore \aleph_\alpha \ge \sup_n \alpha_n = \alpha$ ... ①

$\aleph_\alpha = \sup_{\beta < \alpha} \aleph_\beta$. $\quad \because \sup_n \alpha_n = \alpha$ $\therefore \forall \beta < \alpha$ $\exists \alpha_n$ s.t. $\beta < \alpha_n$

$\therefore \aleph_\beta \le \aleph_{\alpha_n} = \alpha_{n+1} \le \alpha$ $\forall \beta < \alpha$

$\therefore \aleph_\alpha \le \alpha$ ... ②

①+② $\Rightarrow \alpha = \aleph_\alpha$, $cf(\alpha) = \omega$ since $\{\alpha_n\}$ cofinal in α, α is singular

定义复习：

(1). $\aleph_\alpha$: the α-th infinite cardinality

generated by transfinite recursion: $\aleph_0 := \omega$ 第一个无限基数，自然数集的势

$\aleph_{\alpha+1} :=$ the least cardinality greater by $\aleph_\alpha$ (successor card of $\aleph_\alpha$)

$\aleph_\lambda = \sup_{\lambda>\alpha} \aleph_\alpha$ 若 λ is limit ordinal

(2) ω_α: the least ordinal whose cardinality is greater than ω_β, for $\forall \beta<\alpha$

$\rightleftarrows$ the α-th infinite ordinal that is a cardinal ...(*)

α is limit ordinal, then $\omega_\alpha = \sup_{\beta<\alpha} \omega_\beta$

(3): $\aleph_\alpha = |\omega_\alpha|$, the cardinality of ordinal ω_α

"(*), ω_α is a cardinal" $\Rightarrow$ need to be proved! then $\aleph_\alpha = |\omega_\alpha| = \omega_\alpha$

TH10.12 (König). κ is a cardinal, then $\kappa < \kappa^{cf(\kappa)}$

用 κ 个函数 $f_\alpha: cf(\kappa)\to\kappa$ 试图枚举所有函数，总能构造出新函数 h，h 与 f_α 到在一点不同 ($\forall\alpha$)

$\therefore$ κ 个函数不够，$\kappa^{cf(\kappa)} > \kappa$，即 $cf(\kappa)\to\kappa$ 函数的数量严格更多

Pf: WTS: for $\forall$ sequence $\langle f_\alpha: cf(\kappa)\to\kappa, \alpha<\kappa\rangle$, $\exists h: cf(\kappa)\to\kappa$ s.t. for $\forall\alpha<\kappa$, $h\neq f_\alpha$

$cf(\kappa) = \min\{\beta: \exists$ non-dec $f:\beta\to\kappa$ s.t. range(f) cofinal in $\kappa\}$

$\therefore \exists g$ non-dec, s.t. $g: cf(\kappa)\to\kappa$, $\sup g(cf(\kappa)) = \kappa$

$\therefore \forall\alpha<\kappa, \exists\beta<cf(\kappa)$ s.t. $\alpha\le g(\beta)$ ... ①

for $\forall\beta<cf(\kappa)$, let $S_\beta := \{f_\alpha(\beta): \alpha\le g(\beta)\}$

$|S_\beta| = |\{\alpha: \alpha\le g(\beta)\}| \le |g(\beta)| < \kappa$ "$<\kappa$"是因为 $g(cf(\kappa))\le\kappa$ $\therefore \forall\beta\in cf(\kappa)$, $g(\beta)<\kappa$

$g(\beta) < g(cf(\kappa)) \le \sup g(cf(\kappa)) = \kappa$

let $h: cf(\kappa)\to\kappa$ ①: $\forall\alpha<\kappa, \exists\beta$ s.t.

$h(\beta) = \inf\{\kappa - S_\beta\}$ $h(\beta)\neq f_\alpha(\beta)$. $\Rightarrow h\neq f_\alpha$ $\forall\alpha$ 证毕

即找到 $S_\beta = \{$所有 $f_\alpha(\beta)\}$，发现 $S_\beta \subsetneq \kappa$，故 $\exists h(\beta)$ 取其它值

Coro 10.13 $cf(2^\kappa) > \kappa$

Pf: $(2^\kappa)^\kappa = 2^{\kappa\cdot\kappa} = 2^\kappa$

$(2^\kappa)^{cf(2^\kappa)} > 2^\kappa$ by TH 10.12 $\Big\} \Rightarrow (2^\kappa)^\kappa < (2^\kappa)^{cf(2^\kappa)}$, ie. $cf(2^\kappa) > \kappa$

def 10.14. gimel function $\gimel(\kappa) = \kappa^{cf(\kappa)}$, κ is infinite cardinal