2.9 Show that if (X, d_X) and (Y, d_Y) are separable, then $(X \times Y, \varrho_p)$ is separable, where ϱ_p is any one of the metrics from Exercise 2.2.

lemma: 数分HW 2.25: cpt K has ct base, therefore separable

其实可以证 metric space with ct base $\Leftrightarrow$ separable, HW2.23

$$\{V_{mr}\} = \left[\begin{array}{ccc} \vdots & \vdots & \vdots \\ B_{\frac{1}{3}}(x_1) & & \\ B_{\frac{1}{2}}(x_1) & B_{\frac{1}{2}}(x_2) & B_{\frac{1}{2}}(x_n) \\ B_1(x_1) & B_1(x_2) & B_1(x_n) \end{array}\right.$$

$\{V_\alpha\}$. $\forall x \in X$, $\forall$ open set $x \in G \subset X$; $\exists x \in V_\alpha \subset G$

即: $\forall$ open set G is union of subcollection of $\{V_\alpha\}$

$\Leftrightarrow \forall G \neq \emptyset$. $\forall x \in G$, $\exists\, x \in V_\alpha \subset G$

"$\Leftarrow$": let $\{x_1, x_2, \dots x_n \dots\}$ 为 metric space E 的 ct dense set

let $V_{mr} = \{y: d(y, x_m) < r\}$, $\{V_{mr}\}$ $m = 1, 2, \dots$, $r = \frac{1}{1}, \frac{1}{2}, \dots$ is countable

$\forall x \in X$, $\forall$ open $G \ni x$, $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq G$ since G open

if $x = x_i$ some i, V_{ir} with $r < \delta$ satisfies: $x \in V_{ir} \subseteq B_\delta(x) \subseteq G$

if $x \neq x_i$. x is limit pt of E, $\therefore$ for $r < \frac{1}{2}\delta$. $\exists V_{mr} \ni x$, $V_{mr} \subseteq B_\delta(x) \subseteq G$

"$\Rightarrow$": $\{V_n\}$ be the countable base in E 这一条甚至不用metric，一般topo即可

randomly take $x_i \in V_i$, $\{x_n\}$ countable $\Rightarrow$ base的要求实际上很高

$\forall x \in E$, $x \in B_r(x)$ for $\forall r > 0$, $x = x_n$ some n 点和开球之间以有 one

or: $\exists$ some V_i: $x \in V_i \subseteq B_r(x)$ $\therefore d(x_i, x) < r$.

that is $\forall x \notin \{x_n\}$, $\forall r > 0$ $\exists x_i$ s.t. $d(x_i, x) < r$, x limit pt

$\therefore \{x_n\}$ dense and countable

proof:

X, Y separable, 设 X, Y has countable base $\{X_n\}$ $\{Y_n\}$

$\begin{cases} \forall x \in X,\ x \in G.\ G \text{ is any open set in } X.\ \exists X_n \text{ s.t. } x \in X_n \subseteq G \\ \forall y \in X.\ y \in H. \cdots \text{同上} \cdots .\ \exists Y_m \text{ s.t. } y \in Y_m \subseteq H \end{cases}$

the set $\{(X_n, Y_m)\}_{n=1}^{\infty}{}_{m=1}^{\infty}$ is countable since $\{X_n\}$, $\{Y_n\}$ countable

$\forall (x, y) \in X \times Y$, $(x, y) \in G \times H$, $G \times H$ is open set in $X \times Y$ $\Rightarrow$ G open in X. H open in Y

$\therefore \exists X_n, Y_m$ s.t. $(x, y) \in X_n \times Y_m \subseteq G \times H$

$\therefore \{(X_n, Y_m)\} = \{X_n\} \times \{Y_m\}$ is countable dense base of $X \times Y$

2.10 Suppose that $\{F_\alpha\}_{\alpha\in\mathbb{A}}$ are a family of closed subsets of a compact metric space (X,d) with the property that the intersection of any finite number of the sets has non-empty intersection. Show that $\cap_{\alpha\in\mathbb{A}} F_\alpha$ is non-empty.

给了一族cpt集

有限交 $\neq\phi$

proof: 反证: if $\bigcap_{\alpha\in A} K_\alpha=\phi$, then pick any $K_0\in\{K_\alpha\}$; $\forall k\in K_0$, $k\notin\bigcap_{\alpha\in A,\alpha\neq 0} K_\alpha \Rightarrow k\in\bigcup_{\alpha\in A,\alpha\neq 0} K_\alpha^c$

核心: 设 $G_\alpha=K_\alpha^c$, $\bigcup_{\alpha\in A,\alpha\neq 0} G_\alpha \supseteq K_0$, G_α open

$k\subseteq(\cap K_\alpha)^c$ other

K_i is closed subset of compact set $\therefore K_0$ compact

$\Rightarrow \exists$ finite $G_1,\cdots G_n$ s.t. $\bigcup_{i=1}^{n} G_i\supseteq K_0$

$\therefore \forall k\in K_0, k\notin\bigcap_{i=1}^{n}K_i \Rightarrow K_0\cap(\bigcap_{i=1}^{n}K_i)=\phi$

finite intersection $=\phi$ contradiction! 假设不成立

$\exists F_0$ $F_0\cap(\bigcap_{A'} F_\alpha)=\phi \Leftrightarrow F_0\subseteq(\cap F_\alpha)^c=\cup F_\alpha^c=\bigcup_{i=1}^{n}F_i^c$

2.11 Suppose that (F_j) is a decreasing sequence $[F_{j+1}\subseteq F_j]$ of non-empty closed subsets of a compact metric space (X,d). Use the result of the previous exercise to show that $\cap_{j=1}^{\infty}F_j\neq\varnothing$.

proof: $F_1, F_2\cdots F_n\cdots$ compact obv. $F_i\neq\phi\ \forall i$

by T2.10, $F_{a_1}\cap F_{a_2}\cap\cdots F_{a_j}=F_a$, $a=\max(a_1,a_2\cdots a_j)$, $F_a\neq\phi$, $\forall$ finite $\{a_1,\cdots a_j\}\subseteq\{1,2,\cdots n,\cdots\}$

$\therefore \bigcap_{n=1}^{\infty}F_i\neq\phi$

2.12 Show that if S is a closed subset of $\mathbb{R}$, then $\sup(S)\in S$.

① if $\max(S)$ 存在, then $\sup(S)=\max(S)\in S$

② if not S中有 limit point since:

let $\sup(S)=x$, $y\leq x$ $\forall y\in S$

claim: $\forall\varepsilon>0$, $\exists y$ s.t. $y>x-\varepsilon$, $y\neq x$, $y\in S$

反证. $\exists\varepsilon$, $y>x-\varepsilon$ only if $y=x \Rightarrow x$ is the maximal of S, $x\in S$. (case ①)

by the claim: $\forall r>0$, $B_r(x)\setminus\{x\}\cap S\neq\phi$

x is limit point of S, S is closed set $\Rightarrow x\in S$

①+②: $\sup(S)\in S$ if S is closed in $\mathbb{R}$

2.13 Show that if $f : (X, d_X) \to (Y, d_Y)$ is a continuous bijection and X is compact, then f^{-1} is also continuous (i.e. f is a homeomorphism).

proof: 设 $f^{-1} = g : Y \to X$. WTS: $\forall$ closed set $G \subseteq X$, $g^{-1}(G) = f(G)$ is closed in Y

(已证 $f : X \to Y$ cts $\rightleftarrows$ f^{-1}(open set $\subseteq Y$) is open in X …… (#))

Lem1: $f : X \to Y$ cts $\rightleftarrows$ $\forall$ closed set $F \subseteq Y$, $f^{-1}(F)$ is closed in X

"$\to$" F is closed in Y. $\therefore F^c$ is open in Y

f is cts $\therefore f^{-1}(F^c)$ is open in Y …… by (#) 故证明 $f^{-1}(F^c) = f^{-1}(F)^c$ 即可

$x \in f^{-1}(F)^c \iff f(x) \notin F.\ f(x) \in F^c \iff x \in f^{-1}(F^c) \therefore f^{-1}(F^c) = f^{-1}(F)^c$

$\therefore f^{-1}(F)^c$ is open, $f^{-1}(F)$ closed in X

"$\leftarrow$" 已知右侧条件, then $\forall$ open set $E \subseteq Y$, $f^{-1}(E^c) = f^{-1}(E)^c$ is closed

$\therefore f^{-1}(E)$ is open in $X \Rightarrow f$ is cts

证毕. $\therefore$ (#) 中 "open" 换成 "closed" 也行

Lem2: X is compact, $\Rightarrow$ subset closed $\overset{G}{}\subseteq X$, G is also compact (在 metric 和 $\mathbb{R}^n$ 均成立的)

$\forall \{x_n\} \subseteq G$, $\{x_n\} \subseteq X$ $\therefore \exists \{x_{n_k}\} \subseteq G$. $x_{n_k} \to x_0$

x_0 is limit point of G (obv), G closed $\Rightarrow x_0 \in G$. G compact

Lem1 + Lem2: G closed in compact $X \Rightarrow G$ compact

f cts. $\therefore f(G) = g^{-1}(G)$ is compact thus closed Y (cpt 必定 closed, 由子列极限点在空间中可知)

$\Rightarrow g$ cts $\therefore f^{-1}$ cts

2.14 Any compact metric space (X, d) is separable. Prove the stronger result that in any compact metric space there exists a countable subset $(x_j)_{j=1}^{\infty}$ with the following property: for any $\varepsilon > 0$ there is an $M(\varepsilon)$ such that for every $x \in X$ we have

?

$$d(x_j, x) < \varepsilon \quad \text{for some } 1 \le j \le M(\varepsilon).$$

$\Rightarrow \forall x \in X$, $x \in G$ (G is open set) $\exists B_\varepsilon(x_j)$ s.t. $x \in B_\varepsilon(x_j) \subseteq G$

$\therefore$ let $\varepsilon = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \cdots$ $\{B_{\frac{1}{m}}(x_n)\}_{m,n \in \mathbb{N}^+}$ is countable set

证明下一面

proof: $\exists \{x_n\}_{n=1}^{+\infty}$, $\forall \varepsilon>0$, $\exists M_\varepsilon$ s.t. $\forall x \in X$, $d(x_j, x) < \varepsilon$ some $j = 1, 2, \cdots M_\varepsilon$

反证，假设 $\forall \{x_n\}_{n=1}^{+\infty}$, $\exists \varepsilon$, $\forall M$ $\exists x \in X$ $d(x_j, x) > \varepsilon$, $\forall j \in \{1, 2, \cdots M_\varepsilon\}$

即：$\exists \varepsilon$, take any x_1, let $M=1$, $\exists x$ $d(x, x_1) > \varepsilon$, let $x_2 := x$

then let $M=2$, $\exists x'$ $d(x, x_i) > \varepsilon$. $i=1,2$, let $x_3 := x'$

then let $M=3$, ...

由上过程可以不断取出 x_{n+1}, s.t. $d(x_{n+1}, x_j) < \varepsilon$ $\forall j \le n$，构造出 $\{x_n\}_{n=1}^{+\infty}$

$\{x_n\}_{n=1}^{+\infty}$ do not have limit point by construction

$\therefore$ contradict with def "compact"，反证假设不成立.

j 不用固定． $\forall \varepsilon$, $\exists M$. $\forall x$, 有某个 x_j s.t. $j \in \{1, M\}$. $d(x_j, x) < \varepsilon$.

$$\bigcup_{i=1}^{+\infty} B(y_i^n, \tfrac{1}{n}) = X \Rightarrow \bigcup_{i=1}^{N_n} B(y_i^n, \tfrac{1}{n}) = X$$

let $\{x_n\} = y_1^1, y_2^1, \cdots y_{N_1}^1; y_1^2, y_2^2, y_3^2, \cdots y_{N_2}^2; \cdots \underbrace{y_1^n, y_2^n, \cdots}_{} y_{N_n}^n; | \cdots$

if $\varepsilon > \frac{1}{n}$.

$x \in B_{\frac{1}{n}}(y_i^n)$ $d(y_i^n, x) < \frac{1}{n}$