- Method of characteristics: characteristic ODE, Hamilton's ODE.
- Intersection of characteristics: examples of intersection and non-intersection, relation between the occurrence of intersection and the well-posedness of the equation.

Examples of Method of Characteristic

适用于：$\partial_t u + b(t,x,u)\cdot \partial_x u = f(t,x,u)$；

- $b(t,x,u)$ irrelevent to u

例 (HW1.1) $\begin{cases} \partial_t u + \partial_x u = -u + e^{x+2t} \\ u(0,x) = 0 \end{cases}$

特征线 $\dot{x} = 1$, $x = x_0 + t$

设 $\eta(t) = u(t,x) = u(t, x_0+t)$

$\eta'(t) = \partial_t u + \partial_x u \cdot \dot{x} = -u + e^{x+2t} \xrightarrow{\text{代入}u} -\eta(t) + e^{x_0+3t}$

转换成特征线上的ODE: $\eta'(t) = -\eta(t) + e^{x_0+3t}$

$\eta'(t)\cdot e^t + \eta(t)\cdot(e^t)' = e^{x_0+4t} = (\eta\cdot e^t)'$

$\therefore \eta e^t = \int e^{x_0}\cdot e^{4t} dt$

$u(t, x_0+t) = \frac{1}{4} e^{3t+x_0} + Ce^{-t}$

代入 $u(0,x) = 0$, $\therefore u(t, x_0+t) = \frac{1}{4} e^{3t+x_0} - \frac{1}{4} e^{-t+x_0}$

将"x_0+t"顶回 x $\quad u(t,x) = \frac{1}{4} e^{2t}\cdot e^x - \frac{1}{4} e^{-2t}\cdot e^x$ $\Rightarrow$ 这个方程在特征线上成立，∴在R成立

(Details in the last note) since实函数的值可以由Q决定

- $b(x,t,u)$ relevent to u
  - discontinues characteristic

例: $\begin{cases} \partial_t u + u\cdot \partial_x u = 0 \\ u(x,0) = \phi(x) = \begin{cases} -1, & x<0 \\ 1, & x>0 \end{cases} \end{cases}$

特征线 $\dot{x} = u$, $x = ut + x_0$

设 $\eta(t) = u(t,x)$; $\eta'(t) = \partial_t u + \partial_x u \cdot \frac{dx}{dt} = \partial_t u + u\cdot \partial_x u = 0$

$\Rightarrow$ 在特征线上 u 只与 x 有关

$u(t,x) = u(t, ut+x_0) = u(0,x_0) = u(0, x-ut)$

例：若虑 $\phi(x)=\begin{cases}-1 & x<-\varepsilon\\ \frac{x}{\varepsilon} & |x|\le\varepsilon\\ 1 & x>\varepsilon>0\end{cases}$

$$u(t,x)=\phi(x-ut)=\begin{cases}-1 & ,\ x-ut<-\varepsilon\\ (x-ut)/\varepsilon & ,\ |x-ut|\le\varepsilon\\ 1 & ,\ x-ut>\varepsilon\end{cases}$$

将u值代入：$\Rightarrow\begin{cases}-1 & ,\ x+t<-\varepsilon\\ x/(t+\varepsilon) & ,\ -t-\varepsilon\le x\le t+\varepsilon\\ 1 & ,\ x>t+\varepsilon\end{cases}$

Let $\varepsilon\to0$, $u(t,x)=\begin{cases}-1 & ,\ x+t<0\\ \frac{x}{t} & ,\ -t\le x\le t\\ 1 & ,\ x>t\end{cases}$; $\forall x,t$

$$=\begin{cases}-1 & ,\ x-ut<0\\ 1 & \ x-ut>0\end{cases}$$

$$\Rightarrow\begin{cases}-1 & ,\ x+t<0\\ 1 & ,\ x-t>0\end{cases}$$

若不考虑加入 ε，"$-t\le x\le t$"缺失

• intersection of characteristic

例：HW1.2 $\begin{cases}\partial_t u+u\cdot\partial_x u=0\\ u(0,x)=\phi(x)=\begin{cases}1 & x\le0\\ 1-x & 0<x\le1\\ 0 & x>1\end{cases}\end{cases}$，只有在 $t<t_s$ 下，特征线不相交

特征线为 $\dot{x}=u$，$x(t)=ut+x_0$

$\frac{du}{dt}=\partial_t u+\partial_x u\cdot\dot{x}(t)=0$ $\therefore u=(t,x(t))$ 在此时为定值

$\Rightarrow$ 让 $\dot{x}=u$，u与t无关，$u=u(t,x=ut+x_0)$ $\forall t$

$$u(x,t)=u(t,x_0+u_0t)=u(0,x_0)=\phi(x_0)=\begin{cases}1, & x\le0\\ 1-x_0, & 0<x\le1\\ 0, & x>1\end{cases}$$

$$u(t,x)=\phi(x-ut)\ 成立\ \forall\dot{x}=u\Rightarrow u(t,x)=\phi(x-ut)=\begin{cases}1 & x-ut\le0\\ 1-(x-ut) & x-ut\in[0,1]\\ 0 & x-ut>1\end{cases}$$ 为通解

(Cts 函数可以由 Ω 中的元素及对应值确定)

将u的具体值代入：$u(t,x)=\begin{cases}1 & x-1\cdot t\le0\\ 1-(x-ut) & x-ut\in[0,1]\\ 0 & x-0t>1\end{cases}$ $\rightarrow u=1-x+ut$，$u=\frac{x-1}{t-1}$，$x-ut=\frac{t-x}{t-1}\in[0,1]$ here $x-t>0$

$\therefore 1-t>0,\ x<1$

(1): $u(t,x)=1$ if $x\le0$ $u(t,x)=\frac{x-1}{t-1}$ if $t<1, x<1$

若 $t>1$ $u(t,x)=1$ 和 $\frac{x-1}{t-1}$ 部分重合 $\Rightarrow t<1$ $\therefore t_s=1$

(2): $t<1$ 时 $u(t,x)=\begin{cases}1 & x\le0\\ \frac{x-1}{t-1} & x<1,\ x>0\\ 0 & x>1\end{cases}$