Abstract Algebra

: Lecture 8

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Lemma 1. Let G be a abelian group with $|G| = p^e$. Let $g \in G$ which has the largest order. Then $G = \langle g \rangle \times H$ for some subgroup H < G.

证明. Assume G is not cyclic.

Claim: $\exists h \in G \text{ s.t. } \langle h \rangle \cap \langle g \rangle = 1 \text{ and } o(h) = p.$

Since $G \neq \langle g \rangle$, then $\exists y \in G - \langle g \rangle$ s.t. $y^p \in \langle g \rangle$. Then $\exists k \in \mathbb{Z}_{\geqslant 1}$ s.t. $y^p = g^{kp}$. Let $h = y^{-1}g^k$, then $h \neq 1$ and $h^p = 1$. Thus o(h) = p. And if $h \in \langle g \rangle$ then $y^{-1}g^k \in \langle g \rangle \Rightarrow y \in \langle g \rangle$, contradiction.

Let $\bar{G} = G/\langle h \rangle$. Then $|\bar{G}| < |G|$, and $o(g) = o(\bar{g})$ is the largest order of element in \bar{G} . By indction, Let $\bar{G} = \langle \bar{g} \rangle \times \bar{H}$. Let H be the full preimage of \bar{H} in G.

Claim: $\langle g \rangle \cap H = 1$.

Suppose there exists $x \in \langle g \rangle \cap H$. Then $\bar{x} = \bar{1}$ as $\langle \bar{g} \rangle \times \bar{H} = \bar{1}$, so $x \in \langle h \rangle$ and $x \in \langle g \rangle \Rightarrow \langle h \rangle \subseteq \langle g \rangle$, contradiction.

Therefore $G = \langle g \rangle \times H$.

Theorem 2. (Fundamental Theorem of Finite Abelian Groups) Let G be a finite abelian group of order n. Let $n = p_1^{e_1} \dots p_r^{e_r}$. Then:

- (1). $G = G_1 \times \cdots \times G_r \text{ where } |G_i| = p_i^{e_i}$.
- (2). G is a direct product of cyclic groups.

延明. (1). Let $n = p^e m$ s.t. p is a prime and (p, m) = 1. Let $H = \{g^m | g \in G\}$. Then H is a subgroup and every element of H has order p-power. Moreover $|H| = p^e$, and $G = H \times K$ where K has order m. By induction on K we can prove (1).

(2). By lemma 1.
$$\Box$$

Solvable Groups:

Definition 3. Let G be a finite group. For $x, y \in G$, let $[x, y] = x^{-1}y^{-1}xy$, called the commutator of x and y.

Definition 4. Let $H = \langle [x,y]|x,y \in G \rangle$. Then H is called the commutator subgroup of G, denoted by G'.

Proposition 5. $H \triangleleft G$.

证明.
$$\forall g \in G, [x,y]^g = g^{-1}[x,y]g = [x^g,y^g].$$

Example 6. 1. Let
$$G = \langle a, b \rangle = D_{2n}$$
, where $o(a) = n$, $o(b) = 2$ and $a^b = a^{-1}$. Then $G' = \begin{cases} \langle a \rangle, if \ o(a) \ odd, \\ \langle a^2 \rangle, if \ o(a) \ even. \end{cases}$ and $G/G' = \begin{cases} \langle \bar{a} \rangle = C_2, if \ o(a) \ odd, \\ \langle \bar{a}, \bar{b} \rangle = C_2 \times C_2, if \ o(a) \ even. \end{cases}$

Example 7. Let G be abelian. Then G' = 1.

Lemma 8. G/G' is abelian.

证明. Let
$$\bar{x}, \bar{y} \in \bar{G}$$
, let x, y be preimages of \bar{x}, \bar{y} under $\pi : G \to \bar{G}$ respectively. Then by definition $[x.y] = x^{-1}y^{-1}xy \in G'$, hence $[\bar{x}, \bar{y}] = \bar{x}^{-1}\bar{y}^{-1}\bar{x}\bar{y} = \overline{x^{-1}y^{-1}xy} = \bar{1} \Rightarrow \bar{G}$ is abelian.

Lemma 9. For any $H \triangleleft G$, G/H is abelian $\Leftrightarrow G' \leqslant H$.

证明. (\Leftarrow) If $G' \leqslant H$, then $G/H \simeq \frac{G}{G'}/\frac{H}{G'}$ is abelian.

(
$$\Rightarrow$$
) Assume G/H is abelian, then foe any $\bar{x}, \bar{y} \in \bar{G} = G/H$, $\bar{x}\bar{y} = \bar{y}\bar{x}$ i.e. $[\bar{x}, \bar{y}] = \bar{1}$, it shows $\forall x, y \in G, [x, y] \in H$. Thus $G' \leq H$.

We can come to the conclusion that G' is the smallest subgroup of G s.t. G/G' is abelian.

Definition 10. $G \triangleright G' \triangleright G'' \triangleright \cdots \triangleright G^{(n)} \triangleright \cdots$ where $G^{(n)} = (G^{(n-1)})'$. Since G is finite, there exists n s.t. $G^{(n)} = G^{(n+1)} = \cdots$

Example 11. If G is nonabelian simple, then G = G'.

Definition 12. If G = G', then G is called a perfect group.

Definition 13. A finite group G is called a solvable group if $G = G^{(n)}$ for some n. Otherwise, G is called a nonsolvable group.

Proposition 14. A group G is solvable iff there exists a subgroup chain:

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots \triangleright G_s = 1$$

s.t. G_{i+1} is normal in G_i and G_i/G_{i+1} is abelian for all i.

证明. (\Rightarrow) obviously.

 (\Leftarrow) Assume $G = G_0 \rhd G_1 \rhd \cdots \rhd G_s = 1$ s.t. G_i/G_{i+1} is abelian.

Claim: $G^{(i)} \leq G_i$ for all i.

First, $G' \leqslant G_1$ as G/G_1 is abelian. Suppose $G^{(i)} \leqslant G_i$ for some $i \geqslant 1$. Since G_i/G_{i+1} is abelian, $(G_i)' \leqslant G_{i+1}$. Thus $G^{(i+1)} = (G^{(i)})' \leqslant (G_i)' \leqslant G_{i+1}$.

Definition 15. Let $x, g \in G$, consider the conjugation action $x^g = g^{-1}xg$. It induces an automorphism of G s.t. $x \mapsto g^{-1}xg$, called the inner automorphism induced by g.