一、求解的其它方式. Another method to homogeneous equation

HW1 T3. $\begin{cases} \partial_t u = \partial_{xx} u + \partial_t u \\ u|_{t=0} = \delta(x) \end{cases}$

$\downarrow$ if $u(t,x)$ is a solution, $u(\lambda^2 t, \lambda x)$ is also a solution

$\begin{cases} \partial_t u = \partial_{xx} u \\ u|_{t=0} = \delta(x) \end{cases}$; this is a insymmetric problem ; $u \to \lambda^{\alpha} u(\lambda^{\beta} x, \lambda t)$

(1): "$\partial_t u = \partial_{xx} u$" Fourier transformation on "x"

$\partial_t \hat{u} = (\partial_x^2 u)^{\wedge} = (2\pi i \xi)^2 \hat{u}$ $\forall x$ $\Rightarrow$ "ODE" problem, $u(t,x) = g(t)$, fix x

(2): $u(t,x)$ invariant under: "fix the first slot"

$\begin{cases} u(t,x) \to \lambda^{\alpha} u(\lambda t, \lambda^{\beta} x) \ \forall \lambda \ \therefore \text{let } \lambda = \frac{1}{t} \text{ obtain: } (\frac{1}{t})^{\alpha} u(1, (\frac{1}{t})^{\beta} x) \\ u(t,x) = \frac{1}{t^{\alpha}} v(\frac{x}{t^{\beta}}), \text{ should also be a solution: } u(1,r) = v(r) \end{cases}$

$\Rightarrow \partial_t u = -\alpha t^{-\alpha-1} v(\frac{x}{t^{\beta}}) + t^{-\alpha} (-\beta) \cdot \frac{1}{t^{\beta+1}} \cdot x \cdot v'(\frac{x}{t^{\beta}}) = t^{-\alpha-1}\left(-\alpha v(\frac{x}{t^{\beta}}) - \beta \cdot \frac{1}{t^{\beta}} \cdot x \cdot v'(\frac{x}{t^{\beta}})\right) \cdots (*)$

$\partial_{xx} u = t^{-\alpha} \cdot t^{-2\beta} \cdot v''(\frac{x}{t^{\beta}})$ 不用这么麻烦通过v, 直接用 $\lambda^{\alpha} u(\lambda t, \lambda^{\beta} x)$ 可以

$\partial_t u = \partial_{xx} u$ $\forall x, t$, $\therefore t^{-\alpha-2\beta} = t^{-\alpha-1}$ thus "$\beta = \frac{1}{2}$" $\cdots$ ①

$\Rightarrow$ "$v(r) = u(1, r)$" fundamental 定义

$u_{\lambda}(0,x) = \lambda^{\alpha} u(\lambda t, \lambda^{\beta} x) = \lambda^{\alpha} \delta(\lambda^{\beta} x) \overset{\text{need}}{=} \delta(x)$ x换成$\lambda^{\beta} x$

不是真的要$\delta(x)$, 要有Dirac性质即可

$\int \delta(x) dx = 1$, $\therefore \alpha = \beta = \frac{1}{2} \cdots$ ②

then: $\lambda^{\frac{1}{2}} u(\lambda^{\frac{1}{2}} x, \lambda t)$ is also solution for: $\begin{cases} \partial_t u = \partial_{xx} u \\ u|_{t=0} = \delta(x) \end{cases}$

$(*) \Rightarrow$: $-\alpha v(\frac{x}{t^{\beta}}) - \beta \frac{1}{t^{\beta}} \cdot x \cdot v'(\frac{x}{t^{\beta}}) = v''(\frac{x}{t^{\beta}})$

$-\alpha v(r) - \beta \cdot r \cdot v'(r) = v''(r)$, $r = \frac{x}{t^{\beta}}$

$\frac{1}{2} v + \frac{1}{2} r v' + v'' = (\frac{1}{2} r v + v')' = 0$ $\therefore v = C \cdot e^{-\frac{1}{4} r^2}$

$\therefore u(t,x) = \frac{1}{t^{\frac{1}{2}}} v(r) = \frac{1}{t^{\frac{1}{2}}} \times C \cdot e^{-\frac{1}{4}(\frac{x}{t^{1/2}})^2}$

二. Understand $u(t,x)=\int G_t(x-y)\phi(y)dy$

$G_t(x)=\sqrt{\frac{1}{4\pi t}}\cdot e^{-\frac{1}{4}x^2/t}$;

let $\Gamma(t,x;s,y)=G_{t-s}(x-y)=\sqrt{\frac{1}{4\pi(t-s)}}\cdot e^{-\frac{1}{4}(x-y)^2/(t-s)}$

for fixed "s, y": $\Gamma(\cdot,\cdot;s,y)$ solves: $\begin{cases}\partial_t\Gamma=\Delta\Gamma & t>s,\ x\in\mathbb{R}\\ \lim_{t\to s}\Gamma(t,x;s,y)=\delta(x-y)\end{cases}$

if initial conditions: $\phi(x)=\int\delta(x-y)\phi(y)dy=(\delta*\phi)(x)$ linear combination of δ

then solution is: linear conbination of $\Gamma(t,x;s,y)$; and Γ is the fundamental solution

$$\Rightarrow u(t,x)=\int\Gamma(t,x;s,y)\phi(y)dy=(G_{t-s}*\phi)(x)$$

易知: $\begin{cases}(\delta)^\wedge(\xi)=\int\delta(x)e^{-2\pi i x\xi}dx=e^{-2\pi i x\xi}\big|_{x=0}=1\ ;\ (1)^\vee(\xi)=\delta\\ \hat{G}_t(\xi)=e^{-4\pi^2\xi^2 t}\end{cases}$

实际上我们目前常用的: let $s=0$, $\phi=\delta*\phi$

三. non-homogeneous problem

Th: $\begin{cases}\partial_t u=\Delta u+f, & t>0\ x\in\mathbb{R}^d\\ u|_{t=0}=\varphi\end{cases}$ $\Rightarrow$ $u=\int_0^t\int_{\mathbb{R}^n}G_{t-s}(x-y)f(s,y)dsdy+\int_{\mathbb{R}^n}G_t(x-y)\phi(y)dy$

• 在ODE中: $\begin{cases}\dot{x}(t)=AX(t) \Rightarrow X(t)=e^{At}\cdot x_0=\Phi(t)\cdot X_0\ ; & \Phi(t)\text{: fundamental set}\\ \dot{x}(t)=AX(t)+f(t) & \text{by: variation}\end{cases}$

$X(t)=\Phi(t)\cdot C(t)$

$\Rightarrow \dot{X}(t)=\dot{\Phi}(t)\cdot C(t)+\Phi(t)\dot{C}(t)$ $\qquad \dot{\Phi}(t)=A\Phi(t)$

$=A\Phi(t)\cdot C(t)+\Phi(t)\dot{C}(t)$

$\dot{X}(t)=AX(t)+f(t)=A\Phi(t)\cdot C(t)$ $\qquad \dot{C}(t)=\Phi(t)^{-1}f(t)=\Phi(-t)f(t)$

$\Rightarrow C(t)=\int_0^t\Phi(-y)f(y)dy+C(0)$;

and: $X(t)=\Phi(t)X_0+\Phi(t)\cdot\int_0^t\Phi(-y)f(y)dy$ $\qquad \Phi(t)\cdot\Phi(s)=\Phi(t+s)\ \forall t,s$, here t fixed

$=\Phi(t)X_0+\int_0^t e^{(t-y)A}\cdot f(y)dy$

☆ ODE $\Rightarrow$ Duhamel's principle

Th: $f\in C^{1,2}((0,+\infty)\times\mathbb{R}^d)\cap C_0([0,+\infty)\times\mathbb{R}^d)$; then for NHE:

$\begin{cases}u(t,x)=\int_0^t\int_{\mathbb{R}^d}\Gamma(t,x;s,y)f(s,y)dyds\ ; & u\in C^{1,2}((0,+\infty)\times\mathbb{R}^d)\\ \lim_{(t,x)\to(0,x^0)}u(t,x)=0, & \forall x^0\in\mathbb{R}^d\end{cases}$

Dehamel's proof: (the proof for "$\phi \neq 0$" is obv), I stated it again in L6

设 $\phi=0$: $u(t,x)=\int_0^t\int_{\mathbb{R}^d} G_{t-s}(x-y)\cdot f(s,y)\,dy\,ds$

$= \int_0^t\int_{\mathbb{R}^d} G_s(y)\cdot f(t-s,\, x-y)\,dy\,ds$

"$\int_a^b g_1(a-\beta)g_2(\beta)\,d\beta=\int_a^b g_1(\beta)g_2(a-\beta)\,d\beta$"

then: $\partial_t u-\Delta u \overset{①}{=} \int_{\mathbb{R}^d} G_t(y)\cdot f(0,x-y)\,dy + \int_0^t\int_{\mathbb{R}^d} G_s(y)\cdot(\partial_t-\Delta_x) f(t-s,x-y)\,dy\,ds$ $\begin{cases} I_2 \\ I_3 \end{cases}$

(期中复习笔记写了) $= I_1 + \int_\varepsilon^t\int_{\mathbb{R}^d} G_s(y)(-\partial_s-\Delta_y) f(t-s,x-y)\,dy\,ds + \int_0^\varepsilon\int_{\mathbb{R}^d} G_s(y)(\partial_t-\Delta_x) f(t-s,x-y)\,dy\,ds$

用分部积分一样可证

where $|I_3|\le \int_0^\varepsilon(\|\partial_t f\|_{L^\infty}+\|D^2 f\|_{L^\infty})\int_{\mathbb{R}^d} G_s(y)\,dy\,ds$

马错了,是 $G(\partial_s-\Delta y)f$

$=(\|\partial_t f\|_{L^\infty}+\|D^2 f\|_{L^\infty})\cdot C\times\varepsilon \le C'\cdot\varepsilon$

$I_2=\int_\varepsilon^t\int_{\mathbb{R}}(-\partial_s G_s(y)-\Delta_y G_s(y))\cdot f(t-s,x-y)\,dy\,ds$

$②\begin{cases} =\int_{\mathbb{R}}\int_\varepsilon^t(-\partial_s G_s(y)-\Delta_y G_s(y))\cdot f(t-s,x-y)\,ds\,dy \\ =\int_{\mathbb{R}}\left[\int_\varepsilon^t -f(t-s,x-y)\cdot\partial_s G_s(y)\,ds+\int_\varepsilon^t-\Delta_y G_s(y)\cdot f(t-s,x-y)\,ds\right]dy \\ =\left(\int_{\mathbb{R}}\left[-f(t-s,x-y)G_s(y)\Big|_\varepsilon^t+\int_\varepsilon^t G_s(y)\,\partial_s f(t-s,x-y)\,ds\right)+\int_\varepsilon^t-\Delta_y G_s(y)\cdot f\,ds\right]dy \end{cases}$

$=-\underbrace{\int_{\mathbb{R}^d} G_t(y) f(0,x-y)\,dy}_{\text{消去}I_1} + \underbrace{\int_{\mathbb{R}^d} G_\varepsilon(y)\cdot f(t-\varepsilon,x-y)\,dy}_{(G_\varepsilon * f(t-\varepsilon,\cdot))(x)\to f \text{ as } \varepsilon\to 0}$

$\therefore \partial_t u-\Delta_x u=f(t,x)$

$u|_{t=0}=0$ easy to see $\Rightarrow$ Dehamel 可以求出 NHE 的解

①: $\frac{d}{dx}\int_0^x f(t,x)\,dt=f(x,x)+\int_0^x\frac{\partial f(t,x)}{\partial x}\,dt$

②: G 为 "$\partial_t u=\Delta_x u$" solution $\therefore \partial_s G_s(y)=\Delta_y G_s(y)$

$\Rightarrow$ 应该是 $2\times(\cdots+\cdots)$, 后面那段怎么消没的?

$G=(4\pi a t)^{-\frac{1}{2}n}\times e^{-|x|^2/4at}$ for: $\partial_t u-a\cdot\Delta u=0$

example

例: HW2-1 $\begin{cases} \partial_t u = \partial_{xx} u + b\partial_x u + cu + f(t,x) & t>0\ x\in\mathbb{R} \\ u(0,x) = \varphi(x) \end{cases}$

Step1: $\begin{cases} \partial_t \hat{u} = -4\pi^2\xi^2\hat{u} + b\cdot 2\pi i\xi\hat{u} + c\hat{u} + \hat{f}\cancel{(t,x)} \\ \hat{u}(0,x) = \hat{\varphi}(x) \end{cases}$

$\begin{cases} \partial_t \hat{u} = (-4\pi^2\xi^2 + b\cdot 2\pi i\xi + c)\hat{u} \\ \hat{u}(0,x) = \hat{\varphi}(x) \end{cases}$ for $\forall x$, then let $\hat{u}(t,x) = g(t)$, solve "ODE"

$\hat{u} = g(t) = e^{(-4\pi^2\xi^2 + b\cdot 2\pi i\xi + c)t} \times \hat{\varphi}$

$u = \left(e^{(-4\pi^2\xi^2 + b\cdot 2\pi i\xi + c)t}\right)^{\vee} * \varphi(x)$ → fundamental solution "G_t"

Step2: $u = \int_0^t \int_{\mathbb{R}} \Gamma(t,x;s,y) f(s,y)\,dy\,ds$

$\Gamma(t,x;s,y) = G_{t-s}(x-y) = \left(e^{(-4\pi^2\xi^2 + b\cdot 2\pi i\xi + c)(t-s)}\right)^{\vee}(x-y) = h(z)$

在step2中 $\hat{u}: x \to \xi$; here $(e^{t\cdots})^{\vee}$: $\xi \to x$, replace x by $x-y$

$e^{(-4\pi^2\xi^2 + b\cdot 2\pi i\xi + c)(t-s)} = \int_{\mathbb{R}^d} h(z)\, e^{-2\pi i\xi\cdot z}\,dz$

$\therefore h(z) = \int_{\mathbb{R}^d} e^{(-4\pi^2\xi^2 + b\cdot 2\pi i\xi + c)(t-s)} \times e^{-2\pi i z\cdot\xi}\,d\xi$;

$\Rightarrow \Gamma(t,x;s,y) = h(x-y)$

$\therefore u = \int_0^t \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{(-4\pi^2\xi^2 + b\cdot 2\pi i\xi + c)(t-s)} \cdot e^{-2\pi i(x-y)\cdot\xi}\,d\xi\right) \times f(t,s)\,dy\,ds$

在step1中也可以直接当成含f的ODE: $\partial_t \hat{u} = (\cdots)\hat{u} + \hat{f}(t,x)$

$\Rightarrow \hat{u} = e^{\int(-4\pi^2\xi^2 + b\cdot 2\pi i\xi + c)t\,dt} \times \int \left(e^{-\int\cdots dt} \cdot \hat{f}\right)dt$

这样是不是不用证明"这是解"? Dehamel公式要证

ODE的Dehamel公式:

$\begin{cases} \partial_t \hat{u} = -4\pi^2\xi^2\cdot \hat{u}(t,x) + \hat{f}(t,x) \\ \hat{u}|_{t=0} = 0 \end{cases}$

$\hat{u}(t,s) = \int_0^t \overset{= G}{e^{-4\pi^2\xi^2(t-s)}} \cdot \overset{= g}{\hat{f}(s,x)}\,ds$; $e^{-4\pi^2\xi^2 t}$ 为ODE fundametal solution

$\Rightarrow u = \int_0^t (G_{t-s} * f(s,\cdot))(x)\cdot ds$

对 $\hat{u}$: $\int_{\mathbb{R}^d} G_{t-s}(y) \times g(s,x-y)\,dy = \int_{\mathbb{R}^d} G_{t-s}(x-y) \times g(s,y)\,dy$

example

例 HW2.2: $g:[0,+\infty)\to\mathbb{R}$; $g(0)=0$; prove: $u(t,x)=\frac{1}{\sqrt{4\pi}}x\int_0^t \frac{1}{(t-s)^{3/2}}e^{-\frac{x^2}{4}/(t-s)}g(s)ds$

$$\begin{cases}\partial_t u=\partial_{xx}u & t>0,x>0 \cdots ① \\ u=0 & t=0,x>0 \cdots ② \\ u=g & x=0,t\geq 0 \cdots ③\end{cases}$$

let $v(t,x)=u(t,x)-g(t) \Rightarrow v(t,0)=u(t,0)-g(t)=0 \cdots ③'$

$\therefore$ can extend v to odd function about x.

$$\tilde{v}(t,x)=\begin{cases}v(t,x) \\ -v(t,-x)=-u(t,-x)+g(t), & x\leq 0\end{cases}$$

代回: $\partial_t\tilde{v}=\begin{cases}\partial_t u-g'(t) & x>0 \\ -\partial_t u+g'(t) & x\leq 0\end{cases}$, $\partial_{xx}\tilde{v}=\begin{cases}\partial_{xx}u & x>0 \\ -\partial_{xx}u & x\leq 0\end{cases}$, then $\partial_t\tilde{v}=\partial_{xx}\tilde{v}+\begin{cases}g'(t) & x\leq 0 \\ -g'(t) & x>0\end{cases}$ $\cdots ①'$

$$\tilde{v}=\begin{cases}-g(t) & t=0\ x>0 \\ g(t) & t=0\ x\leq 0\end{cases}=0 \cdots ②'$$

$①'+②'+③' \Rightarrow$: $\begin{cases}\partial_t\tilde{v}=\partial_{xx}\tilde{v}+f(t,x) & t>0 \\ \tilde{v}=0 & t=0,\ \forall x \\ \tilde{v}=0 & t\geq 0,\ x=0\end{cases}$ $\Leftarrow$ 要验证的 $\qquad f(t,x)=\begin{cases}g'(t) & x\leq 0 \\ -g'(t) & x>0\end{cases}$

对于 Duhamel principle 的 NHE: $\begin{cases}\partial_t u=\partial_{xx}u+f, & t>0 \\ u|_{t=0}=0; & \forall x\end{cases}$

$$\begin{aligned}\tilde{v}(t,x)&=\int_0^t\int_{\mathbb{R}} G_{t-s}(x-y)\cdot f(s,y)\,dy\,ds \\ &=\int_0^t\int_{-\infty}^0 G_{t-s}(x-y)\cdot g'(s)\,dy\,ds-\int_0^t\int_0^{+\infty}G_{t-s}(x-y)\cdot g'(s)\,dy\,ds \\ &=2\int_0^t g'(s)\int_{-\infty}^0 G_{t-s}(x-y)\,dy\,ds-\int_0^t g'(s)\cdot\int_{-\infty}^{+\infty}G_{t-s}(x-y)\,dy\,ds \\ &=2\int_0^t g'(s)(\cdots)ds-g(t)+g(0)\end{aligned}$$

$\to$ $t\to s$ 时 G 为 Approxiate identity

但 G 本身就是一个 pdf, $\int G\,dx=1$ 必成立

$\therefore$ 在 $x>0$: $u=\tilde{v}+g(t)$

$$\begin{aligned}&=2\int_0^t\left(\int_{-\infty}^0 G_{t-s}(x-y)dy\right)d\cdot g(s) \\ &=2\left(\int_{-\infty}^0 G_{t-s}(x-y)dy\right)\times g(s)\Big|_0^t-2\int_0^t g(s)\times\frac{d\left(\int_{-\infty}^0 G_{t-s}(x-y)dy\right)}{ds}ds \\ &=2g(t)\times\int_{-\infty}^0 G_{t-s}(x-y)dy\Big|_{s=t}-2\int_0^t\cdots ds\end{aligned}$$

Let $t-s=z$, $G=\frac{1}{\sqrt{4\pi(t-s)}}e^{-\frac{1}{4}(x-y)^2/(t-s)}=\frac{1}{\sqrt{4\pi}}z\cdot e^{-\frac{1}{4}(x-y)^2\cdot z^2}\to 0$ as $z\to+\infty$

$$\therefore u=-2\int_0^t g(s)\cdot\frac{d\int_{-\infty}^0\frac{1}{\sqrt{4\pi(t-s)}}\cdot e^{-\frac{1}{4}(x-y)^2/(t-s)}dy}{ds}\cdot ds$$

$\|$ 这一部分积分和 ds 换顺序, 即为要求的式子