29: Thm: A_n is simple group if $n\geq 5$

(1) 说明 if $N \lhd A_n$, N contains 3-cycle 即为 g

$Fix(g) = \{w\in\Omega;\ w^g = w\}$, 在 g 变换下的不动点

Let $g\in N\setminus\{1\}$, s.t. $|Fix(g)| \geq |Fix(x)|$, $\forall$ other $x\in N$ 写成2个cycle不同

Since $g\in A_n$ is even permutation $|Fix(g)|\leq n-3$ (∴)(∴)写成说明3个数字

① if $|Fix(g)| = n-3$, $g \overset{定}{=} (ab)(ac)$ $b\neq c$ 形式 $= (acb)$ ✓

② if $|Fix(g)| < n-3$

$g\in A_n$ $n\geq 5$, 形式可能为 $g = \begin{cases} (1\,2\,3\,4\,5\cdots)(\cdots & \text{不用写这个 已有 3-cycle 了} \\ (1\,2\,3\,4\cdots)(\cdots & \\ (1\,2\,3\cdots)(4\,5\cdots) & (1\,2\,3\cdots)(4\cdots)(5\cdots) \\ (1\,2)(3\,4\cdots)(5\cdots & (1\,2)(3\cdots)(4\cdots)(5\cdots) \end{cases}$

这里可以写5的 Why 不写

$Fix(g)\cap\{1,2,\dots,5\} = \phi$ ∴ $Fix(g)\subseteq\{6,7,\dots,n\}$

设 $\sigma = (3\,4\,5)$, $h = \sigma g \sigma^{-1} g^{-1}$

$h\in N$

$\begin{cases} 1^h = 1^{\sigma g \sigma^{-1} g^{-1}} = 1^{g\sigma^{-1}g^{-1}} = 2^{\sigma^{-1}g^{-1}} = 1 \quad ∴ 1\in Fix(h) \\ \text{and } \forall w\in Fix(g)\ w^h = w^{\sigma g\sigma^{-1}g^{-1}} = w, \text{ since } w\subseteq\{6,7,\dots,n\} \text{ 不受 } \sigma \text{ 影响} \end{cases}$

∴ $w\in Fix(h)$ then

$1\notin Fix(g)$, $\forall w\in Fix(g)$, $w\in Fix(h)$

∴ $|Fix(h)| > |Fix(g)|$

Assumed $|Fix(g)|$ is Largest in $N\setminus\{1\}$ $\Big\}$ ∴ $h = \sigma g\sigma^{-1}g^{-1} = 1$, $g = \sigma g \sigma^{-1}$

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lem 1.2 $g^a = a^{-1}ga$ 共轭

$l = (i_1, i_2, \dots, i_n)^\sigma = (i_1^\sigma, i_2^\sigma, \dots, i_n^\sigma) = r$

proof: $\forall w\in\Omega$, $w^l = w^r$

$w\in\Omega = \{1,2,\dots,n\}$, 上面 n 个 slot 的可以 $m\leq n$ slot 的也可以

$w^l = w^{\sigma^{-1}(i_1,\dots,i_n)\sigma} = \begin{cases} i_i^{(i_1,i_2,\dots,i_n)\sigma} = i_{i+1}^\sigma & \text{if } w = i_i^\sigma \text{ some } i \\ w & \text{if } w\neq i_i^\sigma \text{ 时 即: } w^{-\sigma}\neq i_i \end{cases}$

$w^r = w^{(i_1^\sigma, i_2^\sigma,\dots, i_n^\sigma)} = \begin{cases} w = w^l & w\neq i_i^\sigma \text{ 时} \\ i_{i+1}^\sigma = w^l & w = i_i^\sigma \end{cases}$

∴ $\forall w\in\Omega = \{1,2,\dots,n\}$, $(i_1, i_2, \dots, i_n)^\sigma = (i_1^\sigma, i_2^\sigma, \dots, i_n^\sigma)$

n 改成更小的 $m\leq n$ 也可以

PS: $\text{permutation}^{\text{permutation}}$ 表示共轭; $\text{element}^{\text{permutation}}$ 表示对元素操作

接上面: $g = \sigma g \sigma^{-1}$, $\sigma = (345)$ $\quad (1234)^{(345)} = (1^{(345)}\ 2^{(345)}\ 3^{(345)}\ 4^{(345)})$

共轭 ⇒ 元素操作

$$g = \begin{cases} (12345)(\cdots \\ (1234)(5\cdots) \\ (123)(45)(\cdots,\ (123)(4\cdots)(5\cdots) \\ (12)(34)(5\cdots),\ (12)(3\cdots)(4\cdots)(5\cdots) \end{cases}$$

$$\sigma g \sigma^{-1} = \begin{cases} ((12345)(\cdots))^{\sigma} = (12345)^{\sigma} \overset{\text{不变}}{(\cdots)^{\sigma}} = (12453)(\cdots \\ (1245)(3\cdots) \\ \cdots \\ \cdots \end{cases}$$

共轭

$\neq g$ 矛盾 $\therefore \sigma g \sigma^{-1} g^{-1} = h \neq 1$

∴ ②不成立 ①: $|Fix(g)| = n-3$ "$<n-3$"条件体现为不出现这种 $g = (12)(23)$ cycle disjoint, 必取不到=, 只能<

(2): $\forall$ 3-cycle conjugate to (123) by even permutation

$\Vert$

可以换成任意 (xyz) 与 (ijk) 共轭

证: $(ijk)^a = (123)$ $\exists$:

① $i=1, j=2, k\neq 3$

$a = (34k) = (3k)(34)$ $\quad (ijk)^a = (i^a j^a k^a) = (123)$

② $(ijk) \cap (123) = \phi$. ③ $i=1, j\neq 2, k\neq 3$

(3): $\forall$ element of A_n is product of 3-cycles (HW3)

(1)+(2)+(3) ⇒ (1) $g = (abc)$ is a 3-cycle in N

$\forall y \in A_5$, $y^{-1} g y \in N$, y even permutation

+(2) 任意 $(xyz) \sim (ijk)$ by even permutation 偶数个2-cycle

$(abc)^y = (ijk)$ this y exists for $\forall (ijk)$

$\therefore \forall (ijk) \in N$

+(3) $(i_1 j_1 k_1) \in N$ $(i_2 j_2 k_2) \in N$ 群运算为复合。

$(i_1 j_1 k_1)(i_2 j_2 k_2) \in N$

这样可以构造出 A_n 中所有元素, since A_n 为3-cycle所构

若 $A_5 \to A_6$, 把 $g = \begin{cases} (12345) \\ \cdots \end{cases}$ 修改一下列出来就行了, A_5 不行 A_6 必然也不行

g 就看出来了

Thz. G is simple $\Leftrightarrow$ $\forall e\neq g\in G$, $g^G=\langle g^x=x^{-1}gx : x\in G\rangle = G$;

normal closure $\Rightarrow$ 不是轨道，这个比轨道大！

$\Rightarrow$ "生成"的意思，此处$=\{\cdots\}$ 因为内部已经是 group 了

"$\Rightarrow$" $x^{-1}gx\in g^G$

$\forall$: $y^{-1}x^{-1}gxy=(xy)^{-1}g(xy)\in g^G$ since $y\in G$, $\therefore g^G \trianglelefteq G$

G is simple, $g\neq e$ $\therefore g^G \trianglelefteq G$

"$\Leftarrow$" $\forall g\neq e$. $g^G=G$

任取一个 $N\trianglelefteq G$, $n\in N$, $n^G=G$

$n^G=\langle n^x=x^{-1}nx : x\in G\rangle \subseteq N$ since $x^{-1}nx\in N$ $\forall x$.

$\therefore N=G$ $\therefore G$ is simple

HW3写过要会

补充: A_n is the only proper non-trivial normal subgroup of S_n, $n\geq 5$

$\forall$ permutation in S_n = product of 2-cycle $(\cdot\cdot)(\cdot\cdot)\cdots(\cdot\cdot)$ 记为 x, 有 x' 个 2-cycle

$\forall y\in A_n$ $x^{-1}A_nx=A_n$ since $x^{-1}yx$ 仍为偶数个, 因为加了 $2\cdot x'$ 个

或者 by $[S_n:A_n]=2$ ✓

$\therefore A_n \trianglelefteq S_n$

若 $\exists H\neq\{1\}, \neq A_n$, $H\trianglelefteq S_n$, $H\cap A_n \trianglelefteq A_n$ by iso-2 Th

$n\geq 5$, A_n is simple

$\therefore H\cap A_n=A_n$ or $H\cap A_n=\{1\}$

① $H\cap A_n=A_n$

$|A_n|=\frac{1}{2}n!$, $|S_n|=n!$ $A_n\leq H\leq S_n$ since $A_n\subseteq H$

but $\nexists$ $|H|$ s.t. $\frac{1}{2}n! \,|\, |H|$, $|H| \,|\, n!$ $\therefore$ 这种不行 ($|H|=n!$ $H=S_n$ ✓)

② $H\cap A_n=\{1\}$

Lem1: if $H\trianglelefteq G$, $H\cap G'=\{e\}$, then $H\leq Z(G)$

$G'=\{x^{-1}y^{-1}xy : x,y\in G\} \trianglelefteq G$

fix $h\in H$, $\forall g\in G$ $g^{-1}hg\in H$ by normality

$g^{-1}hgh^{-1}\in H$ by closure, $g^{-1}hgh^{-1}\in G'$ by def.

$\therefore g^{-1}hgh^{-1}\in G'\cap H=\{e\}$

$\therefore$ for $\forall h\in H$, $\forall g\in G$, $hg=gh$; $H\leq Z(G)$

Lem2: $n\geq 3$, $Z(S_3)=\langle 1\rangle$ HW3

Lem3: $S_n'=A_n$, $\forall n\geq 5$

① A_n is generated by 3-cycles ... HW3

接 Lem3: ②': $\forall$ 3-cycle is commutator: $(abc) = (ab)(ac)(ab)(ac)$

~~③ AESn An~~ $\Rightarrow$ ①+② A_n 由 commutator 生成 $\therefore A_n \le S_n'$ (定义) ---(*)

对于 S_n/A_n x = 奇数 2-cycle y = 奇数 2-cycle, $x y \in S_n$

$xA_n \, yA_n = xyA_n = A_n = yA_n \, xA_n$; 同理其它奇偶性

$\therefore S_n/A_n$ Abel

$\forall H \triangleleft G$. G/H Abel $\Leftrightarrow G' \le H$ (Th 已证) $\Rightarrow S_n' \le A_n$ ---(**)

(*)+(**): $S_n' = A_n$

代回②中, $H \cap A_n = H \cap S_n' = \{1\}$

$H \triangleleft S_n$

$\therefore H \le Z(S_n) = \{1\}$ since $n \ge 5 \ge 3$

$\therefore \forall H \triangleleft S_n$, $H \ne A_n \Rightarrow$ then $H = \{e\}$ or $H = S_n$

即 A_n 为 S_n 仅有的 proper non-trivial normal subgroup

(Cayley)
Th3: $\forall$ group G with order n, isomorphic to a permutation group: S_n

记 $G = \{g_1, g_2, \cdots g_n\}$

对于 $\forall x = g_k \in G$. define a permutation $\hat{x}\begin{cases} G \to G \\ g_i \to g_i x \end{cases}$

易知: $\hat{x}$ bij. $G \to G$ $\therefore \hat{x} \in S_n$

定义

设: $\hat{G} = \{\hat{x} : x \in G\}$

$\hat{x}: g_i \to g_i x$ $\hat{y}: g_i \to g_i y$; $\hat{x}\hat{y}(g_i) = \hat{x}(g_i y) = g_i y x = \widehat{xy}(g_i)$ $\therefore$ closed

$\hat{G}$ is group

显然, $\hat{G} \cong G$ (homo 证一下)

$\hat{G} \le S_n$ since $\forall \hat{x} \in \hat{G}$, $\hat{x} \in S_n$ + $|\hat{G}| = |S_n|$ $\Rightarrow$ 不对!

Rmk: $\forall g_i, g_{i'} \in G$, $\exists \hat{x} \in \hat{G}$ s.t. $g_i x = g_i^{\hat{x}} = g_{i'}$ since G finite

$\therefore \hat{G}$ is transitive on $\Omega = G$ $\Rightarrow G, \hat{G}$ 均 transitive

and fix $g_i, g_{i'}$ $\hat{x}$ is unique

$\therefore \hat{G}$ is regular on G (即仅有 identity permutation fix some point of Ω)

$\Downarrow$

铺垫 UD 讲群运算了!