20.10 Show that a point $z \in X$ belongs to $\operatorname{clin}(E)$ if and only if $f(z) = 0$ for every $f \in X^*$ that vanishes on E, i.e. $f(x) = 0$ for every $x \in E$ implies that $f(z) = 0$. (Taking $E = Y$ with Y a linear subspace shows

T20.10 pf: $\operatorname{clin} E = \overline{\operatorname{span} E}$,

"$\Rightarrow$": "$f \equiv 0$ on E then $f(z) = 0 \Rightarrow z \in \operatorname{clin} E$" if not, i.e. $z \notin \operatorname{clin} E$.

$\operatorname{clin} E$ is a closed normed subspace in X

$\therefore \exists$ distance function s.t. $\phi \equiv 0$ on $\operatorname{clin} E$, $\|\phi\| = 1$, $\phi(z) = \operatorname{dist}(z, \operatorname{clin} E) > 0$ 矛盾!

$\therefore$ assumption fails. ; z should in $\operatorname{clin} E$

"$\Leftarrow$": 已知 $f \equiv 0$ on E. $z \in \operatorname{clin} E$, then WTS $f(z) = 0$.

设 $f \equiv 0$ on E, then $f \equiv 0$ on $\operatorname{span}(E)$ since f is linear.

Lem (HW12, T19.3). $\exists$ unique $\hat{f}$ on $\overline{\operatorname{span}(E)}$ s.t. $\|\hat{f}\|_{\operatorname{clin} E} = \|f\|_{E^*} = 0$ ~~$\hat{f}$~~ $\lim_{n\to\infty} f_n(x)$

$\therefore \hat{f} \equiv 0$ on $\overline{\operatorname{span}(E)} = \operatorname{clin} E$. $\Rightarrow f(z) = 0$

证毕

21.1 If $(X, \|\cdot\|)$ is a normed space show that the Minkowski functional of B_X is $\|\cdot\|$. (Lax, 2002)

21.2 Show that if $(K_\alpha)_{\alpha \in \mathbb{A}}$ are a family of convex subsets of a vector space X, then $\bigcap_{\alpha \in \mathbb{A}} K_\alpha$ is also convex.

21.3 Show that if K is a convex subset of a normed space X, then $\overline{K}$ is also convex.

T21.1 pf: $p(x) = \inf\{\lambda > 0 : \frac{1}{\lambda} x \in B_X\} = \inf\{\lambda > 0 : \|x \cdot \frac{1}{\lambda}\| < 1\} = \|x\|$

T21.2 pf: $\forall y, x \in \bigcap K_\alpha$, $y, x \in K_\alpha$ $\forall \alpha \in \Lambda$

$\therefore ty + (1-t)x \in K_\alpha$ $\forall \alpha \in \Lambda$, $t \in [0,1]$ $\Rightarrow ty + (1-t)x \in \bigcap K_\alpha$, $\forall t \in [0,1]$

T21.3 pf: WTS: K convex $\Rightarrow \overline{K}$ convex

$\forall x, y \in \overline{K}$. 设 $x, y \in K$. $tx + (1-t)y \in K$ $\forall t \in [0,1]$ obv

设 $x \in K'$, $y \in K'$, then $\exists \{x_n\}_{n=1}^{\infty}$ $\{y_n\}_{n=1}^{\infty} \subseteq K$ s.t. $x_n \to x$. $y_n \to y$

$tx_n + (1-t)y_n \to tx + (1-t)y$ $\forall t \in [0,1]$

$tx_n + (1-t)y_n \in K$ since K convex; $\overline{K}$ is closed $\Big\} \Rightarrow tx + (1-t)y \in \overline{K}$

设 $x \in K$, $y \in K'$, then $\exists \{y_n\}_{n=1}^{\infty} \subseteq K$, s.t. $y_n \to y$

$tx + (1-t)y_n \to tx + (1-t)y$, $\forall t \in [0,1]$

$tx + (1-t)y_n \in K$ since K convex; $\overline{K}$ is closed $\Big\} \Rightarrow tx + (1-t)y \in \overline{K}$ 证毕

22.1 Let X be an infinite-dimensional Banach space and let $(x_i)_{i=1}^{\infty}$ be a sequence in X. Let $Y_n = \mathrm{Span}(x_1, \ldots, x_n)$. Using the Baire Category Theorem show that the linear span of (x_i) is not the whole of X. (No infinite-dimensional Banach space can have a countable Hamel basis.)

T22.1 pf: let $Y_n = \mathrm{span}(e_1, e_2, \cdots e_n)$

if $X = \mathrm{span}\{e_i\}_{i=1}^{+\infty}$, $X = \bigcup_{n=1}^{+\infty} Y_n \xRightarrow{\text{Baire}} \exists N, Y_N^{\circ} \neq \phi$

finite-dim subspace are complete ($\cong \mathbb{R}^n$) thus closed

$\therefore$ some $y = \sum^n a_i e_i \in Y_n$. $\exists r$ s.t. $\sum^n a_i e_i + r \cdot v = \sum^n b_i e_i$, $\|v\| \le 1$

$\Rightarrow rv = \sum_{i=1}^n c_i e_i$.

let $v = \frac{1}{r} \cdot e_{n+1}$ contradict

$\Rightarrow$ finite-dimensional subspace has no open balls in it (无限维的)

22.4 Suppose that X, Y, and Z are normed spaces, and that one of X and Y is a Banach space. Suppose that $b\colon X \times Y \to Z$ is bilinear and continuous. Use the Principle of Uniform Boundedness to show that there exists an $M > 0$ such that

$$|b(x, y)| \le M \|x\|_X \|y\|_Y \qquad \text{for every } x \in X,\ y \in Y.$$

pf: $b(x,y)$ cts. linear at $x \Rightarrow b(x,y) = b_y(x)$ bounded for $\forall$ fixed y

$\therefore \|b_y(x)\| \le |M(y)| \cdot \|x\|$

$\|b_x(y)\| \le |M(x)| \cdot \|y\|$ similarly

if X is Banach, consider linear operator $\{b_y\}_{y \in Y}$ $\quad \sup_{y \in Y / y \in B_Y} \|b_y(x)\|$ bounded if fixing x 均可以的

$\Rightarrow \sup_{y \in Y / y \in B_Y} \|b_y\| \le M < +\infty$ ... by Principle of Uni-boundness

补充:

$\therefore |b(x,y)| \le \|x\| \cdot \|b_y\| \le M \|x\| \cdot \|y\|$

$b_y: x \mapsto (x, y) \Rightarrow b: y \to b_y$

$\sup_{y \in Y \; y \in B_Y} \|b_y\| \le M \qquad \|b\| = \sup_{y \in B_Y} \|b_y\| \le M$, $\therefore \|b_y\| \le M \cdot \|y\|$

☆ $\sup_{y \in Y} \|b_y\| \le M < +\infty \Rightarrow \|b\| = \sup_{y \in B_Y} \|b_y\| \le \sup_{y \in Y} \|b_y\| \le M$ 拆开by

$\|b_y\| \le M \cdot \|y\|$