Th1. $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots A_n \supseteq \cdots$, A_i is non-empty closed set in complete metric space (X,d)

$\operatorname{diam}(A_n) \to 0$, then $\bigcap_{n=1}^{\infty} A_n$ contains one point $\Rightarrow$ 可知闭区间套定理

proof1: pick any $a_i \in A_i$, $i=1,2,3\cdots$

$a_i, a_{i+1}, a_{i+2}, \cdots \in A_i$, since $A_{i+j} \subseteq A_i$ $\forall j$.

$\operatorname{diam}(A_n) \to 0$ $\therefore \forall \varepsilon > 0$ $\exists N$ s.t. $\forall n \geq N$, $\operatorname{diam}(A_n) \leq \varepsilon$

$\therefore \forall n, m \geq N$. $d(a_n, a_m) \leq \operatorname{diam}(A_n) \leq \varepsilon \Rightarrow \{a_n\}_{n=1}^{\infty}$ is Cauchy

核心步

X complete $\therefore a_n \to a_0$

a_0 is limit point of $\bigcap_{n=1}^{\infty} A_n$ since $\forall \varepsilon > 0$. $\exists N'$ s.t. $a_n \in B_\varepsilon(a_0)$ $\forall n \geq N'$, then $B_\varepsilon(a_0) \setminus \{a_0\} \cap A_n \neq \phi$ $\forall n \geq N$

$n<N$也成立

$\bigcap_{n=1}^{\infty} A_n$ is closed $\therefore a_0 \in \bigcap_{n=1}^{\infty} A_n$ (existence)

if $\exists \bar{a}_0, a_0 \in \bigcap_{n=1}^{\infty} A_n$. $d(\bar{a}_0, a_0) \leq d(\bar{a}_0, a_n) + \underset{n \geq N}{d(a_n, a_0)} \leq \operatorname{diam}(A_n) \times 2$ $\forall n = 1, 2, \cdots \Rightarrow a_0 = \bar{a}_0$ (uniqueness)

Th2: [contracted mapping theorem], (X,d) is complete, the mapping $T: X \to X$ satisfies:

$\exists \theta \in [0,1)$ s.t. $\forall x \in X$, $\forall y \in X$, $d(Tx, Ty) \leq \theta d(x,y)$

then: $\exists$ unique $\bar{x} \in X$ s.t. $T\bar{x} = \bar{x}$

proof2: pick any $x_1 \in X$; $x_2 = Tx_1$, $x_3 = Tx_2$, $\cdots$ $x_{n+1} = Tx_n$, $\cdots$

$$d(x_n, x_{n+m}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{n+m-1}, x_{n+m})$$

$$\leq (\theta^{n-1} + \theta^{n-2} + \cdots + \theta^{m+n-2}) \cdot d(x_1, x_2)$$

$$\leq \frac{\theta^{n-1}}{1-\theta} \cdot d(x_1, x_2), \quad x_2 = Tx_1$$

obv $\{x_n\}$ is Cauchy, X is complete $\therefore x_n \to x_0 \in X$

$\lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = x_0$; $\lim_{n\to\infty} T(x_n) = x_0$ since T cts (这里就不证cts的性质了,用定义可以证的)

$\therefore \bar{x}$ can be x_0 (existence)

if $\bar{x} = x_0$, $\bar{x}$ can be x_0', $d(Tx_0, Tx_0') = d(x_0, x_0') \leq \theta d(Tx_0, Tx_0')$, $\theta < 1$ contradict (uniqueness)

def1: X is metric space. X is compact if: $\forall$ sequence $\{x_n\} \subseteq X$, $\exists$ convergent subsequence $\{x_{n_k}\}$

$\overset{\text{obv}}{\Leftrightarrow}$ $\forall$ infinite subset $Y \subseteq X$, $\exists$ limit point $y \in X$

Th3: X is compact (def1) $\Leftrightarrow$ $\forall$ open cover $\bigcup A_\alpha \supseteq X$, $\exists$ finite subcover $\bigcup_{i=1}^{n} A_i \supseteq X$ $\Leftrightarrow \cdots$ (还有其它情况)

proof3: "$\Leftarrow$" 设 $Y \subseteq X$, Y has no limit point

$\forall x \in X$, $\exists r_x > 0$ s.t. $B_{r_x}(x) \setminus \{x\} \cap Y = \emptyset$

consider $\bigcup_{x\in X} B_{r_x}(x) \supseteq X$, r_x as above

$\Rightarrow \bigcup_{i=1}^{n} B_{r_x}(x_i)$ not cover $X \setminus \{x_1, \cdots x_n\}$, $\forall n$

$\therefore \bigcup_{x\in X} B_r(x)$ not has finite subcover 矛盾!

"$\Rightarrow$" lem1: $\{U_\alpha\}$ is an open cover of cpt $X \Rightarrow \exists\, \delta > 0$, s.t. $\forall x \in X$, $B_\delta(x) \subseteq U_\alpha$ some α

pf1: 反证法

assume $\forall \delta > 0$, $\exists x \in X$ s.t. $B_\delta(x) \nsubseteq U_\alpha$ $\forall \alpha$;

pick $\delta_1 = \frac{1}{1}, x_1$; $\delta_2 = \frac{1}{2}, x_2$; $\cdots$ $\delta_n = \frac{1}{n}, x_n$; $\cdots$ δ_i corresponds to x_n 核心步骤

$\exists x_{n_k} \to x_0$, $x_0 \in X$ (闭集就行吗?)

$x_{n_k} \in (\bigcup U_\alpha)^c$ $\forall n_k$, x_0 is limit pt of $(\bigcup U_\alpha)^c$, $(\bigcup U_\alpha)^c$ is closed $\Rightarrow x_0 \notin U_\alpha$ $\forall \alpha$

$\Rightarrow$ contradict with $\bigcup U_\alpha \supseteq X$

lem2. for δ in lem1, $\exists n$ s.t. $\bigcup_{i=1}^{n} B_\delta(x_i) \supseteq \overset{cpt}{X}$ for some $\{x_1, \cdots x_n\} \subseteq X$. ✓ 完全有界

$\Updownarrow$

$\forall \delta > 0$, $\forall x_1 \in X$, construct $\{x_j\}$ s.t. $d(x_i, x_j) > \delta$ for $\forall i < j$

即 find x_2 s.t. $d(x_1, x_2) > \delta$, find x_3 s.t. $d(x_i, x_3) > \delta$. $i = 1, 2$. find $\cdots$

this sequence is finite! $\{x_j\} = \{x_1, x_2, \cdots x_n\}$, $\bigcup_{i=1}^{n} B_\delta(x_i) \supseteq X$ ✓

否则与 $x_{n_k} \to x_0$ 矛盾, $d(x_i, x_j) > \delta$ 的方式必然没有极限点

def2. continuous, uniformly cts, lipschitz cts 接下来,将数值结论(数分内容)放到空间中证明

Th4: T is continuous, X is compact metric space with d, $T(X)$ is cpt with $\hat{d}$

lem3: $x_n \to x_0$, T is cts, then $\lim_{n\to+\infty} T(x_n) = T(x_0)$

$\Leftarrow$ lem3 + convergence $\{x_{n_k}\}$
或 Th5 + open cover 均可证

proof4: $\forall \{y_n\} \subseteq T(X)$, $\exists \{x_n\} \subseteq X$ s.t. $T(x_n) = y_n$

$\exists \{x_{n_k}\} \subseteq \{x_n\}$, $x_{n_k} \to x_0$, $x_0 \in X$ then $T(x_0) \in T(X)$

$\therefore \lim_{k\to+\infty} T(x_{n_k}) = T(x_0) = \lim_{k\to+\infty} y_{n_k} \in T(X)$ $\Rightarrow \forall \{y_n\} \subseteq T(X)$, $\exists \{y_{n_k}\} \subseteq \{y_n\}$ s.t. $y_{n_k} \to y_0 = T(x_0) \in T(X)$

Th5: $T: X \to Y$ is continuous $\rightleftarrows$ $\forall$ open set $G \subseteq Y$, $T^{-1}(G)$ is also open in X ("close" 也行)

proof5: "$\Rightarrow$" G is open. $\forall y \in G$, $\exists r>0$ $B_r(y) \subseteq G$

T is cts $\therefore \forall z \in X$, $\forall \varepsilon>0$, $\exists \delta_z>0$ s.t. $\forall x \in B_{\delta_z}(z)$, $d(Tx, Tz) < \varepsilon$.

任取: if $a \in T^{-1}(G)$, $T(a) \in G$ $\exists r_a$ $B_{r_a}(T(a)) \subseteq G$

$\therefore$ let $\varepsilon < r_a$, $\exists \delta_a>0$ s.t. $\forall x \in B_{\delta_a}(a)$, $Tx \in B_\varepsilon(Ta) \subseteq B_{r_a}(Ta) \subseteq G \Rightarrow x \in T^{-1}(G)$

$\therefore \forall a \in T^{-1}(G)$ $B_{\delta_a}(a) \subseteq T^{-1}(G) \Rightarrow T^{-1}(G)$ open

"$\Leftarrow$" let $G = B_r(y)$ for any $r>0$, $y \in Y$,

$x \in T^{-1}(G)$, $\exists \delta>0$ s.t. $B_\delta(x) \subseteq T^{-1}(G)$, $\Rightarrow f$

$\Rightarrow$ for this $r>0$, $\exists \delta>0$, $\forall z \in B_\delta(x)$, $d(Tz, Tx) \le 2r$

$\therefore T$ cts at x, x, y, r are arbitary

Th6. $T: X \to Y$ is cts, (X, d_1) is compact $\Rightarrow T$ is uniformly cts

proof6: WTS: $\forall \varepsilon>0$, $\exists \delta>0$ s.t. $d(x_1, x_2) < \delta \Rightarrow d(Tx_1, Tx_2) < \varepsilon$ for $\forall x_1, x_2 \in X$ 条件没用全!

× $\left[\begin{cases} (X,d) \text{ compact}, \therefore \forall \delta>0, \exists \text{ finite point } z_1 \cdots z_n \overset{\in X}{} \text{ s.t. } \bigcup_{i=1}^n B_\delta(z_i) \supseteq X \\ \text{for } \varepsilon>0, \exists \delta_i>0 \text{ s.t. } d(z_i, x) < \delta_i \Rightarrow d(Tz_i, Tx) < \varepsilon/3 \cdots T \text{ cts at } x_i \end{cases}\right]$ --- cpt $\Rightarrow$ totally bdd

× [fix $\varepsilon>0$. let $\delta < \delta_i$, $i=1,2,\cdots n$ $\sigma + 2\delta < \delta_i$, $i=1,2,\cdots n$ $\cdots (*)$ ✗ δ_i is decided by δ!

✓ [for any $x_1, x_2 \in X$, $x_1 \in B_\delta(z_1)$, $x_2 \in B_\delta(z_2)$

$d(z_1, x_1) + d(x_1, x_2) + d(x_2, z_2) > d(z_1, z_2) > d(x_1, x_2) - d(x_2, z_2) - d(x_1, z_1) \Rightarrow \overset{(<\sigma)}{d(x_1, x_2)} + 2\delta > d(z_1, z_2) > d(x_1, x_2) - 2\delta$

then: $d(Tx_1, Tx_2) \le d(Tx_1, Tz_1) + d(Tz_1, Tz_2) + d(Tz_2, Tx_2)$

cpt + cts + $(*)$ $\Rightarrow \begin{cases} d(Tx_i, Tz_i) < \varepsilon/3, \ i=1,2 \text{ since } \delta < \delta_i, T \text{ cts at } z_i \cdots ① \\ d(Tz_1, Tz_2) < \varepsilon/3, \text{ since } d(z_1, z_2) < \sigma + 2\delta < \min(\delta_1, \cdots \delta_n) \cdots ② \end{cases}$

$\therefore d(Tx_1, Tx_2) < \varepsilon$]

Correction:

$\left.\begin{cases} \forall \varepsilon>0, \exists \delta_z \text{ s.t. } d(z, x) < \delta_z \Rightarrow d(Tz, Tx) < \varepsilon/3 \\ \bigcup_{z \in X} B_{\frac{\delta_z}{4}}(z) \text{ is open cover of } X \xRightarrow{X \text{ cpt}} \exists n, \bigcup_{i=1}^n B_{\delta_i}(z_i) \supseteq X, \ \delta_i = \frac{1}{4}\delta_{z_i} \end{cases}\right]$ 核心步骤

$\Rightarrow \begin{cases} ① \text{ holds. } \delta_i < \delta_{z_i} \\ ② \text{ holds if } \delta_1 + \delta_2 + \sigma < \max(\delta_{z_1}, \delta_{z_2}). \text{ just let } \sigma < \frac{1}{4}\min(\delta_{z_1}, \delta_{z_2}, \cdots \delta_{z_n}) \end{cases}$

$\therefore \forall \varepsilon>0$, $\exists \sigma>0$. $d_x(x_1, x_2) < \sigma \Rightarrow d_Y(Tx_1, Tx_2) < \varepsilon$

Rmk: 没有用 totally bdd, 只用了 cts 和 cpt finite cover 的原定义; 先用 cpt 再 cts 不行, 就应该换一个顺序!
totally bdd = equicts 条件中常用

def separable: $\exists$ countable subset E, E dense in X (finite is OK too)

例: $Q^n \subseteq \mathbb{R}^n$, $\overline{Q^n} = \mathbb{R}^n$ $\therefore \mathbb{R}^n$ separable

· $E = \{Q[x]$, 即系数在Q中的多项式$\} = \{a_0 + a_1x^1 + \cdots a_nx^n + \cdots : a_i \in Q\}$

$E \cong Q \times \mathbb{N}$, ~~$E = \mathbb{R}[x]$, 前面 term…~~ 不只多项式

$\therefore \overline{E} = C([a,b])$, $C([a,b])$ 可分

· $X \subset \mathbb{R}$, $d(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$ X is not separable

Th: K is cpt, K separable (proof by def)

K is cpt $\Rightarrow$ thus totally bounded & complete / $\forall$ infinite set has limit pt in K

$\forall \varepsilon > 0$. $\forall x_1 \in K$, choose x_2 to let $d(x_1, x_2) > \varepsilon$

…

choose x_n to let $d(x_i, x_n) > \varepsilon$ $\forall i < n$

若以上过程无限进行, $\{x_1, x_2, \cdots x_n\}$ 在K中无 limit pt. 矛盾

$\therefore \forall \varepsilon > 0$ $\exists$ finite $x_1, x_2 \cdots x_N \in K$ $\bigcup_{i=1}^{N} N_\varepsilon(x_i) \supseteq K$

数分HW 2.24 用 $X = \bigcup_{i=1}^{N} N_\varepsilon(x_i)$

Let $\varepsilon = \frac{1}{1}, \frac{1}{2}, \cdots \frac{1}{n}, \cdots$ $X = \bigcup_{n=1}^{+\infty} \bigcup_{i=1}^{N_n} B_{\frac{1}{n}}(x_{ni})$

$\{x_{ni}: n = 1, 2, \cdots N_n, i = 1, 2 \cdots\}$ dense since $\forall x \in X$. $x \in B_{\frac{1}{n}}(x_{ni})$ some i

$\therefore$ the set $E = \{x_{ni}: \forall n, i\}$ dense. ct $\Rightarrow$ X separable

把X改成K也一样, 只是写成 $K \subset \bigcup_{i=1}^{N} N_\varepsilon(x_i)$

数分HW 2.25: cpt K has ct base, therefore separable

其实可以证 metric space with ct base $\Leftrightarrow$ separable. HW 2.23

$\{V_a\}$. $\forall x \in X$, $\forall$ open set $x \in G \subset X$; $\exists x \in V_a \subset G$

即: $\forall$ open set G is union of subcollection of $\{V_a\}$

$\Leftrightarrow \forall G \neq \phi$. $\forall x \in G$, $\exists$ $x \in V_a \subset G$

$$\{V_{mr}\} = \begin{bmatrix} \vdots & \vdots & \vdots \\ B_{\frac{1}{3}}(x_1) & & \\ B_{\frac{1}{2}}(x_1) & B_{\frac{1}{2}}(x_2) & B_{\frac{1}{2}}(x_3) \\ B_1(x_1) & B_1(x_2) & B_1(x_3) \end{bmatrix}$$

"$\Leftarrow$". Let $\{x_1, x_2, \cdots x_n \cdots\}$ 为 metric space E 的 ct dense set

Let $V_{mr} = \{y: d(y, x_m) < r\}$, $\{V_{mr}\}$ $m = 1, 2, \cdots$, $r = \frac{1}{1}, \frac{1}{2}, \cdots$ is countable

$\forall x \in X$, $\forall$ open $G \ni x$, $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq G$ since G open

if $x = x_i$ some i, V_{ir} with $r < \delta$ satisfies: $x \in V_{ir} \subseteq B_\delta(x) \subseteq G$

if $x \neq x_i$. x is limit pt of E, $\therefore$ for $r < \frac{1}{2}\delta$. $\exists V_{mr} \ni x$, $V_{mr} \subseteq B_\delta(x) \subseteq G$

"$\Rightarrow$" $\{V_n\}$ be the countable base in E　这一条甚至不用metric，一般topo即可

randomly take $x_i \in V_i$, $\{x_n\}$ countable ($\Rightarrow$ base的要求实际上很高)

$\forall x \in E$, $x \in B_r(x)$ for $\forall r > 0$,　$x = x_n$ some n 点和开球之间必有one

or: $\exists$ some V_i: $x \in V_i \subseteq B_r(x)$ $\therefore d(x_i, x) < r$.

that is $\forall x \notin \{x_n\}$, $\forall r > 0$ $\exists x_i$ s.t. $d(x_i, x) < r$, x limit pt

$\therefore \{x_n\}$ dense and countable