CH11. Cardinal arithemetic in ZFC

① this section, we heavily use AC, which implies every set can be well-ordered (equivalent)

given any well-ordered set $(X,\leq)$, $\exists$ a unique ordinal $\alpha = otp(X,\leq)$ s.t. $(X,\leq) \cong (\alpha,\in)$

$\therefore \exists$ bijective $f: X \to \alpha$, $|X| = |\alpha|$, that shows any set has cardinality

② recall: $\aleph_{\beta+1} :=$ the least cardinal which is strictly larger than $\aleph_\beta$ (successive step)

$\therefore$ if $\kappa > \aleph_\alpha \rightleftarrows \kappa \geq \aleph_{\alpha+1}$, which also applies to ORD

Our enventual goal is to get an inductive understanding of the value of κ^λ for infinite cardinal κ, λ.
In Chapter 9, we defined sums, products of pairs of cardinals, More generally, we can add or multiply any set of cardinals.

def 11.1 κ_i is cardinal $\forall i \in I$, then $\sum_{i\in I} \kappa_i = \left|\bigcup_{i\in I} \{i\}\times\kappa_i\right|$ … (1)

$$\prod_{i\in I} \kappa_i = \left|\{f : (\mathrm{dom}(f) = I) \wedge (\forall i \in I)\, f(i) \in \kappa_i\}\right| \quad \cdots (2)$$

rmk: (1) it's like "$A \dot\cup B := \{1\}\times A \cup \{2\}\times B$", we use $\{i\}$ to label the element of κ_i and make "disjoint union"

(2) i.e. choose a value for $f(i)$ from every κ_i; i.e. choose an element from every κ_i

Lem 11.3 if $\kappa_i > 0$ for $\forall i < \lambda$; $\lambda \geq \aleph_0$ then $\sum_{i<\lambda} \kappa_i = \lambda \cdot \sup_{i<\lambda} \kappa_i = \max(\lambda, \sup_{i<\lambda} \kappa_i)$

Pf: $\sum_{i<\lambda} \kappa_i \leq \sum_{i<\lambda} (\sup_{i<\lambda} \kappa_i) \underset{\kappa_i>0}{\leq} \lambda \cdot \sup_{i<\lambda} \kappa_i \underset{CH9}{=} \max(\lambda, \sup_{i<\lambda} \kappa_i)$ … ①

② … $\begin{cases} \sum_{i<\lambda} \kappa_i \geq \sum_{i<\lambda} 1 = \lambda & \text{since } \lambda = \{i : i<\lambda\} \text{ as an ordinal, } \sum_{i<\lambda} 1 = \left|\bigcup_{i<\lambda} \{i\}\times 1\right| = |\{(i,1) : i<\lambda\}| = |\{i : i<\lambda\}| = \lambda \\ \sum_{i<\lambda} \kappa_i = \left|\bigcup_{i<\lambda} \{i\}\times\kappa_i\right| \geq \left|\bigcup_{i<\lambda} \kappa_i\right| = \sup_{i<\lambda} \kappa_i & \end{cases}$

序数的性质：$\bigcup X = \sup X$,

①+②: $\sum_{i<\lambda} \kappa_i = \max(\lambda, \sup_{i<\lambda} \kappa_i)$

用基数性质 $\Rightarrow$ 用序数性质, $\begin{cases} \sum_{i<\lambda} 1 = |\{i : i<\lambda\}| = \lambda \\ \left|\bigcup_{i<\lambda} \kappa_i\right| = \sup_{i<\lambda} \kappa_i, \end{cases}$

注: $\sum_{i<\lambda} \kappa_i = \sum_{i<\bar\lambda} \kappa_i + \kappa_{\bar\lambda}$. if λ is successive, $\lambda = \bar\lambda + 1$ some $\bar\lambda$, $\bar\lambda = \bar{\bar\lambda} + 1$ …

$= \max(\sum_{i<\bar\lambda} \kappa_i, \kappa_{\bar\lambda}) = \max(\sum_{i<\bar{\bar\lambda}} \kappa_i, \kappa_{\bar{\bar\lambda}}, \kappa_{\bar\lambda}) = \cdots = \max_{i<\lambda} \kappa_i$

if λ is limit ordinal, $\sum_{i<\lambda} \kappa_i$ is related to λ's "sup"

not max

prop: if $0 < \kappa_i \le \kappa$, for $\forall i < \lambda$; then $\prod_{i<\lambda} \kappa_i \le \kappa^\lambda$

pf: $\begin{cases} \prod_{i\in I} \kappa_i = |\{f : (\mathrm{dom} f = I) \wedge (f(i) \in \kappa_i \ \forall i < \lambda)\}| \\ \kappa^\lambda = |\{h : \lambda \to \kappa\}| \end{cases}$

$\kappa_i \le \kappa$ $\therefore$ $\exists$ injective $g_i : \kappa_i \to \kappa$ $\forall i < \lambda$ (here we use AC since λ may be large)

def $\Phi : \prod_{i\in I} \kappa_i \to \kappa^\lambda$ $\cdots (*)$

$f \mapsto \Phi(f) : \lambda \to \kappa$, $\Phi(f)(i) := g_i(f(i))$

if $f \ne \tilde{f}$, then $\exists\, i < \lambda$ s.t. $f(i) \ne \tilde{f}(i)$

g_i is injective $\forall i$. $\therefore g_i \circ f(i) \ne g_i \circ \tilde{f}(i)$, i.e. $\Phi(f)(i) \ne \Phi(\tilde{f})(i)$. $\therefore \Phi$ injective

$\therefore |\prod_{i\in I} \kappa_i| \le |\kappa^\lambda|$

rmk: modification $(*)$: $\Phi : \{f : (\mathrm{dom} f = I) \wedge (f(i) \in \kappa_i \ \forall i < \lambda)\} \to \{h : \lambda \to \kappa\}$

$\therefore |\{f : \cdots\}| \le |\{h : \lambda \to \kappa\}|$, thus $\prod_{i<\lambda} \kappa_i \le \kappa^\lambda$

Exe 11.4. if κ, λ is cardinal, then $\prod_{i<\lambda} \kappa = \kappa^\lambda$

pf: $\prod_{i<\lambda} \kappa = |\{f : (\mathrm{dom} f = \lambda) \wedge (f(i) \in \kappa \ \forall i < \lambda)\}| = |\{f : \lambda \to \kappa\}| = \kappa^\lambda$ obv

lem 11.5. infinite cardinal κ is singular iff it's a sum of fewer than κ cardinals smaller than κ

i.e. $\exists\, \lambda < \kappa$, $\{\kappa_i : i < \lambda\}$ where $\kappa_i < \kappa$ $\forall i$, s.t. $\kappa = \sum_{i<\lambda} \kappa_i$

pf: suppose $\{\alpha_i : i \in \lambda\}$ is any sequence of ordinals. $\alpha_i < \kappa$ $\forall i \in \lambda$, $\lambda < \kappa$

if $\{\alpha_i : i \in \lambda\}$ is cofinal in κ, then $\sup_{i<\lambda} \alpha_i = \kappa$

$\therefore \sum_{i<\lambda} |\alpha_i| = \lambda \cdot \sup_{i<\lambda} \alpha_i = \lambda \cdot \kappa = \max(\lambda, \kappa) = \kappa$

$\Rightarrow$ any sequence $\{\alpha_i : i \in \lambda\}$, $\alpha_i < \kappa$ $\forall i \in \lambda$, $\lambda < \kappa$; $\{\alpha_i : i \in \lambda\}$ cofinal in κ then $\kappa = \sum_{i<\lambda} \alpha_i$

recall TH(10.21 König): κ is cardinal, then $\kappa < \kappa^{cf(\kappa)}$

pf: consider $\{f_\alpha : cf(\kappa) \to \kappa, \alpha<\kappa\}$ 为什么不能直接说 $\kappa \cdot \kappa^{cf(\kappa)} > \kappa \cdot 1 = \kappa$?

$cf(\kappa) = \min\{\beta : \exists$ non-decreasing $f : \beta \to \kappa$ s.t. range(f) cofinal in $\kappa\}$

$\therefore \exists\, g$ non-decreasing, s.t. $g : cf(\kappa) \to \kappa$ 实数运算法则完全不能用?

$\sup \text{range}(g) = \sup g(cf(\kappa)) = \kappa \quad \cdots (*)$

具体构造过程

for $\forall \beta < cf(\kappa)$, let $S_\beta := \{f_\alpha(\beta) : \alpha \le g(\beta)\}$

$|S_\beta| = |\{\alpha : \alpha \le g(\beta)\}| = |g(\beta)| < |g(cf(\kappa))| \le \kappa, \quad \therefore \kappa - S_\beta \neq \phi$

def $h : cf(\kappa) \to \kappa$

$h(\beta) = \inf\{\kappa - S_\beta\}$

$\forall \alpha<\kappa, \exists \beta < cf(\kappa)$ s.t. $\alpha \le g(\beta)$ by $(*)$; ie. $\forall \alpha<\kappa$, $f_\alpha(\beta) \in S_\beta$ some β

$\therefore h(\beta) = \inf\{\kappa - S_\beta\} \neq f_\alpha(\beta)$, ie. $f_\alpha \neq h$

$\Rightarrow h \neq f_\alpha$ for $\forall \alpha<\kappa$

Q: 为什么 $\alpha<\kappa$? 只说明 $\kappa \le \kappa^{cf(\kappa)}$. 我觉的不够啊?

TH11.6 (König). $\kappa_i < \lambda_i \ \forall i \in I$, then $\sum_{i\in I} \kappa_i < \prod_{i\in I} \lambda_i$ (if $\kappa_i < \lambda_i$, then $\sum_{i\in I}\kappa_i \le \prod_{i\in I}\lambda_i$)

pf: any function $f : \sum_{i\in I}\kappa_i \to \prod_{i\in I}\lambda_i$, WTS f is not a surjection

given $i\in I$, consider the set $\{f((i,\alpha))(i) : \alpha\in\kappa_i\}$; this is a κ_i size subset of λ_i $\cdots$ (*)

这两个都用 i 是为了用 $\kappa_i<\lambda_i$, 第一个 i 和 α 对应 κ_i size, 第二个对应 λ_i

$\kappa_i < \lambda_i$, $\therefore \lambda_i - \{f((i,\alpha))(i) : \alpha\in\kappa_i\} \neq \phi$

def $h(i) = \inf\{\lambda_i - \{f((i,\alpha))(i) : \alpha\in\kappa_i\}\} \neq f((i,\alpha))(i) \ \forall i\in I, \alpha\in\kappa_i$

$\therefore h \neq f((i,\alpha))$, $h \notin$ range(f)

Corollary (Cantor's): let $\kappa_i = 1, \lambda_i = 2 \ \forall i<\kappa$, then $\kappa = \sum_{i<\kappa} 1 < \prod_{i<\kappa} 2 = 2^\kappa$

corollary (König TH10.2): let $I = cf(\kappa)$, $f : I \to \kappa$ is cofinal, $\kappa_i = f(i)$, $\lambda_i = \kappa$:

$$\kappa = \max(cf(\kappa), \sup_{i\in I} f(i)) = \sum_{i<cf(\kappa)} f(i) < \prod_{i<cf(\kappa)} \kappa = \kappa^{cf(\kappa)}, \quad \therefore \kappa < \kappa^{cf(\kappa)}$$

(*): $\sum_{i\in I}\kappa_i \cong \bigcup_{i\in I}\{i\}\times\kappa_i = \bigcup_{i\in I}(i,\kappa_i)$

$\prod_{i\in I}\lambda_i \cong \{g : (\text{dom} g = I) \wedge g(i)\in\lambda_i \ \forall i\in I\}$ $\quad \therefore f : \sum_{i\in I}\kappa_i \longrightarrow \prod_{i\in I}\lambda_i$

(是否是因为: κ_i 在这里应表示集合 $(i,\kappa_i) \longmapsto f((i,\kappa_i)) : I \longrightarrow \lambda_i$ for $i\in I$

而非具体值? 我认为是) $\quad j \longmapsto f((i,\kappa_i))(j) \in \lambda_i$ if $j=i$

$\therefore \{f((i,\alpha))(i) : \alpha\in\kappa_i\} \subseteq \lambda_i$

问: $f((i,\kappa_i))(i)$ 和 $f((i,\alpha))(i) : \alpha\in\kappa$ 还是不清楚. transfinite recursion 中

$\exists$ unique $F(\alpha) = G(F\upharpoonright\alpha)$ $F\upharpoonright\alpha$ 是像集, 不是 $F(\beta : \beta<\alpha) = F(\alpha)$

Th11.7 fix infinite cardinal λ, $\forall$ infinite cardinals κ, κ^λ is following

1. 若 $\kappa \le \lambda$, $\kappa^\lambda = 2^\lambda$
2. 若 $\kappa > \lambda$, $\exists \mu < \kappa$ s.t. $\mu^\lambda \ge \kappa$, then $\kappa^\lambda = \mu^\lambda$
3. 若 $\kappa > \lambda$, $\forall \mu < \kappa$, $\mu^\lambda < \kappa$. then (1). if $cf(\kappa) > \lambda$, then $\kappa^\lambda = \kappa$

   (2) 若 $cf(\kappa) \le \lambda$, then $\kappa^\lambda = \kappa^{cf(\kappa)}$

Pf: 1: $2^\lambda \le \kappa^\lambda \le (2^\kappa)^\lambda = 2^{\kappa\cdot\lambda} = 2^{\max(\kappa,\lambda)} = 2^\lambda$

2: $\mu^\lambda \le \kappa^\lambda \le (\mu^\lambda)^\lambda = \mu^\lambda$

3:(1) $cf(\kappa) > \lambda$, $\therefore$ every function $f: \lambda \to \kappa$ is bounded

$\therefore \kappa^\lambda = |\bigcup_{\alpha<\lambda} \alpha^\lambda| = \sum_{\alpha<\lambda} |\alpha|^\lambda$ ...(*)

$\kappa = \sum_{\alpha<\kappa} 1 \le \sum_{\alpha<\kappa} |\alpha|^\lambda \le \sum_{\alpha<\kappa} \kappa = \kappa\cdot\kappa = \kappa \Rightarrow \kappa = \sum_{\alpha<\lambda} |\alpha|^\lambda$

$\forall \mu<\kappa, \mu^\lambda<\kappa$ Lem11.3

$\Bigg\} \Rightarrow \kappa = \kappa^\lambda$

(2): $cf(\kappa) \le \lambda$, then $\kappa = \sum_{i<cf(\kappa)} \kappa_i$, $1 < \kappa_i < \kappa$ $\forall i < cf(\kappa)$, $\sup \kappa_i = \kappa$

by König: $\sum_{i<cf(\kappa)} \kappa_i \le \prod_{i<cf(\kappa)} \kappa_i$

$\therefore \kappa^\lambda \le \left(\sum_{i<cf(\kappa)} \kappa_i\right)^\lambda = \left(\prod_{i<cf(\kappa)} \kappa_i\right)^\lambda = \prod_{i<cf(\kappa)} (\kappa_i^\lambda) \le \prod_{i<cf(\kappa)} \kappa = \kappa^{cf(\kappa)} \le \kappa^\lambda \Rightarrow \kappa^{cf(\kappa)} = \kappa^\lambda$

$\forall \mu<\kappa, \mu^\lambda<\kappa$ 按定义写出来

注: $\prod_{i<\sigma} \kappa = \kappa^\sigma$

(*): $\min\{\beta : f: \beta \to \kappa,\ \mathrm{range}(\beta)$ cofinal in $\kappa\} > \lambda$

$\therefore \nexists f: \lambda \to \kappa$, $\sup f(\lambda) = \kappa$, $\therefore \sup f(\lambda) \le \sup f(\beta) < \kappa$ (集合角度 $\lambda < \beta$, $\lambda \subseteq \beta$) 注意 ORD, CARD, set 转换

$\therefore f: \lambda \to \kappa$ bounded $\forall f$

$\kappa^\lambda = \{f: \lambda \to \kappa\}$ $\quad \alpha^\lambda = \{f: \lambda \to \alpha\}$

$f: \lambda \to \kappa$ bounded $\therefore \forall f \in \kappa^\lambda$, $f(\lambda) < \kappa$ ie. $f(\lambda) = \alpha < \lambda$ $\therefore f \in \alpha^\lambda$, $\kappa^\lambda \subseteq \bigcup_{\alpha<\lambda} \alpha^\lambda$

若 $f \in \alpha^\lambda$ some $\alpha \in \kappa$, then $\alpha \in \kappa^\lambda$ $\therefore \bigcup_{\alpha<\lambda} \alpha^\lambda \subseteq \kappa^\lambda$

$\Bigg\} \Rightarrow \kappa^\lambda = \bigcup_{\alpha<\lambda} \alpha^\lambda$

$\bigcup_{\alpha<\lambda} \alpha^\lambda = \bigcup_{\alpha<\lambda} \{\alpha\} \times \alpha^\lambda = \sum_{\alpha<\lambda} |\alpha|^\lambda$

CH: $2^{\aleph_0} = \aleph_1$ GCH: for $\forall$ infinite κ, $2^\kappa = \kappa^+$

Exe 11.8. GCH holds, for infinite κ, λ, $\kappa^\lambda = \begin{cases} \kappa & \lambda < cf(\kappa) \\ \kappa^+ & cf(\kappa) \le \lambda < \kappa \\ \lambda^+ & \lambda > \kappa \end{cases}$ by TH11.7

def 11.9 κ, λ cardinals, then $\kappa^{<\lambda} := \bigcup_{\mu < \lambda} \kappa^\mu$

TH11.12 (Erdős) Wetzel's problem has a negative solution iff CH is true

$\Vert$

$\{f_i\}_{i\in I}$ is a family of pairwise distinct analytic function on the complex number, $\forall z \in \mathbb{C}$, the value set $\{f_i(z) : i \in I\}$ is countable. then is the set $\{f_i\}_{i\in I}$ countable?

pf: "$\Rightarrow$" assume $\neg$CH, $2^{\aleph_0} > \aleph_1$; $|\mathbb{C}| = 2^{\aleph_0}$ ... ①

let set $A(i,j) := \{z \in \mathbb{C} : f_i(z) = f_j(z)\}$, claim: $A(i,j)$ countable $\forall i,j$

pf: f_i, f_j analytic, $\therefore f_i - f_j$ analytic

if $A(i,j) \cap (B_n = \{z : |z| \le n\})$ is infinite, must have limit

(recall: g analytic on $\mathbb{C}$, $A \subseteq \mathbb{C}$ is infinite, $A' \neq \emptyset$, $g|_A = 0$, then $g|_{\mathbb{C}} = 0$)

analytic function $f_i - f_j = 0$ in $A(i,j) \cap B_n$

$\therefore f_i - f_j \equiv 0$ on $\mathbb{C}$, contradict

$\therefore \forall B_n = \{z : |z| \le n\}$, $A(i,j) \cap B_n$ is finite

$\mathbb{C} = \bigcup_{n\in\omega} B_n$, $A(i,j) = \bigcup_{n\in\omega} (A(i,j) \cap B_n)$ is countable

let $A = \bigcup_{i,j\in I} A(i,j)$, let $|I| = \aleph_1$

$|A| = |\bigcup_{i,j\in I} A(i,j)| = |\bigcup_{i,j\in I} \{i\}\times\{j\}\times A(i,j)| \le |\bigcup_{i,j\in I} \{i\}\times\{j\}\times \aleph_0| = |I \times I| = \aleph_1 \times \aleph_1 = \aleph_1$ ... ②

①+②: $\exists z_0 \in \mathbb{C} \setminus A$, $f_i(z_0) \neq f_j(z_0)$ $\forall i,j \in I$, $i \neq j$.

$\therefore |\{f_i(z_0) : i \in I\}| = |I| = \aleph_1$, i.e. Wetzel's problem is positive for z_0,

contradict

rmk: 要证明 CH false, Wetzel's problem is positive i.e. $\forall z \in \mathbb{C}$ $\{f_i(z) : i \in I\}$ countable, then $\{f_i\}_{i\in I}$ countable

$\Leftrightarrow$ if $\{f_i\}_{i\in I}$ uncountable, $\exists z_0 \in \mathbb{C}$, s.t. $\{f_i(z_0) : i \in I\}$ uncountable

given $2^{\aleph_0} > \aleph_1$, $|\{f_i\}_{i\in I}| = |I| = \aleph_1$; let $A = \bigcup_{i,j\in I} A(i,j) = \{z : \text{at } z, \exists i,j \in I, f_i(z) = f_j(z)\}$

$|A| < \mathbb{C}$, $\therefore \exists z_0$, $\nexists i \neq j \in I$ s.t. $f_i(z_0) = f_j(z_0)$

$\therefore |\{f_i(z_0) : i \in I\}| = |I|$

"$\Leftarrow$" if CH, $2^{\aleph_0} = \aleph_1 = |\mathbb{C}|$

let $\{z_\alpha : \alpha \in \omega_1\}$ is a well-ordering of $\mathbb{C}$,

def functions $\{f_\alpha : \alpha < \omega_1\}$ s.t. for $\forall \alpha < \beta$, $\begin{cases} f_\alpha(z_\beta) \in D \\ f_\alpha(z_\beta) \neq f_\beta(z_\beta) \end{cases}$, $D := \{p + qi : p, q \in \mathbb{Q}\}$ dense in $\mathbb{C}$ $(*)$

given $\alpha \in ORD$, enumerate all the ordinals less than α: $\beta_0, \beta_1, \cdots \beta_n, \cdots$

let $f_\alpha(x) = \varepsilon_0 + \varepsilon_1 \cdot (x - z_{\beta_0}) + \varepsilon_2 \cdot (x - z_{\beta_0}) \cdot (x - z_{\beta_1}) + \varepsilon_3 (\cdot)(\cdot)(\cdot) + \cdots$

$|\varepsilon_i| \searrow 0$, then f_α is analytic

by transfinite induction: if we have obtained $\varepsilon_0, \varepsilon_1, \cdots \varepsilon_{n-1}$ s.t. $f_\alpha(z_{\beta_i}) \neq f_{\beta_i}(z_{\beta_n})$ $\forall i < n$,

$f_\alpha(z_{\beta_i}) = \varepsilon_0 + \varepsilon_1 (x - z_{\beta_0}) + \cdots + \varepsilon_i \cdot (x - z_{\beta_0})(x - z_{\beta_1}) \cdots (x - z_{\beta_{i-1}})$; $i = n-1$

$f_{\beta_i}(z_{\beta_n}) = \tilde{\varepsilon}_0 + \tilde{\varepsilon}_1 (x - z_{\beta_0}) + \cdots + \tilde{\varepsilon}_i (\cdot)(\cdot)(\cdot) + \cdots + \tilde{\varepsilon}_n (\cdot)(\cdot) \cdots (x - z_{\beta_{n-1}})$

if $\varepsilon_n \neq (f_{\beta_i}(z_{\beta_n}) - f_\alpha(z_{\beta_i})) \div ((x - z_{\beta_0})(x - z_{\beta_1}) \cdots (x - z_{\beta_{n-1}}))$

then $f_\alpha(z_{\beta_n}) \neq f_{\beta_n}(z_{\beta_n})$, $f_\alpha(z_{\beta_n}) \in D$ $\cdots$ ①?

$\therefore$ f_α analytic, f_α satisfy $(*)$

$\{f_\alpha(z) : \alpha \in \omega_1\}$ is countable $\cdots$ ②?

$|\{f_\alpha : \alpha < \omega_1\}| = |\{f_\alpha : \alpha \in \omega_1\}| = |\omega_1|$ uncountable,

$\therefore$ Wetzel's problem is negative

没明白①.②