Th1: X is compact metric space $\Leftrightarrow$ X is complete, totally bounded

proof1: "$\Rightarrow$" $\forall \varepsilon>0$ $\bigcup_{x\in X} B_\varepsilon(x) \supseteq X$

$\therefore \exists x_1, x_2 \cdots x_n$ s.t. $\bigcup_{i=1}^{n} B_\varepsilon(x_i) \supseteq X$. $\therefore$ X totally bounded

$\forall \{x_n\}_{n=1}^{+\infty} \subseteq X$, $\{x_n\}$ is Cauchy

X is compact $\therefore \exists \{x_{n_k}\}_{k=1}^{+\infty} \subseteq X$ s.t. $x_{n_k} \to x_0 \in X$ $\Rightarrow$ $\{x_n\}$ converges to x_0. X complete

(Cauchy $\{x_n\}$ converges $\Leftrightarrow$ $\{x_{n_k}\}$ converges)

"$\Leftarrow$" goal: $\forall \{x_n\} \subseteq X$, X totally bounded, $\Rightarrow$ $\{x_n\}$ has Cauchy subsequence $\{x_{n_k}\}$

对于任意 $\{x_n\} \subseteq X$, X totally bdd $\therefore \forall \varepsilon>0$. $\exists$ finite x_i s.t. $\bigcup_{i=1}^{n} B_\varepsilon(x_i) \supseteq X \supseteq \{x_n\}$

$\therefore \forall \varepsilon>0$, $\exists x \in X$. s.t. $B_\varepsilon(x) \cap \{x_n\}$ is infinite, countable

$\Rightarrow$ let $\varepsilon=\frac{1}{1}$, $\exists y_1 \in X$, $B_{\frac{1}{1}}(y_1) \cap \{x_n\} = \{x_{11}, x_{12}, x_{13}, \cdots x_{1n}, \cdots\}$

let $\varepsilon=\frac{1}{2}$, $\exists y_2 \in X$, $B_{\frac{1}{2}}(y_2) \cap \{x_{1n}\} = \{x_{21}, x_{22}, x_{23}, \cdots\}$

$\cdots$ 由上过程, 得到 $\{x_{1n}\}_{n=1}^{+\infty}$, $\{x_{2n}\}_{n=1}^{+\infty}$, $\{x_{3n}\}_{n=1}^{+\infty}$, $\cdots$. $\{x_{i+1,n}\}_{n=1}^{+\infty} \subseteq \{x_{i,n}\}_{n=1}^{+\infty}$ $\forall i$.

$\therefore$ let $z_i = x_{ii} \in \{x_{in}\}_{n=1}^{+\infty}$, $d(z_i, z_{i+1}) < \frac{1}{i}$ since $z_{i+1}, z_i \in \{x_{in}\}_{n=1}^{+\infty} \subseteq B_{\frac{1}{i}}(y_i)$

$\left.\begin{array}{l} \therefore \{x_{ii}\}_{i=1}^{+\infty} = \{z_i\}_{i=1}^{+\infty} \text{ is Cauchy} \\ X \text{ is complete space} \end{array}\right\} \Rightarrow \{z_i\}$ converge

即 $\forall \{x_n\} \subseteq X$, $\exists \{z_i\} \subseteq \{x_n\} \subseteq X$, $\{z_i\}$ converge $\Rightarrow$ X compact

prop2: X is totally bounded $\Rightarrow$ X is bounded (always)

proof2: $\forall \varepsilon$, $\exists x_1 \cdots x_n \in X$ s.t. $\bigcup_{i=1}^{n} B_\varepsilon(x_i) \supseteq X$

for x_1, let $r > \varepsilon + \max(d(x_1,x_2), d(x_1,x_3), \cdots d(x_1,x_n))$, $B_r(x_1) \supseteq X$ $\Rightarrow$ X bounded

Rmk: bdd $\not\Rightarrow$ totally bdd

反例 · (Y,d) with $d(y_1,y_2) = \begin{cases} 0 & y_1 = y_2 \\ 1 & y_1 \neq y_2 \end{cases}$, $\forall y \in Y$. $B_r(y) \supseteq Y$ if $r>1$, but not totally bdd

· 无限维空间 $\mathbb{R}^{+\infty}$

Th3: $X \subseteq \mathbb{R}^n$, X is compact $\Leftrightarrow$ X is closed, bounded

proof3: "$\Rightarrow$" complete $\Rightarrow$ closed, totally bdd $\Rightarrow$ bdd

"$\Leftarrow$" $\left.\begin{array}{l} \text{X is bounded } \therefore \exists X \subseteq Y, \text{ Y is compact} \\ \text{closed subset of compact set is compact} \end{array}\right\} \Rightarrow$ X compact

Rmk: Th3 + Th1 可知, in $\mathbb{R}^n$, bdd $\Leftrightarrow$ totally bounded

"bdd $\Rightarrow$ totally bdd" 很难用. separable + complete 也有反例

Th4: Y is totally bounded $\rightleftarrows$ $\bar{Y}$ is totally bounded

proof4: "$\Rightarrow$" $\forall \varepsilon>0$, $\exists\, y_1 \cdots y_n$ s.t. $\bigcup_{i=1}^{n} B_\varepsilon(y_i) \supseteq Y$

$\bar{Y} = Y \cup Y' = Y \cup \partial Y$, $\therefore \bigcup_{i=1}^{n} B_{2\varepsilon}(y_i) \supseteq \bar{Y}$ $\Rightarrow$ $\forall\, 2\varepsilon>0$, $\exists\, y_1 \cdots y_n \in \bar{Y}$ s.t. $\bigcup_{i=1}^{n} B_{2\varepsilon}(y_i) \supseteq \bar{Y}$

otherwise, $\exists\, y \in Y'$, $y \notin B_{2\varepsilon}(y_i)$, $Y \subseteq \bigcup_{i=1}^{n} B_\varepsilon(y_i)$, $\Rightarrow$ $B_\varepsilon(y) \setminus \{y\} \cap Y = \phi$ 矛盾

"$\Leftarrow$" $\forall \varepsilon>0$, $\exists\, y_1 \cdots y_n$, s.t. $\bigcup_{i=1}^{n} B_\varepsilon(y_i) \supseteq \bar{Y}$

不妨设 $y_1, y_2, \cdots y_m \in Y$; $y_{m+1}, y_{m+2} \cdots y_n \in \partial Y \setminus Y$;

obv: $\left(\bigcup_{i=1}^{m} B_\varepsilon(y_i)\right) \cup \left(\bigcup_{i=1}^{n-m} \overline{B_\varepsilon(y_{m+i})}\right) \supseteq \bar{Y}$, $\left(\bigcup_{i=1}^{m} B_\varepsilon(y_i)\right) \cup \left(\bigcup_{i=1}^{n-m} (\overline{B_\varepsilon}(y_{m+i}) \cap \bar{Y})\right) \supseteq \bar{Y}$

$\forall\, \overline{B_\varepsilon}(y_{m+i}) \cap \bar{Y}$, denoted $\bar{Z}_{m+i}$, $\bar{Y}$ is totally bounded $\therefore$ bdd $\Rightarrow$ $\bar{Z}_{m+i} \subseteq \bar{Y}$ bdd

$\bar{Z}_{m+i}$ closed

$\Rightarrow$ $\bar{Z}_{m+i}$ compact,

$\bigcup_{z \in Z_{m+i}^{\circ}} B_\varepsilon(z) \supseteq \bar{Z}_{m+i}$, otherwise $\exists\, z \in \partial Z_{m+i} \setminus Z_{m+i}$ s.t. $B_\varepsilon(z) \setminus \{z\} \cap Z_{m+i} = \phi$

$\Rightarrow$ $\exists$ finite $z_{i1}, z_{i2} \cdots z_{ik_i} \in Z_{m+i}^{\circ} \subseteq Y$ s.t. $\bigcup_{i=1}^{k} B_\varepsilon(z_i) \supseteq Z_{m+i}^{\circ}$

$\therefore \exists\, y_1, y_2, \cdots y_m$; $z_{i1}, z_{i2}, \cdots z_{ik_i}$, $i=1,2,\cdots n-m$ s.t. $\bigcup_{i=1}^{m} B_\varepsilon(y_i) \cup \left(\bigcup_{t=1}^{k_i} \bigcup_{i=1}^{n-m} B_\varepsilon(z_{it})\right) \supseteq Y$

$\therefore Y$ is totally bounded

Th5: $f: (X, d_1) \to (Y, d_2)$ is uniformly cts, X is totally bounded $\Rightarrow$ $f(X)$ totally bdd

proof5: $\forall \varepsilon>0$, $\exists\, x_1, \cdots x_n$, $\bigcup_{i=1}^{n} B_\varepsilon(x_i) \supseteq X$ $\cdots$

(goal: $\forall \varepsilon'>0$, $\exists\, y_1, \cdots y_{n'}$, $\bigcup_{i=1}^{n'} B_{\varepsilon'}(y_i) \supseteq f(X)$)

f is uniformly cts, $\therefore \forall \varepsilon'>0$, $\exists\, \varepsilon>0$ s.t. $d_1(x_i, x)<\varepsilon$ $\forall x_i, x$; $d_2(f(x_i), f(x))<\varepsilon'$

$\therefore$ if $y \in f(B_\varepsilon(x_i))$; $y = f(x)$ for $x \in B_\varepsilon(x_i)$ $\Rightarrow$ $d_2(f(x_i), f(x))<\varepsilon'$

$\therefore B_{\varepsilon'}(f(x_i)) \supseteq f(B_\varepsilon(x_i))$

$\therefore \bigcup_{i=1}^{n} B_{\varepsilon'}(f(x_i)) \supseteq \bigcup_{i=1}^{n} f(B_\varepsilon(x_i)) \supseteq X$

$\therefore$ for this ε', let $y_i = f(x_i)$, $\bigcup_{i=1}^{n} B_{\varepsilon'}(y_i) \supseteq X$ $\Rightarrow$ $f(X)$ is totally bounded

def6: $\bar{X}$ is compact if X is pre-compact

prop6: $\begin{cases} X \text{ is pre-cpt, then } X \text{ is totally bounded} \cdots \text{by Th1 + Th4} \\ X \subseteq Y,\ Y \text{ is a compact set, then } X \text{ totally bounded} \Rightarrow X \text{ pre-cpt} \end{cases}$

(Y complete, $\bar{X} \subseteq Y$ is closed, $\therefore \bar{X}$ complete)

Th7. X is separable, $Y \subseteq X$, $\Rightarrow$ Y is separable

proof7: by definition, exist $E = \{x_1, x_2, \cdots x_n, \cdots\} \subseteq X$, $\bar{E} = X \Leftrightarrow \forall x \in X, \varepsilon > 0, B_\varepsilon(x) \cap E \neq \phi$

$\forall \varepsilon = \frac{1}{n}$, $x = x_m \in X$, 若 $B_{\frac{1}{n}}(x_m) \cap Y \neq \phi$. then pick $y_{n,m} \in B_{\frac{1}{n}}(x_m) \cap Y$ randomly

iter through $n, m = 1, 2, 3, \cdots$ let $D = \{y_{n,m}\}_{n=1}^{+\infty}{}_{m=1}^{+\infty}$, $B_{\frac{1}{n}}(x_m) \cap Y \neq \phi$

$\forall y \in Y$, if $y \in E$, then $y \in D$ obv

if $y \notin E$, $y \in E'$, $\forall r > 0$, $\exists x_i \in E$ s.t. $x_i \in B_r(y)$ $\therefore$ $y \in B_r(x_i) \cap Y$

let $r < \frac{1}{n}$, then $B_{\frac{1}{n}}(x_i) \cap Y \neq \phi$, pick $y_{n,i}$ randomly

if $y_{n,i} = y$. $y \in D$.

if $y_{n,i} \neq y$. $d(y_{n,i}, y) < \frac{1}{n}$. $B_{\frac{1}{n}}(y) \cap D \neq \phi$; let $r \to 0$. $\frac{1}{n}$ approximates 0

$\Rightarrow \forall \varepsilon > 0$, $B_\varepsilon(y) \cap D \neq \phi$, $\therefore \bar{D} = Y$. Y separable

L3.2

def1: $U \subseteq X$ is linearly independent if: $\forall$ finite element of U independent

$S \subseteq X$ is hamel basis if: $\begin{cases} \text{span } S = \{\sum_{i=1}^{n} a_i e_i : e_i \in S, a_i \in \mathbb{F}\} = X \\ S \text{ is linearly independent} \end{cases}$

def2: a partially ordered ralation (denoted $\leq$) in X satisfies: $\begin{cases} a \leq a \\ a \leq b, b \leq c \Rightarrow a \leq c \\ a \leq b, b \leq a \Rightarrow a = b \end{cases}$

例2: $P(X) = \{\text{all subset of } X\}$, "$\subseteq$" 即为偏序

but arbitary $E_1, E_2 \in P(X)$, there may have $E_1 \nsubseteq E_2$, $E_2 \nsubseteq E_1$, 不发生关系

def3: $S \subseteq X$ is a chain if: $\forall s_1, s_2 \in S$, $s_1 \leq s_2$ or $s_2 \leq s_1$ or $s_2 = s_1$, 链中元素必存在偏序关系.

$\begin{cases} p \text{ is upper bound if: } \forall q \in S, q \leq p \\ p \text{ is maximum if: } \forall q \in S, p \leq q \Rightarrow p = q \ ; \ p \in S \end{cases}$

Th1 (Zorn Axiom): X is a partially ordered set, $\forall$ chain $S \subseteq X$, S has upper bound,

then: S has maximal element

证明超纲, 集合论选择公理

哪里用到了"normed"这个条件?

Th2: X is a normed space $\Rightarrow$ X must have hamel basis

proof2: let $L = \{S \subseteq X : S \text{ is linearly independent}\}$, the partial ordered relation "$\subseteq$"

arbitrary $C \subseteq L$, C is chain, let $U_C = \bigcup_{S_\alpha \in C} S_\alpha \subseteq X$, U_C is upper bound of C $\therefore$ C has maximum M_C

找出所有 M_C ($\forall$ chain $C \subseteq L$), $M = \bigcup_{C \subseteq L} M_C$ 并去掉可 ω dependent 的元素得 M', $\text{span}(M) = \text{span}(M')$

则 $M \in L$ 是最大的 independent set, $\text{span}(M) = X$ since: $\forall x \in X$, x in some chain

(任意 $x \in X$, 构造 chain C_x, $x \in C_x$; 找到 M_{C_x}, 并入 M 中)

$\therefore$ M' is the hamel basis

def4: norm on vector space X is a map: $\|\cdot\| : X \to [0, +\infty)$ s.t.

$$\left.\begin{cases} \|x\| = 0 \Leftrightarrow x = 0 \\ \|\lambda x\| = |\lambda| \cdot \|x\|, \ \forall \lambda \in \mathbb{F} \\ \|x+y\| \leq \|x\| + \|y\| \end{cases}\right\} \Rightarrow p : X \to [0, +\infty) \text{ is called Semi-norm}$$

normed space $(X, \|\cdot\|)$ is also a metric space, with induced metric $d(x,y) = \|x-y\|$

Banach space is complete normed space with the induced metric ↓

def5: a subset S in vector space V is convex if:

$x, y \in S$, the line joining x, y lies in S; $\Leftrightarrow \forall \lambda \in [0,1]$ $\lambda x + (1-\lambda) y \in S$ if $x, y \in S$

例5.1: in any normed space $(X, \|\cdot\|)$, $\bar{B}_X = \{x \in X : \|x\| \leq 1\}$, $B_X = \{x \in X : \|x\| < 1\}$ are convex

lem5.2: the map $N: X \to [0, +\infty)$ satisfies: $\begin{cases} N(x) = 0 \Leftrightarrow x = 0 \\ N(\lambda \cdot x) = |\lambda| \cdot N(x), \ \forall \lambda \in \mathbb{F} \ x \in X \end{cases}$ $\Rightarrow N$ is a norm on X

"+ $\bar{B}_X = \{x \in X : N(x) \leq 1\}$ is convex set"

Rmk: 单位球 $(N \leq 1)$ convex $\Leftrightarrow$ 三角不等式 $\underset{x}{\Leftrightarrow}$ 单位开球 convex

proof5.2: WTS: $N(x) \leq 1$, $N(y) \leq 1$, $N(\lambda x + (1-\lambda) y) \leq 1 \Rightarrow N(a+b) \leq N(a) + N(b)$

let $a = \frac{x}{N(x)}$, $b = \frac{y}{N(y)}$ $\quad N\left(\frac{x}{N(x)}\right) = \frac{1}{N(x)} \cdot N(x) = 1 \ \therefore N(a) = N(b) = 1$

let $\lambda = N(x) / N(x) + N(y)$, $1-\lambda = N(y) / N(x) + N(y)$; $\lambda a + (1-\lambda) = \frac{x+y}{N(x) + N(y)} \in \bar{B}_X$

$\therefore N(x+y / N(x) + N(y)) \leq 1 \Rightarrow N(x+y) \leq N(x) + N(y)$

def6: $f: [a,b]' \to \mathbb{R}'$, f convex if: $f(\lambda x + (1-\lambda) y) \leq \lambda \cdot f(x) + (1-\lambda) \cdot f(y)$ $\forall \lambda \in [0,1]$

$\Leftrightarrow f''(x) \geq 0$ $x \in (a,b)$, when $f \in C^2(a,b) \cap C[a,b]$

例6.1 $g: s \mapsto |s|^p$ $p \in [1, +\infty)$ convex, $h: s \mapsto e^s$ convex

例6.2 $\|\cdot\|$ Minkowski's norm (in FA1 notes) is convex ... by lem5.2 (obv)

proof6.2: $p = +\infty$ is obv

$p < +\infty$ $\bar{B}_X = \{x \in X : \|x\| = (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} \leq 1\}$

$x, y \in \bar{B}_X$, $\forall \lambda \in [0,1]$, $\|\lambda x + (1-\lambda) y\| \leq \lambda \cdot (\sum_{i=1}^{n} |x_i|^p)^{\frac{1}{p}} + (1-\lambda) \cdot (\sum_{i=1}^{n} |y_i|^p)^{\frac{1}{p}} \leq \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1$

同样的, let $n \to +\infty$, 可以推广到无穷维

例6.3: some normed space:

$\begin{cases} (C[0,1], \|\cdot\|) & \|f\| = \max_{x \in [0,1]} |f(x)| \quad \Rightarrow (C_b(X : \mathbb{F}), \|\cdot\|) \quad \|f\| = \sup_{x \in X} |f(x)| \quad f \text{是} X \to \mathbb{F} \text{的bdd,cts} \\ (C[0,1], \|\cdot\|) & \|f\|_p := (\int_0^1 |f(x)|^p dx)^{\frac{1}{p}} \end{cases}$

def 7: in $(X, \| \|_1)$, $(X, \| \|_2)$, the norms are equivalent if: $\exists\, C_1, C_2 > 0$ s.t. $C_1 \|x\|_2 \leq \|x\|_1 \leq C_2 \|x\|_2$

Th7.1: in X, $\|\cdot\|_1 \sim \|\cdot\|_2$, $\Rightarrow (X, \|\cdot\|_1), (X, \|\cdot\|_2)$ have same topology

i.e. $U \subseteq (X, \|\cdot\|_1)$ is open $\rightleftharpoons U \subseteq (X, \|\cdot\|_2)$ is open

proof 7.1: U is open in $(X, \|\cdot\|_1)$, then $\forall x_0 \in U$, $\exists r$, s.t. $\|x - x_0\|_1 < r$, then $x \in U$ ①

$\|\cdot\|_1 \leq C_2 \cdot \|\cdot\|_2$ ②

$(③ + ② \Rightarrow ①, x \in U) \Rightarrow \forall x_0 \in U, \exists r' = \frac{r}{C_2}$, s.t. $\|x - x_0\|_2 \leq \frac{r}{C_2}$ ③, then $x \in U$

$\therefore U$ open in $(X, \|\cdot\|_1) \Rightarrow U$ open in $(X, \|\cdot\|_2)$

同样可知另一侧，略

Definition 8 $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are isomorphic, write $X \simeq Y$, if $\exists$ a bijective linear map $T : X \to Y$ and $c_1 > 0, c_2 > 0$ such that

$$c_1 \|x\|_X \leq \|T(x)\|_Y \leq c_2 \|x\|_X \ \forall x \in X; \tag{1.14}$$

and are *isometric* if in addition T preserves the norm, i.e.,

$$\|T(x)\|_Y = \|x\|_X \ \forall x \in X,$$

and in this case, T is called an isometry from X to Y.