Lec10:

def1: G is a group. Ω is a set; G act on Ω if:

(1): $\forall g\in G$ is a bijection from Ω to Ω (故也为Ω上的permutation)

(2): $\forall w\in\Omega$, $w^1 = w$, $1\in G$

(3): $\forall w\in\Omega$, $g,h\in G$, $w^{gh}=(w^g)^h$, 记为 $G\curvearrowright\Omega$

例: · $G=GL(V)$, V is vector group

$v^g = gv$, 满足(1)(2)(3), $GL(V)$ acts on V

· $\mathrm{Sym}(\Omega)$, $\mathrm{Alt}(\Omega)$ acts on Ω obv

def2 (1): G is group. $\forall g\in G$: define right multiplication, 记为 $\hat g$

$\hat g: G\to G$　　　　记the group of action $\hat G$

$x \to xg$

(2) left multiplication, 记为 $\check g$, $\check G$

$\check g: G\to G$

$x\to g^{-1}x$　　$x^{\check g\check h} = h^{-1}g^{-1}x = (x^{\check g})^{\check h} = (g^{-1}x)^{\check h} = h^{-1}g^{-1}x$

用g^{-1}才能满足 $w^{gh}=(w^g)^h$ 性质

(3): conjugation, $\tilde g$, $\tilde G$

$\tilde g: G\to G$

$x\to g^{-1}xg$

Prop: ① $\hat g, \check g, \tilde g\in \mathrm{Sym}(\Omega)$, $\hat G, \check G, \tilde G\subseteq \mathrm{Sym}(\Omega)$

② $g,h\in G$　$\hat g\check h = \check h\hat g$　since: $w^{\hat g\check h} = (wg)^{\check h} = h^{-1}wg = w^{\check h\hat g} = (h^{-1}w)^{\hat g}$

③ $\hat g\check g = \tilde g$　since: $w^{\hat g\check g} = (wg)^{\check g} = g^{-1}wg = w^{\check g\hat g} = w^{\tilde g}$

④: $\hat G\cap\check G = Z(\hat G) = Z(\check G)$

$\hat G\cap\tilde G = \check G\cap\tilde G = \{1\}$

⑤: $\langle\hat G,\check G\rangle = \hat G\check G = \hat G\circ\check G = \hat G:\check G = \check G:\hat G$

中心积, 后面有

④: $\bar g\in\hat G\cap\check G$, $w\in\Omega$

~~then~~ $\cancel{w^{\bar g} = w^{\hat g} = wg = w^{\check g} = g^{-1}w}$; WTS: $\hat h\bar g = \bar g\hat h$ $\forall \hat h\in\hat G$, $\check k\bar g = \bar g\check k$ $\forall\check k\in\check G$

$\because \begin{cases} w^{\hat h\bar g} = (wh)^{\bar g} = g^{-1}wh \\ w^{\bar g\hat h} = g^{-1}wh \end{cases}$

原因来自② $\hat g\check h = \check h\hat g$, here $\bar g\in\hat G$ $\therefore \check k\bar g = \bar g\check k$; $\bar g\in\check G$ $\therefore g\in Z(\check G)$ $\therefore \hat G\cap\check G\subseteq Z(\check G)$

$\bar g\in\check G$ $\therefore \bar g\hat h = \hat h\bar g$; $\bar g\in\hat G$ $\therefore g\in Z(\hat G)$ $\therefore \hat G\cap\check G\subseteq Z(\hat G)$

另一侧易证略

设 $g\in\hat{G}\cap\tilde{G}$ 注意：CAA和老师用的定义不同

$w^{\hat{g}}=wg$, $w^{\tilde{g}}=g^{-1}wg$ $\therefore g=1$ $\therefore \hat{G}\cap\tilde{G}=\check{G}\cap\tilde{G}=\{1\}$ $c(a)$ 为 conjugacy class

补充：CAA P388-389 ⇒ 见 P37，很多后面讲了

Thm: G finite, $|cl(a)|=|G:C(a)|$，$cl(a)$ 不一定为 group

$cl(a)=\{xax^{-1}: x\in G\}$ conjugacy class of a

$C(a)=\{g\in G: ga=ag\}$ a 的 centralizer in G；同样可以有 $C(H)$ $H\le G$

define $\varphi: G/C(a)\to cl(a)$

$xC(a)\to xax^{-1}$

surjective 显然

$xax^{-1}=yay^{-1}$, $y^{-1}x\in C(a)$ $\therefore x=y\cdot c$ for some $c\in C(a)$, $xC(a)=yC(a)$

$\therefore$ injective

$\Rightarrow \therefore |G/C(a)|=|cl(a)|$

数量相等证明就行，不要 homo

corol: G finite, $|G|=\sum_{i=1}^{N}|G:C(a_i)|=\sum_{i=1}^{N}|cl(a_i)|$

$cl(a)$ 是 G 中的等价类，$G=cl(a_1)\cup cl(a_2)\cup\cdots cl(a_N)$

Thm: p-groups have nontrivial centers (讲了) ⇒ HW 6.8, 6.3

$|G|=p^n$ some n, p is prime, $G\neq 1$, $Z(G)$ has more than one element

考虑：设 $a\in Z(G)$, $cl(a)=\{xax^{-1}: x\in G\}=\{a\}$；$C(a)=G$

$\therefore |G|=\sum_{i=1}^{N}|G:C(a_i)| = |Z(G)|+\sum_{i=1}^{M}|G:C(b_i)|$ $=|cl(a)|$

$a\in Z(G)$, $|G:C(a)|=1$，放到 $|Z(G)|$ 里

设 $C(b_i)\neq G$, $\forall i$. $|G:C(b_i)|=p^k$；$|C(b_i)|\neq 1$

$|C(b_i)|$ 整数，$|G:C(b_i)|$ 也是整数 ？ 为什么：Thm $\therefore p\mid |G:C(b_i)|$ 即 $p\mid |cl(a)|$

又 $\because p\mid |G|$ $\therefore p\mid Z(G)$，$Z(G)$ 中至少有 p 个元素

不是因为这个，而是若 $\exists\, C(b_i)=G$

$cl(a)$ 不为子群 $|cl(b_i)|=|G:C(b_i)|=1$ 即 $cl(b_i)=\{b\}$ $\therefore b\in Z(G)$ 放到 $Z(G)$ 那一组

corol 2: $|G|=p^2$, p is prime, G is Abel recall p13

$|Z(G)|=p$ or p^2 (证完了)

若 $|Z(G)|=p$. $|G/Z(G)|=p$ $\therefore G/Z(G)$ cyclic by 9.3 G Abel

在 L5 中用 Abel 证 $G\cong\mathbb{Z}_{p^2}$ or $\mathbb{Z}_p\times\mathbb{Z}_p$ ⇒ 这要先证 Abel)

CAA P187: G is a group; $G/Z(G)$ is cyclic $\Rightarrow$ G is Abel

(即说明 $Z(G)=G$. $G/Z(G)$ element 只有 $Z(G)$)

易证: $G/Z(G)=\langle gZ(G)\rangle$; $a\in G$ $a=g^iz$ for some $z\in Z(G)$

$\therefore ab=g^iz_1\cdot g^jz_2=g^i\cdot g^jz_2\cdot z_1=g^jz_2g^iz_1=ba$ $\forall a,b$

$\therefore G$ Abel

def3: group G,H, $\exists C\lesssim Z(G)$, $C\lesssim Z(H)$; 即 $C\neq 1$, $Z_1\leq Z(G)$, $Z_2\leq Z(H)$; $Z_1\cong Z_2\cong C$

(map homo + inj, 不surjective)

ϕ isomorphism from Z_1 to Z_2, $X=(G\times H)/\langle(x,x^{\phi}):x\in Z_1\rangle\cong(G\times H)/C$ 易证

then: X is called a central product of G,H, 记为 $G\circ H$

定义方式, 和 $H\rtimes K$ 一样: pf $H\rtimes K=HK$, $H\cap K=\{e\}$

def4: $H,K\leq G$, $H\trianglelefteq G$, $H\cap K=\{1\}$, then $\langle H,K\rangle=HK=H\rtimes K=H:K$

记为 semi-direct product of H,K

Rmk. iso-2 th: $H,K\leq G$, $H\trianglelefteq G$

$\Rightarrow H\trianglelefteq HK$, $H\cap K\trianglelefteq K$, $\therefore HK/H\cong K/H\cap K$

$A\trianglelefteq G$, $B\trianglelefteq G$, $A\cap B=\{1\}$; $\Rightarrow\forall a\in A, b\in B$. $a\cdot b=b\cdot a$. 可以说明 $AB=A\times B$ direct product

这里只有一个 $H\trianglelefteq G$.

$\langle HK\rangle=HK$ since HK is a group

$H\rtimes K$ $(h_1,k_1)(h_2,k_2)=(h_1h_2^{\phi(k_1)},k_1k_2)$, HK: $(h_1,k_1)\cdot(h_2,k_2)=(h_1h_2,k_1k_2)$

例: $C_3\rtimes C_2=D_6=\langle a,b: a^3=b^2=1, b^{-1}ab=a^{-1}\rangle=\{1,a,a^2,ab,a^2b:\cdots\}$

$C_3\times C_2=C_6$

对于 $C_3\rtimes C_2$, 定义 $\varphi: C_2\to Aut(C_3)$, $C_2=\{e,a\}$ $C_3=\{e,b,b^2\}$

$\begin{cases} e\to \text{id map} & 全\to e 就是直积 \\ a\to \sigma: x\to x^{-1} \end{cases}$

为什么 $H\rtimes K$ 中元素的积要有 HK 元素积的形式

$(h_1,k_1)(h_2,k_2)=(h_1h_2^{\phi(k_1)},k_1k_2)$ $\Rightarrow$ $h_1h_2^{\phi(k_1)}k_1k_2=\underline{h_1k_1}\cdot\overbrace{\underline{h_2^{\phi(k_1)}k_2}}^{\text{want}}$ $\Rightarrow$ 结果

(大致理解即可)

在直积中 $h_1h_2k_1k_2=h_1k_1h_2k_2$ 可交换, $HK=H\times K$

HK is group 只要 H,K 中有一个 normal 就行

半: $h_1k_1\cdot h_2k_2=h_1k_1^{\phi(h_2)}h_2k_2\overset{s.t.}{=}h_1h_2k_1^{\phi(h_2)}k_2$

直: $h_1k_1\cdot h_2k_2=h_1h_2\cdot k_1k_2$

☆def5: G acts on Ω, then G partitions Ω into orbits; an orbit is $\Delta = w^G$ $\overset{\shortparallel}{\text{normal closure}} = \{w^g : g \in G\}$

核心概念 $\Omega = \dot{\bigcup}_{w \in \Omega} w^G$

since w^G 和 x^G 为等价类 设 $\exists\, w^{g_1} = x^{g_2}$, $x = w^{g_1 g_2^{-1}}$

$x^g = w^{g_1 g_2^{-1} g}$ $\therefore x^G \subseteq w^G$ 同理 $w^G \subseteq x^G$

例: G act on G, since $x^{gh} = (g^{-1}xg)^h = h^{-1}g^{-1}xgh = x^{gh}$

$G = \dot{\bigcup}_{x \in G} x^G \cup \{e\}$, x^G 称为 conjugacy class of x, 记为 $C(x)$

$C(x) = \{x^g : g \in G\} = \{g^{-1}xg : g \in G\}$

☆def6: G acts on Ω, $G_w = \{g \in G : w^g = w\}$: stablizer of w in G, $G_w \leq G$

Th3: (Orbit-Stabilizer Th) G acts on Ω, $w \in \Omega$, $|G| = |w^G| \times |G_w|$

$\Delta = w^G = \{\delta_1, \delta_2, \dots \delta_n\}$, $\delta_i = w^{g_i}$, $\delta_1 = w^e = w$

设 $\delta_1^h = \delta_i \Leftrightarrow w^{g_1 h} = w^{g_i}$, $w^{g_1 h g_i^{-1}} = w \Leftrightarrow hg_i^{-1} \in G_{w^{g_1}} = G_{\delta_1}$

$\Leftrightarrow h \in G_{\delta_1} \cdot g_i$ $\quad \delta_1^h = \delta_1,\ \delta_1^h = \delta_2 \quad \dots \quad \delta_1^h = \delta_n$

$\therefore G = \cancel{G_{\delta_1} \dot{\cup} G_{\delta_2} \dots \dot{\cup} G_{\delta_n}} = G_{\delta_1} \dot{\cup} G_{\delta_1} g_2 \dot{\cup} \dots G_{\delta_1} g_n$

$\therefore |G| = \cancel{|G_{\delta_1}| + |G_{\delta_2}| + \dots |G_{\delta_n}|} = |G_{\delta_1}| + |G_{\delta_1} g_2| + \dots |G_{\delta_1} g_n|$

$= |\Delta| \cdot |G_{\delta_1}|$

$= |\Delta| \cdot |G_w|$

$|G_{\delta_1}| = |G_{\delta_1} g_i|$

$\phi: G_{\delta_1} \to G_{\delta_1} g_i$

$x \to x g_i$

obv ϕ is bijective

ϕ: 类似右乘

补充点: 之前在def5证明了 $\Delta = w^G$ 形成等价类

def6中 G_w 是没有等价类的, G 没法写成 $\dot{\bigcup}_{w \in G} G_w$ ×

✓① $G = \{h : \delta_1^h = \delta_1\} \cup \{h : \delta_1^h = \delta_2\} \dots \cup \{h : \delta_1^h = \delta_n\}$

② $G_{\delta_i} = G_{\delta_1} g_i$ ×

$h \in G_{\delta_i}$, $\delta_i^h = \delta_i$ 即 $\delta_1^{g_i h} = \delta_1^{g_i} \Rightarrow \delta_1^{g_i h g_i^{-1}} = \delta_1$

$\therefore h \in g_i^{-1} G_{\delta_1} g_i$ 而不是 $G_{\delta_1} g_i$ $\quad$ 没有 $G_{\delta_1} \dot{\cup} G_{\delta_2} \dot{\cup} G_{\delta_3} \dots$ 等价类

①这么定成是因为 $w^G = \{w^{g_1}, w^{g_2}, \dots w^{g_n}\}$

$= \{\delta_1, \delta_2, \dots \delta_n\}$

$\delta_i^g = (w^{g_i})^g = w^{g_i g} \in w^G = \delta_j$ some j

整理Th3: $w^G=\{w^{g_1}, w^{g_2}, \cdots w^{g_n}\}=\{s_1, s_2, \cdots s_n\}$ $s_1=w^e=w$ → 直接写 $\{w, s_2, s_3 \cdots\}$ 想办法把标记写清楚

w^G closed, $(s_i)^g=w^{g_i g}\in w^G$

$\therefore s_i^g$ 可能取值为 $s_1, s_2, \cdots s_n$

$G=\{g: s_1^g=s_1\}\cup\{g: s_1^g=s_2\}\cdots\cup\{g: s_1^g=s_n\}$

若 $s_1^g=s_i$, $s_1^g=s_1^{g_i}$, $s_1=s_1^{g^{-1}g_i}$ 或 $s_1=s_1^{g g_i^{-1}}$

$\therefore g^{-1}g_i\in G_{s_1}$, $g\in G_{s_1}g_i$

即: $\{g: s_1^g=s_i\}=G_{s_1}g_i$, $i\geq 2$

$\therefore G=G_{s_1}\cup G_{s_1}g_2\cup\cdots G_{s_1}g_n$

$|G|=|G_{s_1}|+|G_{s_1}g_2|+\cdots|G_{s_1}g_n|$

又 $|G_{s_1}|=|G_{s_1}g_i|$ 因 since we have right multiplication $g_i: G_{s_1}\to G_{s_1}g_i$
$x\to xg_i$

$\therefore |G|=|O|\cdot|G_{s_1}|=|O|\cdot|G_w|$

review P34 CAA详细写法

Th4: (Sylow's 1st Th) G finite, $|G|=p^n m$, p is prime, $(p,m)=1$
then $\exists H\leq G$, $|H|=p^n$; H 称为 sylow p subgroup of G; 记为 $H\in Syl_p(G)$

compare to L8: 对比 Recall: finite Abel G 中, $|G|=p^n\cdot m$, let $H=\{x\in G: x^{p^n}=e\}$, $|H|=p^n$ 这个证明用了Abel
实际上学Sylow之后不用Abel条件了

$|G|=|Z(G)|+|C(g_1)|+\cdots|C(g_r)|$ ⋯ P34写了为什么

Induction: 设对于所有比 $|G|$ 更小的群均成立

① 设 $p\mid|Z(G)|$, $Z(G)$ sylow p 子群记为 N, $N\lhd G$, $|N|=p^{n_1}$

设 $\bar{G}=G/N$, $|\bar{G}|<|G|$ $\therefore \bar{G}$ 有 sylow-p 子群

若 $|N|=p^n$, N 为 sylow p subgroup of G

若 not, 考虑 $\bar{G}$ 的 sylow p 子群 $\bar{M}$, $|\bar{M}|=p^{n_2}$, $n_2=n-n_1$

$M=\{x: xN\in\bar{M}\}$

M 是 G 的 sylow-p 子群 since: $|M|=p^{n_1}\cdot p^{n_2}=p^n$

⇓

$N=\{h_1, h_2, \cdots h_{p^{n_1}}\}$ $x\in M$ 则 $xh_i\in M$

这种 x 有 $|\bar{M}|$ 个, 可以看成coset等价类

再理解一下!

② $p\nmid|Z(G)|$, then $p\nmid|C(g_i)|$ some i ⇒ ~~P34此情况应该是不成立的~~ 的 m

$|G|=|g_i^G||G_{g_i}|$

$=|C(g_i)|\times|C_G(g_i)|$ ← $C_G(g_i)$ 表示中心化子 $hg_i=g_ih$

$G_{g_i}=\{h: g_i=h^{-1}g_ih\}$

$\therefore p^n\mid|C_G(g_i)|$ $\therefore C_G(g_i)$ 由 p^n 子群. 这也是 G 的sylow群