CH14. club and stationary set

club filter & stationary set are the most important largeness notion for subsets of cardinal κ.
(to describe how large κ is)

def 14.1 λ is a limit ordinal, $C \subseteq \lambda$, then

① $C \subseteq \lambda$ is closed in λ: for $\forall \nu < \lambda$, if $C \cap \nu$ is cofinal in ν, then $\nu \in C$

② $C \subseteq \lambda$ is unbounded in λ: for $\forall \alpha \in \lambda$, $\exists \beta \in C$ s.t. $\beta \nleq \alpha$ 任何一个$\alpha\in\lambda$，均有C中的元素不能被限制住

a closed, unbounded subset of λ is a club set of λ

rmk: $\sup(C \cap \nu) = \nu$ implies $\nu \in C$ $\underset{\text{"2"}}{\overset{\text{"1"}}{\rightleftarrows}}$ C contains the suprema of all increasing sequence in C (Exe 14.2)

C is infinite by default?

1. $\{\alpha_\xi : \xi < \xi_0\}$ is increasing; $\forall \xi < \xi_0$, $\alpha_\xi \in C$

let $\gamma = \sup_{\xi < \xi_0} \alpha_\xi$, $\{\alpha_\xi : \xi < \xi_0\}$ is cofinal in γ

① if γ is successor, $\gamma = \alpha_\xi + 1$ some $\xi < \xi_0$

then $\alpha_\xi + 1 = \sup\{\alpha_\xi : \xi < \xi_0\}$, $\because \xi + 1 < \xi_0$ otherwise $\gamma = \alpha_\xi$ 实际上序列无限的话 sup ⇒ must be limit ord

$\alpha_{\xi+1}$ is the successor of α_ξ, $\xi + 1 > \xi$, $\therefore \alpha_{\xi+1} \geq \alpha_\xi + 1 = \sup\{\alpha_\xi\} \geq \alpha_{\xi+1}$, i.e. $\gamma = \max\{\alpha_\xi\} \in C$

② if γ is limit ordinal, $\alpha_\xi < \gamma$ $\forall \alpha_\xi$

$\gamma = \sup \alpha_\xi$ $\therefore \forall x < \gamma, \exists \alpha_\xi > x$, $\alpha_\xi \in C \cap \gamma \Rightarrow \sup(C \cap \gamma) \geq \gamma$

$\forall x > \gamma$, $\alpha_\xi < x$ for $\forall \alpha_\xi \in C \cap \gamma \Rightarrow \sup(C \cap \gamma) \leq \gamma$

$\therefore \sup(C \cap \gamma) = \gamma$, thus $\gamma \in C$

rmk: 直接按照 sup的定义列出来！不要想技巧！老实列定义！

2. if $\nu = \sup(C \cap \nu)$

① if ν is successor, $\nu = \bar{\nu} + 1$, if $\nu \notin C$, then $\sup(C \cap \nu) = \sup(C \cap \{1, \ldots \bar{\nu}\}) < \nu$ contradict, $\therefore \nu \in C$

② if ν is limit ordinal,

$\sup(C \cap \nu) = \nu$ implies $\exists$ sequence in $C \cap \nu$ thus in C approach ν. the sequence is increasing

$\therefore \nu \in C$ obv

corollary: $C \subseteq \omega_1$ is the set of limit ordinals of ω_1, then C is closed; any C' with $C \subseteq C' \subseteq \omega_1$ is closed

pf: by Exe 14.2 if $\{\alpha_\xi\}_{\xi \in \lambda}$ is increasing sequence in $\underset{\substack{\downarrow \\ \ldots \quad C'}}{C}$, then $\sup\{\alpha_\xi\}$ is limit ordinal thus $\underset{\substack{\downarrow \\ \ldots \quad \text{in } C \text{ thus in } C'}}{\text{in } C}$

Consider order topology on ordinal λ

subbasis: $\{\gamma: \beta<\gamma<\lambda\} = (\beta, \lambda)$ for $\forall \beta<\lambda$ 左射线, $\{\gamma: 0\le\gamma<\delta\} = [0,\delta)$ for $\forall \delta<\lambda$ 右射线

basis: $(\beta,\delta) = \begin{cases} \{\gamma: \beta<\gamma<\delta\} & \beta\neq 0 \\ \{\gamma: 0\le\gamma<\delta\} & \beta=0 \end{cases}$, the same as in $\mathbb{R}^1$

Exe11.3. $C\subseteq\lambda$ is closed in $\lambda \rightleftarrows C$ is closed in the order topology

* Exe11.4 ω_1 equipped with order topology is sequentially compact, but not compact

pf: let $U_\alpha = [0,\alpha) = \{\xi<\omega_1 : \xi<\alpha\}$

$\{U_\alpha\}_{\alpha<\omega_1}$ is open cover of ω_1, but there's not a finite subcover $\Rightarrow$ not compact

证不动了...

lem11.5 λ is a limit ordinal, $cf(\lambda)>\omega$; $C, C'\subseteq\lambda$ are clubs in λ; then $C\cap C'$ is still a club (可数并)

pf: $\{\alpha_\xi\}_{\xi\in\lambda}$ is an increasing sequence in $C\cap C'$, then $\{\alpha_\xi\}_{\xi\in\lambda}$ is ... in both C and C'

$\therefore \sup\{\alpha_\xi\}_{\xi\in\lambda}\in C, C'$ thus $\sup\{\alpha_\xi\}_{\xi\in\lambda}\in C\cap C'$

$\therefore C\cap C'$ is closed

$\forall\beta\in\lambda, \exists\alpha_1\in C$ s.t. $\alpha_1>\beta$; $\alpha_1\in\lambda$ let α_1 be the least ord satisfy the previous conditional ...

$\therefore \exists\alpha_2\in C'$ s.t. $\alpha_2>\alpha_1$; $\alpha_2\in\lambda$ let α_2 be the least ...

$\therefore$ we obtain $\{\alpha_{2n}: n\in\omega\}\subseteq C'$, $\{\alpha_{2n+1}: n\in\omega\}\subseteq C$,

(*) $\begin{cases} \sup\{\alpha_{2n}: n\in\omega\}\in C', \sup\{\alpha_{2n+1}: n\in\omega\}\in C \text{ since } C, C' \text{ both closed}, \\ \sup\{\alpha_{2n}\cdots\} = \sup\{\alpha_{2n+1}\cdots\} \text{ by construction}. \end{cases}$

$\therefore \forall\beta\in\lambda, \exists \sup\{\alpha_{2n}\cdots\} = \sup\{\alpha_{2n+1}\cdots\}\in C\cap C'$ can't be bounded by β

$\therefore C\cap C'$ unbounded

Exe10.5 $\forall$ ordinal α, $cf(\alpha)$ = the shortest length λ of a strictly increasing sequence cofinal in α

$= \min\{|A| : A\subseteq\alpha, \sup A = \alpha\}$

What if $cf(\lambda)\le\omega$ in Lem11.5? $\begin{cases} cf(\lambda)<\omega, \text{ then } cf(\lambda) \text{ finite}, \alpha \text{ is successor}, \text{ construction of (*) is invalid} \\ cf(\lambda)=\omega\,? \end{cases}$

lem14.6 λ is cardinal, $cf(\lambda)>\omega$; $\{C_\alpha : \alpha<\beta\}$ is a sequence of club where $\beta<cf(\lambda)$; then $\bigcap_{\alpha<\beta} C_\alpha$ is a club.

pf: $\bigcap C_\alpha$ is closed, proof similarly to lem14.5

$\forall \beta\in\lambda$, $\exists \xi_{1,1}\in C_1$ s.t. $\xi_{1,1}>\beta$, $\xi_{1,1}\in\lambda$; let $\xi_{1,1}$ be the least ordinal satisfying ...

$\exists \xi_{2,1}\in C_2$, s.t. $\xi_{2,1}>\xi_{1,1}$, $\xi_{2,1}\in\lambda$; let $\xi_{2,1}$ be the least ...

... we obtain $\{\xi_\alpha\}_{\alpha<\beta\cdot\omega}$ of length $\beta\cdot\omega$, where $\sup\{\xi_{\alpha,i}\}_{\alpha<\beta}=\sup\{\xi_{\bar\alpha,i}\}_{\bar\alpha<\beta}\in C_\alpha\cap C_{\bar\alpha}$

$\therefore \bigcap C_\alpha$ is unbounded

和lem14.5完全一样的证法 　　$\beta<cf(\lambda)$, $cf(\lambda)>\omega$?

Exe14.7 if $cf(\lambda)=\omega$, $\exists$ disjoint clubs in λ ; if $cf(\lambda)>\omega$, $\exists$ sequence of $cf(\lambda)$ many clubs whose intersection is empty

def14.8 (The club filter) the filter F generated by the club sets on a regular cardinal κ is called "club filter" on κ. That is, $C\subseteq\kappa$ is the club filter on κ $\rightleftarrows$ C contains a club set in κ.
(not necessarily club itself)

Note: by lem14.6, the club filter on κ is κ-complete

recall: F is a filter if $\begin{cases} A\in F, B\in F, A\cap B\in F & \text{--- by lem14.6} \\ A\in F, B\supseteq A, B\in F & \cdots \begin{cases} A \text{ unbounded} \Rightarrow B\supseteq A \text{ unbounded obv} \\ A \text{ closed} \Rightarrow B \text{ closed !} \end{cases} \end{cases}$

比club更大的集合一定是club，无界&闭均可"向上推"

recall: κ-complete

$\{A_i : i<\theta\}\subseteq F$, $\bigcap_{i<\theta} A_i\in F$ for $\forall\theta<\kappa$

def14.10. $\kappa>\omega$ is a regular cardinal. $S\subseteq\kappa$ is stationary if S intersects every club in κ

the dual ideal to the club filter is called "non-stationary ideal".

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$\exists C\subseteq\kappa$, s.t. C is a club, $C\cap I=\phi$ 可以避开某一个club

前提是S先要在k中！ => regular cardinal 显然一定 stationary

Exer14.11 $cf(\kappa)=\kappa>\omega$

1. if $C\subseteq\kappa$ is a club, $S\subseteq\kappa$ is stationary, $\Rightarrow$ $C\cap S$ is stationary
2. $S\subseteq\kappa$ is stationary $\Rightarrow$ S is unbounded
3. $S\subseteq\kappa$ is stationary $\Rightarrow$ $\{\lambda\in S:\lambda \text{ is a limit ordinal}\}$ is stationary

Pf 1: D is any club in κ, $D\cap C\subseteq\kappa$ is club

$\therefore D\cap(C\cap S)=(D\cap C)\cap S\neq\emptyset$

2: if S bounded, $\exists\beta<\kappa$ s.t. $S\subseteq\beta$.

let $C=\{\alpha<\kappa:\alpha\geq\beta\}=[\beta,\kappa)$, C is club but $C\cap S=\emptyset$. contradict!

3. $L=\{\lambda<\kappa:\lambda \text{ is limit ordinal}\}$

L is club (easy). $\therefore L\cap S$ is stationary by "1" 用前面的结论!

def14.13 (diagonal intersection) let $\{X_\alpha:\alpha<\kappa\}$ be a sequence of κ many subsets of κ, their diagonal intersection is defined to be $\Delta_{\alpha<\kappa}X_\alpha=\{\beta<\kappa:\beta\in\bigcap_{\alpha<\beta}X_\alpha\}$

普通交集是 $\bigcap_{\alpha<\kappa}X_\alpha$, 对角交考虑 $\beta\to\bigcap_{\alpha<\beta}X_\alpha$ 只需在 $X_0\cap X_1\cap\cdots\cap X_{\beta-1}$ 前 $\beta-1$ 个交集中

TH14.14. κ is a regular cardinal, then the diagonal intersection of κ many clubs in κ is club in κ

ie. $\{C_\alpha:\alpha<\kappa\}$ are clubs in κ, then $\Delta_{\alpha<\kappa}C_\alpha=\{\beta<\kappa:\beta\in\bigcap_{\alpha<\beta}C_\alpha\}$ is club in κ

- $\{C_\alpha:\alpha<\kappa\}$ is a sequence of clubs, assume C_α are decreasing under "$\subseteq$". $C_0\supseteq C_1\supseteq C_2\supseteq\cdots$ —(*)

  let $C_\alpha'=\bigcap_{\gamma\leq\alpha}C_\gamma$. then $\Delta_{\alpha<\kappa}C_\alpha=\{\beta<\kappa:\beta\in\bigcap_{\alpha<\beta}C_\alpha\}$

  $\Delta_{\alpha<\kappa}C_\alpha'=\{\beta<\kappa:\beta\in\bigcap_{\alpha<\beta}\bigcap_{\gamma\leq\alpha}C_\gamma\}$

  if $\beta\in\Delta_{\alpha<\kappa}C_\alpha'$, $\beta\in\Delta_{\alpha<\kappa}C_\alpha$. $\therefore\Delta_{\alpha<\kappa}C_\alpha'\subseteq\Delta_{\alpha<\kappa}C_\alpha$

  if $\beta\in\Delta_{\alpha<\kappa}C_\alpha$, $\alpha<\beta,\gamma\leq\alpha\Rightarrow\gamma<\beta$. $\therefore\beta\in\Delta_{\alpha<\kappa}C_\alpha'$ $\therefore\Delta_{\alpha<\kappa}C_\alpha\subseteq\Delta_{\alpha<\kappa}C_\alpha'$

  $\therefore\Delta_{\alpha<\kappa}C_\alpha=\Delta_{\alpha<\kappa}C_\alpha'$, $C_\alpha'=\bigcap_{\gamma\leq\alpha}C_\gamma$ is still club

  用前α个club的交构造新的C_α', 新C_α'仍然是club, $C_\alpha'\downarrow$且对角交不变, 故可以假设(*)

- $C=\Delta_{\alpha<\kappa}C_\alpha$.

  $\forall\alpha<\nu$ 为什么? 因为$\alpha<\kappa$是针对一般交集, $\alpha<\nu$是对角交!

  if $\nu\in C$, $C\cap\nu$ is cofinal in ν. WTS: $\nu\in C_\alpha$ $\forall\alpha<\kappa$ thus $\nu\in C$

  $\nu\in\bigcap_{\alpha<\nu}C_\alpha$

  $\because C\cap\nu$ cofinal in ν, $\sup(C\cap\nu)=\nu$

  $\therefore\exists$ strictly increasing sequence $\{\beta_n:n\in\omega\}\subseteq C\cap\nu$, s.t. $\sup_{n\in\omega}\beta_n=\nu$, {for $\alpha<\nu$, $\exists N$ s.t. $\beta_n>\alpha$ $\forall n\geq N$

  $\{\beta_n:n\in\omega\}\subseteq C\cap\nu\subseteq C_\alpha\cap\nu$, $\sup\beta_n=\nu$

  C_α is closed, $\therefore\nu\in C_\alpha$ $\forall\alpha$

• WTS: $\forall \gamma<\kappa, \exists \beta>\gamma, \beta\in C$, thus C is unbounded in κ

$C=\triangle_{\alpha<\kappa}C_\alpha=\{\beta<\kappa: \beta\in\bigcap_{\alpha<\beta}C_\alpha\}$, C_α is club

let $\beta_1\in C_0$, $\beta_1>\gamma$. define $\{\beta_n: n\in\omega\}$ by: $\beta_{n+1}=\min\{\gamma\in C_{\beta_n}: \gamma>\beta_n\}$ C_{β_n}均为无界κ-子集，所以可以找$\beta_{n+1}>\beta_n$

C_{β_n} is club, unbounded, for $\beta_n<\kappa$, $\exists\gamma\in C_{\beta_n}$ s.t. $\gamma>\beta_n$

$\therefore \beta_{n+1}$, $\{\beta_n: n\in\omega\}$ well-defined

$C_{\beta_n}\subseteq\kappa$ $\therefore \gamma\in C_{\beta_n}$ $\gamma\le\kappa$, $\therefore \beta_n<\beta_{n+1}<\kappa$ $\forall n\in\omega$

let $\beta=\sup_{n\in\omega}\beta_n$; $\beta\geq\beta_n>\beta_0$ cf(κ)：κ中最短的升序序列，序列在κ中cofinal

$\because \omega<\kappa=cf(\kappa)$, $\beta_n<\kappa$ $\forall n\in\omega$, then $\beta<\kappa$, $\Rightarrow$ 接下来证明$\beta\in C$即可，即$\beta\in\bigcap_{\alpha<\beta}C_\alpha$；$\beta\in C_\alpha$ for $\forall\alpha<\beta$

for $\forall\alpha<\beta$, $\beta=\sup\beta_n$ $\omega<cf(\kappa)$ 说明$\{\beta_n: n\in\omega\}$在κ中不是cofinal的，$\sup\beta_n<\kappa$（因有$\kappa=cf(\kappa)$），β不可能$=\kappa$

$\therefore \exists N\in\omega$ s.t. $\beta_n\geq\alpha$ $\forall n\geq N$, $\Rightarrow C_{\beta_n}\subseteq C_\alpha$ $\forall n\geq N$

$\because \beta_{n+1}\in C_{\beta_n}$ $\forall n\in\omega$, $\Rightarrow \{\beta_{n+1}: n\geq N\}\subseteq C_\alpha$

$\therefore \{\beta_{n+1}: n>N\}$ is increasing sequence in C_α, C_α is club $\Rightarrow \sup\{\beta_{n+1}: n>N\}=\beta\in C_\alpha$

$\therefore \beta\in C_\alpha$ for $\forall\alpha<\beta$, i.e. $\beta\in\{\beta<\kappa: \beta\in\bigcap_{\alpha<\beta}C_\alpha\}=C$

$\therefore C$ is unbounded

rmk: 思路：任取一个 $\gamma\in\kappa$, C_0 unbounded in κ, $\therefore \exists\beta_1\in C_0$, $\beta_1>\gamma$, $\beta_1\in C_0\subseteq\kappa$ thus $\beta_1\in\kappa$

同样的，$\forall\beta_n\in\kappa$ ($\beta_n\in C_{\beta_{n-1}}$); since C_{β_n} unbounded in κ, $\exists\beta_{n+1}\in C_{\beta_n}$, $\beta_{n+1}>\beta_n$; $\beta_{n+1}\in C_{\beta_n}\subseteq\kappa$ thus $\beta_{n+1}\in\kappa$

即可以通过"C_{β_n} unbounded in κ, $\beta_n\in\kappa$"，找到更大的$\beta_{n+1}>\beta_n$, $\beta_{n+1}\in C_{\beta_n}$

$\Rightarrow \{\beta_n: n\in\omega\}$, $\sup\beta_n=\beta>\gamma$.

$\forall\alpha<\beta$, $\beta\in C_\alpha$ thus $\beta\in C$ 原因则是：$\begin{cases} C_\alpha\downarrow, \ C_\alpha \text{ is club} \overset{def}{=} C_\alpha\text{包含所有上升序列的极限值} \\ \text{"}\beta_n>\alpha\ \forall n>N\text{"} \Rightarrow \text{"}C_{\beta_n}\subseteq C_\alpha\ \forall n>N\text{"} \Rightarrow \text{"}\beta_{n+1}\in C_\alpha\ \forall n>N\text{"} \end{cases}$

def: a filter on κ is "normal" if it's closed under diagonal intersection

$\therefore$ Th14.14 states that the club filter is normal

recall: $\kappa=cf(\kappa)$, $A\subseteq\kappa$ in club filter iff A contains a club in κ, ie $CF=\{A\subseteq\kappa: A \text{ contains a club in } \kappa\}$

$\left.\begin{array}{l} CF\text{中的元素 } A_\alpha \text{ 都包含 club of } \kappa\ C_\alpha;\ \triangle_{\alpha<\kappa}A_\alpha\supseteq\triangle_{\alpha<\kappa}C_\alpha \text{ 对角交的单调性} \\ \triangle_{\alpha<\kappa}C_\alpha \text{ is still a club by Th14.4} \end{array}\right\} \Rightarrow \triangle_{\alpha<\kappa}A_\alpha\in CF$. ie. "closed" $\triangle_{\alpha<\kappa}$

lem14.15 (Fodor's lemma) $\kappa > \omega$ is regular cardinal, $S \subseteq \kappa$ is stationary, $f: S \to \kappa$ satisfies: $f(\alpha) < \alpha$, $\forall \alpha \in S$

then $\exists \gamma < \kappa$, $\exists$ stationary set $T \subseteq S \subseteq \kappa$ s.t. $f(\alpha) = \gamma$ for $\forall \alpha \in T$

pf: WTS: $\exists \gamma < \kappa$, s.t. $T_\gamma = \{\alpha \in S : f(\alpha) = \gamma\}$ is stationary 自动找到了 $T \subseteq S$, $f(T) = \gamma$, 即直接用 $f(T) = \gamma$ 构造 T

- 设 $\forall \gamma < \kappa$, $T_\gamma = \{\alpha \in S : f(\alpha) = \gamma\}$ not stationary 反证

  ie. $\exists$ club set $C_\gamma \subseteq \kappa$, s.t. $T_\gamma \cap C_\gamma = \phi$, (stationary: $T_\gamma \cap C_\gamma \neq \phi$ $\forall$ club set $C_\gamma \subseteq \kappa$)

  $\rightleftharpoons$ $\exists$ club set $C_\gamma \subseteq \kappa$, s.t $f(\alpha) \neq \gamma$, $\forall \alpha \in C_\gamma$

  反证 $\nexists$ stationary $T \subseteq S$, s.t. ..., ie. $\forall$ stationary $T \subseteq S$, $f(T) \neq \gamma$ "太空泛了, 转换得很好; 题目也 $\Leftrightarrow$

  $\exists$ set $T \subseteq S \subseteq \kappa$, $f(T) = \gamma$ s.t. T is stationary set.

- consider $\{C_\gamma : \gamma < \kappa\}$, $C^* = \triangle_{\gamma < \kappa} C_\gamma = \{\alpha < \kappa : \alpha \in \bigcap_{\gamma < \alpha} C_\gamma\}$

  C_γ is club in κ, $C_0' \supseteq C_1' \supseteq \cdots$; C^* is club by TH14.4

  S is stationary, $\therefore S \cap C^* \neq \phi$

  $\left.\begin{array}{l} \exists \alpha \neq 0, \alpha \in S \cap C^*, \ f(\alpha) < \alpha \text{ since } \alpha \in S \\ \alpha \in C^*, \therefore \alpha \in \bigcap_{\gamma < \alpha} C_\gamma, \ \alpha \in C_\gamma \text{ for } \forall \gamma < \alpha \end{array}\right\} \Rightarrow \alpha \in C_{f(\alpha)}$ since $f(\alpha) < \alpha$

  $T_\gamma = \{\alpha \in S : f(\alpha) = \gamma\}$, $\therefore \alpha \in T_{f(\alpha)}$ by def "T_γ"

  $\therefore \alpha \in T_{f(\alpha)} \cap C_{f(\alpha)}$; ie. for $\gamma = f(\alpha) < \kappa$, $\alpha \in T_\gamma \cap C_\gamma$, contradict!

rmk: 先转换题目, 反证 $\forall \gamma < \kappa$, $\exists$ club $C_\gamma \subseteq \kappa$ s.t. $C_\gamma \cap T_\gamma = \phi$

对于 $\forall \gamma < \kappa$, 构造出 $\{C_\gamma : \gamma < \kappa\}$, $C^* = \triangle_{\gamma < \kappa} C_\gamma$ is club

$S \cap C^* \neq \phi \Rightarrow f(\alpha) < \alpha$ in S; $\left.\begin{array}{l} \alpha \in C_\gamma \text{ for } \forall \gamma < \alpha \Rightarrow \alpha \in C_{f(\alpha)} \\ \alpha \in T_{f(\alpha)} \text{ obv} \end{array}\right\} \Rightarrow \gamma = f(\alpha)$, $C_\gamma \cap T_\gamma \neq \phi$

不管在最开始构造 $\{C_\alpha : \alpha < \kappa\}$ 时用了哪一个 $C_\gamma = f(\alpha)$ 的可能值, 都会有 $C_\alpha \cap T_\alpha \neq \phi$

rmk: stationary set is extremly "big", intersecting all club sets

define regression function $f(\alpha) < \alpha$, ie. f: try to "press down" values in S

but S is "too big", so f must press $\underbrace{\text{greatly many}}_{\text{a new stationary set } T_\gamma \subseteq S}$ α to a same γ

原值太多了, 压缩必然会重复, 重复的也很多 (也能形成 stationary)