Review for final

LI Liying

January 5, 2025

1 General concepts

- Multi-dimensional integration by parts or the divergence theorem.
- Classical solution (for heat equation, elliptic equation, parabolic equation, first-order equation).
- Boundary conditions: periodic, Dirichlet, Neumann, mixed.
- Well-posedness: existence, uniqueness and stability.
- Weak solutions.

L2.L22

2 First-order PDEs and Hamilton–Jacobi equations

Reference: Evans 2.1, 3.2 - 3.4, 10.1 - 10.3

- Method of characteristics: characteristic ODE, Hamilton's ODE.
- Intersection of characteristics: examples of intersection and non-intersection, relation between the occurrence of intersection and the well-posedness of the equation.
- Important nonlinear equations: Burgers equation, Hamilton–Jacobi equation.
- representation formula: Hopf–Lax formula, variational problem, properties of the variational solutions such as principle of dynamic programming, Lipchitz continuity, etc.
- Entropy condition: conservation laws, Rankien–Hugoniot conditions.
- Viscosity solutions: definitions, vanishing viscosity limit

3 Heat equation

Refrence: Zhou

- Solution of the equation.
  - On the whole space.
    * Find fundamental solutions via Fourier transform or scaling invariance.
    * Solution to Cauchy problem via *convolution*, meaning of the initial condition, smoothness of the solutions

- - * Extend to other domains via symmetry: half-space, periodic intervals.
  - On bounded domains.
    * Principle of superposition, separation of variables, the Sturm–Liouville/eigenvalue problem.
    * In 1d: Fourier series, boundary conditions, Green's function.
  - Inhomogeneous problem.
    * Duhamel's principle.
    * On the whole space: using fundamental solution.
    * On bounded intervals: using Fourier series.

- Uniqueness and stability
  - Maximum principle, generalization to other parabolic operators, comparison principle.
  - Uniqueness via the maximum/comparison principle.
  - L^∞-stability via the maximum/comparison principle.
  - Energy estimates and L^2-stability.

4 Elliptic equation

- Solution theory.
  - Fundamental solutions, solutions to the Laplace's equation, form of the fundamental solutions in $\mathbb{R}^d$.
  - Green's function, solutions to the Poisson's equation. Poisson's kernel in half-space or balls, properties such as smoothness, symmetry, decay, etc.
  - Meaning of the boundary condition

- Harmonic functions.
  - Mean-value property.
  - Weak/strong maximum principle.
  - $\mathcal{C}^\infty$-smoothness of harmonic functions.
  - Derivative estimate, Liouville theorem.
  - Perron's method, sub-harmonic functions, local uniform convergence of harmonic functions, construction of Green's function.

- Dirichlet principle. L12-13
  - Variational problem, convexity and uniqueness of minimizers, first variation, Euler–Lagrange equation.
  - Weak derivatives, Sobolev spaces $H^k(U)$, $H_0^k(U)$, weak solutions, weak convergence, weak compactness.
  - Poincare's inequality.
  - Free boundary condition, compatible condition.

- Stability.
  - Energy estimate, L^2-stability.
  - Comparison principle and L^∞-stability.

5 Wave equation

- Solutions.
  - D'Alembert's formula, finite speed of propagation.
  - Wave equation on half-line with reflecting/absorbing boundary condition.
  - Method of spherical means, method of descent.
  - Duhamel's principle.
  - Separation of variables, resonance.
- Stability.
  - Energy method.
  - Domain of dependence.

6 Linear evolution equations L17-18

Reference: Evans 7.1.1, 7.1.2, 9.2.1.

- Definition of the weak solutions: $\bar{u} \in L^2\big(0, T; H_0^1(U)\big)$, $\bar{u}' \in L^2\big(0, T; H^{-1}(U)\big)$.
- Energy estimates, uniqueness.
- Galerkin approximation, existence of weak solutions.
- Fixed point method.