eigenfunction for "Δ"

$\Delta \cdot e^{2\pi i \xi \cdot x} = -4\pi^2|\xi|^2 \cdot e^{2\pi i \xi x}$; $e^{-2\pi \xi x}$ 也行

$(-4\pi^2|\xi|^2, e^{2\pi i \xi x})$ is eigen-pair of Δ

$e^{2\pi i \xi x} \notin L^2(\mathbb{R}^d)$，不符合 fourier 要求. $\Rightarrow$ 要限定 bdd 范围

· for: $\begin{cases} -\Delta u = \lambda u ; & x \in \Omega \\ \text{homogeneous boundary condition } (\alpha u + \beta \cdot \frac{\partial u}{\partial n} = 0) \end{cases}$ 有如下性质：

① $-\Delta$ is symmetric in $L^2(\mathbb{R})$; s.t. $(-\Delta u, v) = (v, -\Delta u)$

②: $\forall$ eigenvalue of "$-\Delta$" are real ; semi-positive definite

(*) ③: {eigenvalues} countable ; if ordered as $0 \le \lambda_1 \le \lambda_2 \le \cdots$

$\lim_{n \to +\infty} |\lambda_n| = +\infty$

(*) ④ the eigen-function forms ONB (othornormal basis) for $L^2(\mathbb{R})$.

· 即: $\forall f \in L^2(\mathbb{R})$; $\exists \{C_k\}$ s.t $f(x) = \sum_{n=1}^{+\infty} C_n \cdot V_n$

(1) 设 $(\lambda_1, u), (\lambda_2, v)$ are eigen-pairs, $\lambda_1 \ne \lambda_2$

$u \ne 0$, $\lambda \int_\Omega |u|^2 dx = \int_\Omega (-\Delta u) \cdot u \, dx$

$= \int_\Omega \nabla u \cdot \nabla u \, dx - \int_\Omega \nabla \cdot (u \nabla u) dx$

$= \int_\Omega |\nabla u|^2 dx - \int_{\partial\Omega} (u \cdot \nabla u) \cdot \vec{n} \, ds$

$= \int_\Omega |\nabla u|^2 dx - \int_{\partial\Omega} u \cdot \frac{\partial u}{\partial n} ds$

$= \int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} \frac{\beta}{\alpha} \cdot (\frac{\partial u}{\partial n})^2 ds \ge 0$ if: $\alpha\beta > 0$

$\begin{cases} \nabla \cdot (u \nabla u) = |\nabla u|^2 + u \Delta u \\ \int_\Omega \nabla \cdot F dx = \int_{\partial\Omega} F \cdot \vec{n} ds \end{cases}$

$\Rightarrow$ if HBC: $\alpha u + \beta \frac{\partial u}{\partial n} = 0$. $\alpha\beta > 0$. then λ semi positive-definite

(2): $\lambda_1 \int_\Omega uv \, dx = \int_\Omega (-\Delta u) v \, dx$

$= \int_\Omega (-\Delta v) u \, dx - \underbrace{\int_{\partial\Omega} v \frac{\partial u}{\partial n} ds + \int_{\partial\Omega} u \cdot \frac{\partial v}{\partial n} ds}_{0}$

$= \lambda_2 \int_\Omega v u \, dx = \lambda_2 \int_\Omega u \cdot v \, dx$

$\int_\Omega \nabla \cdot (v \cdot \nabla u) dx$
$= \int_\Omega \nabla v \cdot \nabla u dx + \int_\Omega v \Delta u dx$

$\int_\Omega \nabla \cdot F dx = \int_{\partial\Omega} \vec{F} \cdot \vec{n} ds$

$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega \nabla v \nabla u dx$
$= \int_{\partial\Omega} u \cdot \nabla v \cdot \vec{n} ds$
$= \int_{\partial\Omega} u \cdot \frac{\partial v}{\partial n} ds$
similarly $= \int_{\partial\Omega} v \cdot \frac{\partial u}{\partial n} ds$

since $\lambda_1 \ne \lambda_2$ $\therefore \int_\Omega uv \, dx \equiv 0$, $\Rightarrow uv = 0$

(3): $\lambda \int_\Omega |u|^2 dx = \lambda \int_\Omega u \cdot \bar{u} \, dx$ 易证 这条性质只用到 $-\Delta u = \lambda u$，与 BC 无关！

$= \int_\Omega (-\Delta u) \cdot \bar{u} \, dx = \int_\Omega u (-\Delta \bar{u}) dx$

易知 (λ, u) 为 eigen-pair，$(\bar{\lambda}, \bar{u})$ 也是

$= \int_\Omega u (\bar{\lambda} \bar{u}) dx = \bar{\lambda} \int_\Omega |u|^2 dx \Rightarrow \lambda = \bar{\lambda}$

separation of variables

T4: $\begin{cases} ① \ \partial_t u = \partial_{xx} u & x\in(0,l) \ \ t>0 \\ ② \ u(0,x) = x^2(l-x)^2 & x\in[0,l] \\ ③ \ \partial_t u(t,0) = \partial_x u(t,l) = 0 & t\geq 0 \end{cases}$

$\Rightarrow$ x bounded 是之前证过bdd情况下才能用

$x\to e^{2\pi i\xi x}$ 在 $L^2(\mathbb{R}^d)$ 特征向量

①: 设 $u(t,x) = T(t)X(x)$

$\partial_t u = \partial_{xx} u \quad \therefore T'(t)X(x) = X''(x)T(t)$

设 $\frac{T'(t)}{T(t)} = \frac{X''(t)}{X(t)} = -\lambda$, $\begin{cases} X''(x) + \lambda X(x) = 0 \\ T'(t) + \lambda X(t) = 0 \end{cases}$ ($-\Delta$ symmetry $\therefore \lambda$ 应该正实数 $\therefore$ 这么设)

$\Rightarrow \begin{cases} X(x) = C_1\cos(\sqrt{\lambda}t) + C_2\sin(\sqrt{\lambda}t) \\ T(t) = e^{-\lambda t} \end{cases}$

②: $\partial_x u(t,0) = T(t)X'(0)$; $\partial_x u(t,l) = T(t)X'(l)$

$\therefore T(t)X'(0) = T(t)X'(l) = 0$, $X'(0) = X'(l) = 0$

$X'(0) = -C_1\sin(\sqrt{\lambda}x)\cdot\sqrt{\lambda} + C_2\cos(\sqrt{\lambda}x)\cdot\sqrt{\lambda}\Big|_{x=0} = C_2\sqrt{\lambda} = 0 \quad \therefore X(x) = C\cdot\cos(\sqrt{\lambda}x)$

$-C\cdot\sin(\sqrt{\lambda}\times 0)\sqrt{\lambda} = -C\sin(\sqrt{\lambda}l)\cdot\sqrt{\lambda} \quad \therefore \sqrt{\lambda}l = n\pi$, $\lambda = \left(\frac{n\pi}{l}\right)^2$ ✓

$\Rightarrow X(x) = C\cdot\cos\left(\frac{n\pi}{l}x\right)$, $T(t) = e^{-\left(\frac{n\pi}{l}\right)^2 t}$

$u(t,x) = \sum_{n=1}^{+\infty} C_n e^{-\left(\frac{n\pi}{l}\right)^2 t}\cdot\cos\left(\frac{n\pi}{l}x\right)$

③: $\varphi(x) = u(0,x) = \sum_{n=1}^{+\infty} C_n\cos\left(\frac{n\pi}{l}x\right)$

$\int_0^l \cos\left(\frac{n\pi x}{l}\right)\varphi(x)\,dx = \int_0^l \cos\left(\frac{n\pi x}{l}\right)\cdot\sum_{n=1}^{+\infty} C_n\cos\left(\frac{n\pi}{l}x\right)dx$

$= \int_0^l C_n\cos^2\left(\frac{n\pi}{l}x\right)dx = C_n\int_0^l \frac{1+\cos 2\frac{n\pi x}{l}}{2}\,dx = C_n\frac{l}{2}$

$\Rightarrow C_n = \frac{2}{l}\int_0^l \cos\left(\frac{n\pi x}{l}\right)\varphi(x)\,dx$

$= \frac{2}{l}\int_0^l \cos\left(\frac{n\pi x}{l}\right)x^2(l-x)^2\,dx$ 代入 $\varphi(x) = x^2(l-x)^2$

$\Rightarrow u(t,x) = \sum_{n=1}^{+\infty}\left(\frac{2}{l}\int_0^l \cos\left(\frac{n\pi x}{l}\right)x^2(l-x)^2\,dx\right)e^{-\left(\frac{n\pi}{l}\right)^2 t}\times\cos\left(\frac{\pi n}{l}x\right)$

L6中:

$$\begin{cases} \partial_t u = \Delta u + f & ① \\ u|_{t=0} = \varphi(x) & ② \\ u(t,0) = u(t,l) = 0 & ③ \end{cases}$$

③ ⇒ 用了separation消项

Note: G是fundamental solution, 类比L9

$$\Rightarrow u(t,x) = \underbrace{\int_0^l G(t,x;0,y)\cdot\varphi(y)\,dy}_{=I_1} + \underbrace{\int_0^t\int_0^l G(t,x;s,y)\cdot f(s,y)\,dy\,ds}_{=I_2}$$; Duhemel定理式

在本题中 $G(t,x;s,y) = \frac{2}{l}\sum_{n=1}^{\infty} \sin(\frac{n\pi}{l}x)\sin(\frac{n\pi}{l}y)\, e^{-(\frac{n\pi}{l})^2(t-s)}$, $t<s$

- G具体形式与 ∂U 相关, 自动满足③式
- $\int_0^t\int_0^l G(t-s, x-y)\cdot f(s,y)\,dy\,ds$ 满足①式; I_1不起作用 since $\partial_t(I_1结果) - \Delta(\cdot) = 0$
- $\lim_{t\to 0^+} u(t,x) = \lim_{t\to 0^+} I_1 + \lim_{t\to 0^+} I_2 = \varphi(x) +$ [illegible] 0, 满足①式

▲ Introduce green function:

$$\begin{cases} \partial_t u - \Delta u = \delta(t-s, y-x) \quad \leftarrow t\text{-}s\text{设用} \\ u(t,0) = u(t,l) = 0 \\ u(0,x) = 0 \end{cases} \quad \cdots (1)$$

$$\begin{cases} \partial_t u - \Delta u = f \\ u(t,0) = u(t,l) = 0 \\ u(0,x) = \varphi(x) \end{cases} \quad \cdots (2)$$

↙ 这行与(1)构造获得

若 $G(t,x;s,y)$ 满足(1)式 ⇒ called "Green function"

then: $u(t,x) = \int_0^l G(t,x;0,y)\varphi(y)\,dy + \int_0^t\int_0^l G(t,x;s,y) f(s,y)\,dy\,ds$

u solves (2)式

▲ 计 $u(t,0) = g_1(t)$, $u(t,l) = g_2(t)$:

Let $v(t,x) = u(t,x) - \left(\frac{x}{l} g_2(t) + \frac{x-l}{l} g_1(t)\right)$

$v(t,0) = v(t,l) = 0$

同时求出对应的 $\tilde{f}$ 和 $\tilde{\varphi}$ 即可使用上述方法

▲ 《周P97》 fundamental solution: L5的lecture note有错

没抄会自己看!

$$\begin{cases} \partial_t u - \Delta u = \delta(t-s, y-x) \\ u(0,x) = 0 \end{cases} \quad \cdots (3)$$

物理意义: $\Gamma(x,t;y,\tau)$ solves (3)式, 两端在无穷远, 侧表面均绝热(仅截面传热)的均匀杆, 在 τ 时, 在坐标y瞬时放置一个单位点热源, $\Gamma(x,t;y,\tau)$体现了杆上的温度分布在x,t的情况

"green = fundamental + boundary U"

点源函数 $t>\tau$, $\int_{\mathbb{R}} \Gamma\, dy = 1$