L16

TH: X reflexive (must be Banach), $\{x_n\}$ bounded in X $\Rightarrow$ $\{x_{n_k}\} \subseteq \{x_n\}$ s.t. $x_{n_k} \rightharpoonup x \in X$

Lem: X separable, $\{f_n\}$ bdd in X^* $\Rightarrow$ $\{f_{n_k}\} \subseteq \{f_n\}$ s.t. $f_{n_k} \overset{*}{\rightharpoonup} f \in X^*$

Pf: $\{x_n\} \subseteq X$, let $Y = \overline{\text{span}(\{x_n\})}$

Y closed subspace, X reflexive $\Rightarrow$ Y reflexive, $Y \cong Y^{**}$

Y separable $\Rightarrow$ $Y^{**} \cong Y$ separable

$$\left.\begin{array}{l} \Rightarrow Y^* \text{ separable} \\ \{x_n^{**}\} \text{ bdd in } (Y^*)^* \end{array}\right\} \Rightarrow \{x_{n_k}^{**}\} \subseteq \{x_n^{**}\} \text{ s.t. } x_{n_k}^{**} \overset{*}{\rightharpoonup} x^{**} \in Y^{**}$$

$\Rightarrow f(x_{n_k}) \to f(x)\ \forall f \in Y^*$ i.e. $x_{n_k} \rightharpoonup x \in Y$

证略 $\Rightarrow f(x_{n_k}) \to f(x)\ \forall f \in X^*$ since $x_{n_k} \in Y$, $x \in Y$

$\Rightarrow x_{n_k} \rightharpoonup x$ in X

rmk: 若 $x_n^{**} \overset{*}{\rightharpoonup} x^{**}$ $\Rightarrow$ $x_n \rightharpoonup x$

TH: X reflexive, $T: X \to X$ compact linear, $\{x_n\}$ bounded; $\exists\, 0 < c_1$, $c_1 \le \|x_n\| \le c_2\ \forall n$

$\|Tx_n - x_n\| \to 0$ $\Rightarrow$ $\exists x \neq 0$ s.t. $Tx = x$

Lem: $T: X \to Y$ cpt linear, $x_n \rightharpoonup x \Rightarrow Tx_n \to Tx$

Lem: X reflexive, $\{x_n\} \subseteq X$ bounded $\Rightarrow$ $\{x_{n_k}\} \subseteq \{x_n\}$ s.t. $x_{n_k} \rightharpoonup x \in X$

$$\text{Pf: } \left.\begin{array}{l} \Rightarrow Tx_{n_k} \to Tx \\ Tx_{n_k} \to x_{n_k} \rightharpoonup x \end{array}\right\} \Rightarrow Tx = x \neq 0 \text{ since } c_1 > 0$$

关键是记住引理, TH本身不难, 但Lem嵌套容易忘!

Mazur's TH: $Y \subseteq X$ closed convex, X Banach $\Rightarrow$ Y weakly closed

Lem: X Banach, $C \subseteq X$ closed convex $\Rightarrow C = \{x \in X : \operatorname{Re} f(x) \ge \inf_{y \in C} \operatorname{Re} f(y), \forall f \in X^*\}$

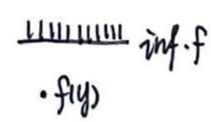
$\Rightarrow X \setminus C = \{x \in X : \operatorname{Re} f(x) < \inf_{y \in C} \operatorname{Re} f(y), \forall f \in X^*\}$

$= \{x \in X : f(x) < \inf f(C), \forall f \in X^*\}$, $\mathbb{F} = \mathbb{R}$

Pf: $y_n \rightharpoonup y \notin Y$, $\Rightarrow f(y) < \inf f(Y)\ \forall f \in X^*$

$\Rightarrow \exists g \in X^*$, $\varepsilon > 0$ s.t. $g(y) < \inf_{n \in \mathbb{N}} g(y_n) - \varepsilon$

$\Rightarrow y_n \not\rightharpoonup y$



Def: X Banach, satisfy Banach-Saks if: $\{x_n\} \subseteq X$ bounded, $\exists \{x_{n_k}\} \subseteq \{x_n\}$ s.t. $\left\| \frac{x_{n_1} + \cdots + x_{n_k}}{k} \right\|$ conv

TH: H satisfies B-H property

Pf: $\{x_n\} \subseteq H$ bounded, H reflexive $\Rightarrow \exists \{x_{n_k}\} \subseteq \{x_n\}$ s.t. $x_{n_k} \rightharpoonup x_0 \overset{设}{=} 0$

$\Rightarrow$ let $x_{n_k} \overset{记}{=} y_k$, $y_k \rightharpoonup 0$ ie. $(y_k, y) \to 0 \ \forall y \in H$ ... ⊕

let $y_{a_1} = y_1$

$(y_k, y_{a_1}) \to 0$, $\exists N_{1,2}$ s.t. $\forall k \geq N_{1,2}$, $|(y_k, y_{a_1})| < \frac{1}{2}$ $\Rightarrow a_2 := N_{1,2}$

$(y_k, y_{a_2}) \to 0$, $\exists N_{2,3}$ s.t. $\forall k \geq N_{2,3}$, $|(y_k, y_{a_2})| < \frac{1}{3}$; similarly $N_{1,3}$ $\Rightarrow a_3 := \mathrm{Max}(N_{2,3}, N_{1,3})$

.... $\Rightarrow a_t := \mathrm{Max}(N_{1,t}, N_{2,t} \cdots N_{t-1,t})$

ie: $|(y_{a_k}, y_{a_i})| < \frac{1}{k}$ $\forall i, \forall k$

$\Rightarrow \left\| \frac{y_{a_1} + y_{a_2} + \cdots y_{a_k}}{k} \right\|^2 \leq \frac{1}{k^2}(k \cdot M + 2(k-1)) \to 0 = x_0$

Fredholm Alternative TH:

X is Banach, $\dim(X) = +\infty$; $T: X \to X$ linear compact $\Rightarrow \mathrm{Ker}(T-I) = \{0\} \Leftrightarrow (T-I)(X) = X$

Pf: "$\Rightarrow$" (1): $\mathrm{Ker}(T-I) = \{0\}$ $\Rightarrow 1 \notin \sigma_p(T)$

Banach, dim $= +\infty$; T cpt $\Rightarrow \sigma(T) = \{0\}$ or $\{0\} \cup \{k_n\}_{n=1}^{+\infty}$ $k_n \in \sigma_p(T)$ $\Big\} \Rightarrow 1 \notin \sigma(T)$

$\Rightarrow 1 \in \rho(T)$, $(T-I)(X) = X$

"$\Leftarrow$" if $\mathrm{Ker}(T-I) \neq \{0\}$, Assume $a \neq 0$, $(T-I)(a) = 0$

T cpt $\Rightarrow T^*: X^* \to X^*$ cpt

$\forall f \in X^*$, $(T^*-I)(f) = 0 \Rightarrow \forall x = 0$, $(T^*-I)(f)(x) = f \circ ((T-I)(x)) = 0$

$(T-I)(X) = X$, surj

$\Rightarrow f(X) = 0$, ie. $f \equiv 0$

$\Rightarrow T^*-I$ injective $\overset{(1)}{\Rightarrow} (T^*-I)(X^*) = X^*$ surj

$\exists g \in X^*, \|g\| = 1$ s.t. $g(a) = \|a\|$

$(T^*-I)g(a) = g((T-I)(a)) = g(0) = 0$

$(T^*-I)g(a) = 0 \Leftrightarrow g(a) = \|a\| = 0$ $\Big\} \Rightarrow$ contradict, $a \neq 0$.

Rmk: $(T-I)$ surj $\overset{关键}{\Rightarrow} (T^*-I)$ inj thus surj, bij, 有结论!

$g(a) = \|a\| \Rightarrow (T^*-I)(g)(a) = g((T-I)(a)) = 0 \Leftrightarrow a = 0$