C15 Applications of Fodor: Δ-System & Silver's Th

in set theory, f on ordinals s.t. $f(\alpha)<\alpha$ can gain many properties by Fodor's lemma.

for the Δ-system lemma, it's the key combinatorial principle behind Cohen's proof of the consistency of $\neg CH$

" 证明 $\neg CH$ 具有一致性

def15.1 X is a collection of sets, r is a set, for $\forall$ distinct $A, B \in X$, $A\cap B = r$ (r不一定在X中)

then X is a Δ-system with root r.

eg. if X is a disjoint collection of sets, then the Δ-system is with root $r=\emptyset$

lem15.2 X is an uncountable set of finite sets, then $\exists$ uncountable subset $X' \subseteq X$, $\exists$ finite set r s.t. X' is a Δ-system with root r. (actually X' is stationary $C\cap S \cap (r,k)$)

(将整个X投影 $\Rightarrow$ 转换成在ω_1上讨论问题, $X\subseteq\omega_1$)

pf: · $|X|$ uncountable, $\therefore |X| \geq \omega_1$;

$\therefore \exists$ subset $Y \subseteq X$ s.t. $|Y| = \omega_1$, it suffices to show that lem15.2 holds on $|Y|$

let $X := Y$, $X = \{x_\alpha : \alpha<\omega_1\}$

x_α finite, $\therefore |\bigcup X| \leq \sum |x_\alpha| = \omega_1 \cdot \sup|x_\alpha| = \omega_1$

$\therefore$ we can inject $\bigcup X$ into ω_1, $x_\alpha' = g(x_\alpha) \subseteq \omega_1$ for injective g. let $x_\alpha := x_\alpha'$

· def $f:\omega_1\to\omega_1$, $f(\alpha) = \begin{cases} \sup(x_\alpha\cap\alpha) & x_\alpha\cap\alpha\neq\emptyset \\ 0 & \text{otherwise} \end{cases}$ (构造回归函数, ω_1 regular, stationary $\Rightarrow$ Fodor's)

$x_\alpha\cap\alpha$ finite $\therefore$ if $x_\alpha\cap\alpha\neq\emptyset$ $\sup(x_\alpha\cap\alpha)<\alpha$

$\therefore f(\alpha)<\alpha$ for $\forall \alpha\neq 0$ $\Rightarrow$ f is regression function on ω_1

Fodor's lem: $\omega_1>\omega$ is regular, $S := \omega_1 - \{0\}$ is stationary

$\Big\} \Rightarrow \exists r<\omega_1, \exists$ stationary $T\subseteq S$, $f(T) = r$.

$\therefore \sup(x_\alpha\cap\alpha) = r$ for $\forall \alpha\in T$, $\alpha\neq 0$

(把T下降到$T_{\bar r}$, 把$\sup = r$限制到$x_\alpha\cap\alpha = \bar r \subseteq r$)

· $x_\alpha\cap\alpha$ is finite in r, thus the possible value of $x_\alpha\cap\alpha$ is finite

let $T_{\bar r} = \{\alpha\in T : x_\alpha\cap\alpha = \bar r\}$, $T = \bigcup_{\bar r\subseteq r} T_{\bar r}$ is a finite, disjoint union, $r<\omega_1$

Exe14.12 S is stationary in regular k, $S = \bigcup_{\alpha<\lambda} S_\alpha$ some $\lambda<k$ is disjoint union $\Rightarrow \exists S_\alpha$ is stationary $\Big\}$

$\Rightarrow \exists \bar r \subseteq r$, s.t. $T_{\bar r}$ is stationary

$\therefore x_\alpha\cap\alpha = \bar r$ for $\forall \alpha\in T_{\bar r}$, $T_{\bar r}$ is stationary, $\bar r\subseteq r$

• def $C = \{\alpha < \omega_1 : (\forall \beta < \alpha) \to X_\beta \subseteq \alpha\}$ (构造一个club set)

by Exe14.9 : $\forall \delta < \omega_1$ | Collect all elements in X_β for all $\beta \le \delta$: let $Y = \bigcup_{\beta \le \delta} X_\beta$; let $\sigma = \sup Y$

$\delta < \omega_1$ thus at most countable, X_β finite $\Rightarrow Y$ countable

$|Y| < cf(\omega_1) = \omega_1$, ∴ $\sigma = \sup Y < \omega_1$

let $\alpha = \max\{\delta+1, \sigma+1\}$, $\alpha > \delta$, $\alpha > \sigma$

$\forall \beta \le \delta$ $X_\beta \subseteq Y \in \sigma < \alpha$, ∴ $X_\beta \subseteq \alpha$

but for $\delta < \beta < \alpha$: ① def $\{\alpha_n : n < \omega\}$ as $\alpha_0 = \delta + 1$

$Z_n := \bigcup_{\beta \le \alpha_n} X_\beta$, is countable, $\sigma_n := \sup Z_n$; $\alpha_{n+1} := \max\{\alpha_n + 1, \sigma_n + 1\}$

∴ $\alpha_{n+1} > \alpha_n$

$\forall \beta \le \alpha_n$, $X_\beta \subseteq Z_n \in \sigma_n \in \alpha_{n+1}$, ∴ $X_\beta \subseteq \alpha_{n+1}$

let $\alpha = \sup\{\alpha_n\}$, $\omega < cf(\omega_1) = \omega_1$ ∴ $\alpha < \omega_1$; $\alpha > \delta$ obv

$\forall \beta < \alpha$, $\exists \alpha_n > \beta$, ∴ $X_\beta \subseteq \alpha_{n+1} \le \alpha$, thus $X_\beta \subseteq \alpha$

$\Rightarrow$ $\forall \delta < \omega_1$, $\exists \alpha \in \omega_1$ s.t. $(\forall \beta < \alpha \to X_\beta \subseteq \alpha)$, $\alpha > \delta$; ie. $\exists \alpha \in \omega_1 \cap C$ s.t. $\alpha > \delta$

∴ C is unbounded in ω_1

② if $\{\alpha_n : n \in \omega\} \subseteq C$ is strictly increasing, $\alpha = \sup \alpha_n$

if $\alpha = \alpha_n$ some n, $\alpha \in C$

if $\alpha \ne \alpha_n$ $\forall n \in \omega$, α is limit ordinal, $\forall \beta < \alpha$, $\exists \alpha_n > \beta$, ∴ $X_\beta \subseteq \alpha_n < \alpha$, thus $X_\beta \subseteq \alpha \Rightarrow \alpha \in C$

∴ C is closed in ω_1

①+② $\Rightarrow$ C is club in ω_1 ... proved in Exe14.9

• let $S := C \cap T_{\bar\gamma} \cap (\gamma, \omega_1)$ is stationary by Exe14.1, S is uncountable ; WTS: $X' = \{X_\alpha \in X : \alpha \in S\}$ is Δ-system with $\bar\gamma$

for $\alpha, \beta \in S$. let $\beta < \alpha$; $X_\beta \subseteq \alpha$ since $\alpha \in C$

∴ $X_\beta \cap X_\alpha = (X_\beta \cap \alpha) \cap X_\alpha = X_\beta \cap (\alpha \cap X_\alpha) = X_\beta \cap \bar\gamma$ since $\alpha \in T_{\bar\gamma}$

$X_\beta \cap \beta = \bar\gamma$ since $\beta \in T_{\bar\gamma}$ ∴ $X_\beta \supseteq \bar\gamma$

$$\left.\begin{array}{l} X_\beta \supseteq \bar\gamma,\ X_\beta \cap X_\alpha = X_\beta \cap \bar\gamma \supseteq \bar\gamma \\ X_\beta \subseteq \bar\gamma,\ X_\alpha \subseteq \bar\gamma \quad ∴\ X_\beta \cap X_\alpha \subseteq \bar\gamma \end{array}\right\} \Rightarrow X_\beta \cap X_\alpha = \bar\gamma \text{ for } \forall \alpha \ne \beta.\ \alpha, \beta \in S$$

Q1: need to intersect (γ, ω_1) since $\sup(X_\alpha \cap \alpha) = \gamma$ for T, and we obtain $X_\alpha \cap \alpha = \bar\gamma$ for $T_{\bar\gamma} \subseteq T$, $\bar\gamma \le \gamma$,

∴ I think we should use $(\bar\gamma, \omega_1)$ instead of (γ, ω_1) (of course (γ, ω_1) is OK too, since it's strictly than $\bar\gamma$)

Q2: $$\left.\begin{array}{l} X_\beta \cap X_\alpha = X_\beta \cap \bar\gamma \\ X_\beta \cap \beta = \bar\gamma \end{array}\right\} \Rightarrow X_\beta \cap \bar\gamma = X_\beta \text{ since } \bar\gamma \subseteq X_\beta$$ Why this conclusion is wrong

Th15.7 κ is singular cardinal of uncountable cofinality; $2^\lambda=\lambda^+$ for $\forall \lambda<\kappa \Rightarrow 2^\kappa=\kappa^+$

(Silver's 1) usage?

recall: λ^+ 后继基数, CH: $2^{\aleph_0}=\aleph_1$ GCH: $\lambda^+=2^\lambda\ \forall\lambda$

即: $\kappa>cf(\kappa)>\omega_1$, GCH性质可以递推

- let $\{u_\alpha:\alpha<cf(\kappa)\}$ be strictly increasing sequence cofinal in κ,

$\begin{cases} \text{if } \alpha \text{ is limit ordinal } u_\alpha=\sup\limits_{\alpha>\beta} u_\beta \\ \sup\limits_{\alpha<cf(\kappa)} u_\alpha=\kappa \text{ by cofinality}\end{cases}$

$\forall\alpha<cf(\kappa),\ u_\alpha<\kappa,\ \therefore 2^{u_\alpha}=u_\alpha^+$ 条件 $2^\lambda=\lambda^+$ 用在这

$\therefore \exists$ bijection $g_\alpha: P(u_\alpha)\mapsto u_\alpha^+$

先处理 $2^\lambda=\lambda^+$; 对于κ下的增序列, 构造 bijection g_α用于编码 $P(\lambda)\to\lambda^+$

- for $A\subseteq\kappa$, def $f_A(\alpha)=g_\alpha(A\cap u_\alpha)$, $\alpha<cf(\kappa)$

$u_\alpha^+<\kappa$ since κ is limit ordinal, $\therefore$ range$(g_\alpha)<\kappa$, $f_A(\alpha)<\kappa\ \forall\alpha<cf(\kappa)$

- let $F=\{f_A: A\subseteq\kappa\}\subseteq\kappa^{cf(\kappa)}$, WTS: $F\leftrightarrow P(\kappa)$

def $\Phi: P(\kappa)\to F$, $\Phi(A)=f_A$, F is surjective for $\forall A\in P(\kappa)$ by def ... ②

$2^\kappa=|P(\kappa)|$, 令 $P(\kappa)\leftrightarrow F$, 双射

$\left[\begin{array}{l} f_A(\alpha)=\cdots\ A\subseteq\kappa \\ F=\{f_A: A\subseteq\kappa\}\leftrightarrow |A\subseteq\kappa\end{array}\right.$

① $A,B\subseteq\kappa$, $A\neq B$ let $\xi=A\triangle B=(A\setminus B)\cup(B\setminus A)=A\cup B-A\cap B\neq\kappa$

$\sup\limits_{\alpha<cf(\kappa)} u_\alpha=\kappa$, $\therefore\exists\alpha<cf(\kappa)$ s.t. $u_\alpha>\xi$

if $\xi\in B\setminus A$, $\xi\in B\cap u_\alpha$, $\xi\notin A\cap u_\alpha$ $\therefore B\cap u_\alpha\neq A\cap u_\alpha$; similarly for $\xi\in A\setminus B$

$\therefore f_A\neq f_B$ since $f_A(\alpha)=g_\alpha(A\cap u_\alpha)\neq g_\alpha(B\cap u_\alpha)=f_B$

$\therefore\Phi$ is injective

②+① Φ bijective, $|P(\kappa)|=2^\kappa\Rightarrow|F|=|\{f_A:A\subseteq\kappa\}|\geq\kappa^+$ ... (*)

- lem Exe15.6 $\leq_L$ is linear order on set L, for $\forall a\in L$, $|\{b\in L: b\leq_L a\}|\leq\kappa\Rightarrow|L|\leq\kappa^+$ 每个元素在$\leq_L$下前驱 $\leq\kappa$

in set F, $f_1,f_2\in F$ $\begin{cases} \text{def } \Delta(f_1,f_2)=\min\{\alpha<cf(\kappa): f_1(\alpha)\neq f_2(\alpha)\} \\ \text{def } f_1<_F f_2 \Leftrightarrow f_1(\Delta(f_1,f_2))<f_2(\Delta(f_1,f_2))\end{cases}$

if $f_1<f_2$ in F, let $\alpha=\Delta(f_1,f_2)$ then: $\begin{cases} f_1(\alpha)<f_2(\alpha) \\ f_1(\beta)=f_2(\beta)\ \forall\beta<\alpha\end{cases}$

$\forall f\in F$, $f(\alpha)=g_\alpha(A\cap u_\alpha)$ some $A\subseteq\kappa$, $\therefore f(\alpha)\leq u_\alpha^+<\kappa$ since $u_\alpha\uparrow$, $f(\alpha)\uparrow$

$|\{f_1:\Delta(f_1,f_2)=\alpha\}|\leq cf(\kappa)\times\kappa=\kappa$ since $|dom(f)|=|dom\, g_{(\cdot)}|<cf(\kappa)$

Predecessor$(f_2)=|\{f_1: f_1<_F f_2\}|\leq\bigcup\limits_{\alpha<\kappa}|\{f_1:\Delta(f_1,f_2)=\alpha\}|\leq\kappa\cdot\kappa=\kappa$

$\Rightarrow$ with $<_F$, $\forall f_2\in L$, $|\{f_1\in L: f_1<_F f_2\}|\leq\kappa$,

$\therefore|F|\leq\kappa^+$ ... (*)'

- (*)+(*)' $2^\kappa=|F|\geq\kappa^+$ since $2^\kappa\geq\kappa^+$; $|F|\leq\kappa^+\Rightarrow 2^\kappa=\kappa^+$

rmk: Th15.7. Silver的初阶版本没用到 Fodor's lem. 证 Silver's Th15.9 时,用了 Exe15.8 + Th15.7

Fodor's 用在这里

Th (claim for Th15.7) for $\forall A \subseteq \kappa$, $|\{B \subseteq \kappa : \{\alpha : f_B(\alpha) < f_A(\alpha)\}$ is stationary$\}| \le \kappa$; $\kappa \cdot f_A$ defined by pf Th15.7

这个用来证 Th15.7中的"predecessor $\le \kappa$"

$f_B(\alpha) < f_A(\alpha)$ on stationary set S. then by Fodor's (可以def regression function)

$\bar{f}_B$ stablizes on a stationary set $T \subseteq S$, $\bar{f}_B(T) = \gamma$ some $\gamma < \kappa$.

B thus has at most κ possibilities

$h_\alpha : \{\beta < f_A(\alpha)\} \to \mathcal{U}_\alpha$ injective

$f_B(\alpha) < f_A(\alpha)$, let $y_\alpha = h_\alpha(f_B(\alpha)) < \mathcal{U}_\alpha$

def $f_B'(\alpha) = \min\{\beta < \alpha : y_\alpha < y_\beta\}$, $f_B'(\alpha) < \alpha$

$h_\alpha(f_B(\alpha)) < \mathcal{U}_\gamma \quad \forall \alpha \in T$

$f_B(\alpha) \le h_\alpha^{-1}(\mathcal{U}_\gamma)$, $\forall \alpha \in T$; $\therefore f_B : T \to \mathcal{U}_\gamma$

$h_\alpha^{-1}(\mathcal{U}_\gamma) < \mathcal{U}_\gamma$

$|T| \le \kappa$, $|\mathcal{U}_\gamma| \le \kappa$ $\Big\} \Rightarrow |\{B : \cdots\}| \le \kappa$

Th15.9 (Silver's) κ is singular cardinal of uncountable cofinality. $2^\lambda = \lambda^+$ for stationary set $\Lambda < \kappa$. then $2^\kappa = \kappa^+$

$$\begin{cases} \text{Th15.9: } 2^\lambda = \lambda^+ \ \forall \lambda \in \Lambda.\ \Lambda \text{ is stationary in } \kappa \\ \text{Th15.7: } 2^\lambda = \lambda^+ \ \forall \lambda < \kappa \end{cases}$$

以上不用记,没有讲证明! 下面要的!

Baire ✓

In topology: def: in a complete metric space, set $A \supseteq G_\delta$ dense set

$\Rightarrow$ element of A is "generic".

$G_\delta = \bigcap_{n=1}^{+\infty} G_n$ for open. dense G_n. n.t.w.

def: A is meager if $A = \bigcup_{n=1}^{+\infty} A_n$. A_n is nowhere dense

$(\bar{A}_n)^\circ = \phi$

$X - A$ is meager $\rightleftharpoons$ A is comeager $\rightleftharpoons$ A contains a G_δ set.

提书上的，补充定理课上讲了！在 Δ-system lemma 之后，我认为这个可能考，方法和其它定理重合部分很多

Th: space $X=\{f:\kappa\to\{0,1\}\}$ with order topology satisfies: any collection of pairwise disjoint non-empty open set in space X is countable. ($\kappa\geq\omega_1$)

recall: base open set in function space $N_s=\{f\in X: f(\alpha)=s(\alpha) \text{ for } \forall\alpha\in dom(s)\}$ in order topology

s is a fixed finite function $s: dom(s)\to\{0,1\}$, $dom(s)\subseteq\kappa$ is finite

即固定有限个坐标的值 ($range(s) :: range(f)$)

Pf: • for any open non-empty set sequence $\{U_\alpha:\alpha<\kappa\}$, $\exists$ basis $N_{s_\alpha}\subseteq U_\alpha$, (consider $\{N_{s_\alpha}:\alpha<\kappa\}$ as well)

if $N_{s_\alpha}\cap N_{s_\beta}\neq\emptyset$, $N_{s_\alpha}\subseteq U_\alpha$, $N_{s_\beta}\subseteq U_\beta$ $\Rightarrow U_\alpha\cap U_\beta\neq\emptyset$

WTS: any uncountable collection of non-empty open set; $\exists\, U_\alpha\cap U_\beta\neq\emptyset$; it suffices to show $N_{s_\alpha}\cap N_{s_\beta}\neq\emptyset$

$\therefore$ replace $\{U_\alpha:\alpha<\kappa\}$ with $\{N_{s_\alpha}:\alpha<\kappa\}$, any s

• $A_\alpha=dom(s_\alpha)\subseteq\kappa$, consider $\{A_\alpha:\alpha<\kappa\}$ uncountable, A_α finite

$\therefore\exists$ a stationary set $S\subseteq\{\cdots\}$, finite set $r\subseteq\kappa$ s.t. S is a Δ-system with root r ... by Δ-system lem 15.2

$\therefore\forall\alpha\neq\beta$, $\alpha,\beta\in S\subseteq\kappa$, $A_\alpha\cap A_\beta=r$

• consider $S_t=\{\alpha\in S: s_\alpha\restriction_r=t\}$, t is a 任意 function $t: r\to\{0,1\}$

$S=\bigcup S_t$ is disjoint union, $\therefore\exists S_t$ is stationary by Exe 14.12 将 s_α 限制在 r 上，S_t: r 上表现相同的函数

• $\forall\alpha<\beta\in S_t$, $dom(s_\alpha)\cap dom(s_\beta)=r$

$s_\alpha\restriction_r=s_\beta\restriction_r$ since $\alpha,\beta\in S_t$

$\Rightarrow$ expand s_α, s_β to $f:\kappa\to\{0,1\}$, $f\restriction_{A_\alpha}=s_\alpha$, $f\restriction_{A_\beta}=s_\beta$

$\Downarrow$ $f\in N_{s_\alpha}$ $\quad$ $\Downarrow$ $f\in N_{s_\beta}$

$\therefore f\in N_{s_\alpha}\cap N_{s_\beta}$ in order topology

$\therefore N_{s_\alpha}\cap N_{s_\beta}\neq\emptyset$; $U_\alpha\cap U_\beta\neq\emptyset$

$\Rightarrow$ any $\{U_\alpha:\alpha<\kappa\}$ of open, non-empty set, $\kappa\geq\omega_1$; $\exists$ stationary set $S_t\subseteq\kappa$ s.t. $\forall\alpha\neq\beta$ in S_t, $U_\alpha\cap U_\beta\neq\emptyset$

$\therefore\exists$ non-disjoint pair U_α, U_β

$\Rightarrow$ "disjoint" implies "countable collection"!

$\{s_\alpha:\alpha<\kappa\}$ $\{A_\alpha:\alpha<\kappa\}$ $A_\alpha=dom(s_\alpha)$ finite $\Rightarrow$ $S\subseteq\{A_\alpha:\alpha<\kappa\}$ is Δ-system with r

在 r 上表现相同为 t 的集 S_t

S_t 的函数 s_α, s_β 可扩张 $f:\kappa\to\{0,1\}$

$f\in N_{s_\alpha}\cap N_{s_\beta}$