Proofs of Theorems about Trace of Matrix

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1 Definitions

Definition The uppercase letters below all denote matrices and the trace of X is denoted trX.

Definition Let $f(A)$, where A is a matrix, be a function $\in R^{m\times n} \to R$, then derivatives of matrix A denoted and defined as:

$$\nabla_A f(A) = \begin{pmatrix} \frac{\partial f}{\partial a_{11}} & \cdots & \frac{\partial f}{\partial a_{1n}} \\ \vdots & & \vdots \\ \frac{\partial f}{\partial a_{m1}} & \cdots & \frac{\partial f}{\partial a_{mn}} \end{pmatrix}$$

2 Theorems Summary

The prerequisite of the following theorems is that matrices multiplication makes sense. For example, AB and BA should both be square matrices. ABC, CAB, BCA should all be square matrices. But they may not be of the same dimension.

1. Commutative Law:
   $trAB = trBA$
   $trABC = trCAB = trBCA$
2. Suppose $f(A) = trAB$, then $\nabla_A trAB = B^T$.
3. If $a \in R$, $tra = a$.
4. $\nabla_A trABA^TC = CAB + C^TAB^T$

3 Theorems and its Proof

Theorem 3.1 *$trAB = trBA$, where we suppose that $A \in R^{m\times n}, B \in R^{n\times m}$.*

Corollary 3.2 *$trABC = trCAB = trBCA$*

Proof First write the expressions of two sides of the equations:

$$trAB = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ji} \tag{1}$$

$$trBA = \sum_{i=1}^{n}\sum_{j=1}^{m} b_{ij}a_{ji} \tag{2}$$

Switch the role of i and j of eq. (2), we get:

$$trBA = \sum_{j=1}^{m}\sum_{i=1}^{n} b_{ji}a_{ij}$$

Change the summation sequence and exchange the position of a and b, we have:

$$trBA = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ji}$$

This is exactly the same with eq. (1).

Proof is done. ∎

Theorem 3.3 *Suppose $f(A) = trAB$, then $\nabla_A trAB = B^T$ where $A \in R^{m\times n}, B \in R^{n\times m}$.*

Proof From eq. (1), we have:

$$\nabla_A trAB = \begin{pmatrix} \frac{\partial \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ji}}{\partial a_{11}} & \cdots & \frac{\partial \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ji}}{\partial a_{1n}} \\ \vdots & & \vdots \\ \frac{\partial \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ji}}{\partial a_{m1}} & \cdots & \frac{\partial \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ji}}{\partial a_{mn}} \end{pmatrix} \tag{3}$$

For $(\nabla_A trAB)_{pq} = \frac{\partial \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{ji}}{\partial a_{pq}}$, there is only one term contains a_{pq}, thus $(\nabla_A trAB)_{pq} = b_{qp}$.

Now, it is easy to see $\nabla_A trAB = B^T$ ∎

Theorem 3.4 *If $a \in R$, $tra = a$*

Proof This one is obvious. ∎

Theorem 3.5 $\nabla_A trABA^TC = CAB+C^TAB^T$, *where $A \in R^{m\times n}, B \in R^{n\times n}, C \in R^{m\times m}$.*

Proof At first, this one seems to be an application of theorem 3.3. However, since A^T is contained in BA^TC, this is not true.

But the good news is, we can do similar steps to prove this theorem.

We denote d_{ij} as the element at i row and j column of BA^TC.

Then, for $trABA^TC$, we have

$$trABA^TC = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}d_{ji}$$

Then, for $\nabla_A trABA^TC$, we have

$$\nabla_A trABA^TC = \nabla_A \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}d_{ji}$$

$$(\nabla_A trABA^TC)_{pq} = d_{qp} + \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}(\nabla_A d_{ji})_{pq}$$

The first term is actually $(BA^TC)_{qp} = (C^TAB^T)_{pq}$

The second term is harder to see.

Let's first figure out what d_{ji} is.

$$\begin{aligned} d_{ji} &= (BA^TC)_{ji} \\ &= \sum_{k=1}^{n} b_{jk}(\sum_{l=1}^{n} a_{lk}c_{li}) \end{aligned}$$

Since only terms contain a_{pq} matter, the second term become this:

$$\begin{aligned} \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}(\nabla_A d_{ji})_{pq} &= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}b_{jq}c_{pi} \\ &= \sum_{i=1}^{m}\sum_{j=1}^{n} c_{pi}a_{ij}b_{jq} \\ &= (CAB)_{pq} \end{aligned}$$

Combine first and second term, we have:

$$\nabla_A trABA^TC = CAB + C^TAB^T$$

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