Algebra

Shuai

December 24, 2013

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1 Start of Journey

1.1 Cartesian Product

Algebra, and much of math deals with domain and mapping between domains. The domain can be natural number, real number, matrix, vector. And the mapping belongs to some universal set – the set of Cartesian product.

Definition 1.1 *Let A and B be sets. Then the set $A \times B$ of all ordered pairs (a, b) where $a \in A, b \in B$ is called the Cartesian product of the sets A and B.*

I first met this concept at the time I leart the Database course. But I do not get a intuitive feeling about it until now.

Actually, the real plane R^2 is a natural example of a Cartesian product. Cartesian product is like a stage where all algebra players must play on such stage.

Remark 1.1 *The definition and examples are all in two dimesion, however, Cartesian product can extend to any dimension.*

1.2 Algebraic Operations

With the objects we can manipulate(this part means set theory, which I did not talk about), and the stage their mappings play on(Cartesian product), we may try to establish some theory upon them.

Remark 1.2 *There are no rules existing yet. Object(set) are just object. Mappings are just mappings.*

With the stage we have now, we may begin to return to the math we are used to.

We are used to the concept of operations, such as addition, subtraction. They are abtracted from our daily life. To generalize them, mathematicians bring forward the idea of binary algebraic operations, which is one of the most fundamental in mathematics[1].

Definition 1.2 *Let M be a set. The mapping $\theta : M \times M \to M$ from Cartesian square of M to M is called binary(algebraic) operation on set M. Thus, corresponding to every ordered pair (a, b) of elements, where $a, b \in M$, there is a uniquely defined element $\theta(a, b) \in M$. The element $\theta(a, b) \in M$ is called composition of the elements a and b relative to this operation.[1]*

Then starting from binary algebraic operation, mathematicians build the algebraic structure bit by bit.

Remark 1.3 *There is one important note given about notation in the book[1]. It is often rather cumbersome to keep referring to the function θ and using the notation $\theta(a, b)$. There are several shorthand symbols that are employed and $\theta(a, b)$ is often written using such special notation. For example, the operation might be denoted by $\diamond$ and we might then write $\theta(a, b) = a \diamond b$. We note that, in general, $\theta(a, b)$ will be different from $\theta(b, a)$. However, quite often, even the notation $a \diamond b$ is confusing, and most often we would rather write the operation $\diamond$*

using something more familiar. The most familiar binary operators are + and · and it is these symbols that are most often useful in writing such operations. Thus, instead of writing $a \diamond b$ we may write $a + b$ or $a \cdot b$. It is important to understand that sometimes these symbols will have familiar meanings, but not always.[1]

1.3 Important Properties of Algebraic Structure

1.3.1 Rules

Definition 1.3 ***Commutativity****: A binary operation on a set M is called commutative if $ab = ba$ for each pair a, b of elements of M.[1]*

Definition 1.4 ***Associativity****:A binary operation on a set M is called associative if $(ab)c = a(bc)$ for each triple a, b, c of elements of M.[1]*

1.3.2 Special Elements

Besides those two rules, the zero and identity element in one algebraic structure are of special status.

Definition 1.5 ***Neutral Element****:Let M be a set with binary operation. The element $e \in M$ is called a neutral element under this operation if $ae = ea = a$ for each element a of the set M.[1]*

Remark 1.4 *If the operation on M is written multiplicatively, then the term identity element is usually ued rather than neutral element and often e is denoted by 1 or 1_m. If we use the additive form, then the neutral element is usually called the zero element and is often denoted by 0_M, so that the definition of the zero element is $a + 0_M = 0_M + a = a$ for each element $a \in M$.[1]*

1.3.3 Algebraic Properties

Properties can also be understanded as restriction put on algebraic structure or a abstraction of natural properties of natural algebraic structure.

Definition 1.6 ***Stable****: Let M be a set with a binary operation. A subset S is called stable under this operation if for each pair of elements $a, b \in S$ the element ab also belongs to S.[1]*

Definition 1.7 ***Invertibility****: Let M be a set with binary operation and suppose that there is an identity element e. The element x is called an inverse of the element a if*

$$ax = xa = e.$$

if a has an inverse then we say that a is invertible.[1]

Remark 1.5 *Invertibility is just a property. We are used to take it as granted if we are so used to the natural operation such as addition or multiplication. Some algebraic structure has it, but some do not.*

1.4 Algebraic Structures

Definition 1.8 ***semigroup****: A nonempty set S is called a semigroup if S has an associative binary operation defined on it. If this operation i commutative, we will say that S is a commutative semigroup.[1]*

Definition 1.9 ***group****: A semigroup G with identity is called a group if every element of G is invertible. Thus, a group is a set G together with a binary algebraic operation $(x, y) \to xy$ where $x, y \in G$, such that the following conditions(the group axiom) holds:[1]*

- ***G 1*** *The operation is associative so that $x(yz) = (xy)z$ for all $x, y, z \in G$.*
- ***G 2*** *G has an identity element, an element e such that $xe = ex = x$ for all $x \in G$; often 1 or 1_G is used in place of e.*
- ***G 3*** *Every element $x \in G$ has an inverse x^{-1} such that $xx^{-1} = x^{-1}x = e$.*

Definition 1.10 ***abelian group****: If the group operation is commutative, then the group is callled abelian(in honor of the great Norwegian mathematician Nielss Henrik Abel(1802 - 1829).[1]*

1.4.1 Mapping Properties

Definition 1.11 *Let M, S be sets with binary operations that we denote by $*$ and $\diamond$, respectively. Let $f : M \to S$ be a mapping. Then f is called a homomorphism, if*

$$f(x * y) = f(x) \diamond f(y)$$

for arbitrary elements $x, y \in M$.[1]

We say that the mapping f respects the operations. An injective homomorphism is called **monomorphism**. A surjective homomorphism is called an **epimorphism** and a bijective homomorphism is called an **isomorphism**.[1]

When two structures M, S are isomorphic in this way, there is no difference between the structures other than the names we give to the elements of the two sets M and S and the names $*$ and $\diamond$ that we give to the names of the operators. Other than this, the structures of M and S are identical.[1]

If M is a set with binary operation, then the study of M has two aspects. The first aspect is concerned with the nature of the elements and the structure of M, while the second one concerns properties of the operation. This enables such a study to be conducted from different points of view. We can sdtudy the relationship between the elements and the subsets of M and also study individual properties with repsect to given operation. Such an approach is feasible for the study of concrete sets, such as permutations, transformations of the plane and space, symmetries, matrices, and so on. However, we can conduct a study of the properties that does not depend on the nature of the elements and which is completely defined by the operation. This approach is the key approach in algebra and it can be covered by very efficienty, thanks to the fundamental notion of isomorphism. Making this more concrete, Gottfired Leibniz(1646 - 1716) introduced the general notion of an isomorphic relation(which he called a similarity) and pointed out the possibility of the identification of isomorphic operations and relations. He brought attention to a classical example of isomorphism, namely the mapping $x \to logx$ from the set of all positive real numbers with operation of multiplication to the set of all real numbers with the operation of addition. A great French mathematician, Evariste Galios(1811 - 1832), was also familiar with the idea of isomorphism. He understood the corresponding elements of isomorphic sets M and S have the same properties with repsect to the given operation. This notion in its general form was developed in the middle of the nineteenth century. In abstract algebra, we study only such properties that are unchanged by isomorphisms.[1]

2 Fields

After becoming familiar with basic algebraic structure and their properties, I finally reach what I want to understand – the algebraic structure, field.

Definition 2.1 ***Division Ring****: A set D with two binary algebraic operations, addition and multiplication, is called a divsion ring if it satisfies the following properties:*

- *the addition is commutative, so*

$$x + y = y + x$$

for all elements $x, y \in D$;

- *the addition is associative, so*

$$x + (y + z) = (x + y) + z$$

for all elements $x, y, z \in D$;

- *D has a zero element, 0_D, an element with the property that*

$$x + 0_D = 0_D + x = x$$

for all elements $x \in D$.

- *each element $x \in D$ has an additive inverse(the opposite or negative element), $-x \in D$, an element with the property that*

$$x + (-x) = 0_D;$$

- *the distributive laws hold in D, so*

$$x(y + z) = xy + xzand(x + y)z = xz + yz$$

for all elements $x, y, z \in D$;

- *the multiplication is associative, so*

$$x(yz) = (xy)z$$

for all elements $x, y, z \in D$;

- *D has a (multiplicative) identity element, $e \neq 0_D$, and element with property that*

$$xe = ex = x$$

for each element $x \in D$.

- *each nonzero element $x \in D$ has a multiplicative inverse(the reciprocal), $x^{-1} \in D$, and element with property*

$$xx^{-1} = x^{-1}x = e$$

[1]

Definition 2.2 ***Field***: *A division ring D is called a field, if the multiplication of its elements is always commutative. Thus a field has the additional property that $xy = yx$ for all elements $x, y \in D$.*

References

[1] M. Dixon, L. Kurdachenko, and I. Subbotin. *Algebra and Number Theory: An Integrated Approach.* Wiley, 2011.