The Natural Numbers

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1 Axiomization of the intuition – Peano Axiom

The notion of a natural number is one of the most fundamental and most important in mathematics. The system of natural numbers was the first abstract scientific concept created by man. Having dealt in everyday life, with certain quantitieso of real things, people noted certain general properties of numbers and developed the notion of counting numbers. This apparently simple concept is in some ways so profound that it has prompted some people to believe that this concept comes directly from God. A great German number theorist, Leopold Kronecker(1823 - 1891) said:"God made the natural numbers, all else is the work of man."[Heinrich Weber. Leopold Kronecker. Jahresberichte DMV 1893; 2:5-31]. Creating the notion of a natural number is first step not only in mathematics, but in the development of all sciences[1].

The history is long, however, the modern axiomatic theory of natural numbers is developed at the end of the nineteenth century and named in honor of a famous Italian mathematician, Giuseppe Peano(1858 - 1932), whose input in the axiomatization of natural numbers was of exceptional mathematical and philosopical value[1].

Mathematics comes from everyday life, but abstracts it, then provides a solid foundation for further reasoning, development of more advanced theory.

The peano axiom is the very foundation of number theory, or in some sense modern mathematics. It clearly defines what we used to count in real life, meaning axiomize it.

Definition 1.1 ***Peano Axiom****: The set N_0 is a nonempty set and for all $a \in N_0$, there is a uniquely defined element a', called the immediate successor of a and for which the following axioms hold:*

- ***(P 1)*** *$a = b$ implies that $a' = b'$.*
- ***(P 2)*** *There is an element 0(the natural number 0) such that 0 is not the immediate successor of any element of N_0. Thus $0 \neq a'$ for all elements $a \in N_0$.*
- ***(P 3)*** *If $a, b \in N_0$ and $a' = b'$, then $a = b$.*
- ***(P 4)*** *(the induction axiom). Let M be a subset of N_0 satisfying the conditions:*
  - *$0 \in M$;*

– *if* $a \in M$*, then* $a' \in M$.

Then $M = N_0$.

The natural number is defined as[1]:

$$0 = \emptyset, 1 = \{\emptyset\}, 2 = 1 \cup \{1\} = \{0, 1\}, 3 = 2 \cup \{2\} = \{0, 1, 2\}...$$

In the book[1], the author does not talk about why natural number is defined this way:

> Such a level of exposition is far beyond the scope of this book and requires significant mathematical maturity.

I just take those definition as natural abstraction of the counting symbol used by people.

With the Peano Axiom, all normal arithmetical properties can be derived.

Ok. I think this is where I should stop digging.

References

[1] M. Dixon, L. Kurdachenko, and I. Subbotin. *Algebra and Number Theory: An Integrated Approach*. Wiley, 2011.