Proofs of Theorems about Trace of Matrix

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Contents

1 Definitions

Definition The uppercase letters below all denote matrices and all are all $N \times N$, and the trace of X is denoted trX.

Definition Let f(A), where A is a matrix, be a function $\in R^{m\times n} \to R$, where

$$\nabla_A f(A) = \begin{pmatrix} \frac{\partial f}{\partial a_{11}} & \cdots & \frac{\partial f}{\partial a_{1n}} \\ \vdots & & \vdots \\ \frac{\partial f}{\partial a_{m1}} & \cdots & \frac{\partial f}{\partial a_{mn}} \end{pmatrix}$$

2 Theorems Summary

1. Commutative Law:
   $trAB = trBA$
   $trABC = trCAB = trBCA$
2. Suppose $f(A) = trAB$, then $\nabla_A trAB = B^T$.
3. If $a \in R$, $tra = a$.
4. $\nabla_A trABA^T C = CAB + C^T AB^T$

3 Theorems and its Proof

Theorem 3.1 *$trAB = trBA$*

Corollary 3.2 *$trABC = trCAB = trBCA$*

Proof First write the expressions of two sides of the equations:

$$trAB = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji} \tag{1}$$

$$trBA = \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}a_{ji} \tag{2}$$

Switch the role of i and j of eq. (2), we get:

$$trBA = \sum_{j=1}^{n}\sum_{i=1}^{n} b_{ji}a_{ij}$$

Change the summation sequence and exchange the position of a and b, we have:

$$trBA = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji}$$

This is exactly the same with eq. (1).

Proof is done. $\blacksquare$

Theorem 3.3 *Suppose* $f(A) = trAB$, *then* $\nabla_A trAB = B^T$

Proof From eq. (1), we have:

$$\nabla_A trAB \quad = \quad \begin{pmatrix} \frac{\partial \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji}}{\partial a_{11}} & \cdots & \frac{\partial \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji}}{\partial a_{1n}} \\ \vdots & & \vdots \\ \frac{\partial \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji}}{\partial a_{n1}} & \cdots & \frac{\partial \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji}}{\partial a_{nn}} \end{pmatrix} \tag{3}$$

Note, m is changed to n, since A and B are square matrix.

For $(\nabla_A trAB)_{pq} = \frac{\partial \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{ji}}{\partial a_{pq}}$, there is only one term contains a_{pq}, thus $(\nabla_A trAB)_{pq} = b_{qp}$.

Now, it is easy to see $\nabla_A trAB = B^T$ ∎

Theorem 3.4 *If* $a \in R$, $tra = a$

Proof This one is obvious. ∎

Theorem 3.5 $\nabla_A trABA^TC = CAB + C^TAB^T$

Proof At first, this one seems to be an application of theorem 3.3. However, since A^T is contained in BA^TC, this is not true.

But the good news is, we can do similar steps to prove this theorem.

We denote d_{ij} as the element at i row and j column of BA^TC.

Then, for $trABA^TC$, we have

$$trABA^TC = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}d_{ji}$$

Then, for $\nabla_A trABA^TC$, we have

$$\nabla_A trABA^TC = \nabla_A \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}d_{ji}$$

$$(\nabla_A trABA^TC)_{pq} = d_{qp} + \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\nabla_A d_{ji})_{pq}$$

The first term is actually $(BA^TC)_{qp} = (C^TAB^T)_{pq}$

The second term is harder to see.

Let's first what d_{ji} is.

$$\begin{aligned} d_{ji} &= (BA^TC)_{ji} \\ &= \sum_{n}^{k=1} b_{jk}(\sum_{l=1}^{n} a_{lk}c_{li}) \end{aligned}$$

Since only terms contain a_{pq} matter, the second term become this:

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(\nabla_A d_{ji})_{pq} &= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}b_{jq}c_{pi} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} c_{pi}a_{ij}b_{jq} \\ &= (CAB)_{pq}\end{aligned}$$

Combine first and second term, we have:

$$\nabla_A trABA^TC = CAB + C^TAB^T$$

$\blacksquare$