1. $\begin{cases} -\epsilon u'' + u = x, & \forall x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}$

Step 1: $f(x) = e^{\lambda x} P_m(x) = x \Rightarrow \lambda = 0,\ m = 1$

character function: $-\epsilon r^2 + 1 = 0 \Rightarrow r = \pm \frac{1}{\sqrt{\epsilon}}$

$\therefore \lambda$ is not the solution of character function

$\therefore$ the special solution is $y^* = C_1 x + C_2$

$\therefore C_1 x + C_2 = x \Rightarrow C_1 = 1,\ C_2 = 0 \quad \therefore y^* = x$

Step 2: the general solution of $-\epsilon u'' + u = 0$ is

$$y = C_1 \cdot e^{\frac{1}{\sqrt{\epsilon}} x} + C_2 \cdot e^{-\frac{1}{\sqrt{\epsilon}} x}$$

$\therefore$ the general solution of $-\epsilon u'' + u = x$ is

$$y = C_1 \cdot e^{\frac{1}{\sqrt{\epsilon}} x} + C_2 \cdot e^{-\frac{1}{\sqrt{\epsilon}} x} + x$$

$\because u(0) = u(1) = 0 \Rightarrow \begin{cases} C_1 + C_2 = 0 \\ C_1 \cdot e^{\frac{1}{\sqrt{\epsilon}}} + C_2 \cdot e^{-\frac{1}{\sqrt{\epsilon}}} + 1 = 0 \end{cases} \Rightarrow C_1 = -\frac{e^{-\frac{1}{\sqrt{\epsilon}}}}{1 - e^{-\frac{2}{\sqrt{\epsilon}}}}$

$$C_2 = \frac{e^{-\frac{1}{\sqrt{\epsilon}}}}{1 - e^{-\frac{1}{\sqrt{\epsilon}}}}$$

$$\therefore y = x - \frac{\exp\left(\frac{x-1}{\sqrt{\epsilon}}\right) - \exp\left(-\frac{x+1}{\sqrt{\epsilon}}\right)}{1 - \exp\left(-\frac{2}{\sqrt{\epsilon}}\right)}$$

$\because$ We can only find this C_1, C_2 in ~~the~~ general solution

$\therefore$ it's the unique solution

$\therefore\ y = x - \frac{\exp(\frac{x}{\sqrt{\epsilon}}) - \exp(-\frac{x}{\sqrt{\epsilon}})}{1-\exp(-\frac{2}{\sqrt{\epsilon}})}$ $\qquad\qquad 1-e^{-\frac{1}{\sqrt{\epsilon}}}$

$\because$ We can only find this C_1, C_2 in ~~this~~ general solution

$\therefore$ it's the unique solution

2. $\because u_i'' \approx \delta_h \delta_{-h} u_i^h = \frac{u_{i+1}^h - 2u_i^h + u_{i-1}^h}{h^2}$, $u_i' \approx \bar{\delta}_h u_i^h \approx \frac{u_{i+1} - u_{i-1}}{2h}$

$\because -\epsilon u'' + u = x$

$$\therefore -\epsilon\left(\frac{u_{i+1}^h - 2u_i^h + u_{i-1}^h}{h^2}\right) + u_i^h = h\cdot i$$

$$\therefore -\frac{\epsilon}{h^2} u_{i+1}^h + \left(\frac{2\epsilon}{h^2}+1\right) u_i^h - \frac{\epsilon}{h^2} u_{i-1}^h = h\cdot i$$

$$\therefore r = \frac{\epsilon}{h^2}, \quad s = \frac{2\epsilon}{h^2}+1, \quad t = \frac{\epsilon}{h^2}$$

$\because u(0) = u(1) = 0 \qquad \therefore u_0^h = u_N^h = 1 \qquad \therefore \begin{cases} u_0^h = 0 \\ -r\cdot u_{i+1}^h + s\, u_i^h - t\, u_{i-1}^h = h\cdot i \\ u_N^h = 0 \end{cases}$

$$\therefore R^h f = \left(0, \frac{1}{N}, \frac{2}{N}, \cdots\cdots, \frac{N-1}{N}, 0\right)$$

$$L^h = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots & 0 & 0 \\ -r & s & -t & & & & \\ & -r & s & -t & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & & -r & s & -t \\ 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 \end{bmatrix}$$

3. $\because L^h R^h V = \begin{bmatrix} 1 & 0 \cdots 0 \\ & L^h & \end{bmatrix} \cdot \begin{bmatrix} V_0 \\ V_1 \\ \vdots \\ V_N \end{bmatrix}$

① $(L^h R^h V)_0 = V_0 = 0$

$(R^h L V)_0 = Lv(0) = 0$

$|(L^h R^h V)_0 - (R^h L V)_0| = 0$

② Similarly $|(L^h R^h V)_N - (R^h L V)_N| = 0$

③ for $1 \le i \le N-1$ $\quad (L^h R^h V)_i = -\epsilon \delta_h \delta_{-h} V_i + V_i$

$(R^h L V)_i = Lv(x_i) = -\epsilon V''(x_i) + V_i$

$\therefore |(L^h R^h V)_i - (R^h L V)_i| = o(h^2)$ $\quad \therefore L^h$ is consistent of $\alpha = 2$

Then NTS the stability of L^h

① if $|V_0| = \|V\|_\infty$, then $\|L^h V\|_\infty \ge |(L^h V)_0| = |V_0| = \|V\|_\infty$

② if $|V_N| = \|V\|_\infty$, then Similarly $\|L^h V\| \ge \|V\|_\infty$

③ if $V_i = \|V\|_\infty$ for some $1 \le i \le N-1$

$(L^h V)_i = -r V_{i-1} + s V_i - t V_{i+1}$

$= r(V_i - V_{i-1}) + t(V_i - V_{i+1}) + (s - r - t) V_i$

$\because V_i = \|V\|_\infty$ $\therefore V_i - V_{i-1} \ge 0$, $V_i - V_{i+1} \ge 0$, $r = \frac{\epsilon}{h^2} > 0$, $s = \frac{2\epsilon}{h^2} + 1 > 0$

$s - r - t = 1$ $\qquad t = \frac{\epsilon}{h^2} > 0$

$\therefore (L^h V)_i \ge V_i = \|V\|_\infty \Rightarrow \|L^h V\|_\infty \ge \|V\|_\infty$

④ if $-V_i = \|V\|_\infty$ for som $1 \le i \le N-1$

$(L^h V)_i = -r(V_{i-1} - V_i) - t(V_{i+1} - V_i) + (s - r - t) V_i$

$\because -r(V_{i-1} - V_i) \le 0$, $-t(V_{i+1} - V_i) \le 0$, $V_i \le 0$ $\therefore V_i \ge (L^h V)_i$

$\therefore |L^h V_i| \ge |V_i|$ $\qquad \therefore \|L^h V_i\| \ge \|V\|_\infty$

$\therefore L^h$ is stable