

Applied Machine Learning

Chapter 3- A Little Linear Algebra



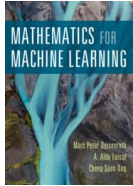
Hossein Homaei

Department of Electrical & Computer Engineering

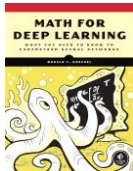


Some resources

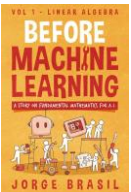
- Books



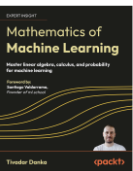
M. P. Deisenroth, A. A. Faisal, and C. S. Ong, *Mathematics for machine learning*. Cambridge University Press, 2020.



R. T. Kneusel, *Math for Deep Learning: What You Need to Know to Understand Neural Networks*. No Starch Press, 2022.



J. Brasil, *Before Machine Learning*, vol. 1, Linear Algebra. 2023.



T. Danka, *Mathematics of Machine Learning: Master Linear Algebra, Calculus, and Probability for Machine Learning*. Packt, 2025.



... Some resources

- Online
 - Linear Algebra for Machine Learning and Data Science
 - Instructor: Luis Serrano
 - DeepLearning.AI
 - **A significant portion of this lecture's content is sourced from this course.**
 - CS229: Machine Learning- The Summer Edition 2019
 - Instructor: Anand Avati
 - Stanford University
 - Essence of linear algebra
 - Grant Sanderson
 - 3Blue1Brown
 - Linear Algebra
 - Instructor: Gilbert Strang
 - MIT
 - Mathematic for AI [In Persian]
 - Instructor: Behrooz Nasihatkon
 - Khajeh Nasir Toosi University of Technology



Linear Algebra

- What is it about?
 - Manipulating vectors and matrices to do calculations
- Why important?
 - It is the backbone of many machine learning techniques
- What is a linear equation?
 - Any equation in this form: $a_1x_1 + a_2x_2 + \dots + a_nx_n + b = 0$
 - Some examples of non-linear equations:
 - $x^2 + y^2 = 0$
 - $\sin(x) + y^2 = 12$
 - $2^x - 3^y = 0$



Solving system of linear equations



... Singularity concept

System 1

$$\begin{aligned}a + b &= 10 \\ a + 2b &= 12\end{aligned}$$

Complete system

Unique solution

Non-singular

System 2

$$\begin{aligned}a + b &= 10 \\ 2a + 2b &= 20\end{aligned}$$

Redundant system

Infinitely many solutions

Singular

System 3

$$\begin{aligned}a + b &= 10 \\ 2a + 2b &= 24\end{aligned}$$

Contradictory system

No solution

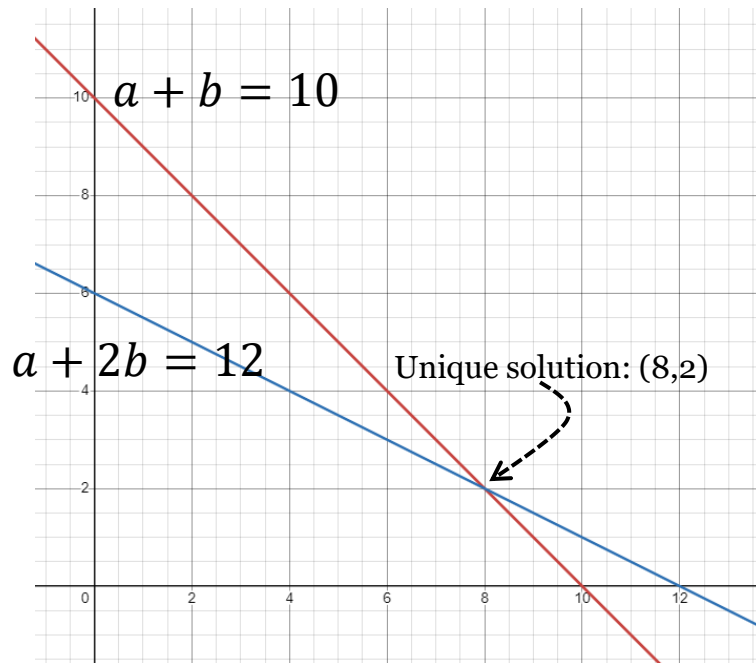
Singular



... Singularity concept

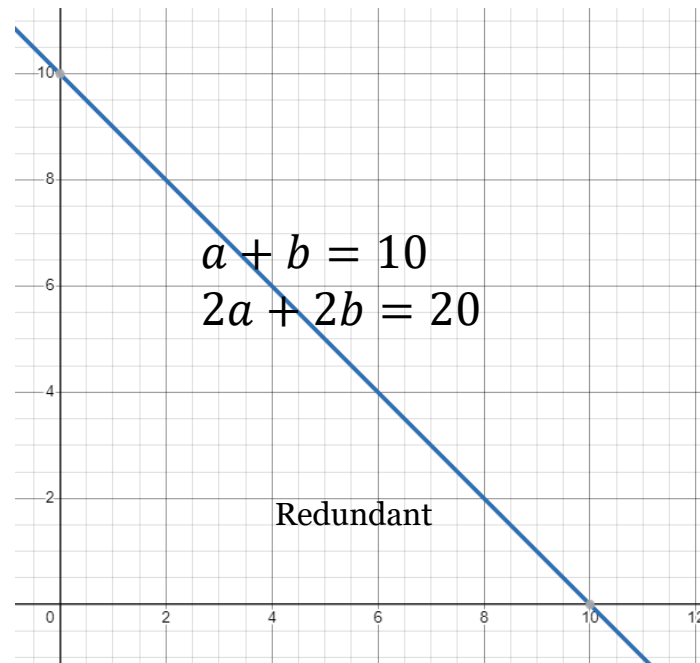
System 1

Non-singular



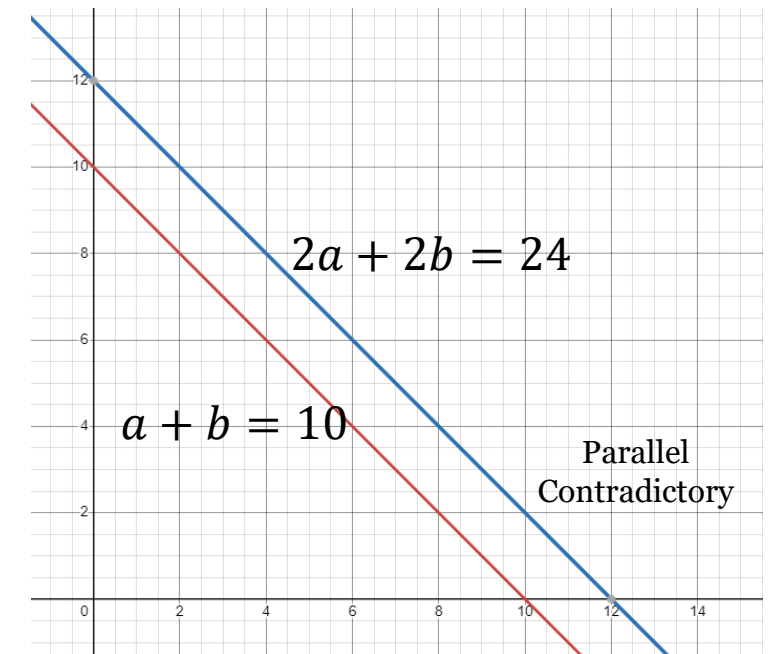
System 2

Singular



System 3

Singular



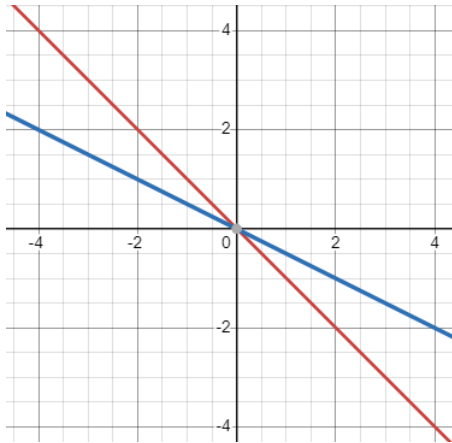
... Singularity concept

System 1

$$a + b = 10 \rightarrow a + b = 0$$

$$a + 2b = 12 \rightarrow a + 2b = 0$$

non-singular \rightarrow non-singular

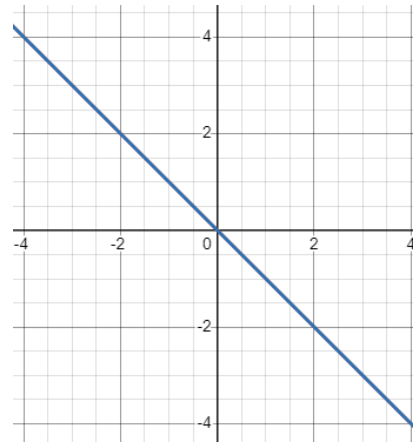


System 2

$$a + b = 10 \rightarrow a + b = 0$$

$$2a + 2b = 20 \rightarrow a + b = 0$$

singular \rightarrow singular

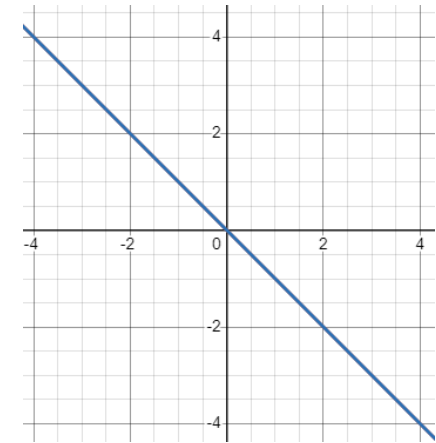


System 3

$$a + b = 10 \rightarrow a + b = 0$$

$$2a + 2b = 24 \rightarrow a + b = 0$$

singular \rightarrow singular



Conclusion

Constants in the system don't matter in determining the singularity.



Matrix Singularity

System of equations

$$\begin{aligned}a + b &= 0 \\ a + 2b &= 0\end{aligned}$$

Non-singular
system

No row is a multiple of the other one

Rows are linearly independent
(columns are also linearly independent)

Corresponding matrix

a	b
1	1
1	2

Non-singular
matrix

System of equations

$$\begin{aligned}a + b &= 0 \\ 2a + 2b &= 0\end{aligned}$$


singular
system

2nd row is a multiple of the 1st row

Rows are linearly dependent

Corresponding matrix

a	b
1	1
2	2



Singular
matrix



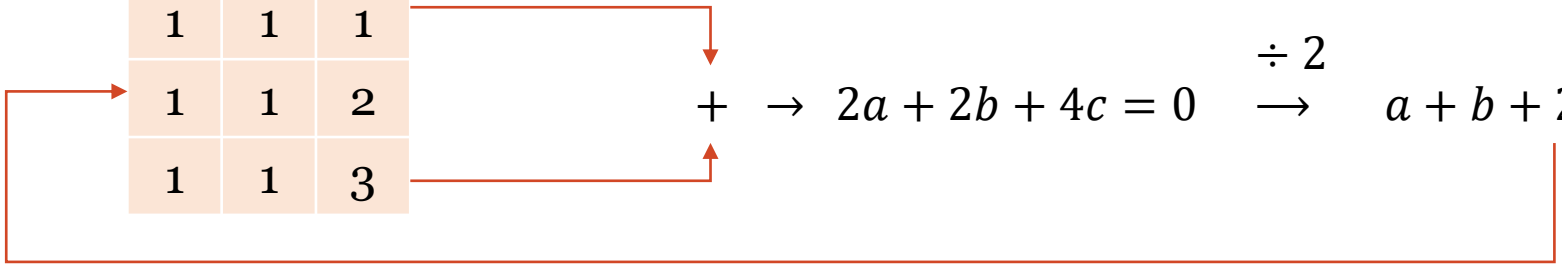
... Matrix Singularity

$$\begin{aligned}a + b + c &= 0 \\a + b + 2c &= 0 \\a + b + 3c &= 0\end{aligned}$$

a	b	c
1	1	1
1	1	2
1	1	3

$+ \rightarrow 2a + 2b + 4c = 0 \xrightarrow{\div 2} a + b + 2c = 0$

2nd row depends of the 1st and 3rd rows



Rows are linearly dependent
The matrix is singular



... Matrix Singularity

$$\begin{aligned}a + b + c &= 0 \\a + 2b + c &= 0 \\a + b + 2c &= 0\end{aligned}$$

a	b	c
1	1	1
1	2	1
1	1	2

There is no relation between rows and columns

Rows and columns are linearly independent
The matrix is non-singular
How can we determine that?



... Matrix Singularity

- Assume that a matrix is singular
 - The 2nd row depends on the 1st row

a	b
c	d

$$\begin{array}{|c|c|} \hline a & b \\ \hline \end{array} \times k = \begin{array}{|c|c|} \hline c & d \\ \hline \end{array} \rightarrow \begin{cases} ak = c \rightarrow k = \frac{c}{a} \\ bk = d \rightarrow k = \frac{d}{b} \end{cases} \rightarrow \frac{c}{a} = \frac{d}{b} \rightarrow ad = bc$$

$$\boxed{ad - bc} = 0$$

Determinant



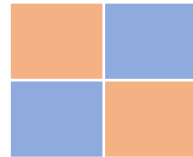
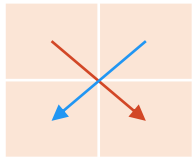
... Matrix Singularity



- We can decide about matrix singularity based on the determinant
 - Non-zero determinant \rightarrow non-singular matrix
 - Zero determinant \rightarrow singular matrix

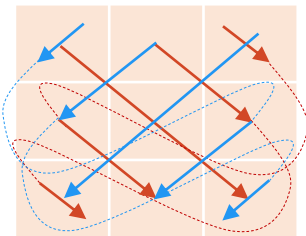


Determinant

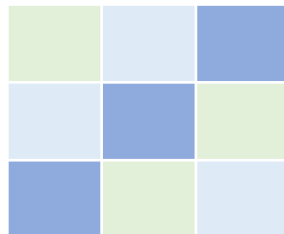
- A method for computation



Add multiplication of  And subtract multiplication of 



Add multiplications of , , and 



And subtract multiplications of , , and 



... Determinant

- More formal formula (for 3×3 matrix)

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \times \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \times \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \times \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$



... Determinant

- More general: Laplace or cofactor expansion
 - The determinant for an $n \times n$ matrix A
 - The Laplace expansion along the i th row ($1 \leq i \leq n$) is

$$\det(A) = \sum_{j=1}^n (-1)^{i+j} a_{i,j} m_{i,j}$$

- Where, $a_{i,j}$ is the entry of the i th row and j th column of A , and $m_{i,j}$ is the determinant of the submatrix obtained by removing the i th row and j th column of A
 - Usually, we set $i = 1$:

$$\det(A) = \sum_{j=1}^n (-1)^{j+1} a_{1,j} m_{1,j}$$



... Determinant

- Example-1

1	1	1
1	2	1
1	1	2

$$1 \times 2 \times 2 + 1 \times 1 \times 1 + 1 \times 1 \times 1 - 1 \times 2 \times 1 - 1 \times 1 \times 2 - 1 \times 1 \times 1 \\ = 4 + 1 + 1 - 2 - 2 - 1 = 1$$

$$1 \times (4 - 1) - 1 \times (2 - 1) + 1 \times (1 - 2) = 3 - 1 - 1 = 1$$

The matrix is non-singular



... Determinant

- Example-2

2	1	3	4
0	-1	2	1
3	2	0	5
-1	3	2	1

$$\begin{aligned} & 2 \times \det(A_{11}) - 1 \times \det(A_{12}) + 3 \times \det(A_{13}) - 4 \times \det(A_{14}) \\ & = 80 + 10 + 57 - 112 = 35 \end{aligned}$$

The matrix is non-singular



Solving system of linear equations

Non-singular system

$$\begin{array}{lcl} \begin{array}{l} 5a + b = 17 \\ 4a - 3b = 6 \end{array} & \xrightarrow{\text{Divide by coefficient of } a} & \begin{array}{l} a + 0.2b = 3.4 \\ a - 0.75b = 1.5 \end{array} \\ & & \begin{array}{r} - \\ \hline 0.95b = 1.9 \end{array} \end{array}$$

Subtract to eliminate a
from one equation

$$\begin{array}{c} \downarrow \\ b = 2 \\ \downarrow \\ a = 3 \end{array}$$

Solved system



... Solving system of linear equations

Singular system

$$\begin{array}{l} a + b = 10 \\ 2a + 2b = 20 \end{array}$$

Divide by coefficient of a

$$\begin{array}{l} a + b = 10 \\ a + b = 10 \end{array}$$

Subtract to eliminate a
from one equation

$$0 = 0$$

$$a + b = 10$$

Solved system
 $a=x, b=10-x$

The solution has **one degree of freedom**
If you vary x, you can get many different solutions to the system.



... Solving system of linear equations

Singular system

$$\begin{array}{lcl} a + b = 10 & \xrightarrow{\text{Divide by coefficient of } a} & a + b = 10 \\ 2a + 2b = 24 & & \underline{a + b = 12} \\ & & 0 = 2 \end{array}$$

Subtract to eliminate a
from one equation

↓

Contradiction

↓

There is no solution



... Solving system of linear equations

System with more variables

$$\begin{aligned}a + b + 2c &= 12 \\ 3a - 3b - c &= 3 \\ 2a - b + 6c &= 24\end{aligned}$$

The Divide by coefficient of a

$$\begin{aligned}a + b + 2c &= 12 \\ a - b - \frac{1}{3}c &= 1 \\ a - \frac{1}{2}b + 3c &= 12\end{aligned}$$

Use the 1st equation to remove a from the others

$$\begin{aligned}a + b + 2c &= 12 \\ -2b - \frac{7}{3}c &= -11 \\ -\frac{3}{2}b + c &= 0\end{aligned}$$

Divide by the coefficient of b

$$\begin{aligned}b + \frac{7}{6}c &= \frac{11}{2} \\ b - \frac{2}{3}c &= 0\end{aligned}$$

Use the 1st equation to remove b from the other

Solve this new system of two equations

$$\frac{11}{6}c = \frac{11}{2} \rightarrow c = 3 \rightarrow b = 2 \rightarrow a = 4$$



Gaussian Elimination

- Solve a system of equations in matrix form
 - Same as we explained, but in a more simplified procedural form
 - Also called Row Reduction



... Gaussian Elimination

Example

Original system

$$\begin{aligned} 5a + b &= 17 \\ 4a - 3b &= 6 \end{aligned}$$

Intermediate system

$$\begin{aligned} a + 0.2b &= 3.4 \\ b &= 2 \end{aligned}$$

Solved system

$$\begin{aligned} a &= 3 \\ b &= 2 \end{aligned}$$

Original matrix

5	1
4	-3

row manipulation

Upper diagonal matrix

1	0.2
0	1

More manipulation

Diagonal matrix

1	0
0	1

Row Echelon Form

Reduced Row Echelon Form



... Gaussian Elimination

- Row Echelon Form
 - Potential zero rows at the bottom
 - The left-most non-zero entry (**pivot**) of every row is on the right of the leading entry of every row above.
 - So, Below the diagonal, everything is zero

Every pivot is to the right of the pivots on the rows above

3	*	*	*	*
0	0	1	*	*
0	0	0	-4	*
0	0	0	0	0
0	0	0	0	0

Potential zero rows at the bottom



... Gaussian Elimination

- Reduced Row Echelon Form
 - Row Echelon Form
 - The pivot in each row is 1
 - Called leading one
 - Each column containing a leading 1 has zeros in all entries above the leading 1

Row echelon form

3	*	*	*	*
0	0	1	*	*
0	0	0	-4	*
0	0	0	0	0
0	0	0	0	0

Reduced row echelon form

1	*	0	0	*
0	0	1	0	*
0	0	0	1	*
0	0	0	0	0
0	0	0	0	0



... Gaussian Elimination

- Transform Row Echelon to Reduced Row Echelon Form
 - Divide each row by its pivot
 - Clear out any number above the pivot

1	2	3		1	0	-5		1	0	0		1	0	0
0	1	4	Subtract 2 times the 2 nd row from the 1 st one	0	1	4	Subtract -5 times the 3 rd row from the 1 st one	0	1	4	Subtract 4 times the 3 rd row from the 2 nd one	0	1	0
0	0	1		0	0	1		0	0	1		0	0	1



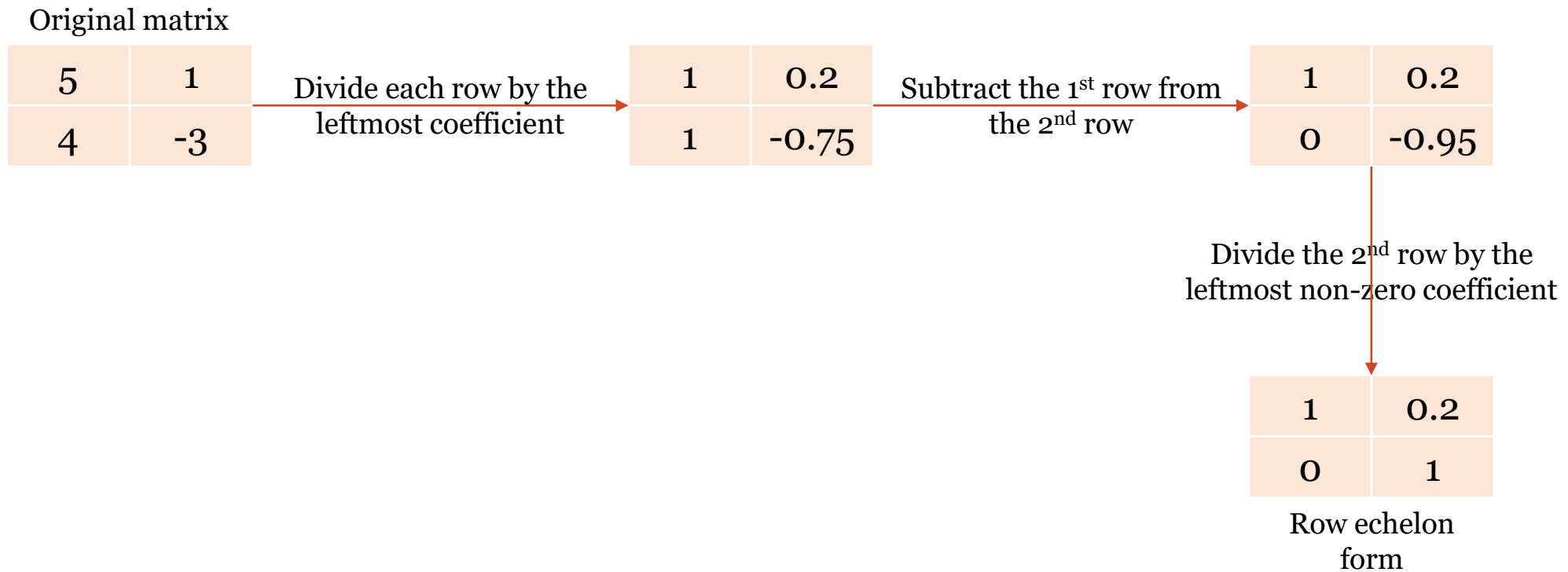
... Gaussian Elimination

- Row operations that preserve singularity
 - Switching rows
 - Multiplying a row by a non-zero scalar
 - Adding a row to another row



... Gaussian Elimination

- Example



... Gaussian Elimination

- Create the augmented matrix
 - Add constants of the equations to the right of the corresponding matrix
- Transform the matrix to the Reduced Row Echelon Form
 - Select the top left cell as the pivot
 - Use row operations to set your pivot to one
 - Use row operations to set all values below the pivot to zero
 - Repeat the above process row by row
- Solve the system using the back substitution process
 - Go from bottom to top rows
 - Use the pivot of each row to cancel the values in the cells above it



... Gaussian Elimination

- Example: non-singular

$$\begin{cases} 2a - b + c = 1 \\ 2a + 2b + 4c = -2 \\ 4a + b = 4 \end{cases}$$

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 2 & 2 & 4 & -2 \\ 4 & 1 & 0 & 4 \end{array}$$

$$\left[\begin{array}{l} R_1 \leftarrow \frac{1}{2}R_1 \\ R_2 \leftarrow R_2 - 2R_1 \\ R_3 \leftarrow R_3 - 4R_1 \\ R_2 \leftarrow \frac{1}{3}R_2 \\ R_3 \leftarrow R_3 - 3R_2 \\ R_3 \leftarrow \frac{-1}{5}R_3 \end{array} \right]$$

$$\begin{cases} a = 1 \\ b = 0 \\ c = -1 \end{cases}$$

$$\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}$$

Identity matrix

$$\left[\begin{array}{l} R_2 \leftarrow R_2 - R_3 \\ R_1 \leftarrow R_1 - \frac{1}{2}R_3 \\ R_1 \leftarrow R_1 + \frac{1}{2}R_2 \end{array} \right]$$

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array} \begin{array}{ccc|c} 1 & -1/2 & 1/2 & 1/2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \end{array}$$



Rank

System 1

The dog is black.
The cat is orange.

Two sentences
Two pieces of information

Rank = 2

System 2

The dog is black.
The dog is black.

Two sentences
One piece of information

Rank = 1

System 3

The dog!
The dog!

Two sentences
Zero piece of information

Rank = 0



... Rank

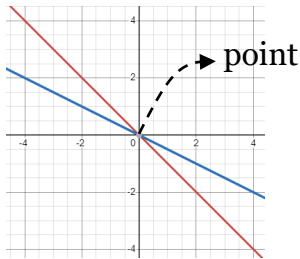
Non-singular system

$$a + b = 0$$

$$a + 2b = 0$$

Two equations

Two pieces of information



Dimension of solution space= 0

1	1
1	2

Rank = 2

Full rank matrix

The rank is equal to the number of rows

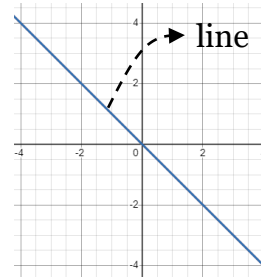
singular system

$$a + b = 0$$

$$2a + 2b = 0$$

Two equations

One piece of information



Dimension of solution space= 1

1	1
2	2

Rank = 1

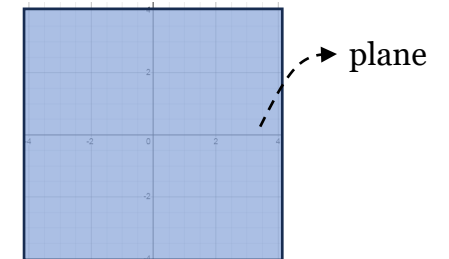
singular system

$$0a + 0b = 0$$

$$0a + 0b = 0$$

Two equations

Zero pieces of information



Dimension of solution space= 2

0	0
0	0

Rank = 0

Rank = (number of rows)-(dimension of solution space)

A matrix is non-singular if and only if it has full rank.



... Rank

- Is there an easier way to calculate the rank?

	Original matrix		Row echelon									
Non-singular matrix	<table><tr><td>5</td><td>1</td></tr><tr><td>4</td><td>-3</td></tr></table>	5	1	4	-3	→	<table><tr><td>1</td><td>0.2</td></tr><tr><td>0</td><td>1</td></tr></table>	1	0.2	0	1	<div><div>Rank = 2 Number of pivots = 2</div><div>Rank = 1 Number of pivots = 1</div><div>Rank = 0 Number of pivots = 0</div></div> <div>Rank = Number of pivots</div>
	5	1										
4	-3											
1	0.2											
0	1											
Singular matrix	<table><tr><td>5</td><td>1</td></tr><tr><td>10</td><td>2</td></tr></table>	5	1	10	2	→	<table><tr><td>1</td><td>0.2</td></tr><tr><td>0</td><td>0</td></tr></table>	1	0.2	0	0	
5	1											
10	2											
1	0.2											
0	0											
Singular matrix	<table><tr><td>0</td><td>0</td></tr><tr><td>0</td><td>0</td></tr></table>	0	0	0	0	→	<table><tr><td>0</td><td>0</td></tr><tr><td>0</td><td>0</td></tr></table>	0	0	0	0	
0	0											
0	0											
0	0											
0	0											

A matrix is non-singular if and only if the row echelon form has no zeros in the main diagonal.



Gaussian Elimination (continue!)

- Example: Singular

1	2	-1	5
2	4	5	1
3	6	4	6

Row reduction

1	2	-1	5
0	0	-7	9
0	0	0	0

$0a + 0b + 0c = 0 \longrightarrow$ Infinitely many solutions

1	2	-1	5
2	4	5	1
3	6	4	10

Row reduction

1	2	-1	5
0	0	-7	9
0	0	0	4

$0a + 0b + 0c = 4 \longrightarrow$ No solution



Vectors and Matrices



Vector

- Vector is a column of numbers
 - Number of coordinates in the vector is the dimension of the space it leaves
 - Example: (4, 3) lives in the plane and (4, 3, 1) lives in the space

Notations

Row vector

$$(x_1 \quad x_2 \quad \dots \quad x_n)$$

$$[x_1 \quad x_2 \quad \dots \quad x_n]$$

Column vector

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Letters

$$\vec{x}$$

$$\mathbf{x}$$



Matrix

- Matrix is an array of numbers

Notation

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}$$



Vector Components

- Size

- **L1-norm**

$$\|\vec{x}\|_1 = |x_1| + |x_2| + \dots + |x_n|$$

$$\text{Example: } |(a, b)|_1 = |a| + |b|$$

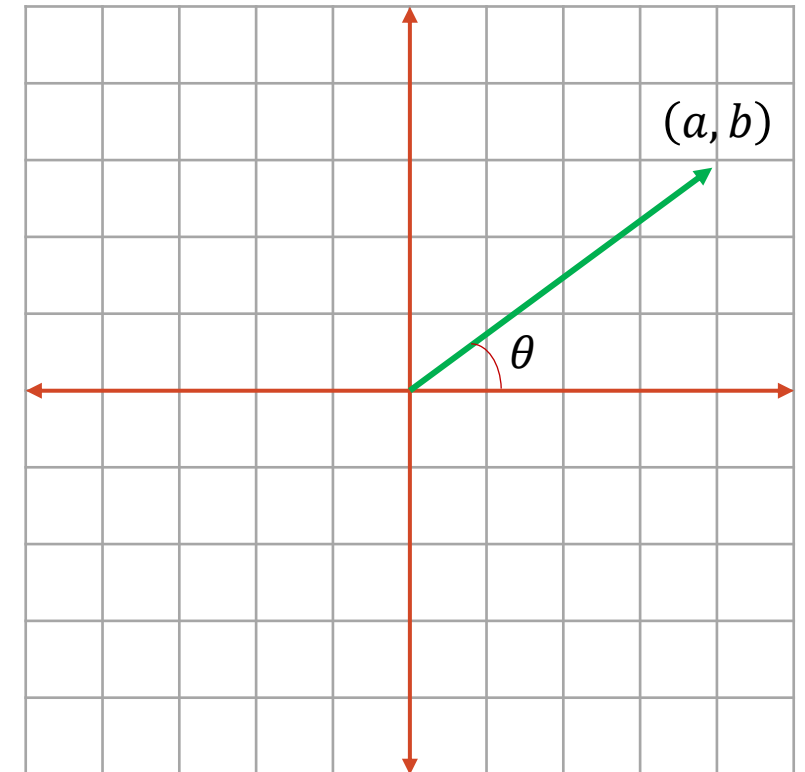
- **L2-norm** (default)

$$\|\vec{x}\|_2 = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

$$\text{Example: } |(a, b)|_2 = \sqrt{a^2 + b^2}$$

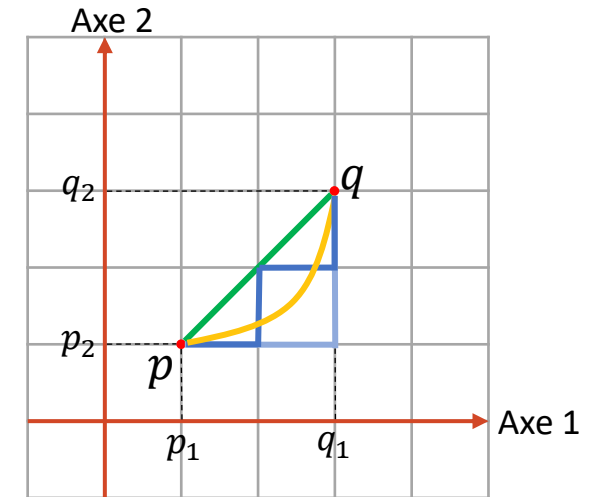
- Direction

$$\tan \theta = \frac{b}{a} \rightarrow \theta = \arctan\left(\frac{b}{a}\right)$$



Distance

- The most common methods to calculate the distance between two points in an n-dimension space:
 - Euclidean distance: $\sqrt{\sum_{i=1}^n (q_i - p_i)^2}$
 - Manhattan distance: $\sum_{i=1}^n |q_i - p_i|$
 - Minkowski distance: $\sqrt[p]{\sum_{i=1}^n |q_i - p_i|^p}$
 - General form of distance
 - If $p = 1 \rightarrow$ Manhattan
 - If $p = 2 \rightarrow$ Euclidean
 - If $p = \infty \rightarrow$ Chebyshev
 - $\lim_{p \rightarrow \infty} (\sum_{i=1}^n |q_i - p_i|^p)^{\frac{1}{p}} = \max_{1 \leq i \leq n} |q_i - p_i|$



Operations

- Assume

- $\vec{u} = (u_1 \quad u_2 \quad \dots \quad u_n)$ is a vector
- $\vec{v} = (v_1 \quad v_2 \quad \dots \quad v_n)$ is a vector
- λ is an scalar

- Sum and subtract between vectors

$$\vec{u} + \vec{v} = (u_1 + v_1 \quad u_2 + v_2 \quad \dots \quad u_n + v_n)$$

$$\vec{u} - \vec{v} = (u_1 - v_1 \quad u_2 - v_2 \quad \dots \quad u_n - v_n)$$

- Scalar-vector multiplication

$$\lambda \vec{x} = (\lambda u_1 \quad \lambda u_2 \quad \dots \quad \lambda u_n)$$



... Operations

- Dot product between two vectors

$$\vec{u} \cdot \vec{v} = u_1 \times v_1 + u_2 \times v_2 + \cdots + u_n \times v_n$$

- Another notation: $\langle \vec{u}, \vec{v} \rangle$
- It is common to represent the first and second vectors in the row and column formats, respectively.

$$[u_1 \quad u_2 \quad \cdots \quad u_n] \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 \times v_1 + u_2 \times v_2 + \cdots + u_n \times v_n$$



... Operations

- ... Dot product

- Connection between dot product and norm

$$\|\vec{u}\|_2 = \sqrt{u_1^2 + u_2^2 + \dots + u_n^2} = \sqrt{u_1 \times u_1 + u_2 \times u_2 + \dots + u_n \times u_n} = \sqrt{\langle \vec{u}, \vec{u} \rangle}$$

- Connection between dot product and the angle between two vectors

$$\langle \vec{u}, \vec{v} \rangle = |\vec{u}| \times |\vec{v}| \times \cos(\theta) = |\vec{u}'| \times |\vec{v}|$$

- Two vectors are orthogonal, if and only if their dot product is zero

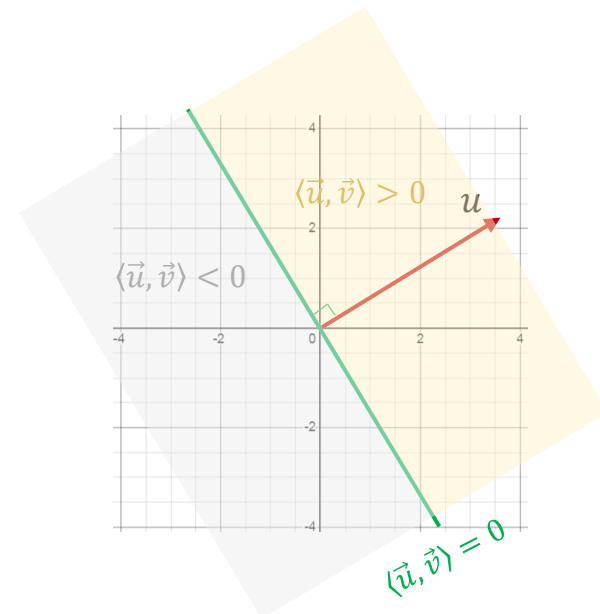
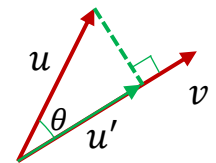
$$\langle \vec{u}, \vec{v} \rangle = 0 \rightarrow \vec{u} \text{ and } \vec{v} \text{ are orthogonal}$$

- If the vectors are in the same direction

$$\langle \vec{u}, \vec{v} \rangle = |\vec{u}| \times |\vec{v}|$$

- The dot product of a vector and itself

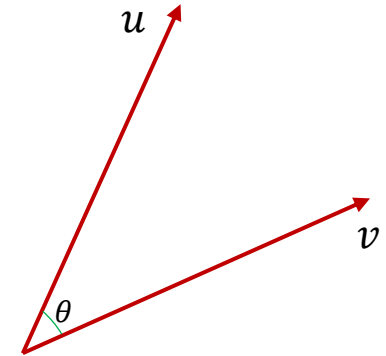
$$\langle \vec{u}, \vec{u} \rangle = |\vec{u}|^2$$



... Operations

- ... Dot product
 - Cosine Similarity
 - The cosine of the angle between the vectors

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$



- Orthogonal vectors have no similarity
- Vectors with the same direction are very similar ($\cos \theta = 1$)



... Operations

- Vector transpose
 - Transforms column to the row or vice versa

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \rightarrow \vec{u}^T = [u_1 \quad u_2 \quad \dots \quad u_n]$$

$$\vec{u} = [u_1 \quad u_2 \quad \dots \quad u_n] \rightarrow \vec{u}^T = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

- Matrix transpose

$$\begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix}_{m \times n}^T = \begin{pmatrix} x_{11} & \cdots & x_{m1} \\ \vdots & \ddots & \vdots \\ x_{1n} & \cdots & x_{mn} \end{pmatrix}_{n \times m}$$



... Operations

- Multiplication

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}_{m \times n} \times \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1p} \\ y_{21} & y_{22} & \cdots & y_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{np} \end{bmatrix}_{n \times p} = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1p} \\ z_{21} & z_{22} & \cdots & z_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ z_{m1} & z_{m2} & \cdots & z_{mp} \end{bmatrix}_{m \times p}$$

Where,

$$z_{ij} = x_{i1}y_{1j} + x_{i2}y_{2j} + \cdots + x_{in}y_{nj} = \sum_{k=1}^n x_{ik}y_{kj}$$



... Operations

- Inverse
 - Identity matrix
 - All elements on the diagonal = one & Any other element = zero

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

- Why called identity?
 - Multiplying it by any vector results in the same vector
 - The inverse matrix is the matrix for which the product of matrices is the identity matrix.

$$AA^{-1} = I$$



... Operations

- ... Inverse
 - How to find the inverse of a matrix?
 - By solving the system of linear equations!
 - Example: What is the inverse of matrix $\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$?

$$\begin{array}{|c|c|} \hline 3 & 1 \\ \hline 1 & 2 \\ \hline \end{array} \times \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array}$$

$$\begin{cases} 3a + c = 1 \\ 3b + d = 0 \\ a + 2c = 0 \\ b + 2d = 1 \end{cases}$$

Use the Gaussian Elimination
method to solve

$$\begin{cases} a = \frac{2}{5} \\ b = -\frac{1}{5} \\ c = -\frac{1}{5} \\ d = \frac{3}{5} \end{cases}$$



... Operations

- ... Inverse
 - Some cases have a straightforward formula
 - Example: 2×2 matrix
 - If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
 - Proof as an exercise!



... Operations

- ... Inverse
 - Which matrices have an inverse?
 - Non-singular matrices (determinant \neq zero) are invertible

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & 2 \\ \hline \end{array} \times \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array}$$

$$\begin{cases} a + c = 1 \\ b + d = 0 \\ 2a + 2c = 0 \\ 2b + 2d = 1 \end{cases}$$

Contradictory!
 $a + c = 1$ and $2a + 2c = 0$

No solution!
The matrix is not invertible



... Operations

- Determinant properties

- Determinant of matrix product: $\det(A \times B) = \det(A) \times \det(B)$
 - Result: singular or non-singular matrix multiplied by a singular matrix \rightarrow singular matrix

- Determinant of inverses: $\det(A^{-1}) = \frac{1}{\det(A)}$

- Proof:

$$\det(A \times A^{-1}) = \det(A) \times \det(A^{-1}) \rightarrow \det(I) = \det(A) \times \det(A^{-1}) \rightarrow \det(A^{-1}) = \frac{1}{\det(A)}$$



... Operations

- Trace

- The trace of a square matrix is the sum of the elements on its main diagonal

- $tr(A) = \sum_{i=1}^n a_{ii}$

- Properties

- $tr(A + B) = tr(A) + tr(B)$
 - $tr(cA) = ctr(A)$
 - $tr(A) = tr(A^T)$

