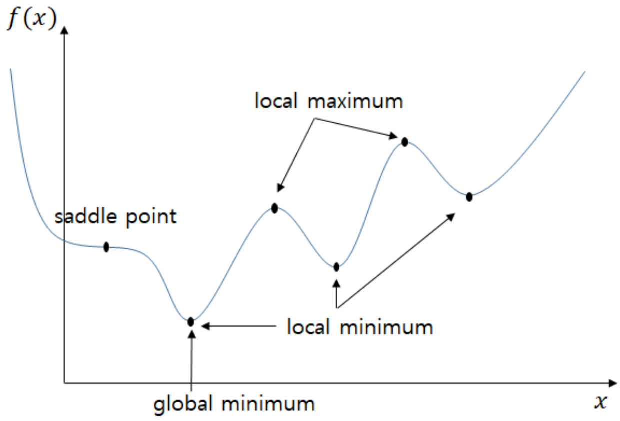
2019.09.19 Lecture Notes – Group 6

Optimization

- Optimization aims to minimize or maximize the cost function (object function).
- In the field of control, the cost function is defined by the purpose.
- For example, the cost function can be formed by purposes such as minimizing energy or control time.

1. Optimization without constraint

1) 1-dimensional case



Suppose f(x) is a cost function, and we can draw the graph of f(x) arbitrarily. The cost function is represented only as a function of x. Points marked with black dots in the figure have the following characteristics. Slope of these points are all zero.

Definition : All x* such that $\frac{\partial f(x*)}{\partial x} = 0$ is called critical points of f(x).

Critical points are classified into 3 categories according to the following conditions.

$\frac{\partial^2 f(x*)}{\partial x^2} > 0$: local minimum

$\frac{\partial^2 f(x*)}{\partial x^2} < 0$: local maximum

$\frac{\partial^2 f(x*)}{\partial x^2} = 0$: saddle point

The above 2 conditions can only determine local maximum and local minimum. To determine whether it is global maximum or minimum, we need to be able to figure out the overall shape of the cost function.

2) n-dimensional case (vector case)

Unlike the 1-dimensional case, the cost function is expressed as a function of vector x. This can be expressed as follows.

Cost function : $f(\underline{x})$, $\quad f: R^n \to R,\ \ \underline{x} = [x_1, \dots, x_n]^T$

The vector case is similar to the 1-dimensional case. In the 1-dimensional case, the critical points have a 0 slope. So, in the n-dimensional case, the critical points have a 0 slope for each variable. To define this, we can define the vector derivative gradient.

Definition : $\frac{\partial f}{\partial \underline{x}} = \left[\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right]^T$: gradient

Critical points in the n-dimensional case have following characteristics.

Definition : All $\underline{x}^*$ such that $\frac{\partial f(\underline{x}^*)}{\partial \underline{x}} = 0$ is called critical points of $f(\underline{x}^*)$.

To divide critical points into 3 cases, we define Hessian, which is the 2nd derivative of the vector.

Definition : $\frac{\partial^2 f}{\partial \underline{x}^2} = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$ **: Hessian**

Using Hessian, we determine the characteristics of the saddle point, local minimum, and local maximum point of the n-dimensional case as follows.

Theorem : $f: R^n \rightarrow R$, f is twice-differentiable, $\frac{\partial f(\underline{x}^*)}{\partial \underline{x}} = 0$

If $\underline{x}^*$ is a local minimum, then $\frac{\partial^2 f}{\partial \underline{x}^2}(\underline{x}^*) \geq 0$ (positive semi definite)

If $\underline{x}^*$ is a local maximum, then $\frac{\partial^2 f}{\partial \underline{x}^2}(\underline{x}^*) \leq 0$ (negative semi definite)

These are the features of the critical point. When we do optimization, we can use the above features to find the critical point of the cost function. We do not consider the case of saddle points, because that would indicate bad modeling.

2. Optimization with Equality constraints

In this case, the optimization point of the cost function must be found satisfying a certain constraint.

We define cost function as $f(\underline{x})$. Also, we define the constraints as $h(\underline{x}) = 0$.

Definition: Any point satisfying the constraints is called a feasible point. The set of all feasible points $\{\underline{x} \in R^n : h(\underline{x}) = 0\}$ is called the feasible set.

If there are many equality constraints like $h_1 = 0$, ..., $h_m = 0$, the surface S is defined as $S = \{x \in R^n : h_1(\underline{x}) = 0, \ldots, h_m(\underline{x}) = 0\}$. Also, we can think of a curve C which passes through a point x* if there exists $t^* \in (a,b)$ such that $x(t^*) = x^*$. We can think of the curve $C = \{x(t) : t \in (a,b)\}$.

Tangent space of x* is given as $T(x^*) = \{y : Dh(x^*)y = 0\}$.

If x* is minimizer of f(x) with constraint $h(\underline{x}) = 0$, it has characteristics such that $\nabla f(x^*) = 0$. It is first order necessary condition. We can parameterize this level set in a neighborhood of x* by a curve C.

Applying it into surface S, surface S is changed to following form.

$$h(x(t)) = 0$$

Indeed, h is constant on the curve C, we have that for all $t \in (a,b)$.

$$\frac{d}{dt} h(x(t)) = \nabla h(x(t))^T \dot{x}(t) = 0$$

So, $\nabla h(x^*)$ is orthogonal to $\dot{x}(t^*)$. It is formulated tangent space.

Applying first order necessary condition of minimizer, the following equation is established.

$$\frac{d}{dt} (f(x(t)) = \nabla f(x^*)^T \dot{x}(t^*) = 0$$

So, $\nabla f(x^*)$ is also orthogonal to $\dot{x}(t^*)$.

Therefore, $\nabla h(x^*)$ and $\nabla f(x^*)$ are parallel. Then there exists a scalar γ^* such that $\nabla f(x^*) + \gamma^* \nabla h(x^*) = 0$. It is called Lagrange's Theorem. Then we make so called Lagrangian function l, given as $l(x,\gamma) \triangleq f(x) + \gamma^T h(x)$. In the Lagrangian function, $Dl(x,\gamma) = 0$ means that $D_x l(x,\gamma) = 0$ and $D_\gamma l(x,\gamma) = 0$. Solving this equation has the following meanings.

$$D_x l(x,\gamma) = \nabla f(x) + \gamma^T \nabla h(x) = 0$$

$$D_\gamma l(x,\gamma) = \nabla h(x) = 0$$

If we can find x* and γ^* satisfying above condition, it is same as solving optimization problem with equality constraints. So, we can use the Lagrangian function to change the optimization problem with equality constraints to optimization problem without constraints.

3. Optimal Control

Optimal control theory deals with operating a dynamic system such that a certain optimality criterion is achieved. This optimality criterion is usually a cost function which we must minimize.

An **optimal control is a set of differential equations** which describes the paths of the control variables that minimize the cost function. All those **control variables are the state of the system** and they allow us to predict the future development of the system.

Let's have a look with an example: given a car traveling in a straight line, how should the driver press the accelerator in order to minimize the traveling time? Here, the system is composed by the car and the road, the cost function is the traveling time, which must be minimized, and the control variables states how the driver presses the accelerator and shifts the gears. In addition to this, an optimal control problem usually has some constraints such as speed limit, in our example.

Full State Feedback (FSF, also known as Pole Placement)

The dynamics of a system can be **designed utilizing the feedback of the state** (control variables).

Assuming a system described by a linear state model that only uses a single input makes it easier to understand this concept at first. The feedback control in this case is called a **full state feedback controller**.

It is designed so that the **closed-loop eigenvalues (poles) are placed in pre-determined, desirable locations**. This allows the controlling of the aspects of how the system responds. However, FSF only works for single-input systems. A multiple-input system raises a linear-quadratic problem.

Linear-Quadratic Regulator (LQR)

The Linear-Quadratic problem represents a system where the dynamics are described by a set of linear differential equations and the cost by a quadratic function. The solution to this problem is given by the LQ regulator (LQR), a feedback controller. LQR is an important part of the solutions to the linear-quadratic-Gaussian problem.

The linear quadratic regulator problem is given as:

$$\min_{u(t)} \frac{1}{2}\int_{t_0}^{t_1} (x^T \mathrm{Q} \mathrm{x} + u^t u)dt \ + \frac{1}{2} x^t(t_1) R x(t_1)$$

subject to $\dot{x}(t) = A(t)x(t) + B(t)u(t)$

Adding and subtracting below,

$$\frac{1}{2}\int_{t_0}^{t_1}\frac{d}{dt}x^T kx\,\mathrm{dt} = \frac{1}{2}\int_{t_0}^{t_1}(\dot{x}^T kx + x^T \dot{k}x + x^T k\dot{x})\,dt$$

where $\dot{k} = -A^t k - kA + kBB^T k - Q$

we get:

$$J = \tfrac{1}{2}x^T Rx + \tfrac{1}{2}\int_{t_0}^{t_1}[x^T \quad u^T][\begin{matrix} Q & 0 \\ 0 & I \end{matrix}][\begin{matrix} x \\ u \end{matrix}]dt + \tfrac{1}{2}\int_{t_0}^{t_1}(\dot{x}^T kx + x^T \dot{k}x + x^T k\dot{x})\,dt - \tfrac{1}{2}x^T kx|_{t_0}{}^{t_1}$$

$$J = \frac{1}{2}x^T(t_1)Rx(t_1) + \frac{1}{2}\int_{t_0}^{t_1}[x^T \quad u^T][\begin{matrix} Q + \dot{k} + A^T k + kA & kB \\ B^T k & I \end{matrix}][\begin{matrix} x \\ u \end{matrix}]dt - \frac{1}{2}x^T kx|_{t_0}{}^{t_1}$$

Simplifying above, we get:

$$J = \tfrac{1}{2}x^T(t_1)Rx(t_1) - \tfrac{1}{2}x^T kx|_{t_0}{}^{t_1} + \int_{t_0}^{t_1}||u + cx||^2 dt$$

where:

$$||u + cx||^2 = u^T u + x^T c^T cx + u^T cx + x^T c^T u\,,\ c = B^T k,\ c^T = kB,$$

$$c^T c = kBB^T k = Q + \dot{k} + A^T k + kA$$

J is optimal when $u = -cx$, $k(t_1) = R$

Combining these formulas we get:

$$u(t) = -B^T(t)k(t)x(t)$$

This formula means that u depends on situation x, which means $\dot{x}$ depends on x and this is the feedback controller.

LQG

The linear-quadratic-Gaussian control problem is one of the most fundamentals optimal control problems. The problem is to find out an output feedback lay that minimizes the expected value of a quadratic cost function. In addition, the output measures are assumed to have some Gaussian noise and the initial state is a Gaussian random vector. This control law is called LQC controller and is a combination of a Kalman filter, which is a linear-quadratic state estimator (LQE), and a linear-quadratic regulator.

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