Lecture Note

CS572 Intelligent Robotics

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Review of Last Lecture

Optimal Control

$$\min_{u(t)} \phi(x(t_1), t_1) + \int_{t_0}^{t_1} L(x, u, t)dt \tag{1}$$

- The optimization form 1 satisfisfies below:
  - $\dot{x} = f(x, u, t)$.
  - $\frac{\partial L}{\partial u} = \lambda^T \frac{\partial f}{\partial u} = 0$.
  - $\dot{\lambda}^T = -(\frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u})$.
  - Boundary Conditions.

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Prerequisite from Wikipedia

Kalman Filter

A Kalman filter is a recursive filter that tracks the state of a linear dynamics system that contains noise.

Linear Quadratic Regulator (LQR)

The theory of optimal control is concerned with operating a dynamic system at minimum cost.

Kalman Linear Quadratic Regulator (LQR)

$$H = L + \lambda^T (Ax + Bu)$$

$$\dot{x} = f(x, u, t)$$

$$\frac{\partial H}{\partial u} = 0$$

$$\lambda^T = -\frac{\partial H}{\partial x} = x^T Q - \lambda^T A$$

$$\dot{x} = Ax + Bu = Ax - BR^{-1}B^T \lambda$$

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

Boundary Conditions

Boundary Conditions

$$\begin{cases} t_0, x(t_0) \text{ fixed} \\ t_1, x(t_1) \text{ free} \end{cases}$$

- You can specify this option using tools provided in programming languages such as Matlab.
- Recall: two-point boundary value problem (TPBVP)

Kalman Linear Quadratic Regulator (LQR)

$$\min_u \frac{1}{2}x^T(t_1)Hx(t_1) + \frac{1}{2}\int_{t_0}^{t_1} x^T(t)Qx(t) + u_t^T Ru(t)dt$$

such that $\dot{x} = Ax(t) + Bu(t), Q, R > 0$

Boundary Conditions

$$\begin{cases} t_0, x(t_0) \text{ fixed} \\ t_1, x(t_1) \text{ free} \end{cases}$$

$$H = L + \lambda^T(Ax + Bu)$$

$$\frac{\partial H}{\partial u} = 0 \rightarrow u^T R + \lambda^T B = 0 \rightarrow u = -R^{-1}B^T\lambda$$

$$L = \frac{1}{2}x^T Qx + \frac{1}{2}u^T Ru$$

Kalman Linear Quadratic Regulator (LQR)

$$
\begin{aligned}
\lambda^T &= -\frac{\partial H}{\partial x} = x^T Q - \lambda^T A \\
\dot{x} &= Ax + Bu = Ax - BR^{-1}B^T\lambda \\
\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} &= \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}
\end{aligned}
$$

- This expression can be seen in this form $\lambda = kx$.

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Linear System Theory

- We need to know about the transition function Φ.
- To solve the problem properly, we have to understand the linear system theory, which is out of scope in our course.

$$\begin{bmatrix} x(t_1) \\ \lambda(t_1) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t,t_0) & \Phi_{12}(t,t_0) \\ \Phi_{21}(t,t_0) & \Phi_{22}(t,t_0) \end{bmatrix} \begin{bmatrix} x(t_0) \\ \lambda(t_0) \end{bmatrix}$$

$$\lambda(t_1) = Hx(t_1)$$

$$(H\Phi_{11} - \Phi_{21})x(t_1) = (\Phi_{22} - H\Phi_{12})\lambda$$

- Also, above equation can be seen as $\lambda = kx$.

Linear System Theory

- Boundary conditions: $k(t_1) = H$.
- k is function of t.

$$
\begin{aligned}
&u = -R^{-1}B^T kx \\
&\lambda = kx \\
&\dot{x} = Ax - BR^{-1}B^T kx \\
&\lambda = -Qx - A^T kx = \dot{k}x + k\dot{x} \\
&\forall X(t), (\dot{k} + Q + A^T k)x = -k\dot{x} = -k(Ax - BR^{-1}B^T kx) \\
&\dot{k} + Q + A^T k + kA - kBR^{-}1B^T k = 0
\end{aligned}
$$

- Therefore, we can get a same result of previous slide.

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Kalman LQR to Tracking Problem

- We will now learn how to reduce the Kalman LQR problem to a tracking problem.
- The method is quite simple: Replace x with (r-x) in the LQR equation.

Tracking Problem

$$\min_u \frac{1}{2}\|r(t_1) - x(t_1)\|_H^2 + \frac{1}{2}\int_{t_0}^{t_1} \|r(t) - x(t)\|_{Q(t)}^2 + \|u(t)\|_{R(t)}^2 dt$$

- Definition of Tracking Problem
    - Looking for a minimal effort u to minimize tracking error.

Definition of Tracking Problem

Tracking Problem

$$\min_u \frac{1}{2}\|r(t_1) - x(t_1)\|_H^2 + \frac{1}{2}\int_{t_0}^{t_1} \|r(t) - x(t)\|_{Q(t)}^2 + \|u(t)\|_{R(t)}^2 dt$$

- $\|r(t_1) - x(t_1)\|_H^2$ is time error.
- $\|r(t) - x(t)\|_Q^2$ is transient error.
- $\|u(t)\|_R^2$ is control effort.
- Tracking errors(time error and transient error) and control effort can be measured by the weight matrices (H, Q(t), R(t)) which correspond to each metric.

- $\|r(t) - x(t)\|_{Q(t)}^2 = (r - x)^T Q(r - x)$

Conditions on Tracking Problem

Tracking Problem

$$\min_u \frac{1}{2}\|r(t_1) - x(t_1)\|_H^2 + \frac{1}{2}\int_{t_0}^{t_1} \|r(t) - x(t)\|_{Q(t)}^2 + \|u(t)\|_{R(t)}^2 dt$$

s.t. $x = Ax + Bu$

$$u = -R^{-1}B^T\lambda$$

$$\lambda(t) = -k(t)'(r - x(t)) = k(t)x(t) - k(t)r = k(t)x(t) - g(t)$$

- $\frac{\partial H}{\partial u} = 0 \rightarrow u = -R^{-1}B^T\lambda$
- $\lambda^T = -\frac{\partial H}{\partial u} = -Q(x - r) - A^T\lambda$
- $\dot{k} = -kA$
- $k(t_1) = H$
- $g = -(A - BR^{-1}B^Tk)g - Qr$
- $k(t)x(t)$ is a feedback term.
- $g(t)$ is tracking term.

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Motivation of Minimum Principle

Example

$$\min_{x \in \mathbb{R}} L(x) \text{ such that } x_0 \le x \le x_1$$

$$\frac{\partial L}{\partial x}(x^*) \neq 0$$

$$\forall \text{admissible } \delta x, L(x) - L(x^*) \simeq \frac{\partial L}{\partial x}(x^*)\delta x \geq 0.$$

Minimum Principle (Pontryagin)

- $u(t)$ is bounded.
- $u^*(t), x^*(t), \lambda^*(t)$ are optimal.
- $H \triangleq L + \lambda^T x$
- Like in the motivation slide, $\forall \delta u$ that is admissible:

$$
\begin{aligned}
J(u^*) &= \int_{t_0}^{t_1} L(x^*, u^*, t) + \lambda^{*T}(f(x^*, u^*, t) - x^*)dt \\
\partial J(u^*, \delta u) &= \int_{t_0}^{t_1} \frac{\partial H(x^*, u^*, \lambda^*, t)}{\partial u} \delta_u H \\
&= \int_{t_0}^{t_1} [H(x^*, u^* + \delta u, \lambda^*, t) - H(x^*, u^*, \lambda^*, t)]dt
\end{aligned}
$$

Solution of Minimum Principle

Minimum Principle

(1) $\dot{x} = f(x, u, t)$
(2) $\dot{\lambda}^T = -\frac{\partial u}{\partial x}$
(3) $\forall$admissible $u, H(x^*, u^*, \lambda^*, t) \leq H(x^*, u, \lambda^*, t)$

Example

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 + u$$
$$\min \frac{1}{2}\int_{t_0}^{t_1} (x_1^2 + u^2)dt$$

Solution of Minimum Principle with Unconstrained u

Solution of Minimum Principle with Unconstrained u

$$H = \frac{1}{2}(x_1^2 + u^2) + \lambda_1 x_2 + \lambda_2(-x_2 + u)$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 + u$$

$$\dot{\lambda}_1 = -x_1$$

$$\dot{\lambda}_2 = -\lambda_1 + \lambda_2$$

$$u + \lambda_2 = 0$$

Solution of Minimum Principle with Constrained u

Constraint u

$$-1 \leq u \leq 1$$
$$\frac{1}{2}u^{*2} + \lambda_2^* u^* \leq \frac{1}{2}u^2 + \lambda_2^* u$$

Solution of Minimum Principle with Constrained u

(1) $\lambda_2^* < -1 \rightarrow u^* = 1$
(2) $\lambda_2^* \geq 1 \rightarrow u^* = -1$
(3) 0 $\leq \lambda_2^* \leq -1 \rightarrow u^* = -\lambda_2^*$
(4) -1 $\leq \lambda_2^* < 0 \rightarrow u^* = \lambda_2^*$

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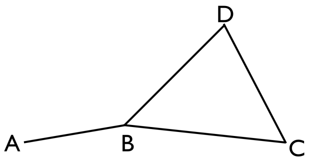
Optimality Principle

- Optimality principle is the basic principle of dynamic programming.



Optimality Principle

If ABC = optimal path from A to C,
then BC = optimal path from B to C

(Proof) BDC is the optimal path from B to C
$\rightarrow J_{AB} + J_{BDC} < J_{AB} + J_{BC} = J^*_{ABC}$
$\therefore BC$ must not be the optimal path from B to C.

Dynamic Programming

Dynamic Programming (From Wikipedia)

The method of how to divide a complex problem into several simple problems.
It uses partial problem iterations and algorithms with optimal sub-structures in less time than usual methods.