Kalman Filter

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1 Kalman Filter

Kalman filter is an optimal recursive data process algorithm that observes a series of measurement and estimates unseen variables. The following is a derivation of the Kalman gain K used in the system.

Assume that we would like to know X_{t+1} in the process:

$$X_{t+1} = A_t X_t + B_t U_t + G_t W_t$$

where X_t is the state vector at time t, A_t is the transition matrix of the process from time t to $t+1$, G_t is the gain matrix for unknown disturbance, and W_t is a zero mean Gaussian process noise vector.

In the same process, the observations on the variable can be modelled in the form:

$$Y_t = H_t X_t + V_t$$

where V_t is a zero mean Gaussian process noise vector, H_t is a noiseless transformation from the state vector to the measurement vector, and Y_t is the actual measured value of X_t.

As the noises are zero mean, we can state that:

$$E[V_t] = 0, E[W_t] = 0$$

Given that the noise processes V_t and W_t are stationary over time, their covariances can be modeled by:

$$Q = E[W_t W_t^T], R = E[V_t V_t^T]$$

Let the error for the process be as follows:

$$e_t = \hat{X}_t - X_t$$

If so, the mean squared error can be represented by:

$$P_t = E[e_t e_t^T] = E[(X_t - \hat{X}_t)(x_t - \hat{X}_t)^T]$$

Assuming that the prior estimate of $\hat{X}_t$ as $\hat{X}_{t|t-1}$, found from the previous state of the system:

$$\hat{X}_{t|t-1} = A\hat{X}_{t-1}$$

The update equation for the estimate at t can be found by combining the prior estimate with measurement data:

$$\hat{X}_t = \hat{X}_{t|t-1} + K_t(Y_t - H\hat{X}_{t|t-1})$$

*Note the term $Y_t - \hat{X}_{t|t-1}$ is known as the *innovation* or *measurement residual.* Using this update equation, the error of the process can be further defined as:

$$e_t = \hat{X}_{t|t-1} + K_t(Y_t - H\hat{X}_{t|t-1}) - X_t$$

$$e_t = (I - K_tH_t)\varepsilon + K_tV_t; \varepsilon = \hat{X}_{t|t-1} - X_t$$

Using the above formula for error, the square of error can be found:

$$\begin{aligned} e_t^T e_t = &\varepsilon^T\varepsilon + \varepsilon^T H^T K^T KH\varepsilon \\ &- 2\varepsilon^T KH\varepsilon + 2\varepsilon^T KV - 2V^T K^T KH\varepsilon + V^T K^T KV \end{aligned}$$

We then differentiate with respect to K,

$$\begin{aligned} \frac{d}{dK}e_t^T e_t = &2KH\varepsilon\varepsilon^T H^T - 2(KH^T + KV\varepsilon^T H^T) \\ &+ 2KVV^T + 2(\varepsilon V^T - \varepsilon\varepsilon^T H^T) \end{aligned}$$

Then,

$$\begin{aligned} 0 &= \frac{d}{dK}E[e_t^T e_t] \\ &= E[\frac{d}{dK}e_t^T e_t] \\ &= KHE[\varepsilon\varepsilon^T]H^T - KHE[\varepsilon V^t] - KE[V\varepsilon^T]H^T + KE[VV^T] + E[\varepsilon V^T] - E[\varepsilon\varepsilon^T]H^T \end{aligned}$$

In the above equation, $E[\varepsilon V^T]$ and $E[V^T\varepsilon]$ are zero terms, $E[VV^T] = R$, and the error covariance of a priori state estimation is:

$$E[\varepsilon\varepsilon^T] \triangleq P_{t|t-1} = E[(\hat{X}_{t|t-1} - X_t)(\hat{X}_{t|t-1} - X_t)^T]$$

So we can simplify the derivative value of mean squared error as:

$$KHP_{t|t-1}H^T + KR - P_{t|t-1}H^T = 0$$

And from this we can get the Kalman gain K_t as:

$$K_t = P_{t|t-1}H_t^T[H_tP_{t|t-1}H_t^T + R_t]^{-1}$$

Next, we can make the update equation for $P_{t|t-1}$,

$$\begin{aligned} P_t &= E[(\hat{X}_t - X_t)(\hat{X}_t - X_t)^T] \\ &= E[((I - KH)\epsilon + KV)((I - KH)\epsilon + KV)^T] \\ &= (I - KH)P_{t|t-1}(I - KH)^T + KR_tK^T \end{aligned}$$

Let $P_t' := P_{t|t-1}$ and consider its expansion,

$$P_t = P_t' - K_tHP_t' - P_t'H^TK_t^T + K_t(HP_t'H^T + R)K_t^T \tag{1}$$

Note that the trace of a matrix is equal to the trace of its transpose, thus it may written as,

$$T[P_t] = T[P_t'] - 2T[K_tHP_t'] + T[K_t(HP_t'H^T + R)K_t^T]$$

In the case of the error covariance matrix the trace is the sum of the mean squared errors. Therefore the mean squared error may be minimized by minimizing the trace of P_t which in turn will minimize the trace of P_t. So differentiating with respect to K_t, we have

$$0 = \frac{\partial T[P_t]}{\partial K_t} = -2(HP_t')^T + 2K_t(HP_t'H^T + R)$$

Solving for K_t, we have

$$K_t = P_t'H^T(HP_t'H^T + R)^{-1} \tag{2}$$

Finally, substitution of equation (2) into (1) gives,

$$P_t = (I - K_tH)P_{t|t-1}$$

Now, we have two equation for P_t,

$$\begin{aligned} P_t &= (I - KH)P_{t|t-1}(I - KH)^T + KR_tK^T \\ &= (I - K_tH)P_{t|t-1} \end{aligned}$$

If $P_{t|t-1}$ is symmetric then P_t has also the property for all t. But numerically it may become asymmetric, leading to divergence in recursive computation. Some round off errors yield an asymmetric posteriori covariance P_t although the a priori covariance $P_{t|t-1}$ is symmetric. To resolve this problem, it is efficient to use the first equation. It is called *Joseph's stabilized form.*

Prior error of next time interval, $t + 1$ is

$$\begin{aligned} \varepsilon_{t+1} &= \hat{X}_{t+1|t} - X_{t+1} \\ &= A_t\hat{X}_t - A_tX_t - G_tW_t \\ &= A_te_t - G_tw_t \end{aligned}$$

We can extend mean squared error term to time $t + 1$,

$$\begin{aligned} P_{t+1|t} &= E[\varepsilon_{t+1}\varepsilon_{t+1}^T] \\ &= E[(Ae_t - G_tW_t)(Ae_t - G_tW_t)^T] \end{aligned}$$

As ε_{t+1} and w_t have zero cross-correlation, so,

$$P_{t+1|t} = A_tP_tA_t^T + G_tQ_tG_t^T$$

Figure 1 is a summary of the Kalman filter Process. One part of the Kalman filter process can be done offline because it is estimation based on model information. The other part has to be online because it is prediction based on observation.

Figure 2 summarize the update process of Kalman filter: once the time update prediction is done, the measurement update collection will do the a posteriori estimation.

Kalman filter can be used for navigation. One of the effect of Kalman filter is dead reckoning: this is the process of estimation current position based on a previously determined position. In Kalman filter, the secondary estimation based on observation will correct the error from the primary information estimation. Figure 3 shows the dead reckoning in Kalman filter.

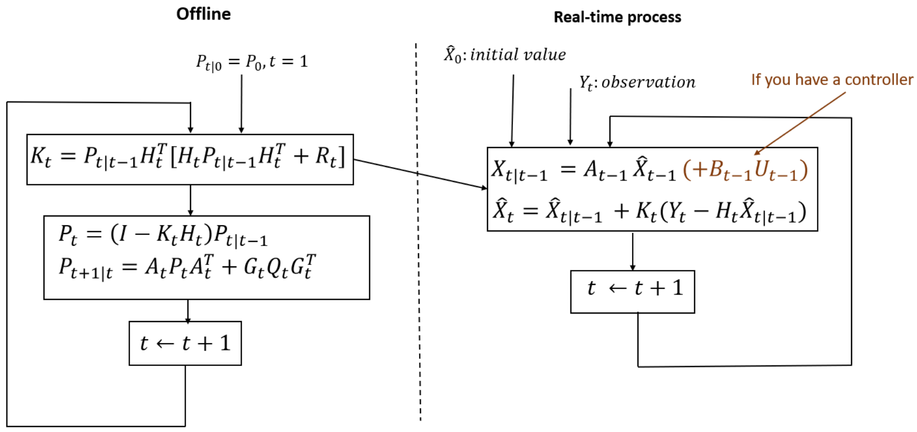

Figure 1: Kalman filter process

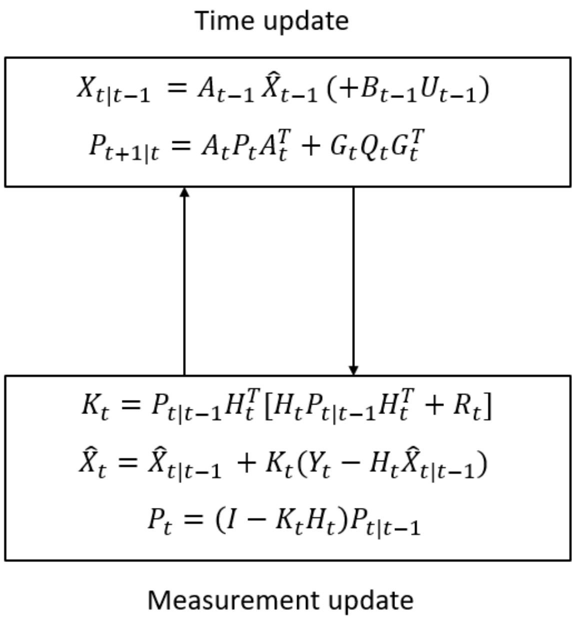

Figure 2: Kalman Filter update

2 Extended Kalman Filter

The extended Kalman Filter is an extension of Kalman filter. It can be applied to non-linear system. Let's consider the same system but this time, it is non-linear.

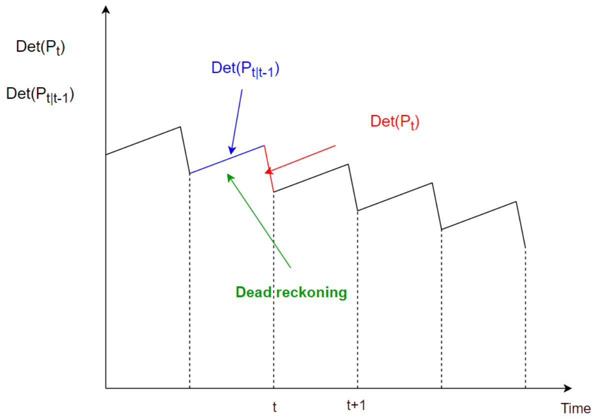

Figure 3: Dead Reckoning

$$\left.\begin{array}{l} X_{t+1} = f(X_t, t) + W_t \\ Y_t = h(X_t, t) + V_t \end{array}\right\} Non - linear$$

Since the system is non-linear, everything must be computed online. At each iteration you must compute:

$$\frac{\partial f}{\partial x}|_{X=\hat{X}_t} = \hat{A}(\hat{X}_t, t) = \hat{A}_t$$

$$\frac{\partial h}{\partial x}|_{X=\hat{X}_t} = \hat{H}(\hat{X}_t, t) = \hat{H}_t$$

Figure 4 shows an example of the Extended Kalman Filter estimation.

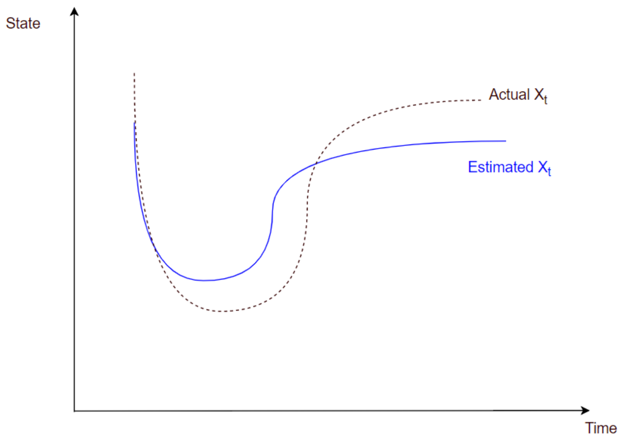

Figure 4: Extended Kalman Filter