Lecture Note 4

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$$\min L(x) \quad \text{s.t. } \underbrace{f(x) = 0}_{\text{constraint}}$$

$$\rightarrow$$

$$\min L(x) + \lambda^T f(x)$$

Example

$$\min \frac{1}{2} x^T x$$

$$f(x) = Ax - b = 0$$

$$\min_x \frac{1}{2} x^T x + \lambda^T (Ax - b)$$

$$\frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} = x^T + \lambda^T A = 0$$

$$Ax - b = 0$$

Unknowns: $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^m$

$$x = -A^T \lambda, Ax = b$$

$$x + A^T \lambda = 0, Ax + AA^T \lambda = 0, \lambda^* = -(AA^T)^{-1} Ax$$

$$x^* = A^T (AA^T)^{-1} Ax = \underbrace{A^T (AA^T)^{-1}}_{\textbf{pseudo-inverse}} b$$

***: optimal solution**

x Changes Over Time Case

find $x(t) \in \mathbb{R}^n$ that minimizes

$$\int_{t_0}^{t_1} L(x(t), \dot{x}(t), t) dt \quad \text{s.t. } x(t_0) = x_0, x(t_1) = x_1$$

$$\mathbf{x(t) = x^*(t)} + \boldsymbol{\varepsilon \delta} \mathbf{x(t)}$$

$$J(\varepsilon) = \int_{t_0}^{t_1} L(x^* + \varepsilon\delta x, \dot{x^*} + \varepsilon\dot{x}, t)dt$$

$$\frac{\partial J}{\partial \varepsilon}(\varepsilon)\Big|_{\varepsilon=0} = 0 = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x}\Big|_{\varepsilon=0} \delta x + \frac{\partial L}{\partial \dot{x}}\Big|_{\varepsilon=0} \delta\dot{x} \right) dt$$

partial integration: $\int_{t_0}^{t_1} \frac{\partial L}{\partial \dot{x}} \delta\dot{x} dt = \underbrace{\frac{\partial L}{\partial \dot{x}}\delta x\Big|_{t_0}^{t_1}}_{=0} - \int_{t_0}^{t_1} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \partial x dt$

Fixed: $x(t_0) = 0, x(t_1) = x_1$

$\rightarrow \delta x(t_1) = \delta x(t_0) = 0$

$$0 = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right)\Big|_{\varepsilon=0} \delta x dt, \quad \forall \delta x(t)$$

Euler-Lagrange Equation: $\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0$

2nd order nonlinear ODE

t_0, x_0 given, t_1, x_1 free.

$$J(\varepsilon) = \int_{t_0}^{t_1} L(x^* + \varepsilon\delta x, \dot{x}^* + \varepsilon\delta\dot{x}, t)dt$$

$$\frac{\delta J}{\delta \varepsilon}(\varepsilon)\Big|_{\varepsilon=0} = 0 = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial x}\Big|_{\varepsilon=0} \delta x + \frac{\partial L}{\partial \dot{x}}\Big|_{\varepsilon=0} \delta\dot{x} \right) dt + L|_{t=t_1} \delta t_1$$

$$= \int_{t_0}^{t_1} \underbrace{\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} \right)}_{=0} \delta x dt + \underbrace{\frac{\partial L}{\partial \dot{x}}\Big|_{t=t_1}}_{=0} \delta x_1 + \underbrace{L|_{t=t_1}}_{=0} \delta t_1$$

With λ

$$\min \int_{t_0}^{t_1} L + \lambda^T f dt$$

$$L + \lambda^T f \triangleq \bar{L}(x, \dot{x}, \lambda, t)$$

$$\delta J = \int_{t_0}^{t_1} \frac{\partial \bar{L}}{\partial \bar{x}}\delta x + \frac{\partial \bar{L}}{\partial \dot{\bar{x}}}\delta\dot{x} + \frac{\partial \bar{L}}{\partial \lambda}\delta\lambda dt$$

$$\frac{\partial \bar{L}}{\partial x} - \frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{x}}\right) = 0$$

$$f(x_1, \dot{x_1}, t) = 0 \quad \text{(BCs)}$$

Optimal Control Formulation

$$\min_{u(t)} \phi(x(t_1), t_1) + \int_{t_0}^{t_1} L(x,u,t)dt \quad \text{s.t. } \dot{x} = f(x,u,t)$$

given boundary conditions

If $u = \dot{x} = f(x,u,t) \Rightarrow$ Calculas of Variations **(No system like this)**

Suppose t_0, x_0 is given, t_1, x_1 is free

$$J = \int_{t_0}^{t_1} \left(L + \frac{d}{dt}\phi \right) dt + \underbrace{\phi(x_0, t_0)}_{\textbf{can be ignored}}$$

$$J = \int_{t_0}^{t_1} (L + \frac{d}{dt}\phi)dt = \int_{t_0}^{t_1} L + \frac{\partial \phi}{\partial x}\dot{x} + \frac{\partial \phi}{\partial t} + \lambda^T(f - \dot{x})dt$$

$$= \int_{t_0}^{t_1} \bar{L}dt$$

$$\delta J = \int_{t_0}^{t_1} \bar{L}_x \delta x + \bar{L}_{\dot{x}} \delta \dot{x} + \bar{L}_u \delta u + \bar{L}_\lambda \delta \lambda dt$$

$$= \int_{t_0}^{t_1} (\bar{L}_x - \frac{d}{dt}\bar{L}_{\dot{x}})\delta x + \bar{L}_u \delta u + \bar{L}_\lambda \delta \lambda dt + \underbrace{\bar{L}_{\dot{x}}|_{t_1} \delta x_1}_{=0} + \underbrace{\{\bar{L}|_{t_1} - \bar{L}_{\dot{x}}|_{t_1} \dot{x}_1\}}_{=0} \delta t_1$$

$$= \int_{t_0}^{t_1} (\underbrace{L_x + \lambda^T f_x + \frac{d}{dt}\lambda^T}_{=0})\delta x + (\underbrace{L_u + \lambda^T f_u}_{=0})\delta u + (\underbrace{f - \dot{x}}_{=0})\delta \lambda dt = 0$$

Summary

$$\begin{cases} \dot{x} = f(x,u,t) \\ \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} = 0 \\ \dot{\lambda}^T = -\left(\frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} \right) \\ BCs \end{cases}$$

$$H =\triangleq L + \lambda^T f$$

$$\begin{cases} \dot{x} = f \\ \frac{\partial H}{\partial u} = 0 \\ \dot{\lambda}^T = -\frac{\partial H}{\partial x} \\ BCs \end{cases}$$