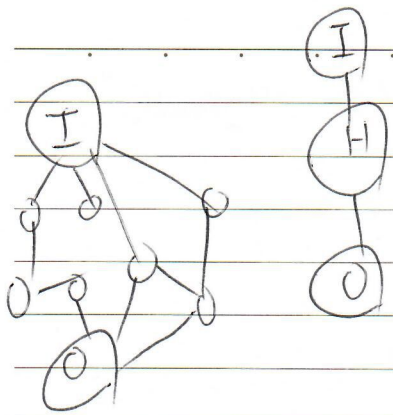
Date 2018. 9. 17 No



: hidden variable.

$\{Y_1, \dots, Y_m\}$

group 1 $\sim N(\mu_1, \sigma_1^2)$

group 2 $\sim N(\mu_2, \sigma_2^2)$

p(group 1 is taken on) $= \pi$

Introduce hidden variables,

$H = \{H_1, \dots, H_m\}$ $H_i = 1$ when Y_i is associated with g1
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$p(H_i=0 \mid \theta_t, D) = p(H_i=0 \mid \theta_t, y_i) = 1 - \gamma_i$ " soft assignment "

$$\theta_{t\,i} = \arg\max_{\theta} \mathop{E}_{p(H \mid \theta_t, D)} \log p(H, D \mid \theta)$$

$$= \arg\max_{\theta} \mathop{E}_{p(H \mid \theta_t, D)} \log p(H \mid \theta) + \log p(D \mid H, \theta)$$

$$= \arg\max_{\theta} \mathop{E}_{p(H \mid \theta_t, D)} \sum_{i=1}^{m} \log \pi^{H_i} (1-\pi)^{1-H_i} + \log N(y_i; \mu_1, \sigma_1)^{H_i} \cdot N(y_i; \mu_2, \sigma_2)^{1-H_i}$$

$$= \arg\max_{\theta} \mathop{E}_{p(H \mid \theta_t, D)} \sum_{i=1}^{m} H_i \log \pi + (1-H_i) \log(1-\pi) + H_i \left(\frac{y_i - \mu_i}{\sigma_1}\right)^2$$

$$\pi = \arg\max_{\pi} \mathop{E}_{p(H \mid \theta_t, D)} H_i \log \pi + (1-H_i) \log(1-\pi)$$

$$= \arg\max_{\pi} \sum_{i=1}^{m} \underbrace{p(H_i=1 \mid \theta_t, y_i)}_{\gamma_i} \log \pi + \underbrace{p(H_i=0 \mid \theta_t, y_i)}_{1-\gamma_i} \log(1-\pi)$$

$$\sum_{i=1}^{m} \left(\frac{r_i}{\pi} - \frac{(1-r_i)}{1-\pi} \right) = 0 \Rightarrow \pi = \frac{1}{m} \sum_{i=1}^{m} r_i$$

$$\mu_1 = \frac{\sum_i r_i \cdot y_i}{r_1}, \quad \sigma_1^2 = \frac{\sum r_i (y_i - \mu_1)^2}{\sum r_i}$$

Error Backpropagation.

Stochastic gradient descent $\sim \frac{dJ}{d\pi} = 0$

$$\theta_i = \theta \cdots \lambda \cdot \frac{d(y - f(x))^2}{d\theta}$$

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E.M is the generalized version of stochastic gradient descent.

Optimal control.

$\dot{x} = f(x, t)$

$$\min \int_{t_0}^{t_1} L(x, u, t)\,dt + \phi(x(t_1), t_1) \quad \text{objective function}$$

linear system w/ quadratic criteria.

$$\min_{u(t)} \frac{1}{2} \int_{t_0}^{t_1} x^T Q x + u^T u \, dt + \frac{1}{2} x^T(t_1) R x(t_1)$$

$$\text{s.t} \quad \dot{x} = A_{(t)} x_{(t)} + B_{(t)} u_{(t)}$$

Riccati eq: $\dot{K}(t) = -A^T(t) K(t) - K(t) A(t) + K(t) B(t) B(t)^T K(t) - Q(t)$

$$\frac{d}{dt} x^T(t) K(t) x(t) = \dot{x}^T K x + x^T \dot{K} x + x^T K \dot{x}$$

$$\int_{t_0}^{t_1} \frac{d}{dt} x^T(t) K(t) x(t)\, dt = \int_{t_0}^{t_1} \dot{x}^T K x + x^T \dot{K} x + x^T K \dot{x} \, dt$$

$$x^T(t) K(t) x(t) \Big|_{t_0}^{t_1} = ? \quad \text{(next time)}$$

Derivation of the E.M.

we want to find θ such that $p(x|\theta)$ is a maximum.

$L(\theta) = \ln p(x|\theta)$ ($\ln$ is a strictly increasing function)

θ_n: n^{th} iteration of the current estimate for θ.

we want to compute an updated estimate θ such that $L(\theta) > L(\theta_n)$

equivalently, $L(\theta) - L(\theta_n) = \ln p(x|\theta) - \ln p(x|\theta_n)$

z: hidden random vector. then

$$p(x|\theta) = \sum_z p(x|z,\theta)\, p(z|\theta)$$

$$L(\theta) - L(\theta_n) = \ln\left(\sum_z \cancel{p(x|z)}\; p(x|z,\theta)\, p(z|\theta)\right) - \ln p(x|\theta_n)$$

$$= \ln\left(\sum_z p(x|z,\theta)\, p(z|\theta) \cdot \frac{p(z|x,\theta_n)}{p(z|x,\theta_n)}\right) - \ln p(x|\theta_n)$$

$$= \ln\left(\sum_z p(z|x,\theta_n) \frac{p(x|z,\theta)\, p(z|\theta)}{p(z|x,\theta_n)}\right) - \ln p(x|\theta_n)$$

using Jensen's inequality,

$$\geq \sum_z p(z|x,\theta_n) \ln\left(\frac{p(x|z,\theta)\, p(z|\theta)}{p(z|x,\theta)}\right) - \ln p(x|\theta_n)$$

$$= \sum_z p(z|x,\theta_n) \ln\left(\frac{p(x|z,\theta)\, p(z|\theta)}{p(z|x,\theta_n)\, p(x|\theta_n)}\right)$$

$$\triangleq \underbrace{\Delta(\theta|\theta_n)}$$

$$L(\theta) \geq L(\theta_n) + \Delta(\theta|\theta_n)$$

$$\ell(\theta|\theta_n) \triangleq L(\theta_n) + \Delta(\theta|\theta_n)$$

$$L(\theta) \geq \ell(\theta|\theta_n)$$

Additionally, observe that,

$$l(\theta_n|\theta_n) = L(\theta_n) + \Delta(\theta_n|\theta_n)$$

$$= L(\theta_n) + \sum_z p(z|x,\theta_n) \ln \frac{p(x|z,\theta_n)\, p(z|\theta_n)}{p(z|x,\theta_n)\, p(x|\theta_n)}$$

$$= L(\theta_n) + \sum_z p(z|x,\theta_n) \ln \frac{p(x,z|\theta_n)}{p(x,z|\theta_n)}$$

$$= L(\theta_n) + \sum_z p(z|x,\theta_n) \ln 1$$

$$= L(\theta_n)$$

So, for $\theta = \theta_n$, the functions $l(\theta|\theta_n)$ and $L(\theta)$ are equal.

Formally we have,

$$\theta_{n+1} = \underset{\theta}{\operatorname{argmax}} \{ l(\theta|\theta_n) \}$$

$$= \underset{\theta}{\operatorname{argmax}} \left\{ L(\theta_n) + \sum_z p(z|x,\theta_n) \ln \frac{p(x|z,\theta)\, p(z|\theta)}{p(x|\theta_n)\, p(z|x,\theta_n)} \right\}$$

Drop the terms which are constant w.r.t θ,

$$= \underset{\theta}{\operatorname{argmax}} \left\{ \sum_z p(z|x,\theta_n) \ln p(x|z,\theta)\, p(z|\theta) \right\}$$

$$= \underset{\theta}{\operatorname{argmax}} \left\{ \sum_z p(z|x,\theta_n) \ln \frac{p(x,z,\theta)}{p(z,\theta)} \frac{p(z,\theta)}{p(\theta)} \right.$$

$$= \underset{\theta}{\operatorname{argmax}} \left\{ \sum_z p(z|x,\theta_n) \ln p(x,z|\theta) \right\}$$

$$= \underset{\theta}{\operatorname{argmax}} \left\{ E_{z|x,\theta_n} \{ \ln p(x,z|\theta) \} \right\}$$

The E.M algorithm consists of iterating the:

1. E step: Determine the conditional expectation $E_{z|x,\theta_n}\{\ln p(x,z|\theta)\}$

2. M step: Maximize this expression with respect to θ.