Optimal Control With End Points

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1 Euler-Lagrange equation

We consider the minimizing problem of the functional

$$\int_{t_0}^{t_1} L(x, \dot{x}, t)dt \tag{1}$$

where $x(t_0) = x_0$ and $x(t_1) = x_1$. Our goal is to find $x(t) \in R^n$ minimizing above functional. Consider an optimal solution $x^* = x - \epsilon\delta x$ and the following modified function

$$J(\epsilon) = \int_{t_0}^{t_1} L(x^* + \epsilon\delta x, \dot{x} + \epsilon\delta\dot{x}, t)dt \tag{2}$$

then for all $\delta x(t)$, we have

$$\begin{aligned}
\frac{\partial J}{\partial \epsilon}|_{t_0}^{t_1} = 0 &= \int_{t_0}^{t_1} [\frac{\partial L}{\partial x}|_{\epsilon=0}\delta x + \frac{\partial L}{\partial \dot{x}}|_{\epsilon=0}\delta\dot{x}]dt \\
&= \int_{t_0}^{t_1} [\frac{\partial L}{\partial x}\delta x - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\delta x]dt + \frac{\partial L}{\partial \dot{x}}\delta x|_{t_0}^{t_1} \\
&= \int_{t_0}^{t_1} [\frac{\partial L}{\partial x}\delta x - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\delta x]dt
\end{aligned}$$

by the partial integration. Then we get a partial differential equation

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0 \tag{3}$$

The equation (2.3) is the Euler-Lagrange equation for our problem. If we can solve this, we get minimal solution of the initial functional.

2 A variable end-point problem

Let's revisit the optimal control problem of the functional

$$\int_{t_0}^{t_1} L(x, \dot{x}, t)dt$$

with only the boundary $x(t_0) = t_0$ and $x(t_1)$ is free. Then by similar process to section 2.1, we have the same equation and an additional term

$$\begin{aligned}
\frac{\partial J}{\partial \epsilon}|_{t_0}^{t_1} = 0 &= \int_{t_0}^{t_1} [\frac{\partial L}{\partial x}|_{\epsilon=0}\delta x + \frac{\partial L}{\partial \dot{x}}|_{\epsilon=0}\delta \dot{x}]dt + L|_{t=t_1}\delta t_1 \\
&= \int_{t_0}^{t_1} [\frac{\partial L}{\partial x}\delta x - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}\delta x]dt + \frac{\partial L}{\partial \dot{x}}|_{t=t_1}\delta x(t_1) + L|_{t=t_1}\delta t_1
\end{aligned}$$

which yields the new boundary conditions for the problem,

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = 0, \qquad \frac{\partial L}{\partial \dot{x}}|_{t=t_1}\delta x(t_1) = 0, \qquad L|_{t=t_1}\delta t_1 = 0 \tag{4}$$

3 Functionals with a term involving endpoints and a control input

Now we consider the functional

$$J = \phi(x(t_1), t_1) + \int_{t_0}^{t_1} L(x, u, t)dt \tag{5}$$

such that $u(t)$ is a control input, $\dot{x} = f(x, u, t)$, the boundary conditions hold and ϕ is continuously differentialble in its arguments. Let's consider the boundaries $x(t_0) = t_0$ and free $x(t_1)$. We can introduce Lagrange multiplier on the (2.5),

$$\begin{aligned}
J &= \int_{t_0}^{t_1} [L(x, u, t) + \frac{d}{dt}\phi(x(t), t)]dt \\
&= \int_{t_0}^{t_1} [L(x, u, t) + \frac{\partial \phi}{\partial x}\dot{x} + \frac{\partial \phi}{\partial t} + \lambda^T(f - \dot{x})]dt
\end{aligned}$$

since $f - \dot{x} = 0$. Let $\bar{L}(x, u, \lambda, t) = L(x, u, t) + \frac{\partial \phi}{\partial x}\dot{x} + \frac{\partial \phi}{\partial t} + \lambda^T(f - \dot{x})$ for simplification. Similarly to the derivation process from section 2.1, we have

$$\delta J = 0 = \int_{t_0}^{t_1} [\bar{L}_x\delta x + \bar{L}_{\dot{x}}\delta x + \bar{L}_u\delta u + \bar{L}_\lambda\delta\lambda]dt \tag{6}$$

$$+ \bar{L}_{\dot{x}}|_{t=t_1}\delta x(t_1) + [\bar{L}|_{t=t_1} - \bar{L}_{\dot{x}}|_{t=t_1}\dot{x}(t_1)]\delta t_1 \tag{7}$$

which yields the many conditions for the problem. For the integral part (2.6), expanding $\bar{L}$ such that

$$
\begin{aligned}
0 &= \int_{t_0}^{t_1} [\bar{L}_x \delta x + \bar{L}_{\dot{x}} \delta x + \bar{L}_u \delta u + \bar{L}_\lambda \delta \lambda] \\
&= \int_{t_0}^{t_1} [(L_x + \lambda^T f_x + \frac{d}{dt}\lambda^T)\delta x + (L_u + \lambda^T f_u)\delta u + (f - \dot{x})\delta\lambda] dt
\end{aligned}
$$

which yields the three conditions such that

$$
L_x + \lambda^T f_x + \frac{d}{dt}\lambda^T = 0, \qquad L_u + \lambda^T f_u = 0 \tag{8}
$$

and the other part (2.7), we get

$$
\bar{L}_{\dot{x}}|_{t=t_1} = 0 \qquad \bar{L}|_{t=t_1} = 0 \tag{9}
$$

then x and u satisfying (2.8) and (2.9) solves our optimal control problem.

4 References

[1] Thiele, Rüdiger (2007). “Euler and the Calculus of Variations”. In Bradley, Robert E.; Sandifer, C. Edward. Leonhard Euler: Life, Work and Legacy. Elsevier. p. 249. ISBN 9780080471297.

[2] Euler-Lagrange Differential Equation, http://mathworld.wolfram.com/Euler-LagrangeDifferentialEquation.html

[3] Vapnyarskii, I.B. (2001) [1994], “Lagrange multipliers”, in Hazewinkel, Michiel, Encyclopedia of Mathematics, Springer Science+Business Media B.V. / Kluwer Academic Publishers, ISBN 978-1-55608-010-4.