Optimal Control

1 Models and Optimal Control Problems

1.1 Modelling and control systems

A *physical system* is a collection of physical objects connected together to serve an objective. Some electrical and mechanical examples are

- car
- satellite
- robot
- chemical reactor

More generally we can speak of a *system*, in which case, the objects, in addition to the electrical and mechanical ones, may be as diverse as chemical, biological, biomedical, environmental, economic, socio-economic or management, and even psychological. (Consider a car, which consists of many objects, such as a motor, wheels, brake, steering wheel, etc, that serve together the objective of moving the whole body together with the occupants inside from one point to another in some fashion.)

A *system model* is an idealized system which is obtained as a result of assumptions, simplifications, etc. (Simplify a car as a point mass on frictionless wheels moving along a straight line, where the only external force acting on it is the acceleration, in both forward and backward direction, and braking force.)

A *mathematical model* is a mathematical representation of the system model through use of pysical or other governing laws. (For the system model of a car, Newton's second law of motion applies.)

A *dynamical system* is a system which evolves with time. A mathematical model of a dynamical system depends on the choice of variables and the coordinate system, so a system model leads to different mathematical models. The particular mathematical model which gives a greater insight into the dynamic behaviour of the system is selected.

The *state variables*, or just *states*, of a dynamical system is a set of physical quantities, the specification of which (in the absence of external excitation) completely determines the evolution of the system. (Typical states of a car are the position and velocity of the car.) The external excitation which can be adjusted to change (hopefully) the behaviour of a dynamical system are referred to as *control variables*, or simply as controls, of the dynamical system. A (dynamical) system model represented in terms of states and controls is called a *control system*. (A car is a control system.)

Most systems, being dynamical, are characterized by differential or difference equations. A system represented by a set of ordinary differential equations (ODEs) is called a *continuous-time system*. Recall that any set of ODEs can be put into the following form

$$\dot{x}(t) = f(x(t), u(t), t) \ , \tag{1.1}$$

where $\dot{x}(t) = dx(t)/dt$, the state vector $x(t) = (x_1(t), x_2(t), \ldots, x_n(t)) \in \mathbb{R}^n$, control vector $u(t) = (u_1(t), u_2(t), \ldots, u_m(t)) \in \mathbb{R}^m$, time $t \in [0, \infty)$, and $f\colon \mathbb{R}^n \times \mathbb{R}^m \times [0, \infty) \to \mathbb{R}^n$. Note that $x_1, x_2, \ldots, x_n$ are the state variables and $u_1, u_2, \ldots, u_m$ are the control variables. The system given above is said to be *non-autonomous*, or *time-varying*, because f explicitly depends on time t.

If the system model is given by

$$\dot{x}(t) = f(x(t), u(t))$$

then it is said to be *autonomous*, or *time-invariant*, because time t does not appear explicitly.

1.2 Optimal control problems

An optimal control problem (OCP) is the problem of finding "best" x and u that solves Equation (1.1), minimizing some cost. (Finding the driving strategy of the car to get from one specified point to another in minimum time constitutes an OCP.)

In other words, an OCP is concerned with minimizing some cost subject to Equation (1.1).

Suppose that the system has a scalar control and a scalar state. One can think of the following typical costs to minimize:

1. Fuel, or control effort:
$$\int_{t_0}^{t_f} u^2(t)\, dt \ ;$$
alternatively,
$$\int_{t_0}^{t_f} |u(t)|\, dt \ .$$
In general, it is not obvious which one of these costs would yield a more favourable solution. It is often best to see the results of both costs, if possible.

2. Fuel (control effort) and magnitude of state:
$$\int_{t_0}^{t_f} (\alpha_1\, u^2(t) + \alpha_2\, x^2(t))\, dt \ ,$$
where α_1 and α_2 are some appropriate weights.

3. Duration of process (i.e. terminal time):

$$t_f \ , \qquad \text{i.e.} \qquad \int_{t_0}^{t_f} 1 \, dt \ ,$$

where the initial time t_0 is fixed and terminal (or final) time t_f is free.

4. Terminal time and fuel:

$$\int_{t_0}^{t_f} (\alpha_1 + \alpha_2 \, u^2(t)) \, dt \ ,$$

where α_1 and α_2 are some appropriate weights, t_0 is fixed and terminal (or final) time t_f is free.

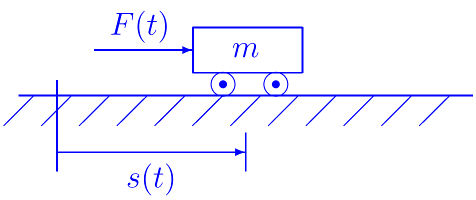
5. Other terminal cost, e.g. magnitude of the terminal state:

$$x^2(t_f)$$

1.3 Examples

In the example below, we summarize the modelling comments we have given so far about a car.

Example 1 We model a car as a point mass m with the acceleration and braking force $F(t)$ acting on it as a function of time.



Newton's 2nd law of motion states that

$$F(t) = m\ddot{s}(t) \ .$$

Let the state variables be defined as the position and velocity of the car, namely that $x_1(t) = s(t)$, $x_2(t) = \dot{s}(t)$. Then

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \frac{F(t)}{m} \end{aligned}$$

Here the term $\dfrac{1}{m} F(t)$ may be designated as the *control variable*, denoted by $u(t)$. In the general form $\dot{x} = f(x, u)$, we have

$$\begin{aligned} x &= \big(x_1, x_2\big)^\top \\ u &= \frac{F}{m} \\ f(x, u) &= \big(x_2, u\big)^\top \ . \end{aligned}$$

Using matrices, we can alternatively write

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{b} u(t);$$

that is

$$\dot{x}(t) = Ax(t) + bu(t) \ ,$$

where A is called the system matrix, b the input vector. Note that the identification of the state (and control) variables is not unique. The variables

$$\begin{aligned} x_1(t) &= s(t) + \dot{s}(t) \\ x_2(t) &= \dot{s}(t) \end{aligned}$$

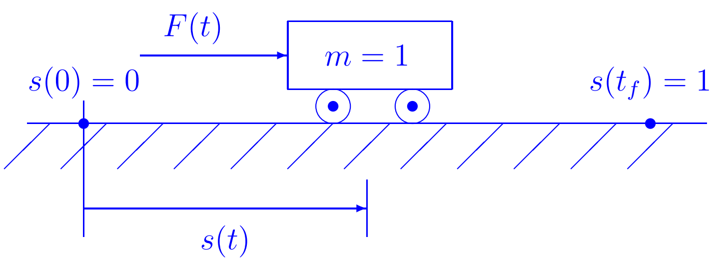
can also be chosen as the states of the system. Then

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) + \frac{F(t)}{m} \\ \dot{x}_2(t) &= \frac{F(t)}{m} \end{aligned}$$

i.e.

$$\dot{x}(t) = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{=A} x(t) + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{b} u(t).$$

Now consider the problem of getting from one point to another, at the same time minimizing a cost:



Suppose also that the force, $F(t)$, is bounded: $-1 \le F(t) \le 1$. Restate the problem as: How to get from $s(0) = 0$ to $s(t_f) = 1$ (from rest to rest) in minimum possible time, t_f? We can neatly write down the OCP for this case as follows.

$$\begin{aligned} \min_{u(\cdot)} \ & t_f \\ \text{s.t.} \ & \dot{x}_1(t) = x_2(t) \ , \\ & \dot{x}_2(t) = u(t) \ , \qquad -1 \le u(t) \le 1 \ , \\ & x(0) = (0, \ 0), \quad x(t_f) = (1, \ 0) \ . \end{aligned}$$

Example 2 If we add a spring to the mass in Example 1, we get a *mass-spring* system, which is also referred to as the *harmonic oscillator*.

$$\begin{aligned}\dot{x}_1(t) &= x_2(t)\\ \dot{x}_2(t) &= -x_1(t) + u(t)\end{aligned}$$

Example 3 Consider a model of an epidemic process in a closed population. At time t:

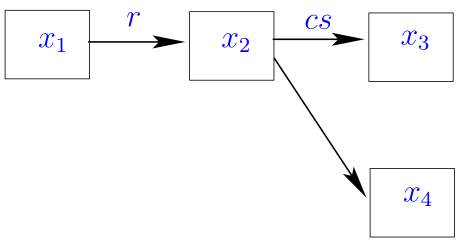
$x_1(t)$: those who are susceptible but not yet infected

$x_2(t)$: those who are ill (those who got the disease)

$x_3(t)$: those who die from it

$x_4(t)$: those who are unaffected (those naturally immune + those who survive the infection and become immune + those who are isolated or quaranteed)

We can schematically describe the transitions as in the diagram below.



The variables $x_i(t)$, $i = 1, \ldots, 4$, are the states of the dynamical system. Suppose that one can decrease the rate of transition from being x_1 to being x_2 by carrying out vaccination. We denote this influence on the process by $u(t)$, the control.

The dynamical system equations modelling the epidemic process are given by

$$\begin{aligned}\dot{x}_1(t) &= -r\, x_1(t)\, x_2(t) - u(t)\\ \dot{x}_2(t) &= r\, x_1(t)\, x_2(t) - s\, x_2(t)\\ \dot{x}_3(t) &= cs\, x_2(t)\end{aligned}$$

Note that once x_1, x_2 and x_3 are known, x_4 can be worked out, because x_4 is the remaining individuals in the closed population.

We wish to minimize the deaths at the end of the epidemic process (at the terminal time t_f) as well as the amount of vaccination throughout the epidemic; namely we want to minimize

$$\int_0^{t_f} bu^2(t)\, dt + (1-b)x_3(t_f)\ , \qquad 0 \le b \le 1\ .$$

Note that the cost can be rewritten as

$$\int_0^{t_f} [bu^2(t) + (1-b)\dot{x}_3(t)]\, dt + (1-b)x_3(0)\ .$$

The additional term after the integral is constant, so can be ignored. After the substitution of $\dot{x}_3(t)$ from the system equations, the problem becomes one of minimizing

$$\int_0^{t_f} [bu^2(t) + (1-b)csx_2(t)]\, dt$$

subject to

$$\begin{aligned} \dot{x}_1(t) &= -r\, x_1(t)\, x_2(t) - u(t) \\ \dot{x}_2(t) &= r\, x_1(t)\, x_2(t) - s\, x_2(t) \end{aligned}$$

In real life applications, the concern to emulate the behaviour of dynamical systems as close to the real situation as possible would often result in very sophisticated models. The next example (very broadly) describes such a complex system.

Example 4 The behaviour of an airplane can be modelled by using the physical laws for rigid body dynamics. The Earth-fixed x-, y- and z-coordinates are used to describe the position of the airplane with respect to the Earth. The angular orientation, however, need the so-called Euler angles θ, ψ and ϕ. The body-fixed coordinates are used to define the pitch, yaw and roll rates, denoted p, q and r, of the airplane. Linear velocities with respect to the body-fixed reference frame introduces three more variables, namely u, v and w, altogether giving rise to 12 differential equations which describe the behaviour of an airplane flying in the air. These equations also contain controls. In one typical case, these controls are the elevator deflection (mainly responsible for pitch control), rudder deflection (mainly for yaw control) and aileron deflection (mainly for roll control).

An OCP for an airplane can be posed as follows. Suppose that because of a gust wind the airplane is perturbed from its nominal (or equilibrium) orientation. Then "stabilizing" the craft and restoring its orientation back to the nominal one, but doing this in minimum possible time is very crucial. This gives rise to a typical OCP for an aircraft.

Flying from one city to another on time, but minimizing the fuel consumption for the total journey, is another typical OCP arising from the operation of airplanes.

2 Calculus of Variations

2.1 Functionals and variational problems

A *functional* J is a rule of correspondence that assigns to each function y (or curve) in some class (or set) of functions Ω a unique number, namely that

$$\begin{aligned} J: \quad & \Omega \to \mathbb{R} \\ & y \mapsto J(y) \end{aligned}$$

We give examples of functionals below.

1. The arc-length of a "smooth" curve $y = y(x)$ is a functional defined on the set of curves represented by continuously differentiable[1] functions y between $x = a$ and $x = b$. Namely
$$J(y) = \int_a^b \sqrt{1 + y'^2(x)}\, dx \ .$$
2. Let y be a continuously differentiable function, defined on the interval $[a, b]$. Then the expression
$$J(y) = \int_a^b y'^2(x)\, dx$$
defines a functional on the set of all such functions.
3. More generally, let F be a continuous function of three variables. Then the expression
$$J(y) = \int_a^b F[x, y(x), y'(x)]\, dx \tag{2.1}$$
where y ranges over the set of all continuously differentiable functions defined on the interval $[a, b]$, defines a functional.
Note that in the first example above, $F(x, y, z) = \sqrt{1 + z^2}$, and in the second example, $F(x, y, z) = z^2$.

We now list *variational problems*:

1. Find the shortest curve between two given points A and B in the plane, i.e., find the curve $y = y(x)$ for which the functional
$$J(y) = \int_a^b \sqrt{1 + y'^2(x)}\, dx \ .$$
is minimized. (The answer turns out to be the straight line segment joining A and B.)

[1] A function y is said to be *continuously differentiable* if has continuous first-order derivative y'.

2. Let A and B be two fixed points. Then the time it takes a particle to slide under the influence of gravity along some path joining A and B depends on the choice of the path (curve), and hence is a functional. The curve such that the particle takes the least time to go from A to B is called the *brachistochrone*[2].

2.2 Function spaces and norms

Points in $\mathbb{R}^n$ can be expressed in terms of a finite number of coordinates: if $y \in \mathbb{R}^n$, then we can write $y = (y_1, y_2, \ldots, y_n)$, where $y_i \in \mathbb{R}$. Therefore $\mathbb{R}^n$ is a finite-dimensional, more precisely, an n-dimensional, space. On the other hand, functions can be regarded as a "point" in a class (or set) of functions, referred to as a *function space*.

In analysis or computations, it is common practice to approximate smooth functions or curves by, say n, polygonal (or secant) lines whose vertices are placed on the curve. This way, an approximation of a smooth curve can be represented by a point in $\mathbb{R}^n$, and so an associated functional becomes a function of finitely many variables. To represent the curve exactly, one has to pass to the limit as $n \to \infty$. In view of this limit, a functional can be regarded as a function of infinitely many variables. Therefore we can regard a function space as an *infinite-dimensional* space.

By a *linear space*, we understand a vector space, satisfying the vector space axioms[3].

A linear space V is said to be *normed*, if each element $x \in V$ is assigned a nonnegative number $\|x\|$, called the *norm* of x, such that

1. $\|x\| = 0$ if and only if $x = 0$, for all $x \in V$,
2. $\|\alpha\, x\| = |\alpha|\, \|x\|$, for all $x \in V$ and all $\alpha \in \mathbb{R}$,
3. $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in V$.

We give examples to norms and normed spaces below.

- $\mathcal{C}[a, b]$ is the space of all continuous functions $y = y(x)$ defined on $[a, b]$.

  We equip $\mathcal{C}[a, b]$ with the following norm.

  $$\|y\|_0 = \max_{a \leq x \leq b} |y(x)| \;.$$

  Norms are useful for describing distances in a space. For example the expression

  $$\|y - y^*\| \leq \delta \;,$$

  where $\delta > 0$, defines the set of all ys which lie inside the strip of width 2δ about the function y^*, which is illustrated in Figure 1 shown

[2]The brachistochrone problem was posed by Johann Bernoulli in 1696. The problem was solved by Johann Bernoulli, James Bernoulli, Newton and l'Hôpital. By many, the brachistochrone problem is regarded as the first optimal control problem.

[3]See a text on Linear Algebra.

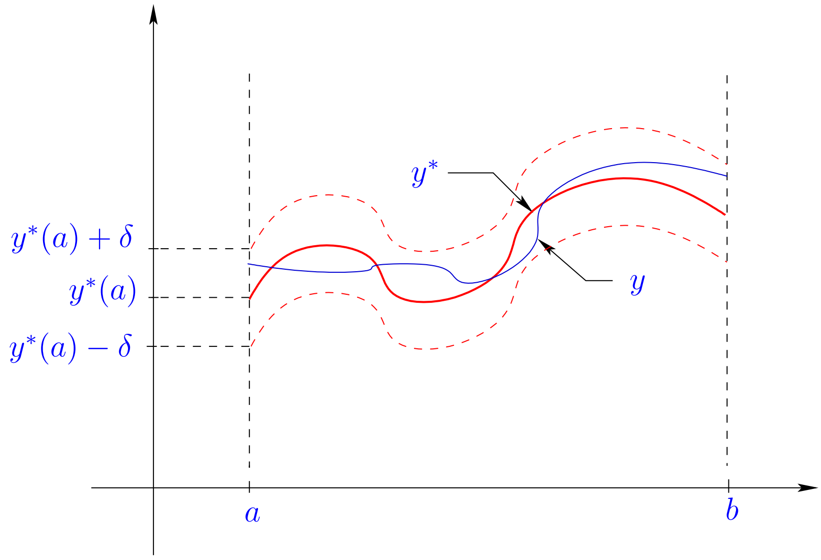


Figure 1: Neighbourhood of a function.

- $\mathcal{D}_1[a,b]$ is the space of all functions $y = y(x)$ defined on $[a,b]$, which are continuous and have continuous first derivatives.

  We equip $\mathcal{D}_1[a,b]$ with the following norm.

  $$\|y\|_1 = \max_{a\le x\le b} |y(x)| + \max_{a\le x\le b} |y'(x)| \ .$$

  Note that $\|y - z\|_1 \le \varepsilon$ implies that $|y(x) - z(x)| \le \varepsilon$ and $|y'(x) - z'(x)| \le \varepsilon$, for all $x \in [a,b]$. Also note that $\mathcal{D}_1[a,b] \subset \mathcal{C}[a,b]$.

The functional J is said to be *continuous at* $\widehat{y} \in V$ if for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|y - \widehat{y}\| < \delta \qquad \Longrightarrow \qquad |J(y) - J(\widehat{y})| < \varepsilon \ .$$

Consider

$$J(y) = \int_a^b F(x, y, y')\, dx$$

as we defined before. J is continuous if $y \in \mathcal{D}_1$ and that we use $\|\cdot\|_1$. In general, J above may not be continuous if we use $\|\cdot\|_0$. For example, the arc-length functional is not continuous in $\mathcal{C}[a,b]$ with norm $\|\cdot\|_0$. Consider the functions depicted in Figure 2 to illustrate this discontinuity. We can always find a function y in a δ-neighbourhood of $\widehat{y}$ (no matter how small δ is) in the sense of the norm of the space $\mathcal{C}[a,b]$, such that the length of y differs from that of $\widehat{y}$ by a factor of, say, 3, because there is no restriction on the magnitude of y'.

Let φ be a functional defined on a normed linear space V, i.e., if $h \in V$, then $h \mapsto \varphi(h)$. The functional φ is said to be a (*continuous*) *linear functional* if

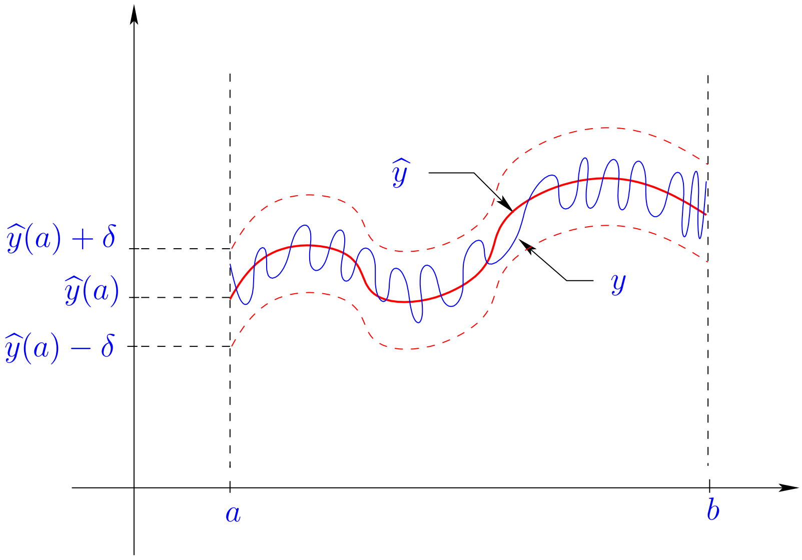


Figure 2: Discontinuity of arc-length functional in $\mathcal{C}[a,b]$ with norm $\|\cdot\|_0$.

1. $\varphi(\alpha h) = \alpha\varphi(h)$, for all $h \in V$ and all $\alpha \in \mathbb{R}$,
2. $\varphi(h_1 + h_2) = \varphi(h_1) + \varphi(h_2)$, for all $h_1, h_2 \in V$,
3. φ is continuous on V.

Let $h \in \mathcal{C}[a,b]$. We list below examples to continuous linear functionals.

1. $\varphi(h) = h(x_0)$, where $x_0 \in [a,b]$ is fixed.
2. $\varphi(h) = \displaystyle\int_a^b h(x)\,dx$.
3. $\varphi(h) = \displaystyle\int_a^b \alpha(x)\,h(x)\,dx$, where $\alpha \in \mathcal{C}[a,b]$.
4. $\varphi(h) = \displaystyle\int_a^b [\alpha_0(x)\,h(x) + \alpha_1(x)\,h'(x)]\,dx$, where $\alpha_i \in \mathcal{C}[a,b]$ and $h \in \mathcal{D}_1[a,b]$. Here φ is a linear functional on $\mathcal{D}_1[a,b]$.

Exercise 1 *Show that the above example functionals are indeed linear.*

2.3 Technical facts

Lemma 1 *If α is continuous in $[a,b]$, and if*

$$\int_a^b \alpha(x)\,h(x)\,dx = 0$$

for every $h \in \mathcal{C}[a,b]$ such that $h(a) = h(b) = 0$, then $\alpha(x) = 0$ for all $x \in [a,b]$.

Proof. (By contradiction.) Suppose $\alpha(x_0)$ is nonzero, say > 0, for some $x_0 \in [a,b]$. Then, by continuity of α, $\alpha(x) > 0$ for all x in some interval $[x_1, x_2]$ containing x_0. Set

$$h(x) = \begin{cases} (x - x_1)\,(x_2 - x), & x \in [x_1, x_2]\ , \\ 0, & \text{otherwise}\ . \end{cases}$$

Then

$$\int_a^b \alpha(x)\,h(x)\,dx = \int_{x_1}^{x_2} \alpha(x)\,(x - x_1)\,(x_2 - x)\,dx > 0\ .$$

Contradiction. □

Lemma 2 *If α is continuous in $[a,b]$, and if*

$$\int_a^b \alpha(x)\,h'(x)\,dx = 0$$

for every $h \in \mathcal{D}_1[a,b]$ such that $h(a) = h(b) = 0$, then $\alpha(x) = c$, c a constant, for all $x \in [a,b]$.

Proof. Let c be defined by

$$\int_a^b (\alpha(x) - c)\,dx = 0\ ,$$

and let

$$h(x) = \int_a^x (\alpha(\xi) - c)\,d\xi$$

so that $h \in \mathcal{D}_1[a,b]$ and that $h(a) = h(b) = 0$. Then

$$\int_a^b (\alpha(x) - c)\,h'(x)\,dx = \int_a^b \alpha(x)\,h'(x)\,dx - c(h(b) - h(a)) = 0\ .$$

Using this, we also get

$$\int_a^b (\alpha(x) - c)\,h'(x)\,dx = \int_a^b (\alpha(x) - c)^2\,dx = 0\ ,$$

which implies $\alpha(x) - c = 0$, i.e., $\alpha(x) = c$, for all $x \in [a,b]$. □

Lemma 3 *If α and β are continuous in $[a,b]$, and if*

$$\int_a^b [\alpha(x)\,h(x) + \beta(x)\,h'(x)]\,dx = 0 \tag{2.2}$$

for every $h \in \mathcal{D}_1[a,b]$ such that $h(a) = h(b) = 0$, then β is differentiable, and

$$\beta'(x) = \alpha(x)$$

for all $x \in [a,b]$.

Proof. Set

$$A(x) = \int_a^b \alpha(\xi)\, d\xi \ .$$

Now

$$\int_a^b \alpha(x)\, h(x)\, dx = h(b)\, A(b) - h(a)\, A(a) - \int_a^b A(x)\, h'(x)\, dx$$

where we have used the integration by parts formula $\int_a^b u\, dv = u(a)v(a) - u(b)v(b) - \int_a^b v\, du$, with $u = h(x)$, $dv = \alpha(x)\, h(x)$, $du = h'(x)\, dx$ and $v = \int_a^b \alpha(\xi)\, d\xi = A(x)$. Using $h(a) = h(b) = 0$, we get

$$\int_a^b \alpha(x)\, h(x)\, dx = -\int_a^b A(x)\, h'(x)\, dx \ .$$

Substitute this into (2.2) to obtain

$$\int_a^b [-A(x) + \beta(x)]\; h'(x)\, dx = 0$$

By Lemma 2, $\beta(x) - A(x) = \text{constant}$. So $\beta'(x) - A'(x) = 0$, which yields

$$\beta'(x) = \alpha(x)$$

for all $x \in [a, b]$. □

2.4 The variation of a functional

Let J be a functional defined on a normed linear space. The *increment* $\Delta J(y, h)$ of J corresponding to the increment h of the independent variable y is defined by

$$\Delta J(y, h) = J(y + h) - J(y) \ .$$

Fix y. Then ΔJ is a functional of h, which is in general nonlinear.

Suppose

$$\Delta J(y, h) = \varphi(y, h) + \varepsilon\, \|h\|$$

where φ is a functional linear in h, and

$$\varepsilon \to 0 \quad \text{as} \quad \|h\| \to 0 \ .$$

Then J is said to be *differentiable*. The functional φ is called the *variation* (or *differential*) of J and is denoted by δJ. The following lemma will be useful for showing that δJ is unique.

Lemma 4 *If φ is a linear functional and if*

$$\frac{\varphi(h)}{\|h\|} \to 0 \quad \text{as} \quad \|h\| \to 0 \tag{2.3}$$

then $\varphi(h) = 0$ for all h.

Proof. Suppose that (2.3) holds, but $\varphi(h_0) \neq 0$ for some $h_0 \neq 0$. Let

$$h_n := \frac{h_0}{n} \quad \text{and} \quad \lambda := \frac{\varphi(h_0)}{\|h_0\|} \neq 0 \; .$$

Note that $\|h_n\| \to 0$ as $n \to \infty$; however

$$\lim_{n\to\infty} \frac{\varphi(h_n)}{\|h_n\|} = \frac{n\,\varphi(h_0)}{n\,\|h_0\|} = \lambda \neq 0 \; ,$$

which contradicts (2.3). □

Theorem 1 *The differential of a differentiable functional is unique.*

Proof. Suppose that the differential of a differentiable functional is unique, so that

$$\Delta J(h) = \varphi_1(h) + \varepsilon_1 \|h\|,$$
$$\Delta J(h) = \varphi_2(h) + \varepsilon_2 \|h\|,$$

where φ_1 and φ_2 are linear functionals, and $\varepsilon_1, \varepsilon_2 \to 0$ as $\|h\| \to 0$. This gives

$$\varphi_1(h) - \varphi_2(h) = \varepsilon_1 \|h\| - \varepsilon_2 \|h\| \; ,$$

or

$$\frac{\varphi_1(h) - \varphi_2(h)}{\|h\|} = \varepsilon_1 - \varepsilon_2 \; .$$

Passing to the limit $\|h\| \to 0$, the above quotient tends to 0 because $\varepsilon_1, \varepsilon_2 \to 0$. Then, by Lemma 4, $\varphi_1(h) - \varphi_2(h) = 0$ for all h, furnishing the required result. □

Recall the following definitions from finite-dimensional calculus: Let F be a differentiable function of n variables, so $F\colon \mathbb{R}^n \to \mathbb{R}$. Then $F(x_1, x_2, \ldots, x_n)$ is said to have a *relative* (or *local*) *extremum* at $(x_1^*, x_2^*, \ldots, x_n^*)$ if

$$\Delta F = F(x_1, x_2, \ldots, x_n) - F(x_1^*, x_2^*, \ldots, x_n^*)$$

has the same sign for all $(x_1, x_2, \ldots, x_n)$ in a neighbourhood of $(x_1^*, x_2^*, \ldots, x_n^*)$. If $\Delta F \geq 0$, then $F(x_1^*, x_2^*, \ldots, x_n^*)$ is called a *minimum*; if $\Delta F \leq 0$, then $F(x_1^*, x_2^*, \ldots, x_n^*)$ is called a *maximum* (see Figure 3).

In function spaces, we make the following analogous definitions for functionals.

The functional J is said to have a *relative* (or *local*) *extremum* for y^* if $J(y) - J(y^*)$ has the same sign for every y in a neighbourhood of y^*. (For the notion of a neighbourhood of y^* refer to Figure 1 in the notes for Week 2.) Depending on what space ($\mathcal{C}[a,b]$ or $\mathcal{D}_1[a,b]$) is considered, one can define two kinds of extrema, as described below.

The functional J is said to have a *weak extremum* for y^* if there exists an $\delta > 0$ such that whenever $\|y - y^*\|_1 < \delta$ the difference $J(y) - J(y^*)$ has the same sign.

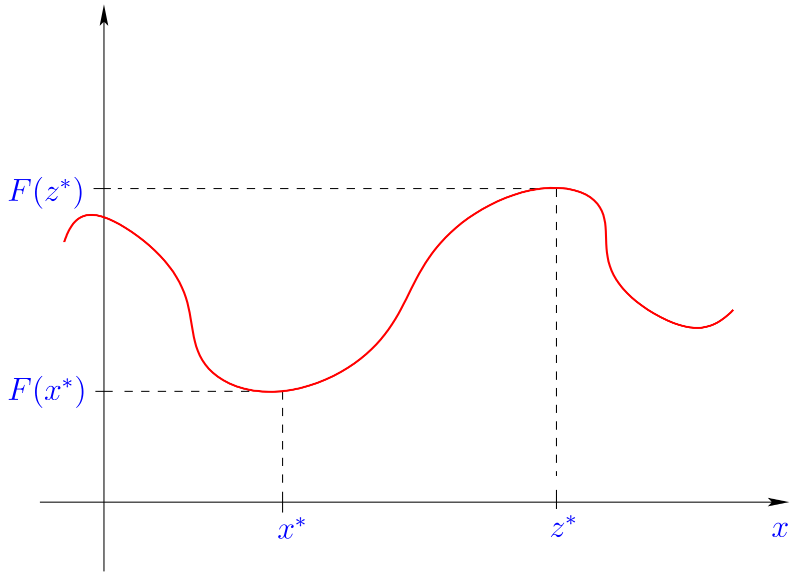


Figure 3: Extrema of a function.

The functional J is said to have a *strong extremum* for y^* if there exists an $\delta > 0$ such that whenever $\|y - y^*\|_0 < \delta$ the difference $J(y) - J(y^*)$ has the same sign.

Recall that $\|y-y^*\|_1 < \delta$ implies that $\|y-y^*\|_0 < \delta$. Therefore every strong extremum is a weak extremum at the same time; but not vice versa. In general, finding a weak extremum is easier than finding a strong extremum. The typical reason for this is that functionals are defined in terms of functions in $\mathcal{D}_1$, which are not necessarily continuous in $\mathcal{C}$ and thus pose difficulties in doing calculations.

Theorem 2 *Suppose that J has an extremum for y^*. Then*

$$\delta J(y^*, h) = 0$$

for all admissible h.

Proof. Without loss of generality, suppose that J has a minimum for y^*, i.e., $\delta J(y, h) \geq 0$ in a neighbourhood of y^* for all h. We have

$$\Delta J(y^*, h) = \delta J(y^*, h) + \varepsilon \|h\|$$

where $\varepsilon \to 0$ as $\|h\| \to 0$. Note that for sufficiently small $\|h\|$,

$$\operatorname{sign}(\Delta J(y^*, h)) = \operatorname{sign}(\delta J(y^*, h)) \ .$$

Suppose $\delta J(y^*, h_0) \neq 0$ for some admissible $\|h_0\|$. Then for any $\alpha > 0$

$$\delta J(y^*, -\alpha h_0) = -\delta J(y^*, \alpha h_0)$$

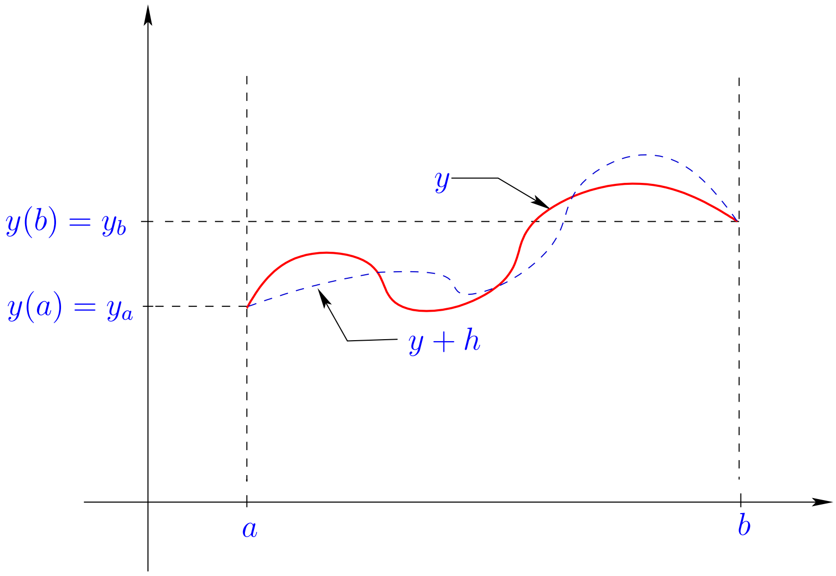


Figure 4: Increment h of a function y.

because δJ is linear. Hence for small enough α (i.e., for small enough $\|h\| = \|\alpha h_0\|$), ΔJ can be made to have either sign! A contradiction with the fact that $\Delta J \geq 0$! $\square$

Observe that Theorem 2 is analogous to a result in calculus: If a function F has an extremum at $(x_1^*, x_2^*, \ldots, x_n^*)$, then the gradient of F at $(x_1^*, x_2^*, \ldots, x_n^*)$ vanishes.

2.5 Euler-Lagrange Equation

Consider the problem of minimizing the functional

$$J(y) = \int_a^b F(x, y, y')\, dx$$

where $y \in \mathcal{D}_1[a, b]$, with the *boundary conditions*

$$\begin{aligned} y(a) &= y_a \ , \\ y(b) &= y_b \ , \end{aligned}$$

y_a and y_b real numbers. The function F is assumed to have continuous first and second partial derivatives with respect to its arguments.

Consider an increment $h \in \mathcal{D}_1[a, b]$ of a function $y \in \mathcal{D}_1[a, b]$, such that $h(a) = h(b) = 0$ (see Figure 4). Now

$$
\begin{aligned}
\Delta J &= J(y+h) - J(h) \\
&= \int_a^b F(x, y+h, y'+h')\, ds - \int_a^b F(x, y, y')\, dx \\
&= \int_a^b \left[F(x, y+h, y'+h') - F(x, y, y')\right]\, dx \\
&= \int_a^b \left[F_y(x, y, y')\, h - F_{y'}(x, y, y')\, h'\right]\, dx + \mathtt{h.o.t.}
\end{aligned}
$$

where we have used Taylor's theorem, and

$$
F_y = \frac{\partial F}{\partial y}\ , \qquad F_{y'} = \frac{\partial F}{\partial y'}\ ,
$$

and "h.o.t." stands for higher-order terms, or, more specifically, terms of order higher than 1 relative to h and h'. So

$$
\delta J = \int_a^b \left[F_y(x, y, y')\, h - F_{y'}(x, y, y')\, h'\right]\, dx
$$

By Theorem 2,

$$
\delta J = \int_a^b \left[F_y\, h - F_{y'}\, h'\right]\, dx = 0\ .
$$

Let $\alpha(x) = F_y(x, y(x), y'(x))$ and $\beta(x) = F_{y'}(x, y(x), y'(x))$. Lemma 3 implies (because $h(a) = h(b) = 0$) that

$$
\alpha(x) - \beta'(x) = 0\ ,
$$

i.e.,

$$
F_y - \frac{d}{dx} F_{y'} = 0\ . \tag{2.4}
$$

Equation (2.4) is referred to as the *Euler-Lagrange Equation.* Note that in the above derivation existence of $\frac{d}{dx}F_{y'}$ is readily implied. The Euler-Lagrange equation is a necessary condition for J to have a weak extremum.

Integral curves of Euler-Lagrange equation are called *extremals.*

Equation (2.4), together with the boundary conditions $y(a) = y_a$ and $y(b) = y_b$ constitute a two-point boundary-value problem (TPBVP), which is in general difficult to solve.

A case study:
Suppose that F does not depend on x explicitly, namely that $F(x, y, y') = F(y, y')$. Then

$$
F_y - \frac{d}{dx} F_{y'} = F_y - F_{y'y}\, y' - F_{y'y'}\, y'' = 0\ .
$$

Multiply by y' to get

$$
F_y\, y' - F_{y'y}\, {y'}^2 - F_{y'y'}\, y'\, y'' = \frac{d}{dx}\left(F - y'\, F_{y'}\right) = 0\ .
$$

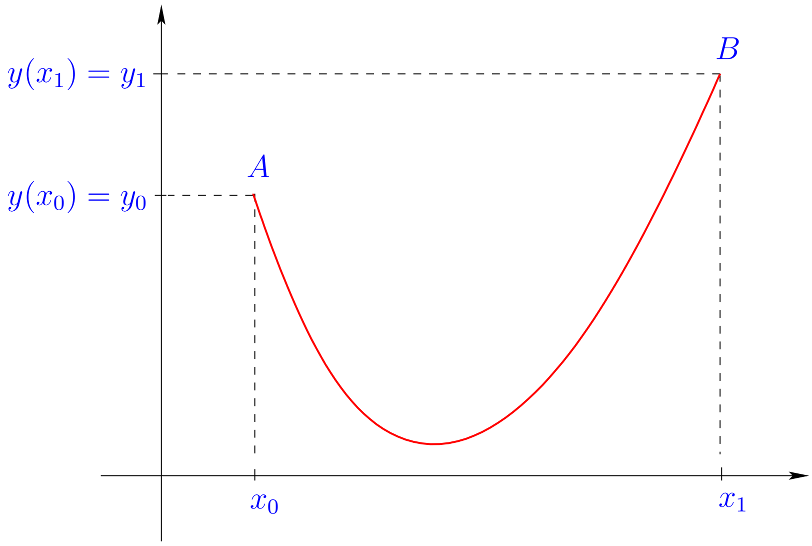


Figure 5: Hanging chain.

Therefore we have the *first integral*

$$F - y' F_{y'} = C \ ,$$

C a constant.

Example 5 (*The catenary problem* — also referred to as *hanging chain*, or *hanging rope, problem*)
Find the shape of an inextensible chain under its own weight, hanging (suspended) between point A (x_0, y_0) and B (x_0, y_0) in planar space (see Figure 5).

Minimize the potential energy

$$\int_{x_0}^{x_1} y \sqrt{1 + y'^2} \, dx \ ,$$

or minimize the area of the surface generated by rotating the chain about the x-axis:

$$2\pi \int_{x_0}^{x_1} y \sqrt{1 + y'^2} \, dx \ .$$

Then

$$F(x, y, y') = y \sqrt{1 + y'^2} \ .$$

Because F does not depend on x, we have the first integral (from the case study) that

$$F - y' F_{y'} = C \ ,$$

namely

$$y \sqrt{1 + y'^2} - y \frac{y'^2}{\sqrt{1 + y'^2}} = C$$

or

$$y = C\sqrt{1+y'^2}\ ;$$

so

$$y' = \sqrt{\frac{y^2 - C^2}{C^2}}\ .$$

Separate variables (using differentials) and integrate:

$$\begin{aligned} dx &= \frac{C\,dy}{\sqrt{y^2 - C^2}} \\ x + C_1 &= C\,\ln\frac{y + \sqrt{y^2 - C^2}}{C} \end{aligned}$$

(May use integral tables in this case.) Further manipulations yield (show this)

$$y = C\,\cosh\frac{x + C_1}{C}\ ,$$

where C and C_1 are found from

$$y(x_0) = y_0 \qquad \text{and} \qquad y(x_1) = y_1\ .$$

Exercise 2 *For the case when the integrand of a functional does not depend on y, namely $F(x, y, y') = F(x, y')$, find the first integral of the Euler-Lagrange equation.*

Exercise 3 *For the case when the integrand of a functional does not depend on y', namely $F(x, y, y') = F(x, y)$, write down the form of the Euler-Lagrange equation. Comment on this form.*

2.6 A variable end-point problem

In deriving the Euler-Lagrange equation in the previous section, we considered a variational problem with fixed end points. Now we will consider the same problem with variable end points, namely we consider the problem of finding a curve $y \in \mathcal{D}_1[a, b]$, for which no boundary conditions are imposed at $t = a$ and $t = b$, and

$$J(y) = \int_a^b F(x, y, y')\,dx \tag{2.5}$$

has an extremum.

Because y has "free" end-points, we consider an increment $h \in \mathcal{D}_1[a, b]$ of a function $y \in \mathcal{D}_1[a, b]$, such that $h(a)$ and $h(b)$ are not specified (see Figure 6). We proceed as in the preceding section and identify $\delta J(y)$:

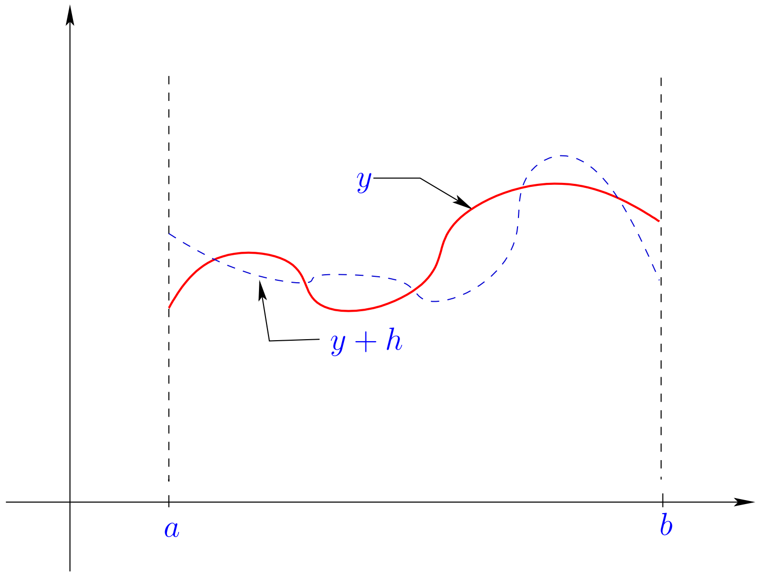


Figure 6: Increment h of a function y with variable end-points.

$$\begin{aligned}
\Delta J &= J(y+h) - J(h) \\
&= \int_a^b \left[F(x, y+h, y'+h') - F(x, y, y')\right] dx \\
&= \int_a^b \left[F_y(x, y, y')\, h - F_{y'}(x, y, y')\, h'\right] dx + \texttt{h.o.t.}
\end{aligned}$$

and so

$$\delta J = \int_a^b \left[F_y(x, y, y')\, h - F_{y'}(x, y, y')\, h'\right] dx \tag{2.6}$$

Recall Theorem 2: it is necessary for an extremum that $\delta J = 0$. However, in this case, $h(a)$ and $h(b)$ do not necessarily vanish; therefore we cannot apply Lemma 3. First apply integration by parts on (2.6) to get

$$\begin{aligned}
\delta J &= \int_a^b \left(F_y - \frac{d}{dx} F_{y'}\right) h(x)\, dx + F_{y'}\, h(x)\Big|_{x=a}^{x=b} \\
&= \int_a^b \left(F_y - \frac{d}{dx} F_{y'}\right) h(x)\, dx + F_{y'}\big|_{x=b}\, h(b) - F_{y'}\big|_{x=a}\, h(a) \qquad (2.7)
\end{aligned}$$

Because it is necessary to have $\delta J = 0$ for any h, consider an h such that $h(a) = h(b) = 0$. Then $\delta J = 0$ immediately yields the Euler-Lagrange equation:

$$F_y - \frac{d}{dx} F_{y'} = 0 \ .$$

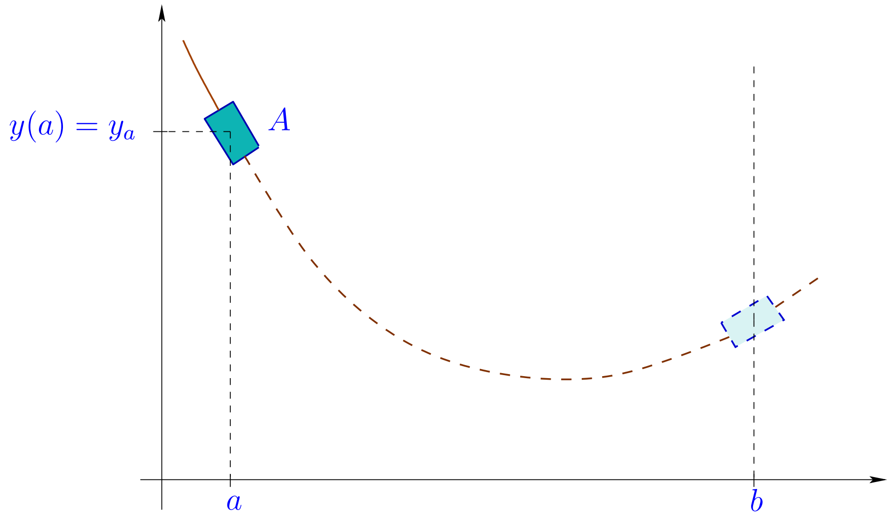


Figure 7: A brachistochrone problem.

So y must be an extremal. The above equation alone, without additional conditions such as boundary conditions, gives a general integral. Now substituting Euler-Lagrange equation into (2.7) we obtain

$$F_{y'}\big|_{x=b} h(b) - F_{y'}\big|_{x=a} h(a) = 0 \tag{2.8}$$

which further yields the new boundary conditions for the variable end-point problem,

$$F_{y'}\big|_{x=a} = 0 \ , \qquad F_{y'}\big|_{x=b} = 0 \ , \tag{2.9}$$

since $h(a)$ and $h(b)$ are arbitrary.

One can also consider a "mixed case" where $y(a)$ is fixed. Then we have a single additional condition

$$F_{y'}\big|_{x=b} = 0 \ .$$

Example 6 (*A brachistochrone problem*)
Figure 7 depicts the brachistochrone problem we would like to consider. The problem is to find the shape of the wire (find a curve $y = y(x)$) which causes the bead to slide under its own weight from the fixed point A to the vertical line at $x = b$, in minimum time. The arc-length s of the wire from A to a point $(x, y(x))$ is given by

$$s(x) = \int_a^x \sqrt{1 + y'^2}\, dx \ ,$$

where $y'(x) = \frac{d}{dx} y(x)$. Let the velocity of the bead along the wire be denoted by v. Then

$$v = \frac{ds}{dt} = \frac{ds}{dx}\frac{dx}{dt} = \sqrt{1 + y'^2}\,\frac{dx}{dt}$$

The velocity v can be written (from physics) as

$$v = \sqrt{2gy}$$

where g is the gravitational acceleration. In terms of the differentials dt and dx we can write

$$dt = \frac{\sqrt{1+y'^2}}{v}\,dx = \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx$$

Integrating both sides over the interval $[a, b]$, we get the *transit time* T, i.e. time for the bead to get from pint A to the vertical line at $x = b$:

$$T = \int_a^b \frac{\sqrt{1+y'^2}}{\sqrt{2gy}}\,dx$$

which is the functional we aim to minimize. Note here that the integrand $F(x, y, y')$ does not depend on x, which result in a simplied form of Euler-Lagrange equation. It can be shown that Euler-Lagrange equation yields the following family of *cycloids*.

$$\begin{aligned} x &= r\,(\theta - \sin\theta) + c\ , \\ y &= r\,(1 - \cos\theta)\ , \end{aligned}$$

where r, c are constants.

Exercise 4 *Show that Euler-Lagrange equation yields the family of cycloids stated above.*

To simplify, suppose that the curve passes through the origin $(0, 0)$. Then $c = 0$, and r is found from the boundary condition

$$F_{y'}\Big|_{x=b} = \frac{y'}{\sqrt{2gy}\,\sqrt{1+y'^2}} = 0$$

i.e.,

$$y' = 0 \qquad \text{for } x = b\ ,$$

which yields

$$r = \frac{b}{\pi}\ .$$

So

$$\begin{aligned} x &= \frac{b}{\pi}\,(\theta - \sin\theta)\ , \\ y &= \frac{b}{\pi}\,(1 - \cos\theta)\ . \end{aligned}$$

For a geometric interpretation of the cycloid curve, see the Week 5 e-beam board notes.

2.7 Functionals with a term involving end-points

In this section we consider the functional

$$J(y) = \varphi(y(a), y(b)) + \int_a^b F(x, y, y')\, dx \tag{2.10}$$

where the end-points $y(a)$ and $y(b)$ are variable, and φ is continuously differentiable in its arguments. The first term in functional J is typically encountered in optimal control. The problem of finding a curve $y \in \mathcal{D}_1[a, b]$ for which J has an extremum is tackled in almost the same way as in Section 2.6.

We consider an increment $h \in \mathcal{D}_1[a, b]$ of a function $y \in \mathcal{D}_1[a, b]$, such that $h(a)$ and $h(b)$ are not specified (see Figure 6). Note that the increment in φ can be expressed as

$$\varphi(y(a) + h(a), y(b) + h(a)) - \varphi(y(a), y(b)) = \varphi_{y(a)}\, h(a) + \varphi_{y(b)}\, h(b) + \text{h.o.t.}$$

where

$$\varphi_{y(a)} = \frac{\partial \varphi}{\partial y(a)} \quad \text{and} \quad \varphi_{y(b)} = \frac{\partial \varphi}{\partial y(b)} \;.$$

The increment in φ can then be incorporated in the increment ΔJ to obtain a modified version of the variation δJ in (2.7) in a straightforward manner:

$$\begin{aligned}\delta J \;=\; & \varphi_{y(a)}\, h(a) + \varphi_{y(b)}\, h(b) \\ & + \int_a^b \left(F_y - \frac{d}{dx} F_{y'} \right) h(x)\, dx + F_{y'}\big|_{x=b}\, h(b) - F_{y'}\big|_{x=a}\, h(a)\end{aligned}$$

or

$$\begin{aligned}\delta J \;=\; & \left(\varphi_{y(a)} - F_{y'}\big|_{x=a}\right) h(a) + \left(\varphi_{y(b)} + F_{y'}\big|_{x=b}\right) h(b) \\ & + \int_a^b \left(F_y - \frac{d}{dx} F_{y'} \right) h(x)\, dx\end{aligned}$$

Using the same arguments as those used in Section 2.6, we get the Euler-Lagrange equation

$$F_y - \frac{d}{dx} F_{y'} = 0$$

with the new boundary conditions

$$\varphi_{y(a)} - F_{y'}\big|_{x=a} = 0 \quad \text{and} \quad \varphi_{y(b)} + F_{y'}\big|_{x=b} = 0 \;. \tag{2.11}$$

2.8 Euler-Lagrange equation for n functions

In the preceding sections we dealt with the case when y is a scalar. Problem of finding more than one, say n, functions for which a functional has an extremum, can be studied in pretty much the same way as in the scalar case. In the case of n functions, a functional can be stated as

$$\int_a^b F(x, y, y')\, dx \;, \tag{2.12}$$

where $y(x) = (y_1(x), \ldots, y_n(x)) \in \mathbb{R}^n$, $y'(x) = (y_1'(x), \ldots, y_n'(x)) \in \mathbb{R}^n$, and $F(x, y(x), y'(x)) \in \mathbb{R}^n$.

Euler-Lagrange equation for the above problem involving n functions is in exactly the same form as before, but with *gradients* of F with respect to the n functions:

$$F_y - \frac{d}{dx} F_{y'} = \mathbf{0} \ , \tag{2.13}$$

where

$$F_y = (F_{y_1}, \ldots, F_{y_n}) \ , \qquad F_{y'} = (F_{y_1'}, \ldots, F_{y_n'}) \ ,$$

and

$$\frac{d}{dx} F_{y'} = (\frac{d}{dx} F_{y_1'}, \ldots, \frac{d}{dx} F_{y_n'}) \ .$$

Finally the boundary conditions are stated in as much the same way as they were stated in the previous sections, but this time using n functions. This include the case of a functional with a term involving the end points in Section 2.7.

3 Unconstrained Optimal Control Problems

Consider the optimal control problem (OCP)

$$\min_{x(\cdot),u(\cdot)} J(x,u) = \int_{t_0}^{t_f} f_0(x(t),u(t),t)\,dt \tag{3.1}$$

$$\text{subject to} \qquad \dot{x}(t) = f(x(t),u(t),t)\;, \qquad \text{for all } t \in [t_0,t_f]\;. \tag{3.2}$$

In the above problem, $x(t) = (x_1(t),\ldots,x_n(t)) \in \mathbb{R}^n$ is the *state vector* , where the *state variables* (or simply the *states*) $x_i \colon [t_0,t_f] \to \mathbb{R}$ have continuous first-order derivatives. The *control variables* (or simply the *controls*) $u_i \colon [t_0,t_f] \to \mathbb{R}$ have continuous first-order derivatives and form the *control vector* $u(t) = (u_1(t),\ldots,u_m(t)) \in \mathbb{R}^m$. Furthermore f_0 and f have continuous first-order partial derivatives with respect to their arguments.

It is easy to see that the functional given in (3.1) and the functional stated in (2.5) are of the same form, where x, y and F in (2.5) correspond to t, (x,u) and f_0 in (3.1), respectively, with $t_0 = a$ and $t_f = b$. However the main difference between the OCP in (3.1)-(3.2) and the calculus of variations problem (CVP) in (2.5) is that the former is constrained by a dynamical equation, or an ODE, given in (3.2). The constraint in (3.2) can be also be written as

$$f(x(t),u(t),t) - \dot{x}(t) = 0\;, \qquad \text{for all } t \in [t_0,t_f]\;.$$

Despite the presence of a constraint, the OCP in (3.1)-(3.2) is referred to as an *unconstrained* OCP, because no constraints are imposed on the states and/or the controls, apart from the dynamical equation in (3.2).

In optimization, it is common practice to reduce a constrained problem into an unconstrained one so that standard unconstrained optimization techniques can be employed. Reduction into an unconstrained problem is achieved by appending the constraint to the cost through a multiplier. We take the same approach for the OCP in (3.1)-(3.2), and form the *augmented cost* functional

$$\begin{aligned} \widetilde{J}(x,\psi,u) &= \int_{t_0}^{t_f} f_0(x(t),u(t),t)\,dt + \int_{t_0}^{t_f} \psi^T(t)\;[f(x(t),u(t),t) - \dot{x}(t)]\;dt \\ &= \int_{t_0}^{t_f} \big\{ f_0(x(t),u(t),t) + \psi^T(t)\;[f(x(t),u(t),t) - \dot{x}(t)] \big\}\;dt \end{aligned} \tag{3.3}$$

where $\psi(t) = (\psi_1(t),\ldots,\psi_n(t)) \in \mathbb{R}^n$ is called the *costate vector* , where the *costate variables* (or simply the *costates*) $\psi_i \colon [t_0,t_f] \to \mathbb{R}$ have continuous first-order derivatives. Costate variables are sometimes referred to as *adjoint variables.*

Now the functional in (3.3) can be treated in as much the same way as we treated the functional in (2.5). One can in fact consider the multi-function generalization of (2.5) given in (2.12), where an extremum of $y = (x,\psi,u)$ is sought. Note that in addition to replacing y in (2.12) by (x,ψ,u), the independent variable x in (2.12) should be replaced by the independent variable t. Then a solution of the multi-function version of Euler-Lagrange equation in (2.13) yields conditions furnishing extremals for the OCP (3.1)-(3.2).

3.1 Necessary conditions of optimality

We can rewrite (3.3) as

$$\widetilde{J}(x,\psi,u) = \int_{t_0}^{t_f} \widetilde{F}(x(t),\dot{x}(t),u(t),t)\,dt$$

where

$$\widetilde{F}(x,\dot{x},\psi,u,t) = f_0(x,u,t) + \psi^T\ [f(x,u,t) - \dot{x}]\ . \tag{3.4}$$

Next, for convenience, we define the *Hamiltonian* function:

$$H(x,\psi,u,t) := f_0(x,u,t) + \psi^T f(x,u,t)\ .$$

The expression in (3.4) then becomes

$$\widetilde{F}(x,\dot{x},\psi,u,t) = H(x,\psi,u,t) - \psi^T\dot{x}\ .$$

Let $y := (x,\psi,u)$, and so $\dot{y} := (\dot{x},\dot{\psi},\dot{u})$. Also let $F(y,\dot{y},t) = \widetilde{F}(x,\dot{x},\psi,u,t)$. Let us re-write the multi-function generalization of the Euler-Lagrange equation in (2.13) for our new setting:

$$F_y - \frac{d}{dt}F_{\dot{y}} = 0\ . \tag{3.5}$$

In other words, componentwise,

$$\begin{aligned}
\widetilde{F}_x - \frac{d}{dt}\widetilde{F}_{\dot{x}} &= H_x + \dot{\psi}^T = 0\ ,\\
\widetilde{F}_\psi - \frac{d}{dt}\widetilde{F}_{\dot{\psi}} &= H_\psi - \dot{x}^T = 0\ ,\\
\widetilde{F}_u - \frac{d}{dt}\widetilde{F}_{\dot{u}} &= H_u = 0\ .
\end{aligned}$$

So we get the following differential algebraic equations as necessary conditions of optimality for the OCP in (3.1)-(3.2), and thus furnishing extremals of the OCP.

$$\dot{x}^T = H_\psi\ , \tag{3.6}$$
$$\dot{\psi}^T = H_x\ , \tag{3.7}$$
$$H_u = 0\ . \tag{3.8}$$

The boundary (or end) conditions for these equations are obtained using the conditions (2.8)-(2.9) for variable end points, whenever applicable:

$$F_{\dot{x}}\big|_{t=t_0} = \psi(t_0) = 0\ , \qquad F_{\dot{x}}\big|_{t=t_f} = \psi(t_f) = 0\ .$$

We summarize all cases for the boundary conditions as follows:

(i) $x(t_0) = x_0$ and $x(t_f) = x_f$ (both end-states $x(t_0)$ and $x(t_f)$ are specified),

(ii) $x(t_0) = x_0$ and $\psi(t_f) = 0$ (only $x(t_0)$ is specified),

(iii) $\psi(t_0) = 0$ and $x(t_f) = x_f$ (only $x(t_f)$ is specified),

(iv) $\psi(t_0) = 0$ and $\psi(t_f) = 0$ (neither $x(t_0)$ nor $x(t_f)$ is specified).

When we say "an end-point is specified" we mean the end-point is "fixed," and when we say "a end-point is not specified" we mean the end-point is "variable" or "free."

The necessary conditions (3.6)-(3.8) can be expanded as

$$\begin{aligned} \dot{x}^T &= f(x, u, t) \ , && (3.9) \\ -\dot{\psi}^T &= \frac{\partial f_0}{\partial x} + \psi^T \frac{\partial f}{\partial x} \ , && (3.10) \\ 0 &= \frac{\partial f_0}{\partial u} + \psi^T \frac{\partial f}{\partial u} \ . && (3.11) \end{aligned}$$

Together with one of the four sets of boundary conditions listed above, these equations constitute a two-point boundary-value problem (TPBVP), which is in general very difficult to solve, especially compared to an initial value problem. In many instances, Equation (3.11) can be solved for the control u in terms of the state x and the costate ψ, such that $u = \phi(x, \psi, t)$. In this case, u can be substituted into (3.9) and (3.10) to get a TPBVP just with the ODEs (3.9) and (3.10).

Example 7 Recall the car model discussed in Section 1. Suppose the car has an initial speed of 1 m/s, namely that $x_2(0) = 1$, and that the initial position is set as $x_1(0) = 0$. The problem is bring the car to rest in 1 s, such that $x_1(1) = 0$ and $x_2(1) = 0$, at the same time spending minimum effort (in some sense). The problem can then be stated in the standard form as

$$\begin{aligned} &\min_{u(\cdot)} \frac{1}{2} \int_0^1 u^2(t)\, dt \\ &\text{subject to } \ \dot{x}_1(t) = x_2(t) \ , \quad x_1(0) = 0, \quad x_1(1) = 0 \ , \\ &\qquad\qquad\quad \dot{x}_2(t) = u(t) \ , \quad x_2(0) = 1, \quad x_2(1) = 0 \ . \end{aligned}$$

Form the Hamiltonian function:

$$H(x, \psi, u, t) = \frac{1}{2} u^2 + \psi_1 x_2 + \psi_2 u \ .$$

Now we start applying the necessary conditions of optimality stated in (3.6)-(3.8).

$$\dot{\psi}_1 = -H_{x_1} = 0, \text{ i.e.,} \quad \psi_1(t) = c_1,$$

where c_1 is a constant. We also have

$$\dot{\psi}_2 = -H_{x_2} = -\psi_1 = -c_1, \text{ i.e.,} \quad \psi_2(t) = -c_1 t + c_2,$$

where c_2 is a constant. An optimal control has to satisfy

$$H_u = u + \psi_2 = 0, \text{ i.e.,} \quad u(t) = -\psi_2(t) = c_1\, t - c_2 \ .$$

Next, the state variables can be easily found using this optimal control candidate:

$$\dot{x}_2 = u(t) = c_1\, t - c_2 \text{ which yields} \quad x_2(t) = \frac{c_1}{2}\, t^2 - c_2\, t + c_3 \ ,$$

and so

$$\dot{x}_1 = x_2(t) = \frac{c_1}{2}\, t^2 - c_2\, t + c_3 \text{ which yields} \quad x_1(t) = \frac{c_1}{6}\, t^3 - \frac{c_2}{2}\, t^2 + c_3\, t + c_4$$

where c_3 and c_4 are constants. Using the boundary conditions, one finds that $c_1 = 6$, $c_2 = 4$, $c_3 = 1$ and $c_4 = 0$. In summary, the extremals of the OCP are given by the following curves (or control and state trajectories),

$$\begin{aligned} u(t) &= 6\, t - 4 \ , \\ x_1(t) &= t^3 - 2\, t^2 + t \ , \\ x_2(t) &= 3\, t^2 - 4\, t + 1 \ , \end{aligned}$$

which are depicted in Figure 8.

Example 8 Consider the problem in Example 7, but with a minimization of the effort "in some other sense," namely consider minimizing the cost

$$\frac{1}{4} \int_0^1 u^4(t)\, dt \ .$$

Through a similar procedure to that in Example 7 conditions for the extremals in this case can also be written down easily. One should note that the constant coefficient 1/4 stands only for algebraic convenience. The integration constants for extremal curves $u(t)$ and $x(t)$ can at least be numerically found, and closed form expressions can be written. However doing this analytically is a little demanding; so instead we solve the problem numerically.

Reward 1 *You will get a 10% bonus for your Assignment 1, if you work out these integration constants (correct to a number of decimal points) in the expressions for $u(t)$ and $x(t)$, explaining how you got them. You may verify your results by checking that the graphs you draw with the expressions you obtain agree with the graphs in Figure 9.*

We show the solution trajectories for $u(t)$ and $x(t)$ in Figure 9. Do you find the extremals of the new problem qualitatively different from those of the one in Example 7?

Now, how about considering an even more different looking cost, but essentially still meaning the same purpose, namely minimizing the control effort? We can do that by taking the cost

$$\frac{1}{p} \int_0^1 u^p(t)\, dt \ . \tag{3.12}$$

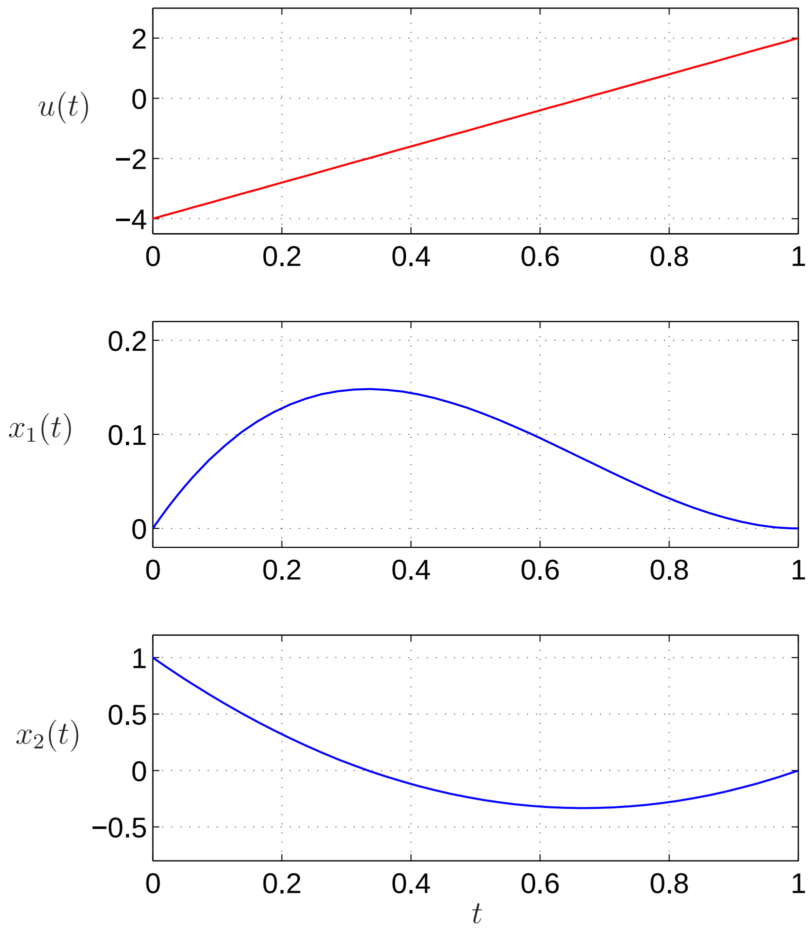


Figure 8: Trajectories for minimum-energy control of the car model — quadratic cost ($p = 2$ in (3.12)).

where $p \geq 2$ is a fixed even integer. Note that, in Example 7 and in this example, we have already treated the cases when $p = 2$ and $p = 4$. It is now natural to ask this question: What happens if we increase p? Figure 10 presents the solution for the case when $p = 18$. What do you observe? Are the extremal curves even more different compared to the previous cases with $p = 2, 4$?

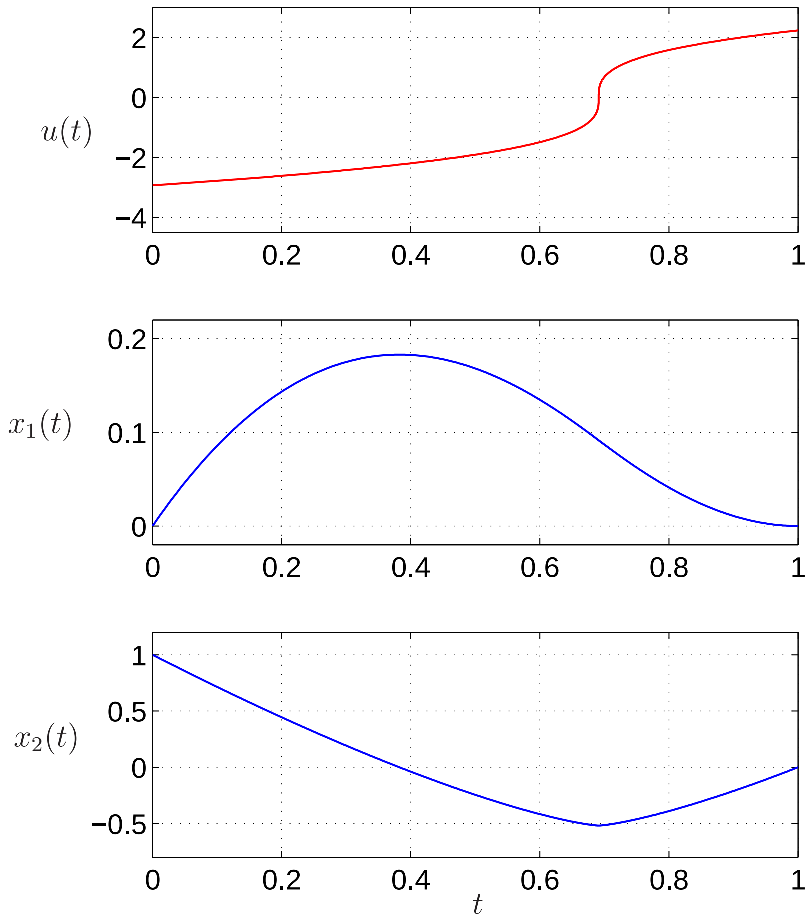


Figure 9: Trajectories for minimum-energy control of the car model — quartic cost ($p = 4$ in (3.12)).

3.2 OCPs with a terminal cost

Consider the OCP

$$\min_{x(\cdot),u(\cdot)} J(x,u) = \varphi(x(t_0), x(t_f)) + \int_{t_0}^{t_f} f_0(x(t), u(t), t)\, dt \tag{3.13}$$

$$\text{subject to} \qquad \dot{x}(t) = f(x(t), u(t), t)\ , \qquad \text{for all } t \in [t_0, t_f]\ . \tag{3.14}$$

In the above problem, $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable in its arguments. It is quite usual to include the cost term $\varphi(x(t_0), x(t_f))$, especially to represent cost on the terminal state; for example, one may like to minimize at the same time the magnitude of the state variables at the terminal time. Dependence of φ on the initial state $x(t_0)$ is considered for the sake of generality.

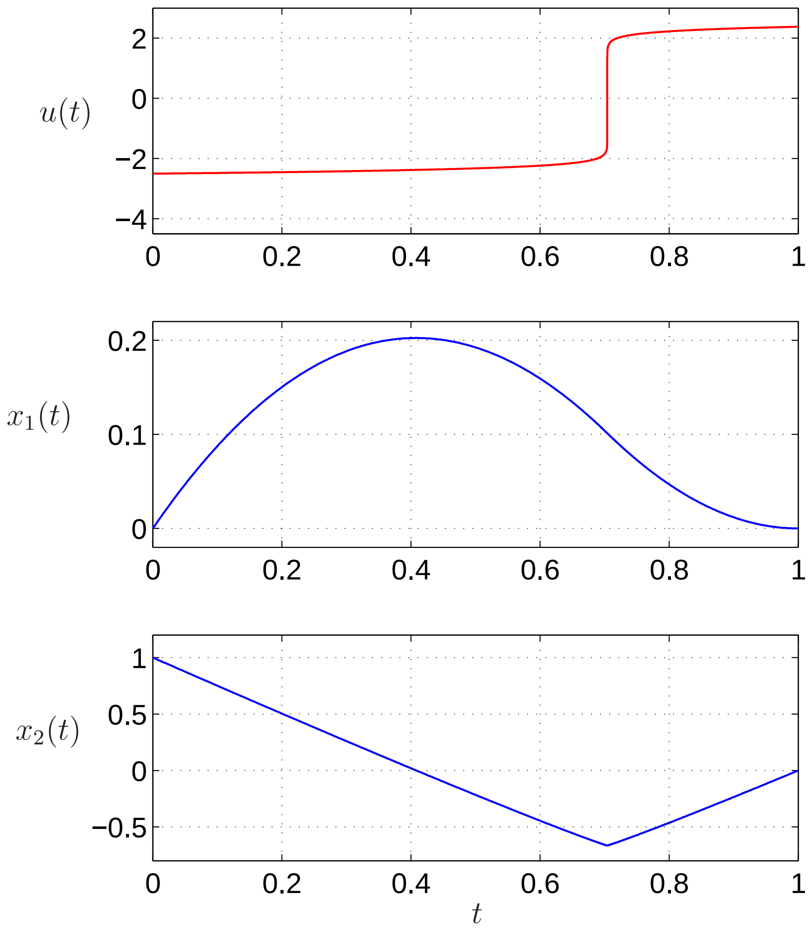


Figure 10: Trajectories for minimum-energy control of the car model — $p = 18$.

As we did in the previous section, first we adjoin the dynamic constraints to the cost and obtain an augmented cost:

$$\begin{aligned}\widetilde{J}(x, \psi, u) &= \varphi(x(t_0), x(t_f)) \\ &\quad + \int_{t_0}^{t_f} \left\{ f_0(x(t), u(t), t) + \psi^T(t) \, [f(x(t), u(t), t) - \dot{x}(t)] \right\} dt \end{aligned} \quad (3.15)$$

With $y := (x, \psi, u)$, $\widetilde{J}$ is in the same form as J in (2.10), for which the Euler-Lagrange equation holds along with the boundary conditions given in (2.11). Then, in this case, the necessary conditions (3.9)-(3.11) holds along with the boundary conditions,

$$\psi(t_0) = -\frac{\partial \varphi(x(t_0), x(t_f))}{\partial x(t_0)} \quad \text{and} \quad \psi(t_f) = \frac{\partial \varphi(x(t_0), x(t_f))}{\partial x(t_f)} \quad (3.16)$$

which have been evaluated using (3.9)-(3.11) with the integrand of (3.15). Note that the above boundary conditions are valid if both end-states $x(t_0)$ and $x(t_f)$ are free, i.e. variable.

A summary of all possible cases for the boundary conditions (for a cost term $\varphi(x(t_0), x(t_f))$) is as follows:

(i) $x(t_0) = x_0$ and $x(t_f) = x_f$ (both end-states $x(t_0)$ and $x(t_f)$ are specified),

(ii) $x(t_0) = x_0$ and $\psi(t_f) = \dfrac{\partial \varphi}{\partial x(t_0)}$ (only $x(t_0)$ is specified),

(iii) $\psi(t_0) = -\dfrac{\partial \varphi}{\partial x(t_f)}$ and $x(t_f) = x_f$ (only $x(t_f)$ is specified),

(iv) $\psi(t_0) = -\dfrac{\partial \varphi}{\partial x(t_0)}$ and $\psi(t_f) = \dfrac{\partial \varphi}{\partial x(t_f)}$ (neither $x(t_0)$ nor $x(t_f)$ is specified).

Fact 3.1 *If the time t does not appear explicitly in the Hamiltonian H, then H is constant along an extremal.*

Proof. We first write down the total derivative of H with respect to t.

$$\frac{dH}{dt} = H_t + H_x\,\dot{x} + H_\psi\,\dot{\psi} + H_u\,\dot{u}\ . \tag{3.17}$$

Recall the notation that $H_t = \partial H/\partial t$, etc. Because t does not appear explicitly in H, $H_t = 0$. Along an extremal (3.6)-(3.8) are satisfied; so $H_x = -\dot{\psi}$, $H_\psi = \dot{x}$ and $H_u = 0$. Substituting these into (3.17), one gets the required result, $dH/dt = 0$. □

Remark 1 *The fact that $H(x, \psi, u) =$ constant is sometimes used to obtain additional information. However it is often not needed for solving the problem.*

Exercise 5 Consider the scalar system

$$\dot{x} = u(t)\ , \qquad x(0) = 1\ ,$$

where $x(1)$ is free, and the objective is to minimize

(i) $J(u) = \displaystyle\int_0^1 \left[x^2(t) + u^2(t)\right]\, dt\ ,$

(ii) $J(u) = \displaystyle\int_0^1 u^2(t)\, dt\ .$

Find the extremals of these two OCPs.

Exercise 6 Consider the scalar system

$$\dot{x} = -x(t) + u(t) \ , \qquad x(0) = 1 \ ,$$

where $x(1)$ is free, and the objective is to minimize

$$J(u) = \int_0^1 \left[x^2(t) + u^2(t)\right] \, dt \ .$$

Find the extremals of the OCP.

Example 9 Consider the controlled van der Pol system

$$\begin{aligned} \dot{x}_1 &= x_2(t) \ , \\ \dot{x}_2 &= -x_1(t) - (x_1^2(t) - 1)\, x_2(t) + u(t) \ , \qquad x(0) = (x_{01}, x_{02}) \ . \end{aligned}$$

The aim is to minimize

$$J(u) = \frac{1}{2} \int_0^1 \left[x_1^2(t) + x_2^2(t) + u^2(t)\right] \, dt \ .$$

We obtain the TPBVP arising from the necessary conditions of optimality (3.9)-(3.11) and suitable boundary conditions, as follows.

First write down the Hamiltonian:

$$H(x, \psi, u) = \frac{1}{2}(x_1^2 + x_2^2 + u^2) + \psi_1 \, x_2 + \psi_2 \left[-x_1 - (x_1^2 - 1)\, x_2 + u\right] \ .$$

Optimal control has to satisfy

$$H_u = u + \psi_2 = 0, \quad \text{i.e.} \quad u = -\psi_2$$

The costate equations are given as Consider the controlled van der Pol system

$$\begin{aligned} \dot{\psi}_1 &= -H_{x_1} = -x_1 + \psi_2 \, (1 + 2\, x_1 \, x_2) \ , \\ \dot{\psi}_2 &= -H_{x_2} = -x_2 + \psi_1 + \psi_2 \, (x_1^2 - 1) \end{aligned}$$

The transversality conditions are furnished by (3.16)-(3.16):

$$\psi_1(1) = 0 \ , \qquad \psi_2(1) = 0 \ .$$

After the substitution of $u = -\psi_2$ in these equations we get the following TPBVP, which can only be solved by using a numerical technique and software.

$$\begin{aligned} \dot{x}_1 &= x_2(t) \ , & x_1(0) &= x_{01} \ , \\ \dot{x}_2 &= -x_1(t) - (x_1^2(t) - 1)\, x_2(t) + u(t) \ , & x_2(0) &= x_{02} \ , \\ \dot{\psi}_1 &= -x_1 + \psi_2 \, (1 + 2\, x_1 \, x_2) \ , & \psi_1(1) &= 0 \ , \\ \dot{\psi}_2 &= -x_2 + \psi_1 + \psi_2 \, (x_1^2 - 1) \ , & \psi_2(1) &= 0 \ . \end{aligned}$$

3.3 Standard forms of cost

There are three most common types of cost used in an OCP, namely

Bolza type: $\varphi(x(t_f)) + \int_{t_0}^{t_f} f_0(x,u,t)\,dt$

Mayer type: $\varphi(x(t_f))$

Lagrange type: $\int_{t_0}^{t_f} f_0(x,u,t)\,dt$

These forms can be transformed into one another as described below. (However, first note that the Mayer and Lagrange type costs are special cases of the Bolza type cost.)

Bolza or Lagrange type to Mayer type:
Introduce a new state variable x_{n+1} such that

$$x_{n+1}(t) = \int_{t_0}^{t} f_0(x(\tau),u(\tau),\tau)\,d\tau \ .$$

Then

$$\dot{x}_{n+1}(t) = f_0(x(t),u(t),t) \ , \quad \text{with } x_{n+1}(t_0) = 0 \ .$$

Next, the state vector is augmented as $\widetilde{x} = (x_1, \ldots, x_n, x_{n+1})$. The Bolza problem can now be rewritten as

$$\min_{u(\cdot)} \ \widetilde{\varphi}(\widetilde{x}(t_f))$$

$$\text{subject to } \dot{\widetilde{x}}(t) = \widetilde{f}(\widetilde{x}(t),u(t),t) \ , \quad \widetilde{x}(t_0) = 0 \ ,$$

where $\widetilde{\varphi}(\widetilde{x}(t_f)) = \varphi(x(t_f)) + x_{n+1}(t_f)$, $\dot{\widetilde{x}} = (\dot{x}, \dot{x}_{n+1})$ and $\widetilde{f}(x,u,t) = (f(x,u,t), f_0(x,u,t))$.

Bolza or Mayer type to Lagrange type:
Consider the terminal cost in either Bolza or Mayer type cost with fixed (i.e. specified)[4] $x(t_0)$:

$$\begin{aligned}
\varphi(x(t_f)) &= \varphi(x(t_f)) - \varphi(x(t_0)) - \varphi(x(t_0)) \\
&= \int_{t_0}^{t_f} \frac{d\varphi(x(t))}{dt}\,dt + \varphi(x(t_0)) \\
&= \int_{t_0}^{t_f} \varphi_x(x(t))\,\dot{x}(t)\,dt + \varphi(x(t_0)) \\
&= \int_{t_0}^{t_f} \varphi_x(x(t))\,f(x,u,t)\,dt + \varphi(x(t_0)) \ .
\end{aligned}$$

[4] In the case when $x(t_0)$ is not specified, i.e. when $x(t_0)$ is free, we can consider the more general $\varphi(x(t_0), x(t_f))$ instead of $\varphi(x(t_f))$ in the Bolza form.

So the Bolza problem can be written as

$$\min_{u(\cdot)} \int_{t_0}^{t_f} \left[f_0(x(t), u(t), t) + \varphi_x(x(t))\, f(x, u, t) \right] dt$$

$$\text{subject to } \dot{x}(t) = f(x(t), u(t), t) \ .$$

Recall that in the case of Mayer problem, $f_0(x, u, t) = 0$.

4 Linear Quadratic Regulator Problem

The linear quadratic regulator (LQR) problem, also referred to as the linear quadratic programming (LQP) problem, is given as

$$\min_{u(\cdot)} \ \frac{1}{2}\int_{t_0}^{t_f} \left[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right] \, dt$$
$$\text{subject to } \dot{x}(t) = A(t)\, x(t) + B(t)\, u(t) \ , \quad x(t_0) = x_0 \ ,$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, t_0 and t_f are fixed, $x(t_f)$ is free, $Q(t)$ an $n \times n$ symmetric positive semi-definite matrix, $R(t)$ an $m \times m$ symmetric positive definite matrix, for all $t \in [t_0, t_f]$. We note that the cost is quadratic in the state and control variables, where the matrices $Q(t)$ and $R(t)$ play the role of appropriate "weights".

In order to find the extremals for this OCP, we apply the theory developed in the previous section. First, we form the Hamiltonian as

$$H(x, \psi, u, t) = \frac{1}{2}\left(x^T Q x + u^T R u\right) + \psi^T(Ax + Bu)$$

where we haven't shown dependence on t of the dependent variables for brevity. Then the necessary conditions of optimality gives the costates equations as

$$\dot{\psi}^T = -H_x = -x^T Q - \psi^T A \ ;$$

in other words,

$$\dot{\psi} = -Qx - A^T\psi \ .$$

The extremal control vector is given by

$$H_u = Ru + B^T\psi = 0 \ ,$$

which can be solved for u as

$$u = -R^{-1}B^T\psi \ ,$$

because R is nonsingular. Then the extremal state vector satisfies

$$\dot{x} = Ax - BR^{-1}B^T\psi \ .$$

The state-costate equations are combined in matrix form as follows.

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\psi}(t) \end{bmatrix} = \begin{bmatrix} A(t) & -B(t)R^{-1}(t)B^T(t) \\ -Q(t) & -A^T(t) \end{bmatrix} \begin{bmatrix} x(t) \\ \psi(t) \end{bmatrix}$$

The solution to this system of linear differential equations is of the form:

$$\begin{bmatrix} x(t_f) \\ \psi(t_f) \end{bmatrix} = \Phi(t_f, t) \begin{bmatrix} x(t) \\ \psi(t) \end{bmatrix} \ , \tag{4.1}$$

where $\Phi(t_f, t)$ is the state transition matrix. Note that $\Phi(t_f, t_f) = I$. Partition the matrix $\Phi(t_f, t)$ such that

$$\Phi(t_f, t) := \begin{bmatrix} \phi_{11}(t_f, t) & \phi_{12}(t_f, t) \\ \phi_{21}(t_f, t) & \phi_{22}(t_f, t) \end{bmatrix} ,$$

where $\phi_{ij}(t_f, t)$, $i, j = 1, 2$, are $n \times n$ matrices.

The transversality condition for the LQR problem is simply $\psi(t_f) = \varphi_{x(t_f)} = 0$. So (4.1) can be rewritten as

$$\begin{aligned} x(t_f) &= \phi_{11}(t_f, t)\, x(t) + \phi_{12}(t_f, t)\, \psi(t) \\ 0 &= \phi_{21}(t_f, t)\, x(t) + \phi_{22}(t_f, t)\, \psi(t) \end{aligned}$$

If $\phi_{22}(t_f, t)$ is nonsingular for all t in $[t_0, t_f]$, then one can solve the second equation above for $\psi(t)$, namely one gets

$$\psi(t) = -\phi_{22}^{-1}(t_f, t)\, \phi_{21}(t_f, t)\, x(t) .$$

We note that, because $\Phi(t_f, t_f) = I$, $\phi_{22}(t_f, t_f) = I$, so $\phi_{22}^{-1}(t_f, t_f)$ exists. It can also be shown that $\phi_{22}^{-1}(t_f, t)$ exists for other t, although this fact is not so easy to prove [cf. Athans and Falb].

This establishes a linear relationship between $\psi(t)$ and $x(t)$:

$$\psi(t) = K(t)\, x(t) , \tag{4.2}$$

where[5] $K(t) = -\phi_{22}^{-1}(t_f, t)\, \phi_{21}(t_f, t)$. Then the optimal control is required to be a linear (state) feedback law:

$$u(t) = -R^{-1} B^T K(t)\, x(t) ,$$

Next, we device a differential equation to compute $K(t)$ as follows. Differentiate both sides of (4.2) and proceed by making the appropriate substitutions to get

$$\begin{aligned} \dot{\psi} &= \dot{K}\, x + K\, \dot{x} \\ &= \dot{K}\, x + K\, (Ax + Bu) \\ &= \dot{K}\, x + K\, A\, x - K\, B\, R^{-1}\, B^T\, K\, x . \end{aligned} \tag{4.3}$$

We also have

$$\begin{aligned} \dot{\psi} &= -Q\, x - A^T\, \psi \\ &= -Q\, x - A^T\, K\, x . \end{aligned} \tag{4.4}$$

Equating the right-hand sides of (4.3) and (4.4), and rearranging, we get

$$(\dot{K} + KA + A^T K - KBR^{-1}B^T K + Q)\, x = \mathbf{0} .$$

[5] After R. E. Kalman.

The above equation must be true for all $t \in [t_0, t_f]$, and for all $x(t)$. Therefore we must have

$$\dot{K} + KA + A^T K - KBR^{-1}B^T K + Q = 0 \tag{4.5}$$

with $K(t_f) = -\phi_{22}^{-1}(t_f, t_f)\,\phi_{21}(t_f, t_f) = 0$. Note that 0 in (4.5) is the $n \times n$ zero matrix. Equation (4.5) is called the (*matrix*) *Riccati equation*, and the $n \times n$ matrix K is referred to as the Riccati matrix.

Exercise 7 *Show that $K(t)$ is a symmetric matrix.*

Because $K(t)$ is symmetric, in the matrix Riccati equation $n(n+1)/2$ first-order differential equations must be solved, but not n^2.

The LQR problem we discussed above can be stated in a slightly more general form, by incorporating a terminal cost:

$$\min_{u(\cdot)} \frac{1}{2}(x^T(t_f)Sx(t_f)) + \frac{1}{2}\int_{t_0}^{t_f} \left[x^T(t)Q(t)x(t) + u^T(t)R(t)u(t)\right]\, dt$$

$$\text{subject to } \dot{x}(t) = A(t)\,x(t) + B(t)\,u(t)\ , \quad x(t_0) = x_0\ ,$$

where S is a positive semi-definite matrix. Then the optimal control for this new case is calculated to be the same as in the case we have outlined above. Furthermore the matrix Riccati equation (4.5) can also be shown to hold in this case, but with the boundary condition $K(t_f) = S$.

Example 10 Consider the control system

$$\dot{x}(t) = -x(t) + u(t)\ ,$$

with the objective of minimizing

$$J(u) = \int_0^T \left[x^2(t) + u^2(t)\right]\, dt\ ,$$

where $T > 0$ is fixed. Find the optimal feedback control $u(t) = -E(t)x(t)$, where $E(t)$ is an $m \times n$ matrix.

In this problem we have

$$A = -1\ , \quad B = 1\ , \quad Q = 1\ , \quad R = 1\ .$$

So the optimal (state) feedback control is given by $u = -K\,x$. The Riccati equation becomes (with the scalar values above)

$$\dot{K} - 2K - K^2 + 1 = 0\ , \quad K(T) = 0\ ,$$

which can be solved by separation of variables:

$$\begin{aligned}
&\frac{dK}{dt} = K^2 + 2K - 1 \\
&\int \frac{dK}{K^2+2K-1} = \int dt \; ; \qquad K^2 + 2K - 1 = (K+1-\sqrt{2})(K+1+\sqrt{2}) \\
&\frac{1}{2\sqrt{2}} \int \left(\frac{1}{K+1-\sqrt{2}} - \frac{1}{K+1+\sqrt{2}} \right) dK = t + C_1 \; , \quad C_1 \text{ a constant,} \\
&\ln \left| \frac{K+1-\sqrt{2}}{K+1+\sqrt{2}} \right| = 2\sqrt{2}\, t + C \; , \quad C \text{ a constant,} \\
&\frac{K+1-\sqrt{2}}{K+1+\sqrt{2}} = De^{2\sqrt{2}t} \; , \quad D \text{ a constant.}
\end{aligned}$$

For simplicity in appearance, let $S := K + 1$. Then

$$\begin{aligned}
\frac{S-\sqrt{2}}{S+\sqrt{2}} &= De^{2\sqrt{2}t} \; , \\
S - \sqrt{2} &= De^{2\sqrt{2}t}(S+\sqrt{2}) \; , \\
S &= \sqrt{2}\, \frac{1+De^{2\sqrt{2}t}}{1-De^{2\sqrt{2}t}}
\end{aligned}$$

Next, solve for the integration constant D using the boundary condition for K:

$$S(T) = K(T) + 1 = 1 = \sqrt{2}\, \frac{1+De^{2\sqrt{2}T}}{1-De^{2\sqrt{2}T}} \; ,$$

$$D = -\frac{\sqrt{2}-1}{\sqrt{2}+1}\, e^{-2\sqrt{2}T} \; .$$

Now

$$S(t) = \sqrt{2}\; \frac{1 - \dfrac{\sqrt{2}-1}{\sqrt{2}+1}\, e^{2\sqrt{2}(t-T)}}{1 + \dfrac{\sqrt{2}-1}{\sqrt{2}+1}\, e^{2\sqrt{2}(t-T)}}$$

and so

$$K(t) = -1 + \sqrt{2}\, \frac{\sqrt{2}+1-(\sqrt{2}-1)e^{2\sqrt{2}(t-T)}}{\sqrt{2}+1+(\sqrt{2}-1)e^{2\sqrt{2}(t-T)}} \; ,$$

which is substituted in the optimal state feedback control expression, $u(t) = -K(t)u(t)$.

4.1 The LQR problem over an infinite time horizon

Stabilization of dynamical systems by means of controls is often a process defined over an infinite time horizon. Here, we consider a version of the LQR problem, where the time horizon is $[0, \infty)$, the cost integrand and dynamics are time-invariant, namely $A(t)$, $B(t)$, $Q(t)$ and $R(t)$ are constant matrices.

$$\min_{u(\cdot)} \frac{1}{2} \int_0^\infty \left[x^T(t) Q x(t) + u^T(t) R u(t) \right] dt$$
$$\text{subject to } \dot{x}(t) = A\,x(t) + B\,u(t) \ , \quad x(t_0) = x_0 \ .$$

As we posed earlier, the matrix R is positive definite. In this case, we further pose that the matrix Q is positive definite, and the system $\dot{x} = A\,x + B\,u$ is controllable[6].

It can be shown that (albeit not so easily), under the controllability assumption, the limiting value of $K(t)$, $\lim_{t\to\infty} K(t)$, exists, is unique, and is positive definite. Denote $\lim_{t\to\infty} K(t) = K$, K a constant positive definite matrix. An additional argument justifies that $K(t) = K$ for all finite $t \in [0, \infty)$, and so $\dot{K}(t) = 0$. Therefore it suffices to solve (what we call) the *algebraic Riccati equation* for finding K:

$$KA + A^T K - KBR^{-1}B^T K + Q = 0 \ .$$

Example 11 Consider the control system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \ , \\ \dot{x}_2(t) &= u(t) \ , \end{aligned}$$

with the objective of minimizing

$$J(u) = \int_0^T \left[x_1^2(t) + x_2^2(t) + u^2(t) \right] dt \ .$$

With this problem, we have

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} , \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} , \qquad R = 1 \ .$$

Let

$$K = \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix} \ .$$

Then the algebraic Riccati equation is rewritten as

$$\begin{bmatrix} 0 & k_1 \\ 0 & k_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} [0 \ \ 1] \begin{bmatrix} k_1 & k_2 \\ k_2 & k_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

[6] The system $\dot{x} = A\,x + B\,u$ is said to be *controllable* if it can be driven from any nonzero initial state to the origin (the zero vector) in any finite time period. The system $\dot{x} = A\,x + B\,u$ is controllable if and only if the matrix $[B \ AB \ A^2B \ \cdots \ A^{n-1}B]$ has n linearly independent columns (see Rugh, "Linear System Theory").

or

$$\begin{bmatrix} 0 & k_1 \\ k_1 & 2k_2 \end{bmatrix} - \begin{bmatrix} k_2^2 & k_2k_3 \\ k_2k_3 & k_3^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

which yields the equations

$$\begin{aligned} -k_2 + 1 &= 0 \\ k_1 - k_2k_3 &= 0 \\ 2k_2 - k_3^2 + 1 &= 0 \ . \end{aligned}$$

The first equation gives $k_2 = \pm 1$. For $k_2 = -1$, solution does not exist for k_1 and k_3. With $k_2 = 1$, one gets (check this!) two solutions for K:

$$K_1 = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix} \qquad \text{and} \qquad K_2 = \begin{bmatrix} -\sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{bmatrix} \ .$$

Of these solutions, only K_1 is positive definite, and should be chosen as the valid Riccati matrix. Note that this choice of K_1 leads to an asymptotically stable closed-loop system. The closed-loop system matrix

$$A - BR^{-1}B^TK_1 = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{3} \end{bmatrix}$$

has eigenvalues $(-\sqrt{3} \pm i)/2$, which clearly have negative real parts.

Exercise 8 *Consider the problem of minimizing*

$$\int_0^\infty \left[2x_1^2(t) + 4x_2^2(t) + 2x_1(t)x_2(t) + \frac{1}{2}u^2(t) \right] dt$$

subject to

$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = u(t)$$

1. *Find the optimal feedback control using the linear quadratic regulator theory.*
2. *Using* MATLAB *Control System Toolbox function* `lqr` *verify the solution you have found in part (a).*

5 Pontryagin Maximum Principle

The Calculus of Variations (COV) techniques for OCPs we discussed earlier allowed the extremal controls to be unbounded and continuously differentiable. These controls can be extended to the case where they belong to the set of all (unbounded) piecewise continuous functions from $[t_0, t_f]$ to $\mathbb{R}$. This case gives rise to the so-called *broken extremals* for the states and costates. This more general class of controls are allowed to have discontinuities at finitely many points with finite right- and left-hand limits at the points of discontinuity.

On the other hand, most processes involve a cost criteria to minimize, such as

$$\int_{t_0}^{t_f} |u(t)|\, dt \ , \tag{5.1}$$

where f_0 is not differentiable in u.

Constraints are very important in real-life applications, for example recources one can allocate for control at any given time are restricted, namely one typically has

$$|u(t)| < M \ , \qquad M \text{ a positive constant,} \tag{5.2}$$

which cannot be handled readily with the COV methods. New theory that can tackle the situations (5.1) and (5.2) is provided by the Pontryagin Maximum Principle (PMP). Before we proceed with PMP, let us pose the following assumptions.

- Given $\Omega \subset \mathbb{R}^m$, we consider the set $\mathcal{U}$ of all *bounded* piecewise continuous functions u on $[t_0, t_f]$, such that

  $$u(t) \in \Omega \qquad \text{for all } t \in [t_0, t_f]$$

  with finite right- and left-hand limits at the points of discontinuity.

- $f_0(x, u, t)$, $\partial f_0/\partial x(x, u, t)$, $\partial f_0/\partial t(x, u, t)$, and $f(x, u, t)$, $\partial f/\partial x(x, u, t)$, $\partial f/\partial t(x, u, t)$ are continuous in x, u, t on $\mathbb{R}^n$, $\overline{\Omega}$, (t_0, t_f).
  Note that f_0, f do not necessarily have continuous partial derivatives with respect to the control u.

- The terminal cost function φ is contiuously differentiable in its arguments.

Pontryagin Maximum Principle
In order that $u \in \mathcal{U}$ be optimal, it is necessary that there exists a *nontrivial* function ψ such that for *almost every* $t \in [t_0, t_f]$,

- $\dot{x}^T = H_\psi = f^T(x, u, t)$,

- $\dot{\psi}^T = -H_x$,

- $H(x(t), \psi(t), u(t), t) = \min_{v(t)\in\Omega} H(x(t), \psi(t), v(t), t)$
  (or, equivalently, $H(x(t), \psi(t), u(t), t) \geq H(x(t), \psi(t), v(t), t)$ for every $v(t) \in \Omega$.)
- $H(x(t_f), \psi(t_f), u(t_f), t_f) = 0$.

Transversality conditions are written as in the case we used COV for the unbounded OCP.

Recall that when f_0 and f do not depend on t explicitly, i.e. when H does not depend on t explicitly, $H(x(t), \psi(t), u(t))$ is constant along extremal trajectories. In this case the final condition of the PMP above becomes

$$H(x(t), \psi(t), u(t)) \equiv 0 \ .$$

In the case when u is unconstrained, the set Ω above can be thought of as being arbitrarily large so that optimal control is contained in the interior of Ω. Then for u to minimize H, it is necessary (but not sufficient) that

$$H_u(x, \psi, u, t) = 0 \ . \tag{5.3}$$

If (5.3) holds and the matrix

$$H_{uu}(x, \psi, u, t)$$

is positive definite, this is sufficient for H to be a local minimum. If H is quadratic in u, then positive definiteness of Huu is a sufficient condition for H to be a global minimum. Namely consider

$$H(x, \psi, u, t) = g(x, \psi, t) + h^T(x, \psi, t)\, u + \frac{1}{2} u^T R u \ .$$

For this H, $Huu = R$. If R is positive definite, then

$$u = -R^{-1} h(x, \psi, t)$$

minimizes the Hamiltonian.

Example 12

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 + u, \qquad x(t_0) = x_0 \ . \end{aligned}$$

The aim is to minimize

$$\frac{1}{2} \int_{t_0}^{t_f} \left(x_1^2 + u^2\right) dt$$

where t_f is specified and $x(t_f)$ is free.

First, form the Hamiltonian:

$$H(x, \psi, u) = \frac{1}{2}\left(x_1^2 + u^2\right) + \psi_1 x_2 + \psi_2(-x_2 + u) \ .$$

Then the costate equations are written as

$$\begin{aligned}\dot{\psi}_1 &= -H_{x_1} = -x_1 \\ \dot{\psi}_2 &= -H_{x_2} = -\psi_1 + \psi_2\end{aligned}$$

The transversality condition is simply $\psi(t_f) = \partial\varphi/\partial x(t_f) = 0$.

Case 1: Unconstrained control

It is necessary that

$$H_u = u + \psi_2 = 0 \implies u = -\psi_2 \ .$$

Note that $H_{uu} = 1 > 0$. So $u = -\psi_2$ does minimize the Hamiltonian. The resulting TPBVP is

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 - \psi_2, \\ \dot{\psi}_1 &= -x_1 \\ \dot{\psi}_2 &= -\psi_1 + \psi_2\end{aligned}$$

or, in matrix form,

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2 \\ \dot{\psi}_1 \\ \dot{\psi}_2\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ \psi_1 \\ \psi_2\end{bmatrix} \ , \quad x(t_0) = x_0 \ , \ \psi(t_f) = 0 \ .$$

Case 2: The control is constrained as

$$-1 \le u(t) \le 1 \quad \text{for all } t \in [t_0, t_f] \ .$$

Select u to minimize $H(x, \psi, u)$:

$$\min_{u\in[-1,1]} H(x,\psi,u) = \min_{u\in[-1,1]} \frac{1}{2}u^2 + \psi_2 u \ .$$

When $-1 < u < 1$,

$$H_u = u + \psi_2 = 0 \implies u = \psi_2 \ .$$

In other words, $u(t) = \psi_2(t)$, whenever $|\psi_2(t)| < 1$. On the other hand, if $\psi_2(t) \ge 1$, then $u(t) = -1$, and if $\psi_2(t) \le -1$, then $u(t) = 1$. In summary,

$$u(t) = \begin{cases} -1, & \text{if } \psi_2(t) \ge 1 \ , \\ -\psi_2(t), & \text{if } |\psi_2(t)| < 1 \ , \\ 1, & \text{if } \psi_2(t) \le -1 \ . \end{cases}$$

Also see the graphical descriptions in the Week 9 Board notes for finding the control minimizing the Hamiltonian.

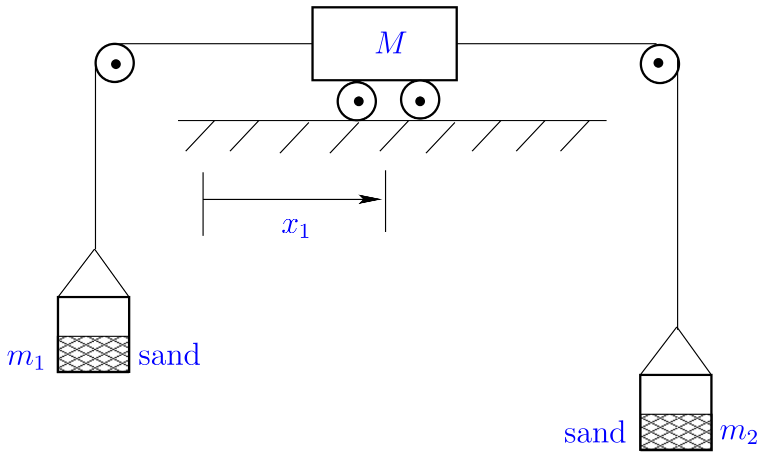


Figure 11: Trolley driven by buckets of sand

Example 13 A trolley of mass M is attached to two buckets containing sand of mass m_1 and m_2, respectively, as shown in Figure 11. When the difference $m_1 - m_2$ is nonzero, the trolley is moved in the direction of the heavier bucket. Let x_1 denote the position of the trolley, and x_2 the velocity. Then the dynamics of the trolley are given by the equations

$$\begin{aligned}\dot{x}_1 &= x_2 \ , \\ M\dot{x}_2 &= (m_1 - m_2)g \ .\end{aligned}$$

The masses m_1 and m_2 can be changed by adding more sand into the buckets, so that

$$\begin{aligned}\dot{m}_1 &= u_1 \ , \\ \dot{m}_2 &= u_2 \ ,\end{aligned}$$

where $0 < u_1 < u_{\max}$ and $0 < u_2 < u_{\max}$. Now define new variables

$$\begin{aligned}x_3 &:= \frac{(m_1 - m_2)g}{M} \ , \\ u &:= \frac{(u_1 - u_2)g}{M} \ .\end{aligned}$$

Then we get the state equations

$$\begin{aligned}\dot{x}_1 &= x_2 \ , \\ \dot{x}_2 &= x_3 \ , \\ \dot{x}_3 &= u \ ,\end{aligned}$$

and $-h \leq u(t) \leq h$, where $h = u_{\max}g/M$. Suppose that the process starts with $x_1(0) = x_{10}$, $x_2(0) = 0$, $x_3(0) = 0$ (i.e. $m_1 = m_2$). The target is $x_1(t_1) = x_2(t_1) = 0$.

The aim is to reach the target quickly, but at the same time use a small amount of sand in the controls (which is effectively given in terms of the difference of the masses of sand). So we define the cost by

$$\int_0^{t_1} (k + |u(t)|)\, dt \ , \quad k \geq 0 \ .$$

The Hamiltonian is formed as

$$H(x, \psi, u) = k + |u| + \psi_1 x_2 + \psi_2 x_3 + \psi_3 u$$

The costate equations are obtained as

$$\begin{aligned} \dot{\psi}_1 &= -H_{x_1} = 0 \implies \psi_1(t) = a \\ \dot{\psi}_2 &= -H_{x_2} = -\psi_1 \implies \psi_2(t) = b - at \\ \dot{\psi}_3 &= -H_{x_3} = -\psi_2 \implies \psi_3(t) = c - bt + \frac{1}{2}at^2 \end{aligned} \tag{5.4}$$

where a, b, c are constants. The optimal control satisfies

$$\min_{u \in [-h,h]} H = \min_{u \in [-h,h]} |u| + \psi_3 u$$

The control minimizing the Hamiltonian is found as

$$u(t) = \begin{cases} h, & \text{for } \psi_3(t) \leq -1 \ , \\ 0, & \text{for } |\psi_3(t)| < 1 \ , \\ -h, & \text{for } \psi_2(t) \geq 1 \ . \end{cases}$$

For a graphical illustration of how one can find the minimum of the Hamiltonian when $\psi_3(t) < -1$, see Week 10 board notes on the course web site. Grahical illustration for the other cases can similarly be obtained.

The optimal control rule above dictates switchings (if at all) from one of the three fixed values of u, namely one of h, 0 and $-h$, to another. Recall from (5.4) that $\psi_3(t)$ is quadratic in t. This makes it possible to deduce the switching patterns for optimal control. Typical patterns are illustrated in Figure 12.

From the paterns given in Figure 12, one concludes that the optimal solution must form *part* or *the whole* of the following sequences:

$$\{h, 0, h\} \ , \quad \{-h, 0, -h\} \ , \quad \{h, 0, -h, 0, h\} \ , \quad \{-h, 0, h, 0, -h\} \ .$$

The optimal control involves at most four switchings.

Example 14 A continuous-time controlled process has the differential equation

$$\dot{x}(t) = u(t)$$

The objective is to find a control $u(t)$, constrained as $0 \leq u(t) \leq 1$, that minimizes the cost functional

$$J(u) = \int_0^1 [x^2(t) + u^2(t)]\, dt$$

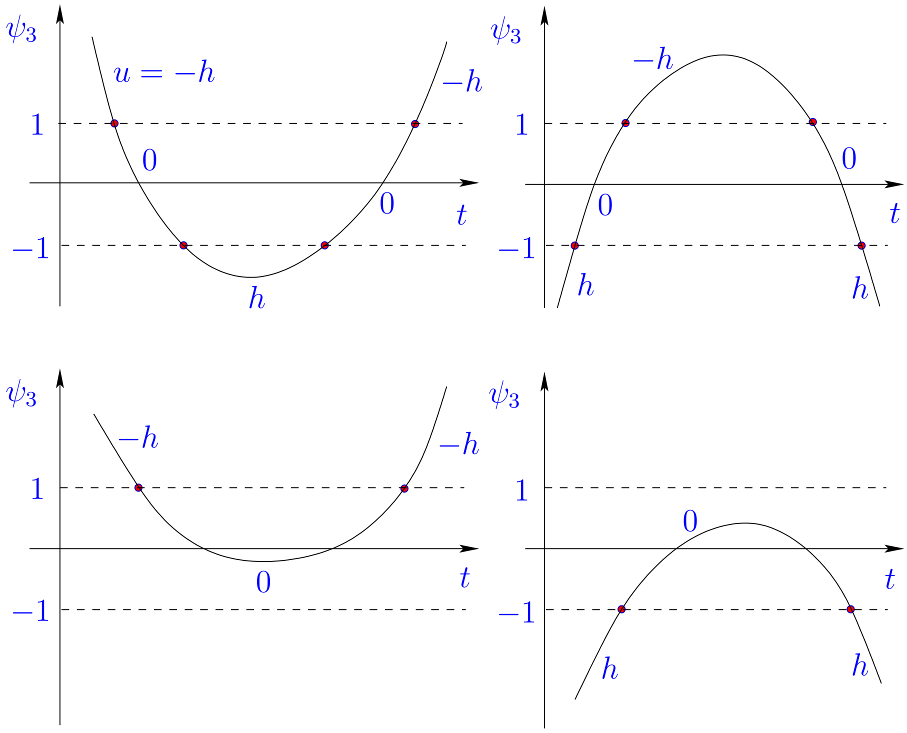


Figure 12: Typical patterns for the quadratic ψ_3. The values the control u takes along trajectory segments are indicated by $-h$, 0 and h.

(a) Apply the Pontryagin Maximum Principle to find the extremal control in terms of the costate.

(b) For each possible choice of the extremal control find the general solution for the state and costate as functions of time. [Hint: Note that no boundary conditions are given on the state, so that is why you are asked to find the "general" solution of the state and costate, leaving unknown coefficients as arbitrary constants.]

(a) Form the Hamiltonian

$$H(x, \psi, u) = x^2 + u^2 + \psi\, u$$

$$\min_u H(x, \psi, u) = \min_u (u^2 + \psi\, u)\ .$$

It is not difficult to show that (by graphing the quadratic $(u^2 + \psi\, u)$ vs u, for example, as we did in the previous example)

If $-\psi/2 \geq 1$, then $u = 1$.

If $-\psi/2 \leq 0$, then $u = 0$.

If $0 < -\psi/2 < 1$, then $u = -\psi/2$.

So the optimal control is required to be of the form

$$u(t) = \begin{cases} 1\ , & \text{if } \psi \le 2\ , \\ 0\ , & \text{if } \psi \ge 0\ , \\ -\psi/2\ , & \text{if } -2 < \psi < 0\ . \end{cases}$$

(b) The costate equation is simply found as

$$\dot{\psi} = -H_x = -2x\ .$$

In summary,

$$\begin{aligned} \dot{x}(t) &= u(t) \\ \dot{\psi}(t) &= -2x(t)\ , \end{aligned}$$

where $u(t)$ is determined as in part(a), giving three types of arcs associated with $u = 1$, $u = 0$ and $u = -\psi/2$. Below, we obtain the general solutions $x(t)$ and $\psi(t)$ for each of these arcs.

(i) $u = 1$:

$$\dot{x} = 1 \implies x(t) = t + c_1\ ,$$
$$\dot{\psi}(t) = -2t - 2c_1 \implies \psi(t) = t^2 - 2c_1 t + c_2\ ,$$

where c_1, c_2 are arbitrary constants.

(ii) $u = 0$:

$$\dot{x} = 0 \implies x(t) = c_1\ ,$$
$$\dot{\psi}(t) = -2c_1 \implies \psi(t) = -2c_1 t + c_2\ ,$$

where c_1, c_2 are arbitrary constants.

(iii) $u = -\psi/2$:

$$\dot{x} = -\psi/2 \implies \ddot{x} = -\dot{\psi}/2 = -(-2x)/2 = x\ .$$

So we get $\ddot{x} - x = 0$, whose general solution is simply obtained as

$$x(t) = c_1 e^{-t} + c_2 e^{t}\ ,$$

where c_1, c_2 are arbitrary constants. Then

$$\dot{\psi}(t) = -2(c_1 e^{-t} + c_2 e^{t}) \implies \psi(t) = 2(c_1 e^{-t} - c_2 e^{t})\ .$$

If initial and/or terminal conditions for the state are specified, then various combinations of these arcs are used to get an extremal solution satisfied the given conditions.

5.1 Bang–bang control

Example 15 Consider the minimum-time (or time-optimal) control of the car model (or double integrator):

$$\begin{aligned} & \min_{u(\cdot)} t_f = \int_0^{t_f} 1\,dt \\ \text{subject to} \quad & \dot{x}_1(t) = x_2(t) \\ & \dot{x}_2(t) = u(t)\ ,\ \text{with } |u(t)| \le 1\ . \end{aligned}$$

The initial and terminal states are specified, namely that $x_1(0) = x_{01}$, $x_2(0) = x_{02}$, $x_1(t_f) = x_{f1}$, $x_2(t_f) = x_{f2}$. The Hamiltonian is

$$H(x, \psi, u) = 1 + \psi_1 x_2 + \psi_2 u\ .$$

Now,

$$\min_{u\in[-1,1]} H = \min_{u\in[-1,1]} \psi_2 u\ .$$

H is minimized when

$$u(t) = \begin{cases} 1, & \text{for } \psi_2(t) < 0\ , \\ -1, & \text{for } \psi_2(t) > 0\ , \\ \text{undetermined}, & \text{for } \psi_2(t) = 0\ . \end{cases}$$

The third condition above is referred to as a *singular control problem* if $\psi_2(t) = 0$ on $[t_1, t_2]$, $0 \le t_1 < t_2 \le t_f$. If $\psi_2(t) = 0$ only at isolated time points, i.e. the problem is not singular, then the optimal control law can be written as

$$u(t) = \begin{cases} 1, & \text{for } \psi_2(t) \le 0\ , \\ -1, & \text{for } \psi_2(t) > 0\ . \end{cases}$$

In this case, the optimal control switches between its upper and lower bounds, which is said to be a *bang–bang* type control. Bang–bang control is encountered in particular in the time-optimal control of systems where the dynamics are desribed as linear in the control variable.

Costate equations are simply written as

$$\begin{aligned} \dot{\psi}_1 &= -H_{x_1} = 0 \implies \psi_1(t) = -a\ , \\ \dot{\psi}_2 &= -H_{x_2} = -\psi_1 = a \implies \psi_1(t) = at + b\ , \end{aligned}$$

where a and b are constants.

From Figure 13, one concludes that the optimal solution must form *part* or *the whole* of the following sequences:

$$\{1, -1\}\ , \quad \{-1, 1\}\ .$$

The optimal control involves at most one switching. The fact that singular control does not exist can be proved simply as follows. Suppose that $\psi_2(t) = 0$ on $[t_1, t_2]$,

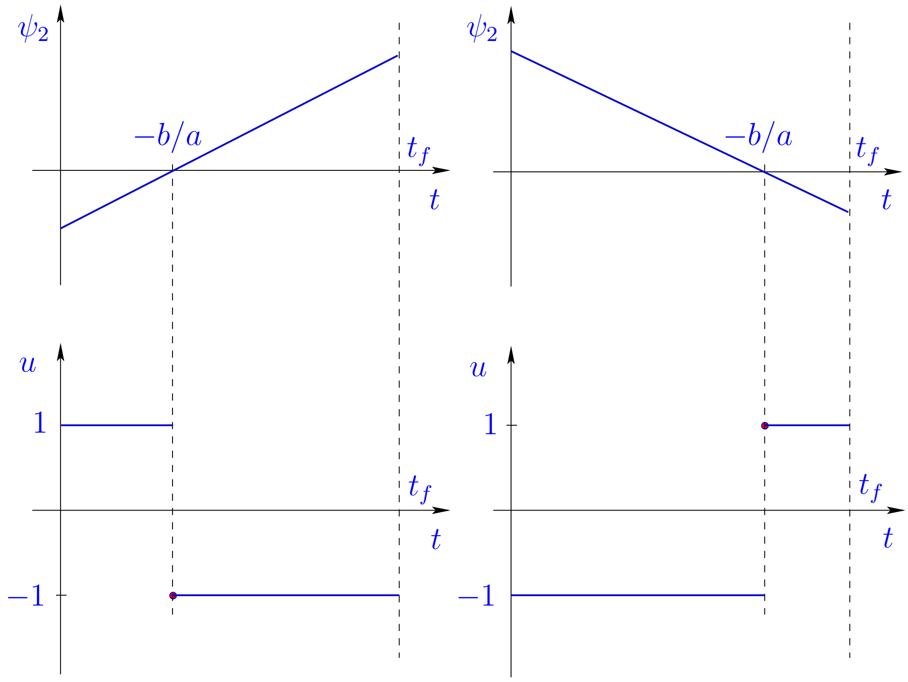


Figure 13: Possible switching structures for the time-optimal control of the car model.

$0 \leq t_1 < t_2 \leq t_f$. Then one must have $a = b = 0$ and that $\psi_1(t) = \psi_2(t) = 0$ on the whole interval $[0, t_f]$, which is a contradicition with the nontriviality of the costate vector $\psi(t) = (\psi_1(t),\ \psi_2(t))$.

Now let us obtain the solution of a *bang-arc* with $u = 1$:

$$\dot{x}_2 = 1 \implies x_2(t) = t + c$$

and so

$$x_1(t) = \frac{1}{2}t^2 + ct + d$$

where c and d are constants. The constants a, b, c and d can be determined from the initial and terminal conditions we have posed for the problem.

Let $t = x_2 - c$. Then one gets

$$x_1 = \frac{1}{2}(x_2 - c)^2 + c(x_2 - c) + d$$

which is a family of quadratic curves in the $x_1 x_2$ phase plane, shown in Figure 14, along with the solution curves for $u = -1$, which are obtained similarly. For another depiction of these phase-plane trajectories, also see Week 10 board notes.

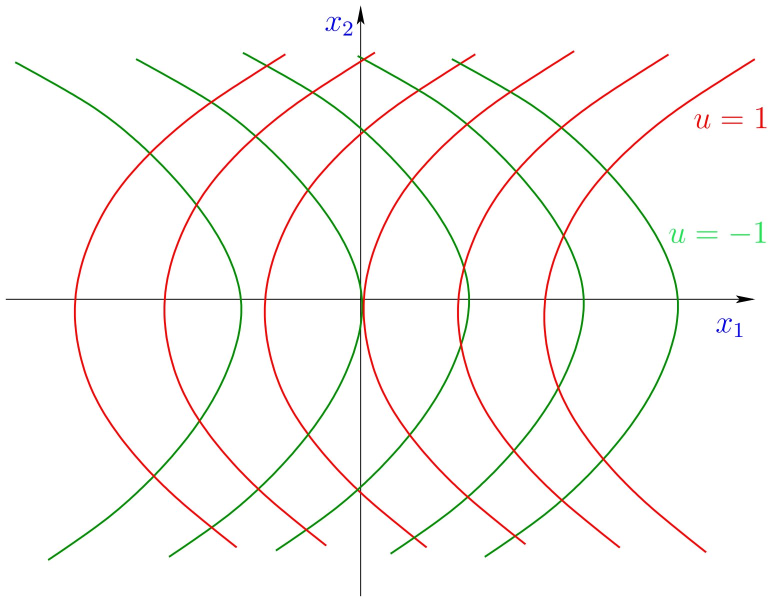


Figure 14: Possible switching structures for the time-optimal control of the car model.

Exercise 9 *Consider a firm which produces a single product and sells it in a market which can absorb no more than M dollars of the product per unit time. It is assumed that if the firm does no advertising, its rate of sales at any point in time will decrease at a rate proportional to the rate of sales at that time. If the firm advertises, the rate of sales will have an additional term that increases at a rate proportional to the rate of advertising, but this increase affects only those parts of the market that have not yet purchased the product.*

Let $x(t) = S(t)/M$ be the rate of sales per saturation rate at time t, $A(t)$ the rate of advertising, γ a positive constant, and $v(t) = A(t)/M$. Then $[1 - x(t)]$ is the part of the market that is influenced by advertising.

Under these assumptions and with a suitable scaling of the time variable, the change in $x(t)$ is given by

$$\dot{x}(t) = -x(t) + \gamma\, v(t)\, [1 - x(t)]\ . \tag{5.5}$$

Furthermore it is assumed that there is an upper bound on the advertising rate, namely that

$$0 \leq A(t) \leq \overline{A}$$

in other words,

$$0 \leq v(t) \leq a$$

where $\overline{A}$ is a positive constant and $a := \overline{A}/M$. The problem is to find an admissible $v(t)$ that will maximize the profit functional

$$J(v) = \int_0^{t_f} [x(t) - v(t)]\, dt$$

subject to the state equation (5.5) *with the initial condition*

$$x(0) = x_0$$

where the final time t_f *is fixed.*

(a) *Write the problem in the standard form as a minimization problem.*

(b) *Write down the Hamiltonian for this problem.*

(c) *Show that the optimal control* $v(t)$ *is of the form*

$$v(t) = \begin{cases} a & \text{if } \gamma\,[x(t)-1]\,\psi(t) > 1 \\ 0 & \text{if } \gamma\,[x(t)-1]\,\psi(t) < 1 \\ \text{undetermined} & \text{if } \gamma\,[x(t)-1]\,\psi(t) = 1 \quad \text{for all} \quad t \in [t_1, t_2] \end{cases}$$

where $t_0 \le t_1 < t_2 \le t_f$.

(d) *Write down the two-point boundary-value problem arising from the Pontryagin Maximum Principle.*

(e) *Show that the optimal advertising policy dictates no advertising near the final time* $t = t_f$.

(f) *Assume that the optimal control is of* bang–bang *type with one switching; namely*

$$v(t) = \begin{cases} a & \text{for } 0 \le t \le \tau \\ 0 & \text{for } \tau < t \le t_f \end{cases}$$

where τ *is the* switching time. *Under this assumption give an explicit solution for* $\psi(t)$ *in the interval* $[\tau, t_f]$.

(g) *Under the assumption stated in part (f) give an explicit solution for* $x(t)$ *in the interval* $[0, \tau]$.

(h) *Hence, or otherwise, show that* τ *is a solution of the equation*

$$\gamma\,\left(1 - e^{-(t_f-\tau)}\right)\left[1 - \left(x_0 - \frac{\gamma a}{1+\gamma a}\right)\, e^{-(1+\gamma a)\tau} - \frac{\gamma a}{1+\gamma a}\right] = 1 \ .$$

6 A Computational Method for Unconstrained OCPs

Consider the OCP

$$\min_{x(\cdot),u(\cdot)} \varphi(x(t_f)) + \int_{t_0}^{t_f} f_0(x(t),u(t),t)\,dt$$

$$\text{subject to} \qquad \dot{x}(t) = f(x(t),u(t),t)\ , \qquad x(t_0) = x_0\ ,$$

where φ is twice continuously differentiable in $x(t_f)$, and f and f_0 are twice continuously differentiable[7] in x and u. Suppose that the terminal state $x(t_f)$ is free. Let $g(x,\psi,u,t) := -H_x(x,\psi,u,t)$, where $H(x,\psi,u,t)$ is the Hamiltonian function. Then the necessary conditions of optimality for the given OCP are simply as follows.

$$\begin{aligned}
\dot{x} &= f(x,u,t)\ , \qquad x(t_0) = x_0\ ,\\
\dot{\psi} &= g(x,\psi,u,t)\ , \qquad \psi(t_f) = \frac{\partial \varphi(x(t_f))}{\partial x(t_f)}\ ,\\
0 &= H_u(x,\psi,u,t)\ . \qquad (6.1)
\end{aligned}$$

Note that g is continuously differentiable in x, ψ and u. Suppose that $H_{uu} > 0$. Then u satisfying (6.1) is a strict minimizer of H. Furthermore (6.1) can be solved for u. Suppose that $u = \phi(x,\psi,t)$. Then ϕ is continuously differentiable (because also H is). Let $h(x,\psi,t) := f(x,\phi(x,\psi,t),t)$ and $s(x,\psi,t) := g(x,\psi,\phi(x,\psi,t),t)$. We note that h and s are continuously differentiable in x and psi. The above conditions reduce to the following TPBVP.

$$\dot{x} = h(x,\psi,t)\ , \qquad x(t_0) = x_0\ , \qquad (6.2)$$

$$\dot{\psi} = s(x,\psi,t)\ , \qquad \psi(t_f) = \frac{\partial \varphi(x(t_f))}{\partial x(t_f)}\ . \qquad (6.3)$$

Simple shooting is a basic method used to solve this TPBVP. It starts with a guess ψ_0 for the unknown initial costate $\psi(t_0)$. With the initial conditions $x(t_0) = x_0$ and $\psi(t_0) = \psi_0$, the system of ODEs in (6.2)-(6.3) are solved to get $x(t_f)$ and $\psi(t_f)$. In general, $\psi(t_f)$ obtained this way will not be equal to $\partial\varphi(x(t_f))/\partial x(t_f)$, the difference amounting to a *discrepancy*. Then, typically, using variation of the discrepancy with respect to changes in $\psi(t_0)$, $\psi(t_0)$ is updated, and iteratively the required $\psi(t_0)$ is obtained within some required accuracy. Figure 15 depicts an instance of simple shooting.

Let the discrepancy be represented by a function $F : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$F(\psi_0) = \psi(t_f) - \frac{\partial \varphi(x(t_f))}{\partial x(t_f)}\ .$$

The function F is also referred to as the *near miss function*. The solution of the problem is obtained when one finds a ψ_0 satisfying

$$F(\psi_0) = 0\ . \qquad (6.4)$$

[7]We keep track of these differentiability conditions, which will be necessary for setting up the numerical procedure.

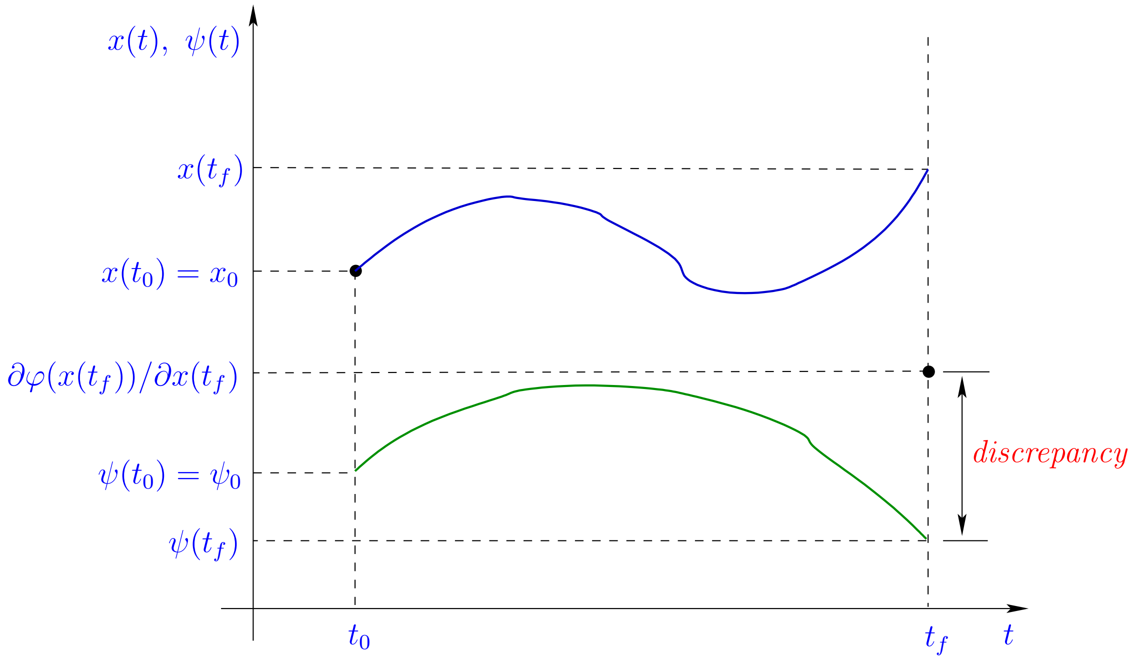


Figure 15: An instance of a shooting.

Because h and s are continuously differentiable, from basic theory of differential equations, the solution of the initial value problem (IVP) above is continuously differentiable in its initial values. Therefore F is continuously differentiable in ψ_0. Now, because $F_{\psi_0} := \partial F/\partial\psi_0$ exists, one can use Newton's method or some of its variants to solve (6.4). The Newton iterate in this case is[8]

$$\psi_0^{(k+1)} = \psi_0^{(k)} - \left[F_{\psi_0}(\psi_0^{(k)})\right]^{-1} F(\psi_0^{(k)}) \ .$$

where

$$F_{\psi_0} = \left[\frac{\partial F_1}{\partial \psi_0}^T \vdots \cdots \vdots \frac{\partial F_n}{\partial \psi_0}^T\right] \ .$$

Derivatives $\partial F_i/\partial\psi_0$, $i = 1, \ldots, n$, are usually only approximately computed using the difference quotient

$$\frac{\partial F_1}{\partial \psi_0} \approx \frac{F_i(\psi_0 + \varepsilon\, e_i) - F_i(\psi_0)}{\varepsilon}$$

where $\varepsilon > 0$ is small and $e_i = (0, \ldots, 1, \ldots, 0)$ is the standard unit vector whose all entries, except the ith entry, are zero. Note that $F_i(\psi_0 + \varepsilon\, e_i)$ is found by solving the IVP

$$\begin{aligned} \dot{x} &= h(x, \psi, t) \ , & x(t_0) &= x_0 \ , \\ \dot{\psi} &= s(x, \psi, t) \ , & \psi(t_0) &= \psi_0 + \varepsilon\, e_i \ . \end{aligned}$$

[8]Note that, in the scalar case, we have $\psi_0^{(k+1)} = \psi_0^{(k)} - F(\psi_0^{(k)})/F'(\psi_0^{(k)})$.

If ε is large, then the quotient approximation is poor, resulting in slow (or no) convergence; if ε too small, then even small errors in computing $F_i(\psi_0 + \varepsilon\, e_i)$ and $F_i(\psi_0)$ may strongly impair the approximation.

Example 16 In this example, we use a computer code (written in MATLAB) implementing simple shooting and numerically solve the OCP involving the van der Pol system given in Example 9. The code has been posted on the course Web site, under Assignment 2.

We suppose that $x(0) = (x_{01}, x_{02}) = (1, 1)$. Then the TPBVP arising from the optimality conditions is

$$\begin{aligned}
\dot{x}_1 &= x_2(t)\ , & x_1(0) &= 1\ ,\\
\dot{x}_2 &= -x_1(t) - (x_1^2(t) - 1)\, x_2(t) + u(t)\ , & x_2(0) &= 1\ ,\\
\dot{\psi}_1 &= -x_1 + \psi_2\, (1 + 2\, x_1\, x_2)\ , & \psi_1(1) &= 0\ ,\\
\dot{\psi}_2 &= -x_2 + \psi_1 + \psi_2\, (x_1^2 - 1)\ , & \psi_2(1) &= 0\ .
\end{aligned}$$

Iterations of the simple shooting as applied to the above TPBVP are listed in the following table, where $\|\cdot\|$ is the Euclidean norm.

k	$\|F(\psi_0^{(k)})\|$	$\psi_0^{(k)}$	cost
0	2.356115	(2.000000, 2.000000)	1.250398
1	0.604453	(0.482042, 0.551462)	0.901786
2	0.006260	(0.771221, 0.708192)	0.895553
3	0.000002	(0.774734, 0.708599)	0.895551
4	0.000000	(0.774734, 0.708601)	0.895551

The state and costate functions in each simple shooting iteration are depicted in Figure 16. It should be noted, simply, $u(t) = \psi_2(t)$.

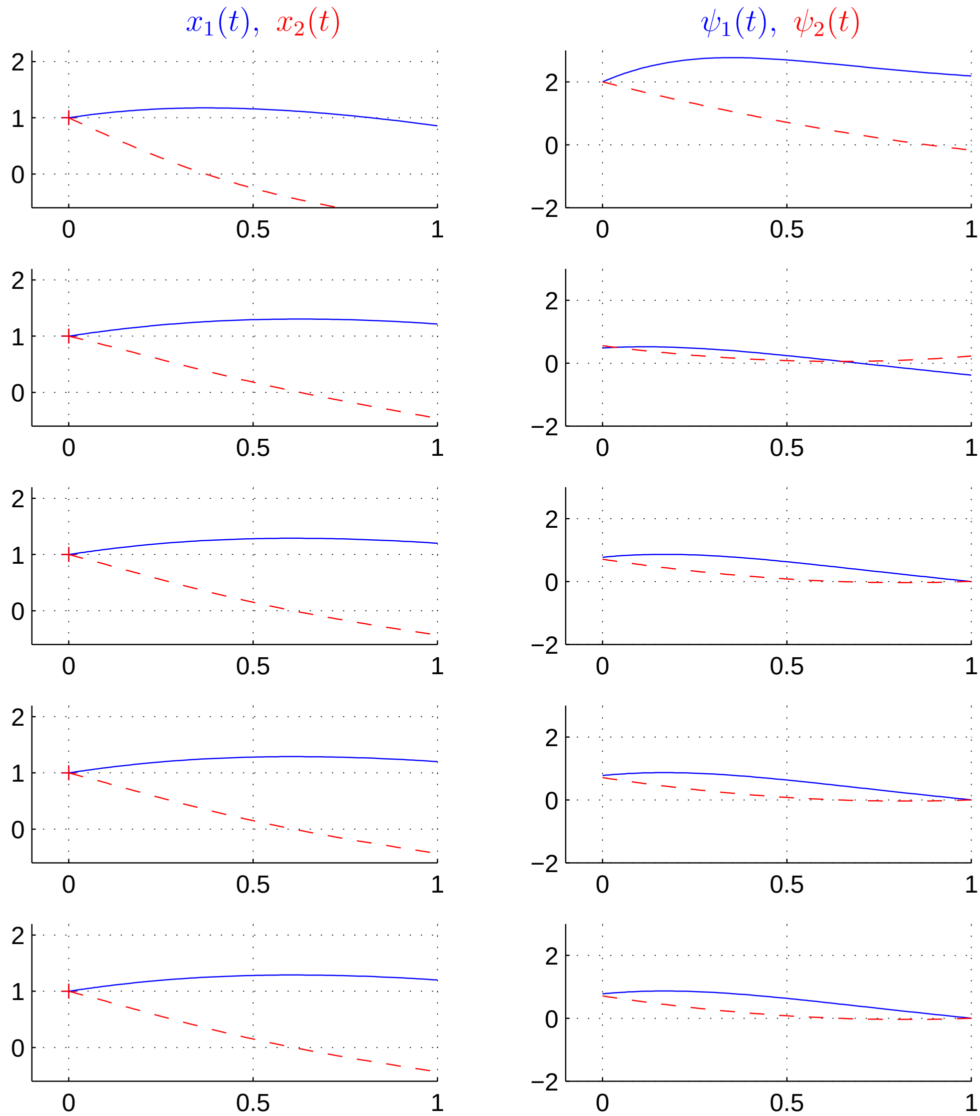
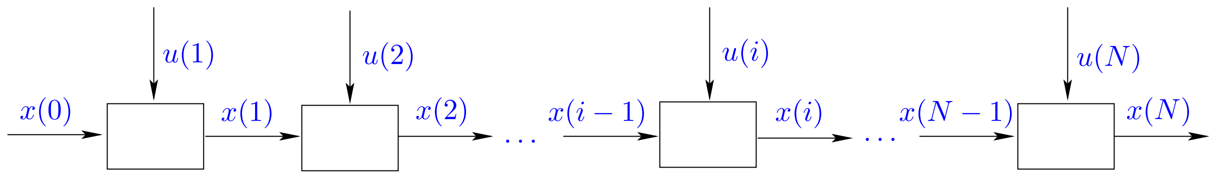


Figure 16: Simple shooting iterations for the OCP with van der Pol system. In the first column, the solid curves represent $x_1^{(k)}(t)$ and the dashed curves $x_2^{(k)}(t)$, $k = 0, 1, \ldots, 4$. In the second column, the solid curves represent $\psi_1^{(k)}(t)$ and the dashed curves $\psi_2^{(k)}(t)$.

7 Discrete-Time OCPs

We can schematically decribe a discrete-time system (also referred to as a multi-stage system) in terms of stages as shown in the figure below.



We assume that a discrete-time system is described by nonlinear difference equations

$$x(i) = f^i(x(i-1), u(i)), \qquad i = 1, 2, \dots, N ,$$

where $x(i) \in \mathbb{R}^n$, $u(i) \in \mathbb{R}^m$, $f^i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. It is assumed that $x(0)$ is known.

The problem is to find the sequence $u(1), u(2), \dots, u(N)$ which produces a sequence $x(1), x(2), \dots, x(N)$ from the above equations and together give a minimum value to the cost (or performance index)

$$J = \varphi(x(N)) + \sum_{i=1}^{N} f_0^i(x(i-1), u(i)) .$$

We adjoin the system equations to J with a sequence of adjoint vectors $\psi(i) \in \mathbb{R}^n$ to give

$$\widetilde{J} = \varphi(x(N)) + \sum_{i=1}^{N} \left[f_0^i(x(i-1), u(i)) + \psi^T(i) \left(f^i(x(i-1), u(i)) - x(i) \right) \right] .$$

Now define a sequence of Hamiltonians

$$H(i) = f_0^i(x(i-1), u(i)) + \psi^T(i)\, f^i(x(i-1), u(i)) ,$$

so that the augmented cost becomes

$$\begin{aligned} \widetilde{J} &= \varphi(x(N)) + \sum_{i=1}^{N} \left[H(i) - \psi^T(i)\, x(i) \right] \\ &= \varphi(x(N)) - \psi^T(N)\, x(N) + \sum_{i=1}^{N-1} H(i) - \sum_{i=1}^{N-1} \psi^T(i)\, x(i) \end{aligned}$$

Now make variations $\delta u(i)$ in $u(i)$ so that there will be consequential variations $\delta x(i)$

in $x(i)$. Then the corresponding variation in $\widetilde{J}$ (or differential of $\widetilde{J}$) may be written as

$$\begin{aligned} \delta\widetilde{J} = & \left[\frac{\partial\varphi(x(N)}{\partial x(N)} - \psi^T(N)\right]\delta x(N) \\ & + \sum_{i=1}^{N-1}\left[\frac{\partial H(i+1)}{\partial x(i)} - \psi^T(i)\right]\delta x(i) \\ & + \sum_{i=1}^{N}\frac{\partial H(i)}{\partial u(i)}\delta u(i) \end{aligned}$$

since $H(N)$ does not contain $x(N)$ explicitly.

Instead of determining the changes $\delta x(i)$ produced by a given sequence of $\delta u(i)$, we choose the adjoint sequence $\psi(i)$, as yet undetermined, such that

$$\begin{aligned} \psi^T(i) &= \frac{\partial H(i+1)}{\partial x(i)} , \qquad i = 1, 2, \ldots, N-1 , \\ \psi^T(N) &= \frac{\partial\varphi(x(N)}{\partial x(N)} . \end{aligned}$$

It follows from this choice that

$$\delta\widetilde{J} = \sum_{i=1}^{N}\frac{\partial H(i)}{\partial u(i)}\delta u(i) .$$

Assuming that each Hamiltonian $H(i)$, $i = 1, 2, \ldots, N$, is differentiable with respect to $x(i)$ and that the $u(i)$ are unconstrained, the differential $\delta\widetilde{J}$ must be zero for arbitrary $\delta u(i)$. For this to be so, it is necessary that

$$\frac{\partial H(i)}{\partial u(i)} = 0 \qquad i = 1, 2, \ldots, N .$$

Thus, to find a sequence $u(i)$ which yields a stationary value of the cost J we must solve the two-point boundary-value problem defined by the equations

$$\begin{aligned} x(i) &= f^i(x(i-1), u(i)), \qquad i = 1, 2, \ldots, N , \\ \psi^T(i) &= \frac{\partial H(i+1)}{\partial x(i)} , \qquad i = 1, 2, \ldots, N-1 , \\ \psi^T(N) &= \frac{\partial\varphi(x(N)}{\partial x(N)} , \\ \frac{\partial H(i)}{\partial u(i)} &= 0 . \end{aligned}$$

These amount to a total of $(2n+m)N$ equations in the $(2n+m)N$ unknown elements of the vectors $x(i)$, $u(i)$, $\psi(i)$, $i = 1, 2, \ldots, N$.

Example 17 Consider the discrete-time control system

$$
\begin{aligned}
x_1(i) &= x_2(i-1) \\
x_2(i) &= -x_1(i-1) + u(i) \ , \\
x_1(0) &= 1 \ , \quad x_2(0) = 1 \ .
\end{aligned}
$$

The aim os to minimize

$$
\frac{1}{2}x_1^2(N) + \frac{1}{2}\sum_{i=1}^{N}\left(x_1^2(i-1) + u^2(i)\right) \ .
$$

Form the sequence of Hamiltonians:

$$
H(i) = \frac{1}{2}\left[x_1^2(i-1) + u^2(i)\right] + \psi_1(i)\, x_2(i-1) + \psi_2(i)\ \left[-x_1(i-1) + u(i)\right] \ ,
$$

or

$$
H(i+1) = \frac{1}{2}\left[x_1^2(i) + u^2(i+1)\right] + \psi_1(i+1)\, x_2(i) + \psi_2(i+1)\ \left[-x_1(i) + u(i+1)\right] \ .
$$

So

$$
\begin{aligned}
\psi_1(i) &= \frac{\partial H(i+1)}{\partial x_1(i)} = x_1(i) - \psi_2(i+1) \ , \\
\psi_2(i) &= \frac{\partial H(i+1)}{\partial x_2(i)} = \psi_1(i+1) \ .
\end{aligned}
$$

The transversality conditions are

$$
\psi_1(N) = x_1(N) \ , \qquad \psi_2(N) = 0 \ .
$$

Furthermore

$$
\frac{\partial H(i)}{\partial u(i)} = u(i) + \psi_2(i) = 0 \implies u(i) = -\psi_2(i) \ .
$$

In summary, the necessary conditions of optimality are given by

$$
\text{For } i = 1, 2, \ldots, N \quad \begin{cases} x_1(i) = x_2(i-1) \ , \quad x_1(0) = 1 \ , \\ x_2(i) = -x_1(i-1) - \psi_2(i) \ , \quad x_2(0) = 1 \ . \end{cases}
$$

$$
\text{For } i = 1, 2, \ldots, N-1 \quad \begin{cases} \psi_1(i) = x_1(i) - \psi_2(i+1) \ , \quad \psi_1(N) = x_1(N) \ , \\ \psi_2(i) = \psi_1(i+1) \ , \quad \psi_2(N) = 0 \ . \end{cases}
$$

Exercise 10 *For $N = 2$, show that the necessary conditions derived in the above example yield*

$\{x_1(i)\} = \{1, 1, -1/2\}$, $i = 0, 1, 2$.

$\{x_2(i)\} = \{1, -1/2, -1\}$, $i = 0, 1, 2$.

$\{\psi_1(i)\} = \{1, -1/2\}$, $i = 1, 2$.

$\{\psi_2(i)\} = \{-1/2, 0\}$, $i = 1, 2$.

$\{u(i)\} = \{1/2, 0\}$, $i = 1, 2$.

Exercise 11

Consider minimizing

$$\frac{1}{2}x_1^2(N) + \frac{1}{2}\sum_{i=1}^{N}\left[x_1^2(i-1) + u^2(i)\right]$$

subject to the the discrete-time (or multi-stage) process

$$x_1(i) = x_2(i-1) \ ,$$
$$x_2(i) = u(i) \ , \quad x_1(0) = 1 \ , \quad x_2(0) = 1 \ .$$

(a) *Form the sequence of Hamiltonians,* $H(i)$.

(b) *Write down the necessary conditions of optimality. Summarize these conditions after eliminating* $u(i)$.

(c) *For* $N = 2$, *find, using part (b), the values of* $x_1(i)$, $x_2(i)$, $\psi_1(i)$, $\psi_2(i)$, *and* $u(i)$, $i = 1, 2$.

8 Singular Control

Consider the problem

$$\begin{aligned}
\min_{u(\cdot)} \quad & \varphi(x(t_f)) \\
\text{s. t.} \quad & \dot{x}(t) = f(x(t)) + u(t)\, g(x(t)) \ , \\
& x(t_0) = x_0 \ , \quad u(t) \in [-1, 1] \quad (i.e. \ |u(t)| \le 1) \ , \\
& x(t) \in \mathbb{R}^n \ , \quad t \in [0, t_f] \ .
\end{aligned}$$

Note that the right-hand side of the differential equations are linear in the control u. Ultimately, the Hamiltonian is linear in u:

$$H(x, \psi, u) = \psi^T \ (f(x) + u\, g(x)) \ ,$$

where $\psi(t) \in \mathbb{R}^n$. Recall the *switching function*[9]:

$$H_u(t) = \psi^T(t)\, g(x(t)) \ .$$

The Pontryagin Maximum Principle states that at a time instant $t \in [0, t_f]$,

$$\begin{aligned}
\psi^T(t)\, g(x(t)) > 0 &\implies u(t) = -1 \ ; \\
\psi^T(t)\, g(x(t)) < 0 &\implies u(t) = 1 \ .
\end{aligned}$$

If $\psi^T(t)\, g(x(t)) = 0$ for every t in some subinterval $[t', t'']$ of $[0, t_f]$, then the original problem is called a *singular control problem* and the corresponding trajectory for $[t', t'']$, a *singular arc*.

In particular, if $0 < t' < t'' < t_f$ (or if either $t' = 0$ or $t'' = t_f$, but not both), then the problem is referred to as *partially singular*. If however, $t' = 0$ and $t'' = t_f$, then the problem is said to be *totally singular*.

In singular case, the Pontryagin Maximum Principle yields no information on the extremal (or stationary) control, because $\psi^T f(x)$.

Example 18 Consider the OCP[10]

$$\begin{aligned}
\min_{u(\cdot)} \quad & x_2(2) \\
& \dot{x}_1 = u \ , \quad x_1(0) = 1 \ , \\
& \dot{x}_2 = \frac{1}{2} x_1^2 \ , \quad x_2(0) = 0 \ , \quad |u(t)| \le 1 \ .
\end{aligned}$$

[9] When introducing bang–bang control, we denoted the switching function by $\sigma(t)$.

[10] This is the celebrated *Goddard's problem* whose cost has been transformed into Mayer-type:

$$\begin{aligned}
\min_{u(\cdot)} \quad & \frac{1}{2} \int_0^2 x^2(t)\, dt \\
& \dot{x} = u \ , \quad x(0) = 1 \ , |u(t)| \le 1 \ .
\end{aligned}$$

The Hamiltonian:

$$H = \psi_1 \, u + \frac{1}{2}\psi_2 \, x_1^2$$

The costate equations:

$$\begin{aligned} \dot{\psi}_1 &= -H_{x_1} = -\psi_2 \, x_1 \ , \\ \dot{\psi}_2 &= -H_{x_2} = 0 \implies \psi_2(0) = C \ , \quad C \text{ a constant.} \end{aligned}$$

Transversality conditions are

$$\psi_1(2) = 0 \ , \qquad \psi_2(2) = 0 \ .$$

Therefore $C = 1$. Now

$$H = \psi_1 \, u + \frac{1}{2} \, x_1^2 \ ,$$

and

$$\begin{aligned} \dot{\psi}_1 &= -x_1 \ , \quad \psi_1(2) = 0 \ , \\ \dot{\psi}_2 &= 1 \ . \end{aligned}$$

The switching function is

$$H_u = \psi_1 \ .$$

Therefore

$$\begin{aligned} &\text{If } \psi_1(t) > 0 \ , \quad \text{then } u(t) = -1 \ ; \\ \textit{or} \quad &\text{if } \psi_1(t) < 0 \ , \quad \text{then } u(t) = 1 \ . \end{aligned}$$

On the other hand, if $\psi_1 = 0$ for all $t \in [t', t'']$, then one has singular control.

In order to find the control on a singular arc, we use the fact that H_u remains zero along the whole arc. Thus, all the time derivatives are zero along such an arc. By successive differentiation of the switching function, one of the time derivatives may contain the control u in which case u can be obtained as a function of x and ψ.

$$\begin{aligned} &\text{If } \psi_1(t) > 0 \ , \quad \text{then } u(t) = -1 \ ; \\ \textit{or} \quad &\text{if } \psi_1(t) < 0 \ , \quad \text{then } u(t) = 1 \ . \end{aligned}$$

Example 19 (Same problem continued.)

$$\begin{aligned} \text{Suppose} \quad & H_u = \psi_1 = 0 \quad \text{for every } t \in [t', t''] \text{ on a singular arc.} \\ & \dot{H}_u = \dot{\psi}_1 = -x_1 = 0 \quad \text{(no control appears explicitly)} \\ & \ddot{H}_u = -u = 0 \ . \end{aligned}$$

So $u(t) = 0$ along a singular arc and also $x_1(t) = 0$.

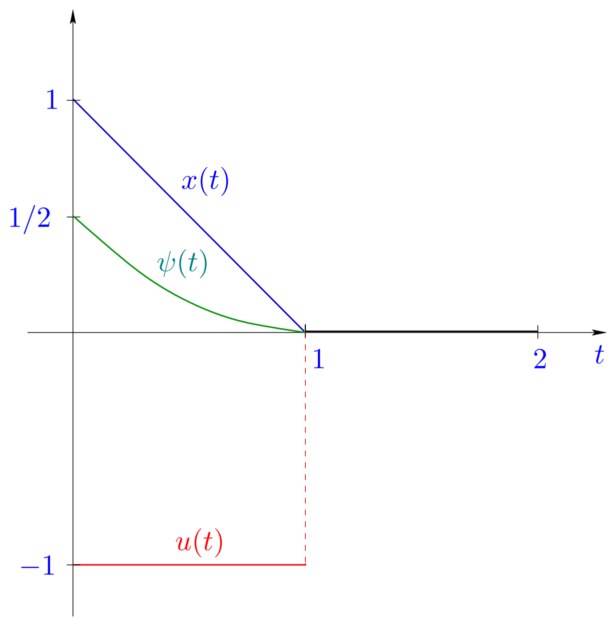
Assume that $\psi(0) > 0$ so that $u(0) = -1$. Then initially

$$\dot{x}_1 = -1 \implies x_1(t) = 1 - t \; .$$

$$\dot{\psi}_1 = -x_1 = -1 + t \implies \psi_1(t) = -t + \frac{1}{2}t^2 + \psi_1(0) \; .$$

Now $x_1(1) = 0$ and $\psi_1(1) = -1 + 1/2 + \psi_1(0) = 0$ if $\psi_1(0) = 1/2$.

With this initial value of ψ_1, all requirements for a singular solution on the interval $1 \le t \le 2$ are established. Transversality condition $\psi_1(2) = 0$ is also satisfied.



Recall the Hamiltonian for the general case:

$$H = \psi^T \; (f(x) + u\, g(x)) \; .$$

Also recall

$$H_u = \psi^T \, g(x) \; .$$

Then

$$\begin{aligned} \dot{H}_u &= \dot{\psi}^T g(x) + \psi^T g_x(x)\dot{x} \\ &= -\psi^T \left(f_x(x) + u\, g_x(x)\right) g(x) + \psi^T g_x(x)(f(x) + ug(x)) \\ &= -\psi^T f_x(x) g(x) + \psi^T g_x(x) f(x) \; . \end{aligned}$$

We note that control u does not appear explicitly after first differentiation. It is needed to differentiate H_u at least once more. In general, one would have

$$\begin{aligned}
H_u &= 0 = H_u(x, \psi) \\
\dot{H}_u &= 0 = \dot{H}_u(x, \psi) \\
\ddot{H}_u &= 0 = \ddot{H}_u(x, \psi) \\
&\vdots \\
H_u^{(2q)} &= 0 = A(x, \psi) + uB(x, \psi) \qquad \text{(the first time that } u \text{ appears)}
\end{aligned}$$

The integer q above is called the *order* of the singular problem.

A necessary condition for a singular arc to be optimal is the *Generalized Legendre-Clebsch (GLC) condition*:

For scalar u,

$$(-1)^q \frac{\partial}{\partial u} \frac{\partial^{2q}}{\partial t^{2q}} H_u \geq 0 \quad \text{for all } t \in [t', t''] \ .$$

Note that this condition is also valid for vector $u(t) \in \mathbb{R}^m$ with ≥ 0 replaced by the positive semi-definiteness of the $m \times m$ matrix on the left-hand side (Bell & Jacobson).

For scalar u, the GLC condition above is nothing but

$$(-1)^q \, B(x, \psi) \geq 0 \quad \text{for all } t \in [t', t''] \ .$$

For vector u,

$$\frac{\partial}{\partial u} H_u^{(p)} = 0 \ , p \text{ odd} \ ,$$

where

$$H_u^{(p)} = \left[H_{u_1}^{(p)}, \ H_{u_2}^{(p)}, \ \ldots, \ H_{u_m}^{(p)} \right]^T \ .$$

This condition is trivially satisfied if u is scalar.

Example 20 (Same problem continued.)

$$\ddot{H}_u = -u \qquad (q = 1)$$

So

$$(-1)^q \frac{\partial}{\partial u} \ddot{H}_u = (-1)(-1) = 1 > 0 \ .$$

Optimal control problems with bounded control usually (or at least often) have three types of subarcs. Two, with the control on the upper or lower bound, come from Pontryagin Maximum Principle (PMP).

Another necessary condition can be stated for a *totally* singular arc to be optimal, which is called the *Jacobson necessary condition*: Consider the system dynamics $\dot{x} = f(x, u)$. Along a totally singular arc

$$(H_{ux} + f_u^T Q) f_u \succeq 0 \quad \text{for all } t \in [0, t_f]$$

where $\succeq 0$ stands for positive semi-definiteness, Q is a time-varying symmetric matrix satisfying

$$\dot{Q} = -H_{xx} - f_x^T Q - Q f_x$$

and $Q(t_f) = \varphi(x(t_f))$.

Example 21

$$\begin{aligned}
\min_{u(\cdot)} \quad & \frac{1}{2}\int_0^{3\pi/2} (x_2^2 - x_1^2)dt \\
s.t. \quad & \dot{x}_1 = x_2 \ , \qquad x_1(0) = 0 \ , \\
& \dot{x}_2 = u \ , \qquad x_2(0) = 1 \ , \quad |u(t)| \leq 1 \ .
\end{aligned}$$

$$H = \frac{1}{2}(x_2^2 - x_1^2) + \psi_1 x_2 + \psi_2 u \ .$$

$$\begin{aligned}
\dot{\psi}_1 &= x_1 \ , & \psi_1(3\pi/2) &= 0 \ , \\
\dot{\psi}_2 &= -x_2 - \psi_1 \ , & \psi_2(3\pi/2) &= 0 \ .
\end{aligned}$$

Switching function: $H_u = \psi_2$.

$$\begin{aligned}
\psi_2(t) > 0 &\implies u(t) = -1 \ ; \\
\psi_1(t) < 0 &\implies u(t) = +1 \ .
\end{aligned}$$

Along a singular trajectory, $\psi_2 = 0$. This implies that

$$\dot{\psi}_2 = 0 \implies \psi_1 = -x_2 \implies -\dot{x}_2 = x_1 \ .$$

Eliminating x_2 we have

$$\ddot{x}_1 = -x_1 \implies x_1(t) = A\cos t + B\sin t$$

where A and B are arbitrary real constants.

$$x_1(0) = 0 \implies A = 0 \implies x_1 = B\sin t$$

So

$$x_2 = \dot{x}_1 = B\cos t$$

$$x_2(0) = 1 \implies B = 1 \ .$$

Therefore

$$\begin{aligned}
x_1(t) &= \sin t \\
x_2(t) &= \cos t \\
u(t) &= -\sin t
\end{aligned}$$

constitute a *totally singular* trajectory, where the transversality condition $\psi_1(3\pi/2) = 0$ is satisfied.

Check GLC condition:

$$\begin{aligned}
&H_u = \psi_2 \\
&\dot{H}_u = \dot{\psi}_2 = -x_2 - \psi_1 \\
&\ddot{H}_u = -\dot{x}_2 - \dot{\psi}_1 = -u - x_1 \ , \qquad \therefore \ \text{order } q = 1 \ .
\end{aligned}$$

$$\text{GLC:} \quad -\frac{\partial}{\partial u}\ddot{H}_u = +1 > 0 \ .$$

So the GLC condition is satisfied.

Now let us check the Jacobson (necessary) condition.

$$\begin{aligned}
&H_{ux} = 0 \ , \qquad H_{x_1} = -x_1 \ , \qquad H_{x_2} = x_2 + \psi_1 \ , \\
&H_{x_1x_1} = -1 \ , \qquad H_{x_1x_2} = 0 \ , \qquad H_{x_2x_2} = 1 \ , \qquad H_{x_2x_1} = 0 \ .
\end{aligned}$$

So

$$H_{xx} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} .$$

We also have

$$f = \begin{bmatrix} x_2 \\ u \end{bmatrix} , \quad f_u = \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \quad f_x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} , \quad \varphi = 0 \ .$$

Let

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} .$$

Then

$$(H_{ux} + f_u^T Q) f_u = [0 \ \ 1] \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = q_{22}$$

where

$$\begin{bmatrix} \dot{q}_{11} & \dot{q}_{12} \\ \dot{q}_{12} & \dot{q}_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} - \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

with $Q(3\pi/2) = 0$.

$$\begin{aligned}
&\dot{q}_{11} = 1 \ , \quad q_{11}(3\pi/2) = 0 \implies q_{11} = t - 3\pi/2 \ . \\
&\dot{q}_{12} = -q_{11} \ , \quad q_{12}(3\pi/2) = 0 \ . \\
&\dot{q}_{22} = -1 - 2q_{12} \ , \quad q_{22}(3\pi/2) = 0 \ .
\end{aligned}$$

Write $\tau = (3\pi/2) - t$ (reverse time) so that $d\tau/dt = -1$. Let $(\cdot)'$ denote $d/d\tau(\cdot)$. Then

$$\begin{aligned}
&q'_{11} = -1 \ , \quad q_{11}(0) = 0 \implies q_{11} = -\tau \ . \\
&q'_{12} = q_{11} = -\tau \ , \quad q_{12}(0) = 0 \implies q_{12} = -\frac{1}{2}\tau^2 \ . \\
&q_{22'} = 1 + 2q_{12} = 1 - \tau^2 \ , \quad q_{22}(0) = 0 \implies q_{22} = \tau - \frac{1}{3}\tau^3 = \tau\left(1 - \frac{1}{3}\tau^2\right) \ .
\end{aligned}$$

Jacobson condition states that $q_{22} \geq 0$ for every $\tau \in [0, 3\pi/2]$ but $q_{22} < 0$ when $\tau > \sqrt{3}$. Therefore the singular arc is *not* optimal.

Exercise 12 *Consider the problem of minimizing* $\frac{1}{2}\int_0^{t_f} x_1^2(t)dt$*, subject to*

$$\dot{x}_1(t) = x_2(t) + u(t) \ ,$$
$$\dot{x}_2(t) = -u(t) \ , \quad |u(t)| \le 1 \ .$$

(a) Show that the Hamiltonian function for this problem is constant along an extremal.

(b) Find the closed-loop control on a singular arc arising from this problem and show that $x_1\, x_2 + \frac{1}{2}\, x_1^2$ *is constant along it. (*Hint *: Use part (a).)*

(c) What is the order of the singular problem.

(d) Check if the Generalized Legendre-Clebsch condition holds.