# CS572 Lecture Note

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# Optimal Control (review)

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Typical formulation of optimal control is as follows.

$$\min_u \phi(x(t_1), t_1) + \int_{t_0}^{t_1} L(x, u, t)dt \text{ such that } \dot{x} = f(x, u, t)$$

General steps to solve the equation are the followings.

1. Define $H \triangleq L + \lambda^T f$
2. Solve $\frac{\partial H}{\partial u} = 0$
3. Integrate $\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} f \\ -\frac{\partial H}{\partial u} \end{bmatrix}$ given $u$

# Kalman Linear Quadratic Regulator (LQR) Problem

# What is LQR problem?

- The aim of optimal control theory is to minimize cost of an operating dynamic system.
- If the dynamic systems consist of a set of linear differential equations and the cost as a quadratic function, the system is called the Linear Quadratic (LQ) problem.
- **Linear Quadratic Regulator (LQR)** is one of the solution in the theory, which is a feedback controller.

# Problem Formulation

The problem formulation is as follows.

$$\min_u \frac{1}{2} x^T(t_1) H x(t_1) + \frac{1}{2} \int_{t_0}^{t_1} x^T(t) Q(t) x(t) + u^T(t) R(t) u(t) dt$$
$$\text{subject to } \dot{x}(t) = A(t)x(t) + B(t)u(t),\ x(t_0) = x_0 \text{ and } Q, R > 0$$

# Solution

We have following equations.

$$H = L + \lambda^T (Ax + Bu)$$
$$L = \frac{1}{2} x^T Q x + \frac{1}{2} u^T R u$$

Then we set derivative of $H$ by $u$ to 0 to get minimum point of $u$.

$$\frac{\partial H}{\partial u} = 0$$
$$u^T R + \lambda^T B = 0$$
$$u = -R^{-1} B^T \lambda$$

# Solution

Also,

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} = -x^T Q - \lambda^T A$$
$$\dot{x} = Ax + Bu = Ax - BR^{-1}B^T\lambda\,.$$

Then by following step 3 of general steps to solve the equation,

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T\lambda \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

We can solve this equation with boundary conditions by applying **Linear System Theory (LST)**.

# Solution - LST

A **linear system** is a mathematical model of a system based on the use of a linear operator. This can be formulated in equations as follows.

$$\begin{bmatrix} x(t_1) \\ \lambda(t_1) \end{bmatrix} = \begin{bmatrix} \Phi_{11}(t_1, t_0) & \Phi_{12}(t_1, t_0) \\ \Phi_{21}(t_1, t_0) & \Phi_{22}(t_1, t_0) \end{bmatrix} \begin{bmatrix} x(t_0) \\ \lambda(t_0) \end{bmatrix}$$

After solving LST, we have the followings.

$$\lambda(t) = k(t)x(t)$$
$$u = -R^{-1}B^T k(t)x(t)$$

# Solution - LST

After that we need to know k(t). And we have

$$\dot{x} = Ax - BR^{-1}B^T kx$$
$$\dot{\lambda} = -Qx - A^T kx = \dot{k}x + k\dot{x} \ .$$

Then with those equations, we can get k(t).

$$(\dot{k} + Q + A^T k)x = -k\dot{x} = -k(Ax - BR^{-1}B^T kx)$$
$$(\dot{k} + Q + A^T k + kA - kBR^{-1}B^T k)x = 0$$

Then we can solve the equation by setting 0 to the front equation in parenthesis and with boundary condition of $k(t_1) = H$.

$$\dot{k} = -Q - A^T k - kA + kBR^{-1}B^T k$$

# Tracking Problem

# Tracking Problem

The problem formulation of tracking problem is as follows.

$$\min_u \frac{1}{2}||r(t_1) - x(t_1)||_H^2 + \frac{1}{2}\int_{t_0}^{t_1} ||r(t) - x(t)||_{Q(t)}^2 + ||u(t)||_{R(t)}^2 dt$$

subject to $\dot{x}(t) = A(t)x(t) + B(t)u(t),\ x(t_0) = x_0$ and $Q, R > 0$

We can solve the equations similarly as LQR problem.

$$\frac{\partial H}{\partial u} = 0$$

$$u = -R^{-1}B^T\lambda$$

$$\lambda^T = -\frac{\partial H}{\partial x} = -Q(x - r) - A^T\lambda$$

Then, $\lambda(t) = k(t)x(t) - g(t)$ and this can be soled by LST shown before.

# Pontryagin Minimum Principle

# Motivating Example

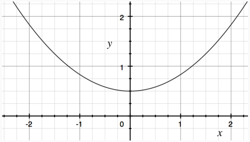


**Figure 1:** Motivating example

Let $x_0 = 1$ and $x_1 = 2$. Then problem formulation is as follow.

$$\min_x L(x) \text{ such that } x_0 \leq x \leq x_1$$

Then the answer is

$$x^* = x_0 \text{ and } \frac{\partial L}{\partial x}(x^*) \neq 0 \ .$$

Also, by Talyor series,

$$L(x) \cong L(x^*) + \frac{\partial L}{\partial x}(x^*)\delta x \ .$$

Suppose $u(t)$ is unbounded. Let $u^*(t)$, $x^*(t)$, $\lambda^*(t)$ be optimal solutions. Then,

$$
\begin{aligned}
F(u^*) &= \int_{t_0}^{t_1} L(x^*, u^*, t) + {\lambda^*}^T (f(x^*, u^*, t) - x^*) dt \\
H &\triangleq L + \lambda^T f \\
\delta F(u^*, \delta u) &= \int_{t_0}^{t_1} \frac{\partial H}{\partial u}(x^*, u^*, \lambda^*, t) \delta u dt \\
&\cong \int_{t_0}^{t_1} H(x^*, u^* + \delta u, \lambda^*, t) - H(x^*, u^*, \lambda^*, t) dt \geq 0
\end{aligned}
$$

where $\forall \delta u$ and $u = u^* + \delta u$. This means that

$$H(x^*, u, \lambda^*, t) \geq H(x^*, u^*, \lambda^*, t) \ \forall u \ .$$

# Optimal Control - example

We present simple example to show optimal control applied to pontryagin minimum principle.

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -x_2 + u$$
$$\min \frac{1}{2} \int_{t_0}^{t_1} x_1^2 + u^2 dt$$

There are two cases of $u$, which are

Unconstrained $u$

Constrained $-1 \leq u \leq 1$ .

# Optimal Control - example

- Unconstrained $u$.

We can solve the problem by using general steps to solve an optimal control problem.

$$
\begin{aligned}
H &= \frac{1}{2}(x_1^2 + u^2) + \lambda_1 x_2 + \lambda_2(-x_2 + u) \\
\dot{x}_1 &= x_2,\ \dot{x}_2 = -x_2 + u \\
\dot{\lambda}_1 &= -x_1,\ \dot{\lambda}_2 = -\lambda_1 + \lambda_2 \\
u &= -\lambda_2
\end{aligned}
$$

- Constrained $-1 \leq u \leq 1$.

$$H(x^*, u^*, \lambda^*, t) \leq H(x^*, u, \lambda^*, t)$$
$$\frac{1}{2}(x_1^{*2} + u^{*2}) + \lambda_1^* x_2^* + \lambda_2^*(-x_2^* + u^*) \leq \frac{1}{2}(x_1^{*2} + u^2) + \lambda_1^* x_2^* + \lambda_2^*(-x_2^* + u)$$
$$\frac{1}{2}u^{*2} + \lambda_2^* u^* \leq \frac{1}{2}u^2 + \lambda_2^* u$$

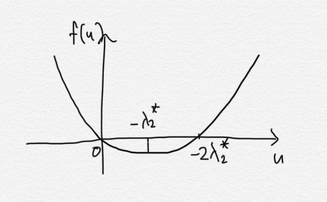


Figure 2: Solution if unconstrained

# Optimal Control - example

- However, we consider constrained $u \in [-1, 1]$, which means we have to do case analysis by $\lambda_2^*$.
- If $\lambda_2^* < -1$, then $u^* = +1$.
- If $\lambda_2^* > 1$, then $u^* = -1$.
- If $-1 \leq \lambda_2^* \leq 1$, then $u^* = -\lambda_2^*$.

# Dynamic Programming

# Dynamic Programming

- **Dynamic programming** is both a mathematical optimization method and a computer programming method where it usually can be though of as simplifying a decision by breaking it down into a sequence of decision steps over time.
- Dynamic programming is a central idea of control theory that is based on the **Principle of Optimality**.
- The core idea of **principle of optimality** is that an optimal path has the property that the path is optimal at any state in some period, which means it can be thought of as formulating dynamic programming.

# Principle of Optimality - example 1

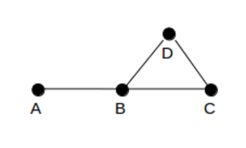


**Figure 3:** The example path to find optimal path from A to C

**Claim** If ABC is optimal path from A to C, then BC is optimal path from B to C.
**Proof** Suppose BDC is optimal path from B to C. Then by assumption, $J_{BDC} < J_{BC}$. By adding $J_{AB}$ to both side, we have $J_{BDC} + J_{AB} < J_{BC} + J_{AB}$. This is contradiction.

# Principle of Optimality - example 2

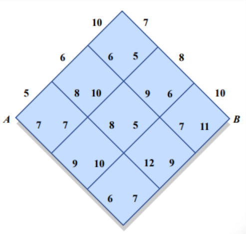


Figure 4: The example of multistage path

We have to find left-to-right only optimal path from A to B. The idea of solving the problem by dynamic programming is to start from B **backward** to A while achieving optimal cost at every stage. However, the approach can be problematic if the size of the problem is large, which implies the **curse of dimensionality**.