Homework 8

Hunter Schwartz

Problem 1

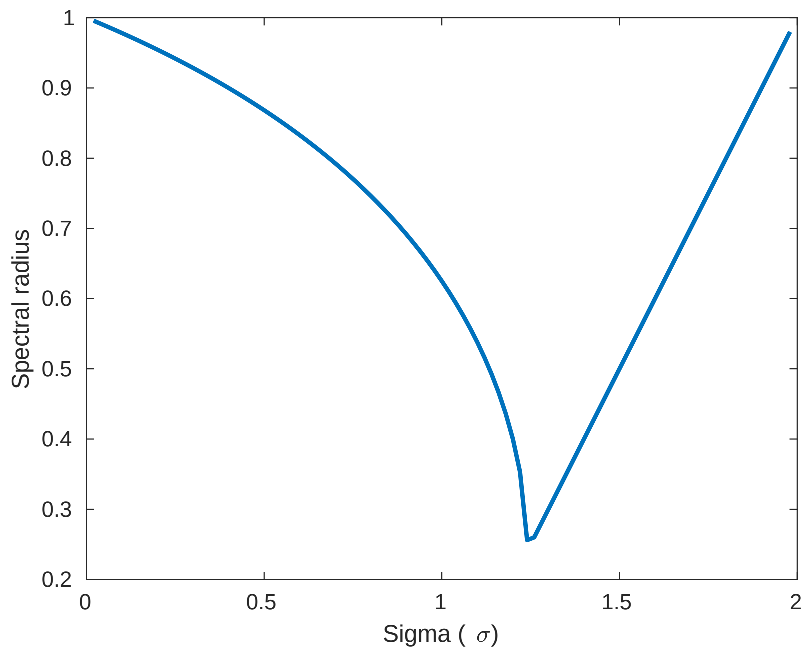

Figure 1: Spectral radius of the S_σ matrix dependent on choice of σ.

This plot depicts a trend common to the Successive Over-Relaxation method and why it is useful. At the endpoints of the domain when $\sigma = 0, 2$, the spectral radius of the S_σ matrix approaches 1, which implies the inability of the method to converge. For intermediate values of σ, the spectral radius $\rho(S_\sigma) < 1$, so the method will in fact converge for any initial condition. When $\sigma = 1$, the method is equivalent to Gauss-Seidel, and the spectral radius is ≈ 0.6, representing the good convergence that we expect. However, the SOR method has the ability to perform much better in this case, reaching a minimum spectral radius of ≈ 0.25 when $\sigma \approx 1.25$.

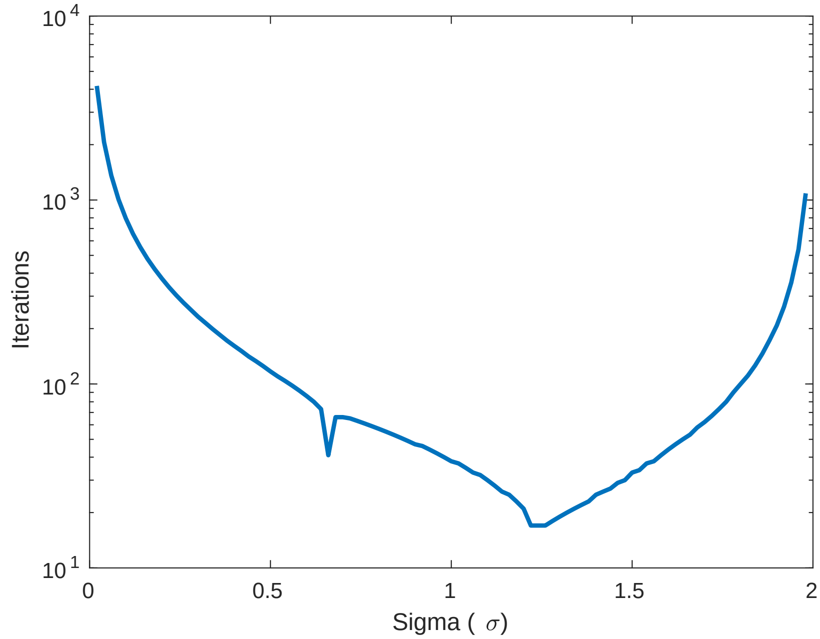

Figure 2: Number of iterations required for SOR method to converge within a tolerance of 10^{-8}.

As expected from our results about how the spectral radius depends of σ, we see that the method converges over the whole range of σ, but performs best at intermediate values. Again, the Gauss-Seidel method performs well, converging in ≈ 40 iterations, but is beat by the method when $\sigma \approx 1.25$, where the spectral radius attains its minimum.

I am uncertain about the brief spike in improvement for $\sigma \approx 0.6$. There is potentially a possibility that our initial condition $x_0 = [1\ 1\ 1]^\top$ happens to be roughly orthogonal to the eigenvector associated with the eigenvalue of maximum magnitude of S_σ in this range. If this were the case, then the convergence of the method would be dominated by the eigenvalue of second-largest magnitude, which would be faster (at least no slower) than for other initial conditions. Regardless, the method still performs better for $\sigma \approx 1.25$.