Homework 4

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Problem 1

The trapezoid and Euler methods are implemented to numerically find the solution to the system as follows.

```python
import numpy as np

# Problem statement
def f(t,Y):
    x,y,z = Y
    return np.array([-y, x, -100*(z - x**2 - y**2)])

def Df(t,Y):
    x,y,z = Y
    return np.matrix([[0, -1, 0],
                      [1, 0, 0],
                      [200*x, 200*y, -100]])

# Function to find root of for trapezoid method
def g(Y, Yj, tj, h):
    return Y - Yj - h/2*f(tj,Yj) - h/2*f(tj+h,Y)

def Dg(Y, Yj, tj, h):
    return np.eye(3) - h/2*Df(tj+h,Y)

# Problem parameters
Y0 = np.array([3, 0, 18])
h = 0.05
t = np.arange(0.2, 5, step=h)

# Trapezoid method
Y = np.empty([len(t), 3])
```

```
Y[0,:] = Y0

for j, tj in enumerate(t[0:-1]):
    Yj = Y[j,:]

    # Newton solve
    Yjp1 = Yj
    niters = 3
    for i in range(niters):
        gi = g(Yjp1, Yj, tj, h)
        Dgi = Dg(Yjp1, Yj, tj, h)
        s = np.linalg.solve(Dgi, gi)
        Yjp1 = Yjp1 - s

    Y[j+1,:] = Yjp1

# Euler's method
Z = np.empty([len(t),3])
Z[0,:] = Y0

for j, tj in enumerate(t[0:-1]):
    Z[j+1,:] = Z[j] + h*f(tj,Z[j])
```

Plots of the resulting solutions are included below.

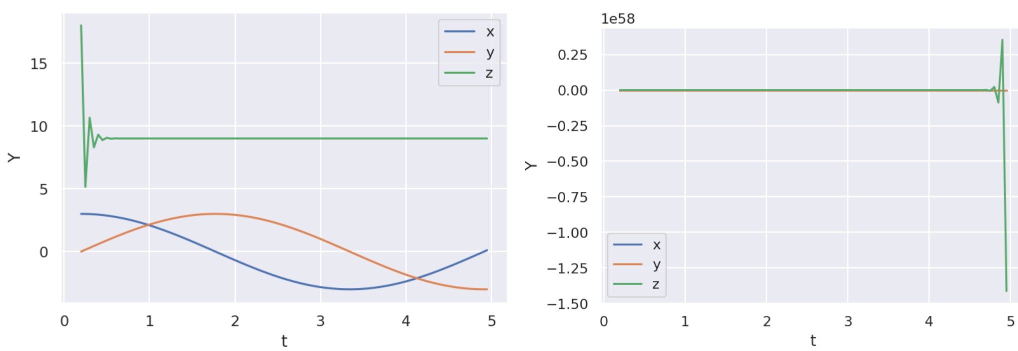

Figure 1: Comparison of the solution found by the trapezoid method (left) and the solution of Euler's method (right).

The equations for x and y are decoupled from z, so they evolve independently in a now familiar way, such that x^2+y^2 is constant. When $z = x^2+y^2$, we have $z' = 0$, so z will be stable if ever $z = x^2 + y^2$. Figure 1 demonstrates

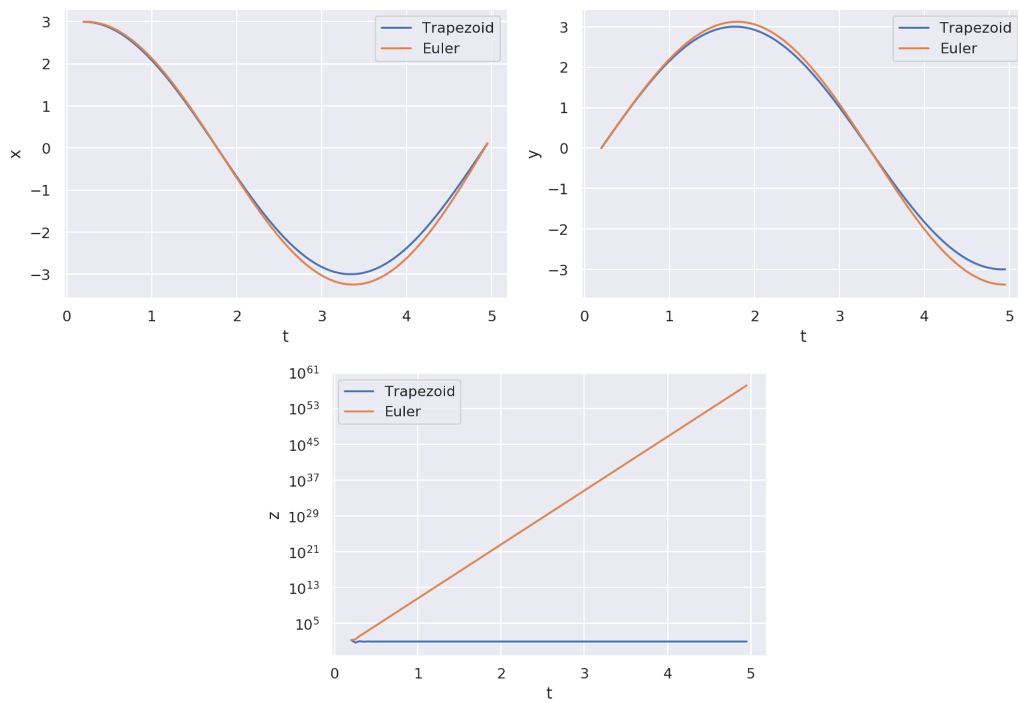

Figure 2: Dimension-wise comparison between the trapezoid and Euler's methods. The z axis is scaled logarithmically.

this in the solution of the trapezoid method, with x, y sinusoidal and z finding its way somewhat erratically to a resting state. In contrast, the solution to Euler's method grows larger with t until z is on the order of 10^{58}. This is a reflection of the fact that Euler's method is stable for a much smaller selection of step sizes for a given problem, and the steps we are taking in this problem are too large for this stiff system.

In Figure 2, we can see that the solutions to x and y are very similar for both methods. We have seen before that Euler's method behaves well for this particular x, y system, and since they are independent of z, we should see that Euler's method still solves for x and y well here. However, it is for the solution of z that Euler's method fails dramatically, with the solution growing exponentially in t whereas the actual solution is just a constant.