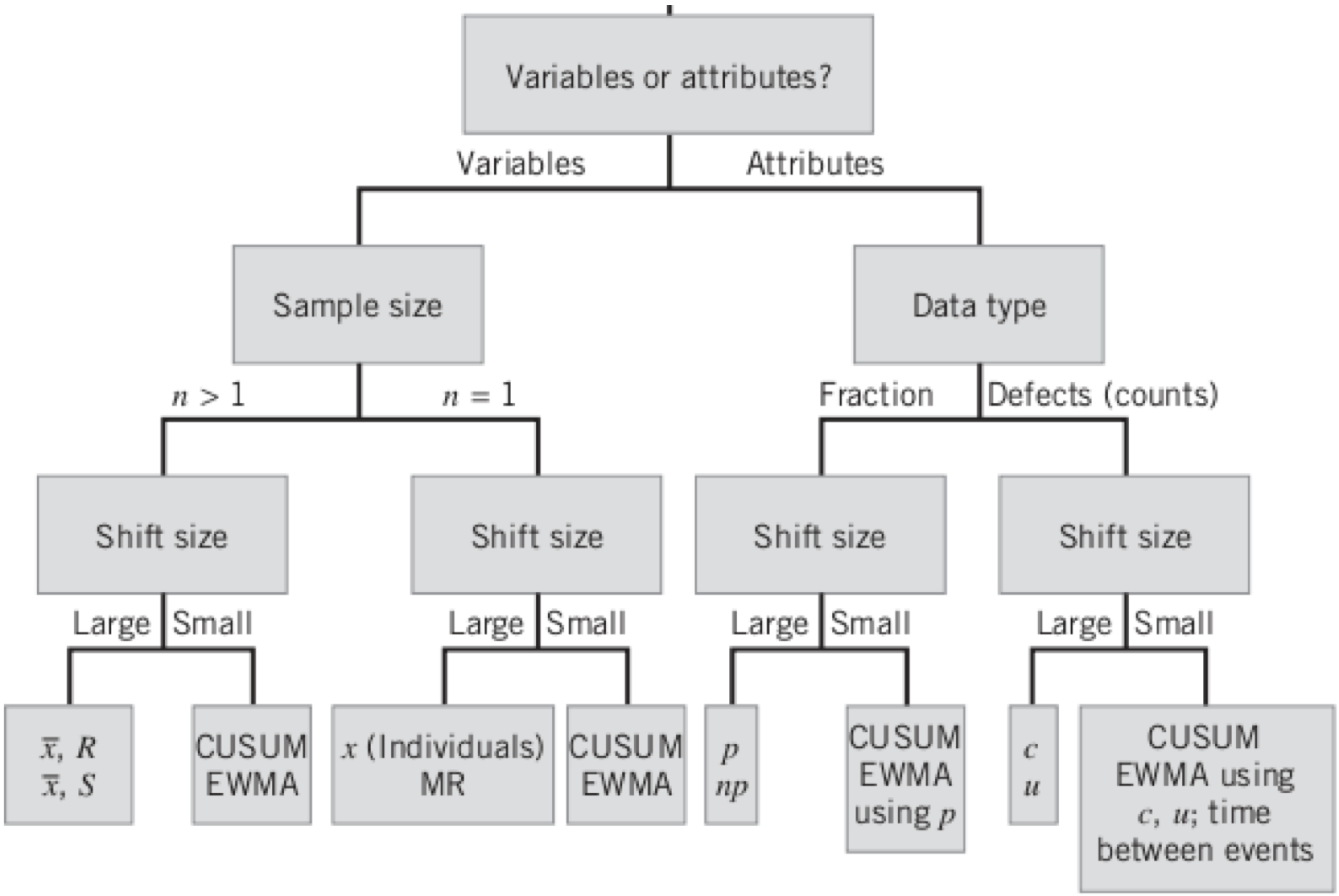
SJSU SAN JOSÉ STATE UNIVERSITY

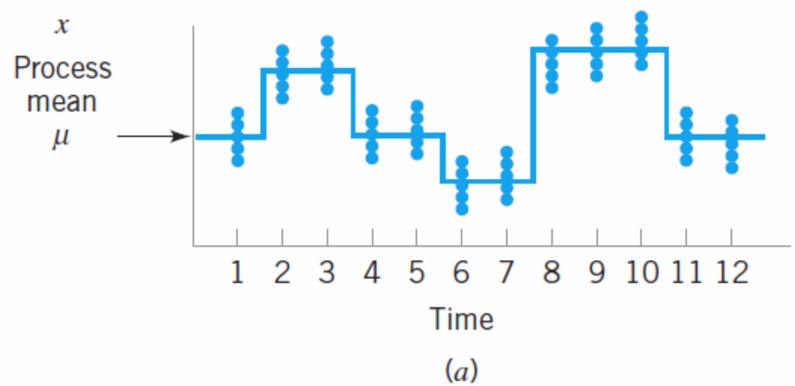
Variable control charts

Type of chart to use



Rational subgroups

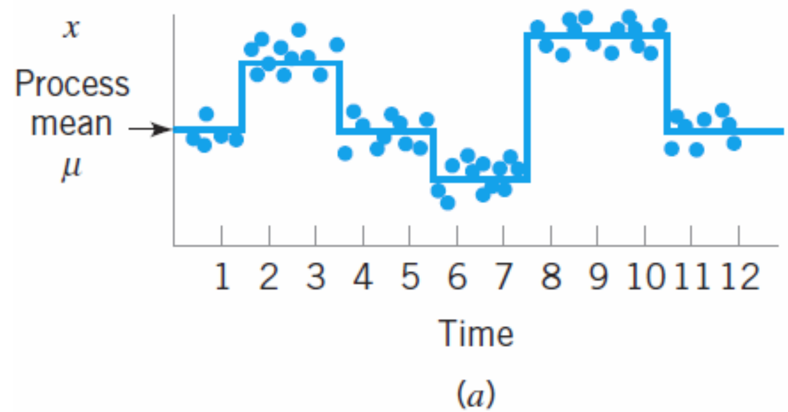
Consecutive units



Sample consists of **consecutive** units within sampling interval

– Primarily to detect process shifts

Random sample



Sample consists of **random sample** of units over sampling interval

– Primarily to decide whether to accept product
– Effective at detecting shifts *between* samples

Montgomery 5

$n > 1$

X-BAR, R

Sample statistics

Sample size n

Sample $x_1, x_2, \ldots, x_n$

Sample mean: $$\overline{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Range:

$$R = x_{\max} - x_{\min}$$

Since we take multiple samples

Number of samples groups *m*

Average sample mean:

$$\bar{\bar{x}} = \frac{\bar{x}_1 + \bar{x}_2 + \cdots + \bar{x}_m}{m} \tag{6.2}$$

Average sample range:

$$\bar{R} = \frac{R_1 + R_2 + \cdots + R_m}{m} \tag{6.3}$$

APPENDIX VI

Factors for Constructing Variables Control Charts

Observations in Sample, n	Chart for Averages			Chart for Standard Deviations						Chart for Ranges						
	Factors for Control Limits			Factors for Center Line		Factors for Control Limits				Factors for Center Line		Factors for Control Limits				
	A	A_2	A_3	c_4	$1/c_4$	B_3	B_4	B_5	B_6	d_2	$1/d_2$	d_3	D_1	D_2	D_3	D_4
2	2.121	1.880	2.659	0.7979	1.2533	0	3.267	0	2.606	1.128	0.8865	0.853	0	3.686	0	3.267
3	1.732	1.023	1.954	0.8862	1.1284	0	2.568	0	2.276	1.693	0.5907	0.888	0	4.358	0	2.574
4	1.500	0.729	1.628	0.9213	1.0854	0	2.266	0	2.088	2.059	0.4857	0.880	0	4.698	0	2.282
5	1.342	0.577	1.427	0.9400	1.0638	0	2.089	0	1.964	2.326	0.4299	0.864	0	4.918	0	2.114
6	1.225	0.483	1.287	0.9515	1.0510	0.030	1.970	0.029	1.874	2.534	0.3946	0.848	0	5.078	0	2.004
7	1.134	0.419	1.182	0.9594	1.0423	0.118	1.882	0.113	1.806	2.704	0.3698	0.833	0.204	5.204	0.076	1.924
8	1.061	0.373	1.099	0.9650	1.0363	0.185	1.815	0.179	1.751	2.847	0.3512	0.820	0.388	5.306	0.136	1.864
9	1.000	0.337	1.032	0.9693	1.0317	0.239	1.761	0.232	1.707	2.970	0.3367	0.808	0.547	5.393	0.184	1.816
10	0.949	0.308	0.975	0.9727	1.0281	0.284	1.716	0.276	1.669	3.078	0.3249	0.797	0.687	5.469	0.223	1.777
11	0.905	0.285	0.927	0.9754	1.0252	0.321	1.679	0.313	1.637	3.173	0.3152	0.787	0.811	5.535	0.256	1.744
12	0.866	0.266	0.886	0.9776	1.0229	0.354	1.646	0.346	1.610	3.258	0.3069	0.778	0.922	5.594	0.283	1.717
13	0.832	0.249	0.850	0.9794	1.0210	0.382	1.618	0.374	1.585	3.336	0.2998	0.770	1.025	5.647	0.307	1.693
14	0.802	0.235	0.817	0.9810	1.0194	0.406	1.594	0.399	1.563	3.407	0.2935	0.763	1.118	5.696	0.328	1.672
15	0.775	0.223	0.789	0.9823	1.0180	0.428	1.572	0.421	1.544	3.472	0.2880	0.756	1.203	5.741	0.347	1.653
16	0.750	0.212	0.763	0.9835	1.0168	0.448	1.552	0.440	1.526	3.532	0.2831	0.750	1.282	5.782	0.363	1.637
17	0.728	0.203	0.739	0.9845	1.0157	0.466	1.534	0.458	1.511	3.588	0.2787	0.744	1.356	5.820	0.378	1.622
18	0.707	0.194	0.718	0.9854	1.0148	0.482	1.518	0.475	1.496	3.640	0.2747	0.739	1.424	5.856	0.391	1.608
19	0.688	0.187	0.698	0.9862	1.0140	0.497	1.503	0.490	1.483	3.689	0.2711	0.734	1.487	5.891	0.403	1.597
20	0.671	0.180	0.680	0.9869	1.0133	0.510	1.490	0.504	1.470	3.735	0.2677	0.729	1.549	5.921	0.415	1.585
21	0.655	0.173	0.663	0.9876	1.0126	0.523	1.477	0.516	1.459	3.778	0.2647	0.724	1.605	5.951	0.425	1.575
22	0.640	0.167	0.647	0.9882	1.0119	0.534	1.466	0.528	1.448	3.819	0.2618	0.720	1.659	5.979	0.434	1.566
23	0.626	0.162	0.633	0.9887	1.0114	0.545	1.455	0.539	1.438	3.858	0.2592	0.716	1.710	6.006	0.443	1.557
24	0.612	0.157	0.619	0.9892	1.0109	0.555	1.445	0.549	1.429	3.895	0.2567	0.712	1.759	6.031	0.451	1.548
25	0.600	0.153	0.606	0.9896	1.0105	0.565	1.435	0.559	1.420	3.931	0.2544	0.708	1.806	6.056	0.459	1.541

For $n > 25$.

$$A = \frac{3}{\sqrt{n}} \qquad A_3 = \frac{3}{c_4\sqrt{n}} \qquad c_4 \cong \frac{4(n-1)}{4n-3}$$

$$B_3 = 1 - \frac{3}{c_4\sqrt{2(n-1)}} \qquad B_4 = 1 + \frac{3}{c_4\sqrt{2(n-1)}}$$

$$B_5 = c_4 - \frac{3}{\sqrt{2(n-1)}} \qquad B_6 = c_4 + \frac{3}{\sqrt{2(n-1)}}$$

Unbiased estimates of μ and σ

$E(\bar{x}) = \mu$

$E\left(\frac{R}{d_2}\right) = \sigma$ where d_2 is in Appendix VI

Unbiased estimates:

$\hat{\mu} = \bar{\bar{x}}$, unbiased estimate of the mean

$\hat{\sigma} = \frac{\bar{R}}{d_2}$, unbiased estimate of the standard deviation

$\hat{\sigma}_R = d_3 \frac{\bar{R}}{d_2}$, unbiased estimate of the range's standard deviation

Control limits on $\bar{x}$

Central Limit Theorem:

As $n \to \infty$, $\bar{x} \to \sim N\left(\mu, {}^{\sigma^2}/_{n}\right)$

Confidence interval on $\bar{x}$:

$$\mu + Z_{\alpha/2}\sigma_{\bar{x}} = \mu + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \text{ and } \mu - Z_{\alpha/2}\sigma_{\bar{x}} = \mu - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \tag{6.1}$$

Choose $Z_{\alpha/_2} = 3$ to get 3-sigma limits (motivation later)

μ and σ are not known and must be estimated

Note: non-normality usually not an issue, especially in Phase I

- Control limits are very robust to normality assumption (Burr 1967)
- Samples of size 4 or 5 are, in most cases, sufficient (Schilling and Nelson 1976)
- The R chart is more sensitive than the $\bar{x}$ chart to departures from normality

- Simple test if you're worried about normality
  - Construct a control chart with no transformation and check the zone rules
  - Construct a control chart with with a normality transformation (e.g., log, Box-Cox) and check the zone rules
  - If both charts have the same zone rule designations, you should be OK using the non-transformed data

$\bar{x}$ Chart's Upper Control Limit (UCL) and Lower Control Limit (LCL)

When using *R* Chart

$$\text{UCL} = \overline{\overline{x}} + \frac{3}{d_2\sqrt{n}}\overline{R}$$

$$\text{Center line} = \overline{\overline{x}} \tag{6.7}$$

$$\text{LCL} = \overline{\overline{x}} - \frac{3}{d_2\sqrt{n}}\overline{R}$$

If we define

$$A_2 = \frac{3}{d_2\sqrt{n}} \tag{6.8}$$

Control Limits for the $\bar{x}$ Chart

$$\text{UCL} = \overline{\overline{x}} + A_2\overline{R}$$

$$\text{Center line} = \overline{\overline{x}} \tag{6.4}$$

$$\text{LCL} = \overline{\overline{x}} - A_2\overline{R}$$

The constant A_2 is tabulated for various sample sizes in Appendix Table VI.

R Chart's Upper Control Limit (UCL) and Lower Control Limit (LCL)

$$\text{UCL} = \overline{R} + 3\hat{\sigma}_R = \overline{R} + 3d_3\frac{\overline{R}}{d_2}$$

$$\text{Center line} = \overline{R} \qquad (6.10)$$

$$\text{LCL} = \overline{R} - 3\hat{\sigma}_R = \overline{R} - 3d_3\frac{\overline{R}}{d_2}$$

If we let

$$D_3 = 1 - 3\frac{d_3}{d_2} \quad \text{and} \quad D_4 = 1 + 3\frac{d_3}{d_2}$$

Control Limits for the *R* Chart

$$\text{UCL} = D_4\overline{R}$$

$$\text{Center line} = \overline{R} \qquad (6.5)$$

$$\text{LCL} = D_3\overline{R}$$

The constants D_3 and D_4 are tabulated for various values of n in Appendix Table VI.

Control Chart Formulae and Coefficients

Montgomery inside front cover

SPC
Calculations for Control Limits

Notation:	UCL	Upper Control Limit	$\bar{x}$	Average of Measurements
	LCL	Lower Control Limit	$\bar{\bar{x}}$	Average of Averages
	CL	Center Line	R	Range
	n	Sample Size	$\bar{R}$	Average of Ranges
	PCR	Process Capability Ratio	USL	Upper Specification Limit
	$\hat{\sigma}$	Process Standard Deviation	LSL	Lower Specification Limit

Variables Data ($\bar{x}$ and R Control Charts)

$\bar{x}$ Control Chart

$\text{UCL} = \bar{\bar{x}} + A_2\bar{R}$

$\text{LCL} = \bar{\bar{x}} - A_2\bar{R}$

$\text{CL} = \bar{\bar{x}}$

R Control Chart

$\text{UCL} = \bar{R}\, D_4$

$\text{LCL} = \bar{R}\, D_3$

$\text{CL} = \bar{R}$

Capability Study

$C_p = (\text{USL} - \text{LSL})/(6\hat{\sigma})$; where $\hat{\sigma} = \bar{R}/d_2$

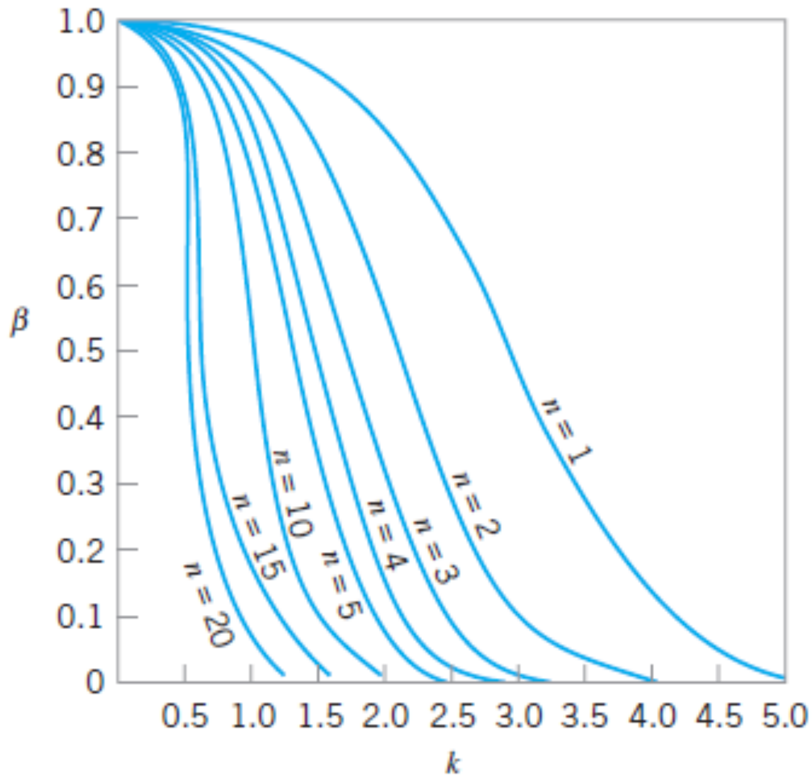
n	A_2	D_3	D_4	d_2
2	1.880	0.000	3.267	1.128
3	1.023	0.000	2.574	1.693
4	0.729	0.000	2.282	2.059
5	0.577	0.000	2.114	2.326
6	0.483	0.000	2.004	2.534
7	0.419	0.076	1.924	2.704
8	0.373	0.136	1.864	2.847
9	0.337	0.184	1.816	2.970
10	0.308	0.223	1.777	3.078

Attribute Data (p, np, c, and u Control Charts)

Control Chart Formulas

	p (fraction)	**np (number of nonconforming)**	**c (count of nonconformances)**	**u (count of nonconformances/unit)**
CL	$\bar{p}$	$n\bar{p}$	$\bar{c}$	$\bar{u}$
UCL	$\bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$	$n\bar{p} + 3\sqrt{n\bar{p}(1-\bar{p})}$	$\bar{c} + 3\sqrt{\bar{c}}$	$\bar{u} + 3\sqrt{\frac{\bar{u}}{n}}$
LCL	$\bar{p} - 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}}$	$n\bar{p} - 3\sqrt{n\bar{p}(1-\bar{p})}$	$\bar{c} - 3\sqrt{\bar{c}}$	$\bar{u} - 3\sqrt{\frac{\bar{u}}{n}}$
Notes	If n varies, use $\bar{n}$ or individual n_i	n must be a constant	n must be a constant	If n varies, use $\bar{n}$ or individual n_i

Operating Characteristic (OC) curve shows n, β and k *tradeoffs*



If the shift is 1.0σ and the sample size is n = 5, then β = 0.75.

As n increases we can detect smaller shifts k with same β
As β increases we can detect smaller shifts k with same n

■ **FIGURE 6.13** Operating-characteristic curves for the $\bar{x}$ chart with three-sigma limits. $\beta = P$ (not detecting a shift of $k\sigma$ in the mean on the first sample following the shift).

Zone Rules

A.K.A. Sensitizing Rules and Run Rules

TABLE 5.1

Some Sensitizing Rules for Shewhart Control Charts

Standard Action Signal:	**1.** One or more points outside of the control limits	Western Electric Rules (rules 1–4)
	2. Two of three consecutive points outside the two-sigma warning limits but still inside the control limits	
	3. Four of five consecutive points beyond the one-sigma limits	
	4. A run of eight consecutive points on one side of the center line	
	5. Six points in a row steadily increasing or decreasing	
	6. Fifteen points in a row in zone C (both above and below the center line)	
	7. Fourteen points in a row alternating up and down	
	8. Eight points in a row on both sides of the center line with none in zone C	
	9. An unusual or nonrandom pattern in the data	

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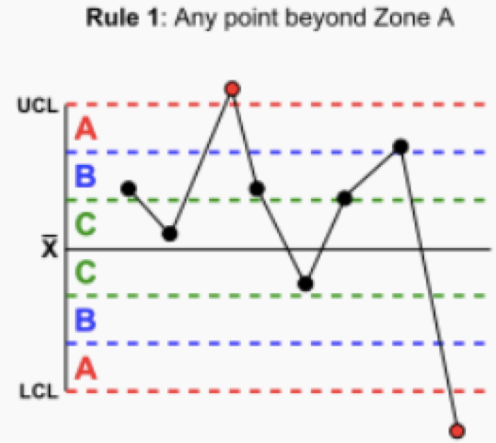
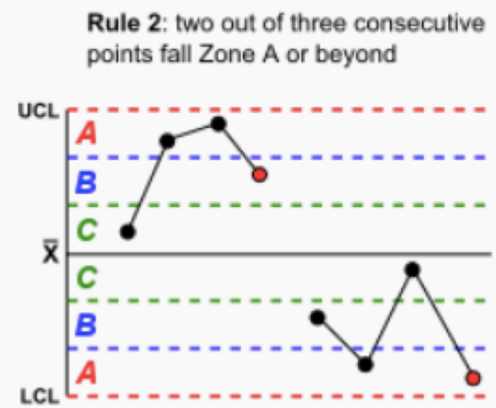
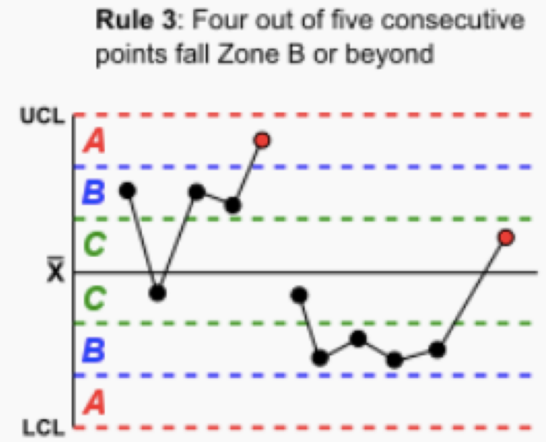
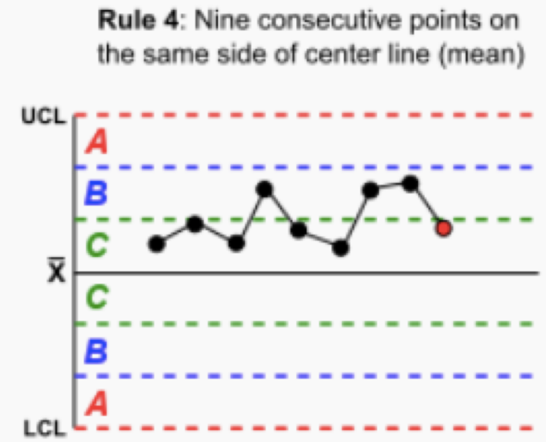
Zone Rules 1-2

Rule	Description	Chart
Rule 1	Any single data point falls outside the 3σ limit from the centerline (i.e., any point that falls outside Zone A, beyond either the upper or lower control limit)	
Rule 2	Two out of three consecutive points fall beyond the 2σ limit (in zone A or beyond), on the same side of the centerline	

Source: https://en.wikipedia.org/wiki/Western_Electric_rules

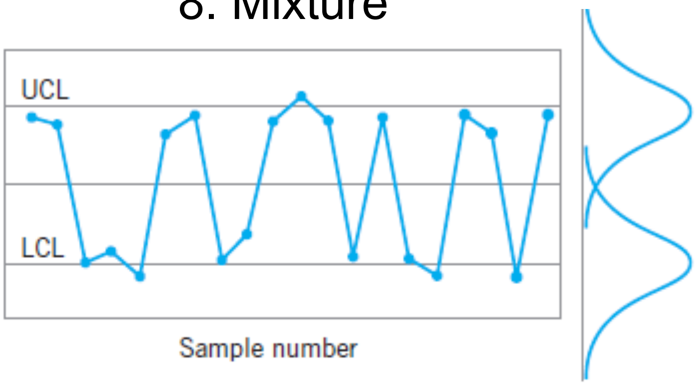
Zone Rules 3-4

Rule 3	Four out of five consecutive points fall beyond the 1σ limit (in zone B or beyond), on the same side of the centerline	
Rule 4	Nine consecutive points fall on the same side of the centerline (in zone C or beyond)	

Source: https://en.wikipedia.org/wiki/Western_Electric_rules

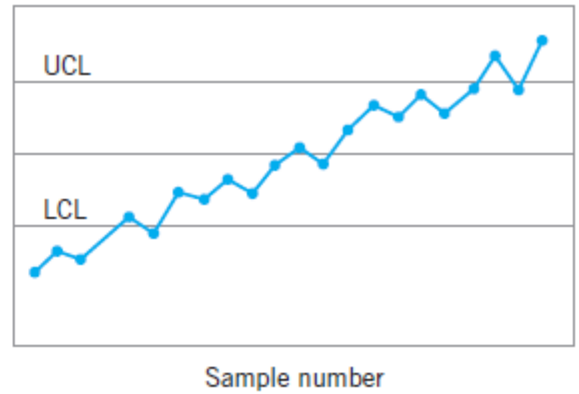
Zone Rules 5, 6 & 8

8. Mixture



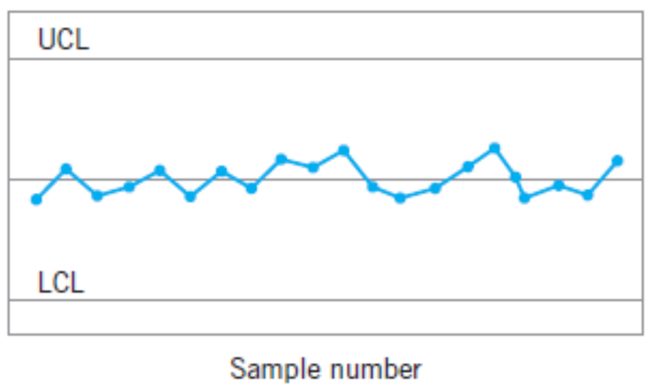
■ FIGURE 6.9 A mixture pattern.

5. Trend



■ FIGURE 6.11 A trend in process

6. Stratification



■ FIGURE 6.12 A stratification pattern.

The Zone Rules are hypothesis tests

- Null hypothesis H_0: The process is in control
- Alternative hypothesis H_1: The process is out-of-control
  - Equivalent notation, $H_1 = H_a$

- High standard of proof to reject the null hypothesis

Analogy: innocent until proven guilty

		Actual state of null hypothesis	
		True *Innocent*	**False** *Guilty*
Test of null hypothesis	**Fail to reject** *"Not guilty"*	**Confidence 1-α** *Innocent person goes free*	**β-risk β** *Guilty person goes free*
	Reject *"Guilty"*	**Significance α** *Innocent person goes to jail*	**Power 1-β** *Guilty person goes to jail*

Source: Neyman-Pearson criterion

Hypothesis testing

Test of null hypothesis (<u>inference</u>)		**<u>Actual</u> state of null hypothesis**	
		True	**False**
	Fail to reject	**Confidence 1-α** *True Positive* No difference is found when there is no difference	**β-risk β** *False Negative* *Type II error* *Consumer's Risk* No difference is found when there is a difference
	Reject	**Significance α** *False Positive* *Type I error* *Producer's Risk* A difference is found when there is no difference	**Power 1-β** *True Negative* A difference is found when there is a difference

Source: Neyman-Pearson criterion

Either we *reject the null hypothesis* or we *fail to reject the null hypothesis*

- Therefore, there is sufficient or insufficient evidence for H_a
- We do not infer that H_0 and H_a are true or false
- This shows some humility that we might have a false positive (Type I error)

Calculations of α and β

Out-of-control limit, *L*=3

$\Phi(L)$ is the cumulative probability distribution

- Take 1-$\Phi(L)$ to get the right tail
- Multiply by 2 to get two tails
- In Excel, use NORM.S.DIST(L,TRUE)

Mean shift to detect, $\Delta\bar{x} = k\sigma$

Sample size, *n*

$$\alpha = 2 * \left(1 - \Phi(L)\right) = 0.0027$$

$$\beta = \Phi\left(L - \frac{\Delta\bar{x}}{\sigma_{\bar{x}}}\right) - \Phi\left(-L - \frac{\Delta\bar{x}}{\sigma_{\bar{x}}}\right) = \Phi\left(L - \frac{k\sigma}{\sigma/\sqrt{n}}\right) - \Phi\left(-L - \frac{k\sigma}{\sigma/\sqrt{n}}\right)$$
$$= \Phi(L - k\sqrt{n}) - \Phi(-L - k\sqrt{n})$$

Calculations of α and β

$$\alpha = 2 * \left(1 - \Phi(L)\right)$$

$$\beta = \Phi(L - k\sqrt{n}) - \Phi(-L - k\sqrt{n})$$

$\Phi(L)$ is the standard normal distribution

- NORM.S.DIST(L, TRUE) in Excel
- pnorm(L, mean=0, sd=1) in R

Average Run Length (*ARL*)

ARL = Average number of points that must be plotted before one is outside control limits

p = Probability a point is outside control limits

$$p = \begin{cases} \alpha \text{ if process is in−control} \\ 1-\beta \text{ if process is out−of−control} \end{cases}$$

For one out-of-control point:

$$ARLp = 1 \text{ so } ARL = \frac{1}{p}$$

False positive run length (actually in control but infer out-of-control):

$$ARL_0 = \frac{1}{\alpha}$$

True negative run length (actually out-of-control and infer out-of-control):

$$ARL_1 = \frac{1}{1-\beta}$$

ARL_0 and why L=3

L=3

$$\alpha = 2 * \left(1 - \Phi(L)\right) = 0.0027$$

$$ARL_0 = \frac{1}{\alpha} = 370$$

370 seems reasonable which is why L=3 is the standard choice

Average Time to Signal (*ATS*)

h = time between samples (sampling interval length)

Time to detect a false positive:

$ATS_0 = hARL_0$

Time to detect a true negative:

$ATS_1 = hARL_1$

As we'll show later, if ATS_1 is too long, we need to decrease h or increase n

Out-Of-Control Action Plan

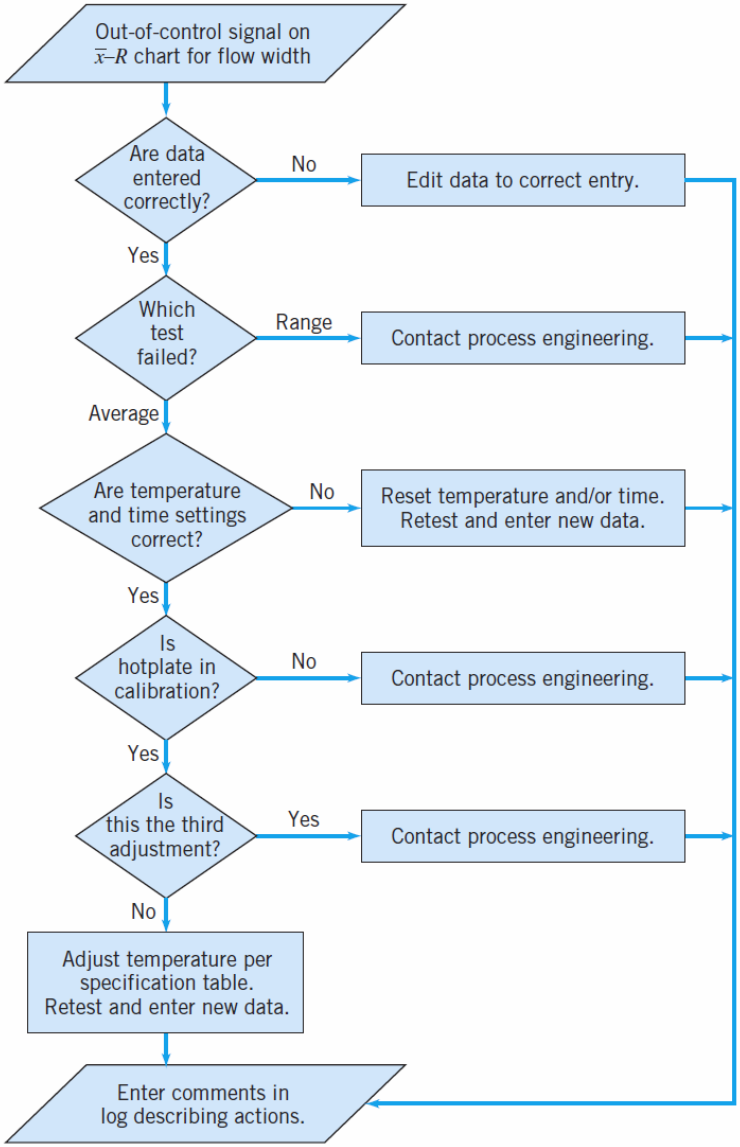


FIGURE 5.6 The out-of-control-action plan (OCAP) for the hard-bake process.

Note: Approaches to Control Charts with Varying Sample Sizes

n_i = sample size of i^{th} sample

A. Variable-width control limits: Use n_i instead of n for UCL_i and LCL_i at each point
   - Hard to calculate zone rules

B. Average-width control limits: Use $\bar{n}$ for *UCL* and *LCL*
   - OK if the n_i's don't vary too much

C. Standardized control limits: $z_i = \frac{x_i - \bar{x}}{\sigma_{\bar{x}}}$, *UCL* = 3, center line = 0, *LCL* = -3
   - Easy to apply, hard to explain
   - Can use for *x, p, np, c* and *u* charts

n > 1; differences from x-bar, R

X-BAR, S

Sample statistics

Sample size *n*

Sample $x_1, x_2, \dots, x_n$

Sample mean: $\bar{x} = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$

Sample variance: $s^2 = \dfrac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1}$

Since we take multiple samples

Number of samples groups *m*

Average sample mean:

$$\bar{\bar{x}} = \frac{\bar{x}_1 + \bar{x}_2 + \cdots + \bar{x}_m}{m} \qquad (6.2)$$

Average sample standard deviation:

$$\bar{s} = \frac{1}{m}\sum_{i=1}^{m} s_i$$

Unbiased estimates of μ and σ

$E(\bar{x}) = \mu$
$E(s^2) = \sigma^2$
$E(s) \neq \sigma$
$E(s) = c_4\sigma$ where c_4 is in Appendix VI

Unbiased estimates:

$\hat{\mu} = \bar{\bar{x}}$, unbiased estimate of the mean
$\hat{\sigma} = \frac{\bar{s}}{c_4}$ unbiased estimate of the standard deviation

$\bar{x}$ Chart's Upper Control Limit (UCL) and Lower Control Limit (LCL)

When using *s* Chart

$$\text{UCL} = \overline{\overline{x}} + \frac{3\overline{s}}{c_4\sqrt{n}}$$

$$\text{Center line} = \overline{\overline{x}}$$

$$\text{LCL} = \overline{\overline{x}} - \frac{3\overline{s}}{c_4\sqrt{n}}$$

$$\begin{aligned} \text{UCL} &= \overline{\overline{x}} + A_3\overline{s} \\ \text{Center line} &= \overline{\overline{x}} \\ \text{LCL} &= \overline{\overline{x}} - A_3\overline{s} \end{aligned} \tag{6.28}$$

s Chart's Upper Control Limit (UCL) and Lower Control Limit (LCL)

$$\text{UCL} = \bar{s} + 3\frac{\bar{s}}{c_4}\sqrt{1 - c_4^2}$$

$$\text{Center line} = \bar{s}$$

$$\text{LCL} = \bar{s} - 3\frac{\bar{s}}{c_4}\sqrt{1 - c_4^2}$$

We usually define the constants

$$B_3 = 1 - \frac{3}{c_4}\sqrt{1 - c_4^2} \quad \text{and} \quad B_4 = 1 + \frac{3}{c_4}\sqrt{1 - c_4^2} \tag{6.26}$$

$$\begin{aligned} \text{UCL} &= B_4\bar{s} \\ \text{Center line} &= \bar{s} \\ \text{LCL} &= B_3\bar{s} \end{aligned} \tag{6.27}$$

n = 1; differences from x-bar, R

XMR

Sample statistics

Sample size *n*

Sample $x_1, x_2, \ldots, x_n$

Sample mean: $$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Moving Range: $$\mathrm{MR}_i = |x_i - x_{i-1}|$$

Average moving range: $\overline{MR} = \sum_{i=2}^{m} MR_i$

Unbiased estimates of μ and σ

$E(\bar{x}) = \mu$

$E\left(\frac{MR}{d_2}\right) = \sigma$ where d_2 = 1.128 is in Appendix VI since n = 2

Unbiased estimates:

$\hat{\mu} = \bar{x}$, unbiased estimate of the mean

$\hat{\sigma} = \frac{\overline{MR}}{d_2}$, unbiased estimate of the standard deviation

$\hat{\sigma}_R = d_3 \frac{\overline{MR}}{d_2}$, unbiased estimate of the range's standard deviation

x Chart's Upper Control Limit (UCL) and Lower Control Limit (LCL)

$$UCL = \bar{x} + 3\hat{\sigma} = \bar{x} + 3\frac{\overline{MR}}{d_2} = \bar{x} + 3\frac{\overline{MR}}{1.128}$$

$$UCL = \bar{x} + A_3\overline{MR} = \bar{x} + 2.659\overline{MR}$$

centerline = $\bar{x}$

$$LCL = \bar{x} - A_3\overline{MR} = \bar{x} - 2.659\overline{MR}$$

MR Chart's Upper Control Limit (UCL) and Lower Control Limit (LCL)

Same as *R* chart with *n*=2

$UCL = D_4\overline{MR} = 3.267\overline{MR}$

Centerline = $\overline{MR}$

$LCL = D_3\overline{MR} = 0\overline{MR}$=0

α, β, *ARL* and *ATS*
Same as with $\bar{x}$, *R* charts with *n*=2

Out-of-control limit, *L*=3

Mean shift to detect, $\Delta\bar{x} = k\sigma$

Sample size, *n*=1

h = time between samples (sampling interval length)

Significance, $\alpha = 2 * \left(1 - \Phi(L)\right) = 0.0027$

β-risk, $\beta = \Phi\left(L - k\sqrt{1}\right) - \Phi\left(-L - k\sqrt{1}\right) = \Phi(L - k) - \Phi(-L - k)$

False positive run length: $ARL_0 = \frac{1}{\alpha}$

True negative run length: $ARL_1 = \frac{1}{1-\beta}$

Time to detect a false positive: $ATS_0 = hARL_0$

Time to detect a true negative: $ATS_1 = hARL_1$

Typically n=1 but also for n>1 applied to x-bar

CUSUM

Cumulative Sum (CUSUM) Chart

Good at detecting small shifts in mean

Better than xMR (n=1) for mean shift < 1.5σ

Let μ_o = target mean
μ_1 = shifted mean
n = sample size

Cumulative Sum: $C_i = \sum_{j=1}^{i}(\bar{x}_j - \mu_o) = C_{i-1} + (\bar{x}_j - \mu_o)$, i=1,…,m

If mean shifts to $\mu_1 \neq \mu_0$, C_i will drift:
- upward if $\mu_1 > \mu_0$
- downward if $\mu_1 < \mu_0$

We'll show for the case n=1, but the results hold for n>1

CUSUM Slack

Let $x_i \sim N(\mu, \sigma^2)$

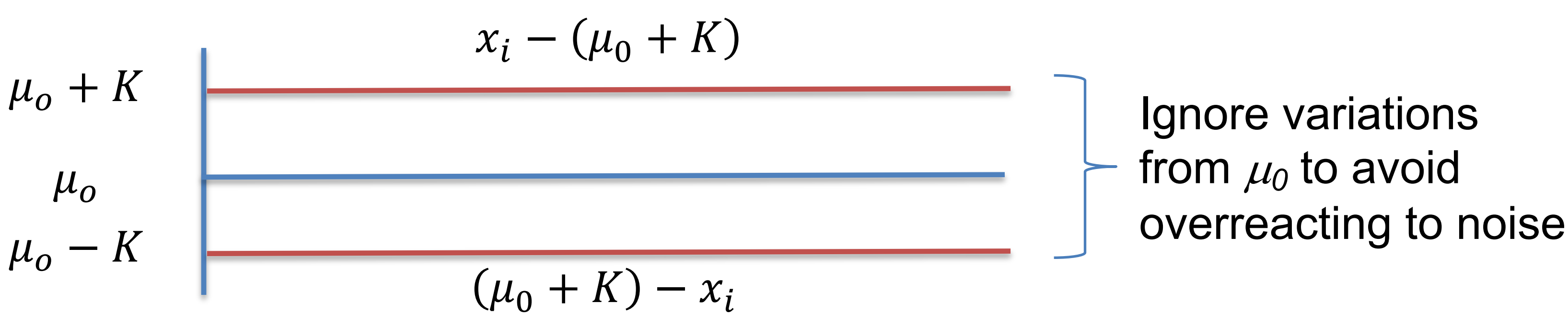
μ_o= target mean

μ_1 = shifted mean

K = “**slack**” (”reference value,” “allowance”)

where we ignore variations from μ_o

= $k\sigma$ WARNING: Not the same k as in $\bar{x}$, R charts!



Two-Sided Tabular CUSUM

$$C_i^+ = \max\left[0, x_i - (\mu_0 + K) + C_{i-1}^+\right] \tag{9.2}$$

$$C_i^- = \max\left[0, (\mu_0 - K) - x_i + C_{i-1}^-\right] \tag{9.3}$$

where the starting values are $C_0^+ = C_0^- = 0$.

N_i^+ = number of points in a row where C_i^+ is > 0
N_i^- = number of points in a row where C_i^- is > 0

H = **decision interval** where process is flagged as out-of-control
= $h\sigma$
If $C_i^+ > H$ or $C_i^- > H$ then process flagged as out-of-control

Two-Sided Tabular CUMUM Chart

UDB = *H* = upper decision bound
K = upper slack
Centerline = 0
-*K* = lower slack
LDB = -*H* = lower decision bound

Note: R's qcc package doesn't use *K* and has a fast initial response start-up

Plot C_i^+ and -C_i^-

If $C_i^+ > UDB$ or -$C_i^- < LDB$ then out-of-control

Often choose *k*=0.5 and *h*=5

TABLE 9.4

Values of k and the Corresponding Values of h That Give $ARL_0 = 370$ for the Two-Sided Tabular CUSUM [from Hawkins (1993a)]

k	0.25	0.5	0.75	1.0	1.25	1.5
h	8.01	4.77	3.34	2.52	1.99	1.61

TABLE 9.3

ARL Performance of the Tabular CUSUM with $k = \frac{1}{2}$ and $h = 4$ or $h = 5$

	Shift in Mean (multiple of σ)	$h = 4$	$h = 5$
ARL_0 ▸	0	168	465
	0.25	74.2	139
	0.50	26.6	38.0
	0.75	13.3	17.0
	1.00	8.38	10.4
	1.50	4.75	5.75
	2.00	3.34	4.01
	2.50	2.62	3.11
	3.00	2.19	2.57
	4.00	1.71	2.01

After a shift, what is a good estimate of the new process mean, $\bar{\mu}$?

$$\hat{\mu} = \begin{cases} \mu_0 + K + \dfrac{C_i^+}{N^+}, & \text{if } C_i^+ > H \\ \mu_0 - K - \dfrac{C_i^-}{N^-}, & \text{if } C_i^- > H \end{cases} \tag{9.5}$$

Fast Initial Response (FIR)

Headstart

Instead of starting with $C_0^+ = C_0^- = 0$

Start with $C_0^+ = C_0^- = H/2$

Makes CUSUM more sensitive at startup

Will quickly go to 0 if in-control

Modified CUSUMs' ARLs

■ **TABLE 9.5**

ARL Values for Some Modifications of the Basic CUSUM with $k = \frac{1}{2}$ and $h = 5$ (If subgroups of size $n > 1$ are used, then $\sigma = \sigma_{\bar{x}} = \sigma/\sqrt{n}$)

Shift in Mean (multiple of σ)	(a) Basic CUSUM	(b) CUSUM–Shewhart (Shewhart limits at 3.5σ)	(c) CUSUM with FIR	(d) FIR CUSUM–Shewhart (Shewhart limits at 3.5σ)
0	465	391	430	360
0.25	139	130.9	122	113.9
0.50	38.0	37.20	28.7	28.1
0.75	17.0	16.80	11.2	11.2
1.00	10.4	10.20	6.35	6.32
1.50	5.75	5.58	3.37	3.37
2.00	4.01	3.77	2.36	2.36
2.50	3.11	2.77	1.86	1.86
3.00	2.57	2.10	1.54	1.54
4.00	2.01	1.34	1.16	1.16

Typically n=1 but also for n>1 applied to x-bar

EWMA

Exponentially Weighted Moving Average (EWMA) Chart

Very robust when data is non-normal
Let x_i be independent with mean μ_0 and variance σ^2

EWMA:

$$z_i = \lambda x_i + (1-\lambda) z_{i-1}$$

$$\lambda \in (0,1)$$

Smaller λ makes EWMA less sensitive to non-normality

Steady state variance of z:

$$\sigma_z^2 = \lambda^2 \sigma^2 + (1-\lambda)^2 \sigma_z^2$$

Solving for σ_z gives:

$$\sigma_z = \sigma \sqrt{\frac{\lambda}{2-\lambda}}$$

Steady state control limits on z_i

$$\mathrm{UCL} = \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}} \tag{9.27}$$

Center line = μ_0, the target mean

$$\mathrm{LCL} = \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}} \tag{9.28}$$

Set $z_0 = \mu_0$, plot z_i, compare vs. control limits and check zone rule 1 to see if out-of-control

Non-steady state control limits

More accurate for initial data points

$$\sigma_{z_i}^2 = \sigma^2 \left(\frac{\lambda}{2-\lambda} \right) \left[1 - (1-\lambda)^{2i} \right]$$

$$\text{UCL} = \mu_0 + L\sigma \sqrt{\frac{\lambda}{(2-\lambda)} \left[1 - (1-\lambda)^{2i} \right]} \tag{9.25}$$

$$\text{Center line} = \mu_0$$

$$\text{LCL} = \mu_0 - L\sigma \sqrt{\frac{\lambda}{(2-\lambda)} \left[1 - (1-\lambda)^{2i} \right]} \tag{9.26}$$

Set $z_0 = \mu_0$, plot z_i, compare vs. control limits and check zone rule 1 to see if out-of-control

Usually choose L=3

■ **TABLE 9.11**

Average Run Lengths for Several EWMA Control Schemes [Adapted from Lucas and Saccucci (1990)]

	Shift in Mean (multiple of σ)	L = 3.054 λ = 0.40	2.998 0.25	2.962 0.20	2.814 0.10	2.615 0.05
ARL_0 ▸	0	500	500	500	500	500
	0.25	224	170	150	106	84.1
	0.50	71.2	48.2	41.8	31.3	28.8
	0.75	28.4	20.1	18.2	15.9	16.4
	1.00	14.3	11.1	10.5	10.3	11.4
	1.50	5.9	5.5	5.5	6.1	7.1
	2.00	3.5	3.6	3.7	4.4	5.2
	2.50	2.5	2.7	2.9	3.4	4.2
	3.00	2.0	2.3	2.4	2.9	3.5
	4.00	1.4	1.7	1.9	2.2	2.7

Note: Approaches to Control Charts with Varying Sample Sizes

n_i = sample size of ith sample

A. Variable-width control limits: Use n_i instead of n for UCL_i and LCL_i at each point
   - Hard to calculate zone rules
B. Average-width control limits: Use $\bar{n}$ for *UCL* and *LCL*
   - OK if the n_i's don't vary too much
C. Standardized control limits: $z_i = \frac{x_i - \bar{x}}{\sigma_{\bar{x}}}$, *UCL* = 3, center line = 0, *LCL* = -3
   - Easy to apply, hard to explain
   - Can use for x, p, np, c and u charts

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