Machine Vision: Homework 1

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1 Problem 1

1.1 Prove Vanishing Lines Geometrically

Please refer to Figure 1 [1] for the answer. Two dotted lines on the plane Φ are parallel, and the point O is a pinhole. Two dotted lines on the plane Π are the projections of the two parallel lines on the plane Φ. The line h is the intersection of the plane Π with the plane parallel to Φ and passing through the pinhole O.

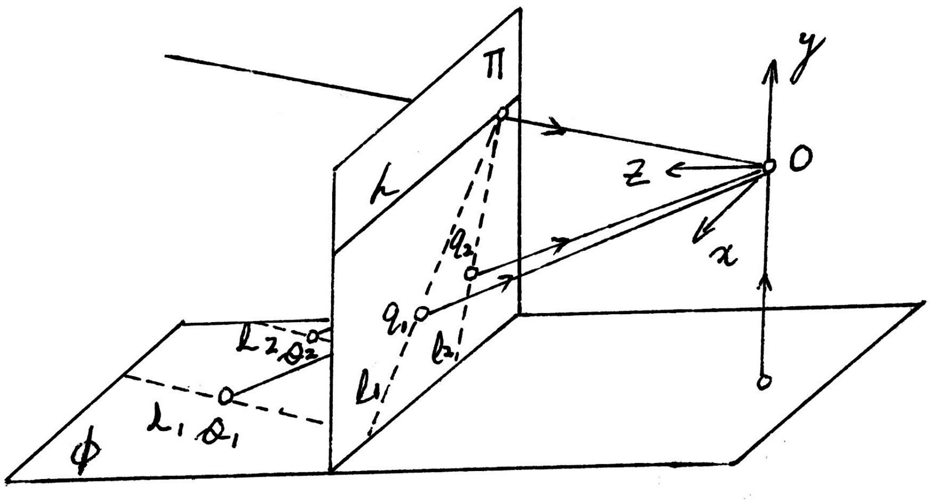

Figure 1: None differentiable function at $x = 0$.

As we can see from the figure, the projections of two parallel lines converge at a point lying on the horizontal line h. Note that we do not make any acclamation that the plane Φ is orthogomal to the plane Π in Figure 1.

1.2 Prove Vanishing Lines Algebraically

In this algebraical proof, we assume Φ is orthogonal to Π for simplicity.

The coordinate system centers on the pinhole O as shown in Figure 1. L_1 and $L2$ are two parallel lines on the plane Φ, l_1 and l_2 are their projections on the plane Π respectively. Q_1 and Q_2 are two points on L_1 and L_2 respectively; q_1 and q_2 are projections of Q_1 and Q_2 respectively.

Suppose the equation for the plane Φ is $y = d$, and the equation for the plane Π is $Z = f$; and in addition, suppose the coordinates for Q_1 and Q_2 are (X_1, d, Z_1) and (X_2, d, Z_2), and the coordinates for q_1 and q_2 are (x_1, y_1, f) and (x_2, y_2, f).

According to the perspective projection law, we have:

$$\begin{cases} x_1 = f\frac{X_1}{Z_1} \\ y_1 = f\frac{Y_1}{Z_1} = f\frac{d}{Z_1} \end{cases} \tag{1}$$

$$\begin{cases} x_2 = f\frac{X_2}{Z_2} \\ y_2 = f\frac{Y_2}{Z_2} = f\frac{d}{Z_2} \end{cases} \tag{2}$$

Let's define the equation of L_1 as:

$$\begin{cases} Z = aX + b \\ Y = d \end{cases} \tag{3}$$

Since L_1 is parallel to L_2, the equation of L_2 is:

$$\begin{cases} Z = aX + b' \\ Y = d \end{cases} \quad (b' \neq b) \tag{4}$$

Plugging euqation (3) into (1), the parametric equation of l_1 can be written as:

$$\begin{cases} x = f\frac{X}{aX+b} \\ y = f\frac{d}{aX+b} \\ z = f \end{cases} \xrightarrow{\text{eliminate } X} \begin{cases} x = -\frac{b}{ad}y + \frac{f}{a} \\ z = f \end{cases} \tag{5}$$

In a similar way, plugging equation (4) into (2), the parametric equation of l_2 can be written as:

$$\begin{cases} x = f\frac{X}{aX+b'} \\ y = f\frac{d}{aX+b'} \\ z = f \end{cases} \xrightarrow{\text{eliminate } X} \begin{cases} x = -\frac{b'}{ad}y + \frac{f}{a} \\ z = f \end{cases} \tag{6}$$

Because h is the intersection of two planes, the equation of line h is:

$$\begin{cases} y = 0 \\ z = f \end{cases} \tag{7}$$

Combing equation (5) and (6) together, the intersection of l_1 and l_2 is computed as $(\frac{f}{a}, 0, f)$. Therefore, the intersection lies on the line h.

Another explanation: from equation (3) and (4), we can see that:

$$Z \to \infty \Leftrightarrow X \to \infty \tag{8}$$

Applying equation (8) to equation (5), the line l_1 will converge at the point $(\frac{f}{a}, 0)$ on the plane Π; applying equation (8) to (6), the line l_2 will converge at the point $(\frac{f}{a}, 0)$ on the plane Π too. Therefore, these two vanishing points are on the same line h as shown in equation (7).

1.3 Vanishing Points

In homogeneous coordinates, suppose there are two points p_i and p_j, the line formed by p_i and p_j are given by $p_i \times p_j$, where $\times$ means cross product. In addition, the intersection of two lines l_i and l_j are given by $l_i \times l_j$.

In this problem, we need to calculate the intersection of two lines formed by p_1, q_1 and p_2, q_2. Suppose we have known the homogeneous coordinates of these four points (If not, these coordinates can be deduced from their Cartesian coordinates).

The line formed by p_1, q_1 is $l_1 = p_1 \times q_1$.

The line formed by p_2, q_2 is $l_2 = p_2 \times q_2$.

Therefore, the intersection (vanishing point) of l_1 and l_2 are given by $l_1 \times l_2$.

2 Problem 2

2.1 Formulation of Least-squares Problem

Suppose we have N pairs of images and corresponding light sources, and we want to compute the normal vector $\mathbf{S}(p)$ of each surface point p. $\mathbf{I}_i(p)$ is the image intensity at the point p of the i-th image, and v_i is source vector of the i-th directional light source. Based on the equation given by the problem description, we have:

$$\mathbf{I}_i(p) = \mathbf{S}(p)^T v_i \tag{9}$$

Considering we have N pairs of images and light sources, rewrite it in a matrix form:

$$\begin{bmatrix} \mathbf{I}_1(p) \\ \vdots \\ \mathbf{I}_i(p) \\ \vdots \\ \mathbf{I}_N(p) \end{bmatrix} = \mathbf{S}(p)^T \begin{bmatrix} v_1 \\ \vdots \\ v_i \\ \vdots \\ v_N \end{bmatrix} \tag{10}$$

Note that equation (10) is build for a single point p on the surface. In addition, $\mathbf{I}_i(p) \in \mathbb{R}$, $\mathbf{S}(p) \in \mathbb{R}^3$, and $v_i \in \mathbb{R}^3$.

Generally, we have more number of pairs of images and light sources than the number of unknown variables in $\mathbf{S}(p)^T$. We need to make the sum of squares of the difference between left term and right term as small as possible:

$$\min_{\mathbf{S}(p)} \sum_{i=1}^{N} (\mathbf{I}_i(p) - \mathbf{S}(p)^T v_i)^2 \tag{11}$$

This is a form of least-squares problem.

2.2 Number of pairs needed

Since there are three unknown variables in $\mathbf{S}(p)$ ($\mathbf{S}(p) \in \mathbb{R}^3$), we only need three pairs of images and light sources to solve this problem.

2.3 Linear basis

Lambertian surface $\mathbf{S}$ is a set of $\mathbb{R}^3$ vectors $\mathbf{S}(i)$, where i represents each point on the surface, and $i \in 1, ..., M$ (M points on the surface in total).

For any image $\mathbf{I}$, it can be represented by all of its points. For any point i in the image, it can be written as $\mathbf{I}(i) = \mathbf{S}(i)^T\mathbf{v}$. If we incorporate indicator function $I(\cdot)$, $\mathbf{I}(i)$ can be rewritten as:

$$\mathbf{I}(i) = \sum_{j=1}^{M} \mathbf{v}^T I(i = j)\mathbf{S}(j) \tag{12}$$

It is a linear combination of $\mathbf{S}(i)$ $(i \in 1, ..., M)$. Since in a Lambertian surface, no three different points lay on the same plane, which means that there are at least three $\mathbf{S}(i), \mathbf{S}(j), \mathbf{S}(k)$ are not coplanar. Therefore, $\mathbf{S}(i)$ $(i \in 1, ..., M)$ are linear independent.

Above all, $\mathbf{S}$ is a linear basis that can represent all images of this surface.

2.4 Limitations

1. Simplistic reflectance. Lambertian surface model only considers diffuse reflection and ignores specular reflection which exists in general surfaces.
2. No interreflections. Lambertian surface model does not consider interreflections between different parts of a surface when the surface is bumpy.

References

[1] Forsyth, D. A., & Ponce, J. (2003). A modern approach. Computer vision: a modern approach, 88.