Convolution and Deconvolution

Hao Huang

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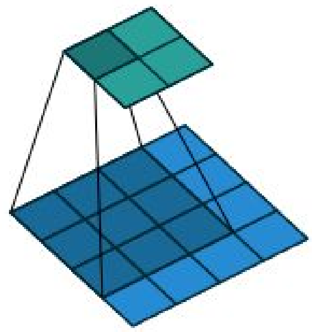
Contents

1 2D Convolution



The blue grid is input image (4×4), and the green grid is output image (2×2). The kernel size is 3×3. Stride is 1 and padding is 0. The weight of the kernel at the localization (i, j) is donated as $w_{i,j}$. The value of the image at the location (i, j) is donated as $x_{i,j}$. The kernel is:

$$w = \begin{bmatrix} w_{0,0} & w_{0,1} & w_{0,0} \\ w_{1,0} & w_{1,1} & w_{1,2} \\ w_{2,0} & w_{2,1} & w_{2,2} \end{bmatrix}$$

Rewrite image X as a vector: $\begin{bmatrix} x_{0,0} & x_{0,1} & \dots & x_{3,2} & x_{3,3} \end{bmatrix}^T$.Therefore, convolution can be written as matrix multiplication:

$$Y = w * X = WX =$$

$$\begin{bmatrix} w_{0,0} & w_{0,1} & w_{0,2} & 0 & w_{1,0} & w_{1,1} & w_{1,2} & 0 & w_{2,0} & w_{2,1} & w_{2,2} & 0 & 0 & 0 & 0 & 0 \\ 0 & w_{0,0} & w_{0,1} & w_{0,2} & 0 & w_{1,0} & w_{1,1} & w_{1,2} & 0 & w_{2,0} & w_{2,1} & w_{2,2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & w_{0,0} & w_{0,1} & w_{0,2} & 0 & w_{1,0} & w_{1,1} & w_{1,2} & 0 & w_{2,0} & w_{2,1} & w_{2,2} & 0 \\ 0 & 0 & 0 & 0 & 0 & w_{0,0} & w_{0,1} & w_{0,2} & 0 & w_{1,0} & w_{1,1} & w_{1,2} & 0 & w_{2,0} & w_{2,1} & w_{2,2} \end{bmatrix} \begin{bmatrix} x_{0,0} \\ x_{0,1} \\ x_{0,2} \\ x_{0,3} \\ x_{1,0} \\ x_{1,1} \\ x_{1,2} \\ x_{1,3} \\ x_{2,0} \\ x_{2,1} \\ x_{2,2} \\ x_{2,3} \\ x_{3,0} \\ x_{3,1} \\ x_{3,2} \\ x_{3,3} \end{bmatrix}$$

W is a doubly block circulant matrix (a special case of Toeplitz matrix). Here we define:

$$W_0 = \begin{bmatrix} w_{0,0} & w_{0,1} & w_{0,2} & 0 \\ 0 & w_{0,0} & w_{0,1} & w_{0,2} \end{bmatrix} \quad W_1 = \begin{bmatrix} w_{1,0} & w_{1,1} & w_{1,2} & 0 \\ 0 & w_{1,0} & w_{1,1} & w_{1,2} \end{bmatrix}$$

$$W_3 = \begin{bmatrix} w_{2,0} & w_{2,1} & w_{2,2} & 0 \\ 0 & w_{2,0} & w_{2,1} & w_{2,2} \end{bmatrix} \; \mathbf{0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

W_0,W_1,W_2, and $\mathbf{0}$ are all Toeplitz matrix. W can be written as:

$$W = \begin{bmatrix} W_0 & W_1 & W_2 & \mathbf{0} \\ \mathbf{0} & W_0 & W_1 & W_2 \end{bmatrix}$$

Therefore, W is a doubly block circulant matrix.

2 2D Deconvolution

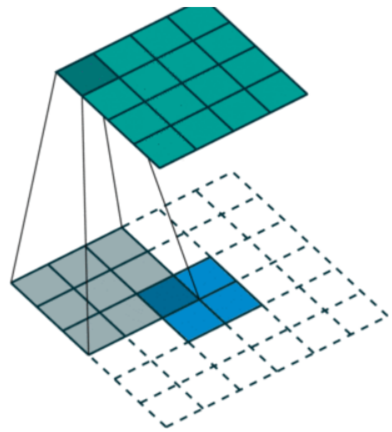
In the above example, the size of X is 16×1 and the size of Y is 4×1. The dimension of W is 4×16. In order to recover the size of Y to the size of X, we just need $W^T Y$ (We cannot recover the value of X, but just size):

$$Y' = W^T Y = W^T (WX)$$

If we donate Y as: $\begin{bmatrix} y_{0,0} & y_{0,1} & y_{1,0} & y_{1,1} \end{bmatrix}^T$, the deconvolution can be written as:

$$Y' = W^T Y = \begin{bmatrix} w_{0,0} & 0 & 0 & 0 \\ w_{0,1} & w_{0,0} & 0 & 0 \\ w_{0,2} & w_{0,1} & 0 & 0 \\ 0 & w_{0,2} & 0 & 0 \\ w_{1,0} & 0 & w_{0,0} & 0 \\ w_{1,1} & w_{1,0} & w_{0,1} & w_{0,0} \\ w_{1,2} & w_{1,1} & w_{0,2} & w_{0,1} \\ 0 & w_{1,2} & 0 & w_{0,2} \\ w_{2,0} & 0 & w_{1,0} & 0 \\ w_{2,1} & w_{2,0} & w_{1,1} & w_{1,0} \\ w_{2,2} & w_{2,1} & w_{1,2} & w_{1,1} \\ 0 & w_{2,2} & 0 & w_{1,2} \\ 0 & 0 & w_{2,0} & 0 \\ 0 & 0 & w_{2,1} & w_{2,0} \\ 0 & 0 & w_{2,2} & w_{2,1} \\ 0 & 0 & 0 & w_{2,2} \end{bmatrix} \begin{bmatrix} y_{0,0} \\ y_{0,1} \\ y_{1,0} \\ y_{1,1} \end{bmatrix}$$

If we rewrite it in convolution format $Y' = w' * Y$, where w' is the top-bottom AND left-right flip of the original filter w.

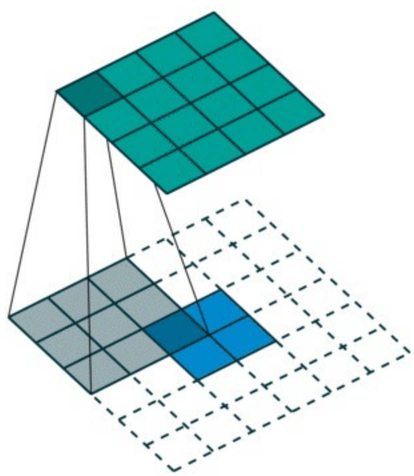


In the above image, the blue grid is Y, and the green grid is Y'. Stride is 1 and padding is 2.
We can write this process as:

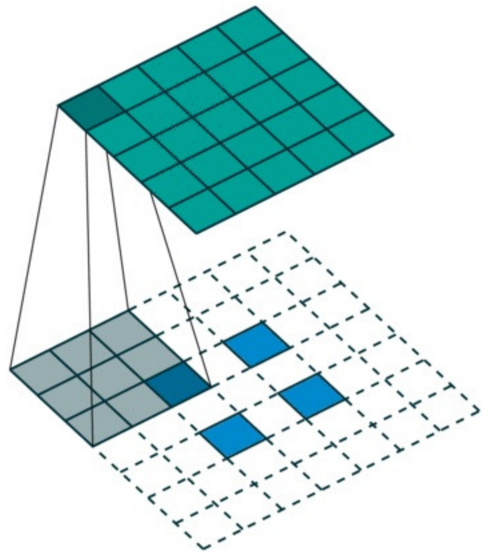
$$X- > Y = conv(X)- > deconv(Y)- > Y'$$

To keep Y' has the same size as X, we need to add paddings to $conv(X)$. Usually we add 0 paddings to $conv(X)$. There are two common padding strategies:

1. Add paddings around an image



2. Add paddings inside an image



These two strategies yield different results. How to choose this two strategies depends on which one can keep the image size.