An Improvement of the Abramov-Petkovšek Reduction for Hypergeometric Terms

Hui Huang and Ziming Li
Key Laboratory of Mathematics Mechanization
AMSS, Chinese Academy of Sciences
Beijing (100190), China
Email: huanghui@amss.ac.cn, zmli@mmrc.iss.ac.cn

The ubiquity of hypergeometric terms in combinatorial enumeration is widely recognized, such as binomial, exponential and factorial functions, etc. Abramov and Petkovšek present an additive decomposition of a hypergeometric term, which allows us to determine hypergeometric summability [1, 2]. Their reduction algorithm needs to compute polynomial solutions of a first-order linear recurrence equation. We show to avoid this step using a new type of polynomial reduction. The new polynomial reduction reduces both the degree and the term number of certain numerators, while Abramov and Petkovšek's merely reduces the degree. Our idea is a difference variant of polynomial reduction for hyperexponential functions given in [3].

Let C be a field of characteristic zero, and n an indeterminate. Define the *shift operator* σ from $C(n)$ to $C(n)$ by mapping $\sigma(R(n)) = R(n+1)$ for all $R(n) \in C(n)$. Also, we set $\Delta = \sigma - 1$, which denotes the *difference operator* on rational functions.

Let $\mathbb{D}$ be a difference ring extension of $C(n)$. A nonzero element T of $\mathbb{D}$ is said to be a *hypergeometric term* over $C(n)$ if $\sigma(T) = rT$ for some $r \in C(n)$. We call r the *shift quotient* of T. A univariate hypergeometric term T is said to be *hypergeometric summable* if there exists a hypergeometric term G such that $T = \Delta(G)$.

We recall some terminologies in [1, §1]. A polynomial $a \in C[n]$ is *shift-free* if $\gcd(a, \sigma^m(a)) = 1$ for every nonzero integer m. Let $R = a/b$ be a rational function in $C(n)$ with $\gcd(a, b) = 1$. We say that R is *shift-reduced* if $\gcd(a, \sigma^m(b)) = 1$ for every integer m. According to [2], a nonzero rational function $R \in C(n)$ has as a rational normal form (K, S) in $C(n) \times C(n)$ such that $R = K \cdot (\sigma(S)/S)$ and K is shift-reduced. We call K and S the *kernel* and the *shell* of R, respectively. For brevity, we write $K = u/v$ for some $u, v \in C[n]$ with $\gcd(u, v) = 1$. We call an irreducible polynomial q in $C[n]$ is *strongly-coprime* with K if it satisfies the following properties:

(i) $\sigma^m(q) \nmid u$ if $m \geq 0$,

(ii) $\sigma^m(q) \nmid v$ if $m \leq 0$.

Moreover, a nonzero polynomial $p \in C[n]$ is said to be *strongly coprime* with K if its irreducible factors are all strongly-coprime with K.

Given a hypergeometric term T over $C(n)$ whose shift quotient has a rational normal form with kernel K and shell S in $C(n)$. Then $T = SH$ for some hypergeometric term H over $C(n)$ with $\sigma(H)/H = K$. The Abramov-Petkovšek Reduction rewrites T as

$$T = \Delta(S_1 H) + S_2 H \quad \text{for some } S_1, S_2 \in C(n) \tag{1}$$

such that

(i) If T is hypergeometric summable, then $S_2 = 0$;

(ii) If T is not hypergeometric summable, then the shift quotient of $S_2 H$ has a rational normal form (F, V), where the denominator of V is minimal in some technical sense.

In order to guarantee the above two properties, one needs to find a polynomial solution of a first-order linear recurrence equation [2, §4].

We extend the idea for polynomial reduction for hyperexponential functions in [3, §4]. Let $K = u/v$ with $u, v \in C[n]$ and $\gcd(u, v) = 1$. Define the *subspace for polynomial reduction w.r.t* K as

$$\mathcal{M}_K = \{u\sigma(f) - vf \mid f \in C[n]\}$$

and the *standard complement* of $\mathcal{M}_K$ as

$$\mathcal{N}_K = \operatorname{span}_C \left\{ n^\ell \mid \ell \in \mathbb{N} \text{ and } \ell \neq \deg(p) \text{ for all } p \in \mathcal{M} \right\}.$$

We show that the dimension of $\mathcal{N}_K$ is bounded by $\max\left(\deg(u), \deg(v)\right)$.

A basis $\mathcal{B}$ for $\mathcal{M}_K$ is called an *echelon* basis if distinct elements in $\mathcal{B}$ have distinct degrees. Using an echelon basis of $\mathcal{M}_K$, one can reduce a polynomial p to a unique polynomial $\tilde{p} \in \mathcal{N}_K$ s.t. $p - \tilde{p} \in \mathcal{M}_K$. After forming a summable hypergeometric term, we obtain

$$T = \Delta\left(gH\right) + \left(\frac{a}{b} + \frac{p}{v}\right) H, \tag{2}$$

where $g \in C(n)$, $a, b \in C[n]$ with $\gcd(a, b) = 1$ and $\deg(a) < \deg(b)$, b is shift-free and strongly-coprime with K, and $p \in \mathcal{N}_K$. Since $p \in \mathcal{N}_K$, the number of terms in p is bounded by the dimension of $\mathcal{N}_K$. Set $S_2 = a/b + p/v$. The decomposition (2) possesses the same properties as decomposition (1) does. This reduction does not need to compute a polynomial solution of any recurrence equation.

Assume that $\tau_K = \operatorname{lc}(v - u)/\operatorname{lc}(u)$. If $\tau_K \in \mathbb{Z}^+$ and $\deg(v - u) = \deg(u) - 1$, then a tight degree bound for p in (2) is

$$\tau_K + \deg(u) - 1.$$

But the number of terms in p is bounded by $\dim(\mathcal{N}_K)$, which does not depend on τ_K. This observation reveals that p is a sparse polynomial.

Note that it is possible that a hypergeometric term T has two additive decompositions

$$T = \Delta(g_1 H) + \underbrace{\left(\frac{a_1}{b_1} + \frac{p_1}{v}\right)}_{r_1} H = \Delta(g_2 H) + \underbrace{\left(\frac{a_2}{b_2} + \frac{p_2}{v}\right)}_{r_2} H.$$

where g_i is in $C(n)$, v is the denominator of K, $\deg(a_i)$ is less than $\deg(b_i)$, b_i is shift-free and strongly coprime with K, and p_i is in $\mathcal{N}_K$ for $i = 1, 2$. We are going to investigate about the relations between r_1 and r_2. So far, we have shown that there is a one-to-one correspondence ϕ between the multisets of nonconstant monic irreducible factors of b_1 and b_2 such that q is shift-equivalent to $\phi(q)$ for all $q \mid b_1$.

We are going to implement the new reduction algorithm and compare it with the maple implementation of the Abramov-Petkovšek reduction in the computer algebra system Maple package SumTools.

References

[1] S.A. Abramov and M. Petkovšek. Minimal decomposition of indefinite hypergeometric sums. In *ISSAC'01: Proceedings of the 2001 International Symposium on Symbolic and Algebraic Computation*, pages 7–14, New York, 2001. ACM.

[2] S.A. Abramov and M. Petkovšek. Rational normal forms and minimal decompositions of hypergeometric terms. *J. Symbolic Comput.*, 33:521–543, 2002.

[3] A. Bostan, S. Chen, F. Chyzak, Z. Li, and G. Xin. Hermite reduction and creative telescoping for hyperexponential functions. In *To appear in ISSAC'13: Proceedings of the 2013 International Symposium on Symbolic and Algebraic Computation*, Boston, Massachusetts, USA, 2013. ACM.