Efficient Rational Creative Telescoping

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Joint work with Mark Giesbrecht, George Labahn and Eugene Zima

Outline

- Technique of creative telescoping
- New algorithm for bivariate rational functions

From 1st year calculus class

Consider

$$\sum_{k=0}^{n} k k!$$

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$$\sum_{k=0}^{n} kk!$$

- Telescoping

$$kk! \; = \; (k+1)! - k!$$

From 1st year calculus class

Consider

$$\sum_{k=0}^{n} kk!$$

- Telescoping

$$\sum_{k=0}^{n} kk! = \sum_{k=0}^{n} \big((k+1)! - k!\big)$$

From 1st year calculus class

Consider

$$\sum_{k=0}^{n} kk!$$

- Telescoping

$$\sum_{k=0}^{n} kk! \;=\; (n+1)! - n! + n! - (n-1)! + \cdots + 1! - 0!$$

From 1st year calculus class

Consider

$$\sum_{k=0}^{n} kk!$$

- Telescoping

$$\sum_{k=0}^{n} kk! \;=\; (n+1)! - \cancel{n!} + \cancel{n!} - \cancel{(n-1)!} + \cdots + \cancel{1!} - 0!$$

From 1st year calculus class

Consider

$$\sum_{k=0}^{n} kk!$$

- Telescoping

$$\sum_{k=0}^{n} kk! \;=\; (n+1)! - 1$$

From 1st year calculus class

Consider

$$\sum_{k=0}^{n} k k!$$

- Telescoping

$$\sum_{k=0}^{n} k k! \; = \; (n+1)! - 1$$

- $F(n) = \sum_{k=0}^{n} k k!$ satisfies

$$F(n) = (n+1)! - 1$$

From 1st year calculus class

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$$\sum_{k=0}^{n} kk! = (n+1)! - 1$$

- Telescoping

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From 2nd year calculus class

Consider

$$\sum_{k=0}^{n} \binom{n}{k}$$

From 2nd year calculus class

Consider

$$\sum_{k=0}^{n} \binom{n}{k}$$

- Creative telescoping

$$-2\binom{n}{k} + \binom{n+1}{k} \quad = \frac{(k+1)}{(k+1)-n-1}\binom{n}{k+1} - \frac{k}{k-n-1}\binom{n}{k}$$

From 2nd year calculus class

Consider

$$\sum_{k=0}^{n} \binom{n}{k}$$

- Creative telescoping

$$\sum_{k=0}^{n} \left(-2\tbinom{n}{k} + \tbinom{n+1}{k} \right) = \sum_{k=0}^{n} \left(\tfrac{(k+1)}{(k+1)-n-1} \tbinom{n}{k+1} - \tfrac{k}{k-n-1} \tbinom{n}{k} \right)$$

From 2nd year calculus class

Consider

$$\sum_{k=0}^{n} \binom{n}{k}$$

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$$-2\sum_{k=0}^{n}\tbinom{n}{k} + \sum_{k=0}^{n}\tbinom{n+1}{k} \;=\; \sum_{k=0}^{n}\left(\tfrac{(k+1)}{(k+1)-n-1}\tbinom{n}{k+1} - \tfrac{k}{k-n-1}\tbinom{n}{k}\right)$$

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From 2nd year calculus class

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- Creative telescoping

$$-2\sum_{k=0}^{n} \binom{n}{k} + \sum_{k=0}^{n} \binom{n+1}{k} = \frac{(n+1)}{(n+1)-n-1}\binom{n}{n+1} - \frac{0}{0-n-1}\binom{n}{0}$$

- $F(n) = \sum_{k=0}^{n} \binom{n}{k}$ satisfies

$$-2F(n) + F(n+1) = 0$$

From 2nd year calculus class

Consider

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

- Creative telescoping

$$-2\sum_{k=0}^{n} \tbinom{n}{k} + \sum_{k=0}^{n} \tbinom{n+1}{k} \;=\; \tfrac{(n+1)}{(n+1)-n-1}\tbinom{n}{n+1} - \tfrac{0}{0-n-1}\tbinom{n}{0}$$

- $F(n) = \sum_{k=0}^{n} \binom{n}{k}$ satisfies

$$-2F(n) + F(n+1) = 0$$

The creative telescoping problem

GIVEN $f(n,k)$, FIND $g(n,k)$ and $c_0(n),\ldots,c_\rho(n)$ such that

$$c_0(n)f(n,k)+\cdots+c_\rho(n)f(n+\rho,k) \;=\; g(n,k+1)-g(n,k)$$

Then $F(n)=\sum_{k=0}^{n} f(n,k)$ satisfies

$$c_0(n)F(n)+\cdots+c_\rho(n)F(n+\rho)=\text{explicit}(n)\ .$$

The creative telescoping problem

GIVEN $kk!$, FIND $k!$ and 1 such that

$$kk! = (k+1)! - k!$$

Then $F(n) = \sum_{k=0}^{n} kk!$ satisfies

$$F(n) = (n+1)! - 1 \qquad .$$

The creative telescoping problem

GIVEN $\binom{n}{k}$, FIND $\frac{k}{k-n-1}\binom{n}{k}$ and $-2, 1$ such that

$$-2\binom{n}{k} + \binom{n+1}{k} = \frac{(k+1)}{(k+1)-n-1}\binom{n}{k+1} - \frac{k}{k-n-1}\binom{n}{k}$$

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The creative telescoping problem

GIVEN $f(n,k)$, FIND $g(n,k)$ and $c_0(n),\dots,c_\rho(n)$ such that

$$\left(c_0(n)+\cdots+c_\rho(n)\sigma_n^\rho\right)(f(n,k)) \;=\; (\sigma_k-1)\big(g(n,k)\big)$$

Then $F(n)=\sum_{k=0}^{n} f(n,k)$ satisfies

$$c_0(n)F(n)+\cdots+c_\rho(n)F(n+\rho)=\text{explicit}(n)\ .$$

Notation. $\sigma_n(f(n,k))=f(n+1,k)$, $\sigma_k(f(n,k))=f(n,k+1)$, and $\Delta_k=\sigma_k-1$.

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The creative telescoping problem

GIVEN $f(n,k)$, FIND $g(n,k)$ and $c_0(n), \ldots, c_\rho(n)$ such that

$$\underbrace{\left(c_0(n) + \cdots + c_\rho(n)\sigma_n^\rho\right)}_{\text{telescoper}}(f(n,k)) \;=\; \Delta_k\underbrace{(g(n,k))}_{\text{certificate}}$$

Then $F(n) = \sum_{k=0}^{n} f(n,k)$ satisfies

$$c_0(n)F(n) + \cdots + c_\rho(n)F(n+\rho) = \text{explicit}(n)\ .$$

Notation. $\sigma_n(f(n,k)) = f(n+1,k)$, $\sigma_k(f(n,k)) = f(n,k+1)$,
and $\Delta_k = \sigma_k - 1$.

Generations of creative telescoping algorithms

1. Elimination in operator algebras / Sister Celine's algorithm (since $\approx$ 1947)
2. Zeilberger's algorithm and its generalizations (since $\approx$ 1990)
3. The Apagodu-Zeilberger ansatz (since $\approx$ 2005)
4. Hermite-like reduction based methods (since $\approx$ 2010)

Generations of creative telescoping algorithms

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4 $\boxed{\text{Hermite-like reduction based methods (since} \approx \text{2010)}}$

Reduction-based approach

Example. $\frac{-10n}{(nk+1)(n(k+10)+1)}+\frac{nk}{(n+2k)^2+2}-\frac{n(k+1)}{(n+2k+2)^2+2}+\frac{n(k+11)}{(n+2k+22)^2+2}$

Reduction-based approach

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$$f = \Delta_k\left(\boxed{g_0}\right) + \frac{nk}{(n+2k)^2+2}$$

$$\sum_{j=1}^{10} \frac{1}{n(k+j)+1} + \sum_{j=1}^{10} \frac{n(k+j)}{(n+2k+2j)^2+2}$$

Reduction-based approach

Example. $\underbrace{\frac{-10n}{(nk+1)(n(k+10)+1)}+\frac{nk}{(n+2k)^2+2}-\frac{n(k+1)}{(n+2k+2)^2+2}+\frac{n(k+11)}{(n+2k+22)^2+2}}_{f}$

$$f = \Delta_k\Big(g_0\Big) + \frac{nk}{(n+2k)^2+2}$$

$$\sigma_n(f) = \Delta_k\Big(g_1\Big) + \frac{(n+1)k}{(n+2k+1)^2+2}$$

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$$c_0(n)\, f = \Delta_k\Big(c_0(n)\, g_0\Big) + c_0(n)\, \frac{nk}{(n+2k)^2+2}$$

$$c_1(n)\, \sigma_n(f) = \Delta_k\Big(c_1(n)\, g_1\Big) + c_1(n)\, \frac{(n+1)k}{(n+2k+1)^2+2}$$

$$c_2(n)\, \sigma_n^2(f) = \Delta_k\Big(c_2(n)\, g_2\Big) + c_2(n)\, \frac{(n+2)(k-1)}{(n+2k)^2+2}$$

$$c_3(n)\, \sigma_n^3(f) = \Delta_k\Big(c_3(n)\, g_3\Big) + c_3(n)\, \frac{(n+3)(k-1)}{(n+2k+1)^2+2}$$

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Reduction-based approach

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$$
+\left\{
\begin{aligned}
c_0(n)\, f &= \Delta_k\big(c_0(n)\, g_0\big) + c_0(n)\, \frac{nk}{(n+2k)^2+2}\\
c_1(n)\, \sigma_n(f) &= \Delta_k\big(c_1(n)\, g_1\big) + c_1(n)\, \frac{(n+1)k}{(n+2k+1)^2+2}\\
c_2(n)\, \sigma_n^2(f) &= \Delta_k\big(c_2(n)\, g_2\big) + c_2(n)\, \frac{(n+2)(k-1)}{(n+2k)^2+2}\\
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c_4(n)\, \sigma_n^4(f) &= \Delta_k\big(c_4(n)\, g_4\big) + c_4(n)\, \frac{(n+4)(k-2)}{(n+2k)^2+2}
\end{aligned}
\right.
$$

$$c_0(n)\, f + \cdots + c_4(n)\, \sigma_n^4(f) = \Delta_k\left(\sum_{\ell=0}^{4} c_\ell(n)\, g_\ell\right) +$$

Reduction-based approach

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\end{aligned}
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$$

$$
c_0(n)\,f+\cdots+c_4(n)\,\sigma_n^4(f) = \Delta_k\Big(\sum_{\ell=0}^{4} c_\ell(n)\,g_\ell\Big) + \boxed{\overset{!}{=}0}
$$

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Example. $$\underbrace{\frac{-10n}{(nk+1)(n(k+10)+1)}+\frac{nk}{(n+2k)^2+2}-\frac{n(k+1)}{(n+2k+2)^2+2}+\frac{n(k+11)}{(n+2k+22)^2+2}}_{f}$$

$$+\left\{\begin{aligned}
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\end{aligned}\right.$$

$$c_0(n)\,f+\cdots+c_4(n)\,\sigma_n^4(f) = \Delta_k\Big(\sum_{\ell=0}^{4} c_\ell(n)\,g_\ell\Big) + \stackrel{!}{=} 0$$

Reduction-based approach

Example. $$\underbrace{\frac{-10n}{(nk+1)(n(k+10)+1)}+\frac{nk}{(n+2k)^2+2}-\frac{n(k+1)}{(n+2k+2)^2+2}+\frac{n(k+11)}{(n+2k+22)^2+2}}_{f}$$

$$\begin{pmatrix} 4n & 4n^2+4n & n^3+2n^2+3n & 0 \\ 4n+4 & 4n^2+4n & n^3+n^2+2n+2 & 0 \\ 4n+8 & 4n^2+8n & n^3-5n-2 & -n^3-4n^2-7n-6 \\ 4n+12 & 4n^2+8n-12 & n^3-n^2-10n+6 & -n^3-3n^2-2n-6 \\ 4n+16 & 4n^2+12n-16 & n^3-2n^2-29n-20 & -2n^3-12n^2-22n-24 \end{pmatrix}^T \begin{pmatrix} c_0(n) \\ c_1(n) \\ c_2(n) \\ c_3(n) \\ c_4(n) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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- A telescoper: $L = \frac{n+4}{n} + \frac{-2(n+4)}{n+2} \cdot \sigma_n^2 + 1 \cdot \sigma_n^4$

Reduction-based approach

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- A telescoper: $L = \frac{n+4}{n} + \frac{-2(n+4)}{n+2} \cdot \sigma_n^2 + 1 \cdot \sigma_n^4$

- A certificate: $g = \frac{n+4}{n} \cdot g_0 + \frac{-2(n+4)}{n+2} \cdot g_2 + 1 \cdot g_4$

Reduction-based approach

Example. $$\underbrace{\frac{-10n}{(nk+1)(n(k+10)+1)}+\frac{nk}{(n+2k)^2+2}-\frac{n(k+1)}{(n+2k+2)^2+2}+\frac{n(k+11)}{(n+2k+22)^2+2}}_{f}$$

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- A telescoper: $L = \frac{n+4}{n} + \frac{-2(n+4)}{n+2} \cdot \sigma_n^2 + 1 \cdot \sigma_n^4$

$$\sum_{j=1}^{10} \frac{1}{n(k+j)+1} + \sum_{j=1}^{10} \frac{n(k+j)}{(n+2k+2j)^2+2}$$

- A certificate: $g = \frac{n+4}{n} \cdot \boxed{g_0} + \frac{-2(n+4)}{n+2} \cdot g_2 + 1 \cdot g_4$

$$= \sum_{j=1}^{10} \frac{1}{n(k+j)+1} + \frac{(n+4)(k+10)}{(n+2k+24)^2+2} - \frac{(n+4)(k+11)}{(n+2k+22)^2+2} - \frac{(n+4)k}{(n+2k+4)^2+2} - \frac{2(n+4)k}{(n+2k+2)^2+2} - \frac{(n+4)k}{(n+2k)^2+2}$$

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Le's direct approach (2002)

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GGSZ reduction (2003)

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Moreover, for $r = \frac{a}{b}$,

- $\deg_k(a) < \deg_k(b)$ and b is σ_k-free;
- f is σ_k-summable $\iff a = 0$;
- g is expressed by a sparse form.

Integer-linear decomposition

Definition.

Integer-linear decomposition

Definition. $p \in C[n,k]$ irreducible, is integer-linear over C if

$$p = P(\lambda n + \mu k)$$

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integer-linear type

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$$P_i(\lambda_i n + \mu_i k) \sim_{n,k} P_j(\lambda_j n + \mu_j k),\ i \neq j$$

$$\Updownarrow$$

$$(\lambda_i, \mu_i) = (\lambda_j, \mu_j)\ \&\ P_i(z) = P_j(z + \nu),\ \nu \in \mathbb{Z}$$

Integer-linear decomposition

Definition. $p \in C[n,k]$ is integer-linear over C if

$$p = \prod_{i=1}^{m} \prod_{j=1}^{n_i} P_i(\lambda_i n + \mu_i k + \nu_{ij})^{e_{ij}}$$

- $P_i(z) \in C[z]$ squarefree, σ_z-free;
- $(\lambda_i, \mu_i) \in \mathbb{Z}^2$ coprime, $\mu_i \geq 0$;
- $e_{ij} \in \mathbb{Z}^+$; $0 = \nu_{i1} < \cdots < \nu_{in_i}$ in $\mathbb{Z}$;
- $P_i(\lambda_i n + \mu_i k) \not\sim_{n,k} P_j(\lambda_j n + \mu_j k)$, $i \neq j$.

Integer-linear decomposition

Definition. $p \in C[n,k]$ admits the integer-linear decomposition

$$p = P_0(n,k) \cdot \prod_{i=1}^{m} \prod_{j=1}^{n_i} P_i(\lambda_i n + \mu_i k + \nu_{ij})^{e_{ij}}$$

- $P_0 \in C[n,k]$ merely having non-integer-linear factors except for constants;
- $P_i(z) \in C[z]$ non-constant, squarefree, σ_z-free;
- $(\lambda_i, \mu_i) \in \mathbb{Z}^2$ coprime, $\mu_i \geq 0$;
- $e_{ij} \in \mathbb{Z}^+$; $0 = \nu_{i1} < \cdots < \nu_{in_i}$ in $\mathbb{Z}$;
- $P_i(\lambda_i n + \mu_i k) \nsim_{n,k} P_j(\lambda_j n + \mu_j k)$, $i \neq j$.

Integer-linear operator

Given $(\lambda, \mu) \in \mathbb{Z}^2$ coprime, $\mu \geq 0$.

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Our new approach

Example. $\frac{-10n}{(nk+1)(n(k+10)+1)}+\frac{nk}{(n+2k)^2+2}-\frac{n(k+1)}{(n+2k+2)^2+2}+\frac{n(k+11)}{(n+2k+22)^2+2}$

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$$\mathrm{LeftQuot}(\sigma_k^{10}-1,\sigma_k-1)\cdot\frac{1}{nk+1}$$

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$$\boxed{L = c_0(n) + c_1(n)\sigma_n + c_2(n)\sigma_n^2 + c_3(n)\sigma_n^3 + c_4(n)\sigma_n^4}$$

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$$\left(c_0(n)nk + c_2(n)(n+2)(k-1) + c_4(n)(n+4)(k-2)\right)$$
$$+ \left(c_1(n)(n+1)k + c_3(n)(n+3)(k-1)\right)\sigma_n$$

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$$\boxed{L \text{ is a telescoper} \quad\Longleftrightarrow\quad \tilde{R}=0}$$

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$$\begin{cases} c_0(n)nk+c_2(n)(n+2)(k-1)+c_4(n)(n+4)(k-2)=0\\ c_1(n)(n+1)k+c_3(n)(n+3)(k-1)=0\end{cases}$$

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$$\begin{pmatrix} 0 & 0 & -n-2 & 0 & -2n-8 \\ n & 0 & n+2 & 0 & n+4 \\ 0 & 0 & 0 & -n-3 & 0 \\ 0 & n+1 & 0 & n+3 & 0 \end{pmatrix} \begin{pmatrix} c_0(n) \\ c_1(n) \\ c_2(n) \\ c_3(n) \\ c_4(n) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

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Our new approach

Example. $\underbrace{\frac{-10n}{(nk+1)(n(k+10)+1)}+\frac{nk}{(n+2k)^2+2}-\frac{n(k+1)}{(n+2k+2)^2+2}+\frac{n(k+11)}{(n+2k+22)^2+2}}_{f}$

$$\begin{pmatrix} 0 & 0 & -n-2 & 0 & -2n-8 \\ n & 0 & n+2 & 0 & n+4 \\ 0 & 0 & 0 & -n-3 & 0 \\ 0 & n+1 & 0 & n+3 & 0 \end{pmatrix} \begin{pmatrix} \frac{n+4}{n} \\ 0 \\ \frac{-2(n+4)}{n+2} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

- A telescoper: $L = \frac{n+4}{n} + \frac{-2(n+4)}{n+2} \cdot \sigma_n^2 + 1 \cdot \sigma_n^4$

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- A telescoper: $L = \frac{n+4}{n} + \frac{-2(n+4)}{n+2} \cdot \sigma_n^2 + 1 \cdot \sigma_n^4$

- A certificate: $g = L \cdot \boxed{g_0} + \mathrm{LeftQuot}(L \cdot M, \sigma_k - 1) \cdot \frac{1}{(n+2k)^2+2}$

  $\mathrm{LeftQuot}(\sigma_k^{10} - 1, \sigma_k - 1) \cdot \frac{1}{nk+1}$

Our new approach

Example. $\underbrace{\frac{-10n}{(nk+1)(n(k+10)+1)}+\frac{nk}{(n+2k)^2+2}-\frac{n(k+1)}{(n+2k+2)^2+2}+\frac{n(k+11)}{(n+2k+22)^2+2}}_{f}$

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Recall: reduction-based approach

$$\begin{pmatrix} 4n & 4n^2+4n & n^3+2n^2+3n & 0 \\ 4n+4 & 4n^2+4n & n^3+n^2+2n+2 & 0 \\ 4n+8 & 4n^2+8n & n^3-5n-2 & -n^3-4n^2-7n-6 \\ 4n+12 & 4n^2+8n-12 & n^3-n^2-10n+6 & -n^3-3n^2-2n-6 \\ 4n+16 & 4n^2+12n-16 & n^3-2n^2-29n-20 & -2n^3-12n^2-22n-24 \end{pmatrix}^T \begin{pmatrix} \frac{n+4}{n} \\ 0 \\ \frac{-2(n+4)}{n+2} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Outline of algorithm (iteration version)

Input. $f \in C(n,k)$.

Output. A minimal telescoper L and a certificate g when exist.

1 $\mathrm{den}(f) = P_0 \prod_{i,j} P_i(\lambda_i n + \mu_i k + \nu_{ij})^{e_{ij}}$.

2 $f = \frac{f_0}{P_0} + \sum_{i,e} \overbrace{\boxed{\sum_{j=1}^{n_i} a_{ije} \sigma_{(\lambda_i,\mu_i)}^{\nu_{ij}}}}^{M_{ie}} \cdot \frac{1}{P_i(\lambda_i n + \mu_i k)^e}$.

3 $\frac{f_0}{P_0} = \Delta_k(g) + r$. If $r \neq 0$, return "No telescoper exists!".

4 $M_{ie} = \Delta_k\big(\cdots\big) + R_{ie}$. If all $R_{ie} = 0$ then return $L = 1$ and $g = g + \sum\limits_{i,e} \mathrm{LeftQuot}(M_{ie}, \sigma_k - 1) \frac{1}{P_i(\lambda_i n + \mu_i k)^e}$.

5 For $\rho = 1, 2, \ldots$ do

Find a telescoper L such that $L \cdot R_{ie} = \Delta_k\big(\cdots\big)$. If succeed return

L and $g = L \cdot g + \sum\limits_{i,e} \mathrm{LeftQuot}(L \cdot M_{ie}, \sigma_k - 1) \cdot \frac{1}{P_i(\lambda_i n + \mu_i k)^e}$.

Worst-case complexity (field operations)

Given $f \in C(n,k)$ with $\deg_n(f) \le d_n$ and $\deg_k(f) \le d_k$.

New_ub	New_it	RCT
$O^{\sim}(\mu^{\omega} d_n d_k^{\omega+1})$	$O^{\sim}(\mu^{\omega+1} d_n d_k^{\omega+2})$	$O^{\sim}(\mu^{\omega+2} d_n d_k^{\omega+3})$

- $\mu \in \mathbb{Z}^+$, $2 \le \omega \le 3$
- Without expanding the certificate
- Size of a minimal telescoper: $O(\mu^2 d_n d_k^3)$

Timings (in seconds)

Test suite: $f(n,k) = \Delta_k\left(\frac{f_0(n,k)}{P_0(n,k)}\right) + \frac{a(n,k)}{P_1(2n+\mu k)\cdot P_2(4n+\mu k)}$.

- $P_i(z) = p_i(z)\cdot p_i(z+2^i)\cdot p_i(z+\mu)\cdot p_i(z+2^i+\mu)$,
- $\mu \in \mathbb{Z}$, $\deg_{n,k}(a) = d_1$, $\deg_{n,k}(P_0) = \deg_z(p_i) = d_2$.

(d_1, d_2, μ)	RCT	New_ub	New_it	Order	Upper
(1, 1, 1)	0.28	0.19	0.19	3	4
(1, 2, 1)	5.86	4.88	2.15	7	8
(1, 3, 1)	283.84	630.61	30.94	11	12
(1, 4, 1)	5734.80	37272.09	448.09	15	16
(10, 2, 1)	7.79	11.89	3.18	7	8
(20, 2, 1)	9.49	25.22	4.21	7	8
(30, 2, 1)	16.57	9.67	10.17	8	8
(30, 2, 3)	807.31	39.37	41.16	12	12
(30, 2, 5)	4875.63	305.16	344.81	20	20
(30, 2, 7)	34430.03	1479.36	1240.54	28	28

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Summary

Result.

- A creative telescoping algorithm for bivariate rational function
  - Avoids need to construct certificates
  - Uses merely "rational" operations
  - Expresses certificates in precise and manipulable sparse forms
  - Has better control in size of intermediate expression
  - Easier to analyze, and more efficient

Future work.

- Generalize to hypergeometric terms