Two Applications of the Modified Abramov-Petkovšek Reduction*

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In this talk, we describe the modified Abramov-Petkovšek reduction developed in [6], and present two applications: one is a reduction-based algorithm to compute telescopers for hypergeometric terms; the other provides lower and upper order bounds for telescopers. The latter application is new.

1 Modified Abramov-Petkovšek reduction

The ubiquity of hypergeometric terms in enumerative combinatorics is widely recognized, such as binomial coefficients, power functions and factorials, etc. Let C be a field of characteristic zero. A univariate term $T(n)$ is said to be *hypergeometric* if its shift quotient $T(n+1)/T(n)$ is in $C(n)$.

A hypergeometric term $T(n)$ is said to be *hypergeometric-summable* if there exists another hypergeometric term $G(n)$ such that

$$T(n) = G(n+1) - G(n). \tag{1}$$

We abbreviate "hypergeometric-summable" as "summable" in the sequel. For two hypergeometric terms $T(n)$ and $H(n)$, we write $T(n) \equiv H(n)$ if $T(n) - H(n)$ is summable.

Based on Abramov and Petkovšek's work in [3, 4], the authors of [6] present a modified Abramov-Petkovšek Reduction, which, for a given hypergeometric term $T(n)$, computes five polynomials a, b, q, u, v in $C[n]$ such that

$$T(n) \equiv \left(\frac{a}{b} + \frac{q}{v}\right) H(n), \tag{2}$$

where $H(n+1)/H(n) = u/v$. Moreover $T(n)$ is summable iff $a = q = 0$ [6, Theorem 4.8].

2 Telescoping via reductions

A bivariate term $T(k,n)$ is said to be *hypergeometric* if its two shift quotients $T(k+1,n)/T(k,n)$ and $T(k,n+1)/T(k,n)$ are in $C(k,n)$. Given a hypergeometric term $T(k,n)$, the computational problem of creative telescoping, a staple of symbolic summation, is to construct a nonzero operator $L \in C(k)\langle\sigma_k\rangle$ with σ_k the shift operator that maps k to $k+1$, s.t.

$$L(T(k,n)) = G(k,n+1) - G(k,n),$$

for some hypergeometric term $G(k,n)$. We call L a *telescoper* for T and G the *certificate* of L. The classical one to do this is known as Zeilberger's algorithm [7, 8], which computes the minimal telescoper and its corresponding certificate simultaneously. It has been implemented in Maple[SumTools] package.

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Based on the modified reduction, the authors of [6] developed a reduction-based telescoping algorithm. The key advantage of this approach is that it separates the computation of telescopers from that of certificates. This is desirable in the typical situation where we are only interested in the telescopers and their size is much smaller than that of certificates. Computational experiments indicate that the reduction-based algorithm is faster than the MAPLE procedure *Zeilberger* if it normalizes the certificates, and much more efficient if it omits the computation of the certificates.

3 Upper and lower bounds

For two bivariate hypergeometric terms $T_1(k,n)$ and $T_2(k,n)$, we write $T_1 \equiv_n T_2$ if $T_1 - T_2$ is summable w.r.t. n. Let $T(k,n)$ be a hypergeometric term. Applying the modified Abramov-Petkovšek's reduction w.r.t. n to T yields

$$T(k,n) \equiv_n \left(\frac{a}{b} + \frac{q}{v}\right) H(k,n), \tag{3}$$

where $a, b, q, u, v \in C(k)[n]$ and $H(k,n+1)/H(k,n) = u/v$. By Theorem 10 in [1], $T(k,n)$ has a telescoper if and only if

$$b = c\prod_{\ell=1}^{m} P_\ell(\lambda_\ell k + \mu_\ell n)^{c_{\ell 0}} \cdot P_\ell(\lambda_\ell k + \mu_\ell n + 1)^{c_{\ell 1}} \cdots P_\ell(\lambda_\ell k + \mu_\ell n + d_\ell)^{c_{\ell d_\ell}}, \tag{4}$$

where $c \in C(k)$, $m, d_\ell, c_{\ell j} \in \mathbb{Z}^+$, λ_ℓ, μ_ℓ are co-prime integers with $\mu_\ell > 0$, $P_1, \ldots, P_m$ are nontrivial monic irreducible univariate polynomials over C and $P_{\ell_1}(\lambda_{\ell_1} k + \mu_{\ell_1} n) \neq P_{\ell_2}(\lambda_{\ell_2} k + \mu_{\ell_2} n + \gamma)$ for any $\gamma \in \mathbb{Z}$ if $\ell_1 \neq \ell_2$. Then we have the following new order bounds of the minimal telescoper for T,

Theorem 3.1 *With the notations introduced in* (3) *and* (4)*, assume further that $L \in C(k)\langle\sigma_k\rangle$ is the minimal telescoper for T w.r.t. n. Then*

$$\operatorname{ord}(L) \leq \max\{\deg_n(u), \deg_n(v)\} - [\![\deg_n(v-u) \leq \deg_n(u) - 1]\!] + \sum_{\ell=1}^{m} \mu_\ell \cdot \max_{0 \leq j \leq d_\ell}\{c_{\ell j}\} \cdot \deg(P_\ell), \tag{5}$$

where $[\![\ldots]\!]$ equals 1 if $\ldots$ is true, otherwise it is 0. Further assume that T is not summable. Then

$$\operatorname{ord}(L) \geq \max_{1 \leq \ell \leq m} \max_{0 \leq j \leq d_\ell} \min_{\substack{0 \leq i \leq d_\ell \\ c_{\ell i} = c_{\ell j}}} \{\rho \in \mathbb{N} \setminus \{0\} : \mu_\ell \mid \lambda_\ell \rho + i - j\}. \tag{6}$$

The upper bound on the order of minimal telescopers was first given by Apagodu and Zeilberger in [5]. Their bound is merely for proper hypergeometric terms, while ours is for general ones. Moreover, our bound is no greater than theirs in all cases, and tighter in some cases.

The lower bound on the order of minimal telescopers was first estimated by the algorithm *LowerBound* given by Abramov and Le in [2], and was implemented in MAPLE[SumTools] package. But for the rational function

$$T(k,n) := \frac{1}{(-5k+4n+4)(-5k+4n+14)},$$

whose minimal telescoper is $\sigma_k^2 + 1$, the result returned by the MAPLE procedure is 4, which is clearly an error. By the formula (6), the lower bound should be 2. We are investigating wether the algorithm *LowerBound* in [2] has an error or the implementation in MAPLE has a bug. We will also compare our lower bound with that in [2].

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