Ultimate Positivity of Diagonals of Quasi-rational Functions *

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1 Introduction

The problem to decide whether a given multivariate (quasi-)rational function has only positive coefficients in its power series expansion has a long history. It dates back to Szegö [10], who showed that $((1-Z_1)(1-Z_2)+(1-Z_1)(1-Z_3)+(1-Z_2)(1-Z_3))^{-\beta}$ for $\beta \geq 1/2$ is positive, in the sense that all its series coefficients are positive, using an involved theory of special functions. In contrast to the simplicity of the statement, the method was surprisingly difficult. This dependency motivated further research for positivity of (quasi-)rational functions. More and more (quasi-)rational functions have been proven to be positive, and some of the proofs are even quite simple [4]. However, there are also others whose positivity are still open conjectures. For instance, the rational function P^{-1} with

$$P = 1 - (Z_1 + Z_2 + Z_3 + Z_4) + \frac{64}{27}(Z_1Z_2Z_3 + Z_1Z_2Z_4 + Z_1Z_3Z_4 + Z_2Z_3Z_4)$$

is conjectured to be positive by Kauers [5], while no proof is available so far. This is equivalent to verify the positivity of the quasi-rational function $P^{-\beta}$ for $\beta \geq 1$ by [4, Proposition 1]. In this talk, we focus on a less difficult but also interesting question to decide whether the diagonal of $P^{-\beta}$ is *ultimately* positive, inspired by [7, 9]. To solve this question, it suffices to compute the asymptotics of the diagonal coefficients, which can be done by the multivariate singularity analysis developed by Baryshnikov, Pemantle and Wilson [2, 8]. Note that the ultimate positivity is a necessary condition for the positivity, and therefore can be used to either exclude the nonpositive cases or further support the conjectural positivity.

2 Multivariate singularity analysis

In this section, we sketch and adapt the precise results for diagonals from [2, 8]. Reader interested in knowing more general cases may refer to the original text.

Let $F(\mathbf{Z}) = \sum_{\mathbf{r}\in\mathbb{N}^d} a_{\mathbf{r}}\, \mathbf{Z}^{\mathbf{r}}$ be a d-variate complex generating function analytic at the origin, where $\mathbf{Z} = (Z_1, \ldots, Z_d)$, $\mathbf{r} = (r_1, \ldots, r_d)$ and $\mathbf{Z}^{\mathbf{r}} = Z_1^{r_1} \ldots Z_d^{r_d}$. Then the *diagonal* of the rational function F is defined to be $\operatorname{diag}(F) = \sum_{n\in\mathbb{N}} a_{n,\ldots,n} Z^n$. For simplicity, we assume F to be a *quasi-rational function* of the form $F(\mathbf{Z}) = P(\mathbf{Z})^{-\beta}$ with β a real number except for nonpositive integers and P a polynomial. The zero set $\mathcal{V}_P$ of P in $\mathbb{C}^d$ is called the *singular variety* of F. Note that $\mathbf{0} \notin \mathcal{V}_P$ since F is analytic at the origin.

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We aim at estimating the coefficient $a_{n,\dots,n}$ asymptotically. Similar to the univariate case, we always start with the multivariate Cauchy integral formula

$$a_{n,\dots,n} = \frac{1}{(2\pi i)^d} \int_T F(\mathbf{Z}) \frac{d\mathbf{Z}}{\mathbf{Z}^{n+1}} \tag{1}$$

where T is a sufficiently small torus around the origin and $\mathbf{Z}^{n+1} = Z_1^{n+1} \dots Z_d^{n+1}$. The essential idea of the method in [2, 8] is to deform the contour T without changing the integral (i.e. avoiding the points on $\mathcal{V}_P$), such that the local behavior of the integrand at so-called *minimal critical points* determines the asymptotics (under certain conditions). To describe minimal critical points, we need the definition of amoebas. Following [2], we let

$$\mathrm{ReLog}(\mathbf{Z}) = (\log|Z_1|, \dots, \log|Z_d|).$$

Then the real set of $\mathrm{ReLog}(\mathbf{Z})$ for all $\mathbf{Z} \in \mathcal{V}_P$ is called the *amoeba* of the polynomial P, denoted by $\mathbf{amoeba}(P)$. Note that amoebas can be computed effectively, see [11]. By [2, Proposition 2.2], there exists a component B of $\mathbb{R}^d \setminus \mathbf{amoeba}(P)$ such that B is convex and the set $\mathrm{ReLog}^{-1} B$ is precisely the open domain of convergence of the power series $F = \sum_{\mathbf{r}\in\mathbb{N}^d} a_{\mathbf{r}} \mathbf{Z}^{\mathbf{r}}$. Assume that there exists a unique point $\mathbf{x}_{\min}$ on the boundary ∂B minimizing the function $-x_1 - \cdots - x_d$ with $\mathbf{x} \in \bar{B}$. We call $\mathbf{x}_{\min}$ the *minimizing point* for the diagonal. Let $\mathbf{tan}_{\mathbf{x}_{\min}}(B)$ denote the tangent cone to B at $\mathbf{x}_{\min}$, that is,

$$\mathbf{tan}_{\mathbf{x}_{\min}}(B) = \{\mathbf{b} : \mathbf{x}_{\min} + \epsilon\mathbf{b} \in B \text{ for all sufficiently small } \epsilon > 0\}.$$

Let $\mathbf{N}^*_{\mathbf{x}_{\min}}(B)$ be the normal cone to $\mathbf{tan}_{\mathbf{x}_{\min}}(B)$, namely the set of vectors $\mathbf{v}$ such that $\mathbf{v}\cdot\mathbf{b} \le 0$ for all $\mathbf{b} \in \tan_{\mathbf{x}_{\min}} B$. Then [2, Definition 2.13] asserts that for each $\mathbf{Z}$ with $\mathrm{ReLog}(\mathbf{Z}) = \mathbf{x}_{\min}$ there is a naturally defined cone $\mathbf{K}(\mathbf{Z})$ (which is too lengthy to give here) that contains $\mathbf{tan}_{\mathbf{x}_{\min}}(B)$. We denote $\mathbf{N}^*(\mathbf{Z})$ for the normal cone of $\mathbf{K}(\mathbf{Z})$ and define the set of *minimal critical points* by

$$\mathbf{crit}(H, \mathbf{x}_{\min}) = \{\mathbf{Z} \in \mathcal{V}_P \cap \mathrm{ReLog}^{-1}(\mathbf{x}) : (1, \dots, 1) \in \mathbf{N}^*(\mathbf{Z})\}.$$

Note that $\mathbf{N}^*(\mathbf{Z}) \subseteq \mathbf{N}^*_{\mathbf{x}_{\min}}(B)$. When P is irreducible, for $\mathbf{Z}$ to be a smooth minimal critical point in the sense that the gradient of P at $\mathbf{Z}$ is nonzero, we must have

$$P(\mathbf{Z}) = 0 \quad \text{and} \quad Z_j \frac{\partial}{\partial Z_j} P(\mathbf{Z}) = Z_k \frac{\partial}{\partial Z_k} P(\mathbf{Z}), \quad j, k = 1, \dots, d. \tag{2}$$

We are mainly interested in the following quadratic case.

Theorem 1 ([2, Proposition 3.7])**.** *Let $F(\mathbf{Z}) = P^{-\beta}(\mathbf{Z})$ be a d-variate quasi-rational function with $\beta \ne 0, -1, -2, \dots$ and P a polynomial. Let B be a component of $\mathbb{R}^d \setminus \mathbf{amoeba}(P)$ so that F has a convergent power series expansion in $\mathrm{ReLog}^{-1}(B)$. Assume that there exists a minimizing point $\mathbf{x}_{\min}$ for the diagonal, and the set $\mathbf{crit}(H, \mathbf{x}_{\min})$ contains only one point $\mathbf{Z}_*$. Further assume that the leading homogeneous part q of $P \circ \exp$ at $\mathbf{z}_*$ with $\mathbf{Z}_* = \exp(\mathbf{z}_*) = (\exp(z_{*1}), \dots, \exp(z_{*d}))$ is an irreducible quadratic with the matrix M_q congruent to the $d \times d$ diagonal matrix $\mathrm{diag}(1, -1, \dots, -1)$. Then, when the Gamma functions in the denominator are finite,*

$$a_{n,\dots,n} \sim \frac{((-1)^{d-1}\det(M_q))^{-1/2}}{2^{2\beta-1}\pi^{d/2-1}\Gamma(\beta)\Gamma(\beta+1-d/2)} \mathbf{Z}_*^{-n} (n^2 q^*(\mathbf{1}))^{\beta-d/2}, \tag{3}$$

where q^* is the dual quadratic form of q with the matrix M_q^{-1}.

3 Asymptotics of diagonals

In this section, we apply the multivariate singularity analysis to two quasi-rational functions. The first example comes from a well-known rational function, which was shown to be positive for $\beta = 1$ in [10, 1].

Example 2. *Consider the quasi-rational function*

$$F(\mathbf{Z}) = \frac{1}{(1-(Z_1+Z_2+Z_3)+\frac{3}{4}(Z_1Z_2+Z_1Z_3+Z_2Z_3))^{\beta}} \quad \text{with } \beta \neq 0,-1,-2,\ldots.$$

We are interested in the asymptotics for the diagonal coefficients of F. For simplicity, we translate each coordinate Z_j to $\frac{2}{3}Z_j$, and then apply the method to $P^{-\beta}$ with

$$P(\mathbf{Z}) = 1 - \frac{2}{3}(Z_1+Z_2+Z_3) + \frac{1}{3}(Z_1Z_2+Z_1Z_3+Z_2Z_3).$$

Identify minimal critical points. *Let $\mathcal{V}_P$ be zero set of P. It is readily seen that $\mathcal{V}_P$ is smooth except for the point $(1,1,1)$. Let B be the component of $\mathbb{R}^3 \setminus$ **amoeba**(P) corresponding to the power series expansion of $P^{-\beta}$ at the origin. Then B contains the negative orthant by [11]. We claim that $(0,0,0)$ is on the boundary of B. Indeed, it suffices to verify that P is nonzero in the open unit polydisk $\{\mathbf{Z} \in \mathbb{C}^3 : |Z_j| < 1\}$. Following [2, Section 4.4], it is equivalent to show that $P(\mathbf{Z}+\mathbf{1})$ is nonzero in the open disk $\{\mathbf{Z} \in \mathbb{C}^3 : |Z_j+1| < 1\}$ by sending $\mathbf{Z}$ to $\mathbf{Z}+\mathbf{1}$. Then further setting $\mathbf{Z}$ to $\mathbf{1}/\mathbf{Z}$ changes the problem to prove that*

$$P\left(\mathbf{1} + \frac{\mathbf{1}}{\mathbf{Z}}\right) = \frac{Z_1+Z_2+Z_3}{3Z_1Z_2Z_3}$$

is nonzero in $\{\mathbf{Z} \in \mathbb{C}^3 : \mathrm{Re}(Z_j) < -1/2\}$, which is trivial since $\mathrm{Re}(Z_1+Z_2+Z_3) < -3/2$. Since the diagonal direction $(1,1,1) \in \mathbf{N}^*_{\mathbf{0}}(B)$, the point $(0,0,0)$ is the minimizing point for the diagonal by the definition of normal cones. At the point $(1,1,1)$, the leading homogeneous term of P composing with the exponential is

$$q(\mathbf{Z}) = \frac{1}{3}(Z_1Z_2+Z_1Z_3+Z_2Z_3)$$

with the matrix congruent to the 3×3 diagonal matrix $\mathrm{diag}(1,-1,-1)$. Then the dual quadratic form of q is

$$q^*(\mathbf{r}) = \begin{pmatrix} r_1 & r_2 & r_3 \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & 0 & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & 0 \end{pmatrix}^{-1} \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = 3(2r_1r_2 + 2r_1r_3 + 2r_2r_3 - r_1^2 - r_2^2 - r_3^2).$$

*By definition, the normal cone $\mathbf{N}^*(\mathbf{1})$ is the set $\{\mathbf{r} \in \mathbb{R}^3 : q^*(\mathbf{r}) > 0\}$ containing the diagonal direction $(1,1,1)$. Hence the point $(1,1,1) \in$ **crit**$(H,\mathbf{0})$. The remaining points in **crit**$(H,\mathbf{0})$ could only be smooth critical points on $\mathcal{V}_P \setminus \{(1,1,1)\}$. Solving (2) implies that **crit**$(H,\mathbf{0})$ has only one point, namely $(1,1,1)$. To get the leading term of the asymptotics, it suffices to compute the contribution from $(1,1,1)$.*

Compute diagonal asymptotics. *So far, we have shown that $P^{-\beta}$ satisfies the hypotheses in Theorem 1. Since $F(\mathbf{Z}) = P(\frac{3}{2}\mathbf{Z})^{-\beta}$, by (3) the n-th diagonal coefficient of F is asymptotic to*

$$\frac{3^{2\beta-\frac{3}{2}} n^{2\beta-3}}{2^{2\beta-2}\sqrt{\pi}\,\Gamma(\beta)\Gamma(\beta-1/2)} \left(\frac{2}{3}\right)^{-3n}, \tag{4}$$

which implies that the diagonal of F is ultimately positive for $\beta > 1/2$.

Now let us turn to the function that we mention in the introduction.

Example 3 ([5]). *Consider the quasi-rational function*

$$F(\mathbf{Z}) = \frac{1}{(1-(Z_1+Z_2+Z_3+Z_4)+\frac{64}{27}(Z_1Z_2Z_3+Z_1Z_2Z_4+Z_1Z_3Z_4+Z_2Z_3Z_4))^\beta}$$

with $\beta \neq 0,-1,-2,\dots$. *We want to know the diagonal asymptotics of* F. *Similar to the previous example, we first translate each coordinate* Z_j *to* $\frac{3}{8}Z_j$ *and work with* $P^{-\beta}$ *where*

$$P(\mathbf{Z}) = 1-\frac{3}{8}(Z_1+Z_2+Z_3+Z_4)+\frac{1}{8}(Z_1Z_2Z_3+Z_1Z_2Z_4+Z_1Z_3Z_4+Z_2Z_3Z_4).$$

Then the diagonal asymptotics of $F(\mathbf{Z}) = P(\frac{8}{3}\mathbf{Z})^{-\beta}$ *can be easily computed.*

Identify minimal critical points. *Let* $\mathcal{V}_P$ *be the zero set of* P. *Then the only non-smooth point on* $\mathcal{V}_P$ *is the point* $(1,1,1,1)$. *Again the component* B *of* $\mathbb{R}^4 \setminus \mathbf{amoeba}(P)$ *corresponding to the power series expansion of* $P^{-\beta}$ *at the origin is the one contains the negative orthant. Again, we claim that* $(0,0,0,0)$ *is on the boundary of* B. *Similarly, it suffices to verify that* P *is nonzero in the open unit polydisk* $\{\mathbf{Z} \in \mathbb{C}^4 : |Z_j| < 1\}$, *which is then equivalent to show that the numerator of*

$$P\left(\mathbf{1}+\frac{\mathbf{1}}{\mathbf{Z}}\right) = \frac{Z_1+Z_2+Z_3+Z_4+2(Z_1Z_2+Z_1Z_3+Z_1Z_4+Z_2Z_3+Z_2Z_4+Z_3Z_4)}{8Z_1Z_2Z_3Z_4}$$

is nonzero in $D = \{\mathbf{Z} \in \mathbb{C}^4 : \mathrm{Re}(Z_j) < -1/2\}$. *This can be done by cylindrical algebraic decomposition (CAD) [3, 6] as follows. Suppose that* $(Z_1,Z_2,Z_3,Z_4) \in D$ *is a zero of the numerator. Then we can represent* Z_4 *as*

$$Z_4 = \frac{-Z_1-Z_2-Z_3-2Z_1Z_2-2Z_1Z_3-2Z_2Z_3}{1+2Z_1+2Z_2+2Z_3},$$

whose denominator cannot be zero since $\mathrm{Re}(Z_j) < -1/2$ *for* $j=1,2,3$. *Applying CAD shows that the real part of* Z_4, *in terms of real and imaginary parts of* Z_1, Z_2, Z_3, *must be greater than* $-1/2$, *a contradiction. Since the diagonal direction* $(1,1,1)$ *belongs to* $\mathbf{N}^*_{\mathbf{0}}(B)$, *the point* $(0,0,0,0)$ *is the minimizing point for the diagonal by the definition of normal cones. At the point* $(1,1,1,1)$, *the leading homogeneous term of* $P \circ \exp$ *is*

$$q(\mathbf{Z}) = \frac{1}{4}(Z_1Z_2+Z_1Z_3+Z_1Z_4+Z_2Z_3+Z_2Z_4+Z_3Z_4)$$

with the matrix

$$\begin{pmatrix} 0 & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & 0 & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & 0 & \frac{1}{8} \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & 0 \end{pmatrix}$$

congruent to the 4×4 *diagonal matrix* $\mathrm{diag}(1,-1,-1,-1)$. *Then the dual quadratic form is*

$$q^*(\mathbf{r}) = \frac{16}{3}(r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4-r_1^2-r_2^2-r_3^2-r_4^2).$$

Then the normal cone $\mathbf{N}^*(\mathbf{1}) = \{\mathbf{r} \in \mathbb{R}^4 : q^*(\mathbf{r}) > 0\}$. *By the same reason as the previous example, the set* $\mathbf{crit}(H,\mathbf{0})$ *only contains the point* $(1,1,1,1)$, *which determines the leading term of the asymptotics.*

Computing diagonal asymptotics. *We have seen that $P^{-\beta}$ satisfies the hypotheses in Theorem 1. Hence by (3), the n-th diagonal coefficient of F is asymptotic to*

$$\frac{2^{3\beta-3}n^{2\beta-4}}{3^{\beta-\frac{3}{2}}\pi\Gamma(\beta)\Gamma(\beta-1)}\left(\frac{3}{8}\right)^{-4n},$$

which implies that the diagonal of F is ultimately positive for $\beta > 1$.

Remark 4. *For the cases when the Gamma functions in the denominator of (3) is infinite, e.g., when $\beta = \frac{1}{2}$ for Example 2 and $\beta = 1$ for Example 3, we need more techniques from [8] to compute the diagonal asymptotics, which will be addressed in future work.*

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