Stat 500 Solutions to Homework #12 Fall 2020

1. 15 points

   Note:

   - Linearity does not necessarily imply identifiability, i.e. each parameter has a unique solution. See part (c) as an example
   - "Linear model" means linear in terms of parameters β, but not necessarily linear in explanatory variables X_i

   (a) Linear model.

   (b) Linear model.

   (c) Linear model, since it is equivalent to $Y_i - log(X_{1i}) = (\beta_0 + log(\beta_1)) + \beta_2 X_{2i} + \epsilon_i$. In this scenario, neither β_0 nor β_1 are identifiable.

   (d) Nonlinear model.

   (e) Intrinsically linear model. $log((1-x)/x)$ transformation to both side. (This is the logistic regression)

   (f) Nonlinear model.

   (g) Intrinsically linear model. Log transformations of the response variable and random error are needed.

2. 5 points

   (Properties of P_X: see notes page 872)

   With $P_X = X(X^TX)^{-1}X^T$, $P_{\mathbf{1}} = \mathbf{1}(\mathbf{1}^T\mathbf{1})^{-1}\mathbf{1}^T$, and $P_X\boldsymbol{u} = \boldsymbol{u}$ for any vector $\boldsymbol{u}$ in the space spanned by the columns of X,

$$\begin{aligned}
(P_X - P_{\mathbf{1}})(P_X - P_{\mathbf{1}}) &= P_XP_X - P_XP_{\mathbf{1}} - P_{\mathbf{1}}P_X + P_{\mathbf{1}}P_{\mathbf{1}} \\
&= X(X^TX)^{-1}X^TX(X^TX)^{-1}X^T - P_X\mathbf{1}(\mathbf{1}^T\mathbf{1})^{-1}\mathbf{1}^T - \\
&\quad \mathbf{1}(\mathbf{1}^T\mathbf{1})^{-1}\mathbf{1}^TP_X + \mathbf{1}(\mathbf{1}^T\mathbf{1})^{-1}\mathbf{1}^T\mathbf{1}(\mathbf{1}^T\mathbf{1})^{-1}\mathbf{1}^T \\
&= X(X^TX)^{-1}X^T - \mathbf{1}(\mathbf{1}^T\mathbf{1})^{-1}\mathbf{1}^T - \mathbf{1}(\mathbf{1}^T\mathbf{1})^{-1}(P_X\mathbf{1})^T + \mathbf{1}(\mathbf{1}^T\mathbf{1})^{-1}\mathbf{1}^T \\
&= X(X^TX)^{-1}X^T - \mathbf{1}(\mathbf{1}^T\mathbf{1})^{-1}\mathbf{1}^T \\
&= P_X - P_{\mathbf{1}}.
\end{aligned}$$

   So $(P_X - P_{\mathbf{1}})$ is idempotent.

3. 5 points To show B satisfy the definition of a generalized inverse of A, you must check if ABA = A.

$$
\begin{aligned}
ABA &= A(BA) = \begin{bmatrix} 1 \\ 2 \\ 5 \\ -2 \end{bmatrix} \cdot ([1,0,0,0] \cdot \begin{bmatrix} 1 \\ 2 \\ 5 \\ -2 \end{bmatrix}) \\
&= \begin{bmatrix} 1 \\ 2 \\ 5 \\ -2 \end{bmatrix} \cdot 1 \\
&= A.
\end{aligned}
$$

B is a generalized inverse matrix for A by definition.

4. 10 points
If there exists matrix A such that $c^T\beta = AEY$, denote $A = [a_1, a_2, a_3, \cdots,]$

(a)

$$
\begin{aligned}
c^T\beta &= \alpha_1 - \frac{1}{2}(\alpha_2 + \alpha_3) \\
&= (\mu + \alpha_1) - \frac{1}{2}[(\mu + \alpha_2) + (\mu + \alpha_3)] \\
&= EY_{11} - \frac{1}{2}(EY_{21} + EY_{31}) \\
&= \left[1, 0, -\frac{1}{2}, 0, 0, 0, -\frac{1}{2}, 0\right] EY.
\end{aligned}
$$

$A = \left[1, 0, \frac{1}{2}, 0, 0, 0, \frac{1}{2}, 0\right]$ is one non-random matrix so that AEY is the specified linear function of the model parameters. Or any A such that $a_1 + a_2 = 1, a_3 + a_4 + a_5 + a_6 = a_7 + a_8 = -\frac{1}{2}$ is correct.

(b)

$$
\begin{aligned}
c^T\beta &= 3\mu + \alpha_1 + 2\alpha_2 \\
&= (\mu + \alpha_1) + 2(\mu + \alpha 2) \\
&= EY_{11} + 2EY_{21} \\
&= [1, 0, 2, 0, 0, 0, 0, 0]EY.
\end{aligned}
$$

One non-random matrix A can be $[1, 0, 2, 0, 0, 0, 0, 0]$ or any A such that $a_1 + a_2 = 1, a_3 + a_4 + a_5 + a_6 = 2, a_7 + a_8 = 0$ is correct.

(c) If there exists a matrix A such that $c^T\beta = AEY$, then

$$
\begin{aligned}
\alpha_2 + \alpha_3 &= AEY = (a_1 + a_2)(\mu + \alpha_1) + (a_3 + a_4 + a_5 + a_6)(\mu + \alpha_2) + (a_7 + a_8)(\mu + \alpha_3) \\
&= (a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8)\mu + (a_1 + a_2)\alpha_1 + (a_3 + a_4 + a_5 + a_6)\alpha_2 + \\
&\quad (a_7 + a_8)\alpha_3,
\end{aligned}
$$

the above equation requires $(a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8)=0$, $(a_1+a_2)=0$, $(a_3+a_4+a_5+a_6)=1$ and $(a_7+a_8)=1$, no solutions exist that can satisfy all the above three equations.

(d) If there exists a matrix A such that $c^T\beta = AEY$, then

$$\begin{aligned} 3\mu-\alpha_1-\alpha_2-\alpha_3 &= AEY = (a_1+a_2)(\mu+\alpha_1)+(a_3+a_4+a_5+a_6)(\mu+\alpha_2)+(a_7+a_8)(\mu+\alpha_3) \\ &= (a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8)\mu+(a_1+a_2)\alpha_1+ \\ &\quad (a_3+a_4+a_5+a_6)\alpha_2+(a_7+a_8)\alpha_3, \end{aligned}$$

the above equation requires $(a_1+a_2+a_3+a_4+a_5+a_6+a_7+a_8)=3$, $(a_1+a_2)=-1$, $(a_3+a_4+a_5+a_6)=-1$ and $(a_7+a_8)=-1$, no solutions exist that can satisfy all the above three equations.

5. 15 points

(a) μ is not estimable. No individual components of the parameter vector β is estimable as they are not identifiable with not full-ranked X.

(b) $\alpha_1-\alpha_2$ is not estimable. We need to find a linear combination of expected values $EY_{ijl}=\mu+\alpha_i+\tau_j+(\alpha\tau)_{ij}$ that equals to $\alpha_1-\alpha_2$, which is not possible because of the interaction terms $(\alpha\tau)_{ij}$.

(c) $(\alpha\tau)_{12}$ is not estimable with the same reason as part (a).

(d) $(\alpha\tau)_{11}-(\alpha\tau)_{12}$ is not estimable. We need to find a linear combination of expected values $EY_{ijl}=\mu+\alpha_i+\tau_j+(\alpha\tau)_{ij}$ that equals to $(\alpha\tau)_{11}-(\alpha\tau)_{12}$, which is not possible because of the terms τ_j.

(e) $(\alpha\tau)_{11}-(\alpha\tau)_{12}-(\alpha\tau)_{21}-(\alpha\tau)_{22}$ is not estimable.
We want to find a,b,c,d such that $E[aY_{11k}+bY_{12k}+cY_{21k}+dY_{22k}]=(\alpha\tau)_{11}-(\alpha\tau)_{12}-(\alpha\tau)_{21}-(\alpha\tau)_{22}$, thus $a=1, b=c=d=-1$ has to hold. But then the coefficient for μ is $1-1-1-1=-2\neq 0$, so such set of a,b,c,d doesn't exist.

(f) $(\alpha\tau)_{11}-(\alpha\tau)_{12}-(\alpha\tau)_{21}+(\alpha\tau)_{22}$ is estimable.

$$\begin{aligned} (EY_{11k}-EY_{12k})-(EY_{21k}-EY_{22k}) &= [(\mu+\alpha_1+\tau_1+(\alpha\tau)_{11})-(\mu+\alpha_1+\tau_2+(\alpha\tau)_{12})]- \\ &\quad [(\mu+\alpha_2+\tau_1+(\alpha\tau)_{21})-(\mu+\alpha_2+\tau_2+(\alpha\tau)_{22})] \\ &= (\alpha\tau)_{11}-(\alpha\tau)_{12}-(\alpha\tau)_{21}-(\alpha\tau)_{22}. \end{aligned}$$