STAT 500

Two-Way ANOVA Effects Model

Effects Model

$$Y_{ijk} = \mu + \alpha_i + \tau_j + (\alpha\tau)_{ij} + \epsilon_{ijk}$$

$$\begin{bmatrix} Y_{111} \\ Y_{112} \\ Y_{121} \\ Y_{122} \\ Y_{131} \\ Y_{132} \\ Y_{211} \\ Y_{212} \\ Y_{221} \\ Y_{222} \\ Y_{231} \\ Y_{232} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \tau_1 \\ \tau_2 \\ \tau_3 \\ (\alpha\tau)_{11} \\ (\alpha\tau)_{12} \\ (\alpha\tau)_{13} \\ (\alpha\tau)_{21} \\ (\alpha\tau)_{22} \\ (\alpha\tau)_{23} \end{bmatrix} + \begin{bmatrix} \epsilon_{111} \\ \epsilon_{112} \\ \epsilon_{121} \\ \epsilon_{122} \\ \epsilon_{131} \\ \epsilon_{132} \\ \epsilon_{211} \\ \epsilon_{212} \\ \epsilon_{221} \\ \epsilon_{222} \\ \epsilon_{231} \\ \epsilon_{232} \end{bmatrix}$$

Effects Model

- Estimate $a \times b$ Treatment means with
  - μ
  - a effects for Factor A
  - b effects for Factor B
  - $a \times b$ interaction effects for Factors A and B
- Impose constraints on main effects and interaction effects to reduce number of parameters to $a \times b$

Effects Model - Baseline Constraints

$$Y_{ijk} = \mu + \alpha_i + \tau_j + (\alpha\tau)_{ij} + \epsilon_{ijk}$$

- Set $\alpha_a = 0$ - level a of Factor A
- Set $\tau_b = 0$ - level b of Factor B
- Set $(\alpha\tau)_{aj} = 0$ for all $j = 1, \ldots, b$
  All interaction effects with level a of Factor A
- Set $(\alpha\tau)_{ib} = 0$ for all $i = 1, \ldots, a$
  All interaction effects with level b of Factor B

Effects Model - Baseline Constraints

- $\mu = \mu_{ab}$
  mean response at level a of Factor A and level b of Factor B

- $\alpha_i = \mu_{ib} - \mu_{ab}$
  simple effect of the ith level of Factor A when Factor B is at level b

- $\tau_j = \mu_{aj} - \mu_{ab}$
  simple effect of the jth level of Factor B when Factor A is at level a

- $(\alpha\tau)_{ij} = \mu_{ij} - \mu_{ib} - \mu_{aj} + \mu_{ab}$
  interaction effect

Effects Model - Baseline Constraints

$$\begin{bmatrix} Y_{111} \\ Y_{112} \\ Y_{121} \\ Y_{122} \\ Y_{131} \\ Y_{132} \\ Y_{211} \\ Y_{212} \\ Y_{221} \\ Y_{222} \\ Y_{231} \\ Y_{232} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \tau_1 \\ \tau_2 \\ (\alpha\tau)_{11} \\ (\alpha\tau)_{12} \end{bmatrix} + \begin{bmatrix} \epsilon_{111} \\ \epsilon_{112} \\ \epsilon_{121} \\ \epsilon_{122} \\ \epsilon_{131} \\ \epsilon_{132} \\ \epsilon_{211} \\ \epsilon_{212} \\ \epsilon_{221} \\ \epsilon_{222} \\ \epsilon_{231} \\ \epsilon_{232} \end{bmatrix}$$

Effects Model - Baseline Constraints

Least squares estimate of $\boldsymbol{\beta}$ is given by

$$
\begin{aligned}
\widehat{\boldsymbol{\beta}} &= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} \\
&= \begin{pmatrix}
\bar{Y}_{23\cdot} \\
\bar{Y}_{13\cdot} - \bar{Y}_{23\cdot} \\
\bar{Y}_{21\cdot} - \bar{Y}_{23\cdot} \\
\bar{Y}_{22\cdot} - \bar{Y}_{23\cdot} \\
\bar{Y}_{11\cdot} - \bar{Y}_{13\cdot} - \bar{Y}_{21\cdot} + \bar{Y}_{23\cdot} \\
\bar{Y}_{12\cdot} - \bar{Y}_{13\cdot} - \bar{Y}_{22\cdot} + \bar{Y}_{23\cdot}
\end{pmatrix}
\end{aligned}
$$

Effects Model - Baseline Constraints

Copper Conc.	Zinc Concentration 0 ppm	750 ppm	1500 ppm
0 ppm	$\bar{Y}_{11.} = 193.5$	$\bar{Y}_{12.} = 167.5$	$\bar{Y}_{13.} = 119.5$
150 ppm	$\bar{Y}_{21.} = 172.5$	$\bar{Y}_{22.} = 170.5$	$\bar{Y}_{23.} = 111$

Effects Model - Sum to Zero Constraints

$$Y_{ijk} = \mu + \alpha_i + \tau_j + (\alpha\tau)_{ij} + \epsilon_{ijk}$$

- Set $\sum_{i=1}^{a} \alpha_i = 0$
- Set $\sum_{j=1}^{b} \tau_j = 0$
- Set $\sum_{i=1}^{a} (\alpha\tau)_{ij} = 0$ for all j
- Set $\sum_{j=1}^{b} (\alpha\tau)_{ij} = 0$ for all i

Effects Model - Sum to Zero Constraints

- $\mu = \bar{\mu}_{\cdot\cdot}$
  overall mean response
- $\alpha_i = \bar{\mu}_{i\cdot} - \bar{\mu}_{\cdot\cdot}$
  related to main effect of the ith level of Factor A
- $\tau_j = \bar{\mu}_{\cdot j} - \bar{\mu}_{\cdot\cdot}$
  related to main effect of the jth level of Factor B
- $(\alpha\tau)_{ij} = \mu_{ij} - \bar{\mu}_{i\cdot} - \bar{\mu}_{\cdot j} + \bar{\mu}_{\cdot\cdot}$
  interaction effect

Effects Model - Sum to Zero Constraints

- Set $\alpha_2 = -\alpha_1$
- Set $\tau_3 = -\tau_1 - \tau_2$
- Set $(\alpha\tau)_{13} = -(\alpha\tau)_{11} - (\alpha\tau)_{12}$
- Set $(\alpha\tau)_{21} = -(\alpha\tau)_{11}$
- Set $(\alpha\tau)_{22} = -(\alpha\tau)_{12}$
- Set $(\alpha\tau)_{23} = (\alpha\tau)_{11} + (\alpha\tau)_{12}$

Effects Model - Sum to Zero Constraints

$$\begin{bmatrix} Y_{111} \\ Y_{112} \\ Y_{121} \\ Y_{122} \\ Y_{131} \\ Y_{132} \\ Y_{211} \\ Y_{212} \\ Y_{221} \\ Y_{222} \\ Y_{231} \\ Y_{232} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 1 & 0 & -1 & 0 \\ 1 & -1 & 0 & 1 & 0 & -1 \\ 1 & -1 & 0 & 1 & 0 & -1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \mu \\ \alpha_1 \\ \tau_1 \\ \tau_2 \\ (\alpha\tau)_{11} \\ (\alpha\tau)_{12} \end{bmatrix} + \begin{bmatrix} \epsilon_{111} \\ \epsilon_{112} \\ \epsilon_{121} \\ \epsilon_{122} \\ \epsilon_{131} \\ \epsilon_{132} \\ \epsilon_{211} \\ \epsilon_{212} \\ \epsilon_{221} \\ \epsilon_{222} \\ \epsilon_{231} \\ \epsilon_{232} \end{bmatrix}$$

Effects Model - Sum to Zero Constraints

$$
\begin{aligned}
\hat{\boldsymbol{\beta}} &= (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{Y} \\
&= \begin{pmatrix}
\bar{Y}_{\cdot\cdot\cdot} \\
\bar{Y}_{1\cdot\cdot} - \bar{Y}_{\cdot\cdot\cdot} \\
\bar{Y}_{\cdot 1\cdot} - \bar{Y}_{\cdot\cdot\cdot} \\
\bar{Y}_{\cdot 2\cdot} - \bar{Y}_{\cdot\cdot\cdot} \\
\bar{Y}_{11\cdot} - \bar{Y}_{1\cdot\cdot} - \bar{Y}_{\cdot 1\cdot} + \bar{Y}_{\cdot\cdot\cdot} \\
\bar{Y}_{12\cdot} - \bar{Y}_{1\cdot\cdot} - \bar{Y}_{\cdot 2\cdot} + \bar{Y}_{\cdot\cdot\cdot}
\end{pmatrix}
\end{aligned}
$$

Effects Model - Sum to Zero Constraints

Copper Conc.	Zinc Concentration 0 ppm	750 ppm	1500 ppm	Mean
0 ppm	$\bar{Y}_{11.} = 193.5$	$\bar{Y}_{12.} = 167.5$	$\bar{Y}_{13.} = 119.5$	$\bar{Y}_{1..} = 160.1667$
150 ppm	$\bar{Y}_{21.} = 172.5$	$\bar{Y}_{22.} = 170.5$	$\bar{Y}_{23.} = 111$	$\bar{Y}_{2..} = 151.3333$
Mean	$\bar{Y}_{.1.} = 182$	$\bar{Y}_{.2.} = 169$	$\bar{Y}_{.3.} = 115.25$	$\bar{Y}_{...} = 155.75$

Effect Tests

Question 2: Are the response means the same for the two levels of copper, averaging across zinc levels?

- $H_o : \bar{\mu}_{1\cdot} = \bar{\mu}_{2\cdot}$

$$\begin{aligned}
\bar{\mu}_{1\cdot} &= \mu + \alpha_1 + \frac{\sum_{j=1}^{b} \tau_j}{b} + \frac{\sum_{j=1}^{b} (\alpha\tau)_{1j}}{b} \\
\bar{\mu}_{2\cdot} &= \mu + \alpha_2 + \frac{\sum_{j=1}^{b} \tau_j}{b} + \frac{\sum_{j=1}^{b} (\alpha\tau)_{2j}}{b}
\end{aligned}$$

Effect Tests

$$H_o : \bar{\mu}_{1.} = \bar{\mu}_{2.}$$

is equivalent to testing

$$\alpha_1 + \frac{\sum_{j=1}^{b} (\alpha\tau)_{1j}}{b} = \alpha_2 + \frac{\sum_{j=1}^{b} (\alpha\tau)_{2j}}{b}$$

Effect Tests

- Baseline Constraints

$$H_0 : \alpha_1 + \frac{\sum_{j=1}^{b} (\alpha\tau)_{1j}}{b} = 0$$

- Sum to Zero Constraints

$$H_0 : \alpha_1 = 0$$

Effect Tests

Question 3: Are the response means the same for the three levels of zinc, averaging across copper levels?

- $H_o : \bar{\mu}_{\cdot 1} = \bar{\mu}_{\cdot 2} = \bar{\mu}_{\cdot 3}$

$$\bar{\mu}_{\cdot 1} = \mu + \frac{\sum_{i=1}^{a} \alpha_i}{a} + \tau_1 + \frac{\sum_{i=1}^{a} (\alpha\tau)_{i1}}{a}$$

$$\bar{\mu}_{\cdot 2} = \mu + \frac{\sum_{i=1}^{a} \alpha_i}{a} + \tau_2 + \frac{\sum_{i=1}^{a} (\alpha\tau)_{i2}}{a}$$

$$\bar{\mu}_{\cdot 3} = \mu + \frac{\sum_{i=1}^{a} \alpha_i}{a} + \tau_3 + \frac{\sum_{i=1}^{a} (\alpha\tau)_{i3}}{a}$$

Effect Tests

$$H_o : \bar{\mu}_{.1} = \bar{\mu}_{.2} = \bar{\mu}_{.3}$$

is equivalent to testing

$$H_0 : \tau_1 + \frac{\sum_{i=1}^{a} (\alpha\tau)_{i1}}{a} = \tau_2 + \frac{\sum_{i=1}^{a} (\alpha\tau)_{i2}}{a} = \tau_3 + \frac{\sum_{i=1}^{a} (\alpha\tau)_{i3}}{a}$$

Effect Tests

- Baseline Constraints

$$H_0 : \tau_1 + \frac{\sum_{i=1}^{a} (\alpha\tau)_{i1}}{a} = \tau_2 + \frac{\sum_{i=1}^{a} (\alpha\tau)_{i2}}{a} = 0$$

- Sum to Zero Constraints

$$H_0 : \tau_1 = \tau_2 = 0$$

Effect Tests

Question 4: Are differences in mean responses between copper levels consistent across zinc levels?

- $H_o : \mu_{11} - \mu_{21} = \mu_{12} - \mu_{22} = \mu_{13} - \mu_{23}$

$$\mu_{11} - \mu_{21} = (\mu + \alpha_1 + \tau_1 + (\alpha\tau)_{11}) - (\mu + \alpha_2 + \tau_1 + (\alpha\tau)_{21})$$

$$\mu_{12} - \mu_{22} = (\mu + \alpha_1 + \tau_2 + (\alpha\tau)_{12}) - (\mu + \alpha_2 + \tau_2 + (\alpha\tau)_{22})$$

$$\mu_{13} - \mu_{23} = (\mu + \alpha_1 + \tau_3 + (\alpha\tau)_{13}) - (\mu + \alpha_2 + \tau_3 + (\alpha\tau)_{23})$$

Effect Tests

$H_o : \mu_{11} - \mu_{21} = \mu_{12} - \mu_{22} = \mu_{13} - \mu_{23}$

is equivalent to

$$H_0 : (\alpha\tau)_{11} - (\alpha\tau)_{21} = (\alpha\tau)_{12} - (\alpha\tau)_{22} = (\alpha\tau)_{13} - (\alpha\tau)_{23}$$

Effect Tests

- Baseline Constraints

$$H_0 : (\alpha\tau)_{11} = (\alpha\tau)_{12} = 0$$

- Sum to Zero Constraints

$$H_0 : (\alpha\tau)_{11} = (\alpha\tau)_{12} = 0$$

Effects Model - Interactions

- When there are no interaction effects, the effects model becomes an additive model:

$$E(Y_{ijk}) = \mu + \alpha_i + \tau_j$$

  - Same effect of factor A at all levels of factor B:
    $E(Y_{ijk}) - E(Y_{rjk}) = (\mu + \alpha_i + \tau_j) - (\mu + \alpha_r + \tau_j) = \alpha_i - \alpha_r$
  - Same effect of factor B at all levels of factor A:
    $E(Y_{ijk}) - E(Y_{isk}) = (\mu + \alpha_i + \tau_j) - (\mu + \alpha_i + \tau_s) = \tau_j - \tau_s$
  - Main effect = Simple effect

STAT 500

Comparing Treatment Means, Simple and Main Effects, Model Diagnostics

Treatment Means

- Treatment mean: $\mu_{ij} = \mu + \alpha + i + \tau_j + (\alpha_i \tau_j)_{ij}$
- Estimate: $\bar{Y}_{ij}.$
- Std. Error: $\sqrt{MS_{\text{error}}/n}$

Simple Effects

- Simple Effects: Difference in treatment means between two levels of one factor at a specific level of the other factor.
  - For Factor A: $\mu_{ij} - \mu_{rj}$
  - For Factor B: $\mu_{ij} - \mu_{is}$
- Estimate:
  - For Factor A: $\bar{Y}_{ij.} - \bar{Y}_{rj.}$
  - For Factor B: $\bar{Y}_{ij.} - \bar{Y}_{is.}$
- Std. Error: $\sqrt{MS\text{error}\left(\frac{2}{n}\right)}$

Marginal Means

Marginal Means: Mean of level of a Factor

Factor	A	B
Marginal Mean	$\bar{\mu}_{i\cdot}$	$\bar{\mu}_{\cdot j}$
Estimate	$\bar{Y}_{i\cdot\cdot}$	$\bar{Y}_{\cdot j\cdot}$
Std. Error	$\sqrt{MS\text{error}\left(\frac{1}{nb}\right)}$	$\sqrt{MS\text{error}\left(\frac{1}{na}\right)}$

Main Effects

Main Effects: Difference in means of two levels of a Factor

Factor	A	B
Main Effect	$\bar{\mu}_{i\cdot} - \bar{\mu}_{r\cdot}$	$\bar{\mu}_{\cdot j} - \bar{\mu}_{\cdot s}$
Estimate	$\bar{Y}_{i\cdot\cdot} - \bar{Y}_{r\cdot\cdot}$	$\bar{Y}_{\cdot j\cdot} - \bar{Y}_{\cdot s\cdot}$
Std. Error	$\sqrt{MS\text{error}\left(\frac{2}{nb}\right)}$	$\sqrt{MS\text{error}\left(\frac{2}{na}\right)}$

Interactions Effects

- Interaction effects are differences of simple effects.
  - e.g. $(\mu_{ij} - \mu_{rj}) - (\mu_{il} - \mu_{rl})$
- Estimate:
  - $(\bar{Y}_{ij.} - \bar{Y}_{rj.}) - (\bar{Y}_{il.} - \bar{Y}_{rl.})$
- Std. Error: $\sqrt{MS\text{error}\left(\frac{4}{n}\right)}$
- Interaction effects are the least precisely estimated effects.

Multiple Comparison

- Adjust for multiple comparisons depending on desired analysis
  - Tukey HSD: all pairs of treatment or marginal means
  - Scheffe: all linear contrasts
  - Bonferroni: predetermined comparisons
  - Dunnett: compare many levels to one baseline

Model Assumptions

ϵ_{ijk} are i.i.d. $N(0, \sigma^2)$

- Independence
- Homogeneous Variance
- Normality

Model Diagnostics

- Independence - check details of data collection
- Homogeneous variance
  - Plot residuals vs estimated means
  - Plot residuals vs levels of each factor
- Normality
  - Normal probability plot for residuals
  - Histogram of residuals
  - Tests for Normality

Model Diagnostics

- Remedies
  - Transformation remedy for non-normality is commonly used
  - Remember, transformation changes
    - Error properties
    - The model for the mean responses
    - Can eliminate (reduce) or introduce (enhance) interactions.
  - Randomization tests
  - Rank tests