STAT 500

ANOVA Effects Model

Cell Means Model

$$Y_{ij} = \mu_i + \epsilon_{ij}$$

- Each observation Y_{ij} can be described by two components:
  - Fixed mean value μ_i
  - Random error term ϵ_{ij}

Effects Model

$$Y_{ij} = \mu + \alpha_i + \epsilon_{ij}$$

- Each observation Y_{ij} can be described by two components:
  - Fixed mean value $\mu_i = \mu + \alpha_i$
    * Overall mean value: μ
    * Treatment effects compared with overall mean: α_i
  - Random error term ϵ_{ij}

Effects Model in Matrix Notation

We can write this system of N equations in matrix notation

$$\begin{pmatrix} Y_{11} \\ Y_{12} \\ \vdots \\ Y_{1n_1} \\ Y_{21} \\ \vdots \\ Y_{2n_2} \\ Y_{31} \\ \vdots \\ Y_{rn_r} \end{pmatrix} = \begin{pmatrix} \mu + \alpha_1 + \epsilon_{11} \\ \mu + \alpha_1 + \epsilon_{12} \\ \vdots \\ \mu + \alpha_1 + \epsilon_{1n_1} \\ \mu + \alpha_2 + \epsilon_{21} \\ \vdots \\ \mu + \alpha_2 + \epsilon_{2n_2} \\ \mu + \alpha_3 + \epsilon_{31} \\ \vdots \\ \mu + \alpha_r + \epsilon_{rn_r} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ 1 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_r \end{pmatrix} + \begin{pmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \vdots \\ \epsilon_{1n_1} \\ \epsilon_{21} \\ \vdots \\ \epsilon_{2n_2} \\ \epsilon_{31} \\ \vdots \\ \epsilon_{rn_r} \end{pmatrix}$$

Effects Model

- Model has too many parameters - estimates r means with $r+1$ parameters
- Design matrix X is not full column rank.
- Usual inverse for $(X^T X)$ does not exist.
- There are an infinite number of least squares estimators.
- Solution - constrain the parameters in the model
  - Set $\alpha_r = 0$ (baseline)
  - Set $\Sigma_{i=1}^{r} \alpha_i = 0$ (sum to zero)

Effects Model

Using the constraint $\alpha_r = 0$, we have

$$
\begin{pmatrix} Y_{11} \\ Y_{12} \\ \vdots \\ Y_{1n_1} \\ Y_{21} \\ \vdots \\ Y_{2n_2} \\ \vdots \\ Y_{r-1,1} \\ \vdots \\ Y_{rn_r} \end{pmatrix}
=
\begin{pmatrix} \mu+\alpha_1+\epsilon_{11} \\ \mu+\alpha_1+\epsilon_{12} \\ \vdots \\ \mu+\alpha_1+\epsilon_{1n_1} \\ \mu+\alpha_2+\epsilon_{21} \\ \vdots \\ \mu+\alpha_2+\epsilon_{2n_2} \\ \vdots \\ \mu+\alpha_{r-1}+\epsilon_{r-1,1} \\ \vdots \\ \mu+\epsilon_{rn_r} \end{pmatrix}
=
\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}
\begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{r-1} \end{pmatrix}
+
\begin{pmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \vdots \\ \epsilon_{1n_1} \\ \epsilon_{21} \\ \vdots \\ \epsilon_{2n_2} \\ \vdots \\ \epsilon_{r-1,1} \\ \vdots \\ \epsilon_{rn_r} \end{pmatrix}
$$

Effects Model

The population or treatment means in model with $\alpha_r = 0$ are:

$$\begin{aligned}
\mu_1 &= \mu + \alpha_1 \\
\mu_2 &= \mu + \alpha_2 \\
\vdots &= \vdots \\
\mu_{r-1} &= \mu + \alpha_{r-1} \\
\mu_r &= \mu
\end{aligned}$$

Effects Model

The least squares estimator of $\boldsymbol{\beta}$ when $\boldsymbol{\alpha_r = 0}$ is:

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y} = \begin{pmatrix} \bar{\boldsymbol{Y}}_{\boldsymbol{r}\cdot} \\ \bar{\boldsymbol{Y}}_{\boldsymbol{1}\cdot} - \bar{\boldsymbol{Y}}_{\boldsymbol{r}\cdot} \\ \bar{\boldsymbol{Y}}_{\boldsymbol{2}\cdot} - \bar{\boldsymbol{Y}}_{\boldsymbol{r}\cdot} \\ \vdots \\ \bar{\boldsymbol{Y}}_{(\boldsymbol{r-1})\cdot} - \bar{\boldsymbol{Y}}_{\boldsymbol{r}\cdot} = \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{\mu}} \\ \hat{\boldsymbol{\alpha}}_{\boldsymbol{1}} \\ \hat{\boldsymbol{\alpha}}_{\boldsymbol{2}} \\ \hat{\boldsymbol{\alpha}}_{\boldsymbol{3}} \\ \vdots \\ \hat{\boldsymbol{\alpha}}_{\boldsymbol{r-1}} \end{pmatrix}$$

Effects Model

Using the constraint $\Sigma_{i=1}^{r}\alpha_i = 0$, we have $\alpha_r = -\Sigma_{i=1}^{r-1}\alpha_i$.

$$\begin{pmatrix} Y_{11} \\ Y_{12} \\ \vdots \\ Y_{1n_1} \\ Y_{21} \\ \vdots \\ Y_{2n_2} \\ \vdots \\ Y_{r-1,1} \\ \vdots \\ Y_{rn_r} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & -1 & -1 & \cdots & -1 \end{pmatrix} \begin{pmatrix} \mu \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{r-1} \end{pmatrix} + \begin{pmatrix} \epsilon_{11} \\ \epsilon_{12} \\ \vdots \\ \epsilon_{1n_1} \\ \epsilon_{21} \\ \vdots \\ \epsilon_{2n_2} \\ \vdots \\ \epsilon_{r-1,1} \\ \vdots \\ \epsilon_{rn_r} \end{pmatrix}$$

Effects Model

The population or treatment means in model with $\Sigma_{i=1}^{r} \alpha_i = 0$ are:

$$\begin{aligned}
\mu_1 &= \mu + \alpha_1 \\
\mu_2 &= \mu + \alpha_2 \\
\vdots &= \vdots \\
\mu_{r-1} &= \mu + \alpha_{r-1} \\
\mu_r &= \mu + \alpha_r = \mu - \sum_{i=1}^{r-1} \alpha_i
\end{aligned}$$

Effects Model

The least squares estimator of $\boldsymbol{\beta}$ when $\sum_{i=1}^{r} \boldsymbol{\alpha}_i = \mathbf{0}$ is:

$$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y} = \begin{pmatrix} \frac{\sum_{i=1}^{r} \bar{Y}_{i\cdot}}{r} \\ \bar{Y}_{1\cdot} - \frac{\sum_{i=1}^{r} \bar{Y}_{i\cdot}}{r} \\ \bar{Y}_{2\cdot} - \frac{\sum_{i=1}^{r} \bar{Y}_{i\cdot}}{r} \\ \vdots \\ \bar{Y}_{(r-1)\cdot} - \frac{\sum_{i=1}^{r} \bar{Y}_{i\cdot}}{r} \end{pmatrix} = \begin{pmatrix} \hat{\mu} \\ \hat{\alpha}_1 \\ \hat{\alpha}_2 \\ \hat{\alpha}_3 \\ \vdots \\ \hat{\alpha}_{r-1} \end{pmatrix}$$

Effects Model

- The above two types of constraints for effects model are not the only ways to model the means.

- The choice of constraint will affect your least squares estimator $\hat{\boldsymbol{\beta}}$.

- You must determine which constraint was applied before interpreting parameter estimates.

- The interpretation of the parameters (elements of $\boldsymbol{\beta}$) depends on the parametrization.

What about ANOVA Table?

- The things that we are interested in (population means, contrasts, etc.) are the same no matter which set of parameters you use.

- **Estimable function**: a quantity that does not depend on the arbitrary choice of constraint.

- The population means are estimable, i.e., $\boldsymbol{P_X}$ is invariant to the choice of constraints (generalized inverse matrix $(\boldsymbol{X^T X})^-$).

- Use of effects model with different constraints does not affect the ANOVA Table.

ANOVA: Fixed or Random Effects?

$$Y_{ij} = \mu + \alpha_i + \epsilon_{ij}$$

Fixed effects

- The r treatments or populations examined in the study are the only treatments under consideration
- Research questions are about treatment means or difference in means
  - e.g., two drugs, four pesticides

ANOVA: Fixed or Random Effects?

$$Y_{ij} = \mu + \alpha_i + \epsilon_{ij}$$

Random effects

- The r treatments (or groups) are a random sample from some larger population of treatments (or groups) that could have been included in the study
- Research questions are about variability in sets of treatments (or groups) the could be selected for different studies
- Additional assumptions that

$$\alpha_i \sim N(0, \sigma_\alpha^2)$$

  and any α_i is independent of any ϵ_{ij}

ANOVA: Random Effects

$$Y_{ij} = \mu + \alpha_i + \epsilon_{ij}$$

where $\alpha_i \sim N(0, \sigma_\alpha^2)$ and $\epsilon_{ij} \sim N(0, \sigma_{error}^2)$
and
any α_i is independent of any ϵ_{ij}

- Parameter of interest: μ, σ_α^2, and σ_{error}^2
- intraclass correlation

$$\frac{\sigma_\alpha^2}{\sigma_\alpha^2 + \sigma_{error}^2}$$

  is the correlation between any pair of observations in same group

ANOVA: Random Effects

Objective: Examine variability in student performance in AP (high school) statistics classes.

- Random sample of 8 AP (high school) statistics classes
- Random sample of 10 students from each class (students are *nested* in classes)
- Give ISU Stat 101 final exam to each student and record the scores
- Not interested in just the 8 classes selected for the study

ANOVA: Random Effects

- Model: $\boldsymbol{Y_{ij} = \mu + \alpha_i + \epsilon_{ij}}$
- $\boldsymbol{\mu}$ measures overall performance on a college exam
- $\boldsymbol{\sigma^2_\alpha}$ measures variability among AP classes
- $\boldsymbol{\sigma^2_{error}}$ measures variability among students within classes
- intraclass correlation measures correlation between any pair of students in same class
- Focus on fixed effects in 500.