STAT 500

MLR with categorical predictors

Motivating Example

- Students in STAT 101 at Iowa State University were asked in a recent semester to provide demographic data for use during the semester. A random sample of 75 students were selected and a few of the variables collected were:

  - Height (in inches)
  - Height of the student’s ideal romantic partner (in inches)

SLR for Motivating Example

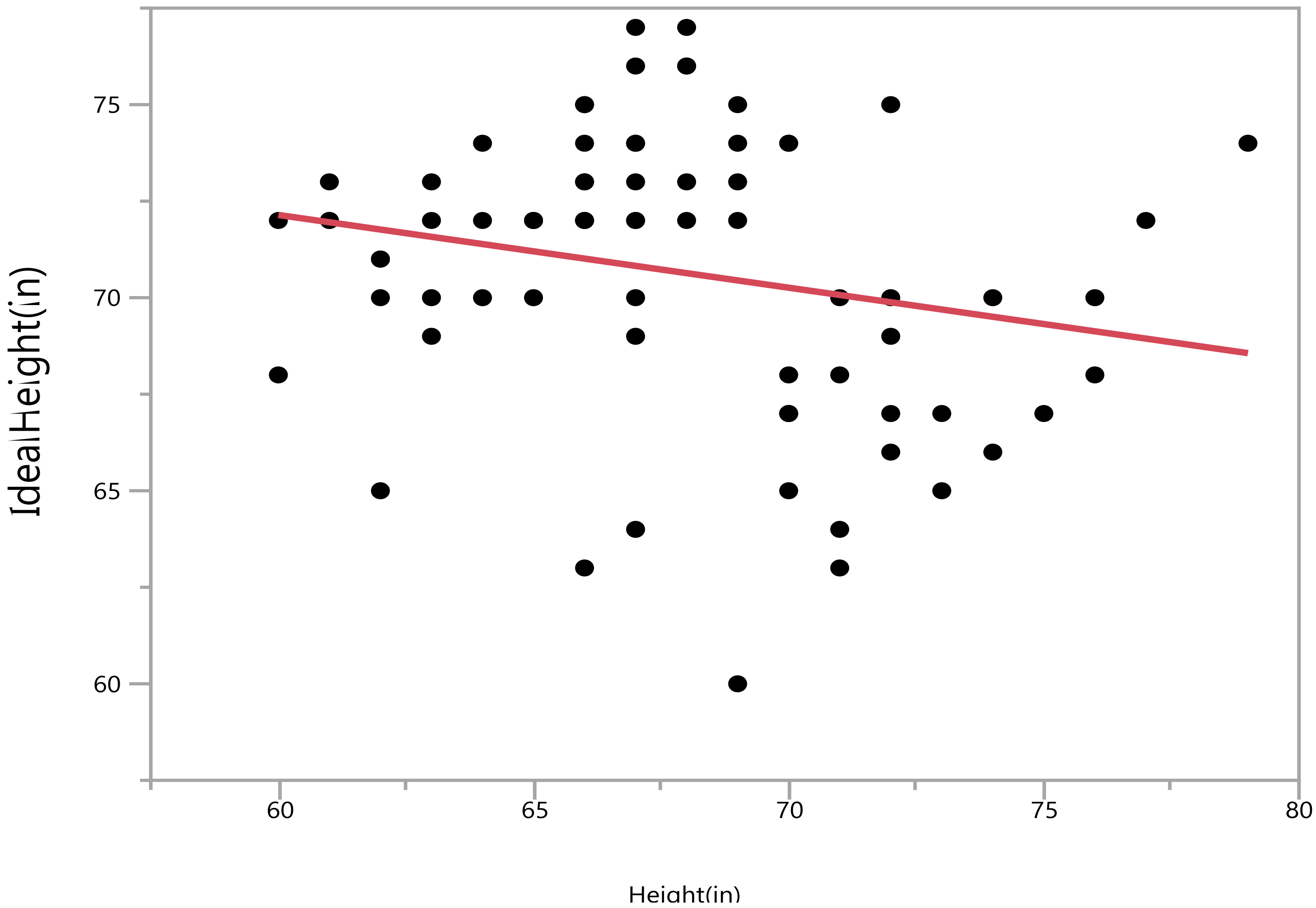
Model

$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$$

where

- Y = height of student's ideal romantic partner
- x = height of student

Scatterplot



SLR Output

Analysis of Variance					
Source	**DF**	**Sum of Squares**	**Mean Square**	**F Value**	**Pr > F**
Model	1	47.08523	47.0852	3.6925	0.0586
Error	73	930.86144	12.7515		
C. Total	74	977.94667			

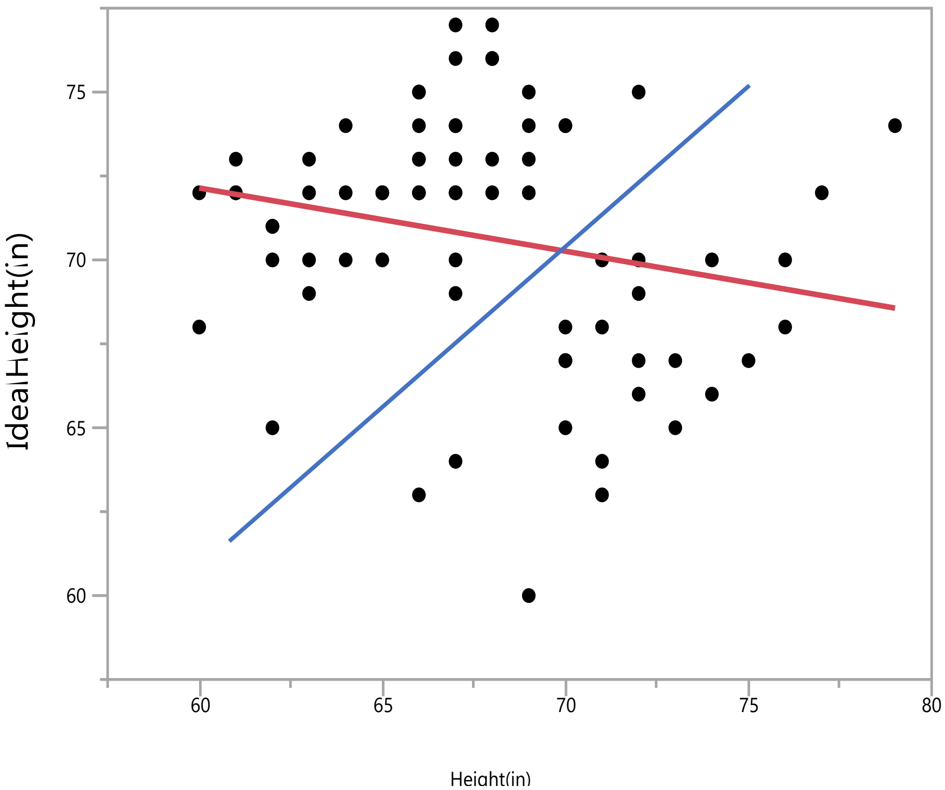
Root MSE	3.570928	**R-Square**	0.04815
Dependent Mean	70.69333	**Adj R-Sq**	0.03511

Variable	**DF**	**Parameter Estimate**	**Standard Error**	**t Value**	**Pr > |t|**
Intercept	1	83.440038	6.646211	12.55	<0.0001
Height(in)	1	-0.188301	0.097992	-1.92	0.0586

Summary of SLR Model

- Linear relationship between student's height and the height of their ideal romantic partner is very weak and negative.
- $R^2 = 4.815\%$; Only 4.815% of the variation in the height of a student's ideal romantic partner can be explained by the simple linear regression with the student's height.
- Student's height is statistically significant at the 10% level, but not the 5% level.

What's going on?



- Two clusters of points in scatterplot.
  - To left of blue line
  - To right of blue line
- What are those clusters related to?

Multiple Linear Regression Model with Categorical Explanatory Variable

- Variable for student's Gender should be added to the model.
- $x_{i1} = \text{student's height}$
- $x_{i2} = \begin{cases} 1 - \text{if student is female} \\ 0 - \text{if student is male} \end{cases}$

Model

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i$$

Two Models

- If student is female ($x_{i2} = 1$)

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 + \varepsilon_i = (\beta_0 + \beta_2) + \beta_1 x_{i1} + \varepsilon_i$$

- If student is male ($x_{i2} = 0$)

$$Y_i = \beta_0 + \beta_1 x_{i1} + \varepsilon_i$$

Comparison of Two Models

- Same Slope (β_1)
  - Assumes relationship between height of ideal romantic partner (response variable Y) and student's height (quantitative explanatory variable x_1) is the same for both groups in the categorical explanatory variable x_2 (Gender = female and Gender = male).

- Different Intercept
  - For females: $\beta_0 + \beta_2$
  - For males: β_0

MLR Output

Analysis of Variance					
Source	**DF**	**Sum of Squares**	**Mean Square**	**F Value**	**Pr > F**
Model	2	675.04314	337.522	80.2287	<0.0001
Error	72	302.90353	4.207		
C. Total	74	977.94667			

Root MSE	2.051096	**R-Square**	0.69027
Dependent Mean	70.69333	**Adj R-Sq**	0.68166

Variable	**DF**	**Parameter Estimate**	**Standard Error**	**t Value**	**Pr > |t|**
Intercept	1	27.664081	5.951059	4.65	<.0001*
Height(in)	1	0.5471394	0.082411	6.64	<.0001*
Gender(1/0)	1	8.9873486	0.735618	12.22	<.0001*

MLR Model

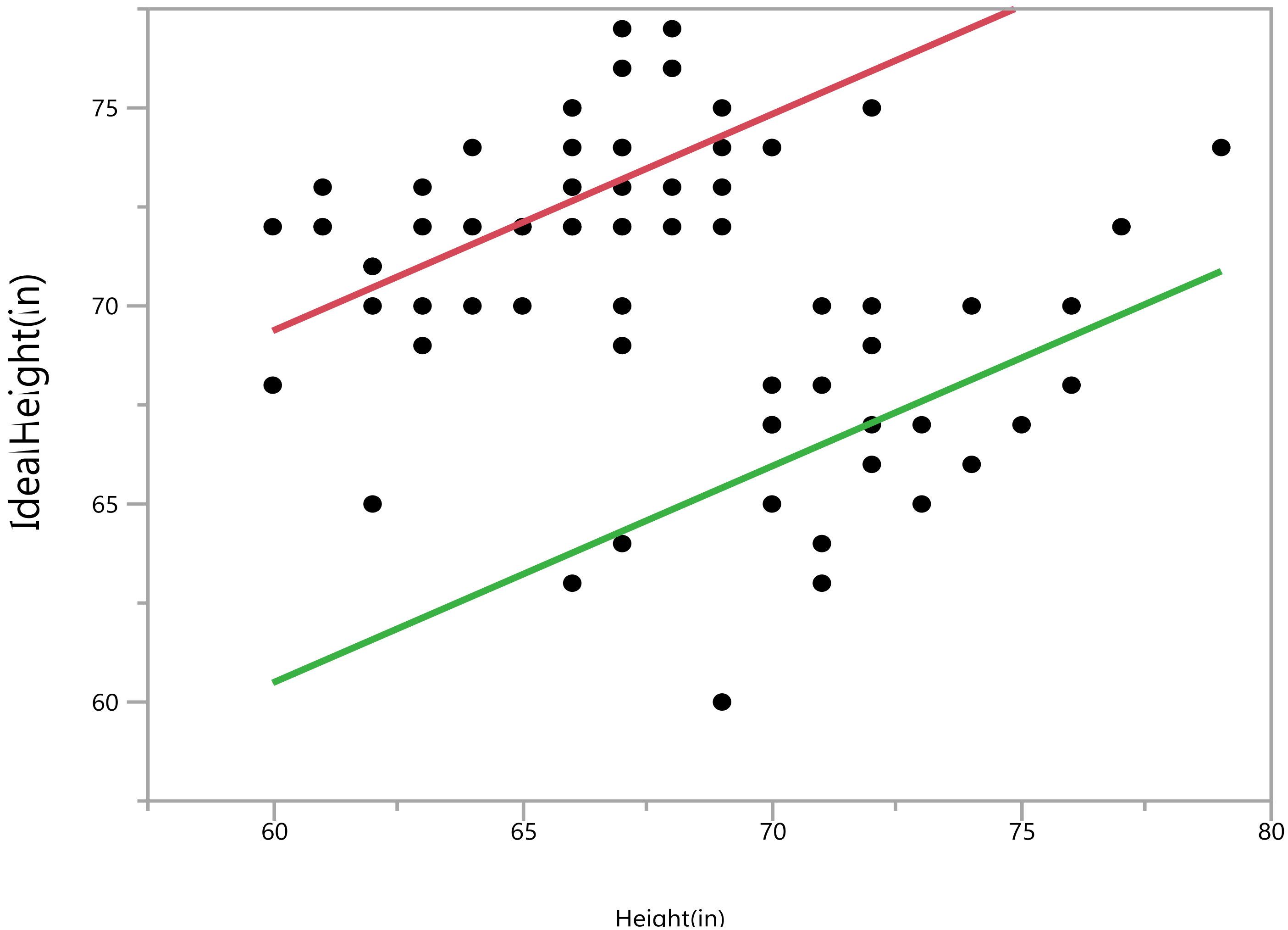
- Model is highly significant in explaining the height of the students' ideal romantic partner (p-value < 0.0001)
- $R^2 = 69.03\%$
  - 69.03% of the variation in the height of the students' ideal romantic partner can be explained by the multiple linear regression model with height and Gender of the student.
- Given height in the model, Gender is highly significant (p-value < 0.0001)
- Given Gender in the model, height is highly significant (p-value < 0.0001)

Two Estimated Models

- Using the parameter estimates from the MLR, we can determine the estimated intercept and slope of the two models – one for females and one for males. The two models are:

$$\hat{Y}_i = \begin{cases} 36.651 + 0.547x_i - \text{for females} \\ 27.664 + 0.547x_i - \text{for males} \end{cases}$$

Scatterplot



Interaction Model

- We can consider a model with an interaction term between x_1 and x_2 - height and Gender.

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i1} x_{i2} + \varepsilon_i$$

- This model will allow for a different relationship between students' height and the height of their ideal romantic partner.

Two Models

- If student is female ($x_{i2} = 1$)

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 + \beta_3 x_{i1} + \varepsilon_i$$

$$= (\beta_0 + \beta_2) + (\beta_1 + \beta_3) x_{i1} + \varepsilon_i$$

- If student is male ($x_{i2} = 0$)

$$Y_i = \beta_0 + \beta_1 x_{i1} + \varepsilon_i$$

Comparison of Two Models

- Different Slope
  - For females: $\beta_1 + \beta_3$
  - For males: β_1

- Different Intercept
  - For females: $\beta_0 + \beta_2$
  - For males: β_0

MLR Model with Interaction

Analysis of Variance					
Source	**DF**	**Sum of Squares**	**Mean Square**	**F Value**	**Pr > F**
Model	3	684.77579	228.259	55.2796	<0.0001
Error	71	293.17088	4.129		
C. Total	74	977.94667			

Root MSE	2.032035	**R-Square**	0.70022
Dependent Mean	70.69333	**Adj R-Sq**	0.68755

Variable	**DF**	**Parameter Estimate**	**Standard Error**	**t Value**	**Pr > |t|**
Intercept	1	15.451772	9.90122	1.56	0.1231
Height(in)	1	0.7166606	0.137325	5.22	<.0001*
Gender(1/0)	1	27.272359	11.93225	2.29	0.0253*
Interaction	1	-0.262206	0.170788	-1.54	0.1292

MLR Model with Interaction

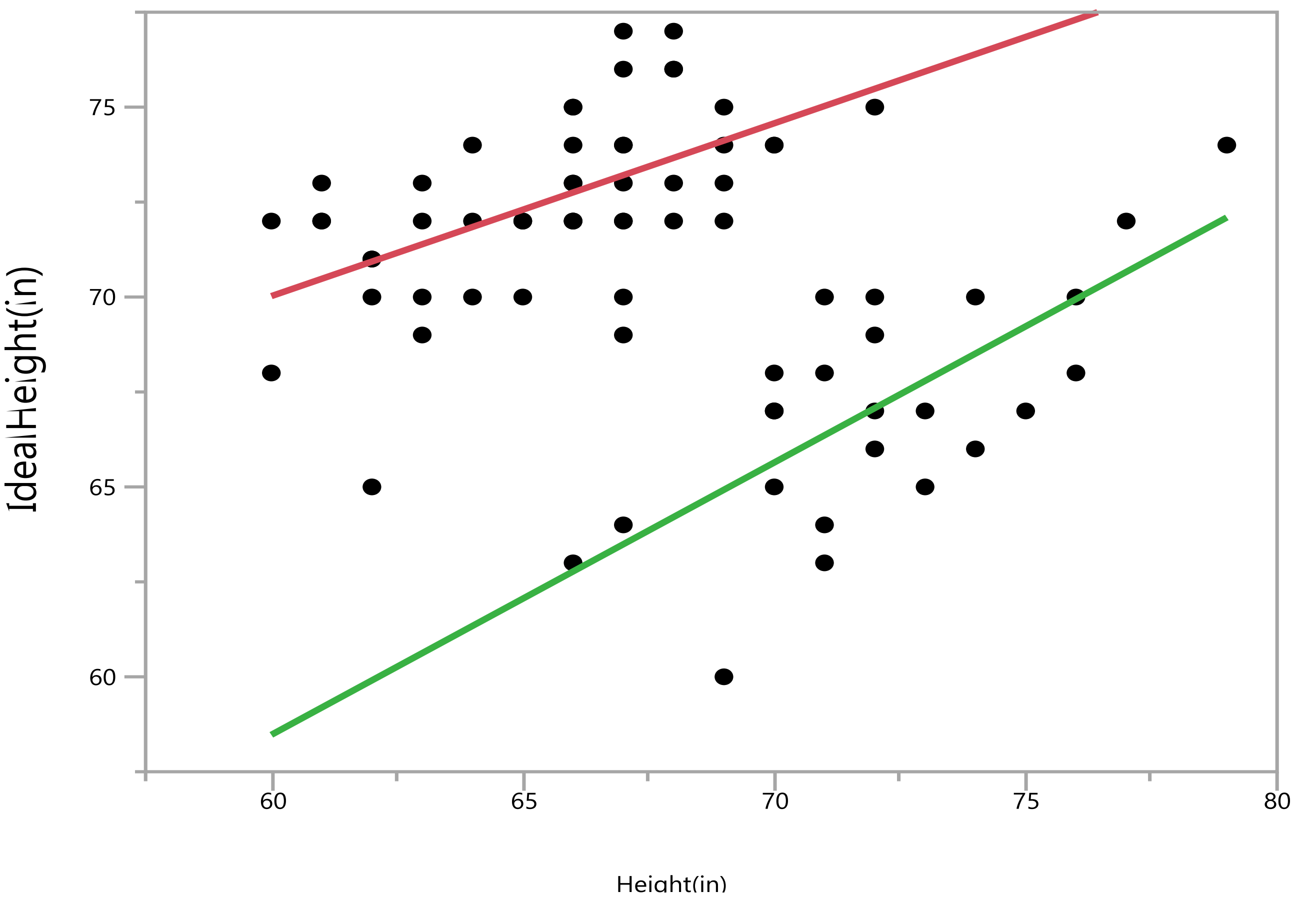
- Model is highly significant in explaining the height of the students' ideal romantic partner (p-value < 0.0001)
- $R^2 = 70.02\%$; 70.02% of the variation in the height of the students' ideal romantic partner can be explained by the multiple linear regression model with height and Gender of the student.
- The interaction term is not statistically significant (p-value = 0.1292).

Two Estimated Models

- Using the parameter estimates from the MLR model, we can determine the estimated intercept and slope of the models for females and for males. The two models are:

$$\hat{Y}_i = \begin{cases} 42.724 + 0.455x_i - \text{for females} \\ 15.452 + 0.717x_i - \text{for males} \end{cases}$$

Scatterplot



Model Comparison

- MLR Model
  - $SS_{\text{error}} = 302.90353$
  - $MS_{\text{error}} = 4.207$
  - $R^2 = 69.03\%$
  - adj $R^2 = 68.17\%$
- MLR Model with Interaction
  - $SS_{\text{error}} = 293.17088$
  - $MS_{\text{error}} = 4.129$
  - $R^2 = 70.02\%$
  - adj $R^2 = 68.76\%$

Which Model to Select?

- The two models are very similar.
- MLR model with interaction term has slightly better values for:
  - MS_{error}
  - adj R^2
- Interaction term is not statistically significant (p-value = 0.1292)
- Estimated MLR model with interaction term appears to fit the points in the scatterplot slightly better than the MLR model.

Alternative Parameterization for Categorical Variables

- Baseline coding of categories
  - 1 = if student is female
  - 0 = if student is male

- Sum to zero coding of categories
  - 1 = if student is female
  - -1 = if student is male

Multiple Linear Regression Model with Categorical Explanatory Variable

- $x_{i1} = \text{student's height}$
- $x_{i2} = \begin{cases} 1 - \text{if student is female} \\ -1 - \text{if student is male} \end{cases}$

Model

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \varepsilon_i$$

Two Models

- If student is female ($x_{i2} = 1$)

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 + \varepsilon_i = (\beta_0 + \beta_2) + \beta_1 x_{i1} + \varepsilon_i$$

- If student is male ($x_{i2} = -1$)

$$Y_i = \beta_0 + \beta_1 x_{i1} - \beta_2 + \varepsilon_i = (\beta_0 - \beta_2) + \beta_1 x_{i1} + \varepsilon_i$$

Comparison of Two Models

- Same Slope (β_1)

- Different Intercept
  - For females: $\beta_0 + \beta_2$
  - For males: $\beta_0 - \beta_2$

MLR Output

Analysis of Variance					
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Dependent Mean	70.69333	**Adj R-Sq**	0.68166

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	32.157755	5.673807	5.67	<.0001*
Height(in)	1	0.5471394	0.082411	6.64	<.0001*
Gender[Female]	1	4.4936743	0.367809	12.22	<.0001*

Two Estimated Models

- Using the parameter estimates from the MLR, we can determine the estimated intercept and slope of the two models – one for females and one for males. The two models are:

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$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i1} x_{i2} + \varepsilon_i$$

- This model will allow for a different relationship between students' height and the height of their ideal romantic partner.

Two Models

- If student is female ($x_{i2} = 1$)

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 + \beta_3 x_{i1} + \varepsilon_i$$

$$= (\beta_0 + \beta_2) + (\beta_1 + \beta_3) x_{i1} + \varepsilon_i$$

- If student is male ($x_{i2} = -1$)

$$Y_i = \beta_0 + \beta_1 x_{i1} - \beta_2 - \beta_3 x_{i1} + \varepsilon_i$$

$$Y_i = (\beta_0 - \beta_2) + (\beta_1 - \beta_3) x_{i1} + \varepsilon_i$$

Comparison of Two Models

- Different Slope
  - For females: $\beta_1 + \beta_3$
  - For males: $\beta_1 - \beta_3$

- Different Intercept
  - For females: $\beta_0 + \beta_2$
  - For males: $\beta_0 - \beta_2$

MLR Model with Interaction

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Variable	**DF**	**Parameter Estimate**	**Standard Error**	**t Value**	**Pr > |t|**
Intercept	1	29.087952	5.966125	4.88	<.0001*
Height(in)	1	0.5855576	0.085394	6.86	<.0001*
Gender(1/0)	1	13.63618	5.966125	2.29	0.0253*
Interaction	1	-0.131103	0.085394	-1.54	0.1292

Two Estimated Models

- Using the parameter estimates from the MLR model, we can determine the estimated intercept and slope of the models for females and for males. The two models are:

$$\hat{Y}_i = \begin{cases} 42.724 + 0.455x_i - \text{for females} \\ 15.452 + 0.717x_i - \text{for males} \end{cases}$$

MLR with Categorical Explanatory Variable

- We can also add categorical variables with more than two categories to our multiple linear regression model.

- Multiple columns to our design matrix or such categorical variables

MLR with Categorical Explanatory Variable

- For example, the type of vehicle has 4 categories: car, truck, minivan, SUV/crossover
- Will the following design matrix work?

Intercept	x_1	x_2	Car	Truck	Minivan	SUV/crossover
1	$\vdots$	$\vdots$	1	0	0	0
1	$\vdots$	$\vdots$	1	0	0	0
1	$\vdots$	$\vdots$	0	1	0	0
1	$\vdots$	$\vdots$	0	1	0	0
1	$\vdots$	$\vdots$	0	0	1	0
1	$\vdots$	$\vdots$	0	0	1	0
1	$\vdots$	$\vdots$	0	0	0	1
1	$\vdots$	$\vdots$	0	0	0	1

MLR with Categorical Explanatory Variable

- The problem is the four columns specifying these four groups will add to the first column (corresponding to the intercept)

Intercept	x_1	x_2	Car	Truck	Minivan	SUV/crossover
1	$\vdots$	$\vdots$	1	0	0	0
1	$\vdots$	$\vdots$	1	0	0	0
1	$\vdots$	$\vdots$	0	1	0	0
1	$\vdots$	$\vdots$	0	1	0	0
1	$\vdots$	$\vdots$	0	0	1	0
1	$\vdots$	$\vdots$	0	0	1	0
1	$\vdots$	$\vdots$	0	0	0	1
1	$\vdots$	$\vdots$	0	0	0	1

MLR with Categorical Explanatory Variable

- To fix this problem, we will constraint the value of one of the groups (called the baseline group).
- For example, let's use the SUV/crossover group as our baseline group

Intercept	x_1	x_2	Car	Truck	Minivan
1	$\vdots$	$\vdots$	1	0	0
1	$\vdots$	$\vdots$	1	0	0
1	$\vdots$	$\vdots$	0	1	0
1	$\vdots$	$\vdots$	0	1	0
1	$\vdots$	$\vdots$	0	0	1
1	$\vdots$	$\vdots$	0	0	1
1	$\vdots$	$\vdots$	0	0	0
1	$\vdots$	$\vdots$	0	0	0

MLR with Categorical Explanatory Variable

- The model is:

$$Y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \beta_5 x_{i5} + \varepsilon_i$$

where

- $x_{i3} = \begin{cases} 1 - \text{if vehicle is car} \\ 0 - \text{otherwise} \end{cases}$
- $x_{i4} = \begin{cases} 1 - \text{if vehicle is truck} \\ 0 - \text{otherwise} \end{cases}$
- $x_{i5} = \begin{cases} 1 - \text{if vehicle is minivan} \\ 0 - \text{otherwise} \end{cases}$

MLR with Categorical Explanatory Variable

The parameters $\beta_3, \beta_4, \text{and } \beta_5$ have special interpretations in this model. They are:

- $\beta_3 =$ the difference in the expected value of the response variable between *cars* and SUV/crossovers.
- $\beta_4 =$ the difference in the expected value of the response variable between *trucks* and SUV/crossovers.
- $\beta_5 =$ the difference in the expected value of the response variable between *minivans* and SUV/crossovers.

MLR with Categorical Explanatory Variable

- Testing for the statistical significance of the type of vehicle requires testing:

$$H_0: \beta_3 = \beta_4 = \beta_5$$

H_a: at least one of $\beta_j \neq 0, j = 3, 4, 5.$

- Use the partial F test to do it.

MLR with Categorical Explanatory Variable

- Reduced model: the model without the categorical variable.
- Full model: the model with the categorical variable.

- The F-test statistic is

$$F = \frac{(SSE_{r.model} - SSE_{f.model})/(m-1)}{MSE_{f.model}}$$

where $m =$ the number of categories in the categorical variable (m-1=3 in our example)

- Reject H_0 if $F > F_{m-1,n-(k+1),1-\alpha}$ distribution where $n-(k+1)$ is the df for error for the full model.