STAT 500 Homework 11

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November 19, 2020

1 Question 1

(a) Linear model.

(b) Linear model.

(c) Intrinsically linear model.

$Y_i = \beta_0 + log(\beta_1 X_{1i}) + \beta_2 X_{2i} + \epsilon_i = \beta_0 + log(\beta_1) + log(X_{1i}) + \beta_2 X_{2i} + \epsilon_i$

$\implies Y_i - log(\beta_1) = \beta_0 + log(X_{1i}) + \beta_2 X_{2i} + \epsilon_i$

(d) Nonlinear model.

(e) Intrinsically linear model.

$Y_i = [1 + exp(\beta_0 + \beta_1 X_{1i} + \epsilon_i)]^{-1}$

$\implies Y_i[1 + exp(\beta_0 + \beta_1 X_{1i} + \epsilon_i)] = 1$

$\implies Y_i + Y_i exp(\beta_0 + \beta_1 X_{1i} + \epsilon_i) = 1$

$\implies 1 - Y_i = Y_i exp(\beta_0 + \beta_1 X_{1i} + \epsilon_i)$

$\implies ln(1 - Y_i) = ln(Y_i) + \beta_0 + \beta_1 X_{1i} + \epsilon_i$

$\implies ln(1 - Y_i) - ln(Y_i) = \beta_0 + \beta_1 X_{1i} + \epsilon_i$

(f) Nonlinear model.

$Y_i = (\beta_0 + \beta_1 X_i)\epsilon_i$

$\implies \dfrac{Y_i}{\epsilon_i} = \beta_0 + \beta_1 X_i$

(g) Intrinsically linear model.

$Y_i = \epsilon_i exp(\beta_0 + \beta_1 X_{1i} + \beta - 2X_{2i})$

$\implies ln(Y_i) = ln(\epsilon_i) + \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i}$

2 Question 2

$P_X = X(X^TX)^{-1}X^T$

$P_X^2 = X(X^TX)^{-1}X^TX(X^TX)^{-1}X^T = XI(X^TX)^{-1}X^T = X(X^TX)^{-1}X^T$

Therefore, P_X is idempotent.

Moreover, P_X is symmetric.

P_1 is an idempotent, square matrix of size $n \times n$ and $P_1 = \begin{pmatrix} \frac{1}{n} & \dots & \frac{1}{n} \\ \dots & \dots & \dots \\ \frac{1}{n} & \dots & \frac{1}{n} \end{pmatrix}$ Thus, $P_XP_1 = P_1P_X$

We have $P_XP_1 = P_1$. Indeed, consider $P_XP_1 - P_1$. Multiply P_X, which is non-zero, to the right, we have $P_XP_1P_X - P_1P_X = P_1P_XP_X - P_1P_X = P_1P_X - P_1P_X = 0$. Hence, $P_XP_1 - P_1 = 0$ and $P_XP_1 = P_1 = P_1P_X$ Now, $(P_X - P_1)(P_X - P_1) = P_X - PXP_1 - P_1P_x + P_1 = P_X - P_1 - P_1 + P_1 = P_X - P_1$

Thus, $P_X - P_1$ is idempotent.

3 Question 3

Prove $AA^gA = A$.

$$AA^g = \begin{pmatrix} 1 \\ 2 \\ 5 \\ -2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix}$$

$$AA^gA = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 5 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \\ -2 \end{pmatrix}$$

4 Question 4

$$X = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \ A = \begin{pmatrix} a_1 & a_2 & \dots & a_8 \end{pmatrix}$$

(a) $C = \begin{pmatrix} 0 & 1 & \frac{-1}{2} & \frac{-1}{2} \end{pmatrix}$

We obtain a system of equations:

$\sum_{i=1}^{8} a_i = 0$

$a_1 + a_2 = 1$

$\sum_{i=3}^{6} a_i = \frac{-1}{2}$

$a_7 + a_8 = \frac{-1}{2}$

Choose $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} & 0 & 0 & 0 & 0 & \frac{-1}{2} \end{pmatrix}$

The linear function is estimable.

(b) $C = \begin{pmatrix} 3 & 1 & 2 & 0 \end{pmatrix}$

We obtain a system of equations:

$\sum_{i=1}^{8} a_i = 3$

$a_1 + a_2 = 1$

$\sum_{i=3}^{6} a_i = 2$

$a_7 + a_8 = 0$

Choose $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 1 & 1 & 0 & 0 & \frac{-1}{2} & \frac{1}{2} \end{pmatrix}$

The linear function is estimable.

(c) $C = \begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix}$

We obtain a system of equations:

$\sum_{i=1}^{8} a_i = 0$

$a_1 + a_2 = 0$

$\sum_{i=3}^{6} a_i = 1$

$a_7 + a_8 = 1$

It is not possible to find such a_i. The linear function is not estimable.

(d) $C = \begin{pmatrix} 3 & -1 & -1 & -1 \end{pmatrix}$

We obtain a system of equations:

$$\sum_{i=1}^{8} a_i = 3$$
$$a_1 + a_2 = -1$$
$$\sum_{i=3}^{6} a_i = -1$$
$$a_7 + a_8 = -1$$

It is not possible to find such a_i. The linear function is not estimable.

5 Question 5

$$X = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, A = \begin{pmatrix} a_1 & a_2 & \dots & a_{12} \end{pmatrix}$$

(a) The linear function is not estimable because X_1 is a linear combination of the other columns.

(b) $C = \begin{pmatrix} 0 & 1 & -1 & 0 & \dots & 0 \end{pmatrix}$

We obtain a system of equations:

$$\sum_{i=1}^{12} a_i = 0$$
$$\sum_{i=1}^{6} a_i = 1 \ (1)$$
$$\sum_{i=7}^{12} a_i = -1$$
$$a_1 + a_2 + a_7 + a_8 = 0$$
$$a_3 + a_4 + a_9 + a_{10} = 0$$
$$a_5 + a_6 + a_{11} + a_{12} = 0$$
$$a_1 + a_2 = 0 \ (2)$$
$$a_3 + a_4 = 0 \ (3)$$
$$a_5 + a_6 = 0 \ (4)$$

$a_7 + a_8 = 0$

$a_9 + a_{10} = 0$

$a_{11} + a_{12} = 0$

(1), (2), (3), and (4) are conflicting with each other. It is not possible to find such a_i. The linear function is not estimable.

(c) $X_8 = X_1 - X_2 - X_3 + X_4 + X_5 + X_6 - X_7 - X_8 - X_9 - X_{10} - X_{11} - X_{12}$. Hence, X_8 is a linear combination of the other columns, so this linear function is not estimable.

(d) C = $\begin{pmatrix} 0 & \dots & 1 & -1 & 0 & \dots & 0 \end{pmatrix}$

We obtain a system of equations:

$\sum_{i=1}^{12} a_i = 0$

$\sum_{i=1}^{6} a_i = 0$

$\sum_{i=7}^{12} a_i = 0$

$a_1 + a_2 + a_7 + a_8 = 0$ (1)

$a_3 + a_4 + a_9 + a_{10} = 0$

$a_5 + a_6 + a_{11} + a_{12} = 0$

$a_1 + a_2 = 1$ (2)

$a_3 + a_4 = -1$ (3)

$a_5 + a_6 = 0$

$a_7 + a_8 = 0$ (4)

$a_9 + a_{10} = 0$

$a_{11} + a_{12} = 0$

(1), (2), (3), and (4) are conflicting with each other. It is not possible to find such a_i. The linear function is not estimable.

(e) C = $\begin{pmatrix} 0 & \dots & 1 & -1 & 0 & -1 & -1 & 0 \end{pmatrix}$

We obtain a system of equations:

$\sum_{i=1}^{12} a_i = 0$

$\sum_{i=1}^{6} a_i = 0$

$\sum_{i=7}^{12} a_i = 0$

$a_1 + a_2 + a_7 + a_8 = 0$

$a_3 + a_4 + a_9 + a_{10} = 0$ (1)

$a_5 + a_6 + a_{11} + a_{12} = 0$

$a_1 + a_2 = 1$

$a_3 + a_4 = -1$ (2)

$a_5 + a_6 = 0$

$a_7 + a_8 = -1$

$a_9 + a_{10} = -1$ (3)

$a_{11} + a_{12} = 0$

(1), (2), and (3) are conflicting with each other. It is not possible to find such a_i. The linear function is not estimable.