STAT 500

Multi-Factor Experiments

Factors and Levels

- A **factor** is an explanatory variable studied in an investigation.

- The different values of a factor are called **levels**.

Multi-Factor Experiments

- Examine effects of two or more factors within a single experiment
- Examples:
  - Vary price (3 levels) and type of advertising media (2 levels) to explore effect on sales
  - Examine the effects of varieties (4 levels) and soil type (3 levels) on corn yield
- Can learn about <u>interactions</u>:

  The effects of changing the levels of one factor are not the same across all levels of another factor

Factorial Designs

- **Factorial treatment designs** use combinations of levels of two or more factors as treatments

- Example

  - Factor A - 3 levels (a_1, a_2, a_3)
  - Factor B - 2 levels (b_1, b_2)
  - Combinations of A and B $\Rightarrow$ 6 Treatments

$$(a_1b_1,\ a_1b_2,\ a_2b_1,\ a_2b_2,\ a_3b_1,\ a_3b_2)$$

Factorial Designs

- Terminology:
  - **Complete (full) factorial** - all possible combinations of factor levels are used
  - Fractional factorial - only a subset used are used

- Complete designs are commonly used. Fractional designs are important in industrial applications.

Factorial Designs: Outline

- Factorial designs with two treatment factors
- Factorial designs with blocking
- Factorial designs with more than two factors
- Factorial designs with no replication
- Unbalanced factorial designs (combinations of factor levels are not all used the same number of times) ANOVA (considered in Stat 510).

Two-Factor Experiments

Example

Example: Examine the effects of different concentrations of copper and zinc in water on the ability of minnow larvae to produce protein

- Factor A: Concentration of copper (0 or 150 ppm)
- Factor B: Concentration of zinc (0, 750 or 1500 ppm)
- Treatments: All 6 combination of 2 levels of copper and 3 levels of zinc (complete/full factorial treatment design)

Copper Conc.	Zinc Concentration		
	0 ppm	750 ppm	1500 ppm
0 ppm			
150 ppm			

Example

- Experimental units: Twelve water tanks containing minnow larvae

- Experimental design: CRD, Experimental units are randomly assigned to the 6 treatments with 2 units per treatment.

- Response Variable: protein content (μg/larva)

Data: Example

Copper Conc.	Zinc Concentration 0 ppm	750 ppm	1500 ppm
0 ppm	$Y_{111} = 201$	$Y_{121} = 173$	$Y_{131} = 115$
	$Y_{112} = 186$	$Y_{122} = 162$	$Y_{132} = 124$
150 ppm	$Y_{211} = 163$	$Y_{221} = 184$	$Y_{231} = 114$
	$Y_{212} = 182$	$Y_{222} = 157$	$Y_{232} = 108$

Notation

- Factor A - $i = 1, \ldots, a$
- Factor B - $j = 1, \ldots, b$
- Replications - $k = 1, \ldots, n$
- Total number of observations $= nab$

Data

- $\boldsymbol{Y_{ijk}}$ = response of the $\boldsymbol{k}$th repetition of level $\boldsymbol{i}$ of factor A and level $\boldsymbol{j}$ of factor B
- $\boldsymbol{\bar{Y}_{ij.} = \frac{1}{n} \Sigma_{k=1}^{n} Y_{ijk}}$ = mean response of observations in level $\boldsymbol{i}$ of factor A and level $\boldsymbol{j}$ of factor B
- $\boldsymbol{\bar{Y}_{i..} = \frac{1}{nb} \Sigma_{k=1}^{n} \Sigma_{j=1}^{b} Y_{ijk}}$ = mean response of observations in level $\boldsymbol{i}$ of factor A
- $\boldsymbol{\bar{Y}_{.j.} = \frac{1}{na} \Sigma_{k=1}^{n} \Sigma_{i=1}^{a} Y_{ijk}}$ = mean response of observations in level $\boldsymbol{j}$ of factor B
- $\boldsymbol{\bar{Y}_{...} = \frac{1}{nab} \Sigma_{k=1}^{n} \Sigma_{i=1}^{a} \Sigma_{j=1}^{b} Y_{ijk}}$ = overall mean response

Cell Means Model

$$Y_{ijk} = \mu_{ij} + \epsilon_{ijk} \qquad \epsilon_{ijk} \text{ are i.i.d. } N(0, \sigma^2)$$

- μ_{ij} = mean response to level i of factor A and level j of factor B
- $\bar{\mu}_{i.} = \frac{1}{b} \Sigma_j \mu_{ij}$ = mean response of factor A at level i, averaging across the levels of factor B
- $\bar{\mu}_{.j} = \frac{1}{a} \Sigma_i \mu_{ij}$ = mean response of factor B at level j, averaging across the levels of factor A
- $\bar{\mu}_{..} = \frac{1}{ab} \Sigma_i \Sigma_j \mu_{ij}$ = mean response, averaging across the levels of both factors
- σ^2 = variance of responses in level i of factor A and level j of factor B

Research Questions

- Questions that could be asked:

  1. Are the 6 response means (μ_{ij}) the same?

  2. Are mean responses to copper levels the same, averaging over zinc levels? $\bar{\mu}_{1\cdot} = \bar{\mu}_{2\cdot}$?

  3. are mean responses to zinc levels the same, averaging over copper levels? $\bar{\mu}_{\cdot 1} = \bar{\mu}_{\cdot 2} = \bar{\mu}_{\cdot 3}$?

  4. Are differences in mean responses between copper levels the same across zinc levels? $(\mu_{11} - \mu_{21}) = (\mu_{12} - \mu_{22}) = (\mu_{13} - \mu_{23})$?

Model: Example

As in the minnow larvae experiment, suppose there are 2 levels of factor A (i=1,2), 3 levels of factor B (j=1,2, 3) and 2 units assigned to each of the combinations of the two factors (k=1,2)

$$\begin{bmatrix} \boldsymbol{Y_{111}} \\ \boldsymbol{Y_{112}} \\ \boldsymbol{Y_{121}} \\ \boldsymbol{Y_{122}} \\ \boldsymbol{Y_{131}} \\ \boldsymbol{Y_{132}} \\ \boldsymbol{Y_{211}} \\ \boldsymbol{Y_{212}} \\ \boldsymbol{Y_{221}} \\ \boldsymbol{Y_{222}} \\ \boldsymbol{Y_{231}} \\ \boldsymbol{Y_{232}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \mu_{13} \\ \mu_{21} \\ \mu_{22} \\ \mu_{23} \end{bmatrix} + \begin{bmatrix} \epsilon_{111} \\ \epsilon_{112} \\ \epsilon_{121} \\ \epsilon_{122} \\ \epsilon_{131} \\ \epsilon_{132} \\ \epsilon_{211} \\ \epsilon_{212} \\ \epsilon_{221} \\ \epsilon_{222} \\ \epsilon_{231} \\ \epsilon_{232} \end{bmatrix}$$

Least Squares Estimate: Example

The model has the form of a linear model $\mathbf{Y} = \boldsymbol{X}\,\boldsymbol{\beta} + \boldsymbol{\epsilon}$

The least squares estimates of the mean responses for the six combinations of the levels of the two factors are

$$\hat{\beta} = \begin{bmatrix} \hat{\mu}_{11} \\ \hat{\mu}_{12} \\ \hat{\mu}_{13} \\ \hat{\mu}_{21} \\ \hat{\mu}_{22} \\ \hat{\mu}_{23} \end{bmatrix} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\mathbf{Y} = \begin{bmatrix} \bar{Y}_{11\cdot} \\ \bar{Y}_{12\cdot} \\ \bar{Y}_{13\cdot} \\ \bar{Y}_{21\cdot} \\ \bar{Y}_{22\cdot} \\ \bar{Y}_{23\cdot} \end{bmatrix} = \begin{bmatrix} 193.5 \\ 167.5 \\ 119.5 \\ 172.5 \\ 170.5 \\ 111.0 \end{bmatrix}$$

ANOVA Table

- The following formulas assume equal sample sizes $n_{ij} = n$. (Note that the total sample size is abn)

- The ANOVA Table for the $a \times b$ treatments is

Source	d.f.	Sum of Squares
Model	$ab - 1$	$SS_{\text{model}} = n \sum_i \sum_j (\bar{Y}_{ij.} - \bar{Y}_{...})^2$
Error	$ab(n-1)$	$SS_{\text{error}} = \sum_i \sum_j \sum_k (Y_{ijk} - \bar{Y}_{ij.})^2$
Total	$abn - 1$	$SS_{\text{total}} = \sum_i \sum_j \sum_k (Y_{ijk} - \bar{Y}_{...})^2$

ANOVA Table: Example

Source	d.f.	SS	MS	F-test	p-value
Model	5	10755.75	2151.15	16.62	0.0019
Error	**6**	776.50	129.42		
Total	**11**	11532.25			

F-test for Treatments

- $H_o : \mu_{ij}$ are equal for all $i = 1, \ldots, a$ and $j = 1, \ldots, b$
- $H_a :$ at least one μ_{ij} is different
- Reject H_0 if

$$F = \frac{MS_{\text{model}}}{MS_{\text{error}}} > F_{ab-1, ab(n-1), 1-\alpha}$$

- For our example
  - F = 16.62 with p-value 0.0019 $\Rightarrow$ Reject H_0
  - Conclude there is sufficient evidence that at least one of the mean responses for the six treatments is different.

Additional Research Questions

- Question 2: Are mean responses to copper levels the same, averaging over zinc levels? ($\bar{\boldsymbol{\mu}}_{\mathbf{1}\cdot} = \bar{\boldsymbol{\mu}}_{\mathbf{2}\cdot}$)
- Question 3: Are mean responses to zinc levels the same, averaging over copper levels? ($\bar{\boldsymbol{\mu}}_{\cdot\mathbf{1}} = \bar{\boldsymbol{\mu}}_{\cdot\mathbf{2}} = \bar{\boldsymbol{\mu}}_{\cdot\mathbf{3}}$)
- Question 4: Are differences in mean responses between copper levels consistent across zinc levels?
  ($\boldsymbol{\mu}_{\mathbf{11}} - \boldsymbol{\mu}_{\mathbf{21}} = \boldsymbol{\mu}_{\mathbf{12}} - \boldsymbol{\mu}_{\mathbf{22}} = \boldsymbol{\mu}_{\mathbf{13}} - \boldsymbol{\mu}_{\mathbf{23}}$)

Contrasts and Factor Effects

Questions 2, 3 and 4 can be addressed with contrasts on the cell means μ_{ij}

- Question 2: Are there equal means for copper levels?
  - H_0: $\bar{\mu}_{1\cdot} - \bar{\mu}_{2\cdot} = 0$
  - One Contrast:

$$\bar{\mu}_{1\cdot} - \bar{\mu}_{2\cdot} = \frac{\mu_{11} + \mu_{12} + \mu_{13}}{3} - \frac{\mu_{21} + \mu_{22} + \mu_{23}}{3}$$

Contrasts and Factor Effects

- Question 3: Are there equal means for zinc levels?
  - H_0: $\bar{\mu}_{\cdot 1} = \bar{\mu}_{\cdot 2} = \bar{\mu}_{\cdot 3}$
  - Two Contrasts:

$$\bar{\mu}_{\cdot 1} - \bar{\mu}_{\cdot 2} = \frac{\mu_{11} + \mu_{21}}{2} - \frac{\mu_{12} + \mu_{22}}{2}$$

$$\frac{\bar{\mu}_{\cdot 1} + \bar{\mu}_{\cdot 2}}{2} - \bar{\mu}_{\cdot 3} = \frac{\mu_{11} + \mu_{21} + \mu_{12} + \mu_{22}}{4} - \frac{\mu_{13} + \mu_{23}}{2}$$

Contrasts and Factor Effects

- Question 4: Are difference in mean response for copper levels consistent across zinc levels?
  - $H_0 : \mu_{11} - \mu_{21} = \mu_{12} - \mu_{22} = \mu_{13} - \mu_{23}$
  - Two Contrasts:

$$(\mu_{11} - \mu_{21}) - (\mu_{12} - \mu_{22})$$

$$\frac{(\mu_{11} - \mu_{21}) + (\mu_{12} - \mu_{22})}{2} - (\mu_{13} - \mu_{23})$$

Contrasts and Factor Effects

- For convenience, orthogonal contrasts are used. This will enable us to add contrast sums of squares to develop an F-test for the null hypothesis

- Five contrasts are orthogonal among the cell means μ_{ij}

	Cell Means					
	μ_{11}	μ_{12}	μ_{13}	μ_{21}	μ_{22}	μ_{23}
C1: Copper effect	1/3	1/3	1/3	-1/3	-1/3	-1/3
C2: Zinc effect 1	1/2	-1/2	0	1/2	-1/2	0
C3: Zinc effect 2	1/4	1/4	-1/2	1/4	1/4	-1/2
C4: Interaction 1	1	-1	0	-1	1	0
C5: Interaction 2	-1/2	-1/2	1	1/2	1/2	-1

Contrasts and Factor Effects

- Obtain sums of squares for each of these orthogonal contrasts.

Contrast	SS
C1: Copper effect	234.08
C2: Zinc effect 1	392.00
C3: Zinc effect 2	9841.50
C4: Interaction 1	288.00
C5: Interaction 2	0.16667

- Because these 5 (=6-1) contrasts are orthogonal,

$$SS_{\text{model}} = SS_{\text{C1}} + SS_{\text{C2}} + SS_{\text{C3}} + SS_{\text{C4}} + SS_{\text{C5}}$$

Contrasts and Factor Effects

- **Main effect**: difference (or contrast) between levels of one factor averaged over all levels of the other factor(s).
- **Simple effect**: difference (or contrast) between levels of one factor at one specific level of the other. e.g., difference between different copper concentration levels when zinc concentration = 0ppm .
- **Interaction** exists when simple effects are not the same.
  - Interaction measures the differences between the simple effects of one factor at different levels of the other factor.
  - Equivalent to non-parallel lines in a plot of means.
  - Could be difference in the magnitude or in direction of responses.

Contrasts and Factor Effects

- Divide model sums of squares into main effects plus interaction

Source	d.f.	SS
Copper	$\mathbf{1}$	$\boldsymbol{SS_{\text{C1}} = 234.08}$
Zinc	$\mathbf{2}$	$\boldsymbol{SS_{\text{C2}} + SS_{\text{C3}} = 10233.50}$
Interaction	$\mathbf{2}$	$\boldsymbol{SS_{\text{C4}} + SS_{\text{C5}} = 288.17}$
Error	$\mathbf{6}$	$\boldsymbol{SS_{\text{error}} = 776.50}$
Total	$\mathbf{11}$	$\boldsymbol{SS_{\text{total}} = 11532.25}$

ANOVA Table: Example

- Including Main Effects and Interaction

Source	d.f.	SS	MS	F-test	p-value
Copper levels	1	234.08	234.08	1.809	0.2272
Zinc levels	2	10233.50	5116.75	39.536	0.0004
Interaction	2	288.17	144.085	1.113	0.3881
Error	6	776.50	129.42		
Total	11	11532.25			

ANOVA Table

In general, partition SS_{model} to assess main effects for each factor and interaction

source of variation	degrees of freedom	sums of squares
Factor A	$a-1$	$nb\sum_i(\bar{Y}_{i\cdot\cdot}-\bar{Y}_{\cdot\cdot\cdot})^2$
Factor B	$b-1$	$na\sum_j(\bar{Y}_{\cdot j\cdot}-\bar{Y}_{\cdot\cdot\cdot})^2$
Interaction AB	$(a-1)(b-1)$	$n\sum_i\sum_j(\bar{Y}_{ij\cdot}-\bar{Y}_{i\cdot\cdot}-\bar{Y}_{\cdot j\cdot}+\bar{Y}_{\cdot\cdot\cdot})^2$
Error	$ab(n-1)$	$\sum_i\sum_j\sum_k(Y_{ijk}-\bar{Y}_{ij\cdot})^2$
Total	$abn-1$	$\sum_i\sum_j\sum_k(Y_{ijk}-\bar{Y}_{\cdot\cdot\cdot})^2$

Expected Mean Squares

$$E(MS_{error}) = \sigma^2$$

$$E(MS_A) = \sigma^2 + nb \textstyle\sum_i (\bar{\mu}_{i.} - \bar{\mu}_{..})^2/(a-1)$$

$$E(MS_B) = \sigma^2 + na \textstyle\sum_j (\bar{\mu}_{.j} - \bar{\mu}_{..})^2/(b-1)$$

$$E(MS_{AB}) = \sigma^2 + n\frac{\sum_i \sum_j (\mu_{ij} - \bar{\mu}_{i.} - \bar{\mu}_{.j} + \bar{\mu}_{..})^2}{(a-1)(b-1)}$$

- All sums of squares (or mean squares) are independent
- Test hypotheses about marginal means and interaction effects using F-tests

F-test for Main Effect of Factor A

$H_o : \bar{\mu}_{1.} = \bar{\mu}_{2.} = \cdots = \bar{\mu}_{a.}$

H_a : at least one $\bar{\mu}_{i.}$ is different, $i = 1, \ldots, a$

Reject H_o if

$$F = MS_A / MS_{\text{error}} \geq F_{a-1, ab(n-1), 1-\alpha}$$

F-test for Main Effect of Factor B

$H_o : \bar{\mu}_{.1} = \bar{\mu}_{.2} = \cdots = \bar{\mu}_{.b}$

H_a : at least one $\bar{\mu}_{.j}$ is different, $j = 1, \ldots, b$

Reject H_o if

$$F = MS_B / MS\text{error} \geq F_{b-1, ab(n-1), 1-\alpha}$$

F-test for Interaction

$H_o : (\mu_{ij} - \mu_{kj}) = (\mu_{ir} - \mu_{kr})$ for all $i \neq k$ and $j \neq r$

$H_o :$ at least one $(\mu_{ij} - \mu_{kj}) \neq (\mu_{ir} - \mu_{kr})$ for some $i \neq k$ and $j \neq r$

Reject H_o if

$$F = MS_{AB}/MS\text{error} \geq F_{(a-1)(b-1),ab(n-1),1-\alpha}$$

Example: F-test for Main Effect of Copper

$\boldsymbol{H_o : \bar{\mu}_{1.} = \bar{\mu}_{2.}}$

$\boldsymbol{H_a : \bar{\mu}_{1.} \neq \bar{\mu}_{2.}}$

$\boldsymbol{F = 1.809}$ with p-value $= \mathbf{0.2272}$

Fail to reject $\boldsymbol{H_o}$ and conclude we do not have significant evidence of a main effect of copper.

Example: F-test for Main Effect of Zinc

$H_o : \bar{\mu}_{.1} = \bar{\mu}_{.2} = \bar{\mu}_{.3}$

$H_a :$ at least one $\bar{\mu}_{.j}$ is different, $j = 1, 2, 3$

$F = 39.536$ with p-value $= 0.0004$

Reject H_o and conclude we do have significant evidence of a main effect of zinc.

F-test for Interaction of Copper and Zinc

$H_o : (\mu_{ij} - \mu_{kj}) = (\mu_{ir} - \mu_{kr})$ for all $i \neq k$ and $j \neq r$

H_a : at least one $(\mu_{ij} - \mu_{kj}) \neq (\mu_{ir} - \mu_{kr})$ for some $i \neq k$ and $j \neq r$

$F = 1.113$ with p-value $= 0.3881$

Fail to reject H_o and conclude we do not have a significant interaction effect between copper and zinc. (The effect of copper is the same for all levels of zinc and the effect of zinc is the same for all levels of copper).

No interaction $\Rightarrow$ homogeneous simple effects

Interpretation of Results When There Is NO Interaction

Interpretation of marginal means is straightforward

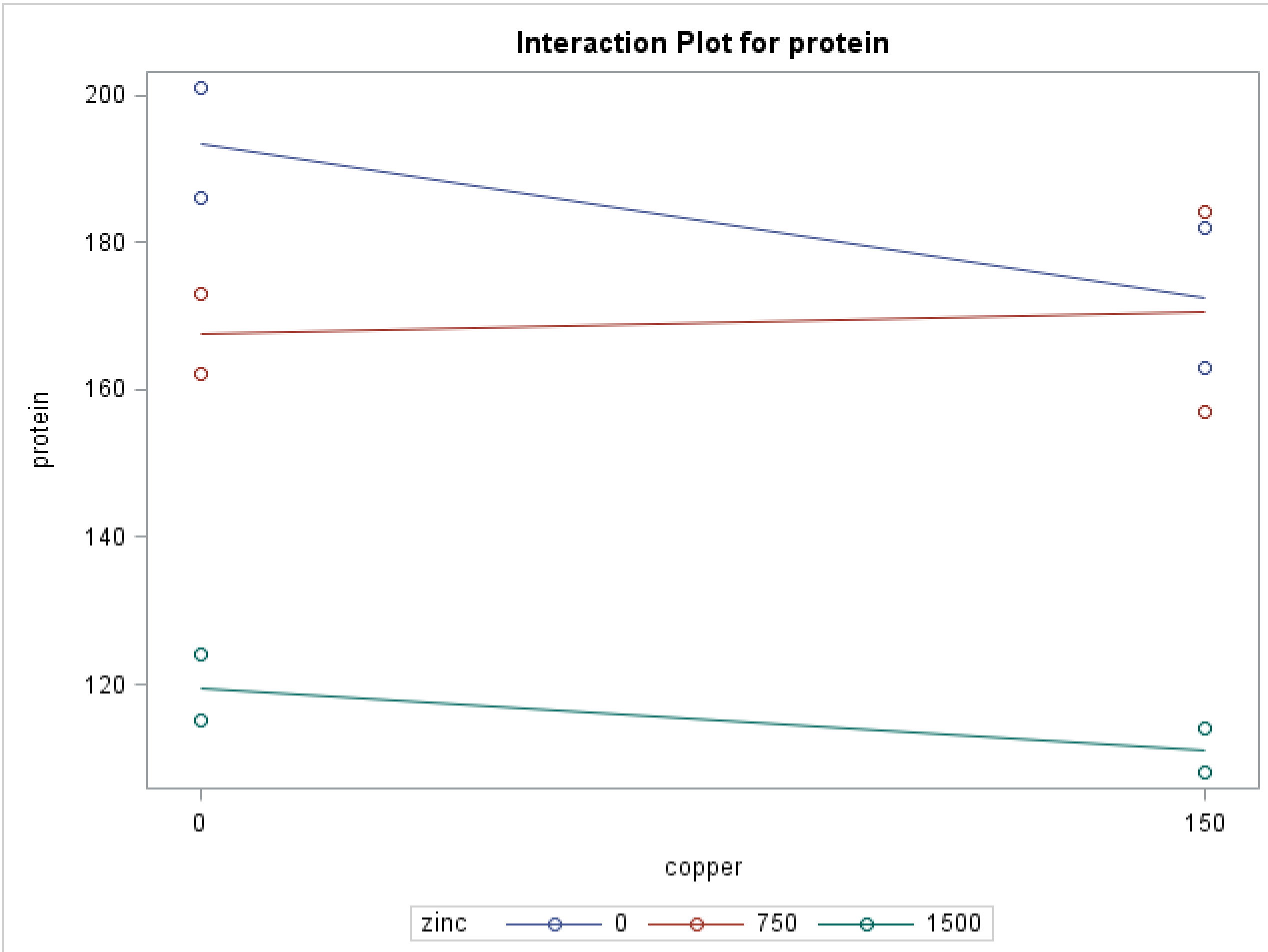
- F test for each factor: Are there differences (effects) in response means for different levels of the factor, averaging across all levels of other factors?

- Contrasts in marginal means: estimate contrast of mean responses across levels of one factor averaging across all levels of any other factors

Two Factor Study–Interactions

- When interactions are present:
  - The effect of factor A is not the same at every level of factor B
  - The effect of factor B is not the same at every level of factor A
- Can see interactions by plotting the sample response means versus levels of factor A and connect points within each level of factor B
- Can also see interactions in tables of sample means. Look at differences between two levels of one factor at each level of the other factor.

Effects Model - Interactions



What if there is an interaction?

- Main effect, $\bar{\mu}_{\cdot 1} - \bar{\mu}_{\cdot 2}$, is not the same as some simple effect, $\mu_{i1} - \mu_{i2}$
- Each simple effect for the first factor ($\mu_{ij} - \mu_{rj}$) is conditional on a level of the other factor
- Why is there an interaction? Are effects additive on some other scale?
- Is the interaction effect practically significant?
- Should I report main effect or simple effect? Do differences in marginal means have any practical importance?