STAT 500

Wilcoxon Rank-Sum Test:
Non-Parametric Test for Two Samples

Wilcoxon Rank-Sum Test

- Assumptions
  - Independence
- Null hypothesis: two populations have the same distribution
  - Distribution is not required to be normal.
  - Implies equal medians, percentiles, means and variances

Wilcoxon Rank-Sum Test

- Combine the data from the two groups into a single data set
- Order the $n_1 + n_2$ observations from smallest to largest
- Assign ranks
  - The smallest observation gets rank=1, the second smallest gets rank=2, etc...
  - For tied observations, average the ranks
- Compute the sum of the ranks for one group (call it W)

Wilcoxon Rank-Sum Test

- Assuming the null hypothesis of equivalent distributions is true, compute the expectation and variance of W (for group 1)

$$E_0(W) = \frac{n_1(n_1 + n_2 + 1)}{2}$$

$$V_0(W) = \frac{n_1 n_2(n_1 + n_2 + 1)}{12}$$

- Large sample Z-test:

$$z = \frac{|W - E_0(W)| - 0.5}{\sqrt{V_0(W)}}$$

- Approximate p-value $= 2 * P(Z > |z|)$

Wilcoxon Rank-Sum Test

- Can also test against a one-sided alternative
- Can compute "exact" p-values based on the null distribution of the ranks

Small Example

Randomized experiment with two treatments

- Animals A, B, and C receive a standard drug:
  observations are 3,8,4

- Animals D, E, and F receive the new drug:
  observations are 7,9,11

Null Hypothesis: Distribution of response variable is the same for both groups.

Alternative Hypothesis: Distribution of response variable is different between the two groups.

Small Example

Order the combined sample:

Observation Y	Animal ID	Treatment group	rank
3	A	standard	1
4	C	standard	2
7	D	new drug	3
8	B	standard	4
9	E	new drug	5
11	F	new drug	6

- Test statistic: $\boldsymbol{W = 3 + 5 + 6 = 14}$

Small Example

- $E_0(W) = (3)(7)/2 = 10.5$

  $V_0(W) = (3)(3)(7)/12 = 5.25$

- $z = \dfrac{|14 - 10.5| - 0.5}{\sqrt{5.25}} = 1.31$

- Approximate p-value is $2 * P(Z > 1.31) = 0.19$

- Fail to reject the null hypothesis

- We do not have significant evidence that the response distributions for the new drug and the standard drug are different.

Exact P-value

- There are $\binom{6}{3} = 20$ ways to choose 3 units out of 6 to assign to the treatment group.
- The observed set of ranks for the members of the treatment group are 3, 5, 6
- There is only one other random assignment that could produce a larger value of W: subjects with ranks 4, 5, 6 are assigned to the treatment group

Exact P-value

- Two other random assignments are equally extreme in the other direction
  - ranks 1, 2, 4 (observed)
  - ranks 1, 2, 3 (more extreme than observed)
- The exact two-sided p-value is

$$= \frac{\#\text{ of sets as extreme or more extreme than observed}}{\#\text{ of possible sets of ranks}}$$

$$= \mathbf{4/20 = 0.20}$$

Creative Writing Study

```
/* Evaluate the Wilcoxon test.  The exact
   statement requests the exact randomization
   p-value for the test.  After the slash you
   can include the MC option to get a Monte
   Carlo approximation to the exact p-value. */

proc npar1way data=set1 wilcoxon;
  class trt;
  var y;
  exact wilcoxon / alpha=.05 maxtime=20
                   MC N=20000 Seed=7892441;
  run;
```

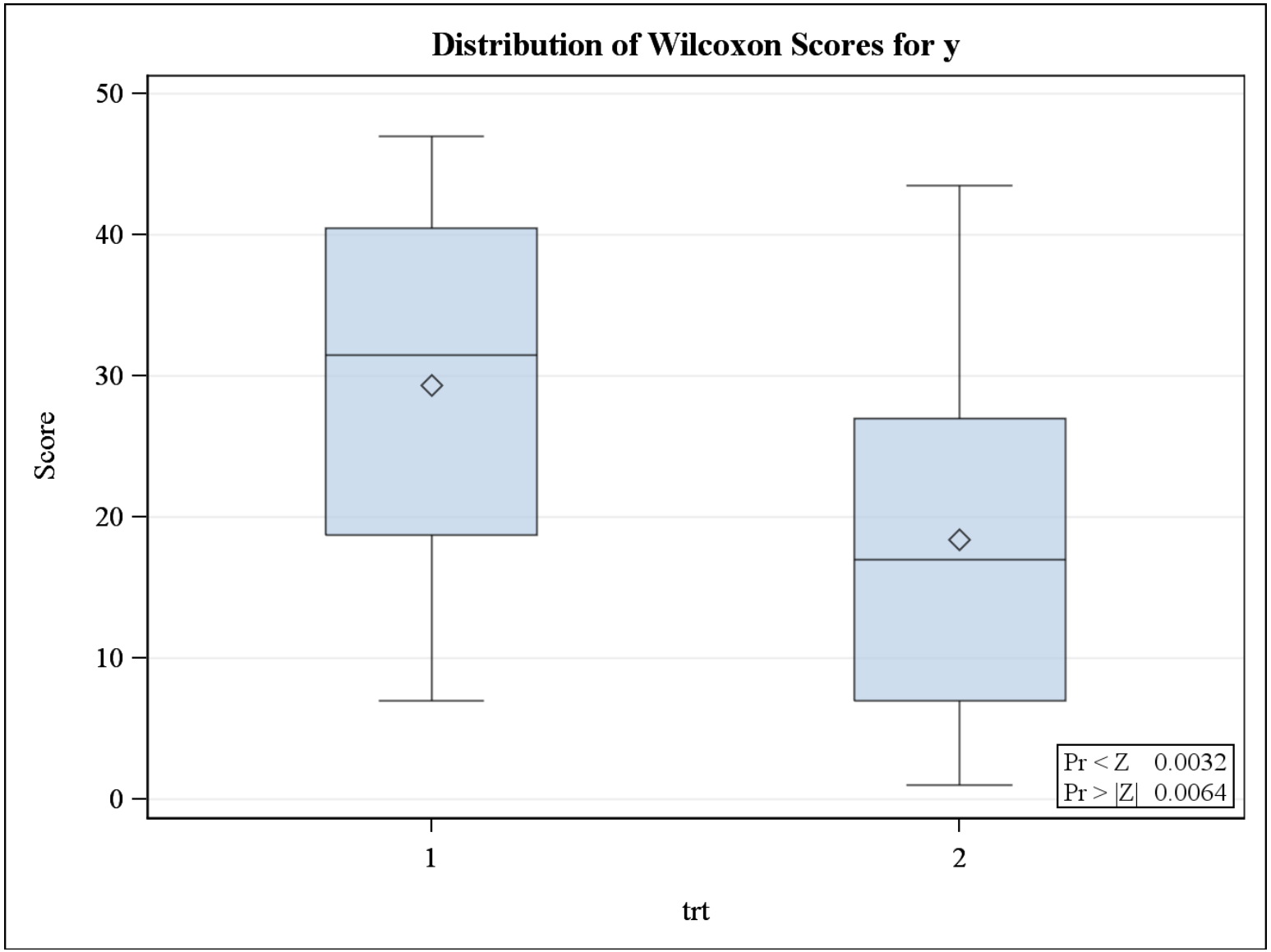
The NPAR1WAY Procedure

Wilcoxon Scores (Rank Sums) for Variable y Classified by Variable trt					
trt	**N**	**Sum of Scores**	**Expected Under H0**	**Std Dev Under H0**	**Mean Score**
1	24	704.50	576.0	46.973057	29.354167
2	23	423.50	552.0	46.973057	18.413043
Average scores were used for ties.					

Wilcoxon Two-Sample Test	
Statistic (S)	423.5000
Normal Approximation	
Z	-2.7250
One-Sided Pr < Z	0.0032
Two-Sided Pr > |Z|	0.0064
t Approximation	
One-Sided Pr < Z	0.0045
Two-Sided Pr > |Z|	0.0091
Z includes a continuity correction of 0.5.	

Monte Carlo Estimates for the Exact Test	
One-Sided Pr <= S	
Estimate	0.0029
95% Lower Conf Limit	0.0021
95% Upper Conf Limit	0.0036
Two-Sided Pr >= |S - Mean|	
Estimate	0.0051
95% Lower Conf Limit	0.0041
95% Upper Conf Limit	0.0060
Number of Samples	20000
Initial Seed	7892441

Kruskal-Wallis Test	
Chi-Square	7.4836
DF	1
Pr > Chi-Square	0.0062


Distribution of Wilcoxon Scores for y
Score
50
40
30
20
10
0
1
2
trt
Pr < Z 0.0032
Pr > |Z| 0.0064