STAT 500

More on Blocking Designs

Efficiency of Blocking

- Is RCBD better than CRD?
  - If the experiment was repeated on similar e.u.'s, should you block?
  - Not a question about how to analyze the observed data. Analysis should match the design.
- How to measure "better"?
  - Consider the **error variance** for each design:

$$\boldsymbol{\sigma^2_{CRD}} \text{ versus } \boldsymbol{\sigma^2_{RCBD}}$$

  - Efficiency of RCBD relative to CRD is $\boldsymbol{\sigma^2_{CRD}/\sigma^2_{RCBD}}$.
  - Efficiency $> 1 \Rightarrow$ RCBD provides more precise estimates of treatment mean contrasts.

Randomized Complete Block Designs

- Can also express Efficiency in terms of sample sizes

$$Var(\bar{Y}_{\cdot j} - \bar{Y}_{\cdot k}) = \sigma_e^2(\frac{2}{n})$$

To have $Var(\bar{Y}_{\cdot j} - \bar{Y}_{\cdot k})$ the same for both designs, we need

$$\sigma^2_{CRD}\left(\frac{2}{n_{CRD}}\right) = \sigma^2_{RCBD}\left(\frac{2}{n_{RCBD}}\right)$$

$\Rightarrow$

$$\text{Efficiency} = \frac{\sigma^2_{CRD}}{\sigma^2_{RCBD}} = \frac{n_{CRD}}{n_{RCBD}}$$

- i.e. Efficiency $= 1.5 \Rightarrow$ CRD requires 50% more units per treatment than the RCBD

Randomized Complete Block Designs

- In the randomized block design that provided the data $\hat{\sigma}^2_{RCBD} = MS_{error}$
- Snedecor and Cochran give

$$\hat{\sigma}^2_{CRD} = \frac{(n-1)MS_{blocks} + n(J-1)MS_{error}}{nJ-1}$$

  as an unbiased estimate of the error variance if a completely randomized design had been used instead (proof in Cochran and Cox, 1957)

- One complication is that $\hat{\sigma}^2_{CRD}$ and $\hat{\sigma}^2_{RCBD}$ have different degrees of freedom.

RCBD: Efficiency

- Fisher used "relative amount of information", an estimated efficiency,

$$\frac{(\text{df}_{RCBD}+1)(\text{df}_{CRD}+3)\hat{\sigma}^2_{CRD}}{(\text{df}_{RCBD}+3)(\text{df}_{CRD}+1)\hat{\sigma}^2_{RCBD}}$$

  to adjust for d.f.

- Typical values of efficiency depend on the subject matter

- Values of 1.10 to 1.30 are common (eg, blocking often reduces the number of units by 10 to 30 percent)

Efficiency of Penicillin Experiment

ANOVA Table

Source	d.f.	SS	MS	F	p-value
Blocks	4	264	66.000	3.50	0.0407
Processes	3	70	23.333	1.24	0.3387
Error	12	226	18.833		
Total	19	560			

Efficiency of Penicillin Experiment

- $\hat{\sigma}^2_{RCBD} = MS_{error} = 18.833$

$$\begin{aligned}\hat{\sigma}^2_{CRD} &= \frac{(n-1)MS_{blocks} + n(J-1)MS_{error}}{nJ-1} \\ &= \frac{(4)(66.0) + (5)(3)(18.833)}{19} = 28.76289\end{aligned}$$

$$\text{estimated efficiency} = \frac{(12+1)(16+3)(28.76289)}{(12+3)(16+1)(18.833)} = 1.48$$

- To have the same efficiency, $n_{\text{CRD}} = 1.48 n_{\text{RCBD}}$

RCBD: Diagnostics

Assumptions (each treatment used equally often in each block)

- Independence of errors
- Homogeneous error variance
- Normality of errors
- Block and treatment effects are additive (no interaction)
- Relative importance of first three assumptions and diagnoses are similar to before
- Non-parametric test: Friedman test performs an ANOVA with observed data replaced by ranks within blocks

Additive Model

- Additivity - treatment effect is the same within each block.

$$\text{Additive Model: } \boldsymbol{Y_{ij} = \mu + \beta_i + \tau_j + \epsilon_{ij}}$$

- Non-additivity - treatment effect varies depending on block.

$$\text{Non-Additive Model: } \boldsymbol{Y_{ij} = \mu + \beta_i + \tau_j + (\beta\tau)_{ij} + \epsilon_{ij}}$$

- Unless replicates of treatments within blocks, we cannot test for significance of $\boldsymbol{(\beta\tau)_{ij}}$

Tukey's Test for Non-Additivity

- Used when no replicates of treatments within blocks

- Detects one specific pattern of non-additivity: multiplicative interaction between block and treatment effects.

$$\text{Tukey Model: } Y_{ij} = \mu + \beta_i + \tau_j + \kappa\beta_i\tau_j + \epsilon_{ij}$$

- Tukey constructed an F test for $H_0 : \kappa = 0$ vs. $H_a : \kappa \neq 0$

More than One Blocking Factor

- Can use a broader definition of blocks
- Example: if gender and age are both blocking factors, then one could use as blocks: males 20-29, males 30-39, females 20-29, etc.
- Problem - have many blocks = need many experimental units
- Special case: Latin Square Designs

Latin Square Designs

- Two blocking variables
- Number of levels for each blocking factor = number of treatments (or its multiple)
  - Three treatments - each block has three levels (or 6, 9, 12, etc.)
  - Four treatments - each block has four levels (or 8, 12, 16, etc.)
- Each block contains only one unit for each treatment
- Each level of each blocking variable gets all treatments

Latin Square Designs: Example

- Fuel efficiency study
  - Block 1 (row blocks) = Drivers (1 through 4)
  - Block 2 (column blocks) = Cars (1 through 4)
  - Treatments = Fuel Additives (A, B, C, D)
  - Each treatment occurs once in each row and once in each column

Latin Square Designs: Example

- Example Latin Square Design

	Cars			
Drivers	1	2	3	4
1	B	A	C	D
2	A	C	D	B
3	C	D	B	A
4	D	B	A	C

Latin Square Designs

- Advantages
  - Can estimate treatment effects in a small study
  - Can use two blocking factors to reduce variability
- Limitations
  - Levels of each blocking variable must equal (or be a multiple of) the number of treatments
  - Analysis assumes no interactions between blocking factors and treatments: critical, because each block contains only one unit for each treatment
  - Few degrees of freedom for error, can increase by using multiple Latin squares

Latin Square Designs

Model

$$Y_{ijk} = \mu + \beta_i + \gamma_j + \tau_k + \epsilon_{ijk}$$

where $i, j, k = 1, 2, \ldots, r$

- β_i first blocking factor effect
- γ_j second blocking factor effect
- τ_k is a fixed treatment effect
- k is the treatment and is determined by (i, j)
- $\epsilon_{ijk} \sim NID(0, \sigma^2)$

Latin Square Designs

ANOVA table

source	d.f.	SS
Block 1	$r-1$	$r\sum_i(\bar{Y}_{i..}-\bar{Y}_{...})^2$
Block 2	$r-1$	$r\sum_j(\bar{Y}_{.j.}-\bar{Y}_{...})^2$
Treatment	$r-1$	$r\sum_k(\bar{Y}_{..k}-\bar{Y}_{...})^2$
Error	$(r-1)(r-2)$	SS_{error}
Total	r^2-1	$\sum_i\sum_j(Y_{ij.}-\bar{Y}_{...})^2$

where $\bar{Y}_{...} = r^{-2}\sum_i\sum_j Y_{ij.}$ and

SSerror = SS_{total} - $SS_{block\ 1}$ - SS_{block2} - SS_{trt}

Latin Square Designs

- Inference
  - Tests for treatment based on usual F-test
  - CIs for means, pairwise comparisons, contrasts as in one-way ANOVA