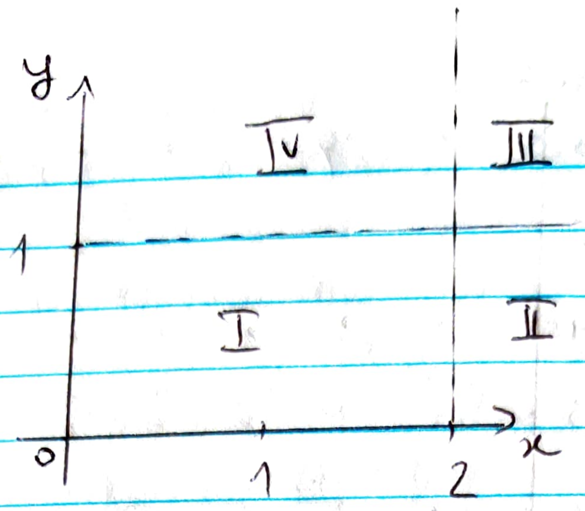
HW 2 Q 8

a)



If either $x < 0$ or $y < 0$, it immediately follows that $F(x,y) = 0$.

For $0 < x < 2$ and $0 < y < 1$ (Region I).

$$F(x,y) = \int_0^x \int_0^y \frac{1}{4}(s + 2t)\, ds\, dt$$

$$= \frac{1}{4}\int_0^x (st + t^2)\Big|_0^y \, ds$$

$$= \frac{1}{4}\int_0^x (sy + y^2)\, ds$$

$$= \frac{1}{4}\left(y\frac{s^2}{2} + y^2 s\right)\Big|_0^x$$

$$= \frac{1}{4}\left(y\frac{x^2}{2} + y^2 x\right) = \frac{xy}{4}\left(\frac{x}{2} + y\right)$$

For $x > 2$ and $0 < y < 1$ (Region 2):

$$F(x,y) = \frac{1}{4}\int_0^2 \int_0^y (s + 2t)\, ds\, dt$$

$$= \frac{1}{4}\int_0^2 (st + t^2)\Big|_0^y \, ds$$

$$= \frac{1}{4}\int_0^2 (sy + y^2)\, ds = \frac{1}{4}\left(y\frac{s^2}{2} + y^2 s\right)\Big|_0^2$$

$$= \frac{1}{4}\left(y \cdot \frac{4}{2} + y^2 \cdot 2\right) = \frac{y^2}{2} + \frac{y}{2} = \frac{y}{2}(y+1)$$

For $x > 2$, $y > 1$ (Region 3),

$$F(x,y) = \int_0^2 \int_0^1 \frac{1}{4}(s + 2t)\, ds\, dt = 1.$$

For $0 < x < 2$, $y > 1$ (Region IV),

$$F(x,y) = \int_0^x \int_0^1 \frac{1}{4}(s + 2t)\, ds\, dt$$

$$= \frac{1}{4} \int_0^x \left(st + t^2\right)\Big|_0^1 ds$$

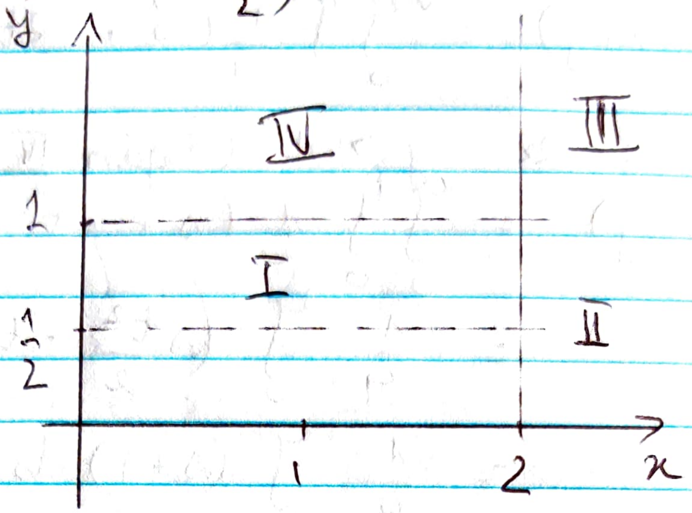
$$= \frac{1}{4} \int_0^x (s+1)\, ds$$

$$= \frac{1}{4} \left(\frac{s^2}{2} + s\right)\Big|_0^x = \frac{x}{4}\left(\frac{x}{2} + 1\right)$$

Since the distribution ~~is~~ function is everywhere continuous, we can include boundaries ~~in~~ between 2 regions in any of them, and therefore,

$$F(x,y) = \begin{cases} 0 & \text{for } x \le 0,\ y \le 0. \\ \frac{1}{4} xy\left(\frac{x}{2} + y\right) & \text{for } 0 < x < 2,\ 0 < y < 1 \\ \frac{1}{2} y (y+1) & \text{for } x \geq 2,\ 0 < y < 1 \\ \frac{1}{4} x\left(\frac{x}{2} + 1\right) & \text{for } 0 < x < 2,\ y \geq 1 \\ 1 & \text{for } x \geq 2,\ y \geq 1. \end{cases}$$

b) $P(X \le x,\ Y \geq \frac{1}{2})$.



For $x < 0$ or $y < 0$, $F(x,y) = 0$.

For $0 < x < 2$ and $0 < y < 1$:

$$F(x,y) = \int_0^x \int_0^y \frac{1}{4}(s + 2t)\, ds\, dt$$

$$= \int_0^x \int_{\frac{1}{2}}^y \frac{1}{4}(s + 2t)\, ds\, dt$$

$$= \int_0^x \frac{1}{4}\left(st + t^2\right)\Big|_{\frac{1}{2}}^{y} ds$$

$$= \frac{1}{4}\int_0^x \left(sy + y^2 - \frac{s}{2} - \frac{1}{4}\right) ds$$

$$= \frac{1}{4}\left(y\frac{s^2}{2} + y^2 s - \frac{s^2}{4} - \frac{1}{4}s\right)\Big|_0^x$$

$$= \frac{1}{4}\left(\frac{x^2 y}{2} + xy^2 - \frac{x^2}{4} - \frac{x}{4}\right)$$

$$= \frac{x}{4}\left(\frac{xy}{2} + y^2 - \frac{x}{4} - \frac{1}{4}\right)$$

For $x > 2$ and $0 < y < 1$:

$$F(x,y) = \frac{1}{4}\int_0^2 \int_{\frac{1}{2}}^y (s + 2t)\, ds\, dt = \frac{1}{4}\int_0^x (st + t^2)\Big|_{\frac{1}{2}}^{y} ds$$

$$= \frac{1}{4}\int_0^2 \left(sy + y^2 - \frac{s}{2} - \frac{1}{4}\right) ds = 2y^2 + 2y - \frac{3}{2}$$

For $x > 2$, $y > 1$, $F(x,y) = 1$.

For $0 < x < 2$, $y > 1$, $F(x,y) = \frac{1}{2}\frac{x}{4}\left(\frac{x}{2}+1\right)$

Then:

$$P\left(X \leq x, Y \geq \frac{1}{2}\right) = \begin{cases} 0 & \text{if } x < 0 \text{ or } y < 0 \\ \frac{x}{4}\left(\frac{xy}{2} + y^2 - \frac{x}{4} - \frac{1}{4}\right) & \text{if } 0 < x < 2;\ \frac{1}{2} \leq y < 1 \\ 2y^2 + 2y - \frac{3}{2} & \text{if } x > 2;\ \frac{1}{2} \leq y < 1 \\ \frac{x}{8}\left(\frac{x}{2} + 1\right) & \text{if } 0 \leq x \leq 2,\ y \geq 1 \\ 1 & \text{if } x \geq 2,\ y \geq 1 \end{cases}$$