STAT 588 Homework 2

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1 Question 1

Elements of sample space	x	y
HHHH	4	2
HHHT	3	$\sqrt{2}$
HHTH	3	$\sqrt{2}$
HTHH	3	$\sqrt{2}$
THHH	3	$\sqrt{2}$
HHTT	2	0
HTHT	2	0
HTTH	2	0
THHT	2	0
THTH	2	0
TTHH	2	0
HTTT	1	$\sqrt{2}$
THTT	1	$\sqrt{2}$
TTHT	1	$\sqrt{2}$
TTTH	1	$\sqrt{2}$
TTTT	0	2

$Y =$ the square root of the absolute difference between the number of heads and the number of tails obtained in the four tosses. Then $P(Y=0) = \dfrac{6}{16}$, $P(Y=\sqrt{2}) = \dfrac{8}{16}$, $P(Y=2) = \dfrac{2}{16}$. The formula for the probability distribution of Y is

$$f(y) = \begin{cases} \frac{6}{16}, & \text{if } y = 0 \\ \frac{8}{16}, & \text{if } y = \sqrt{2} \\ \frac{2}{16}, & \text{if } y = 2 \end{cases}$$

2 Question 2

$f(0) + f(1) + f(2) + f(3) + f(4) = 0 + \dfrac{1}{30} + \dfrac{4}{30} + \dfrac{11}{30} + \dfrac{15}{30} = \dfrac{31}{30} > 1$. Hence, the function $f(x)$ is not a valid probability distribution.

3 Question 3

$$f(0) = \frac{\binom{3}{0}\binom{3}{3}}{\binom{6}{3}} = \frac{1}{20}$$
$$f(1) = \frac{\binom{3}{1}\binom{3}{2}}{\binom{6}{3}} = \frac{9}{20}$$
$$f(2) = \frac{\binom{3}{2}\binom{3}{1}}{\binom{6}{3}} = \frac{9}{20}$$
$$f(3) = \frac{\binom{3}{3}\binom{3}{0}}{\binom{6}{3}} = \frac{1}{20}$$

The distribution function of X is

$F(0) = f(0) = \dfrac{1}{20}$
$F(1) = f(0) + f(1) = \dfrac{10}{20}$
$F(2) = f(0) + f(1) + f(2) = \dfrac{19}{20}$
$F(3) = f(0) + f(1) + f(2) + f(3) = \dfrac{20}{20}$

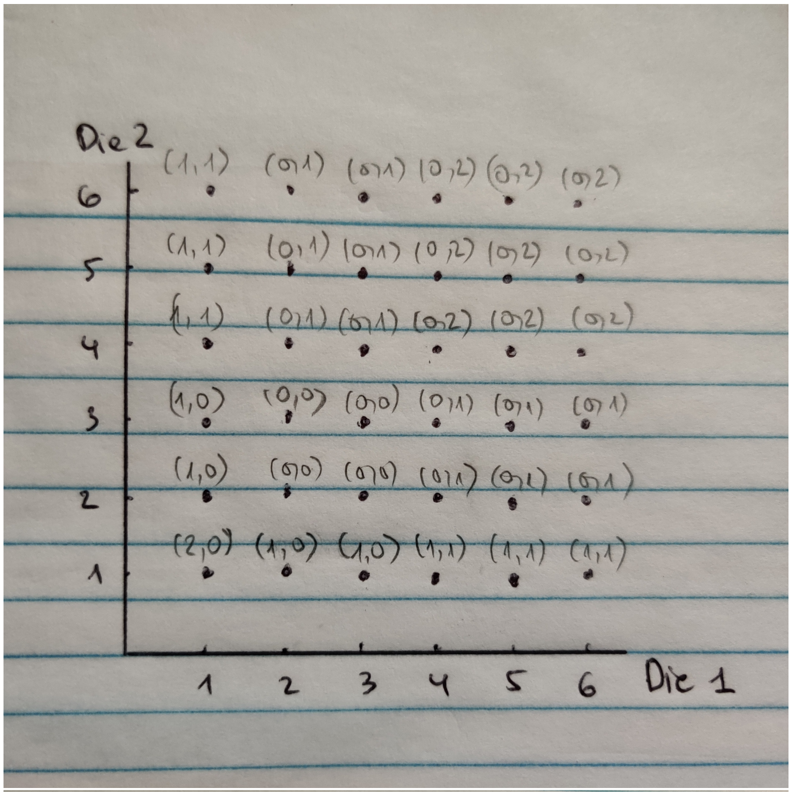
4 Question 4

The distribution function of X is
For $-9 < x < 9$
$F(x) = \displaystyle\int_{-\infty}^{x} f(t)dt = \int_{-9}^{x} \frac{1}{972}(81 - t^2)dt = \frac{1}{972}(81x - \frac{x^3}{3} + 486)$
Otherwise, $F(x) = 0$
Hence, we can write
$$F(x) = \begin{cases} \frac{1}{972}(81x - \frac{x^3}{3} + 486), & \text{if } -9 < x < 9 \\ 0, & \text{otherwise.} \end{cases}$$
The probability that one of these flights is at least one minute late:
$P(X \geq 1) = 1 - P(X < 1) = 1 - F(1) = 1 - \frac{1}{972}(81 - \frac{1}{3} + 486) = 0.417$

5 Quesion 5

Since the given distribution function is differentiable everywhere, we differentiate it and get
$$f(x) = \begin{cases} xe^{-x}, & \text{for } x > 0 \\ 0, & \text{for } x \leq 0 \end{cases}$$
Verify $F(x)$ is a valid CDF:
$\lim\limits_{x \to +\infty} F(x) = \lim\limits_{x \to +\infty} [1 - \dfrac{1+x}{e^x}] = 1 + \lim\limits_{x \to +\infty} \dfrac{1}{e^x} = 1$ (L'Hospital Rule).
$\lim x \to -\infty F(x) = 0$.
$F(x)$ is monotonically increasing.

6 Question 6



		x		
		0	1	2
y	0	$\frac{4}{36}$	$\frac{4}{36}$	$\frac{1}{36}$
	1	$\frac{12}{36}$	$\frac{6}{36}$	
	2	$\frac{9}{36}$		

7 Question 7

$F(x,y) = P(X \le x, Y \le y) = \sum_{s \le x} \sum_{t \le y} f(s,t)$ for $x = 0,1,2,3$ and $y = 0,1,2$.

$F(x,y) = f(0,0)+f(0,1)+f(0,2)+f(1,0)+f(1,1)+f(1,2)+f(2,0)+f(2,1)+f(2,2)+f(3,0)+f(3,1)+f(3,2) =$
$0+\frac{1}{48}+\frac{2}{48}+\frac{2}{28}+\frac{3}{48}+\frac{4}{48}+\frac{4}{48}+\frac{5}{48}+\frac{6}{48}+\frac{6}{48}+\frac{7}{48}+\frac{8}{48} = 1$ is a valid CDF.

8 Question 8

9 Question 9

$$\frac{\partial^2}{\partial x \partial y} F(x,y) = \frac{\partial^2}{\partial x \partial y}\left(1-e^{-x}-e^{-y}+e^{-x-y}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\left(1-e^{-x}-e^{-y}+e^{-x-y}\right)\right)$$

$$= \frac{\partial}{\partial y}\left[-(-1)e^{-x}+(-1)e^{-x-y}\right]$$

$$= \frac{\partial}{\partial y}\left(e^{-x}-e^{-x-y}\right) = -(-1)e^{-x-y} = e^{-x-y}$$

Then, $f(x,y) = \begin{cases} e^{-x-y} & \text{for } x>0,\ y>0 \\ 0 & \text{else where} \end{cases}$

10 Question 10

	z = 0	z = 1	
w = 0	$\frac{188}{221}$	0	$\frac{188}{221}$
w = 1	$\frac{16}{221}$	$\frac{16}{221}$	$\frac{32}{221}$
w = 2	0	$\frac{1}{221}$	$\frac{1}{221}$
	$\frac{204}{221}$	$\frac{17}{221}$	1

Marginal distribution of Z is $g(z) = \sum_{w=0}^{2} f(z,w)$ for $z = 0,1$.

Conditional distribution of W given Z = z is $f(w|z) = \frac{f(w,z)}{g(z)}$

$f(w=0|z=0) = \frac{f(w=0,z=0)}{f(z=0)} = \frac{\frac{188}{221}}{\frac{204}{221}} = \frac{188}{204}$

$f(w=1|z=0) = \frac{f(w=1,z=0)}{f(z=0)} = \frac{\frac{16}{221}}{\frac{204}{221}} = \frac{16}{204}$

$f(w=2|z=0) = \frac{f(w=2,z=0)}{f(z=0)} = \frac{0}{\frac{204}{221}} = 0$

$f(w=0|z=1) = \frac{f(w=0,z=1)}{f(z=1)} = \frac{0}{\frac{17}{221}} = 0$

$f(w=1|z=1) = \frac{f(w=1,z=1)}{f(z=1)} = \frac{\frac{16}{221}}{\frac{17}{221}} = \frac{16}{17}$

$f(w=2|z=1) = \frac{f(w=2,z=1)}{f(z=1)} = \frac{\frac{1}{221}}{\frac{17}{221}} = \frac{1}{17}$

Marginal distribution of Z is $h(w) = \sum_{z=0}^{1} f(z, w)$ for $w = 0, 1, 2$.

Conditional distribution of Z given W = w is $f(z|w) = \dfrac{f(w, z)}{h(w)}$

$f(z=0|w=0) = \dfrac{f(z=0, w=0)}{f(w=0)} = \dfrac{\frac{188}{221}}{\frac{188}{221}} = 1$

$f(z=1|w=0) = \dfrac{f(z=1, w=0)}{f(w=0)} = \dfrac{0}{\frac{188}{221}} = 0$

$f(z=0|w=1) = \dfrac{f(z=0, w=1)}{f(w=1)} = \dfrac{\frac{16}{221}}{\frac{32}{221}} = \dfrac{1}{2}$

$f(z=1|w=1) = \dfrac{f(z=1, w=1)}{f(w=1)} = \dfrac{\frac{16}{221}}{\frac{32}{221}} = \dfrac{1}{2}$

$f(z=0|w=2) = \dfrac{f(z=0, w=2)}{f(w=2)} = \dfrac{0}{\frac{32}{221}} = 0$

$f(z=1|w=2) = \dfrac{f(z=1, w=2)}{f(w=2)} = \dfrac{\frac{1}{221}}{\frac{1}{221}} = 1$

11 Question 11

$g(p) = \int_{-\infty}^{+\infty} 2.5pe^{-ps}ds = \int_{0}^{+\infty} 2.5pe^{-ps}ds = 2.5(e^{-\infty} - 1) = 2.5$
for $0.1 < p < 0.5$ and $g(p) = 0$ otherwise.
$f(s|p) = \dfrac{f(s,p)}{g(p)} = \dfrac{2.5pe^{-ps}}{2.5} = pe^{-ps}$ for $0.1 < p < 0.5; s > 0$ and $f(s|p) = 0$ otherwise.