Homework 4 - Stat 588

① a) $X \sim$ uniform distribution

$$\mu_1' = \mathbb{E}[X] = \sum_x x.f(x)$$

X follows uniform distribution, so $f(x) = \frac{1}{k}$ for $x = 1, 2, \ldots, k$.

Then, $$\mu_1' = \sum_x x.\frac{1}{k} = \frac{1}{k}\sum_{x=1}^{k} x = \frac{1}{k}\cdot\frac{k(k+1)}{2}$$

$$= \frac{k+1}{2}$$

b) $$\mu_2' = \mathbb{E}[X^2] = \sum_x x^2 f(x) = \sum_{x=1}^{k} x^2 . \frac{1}{k}$$

$$= \frac{1}{k}\cdot\frac{k(k+1)(2k+1)}{6} = \frac{(k+1)(2k+1)}{6}$$

$$Var(X) = \mu_2' - \mu^2 = \frac{(k+1)(2k+1)}{6} - \left[\frac{(k+1)}{2}\right]^2$$

$$= \frac{k^2}{12} - \frac{1}{12}$$

c) $$M_X(t) = \sum_x e^{tx} f(x) = \sum_{x=1}^{k} (e^t)^x . \frac{1}{k}$$

$$= \frac{1}{k}\left[\sum_{x=1}^{k-1} (e^t)^x + e^{tk}\right]$$

$$= \frac{1}{k}\cdot\left[\frac{e^t - (e^t)^k}{1-e^t} + e^{tk}\right]$$

$$= \frac{1}{k}\left[\frac{e^t - e^{tk} + e^{tk} - e^{t+k}}{1-e^t}\right]$$

$$= \frac{1}{k}\cdot\frac{e^t(1-e^{tk})}{1-e^t}$$

(2) $X \sim$ Geometric $g(x, \theta)$

$g(x, \theta) = \theta(1-\theta)^{x-1}$ for $x = 1, 2, 3.$

a) $\mathbb{E}[X] = \sum_{x} x\,\theta(1-\theta)^{x-1}$

$= \sum_{x=1}^{\infty} x\,\theta(1-\theta)^{x-1}$

We have $|1-\theta| < 1$

$$\sum_{x=0}^{\infty} (1-\theta)^x = \frac{1}{1-(1-\theta)} = \frac{1}{\theta}$$

$$\frac{\partial}{\partial x} \sum_{x=0}^{\infty} (1-\theta)^x = \sum_{x=0}^{\infty} x(1-\theta)^{x-1}$$

$$= \sum_{x=1}^{\infty} x(1-\theta)^{x-1} = \frac{1}{[1-(1-\theta)]^2} = \frac{1}{\theta^2}.$$

Hence, $\mathbb{E}[X] = \theta \cdot \frac{1}{\theta^2} = \frac{1}{\theta}.$

b) $\mathbb{E}[X^2] = \sum_{x=1}^{\infty} x^2\theta(1-\theta)^{x-1} = \sum_{x=0}^{\infty} x^2\theta(1-\theta)^{x-1}$

$$S_2 = \sum_{x=0}^{\infty} x^2(1-\theta)^{x-1} = 1 + 4(1-\theta) + 9(1-\theta)^2 + \dots$$

$$(1-\theta)S_2 = (1-\theta) + 4(1-\theta)^2 + 9(1-\theta)^3 + \dots$$

So,

$$S_2 - (1-\theta)S_2 = 1 + 3(1-\theta) + 5(1-\theta)^2 + \dots$$

$$= (2 + 4(1-\theta) + 6(1-\theta)^2 + \dots) - (1 + (1-\theta) + (1-\theta)^2 + \dots)$$

$$= 2\sum_{x=0}^{\infty} x(1-\theta)^{x-1} - \sum_{x=0}^{\infty} (1-\theta)^x$$

$$= \frac{2}{\theta^2} - \frac{1}{\theta}$$

Hence, $S_2 = \left(\frac{2}{\Theta^2} - \frac{1}{\Theta}\right) \cdot \frac{1}{\Theta}$

$$\mathrm{Var}(X) = \Theta \cdot \left(\frac{2}{\Theta^2} - \frac{1}{\Theta}\right) \frac{1}{\Theta} - \frac{1}{\Theta^2}$$

$$= \Theta \cdot \left(\frac{2}{\Theta^3} - \frac{1}{\Theta^2}\right) - \frac{1}{\Theta^2} = \frac{2}{\Theta^2} - \frac{1}{\Theta^2} - \frac{1}{\Theta}$$

$$= \frac{1}{\Theta^2} - \frac{1}{\Theta} = \frac{1-\Theta}{\Theta^2}$$

③ $X \sim \text{Poi}(x, \lambda)$

$$p(x, \lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \quad \text{for } x = 0, 1, 2, \ldots$$

a) $$\mathbb{E}[X] = \sum_{x=0}^{\infty} x \cdot \frac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{x \lambda^x}{x!}$$

$$= e^{-\lambda} \cdot \lambda \cdot e^{\lambda} = \lambda$$

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b)

$$\mathbb{E}[X(X-1)] = \sum_{x=0}^{\infty} x(x-1)\frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=0}^{\infty} \frac{\lambda^x e^{-\lambda}}{(x-2)!}$$

$$= \sum_{x=0}^{\infty} \frac{\lambda^{(x-2)} e^{-\lambda}}{(x-2)!} \cdot \lambda^2 \overset{t:=x-2}{=\!=} \sum_{t=0}^{\infty} \frac{\lambda^t e^{-\lambda}}{t!} \cdot \lambda^2$$

$$= \lambda^2 \cdot e^{\lambda} \cdot e^{-\lambda} = \lambda^2.$$

$$\mathbb{E}[X^2] = \mathbb{E}[X^2 - X) + E[X] = \lambda^2 + \lambda.$$

$$Var(X) = \lambda + \lambda^2 - \lambda^2 = \lambda.$$

(4) $X \sim$ Bernoulli

$f(x,\theta) = \theta^x (1-\theta)^{1-x}$ for $x = 0, 1$.

a) $M_X(t) = \sum_{x=0}^{1} e^{xt} f(x)$

$= \theta^0 (1-\theta)^1 . e^{0.t} + e^t \theta^1 (1-\theta)^{1-1}$

$= 1-\theta + \theta . e^t$

b) $\mathbb{E}[X] = \frac{\partial}{\partial x}\Big|_{t=0} M_X(t) = \theta e^t \Big|_{t=0} = \theta$.

c) $\mu_2' = \frac{\partial^2}{\partial x^2}\Big|_{t=0} M_X(t) = \theta . e^t \Big|_{t=0} = \theta$

$Var(X) = \mu_2' - E[X]^2 = \theta - \theta^2 = \theta(1-\theta)$.

(5) $X_1, X_2, \dots, X_k \sim$ Multinomial distribution

$f(x_1, x_2, \dots x_k ; n, \theta_1, \dots, \theta_k)$

$= \binom{n}{x_1, x_2, \dots x_k} . \theta_1^{x_1} . \theta_2^{x_2} \dots \theta_k^{x_k}$

a) $f_{X_i}(x_i) = \binom{n}{x_i} \theta_i^{x_i} (1-\theta_i)^{n-x_i}$

$\mathbb{E}[X_i] = \sum_{x_i=0}^{n} x_i \binom{n}{x_i} \theta_i^{x_i} (1-\theta)^{n-x_i}$

$= \sum_{x_i=1}^{n} \frac{n!}{(x_i-1)!(n-x_i)!} \theta_i^{x_i} (1-\theta_i)^{n-x_i}$

$= n\theta_i \sum_{x_i=1}^{n} \binom{n-1}{x_i-1} \theta_i^{x_i-1} (1-\theta_i)^{n-x_i}$

let $y = x_i - 1$; $m = n-1$) $= n\theta_i \sum_{y=0}^{m} \binom{m}{y} \theta_i^y (1-\theta_i)^{m-y} = n\theta_i$.

$$E[X_i(X_i-1)] = \sum_{x_i=0}^{n} x_i(x_i-1)\binom{n}{x_i}\theta_i^{x_i}(1-\theta_i)^{n-x_i}$$

$$= \sum_{x_i=2}^{n} \frac{n!}{(x_i-2)!\,(n-x_i)!}\theta_i^{x_i}(1-\theta_i)^{n-x_i}$$

$$= n(n-1)\theta_i^2 \sum_{x_i=2}^{n}\binom{n-2}{x-2}\theta_i^{x_i-2}(1-\theta_i)^{n-x_i}$$

(let $y=x-2$, $m=n-2$) $$= n(n-1)\theta_i^2\sum_{y=0}^{m}\binom{m}{y}\theta_i^{y}(1-\theta_i)^{m-y}$$

$$= n(n-1)\theta_i^2$$

Therefore,

$$\mu_2' = E[X_i(X_i-1)] + E[X_i]$$

$$= n(n-1)\theta_i^2 + n\theta_i$$

Hence,

$$Var(X_i) = n(n-1)\theta_i^2 + n\theta_i - n^2\theta_i^2$$

$$= n(1-\theta_i)\theta_i$$

b) Let $Y = n - X_i - X_j$ for $i \neq j$, where y is observation of Y. Then, Y follows binomial distribution with parameters n and $1-\theta_i-\theta_j$, because Y is a linear combination of binomial random variables.

We also have $x_i + x_j + y = n$ and

$\theta_i + \theta_j + (1-\theta_i-\theta_j) = 1$

Moreover,

$$P(X_i = x_i, X_j = x_j, Y = y) = \binom{n}{x_i, x_j, y}\theta_i^{x_i}\theta_j^{x_j}(1-\theta_i-\theta_j)^{y}$$

because by independence, any sequence of trails in which outcome X_i occurs exactly x_i times (X_j : x_j times, Y: y times) is $\theta_i^{x_i}\,\theta_j^{x_j}\,(1-\theta_i-\theta_j)^y$

Also, the probability of any specific order is $\binom{n}{x_i, x_j, y}$

Hence, X_i, X_j, Y follow multinomial distribution with parameters $n, \theta_i, \theta_j, 1-\theta_i-\theta_j$

6 a) Binomial distribution.

b) $n = 6$, $\theta = 0.7$

c) $P(x=5) = b(5, 6, 0.7)$

$= \binom{6}{5} 0.7^5\, 0.3^1 = 0.3025.$

7 a) Negative binomial distribution

b) $k = 5$; $\theta = 0.75$.

c) $P(x=8) = b^*(8; 5; 0.75) =$

$\binom{8-1}{5-1} 0.75^5 (1-0.75)^{8-5}$

$= \binom{7}{4} 0.75^5\, 0.25^3 = 0.1298$

8 a) Hypergeometric distribution

b) $N = 80$, $M = 4$, $n = 3$

c) $x = 1$

$h(1; 3, 80, 4) = \dfrac{\binom{4}{1}\binom{76}{2}}{\binom{80}{3}} = 0.1388.$