Stat 588 HW5.

① $Z \sim g(z, \alpha, \beta)$

$$g(z,\alpha,\beta) = \begin{cases} \dfrac{1}{\beta^\alpha \Gamma(\alpha)} z^{\alpha-1} e^{-\frac{z}{\beta}} & \text{for } z > 0. \\ 0 & \text{elsewhere.} \end{cases}$$

where $\alpha > 0$, $\beta > 0$.

a) Let $\alpha = 1$ and $\beta = \theta$. Then

$\beta^\alpha = \theta^1 = \theta$

$\Gamma(1) = 1$

$z^{\alpha-1} = z^0 = 1$.

$$M_Z(t) = \int_{-\infty}^{+\infty} e^{tz} g(z,\alpha,\beta)\, dz$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_{-\infty}^{+\infty} z^{\alpha-1} e^{tz} e^{-z/\beta}\, dz$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_{0}^{+\infty} z^{\alpha-1} e^{-z(\frac{1}{\beta}-t)}\, dz$$

(Denote $z(\frac{1}{\beta}-t) := h$) $$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{+\infty} \left(\frac{h}{\frac{1}{\beta}-t}\right)^{\alpha-1} \cdot e^{-h} \frac{1}{\frac{1}{\beta}-t}\, dh$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \frac{1}{(\frac{1}{\beta}-t)^\alpha} \int_0^{+\infty} h^{\alpha-1} e^{-h}\, dh$$

$$= \frac{1}{\beta^\alpha \Gamma(\alpha)} \frac{\beta^\alpha}{(1-\beta t)^\alpha} \Gamma(\alpha)$$

$$= \left(\frac{1}{1-\beta t}\right)^\alpha \quad \text{if } t < \frac{1}{\beta}.$$

Then,

$$M_Y(t) = \frac{1}{1-\theta t} \quad \forall\, t < \frac{1}{\theta}$$

b) When $\alpha = \frac{v}{2}$ and $\beta = 2$:

$$M_X(t) = \left(\frac{1}{1-2t}\right)^{\frac{v}{2}} \quad \text{if } t < \frac{1}{2}$$

2) $X \sim u(x, \alpha, \beta)$

$$u(x, \alpha, \beta) = \begin{cases} \frac{1}{\beta-\alpha} & \text{for } \alpha < x < \beta \\ 0 & \text{otherwise} \end{cases}$$

a) $$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x \frac{1}{\beta-\alpha} dx = \frac{1}{\beta-\alpha} \int_{\alpha}^{\beta} x \, dx$$

$$= \frac{1}{\beta-\alpha} \cdot \frac{1}{2} \cdot x^2 \Big|_{\alpha}^{\beta} = \frac{1}{\beta-\alpha} \frac{\beta^2-\alpha^2}{2}$$

$$= \frac{\beta+\alpha}{2}$$

b) $$\mathbb{E}[X^2] = \int_{-\infty}^{+\infty} x^2 \frac{1}{\beta-\alpha} dx = \int_{\alpha}^{\beta} \frac{1}{\beta-\alpha} x^2 dx$$

$$= \frac{1}{\beta-\alpha} \cdot \frac{1}{3} x^3 \Big|_{\alpha}^{\beta} = \frac{1}{\beta-\alpha} \frac{\beta^3-\alpha^3}{3}$$

$$= \frac{\alpha^2+\alpha\beta+\beta^2}{3}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$= \frac{\alpha^2+\alpha\beta+\beta^2}{3} - \frac{\alpha^2+2\alpha\beta+\beta^2}{4}$$

$$= \frac{\alpha^2}{12} - \frac{\alpha\beta}{6} + \frac{\beta^2}{12} = \frac{1}{12}(\alpha-\beta)^2.$$

c) $F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{\alpha}^{x} \frac{1}{\beta-\alpha}\,dt = \frac{t}{\beta-\alpha}\Big|_{\alpha}^{x}$

$= \frac{x-\alpha}{\beta-\alpha}$ for $\alpha < x < \beta$

We have

$$F(x) = \begin{cases} 0 & \text{for } x < \alpha \\ \frac{x-\alpha}{\beta-\alpha} & \text{for } \alpha \leq x \leq \beta \\ 1 & \text{for } x > \beta \end{cases}$$

(3) $f(x, v) = \begin{cases} \frac{1}{2^{v/2}\,\Gamma(v/2)}\, x^{\frac{v-2}{2}}\, e^{-x/2} & \text{for } x > 0 \\ 0 & \text{otherwise.} \end{cases}$

a) $M_X(t) = \int_{-\infty}^{+\infty} \frac{e^{tx}}{2^{v/2}\,\Gamma(v/2)}\, x^{\frac{v-2}{2}}\, e^{-x/2}\,dx$

$= \frac{1}{2^{v/2}\,\Gamma(v/2)} \int_{0}^{+\infty} x^{\frac{v-2}{2}}\, e^{-x\left(\frac{1-2t}{2}\right)}\,dx$

(Denote $x\left(\frac{1-2t}{2}\right) := y$) $= \frac{1}{2^{v/2}\,\Gamma(v/2)} \int_{0}^{+\infty} \left(\frac{y}{\frac{1-2t}{2}}\right)^{\frac{v}{2}-1} e^{-y} \cdot \left(\frac{1-2t}{2}\right)^{-1} dy$

$= \frac{1}{2^{v/2}\,\Gamma\left(\frac{v}{2}\right)} \cdot \frac{1}{\left(\frac{1-2t}{2}\right)^{\frac{v}{2}}} \int_{0}^{+\infty} y^{\frac{v}{2}-1}\, e^{-y}\,dy$

$= \frac{1}{(1-2t)^{v/2}}$ if $t < \frac{1}{2}$

$\mu_1' = \frac{\partial}{\partial t} M_X(t)\Big|_{t=0} = 2 \cdot \frac{v}{2} \cdot (1-2t)^{-\frac{v}{2}-1}\Big|_{t=0} = v$

Hence, $\mathbb{E}[X] = v$

b) $\mu_2' = \frac{\partial^2}{\partial x^2} M_X(t) \Big|_{t=0} = \frac{\partial}{\partial x}\left(v(1-2t)^{\frac{-v}{2}-1} \right)\Big|_{t=0}$

$= v\left(\frac{-v}{2}-1\right)(1-2t)^{\frac{-v}{2}-2}(-2)\Big|_{t=0}$

$= v^2 + 2v$

$Var(X) = v^2 + 2v - v^2 = 2v$

(4) $Y \sim exp(x, \theta)$

$$exp(x,\theta) = \begin{cases} \frac{1}{\theta} e^{-x/\theta} & (x > 0) \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} \frac{x}{\theta} e^{-x/\theta}\, dx = \int_{0}^{+\infty} \frac{x}{\theta} e^{-x/\theta}\, dx$$

Denote $\begin{cases} u = \frac{x}{\theta} \\ dv = e^{-x/\theta}\, dx \end{cases} \Rightarrow \begin{cases} du = \frac{dx}{\theta} \\ v = -\theta e^{-x/\theta} \end{cases}$

$$\mathbb{E}[X] = uv\Big|_0^{+\infty} - \int_0^{+\infty} -\theta e^{-x/\theta} \frac{dx}{\theta}$$

$$= -\frac{x}{\theta}\theta e^{-x/\theta}\Big|_0^{+\infty} + (-\theta) e^{-x/\theta}\Big|_0^{+\infty}$$

$$= (0-0) + (-\theta)(0-1) = \theta.$$

(5) $X \sim N(\mu, \sigma^2)$

a) $\mathbb{E}[(X-\mu)^3] = \mathbb{E}[X^3 - 3X^2\mu + 3X\mu^2 - \mu^3]$

$= \mathbb{E}[X^3] - 3\mu\mathbb{E}[X^2] + 3\mu^2\mathbb{E}[X] - \mu^3$

$= \mathbb{E}[X^3] - 3\mu(\mu^2 + \sigma^2) + 3\mu^3 - \mu^3$

$= \mathbb{E}[X^3] - \mu^3 - 3\mu\sigma^2$

b) $\mathbb{E}[X^3] = \mu_3' = \frac{\partial^3}{\partial t^3} M_X(t)\Big|_{t=0}$

$$M_X(t) = \int_{-\infty}^{+\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(\frac{-1}{2\sigma^2}[-2xt\sigma^2 + (x-\mu)^2]\right) dx$$

We have $-2xt\sigma^2 + (x-\mu)^2$

$= -2xt\sigma^2 + x^2 - 2x\mu + \mu^2$

$= [x - (\mu + t\sigma^2)]^2 - 2\mu t\sigma^2 - t^2\sigma^4$

Then

$$\exp\left[\frac{-1}{2\sigma^2}[-2xt\sigma^2 + (x-\mu)^2]\right]$$

$$= \exp\left\{[(x-(\mu+t\sigma^2))^2 - 2\mu t\sigma^2 - t^2\sigma^4]\frac{-1}{2\sigma^2}\right\}$$

$$= \exp\left[\frac{-(x-(\mu+t\sigma^2))^2}{2\sigma^2}\right] \cdot \exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$$

$$M_X(t) = \exp\left(\mu t + \frac{t^2\sigma^2}{2}\right) \underbrace{\frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left[\frac{-1}{2}\left(\frac{x-(\mu+t\sigma^2)}{\sigma}\right)^2\right] dx}_{=1}$$

(because it's the PDF of a normal distribution)

$$= \exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$$

$$\frac{\partial}{\partial t} M_X(t) = (\mu + \sigma^2 t)\exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$$

$$\frac{\partial^2}{\partial t^2} M_X(t) = [(\mu + \sigma^2 t)^2 + \sigma^2]\exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$$

$$\frac{\partial^3}{\partial t^3} M_X(t) = 2(\mu + \sigma^2 t)\sigma^2 \exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$$
$$+ (\mu + \sigma^2 t)\exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)[(\mu+\sigma^2 t)^2 + \sigma^2]$$

$$\frac{\partial^3}{\partial t^3} M_X(t) \Big|_{t=0} = 3\mu\sigma^2 + \mu^3$$

Hence,

$$\mathbb{E}[(X-\mu)^3] = 3\mu\sigma^2 + \mu^3 - \mu^3 - 3\mu\sigma^2$$
$$= 0$$

c) $$\alpha_3 = \frac{\mathbb{E}[(X-\mu)^3]}{\sigma^3} = 0$$

(6)
$$f_X(x) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu_1}{\sigma_1}\right)^2}$$
$$f_Y(y) = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-1/2\left(\frac{y-\mu_2}{\sigma_2}\right)^2}.$$
$$f(x,y) = \frac{e^{-\frac{1}{2(1-\rho)^2}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

a) $\mu_{Y|x}$?

$$w(y|x) = \frac{f(x,y)}{f_X(x)}. \quad \text{let } u = \frac{x-\mu_1}{\sigma_1}, \; v = \frac{y-\mu_2}{\sigma_2}$$

$$w(y|x) = \frac{\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)}[u^2 - 2\rho uv + v^2]}}{\frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-1/2 u^2}}$$
$$= \frac{1}{\sigma_2\sqrt{2\pi}\sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)}[v^2 - 2\rho uv + u^2\rho^2]}$$
$$= \frac{1}{\sigma_2\sqrt{2\pi}\sqrt{1-\rho^2}} e^{\frac{-1}{2}\left[\frac{v-\rho u}{\sqrt{1-\rho^2}}\right]^2}$$

Then $$w(y|x) = \frac{1}{\sigma_2 \sqrt{2\pi} \sqrt{1-\rho^2}} \exp \left\{ \frac{-1}{2} \left[\frac{y - (\mu_2 + \rho \frac{\sigma_2}{\sigma_1} (x - \mu_1))}{\sigma_2 \sqrt{1-\rho^2}} \right]^2 \right\}$$

We can see that $w(y|x)$ is a normal distribution function with mean $\mu_{Y|x} = \mu_2 + \rho \frac{\sigma_2}{\sigma_1} (x - \mu_1)$

$\Rightarrow \mu_{Y|x} = 1 + 0.35(x+2)$

b) $\sigma^2_{Y|x} = \sigma_2^2 (1 - \rho^2) = 12.75.$

(7) Suppose $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$.

a) $U = X + Y$

$\mathbb{E}[U] = \mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] = \mu_1 + \mu_2$.

Denote ρ to be the correlation coefficient between X and Y. Then $\mathbb{E}[XY] = \rho\sigma_1\sigma_2 + \mu_1\mu_2$.

$\mathbb{E}[U(U-1)] = \mathbb{E}[(X+Y)(X+Y-1)]$

$= \mathbb{E}[X^2 + 2XY + Y^2 - X - Y]$

$= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - \mathbb{E}[X] - \mathbb{E}[Y]$

$= \sigma_1^2 + \mu_1^2 + 2\rho\sigma_1\sigma_2 + 2\mu_1\mu_2 + \sigma_2^2 + \mu_2^2 - \mu_1 - \mu_2$

$$\mu'_{2,u} = \mathbb{E}[U(U-1)] + E[U]$$
$$= \sigma_1^2 + \mu_1^2 + 2\rho\sigma_1\sigma_2 + 2\mu_1\mu_2 + \sigma_2^2 + \mu_2^2.$$

$$Var(U) = \mu'_{2,u} - (\mathbb{E}[U])^2$$
$$= \sigma_1^2 + \mu_1^2 + 2\rho\sigma_1\sigma_2 + 2\mu_1\mu_2 + \sigma_2^2 + \mu_2^2$$
$$- \mu_1^2 - 2\mu_1\mu_2 - \mu_2^2$$
$$= \sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2.$$

b) $V = X - Y.$

$$\mathbb{E}[V] = \mu_1 - \mu_2.$$
$$\mathbb{E}[V(V-1)] = \mathbb{E}[(X-Y)(X-Y-1)]$$
$$= \mathbb{E}[X^2 - XY - X - XY + Y^2 + Y]$$
$$= \mathbb{E}[X^2] - 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - \mathbb{E}[X] + \mathbb{E}[Y]$$
$$= \sigma_1^2 + \mu_1^2 - 2\rho\sigma_1\sigma_2 - 2\mu_1\mu_2 + \sigma_2^2 + \mu_2^2 - \mu_1 + \mu_2$$

$$\mu'_{2,v} = \mathbb{E}[V(V-1)] + \mathbb{E}[V]$$
$$= \sigma_1^2 + \mu_1^2 - 2\rho\sigma_1\sigma_2 - 2\mu_1\mu_2 + \sigma_2^2 + \mu_2^2$$

$$Var(V) = \mu'_{2,v} - (\mathbb{E}[V])^2$$
$$= \sigma_1^2 + \mu_1^2 - 2\rho\sigma_1\sigma_2 - 2\mu_1\mu_2 + \sigma_2^2 + \mu_2^2$$
$$- \mu_2^2 + 2\mu_1\mu_2 - \mu_2^2$$
$$= \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2$$

c) $Cov(U, V) = \mathbb{E}[UV] - \mathbb{E}[U]\mathbb{E}[V]$

$$\mathbb{E}[UV] = \mathbb{E}[(X+Y)(X-Y)] = \mathbb{E}[X^2] - \mathbb{E}[Y^2]$$
$$= \sigma_1^2 + \mu_1^2 - \sigma_2^2 - \mu_2^2.$$
$$Cov(U,V) = \sigma_1^2 + \mu_1^2 - \sigma_2^2 - \mu_2^2 - (\mu_1+\mu_2)(\mu_1-\mu_2)$$
$$= \sigma_1^2 + \mu_1^2 - \sigma_2^2 - \mu_2^2 - \mu_1^2 + \mu_2^2$$
$$= \sigma_1^2 - \sigma_2^2.$$

a) $$\mathrm{Corr}(U,V) = \frac{\mathrm{Cov}(U;V)}{\sqrt{\mathrm{Var}(U)\,\mathrm{Var}(V)}}$$

$$= \frac{\sigma_1^2 - \sigma_2^2}{\sqrt{(\sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2)(\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2)}}$$

(8) $X \sim g(x, \alpha = 3, \beta = 2)$

$$g(x,\alpha,\beta) = \begin{cases} \dfrac{1}{\beta^{\alpha}\,\Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{P}(X > 12) = 1 - P(X \le 12)$$
$$= 1 - \int_0^{12} \frac{1}{2^3\,\Gamma(3)} x^2 e^{-x/2}\,dx$$

(9) $X \sim f(x, \alpha, \beta)$

$$f(x,\alpha,\beta) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,.\,\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & (0 < x < 1) \\ 0 & \text{elsewhere} \end{cases}$$

$(\alpha > 0, \beta > 0)$

$$\mathbb{E}[X] = \int_0^1 \frac{x\,\Gamma(\alpha+\beta)}{\Gamma(\alpha)\,\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}\,dx$$

Given that $\alpha = 1$, $\beta(4)$: $\Gamma(1) = 1$, $\Gamma(4) = 6$;
$\Gamma(5) = 24$

$$\mathbb{E}[X] = \int_0^1 \frac{24}{1 \cdot 6} x (1-x)^3 \, dx$$

$$= 4 \int_0^1 (-x^4 + 3x^3 - 3x^2 + x) \, dx$$

$$= 4 \left(\frac{-x^5}{5} + 3 \cdot \frac{x^4}{4} - 3 \cdot \frac{x^3}{3} + \frac{x^2}{2} \right) \Big|_0^1$$

$$= 4 \left(\frac{-1}{5} + \frac{3}{4} - 1 + \frac{1}{2} \right) = \frac{1}{5}$$