STAT 588 HOMEWORK #2
SPRING 2020 RUBRIC

1. **[6 pts.]** Let X denote the number of heads in four tosses of a fair coin. Let $Y = \sqrt{|2X-4|}$. Find the probability distribution (PMF) of Y, denoted by $f(y)$, where Y is the square root of the absolute difference between the number of heads and the number of tails obtained in the four tosses.

S	Y
(H,H,H,H)	$\sqrt{\lvert 4-0\rvert} = 2$
(H,H,H,T), (H,H,T,H), (H,T,H,H), (T,H,H,H)	$\sqrt{\lvert 3-1\rvert} = \sqrt{2}$
(H,H,T,T), (H,T,H,T), (H,T,T,H), (T,T,H,H), (T,H,T,H), (T,H,H,T)	$\sqrt{\lvert 2-2\rvert} = 0$
(H,T,T,T), (T,H,T,T), (T,T,H,T), (T,T,T,H)	$\sqrt{\lvert 1-3\rvert} = \sqrt{2}$
(T,T,T,T)	$\sqrt{\lvert 0-4\rvert} = 2$

y	0	$\sqrt{2}$	2
$f(y)$	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{1}{8}$

Note: The first table is not required

+1 for each cell of the second table

2. **[3 pts.]** Determine whether the function $f(x)$ is a valid probability distribution (PMF) for a random variable with the range $x = 0, 1, 2, 3, 4$.

x	0	1	2	3	4
$f(x)$	0	$\frac{1}{30}$	$\frac{4}{30}$	$\frac{11}{30}$	$\frac{15}{30}$

$f(x) \geq 0, \forall x$

$\sum_x f(x) = \frac{0+1+4+11+15}{30} = \frac{31}{30} > 1$

No, $f(x)$ is **not** a valid PMF

+1 for including discussion of $f(x) \geq 0$

+1 for including discussion of the values for $f(x)$ summing to 1, or not

+1 for correct answer "no, not valid"

3. **[8 pts.]** Suppose X is a random variable with probability distribution (PMF) given by $f(x)$ and a range of $x = 0, 1, 2, 3$. Find the distribution function (CDF) for X. $f(x) = \frac{\binom{3}{x}\binom{3}{3-x}}{\binom{6}{3}}$

$$f(0) = \frac{\binom{3}{0}\binom{3}{3}}{\binom{6}{3}} = \frac{1}{20}$$

$$f(1) = \frac{\binom{3}{1}\binom{3}{2}}{\binom{6}{3}} = \frac{9}{20}$$

$$f(2) = \frac{\binom{3}{2}\binom{3}{1}}{\binom{6}{3}} = \frac{9}{20}$$

$$f(3) = \frac{\binom{3}{3}\binom{3}{0}}{\binom{6}{3}} = \frac{1}{20}$$

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{20}, & 0 \le x < 1 \\ \frac{1}{2}, & 1 \le x < 2 \\ \frac{19}{20}, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

+1 for showing work

+1 for each correct value in the 5 pieces of $F(x)$

+1 for having closed left endpoints in $F(x)$

+1 for having open right endpoints in $F(x)$

4. The number of minutes that a flight from Phoenix to Tucson is late is a random variable, X, with probability density (PDF) given by

$$f(x) = \begin{cases} \frac{1}{972}(81 - x^2), & -9 < x < 9 \\ 0, & \text{otherwise} \end{cases}$$

where negative values indicate the flight was early and positive values indicate the flight was late.

(a) **[3 pts.]** Find the distribution function (CDF) for X.

$$\begin{aligned} F(x) &= \int_{-9}^{x} \frac{1}{972}(81 - s^2)\, ds \\ &= \frac{1}{972}(81s - \frac{1}{3}s^3)\Big|_{-9}^{x} \\ &= \frac{1}{972}(81x - \frac{1}{3}x^3) - \frac{1}{972}\left[81(-9) - \frac{1}{3}(-9)^3\right] \\ &= \frac{1}{972}(81x - \frac{1}{3}x^3 + 486) \end{aligned}$$

+1 for correct limits of integration

+1 for showing work

+1 for correct final answer

(b) **[4 pts.]** Find the probability that one of these flights will be at least 1 minute late.

$$\begin{aligned} P(X > 1) &= 1 - P(X \leq 1) \\ &= 1 - F(1) \\ &= 1 - \frac{1}{972}\left(81 - \frac{1}{3} + 486\right) \\ &= 0.4170 \end{aligned}$$

+1 for first line of derivation
+1 for second line of derivation
+1 for third line of derivation
+1 for correct final answer

5. **[8 pts.]** The distribution function (CDF) for the random variable X is given by $F(x)$ below. Find the probability density function (PDF) of X, denoted $f(x)$. Then, verify $F(x)$ is a valid CDF.

$$F(x) = \begin{cases} 1 - (1+x)e^{-x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

$$\begin{aligned} f(x) &= \frac{d}{dx}F(x) \\ &= -(1+x)(-e^{-x}) + (e^{-x})(-1) \end{aligned}$$

$$f(x) = \begin{cases} xe^{-x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

$F(0) = 1 - (1+0)e^{-0} = 1 - 1 * 1 = 0$

$f(x) > 0 \Rightarrow F(x)$ increases in x

$\lim_{x\to\infty} F(x)$

$= \lim_{x\to\infty} 1 - e^{-x} - xe^{-x}$

$= 1 - 0 - 0 = 1.$

+1 for showing work
+1 for doing the derivative of $F(x)$
+1 for correctly using the chain rule
+1 for correct function values in $f(x)$
+1 for correct limits for x in $f(x)$
+1 for finding $F(0) = 0$
+1 for noting $F(x)$ is increasing
+1 for finding $\lim_{x\to\infty} F(x) = 1.$

6. Suppose that we roll a pair of fair dice. Let X be the number of dice that show 1 and Y be the number of dice that show either 4, 5, or 6.

(a) **[3 pts.]** Draw a diagram like Figure 3.1 on page 62 showing the values of the pair (X, Y) associated with each of the 36 equally likely points of the sample space.

(x,y)	1	2	3	4	5	6
1	(2,0)	(1,0)	(1,0)	(1,1)	(1,1)	(1,1)
2	(1,0)	(0,0)	(0,0)	(0,1)	(0,1)	(0,1)
3	(1,0)	(0,0)	(0,0)	(0,1)	(0,1)	(0,1)
4	(1,1)	(0,1)	(0,1)	(0,2)	(0,2)	(0,2)
5	(1,1)	(0,1)	(0,1)	(0,2)	(0,2)	(0,2)
6	(1,1)	(0,1)	(0,1)	(0,2)	(0,2)	(0,2)

+1 for correct row 1

+1 for correct row 2, 3 (should be same!)

+1 for correct row 4, 5, 6 (should be same!)

(b) **[3 pts.]** Construct a table showing the values of the joint probability distribution (PMF) of X and Y.

		x		
		0	1	2
	0	$\frac{4}{36}$	$\frac{4}{36}$	$\frac{1}{36}$
y	1	$\frac{12}{36}$	$\frac{6}{36}$	0
	2	$\frac{9}{36}$	0	0

+1 for correct row 1

+1 for correct row 2

+1 for correct row 3

7. **[4 pts.]** Consider the function $f(x, y)$ for $x = 0, 1, 2, 3$ and $y = 0, 1, 2$. Find the joint distribution function (CDF) corresponding to $f(x, y)$. $f(x, y) = \frac{1}{48}(2x + y)$

f(x,y)		x			
		0	1	2	3
	0	0	$\frac{2}{48}$	$\frac{4}{48}$	$\frac{6}{48}$
y	1	$\frac{1}{48}$	$\frac{3}{48}$	$\frac{5}{48}$	$\frac{7}{48}$
	2	$\frac{2}{48}$	$\frac{4}{48}$	$\frac{6}{48}$	$\frac{8}{48}$

F(x,y)		x			
		0	1	2	3
	0	0	$\frac{2}{48}$	$\frac{6}{48}$	$\frac{12}{48}$
y	1	$\frac{1}{48}$	$\frac{6}{48}$	$\frac{15}{48}$	$\frac{28}{48}$
	2	$\frac{3}{48}$	$\frac{12}{48}$	$\frac{27}{48}$	$\frac{48}{48}$

Note: the first table is not required

+1 for correct column 1

+1 for correct column 2

+1 for correct column 3

+1 for correct column 4

8. Consider the random variables X and Y with joint probability density (PDF) given by $f(x, y)$ below.

$$f(x,y) = \begin{cases} \frac{1}{4}(x + 2y), & 0 < x < 2,\ 0 < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) **[3 pts.]** Integrate the PDF to find $F(x,y)$ the cumulative distribution function of (X,Y).

$$F(x,y) =$$

$$\int_{s=0}^{s=x}\int_{t=0}^{t=y} \tfrac{1}{4}(s+2t)\ dt\ ds$$

$$= \int_{s=0}^{s=x} \tfrac{1}{4}(sy+y^2)ds$$

$$= \tfrac{1}{8}x^2y + \tfrac{1}{4}xy^2$$

+1 for setting up integral and bounds correctly

+1 for correct answer for showing work

+1 for correct answer for $F(x,y)$

(b) **[3 pts.]** Integrate the PDF to find $P(X \leq x, Y \geq 1/2)$.

$$P(X \leq x, Y \geq 1/2) =$$

$$\int_{s=0}^{s=x}\int_{t=1/2}^{t=1} \tfrac{1}{4}(s+2t)\ dt\ ds$$

$$= \int_{s=0}^{s=x} \tfrac{1}{4}(\tfrac{1}{2}s+\tfrac{3}{4})ds$$

$$= \frac{x^2+x}{16}$$

+1 for setting up integral and bounds correctly

+1 for correct answer for showing work

+1 for correct answer for $P(X \leq x, Y \geq 1/2)$

9. **[5 pts.]** Find the joint probability density (PDF) of two random variables X and Y whose joint distribution function (CDF) is given by $F(x,y)$ below.

$$F(x,y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-x-y}, & x > 0,\ y > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\tfrac{\partial}{\partial x}F(x,y) = e^{-x} - e^{-x-y}$$
$$\tfrac{\partial^2}{\partial x \partial y}\left(e^{-x} - e^{-x-y}\right) = e^{-x-y}$$

$$f(x,y) = \begin{cases} e^{-x-y}, & x > 0,\ y > 0 \\ 0, & \text{otherwise} \end{cases}$$

+1 for showing work

+1 for correct first partial derivative

+1 for correct second partial derivative

+1 for correct function values in both pieces of $f(x,y)$

+1 for correct limits on x and y in both pieces of $f(x,y)$

10. If two cards are randomly drawn (without replacement) from an ordinary deck of 52 playing cards, let Z be the number of Kings obtained from the first draw and let W be the total number of Kings obtained from both draws. The table below provides values for $f(z,w)$, the joint distribution (PMF) of Z and W.

	$z=0$	$z=1$
$w=0$	$\frac{188}{221}$	0
$w=1$	$\frac{16}{221}$	$\frac{16}{221}$
$w=2$	0	$\frac{1}{221}$

(a) **[2 pts.]** Find the marginal distribution (PMF) of Z.

z	0	1
$f(z)$	$\frac{204}{221}$	$\frac{17}{221}$

+1 for each correct probability the table

(b) **[6 pts.]** Find the conditional distribution (PMF) of W given $Z = z$.

$f(w\|z)$		z = 0	z = 1
w	0	$\frac{188}{204}$	0
	1	$\frac{16}{204}$	$\frac{16}{17}$
	2	0	$\frac{1}{17}$

+1 for each correct probability in the table

(c) **[3 pts.]** Find the marginal distribution (PMF) of W.

w	0	1	2
$f(w)$	$\frac{188}{221}$	$\frac{32}{221}$	$\frac{1}{221}$

+1 for each correct probability in the table

(d) **[6 pts.]** Find the conditional distribution (PMF) of Z given $W = w$.

$f(z\|w)$		z = 0	z = 1
w	0	1	0
	1	$\frac{1}{2}$	$\frac{1}{2}$
	2	0	1

+1 for each correct probability in the table

11. Suppose that P, the price of a certain commodity (in dollars), and S, the commodity's total sales (in 10,000 units), are random variables with joint probability density function (PDF) given by $f(p, s)$ below.

$$f(p,s) = \begin{cases} 2.5pe^{-ps}, & 0.1 < p < 0.5,\ s > 0 \\ 0, & \text{otherwise} \end{cases}$$

(a) **[6 pts.]** Find the marginal density (PDF) of P.

$\int_0^\infty 2.5pe^{-ps}\ ds = \lim_{s\to\infty} -2.5e^{-ps} + 2.5 = 2.5$

$$f(p) = \begin{cases} 2.5, & 0.1 < p < 0.5 \\ 0, & \text{otherwise} \end{cases}$$

+1 for showing work
+1 for setting up integral correctly
+1 for correct limit to evaluate integral
+1 for correct evaluation of integral at 0
+1 for correct function values for $f(p)$
+1 for correct limits on p for $f(p)$

(b) **[4 pts.]** Find the conditional density (PDF) of S given $P = p$.

$$f(s|p) = \frac{f(s,p)}{f(p)} = \frac{2.5pe^{-ps}}{2.5} = pe^{-ps}$$

$$f(s|p) = \begin{cases} pe^{-ps}, & s > 0 \\ 0, & \text{otherwise} \end{cases}$$

+1 for showing work

+1 for dividing the joint density by the marginal density

+1 for correct function values for $f(s|p)$

+1 for correct domain for $f(s|p)$