HOMEWORK 7

① $\mathbb{E}\left[n\left(\frac{X}{n}\right)\left(1-\frac{X}{n}\right)\right] = n\,\mathbb{E}\left[\frac{X}{n} - \left(\frac{X}{n}\right)^2\right] = n\left(\frac{\mathbb{E}[X]}{n} - \frac{\mathbb{E}[X^2]}{n^2}\right)$

$\mathbb{E}[X] = n\theta$; $\mathbb{E}[X^2] = n\theta - n\theta^2 + n^2\theta^2$.

Then, $\mathbb{E}\left[n\left(\frac{X}{n}\right)\left(1-\frac{X}{n}\right)\right] = n\left(\theta - \frac{\theta}{n} + \frac{\theta^2}{n} - \theta^2\right)$

$= n\theta - \theta + \theta^2 - n\theta^2 = n\theta(1-\theta) + \theta^2 - \theta$.

For $\theta^2 - \theta \neq 0$, $\mathbb{E}\left[n\left(\frac{X}{n}\right)\left(1-\frac{X}{n}\right)\right] \neq n\theta(1-\theta)$.

Hence, $\mathbb{E}\left[n\left(\frac{X}{n}\right)\left(1-\frac{X}{n}\right)\right]$ is a biased estimator of

the variance $n\theta(1-\theta)$

(2) a) $\mathbb{E}\left[\frac{\bar{X}+1}{n+2}\right] = \frac{\mathbb{E}[\bar{X}+1]}{n+2} = \frac{\mathbb{E}[\bar{X}]+1}{n+2} = \frac{\theta+1}{n+2}$

For $\theta \neq \frac{1}{n+1}$, $\mathbb{E}\left[\frac{\bar{X}+1}{n+2}\right] \neq \theta$. Hence, $\mathbb{E}\left[\frac{\bar{X}+1}{n+2}\right]$ is a biased estimator of θ.

b) let $b_n(\theta) = \mathbb{E}\left[\frac{\bar{X}+1}{n+2}\right] - \theta = \frac{1-(n+1)\theta}{n+2}$

$\lim_{n\to\infty} b_n(\theta) = \lim_{n\to\infty}\left(\frac{1}{n+2} - \frac{n+1}{n+2}\theta\right) = \lim_{n\to\infty}\left(\frac{1}{n+2}\right) - \theta \lim_{n\to\infty}\left(\frac{n+1}{n+2}\right)$

$= \lim_{n\to\infty}\left(\frac{1}{n+2}\right) - \theta \lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{2+\frac{1}{n}}\right) = -\frac{\theta}{2}$

As $\lim_{n\to\infty} b_n(\theta)$ is generally not 0, we have $\frac{\bar{X}+1}{n+2}$ is not asympototically unbiased

(3) a) Suppose X_1, X_2 and X_3 are independent

$Var(Y) = Var\left(\frac{X_1}{4} + \frac{X_2}{2} + \frac{X_3}{4}\right) = \frac{1}{16}Var(X_1) + \frac{1}{4}Var(X_2)$

$+ \frac{1}{16}Var(X_3) = \frac{3}{8}\sigma^2$

b) $Var(Z) = Var\left(\frac{X_1+X_2+X_3}{3}\right) = \frac{Var(X_1)+Var(X_2)+Var(X_3)}{9}$

$= \frac{1}{3}\sigma^2$

c) As both Y and Z are unbiased estimators of μ ($\mathbb{E}[Y] = \mu$ and $\mathbb{E}[Z] = \mu$), but $\frac{Var(Z)}{Var(Y)} < 1$, we have Z is relatively more efficient than Y

(4) $\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2\right] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n}(X_i^2-2X_i\mu+\mu^2)\right]$

$= \frac{1}{n}\left(\sum_{i=1}^{n}\mathbb{E}[X_i^2] - 2\mu\sum_{i=1}^{n}\mathbb{E}[X_i] + \sum_{i=1}^{n}\mu^2\right)$

$= \frac{1}{n}\left[n(\sigma^2+\mu^2) - 2\mu^2 n + n\mu^2\right]$

$= \frac{1}{n}\left(n\sigma^2 + n\mu^2 - 2n\mu^2 + n\mu^2\right) = \sigma^2$

Hence $\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2$ is an unbiased estimator of σ^2

(5) $X_i \sim \text{Poisson}(\lambda, n)$

$\mathbb{E}[\bar{X}] = \mathbb{E}\left[\frac{\sum^n X_i}{n}\right] = \frac{n\lambda}{n} = \lambda.$

Hence, $\bar{X}$ is an unbiased estimator of μ.

$\text{Var}(\bar{X}) = \text{Var}\left(\frac{\sum X_i}{n}\right) = \frac{n.\lambda}{n^2} = \frac{\lambda}{n}.$

Moreover, $f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$ for $x = 0, 1, 2, \ldots$

$\ln f(x) = \ln(\lambda^x e^{-\lambda}) - \ln(x!)$

$= x\ln\lambda - \lambda\ln(e) - \ln(x!)$

$= x\ln\lambda - \lambda - \ln(x!)$

$\frac{\partial \ln(f(x))}{\partial\lambda} = \frac{x}{\lambda} - 1.$

$\mathbb{E}\left[\left(\frac{\partial\ln(f(X))}{\partial\lambda}\right)^2\right] = \mathbb{E}\left[\left(\frac{X}{\lambda}-1\right)^2\right] = \mathbb{E}\left[\frac{X^2}{\lambda^2} - \frac{2X}{\lambda} + 1\right]$

$= \frac{\mathbb{E}[X^2]}{\lambda^2} - \frac{2\,\mathbb{E}[X]}{\lambda} + 1 = \frac{\lambda+\lambda^2}{\lambda^2} - \frac{2\lambda}{\lambda} + 1 = \frac{1}{\lambda}$

Hence, $n\,\mathbb{E}\left[\left(\frac{\partial\ln(f(X))}{\partial\lambda}\right)^2\right] = \frac{n}{\lambda} = \frac{1}{\text{Var}(\bar{X})}$

Therefore, $\bar{X}$ is a minimum variance unbiased estimator of λ

⑥ The system of equations we have to solve is

$$m_1' = \mu_1' \quad \text{and} \quad m_2' = \mu_2'$$

where $\mu_1' = \mu$ and $\mu_2' = \sigma^2 + \mu^2$. Thus,

$$m_1' = \mu \quad \text{and} \quad m_2' = \mu^2 + \sigma^2,$$

and, solving for μ and σ^2, we get the following formulas for estimating the two parameters

$$\hat{\mu} = m_1' \quad \text{and} \quad \hat{\sigma}^2 = m_2' - (m_1')^2$$

Since $m_1' = \frac{\sum x_i}{n} = \bar{x}$ and $m_2' = \frac{\sum x_i^2}{n}$, we can write

$$\hat{\mu} = \bar{x} \quad \text{and} \quad \hat{\sigma}^2 = \frac{\sum x_i^2}{n} - (\bar{x})^2$$

⑦ a) The system of equations we have to solve is

$$m_1' = \mu_1' \quad \text{and} \quad m_2' = \mu_2'$$

where $\mu_1' = \frac{\alpha}{\alpha+\beta}$ and $\mu_2' = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} + \frac{\alpha^2}{(\alpha+\beta)^2}$

We also have $\beta = 1$, so

$$m_1' = \frac{\alpha}{\alpha+1} \quad \text{and} \quad m_2' = \frac{\alpha}{(\alpha+1)^2(\alpha+2)} + \frac{\alpha^2}{(\alpha+1)^2}$$

Hence, $(\alpha+1)\, m_1' = \alpha$

$$\Rightarrow (m_1' - 1)\alpha = -m_1' \Rightarrow \hat{\alpha} = \frac{m_1'}{1 - m_1'} \quad (m_1' \neq 1)$$

As $m_1' = \frac{\sum^n x_i}{n} = \bar{x}$, we can write

$$\hat{\alpha} = \frac{\bar{x}}{1 - \bar{x}} \quad (\bar{x} \neq 1)$$

b) $$f(\alpha, \beta) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} & \text{for } 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

As $\beta = 1$, we can write $f(\alpha, 1) = \begin{cases} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(1)} x^{\alpha-1} & (0 < x < 1) \\ 0 & \text{elsewhere} \end{cases}$

We have $\Gamma(1) = 1$ and $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$.

$$f(\alpha, 1) = \begin{cases} \dfrac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)\cdot 1} x^{\alpha-1} & (0 < x < 1) \\ 0 & \text{elsewhere} \end{cases}$$

$$= \begin{cases} \alpha x^{\alpha-1} & (0 < x < 1) \\ 0 & \text{elsewhere} \end{cases}$$

~~We have to find α that maximizes $\alpha x^{\alpha-1}$~~

~~Use $\ln L(\alpha, 1) = \ln(\alpha x^{\alpha-1}) = \ln(\alpha) + (\alpha-1)\ln x$~~

$$\cancel{\frac{\partial \ln L(\alpha, 1)}{\partial \alpha} = \frac{1}{\alpha} + \ln x.}$$

Then,

$$L(\alpha) = \prod_{i=1}^{n} f(x_i; \alpha) = \alpha^n \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} \quad (0 < x_i < 1)$$

Use

$$\ln(L(\alpha)) = n\ln(\alpha) + (\alpha-1)\ln\left(\prod_{i=1}^{n} x_i\right)$$

$$= n\ln\alpha + (\alpha-1)\sum_{i=1}^{n} \ln(x_i)$$

$$\frac{\partial \ln(L(\alpha))}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln(x_i)$$

Hence,

$$\hat{\alpha} = -\frac{n}{\sum_{i=1}^{n} \ln(x_i)}$$

(8) a)

$$f(x, \theta, \delta) = \begin{cases} \dfrac{1}{\theta} e^{\frac{-(x-\delta)}{\theta}} & x > \delta \\ 0 & x \leq \delta. \end{cases}$$

Mean : $\mu_1' = \delta + \theta$; Variance ; $\mu_2' - (\mu_1')^2 = \theta^2$

$\Rightarrow \mu_2' = 2\theta^2 + \delta^2 + 2\delta\theta$

The system of equations we have to solve is

$$m_1' = \mu_1' \quad \text{and} \quad m_2' = \mu_2'$$

where $\mu_1' = \delta + \theta$ and $\mu_2' = 2\theta^2 + \delta^2 + 2\theta\delta$

Thus,

$$m_1' = \delta + \theta \quad \text{and} \quad m_2' = 2\theta^2 + \delta^2 + 2\theta\delta$$

Solving for $\hat{\delta}$ and $\hat{\theta}$ we have

$$\begin{cases} \hat{\delta} = \dfrac{2(m_1')^2 - m_2'}{2m_1'} = m_1' - \dfrac{m_2'}{2m_1'} \\ \\ \hat{\theta} = m_1' - \dfrac{2(m_1')^2 - m_2'}{2m_1'} = \dfrac{m_2'}{2m_1'} \end{cases}$$

where $m_1' = \bar{x}$ and $m_2' = \dfrac{\sum^n x_i^2}{n}$

e) $$L(\theta, \delta) = \begin{cases} \prod_{i=1}^{n} \left(\dfrac{1}{\theta} e^{-\frac{(x_i - \delta)}{\theta}} \right) & (x_i > \delta) \\ 0 & (x_i \leq \delta) \end{cases}$$

Consider $x_i > \delta$:

$$L(\theta, \delta) = \frac{1}{\theta^n} e^{-\frac{\sum_{i=1}^{n}(x_i - \delta)}{\theta}}$$

$$\Rightarrow \ln(L(\theta,\delta)) = -n\ln\theta - \frac{\sum^n (x_i - \delta)}{\theta} \ln(e)$$

$$= -n\ln\theta - \frac{\sum^n (x_i - \delta)}{\theta}$$

$$\frac{\partial \ln(L(\theta,\delta))}{\partial\theta} = \frac{-n}{\theta} + \sum_{i=1}^{n} (x_i - \delta) \frac{1}{\theta^2}$$

Setting this derivative to 0. $\dfrac{-n}{\theta} + \sum_{i=1}^{n} (x_i - \delta) \dfrac{1}{\theta^2} = 0$.

$$\Rightarrow \sum_{i=1}^{n} (x_i - \delta) \frac{1}{\theta} = n \Rightarrow \hat{\theta} = \frac{\sum^n (x_i - \delta)}{n} = \bar{x} - \delta.$$

(*) $$\frac{\partial \ln(L(\theta,\delta))}{\partial \delta} = \frac{\partial}{\partial \delta}\left(-n\ln\theta - \frac{\sum^n x_i}{\theta^2} + \frac{n\delta}{\theta^2}\right)$$

$$= \frac{n}{\theta^2} > 0.$$

Hence, $L(\theta,\delta)$ is an increasing function of δ given that $x_i > \delta$. Otherwise, $L(\theta,\delta) = 0$
$\ln(L(\theta,\delta))$ will achieve its maximum value when the smallest number of x_i (denote $x_{(1)}$) is greater or equal to δ.
Choose $\hat{\delta} = x_{(1)}$. Then $\hat{\theta} = \bar{x} - \hat{\delta}$.

9) $$p(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!} \quad \text{for } x = 0, 1, 2\ldots$$

a) The system of equations we have to solve is
$$m_1' = \mu_1' = \lambda$$
Thus, $\hat{\lambda} = m_1' = \bar{x}$.

b) $$L(\lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} = \frac{e^{-\lambda n}\lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!}$$

$$\ln L(\lambda) = \sum_{i=1}^{n} x_i \ln\lambda - n\lambda - \sum_{i=1}^{n}\ln(x_i!)$$

$$\frac{\partial \ln L(\lambda)}{\partial \lambda} = \frac{\sum_{i=1}^{n} x_i}{\lambda} - n$$

Setting this derivative to 0, we get $\hat{\lambda} = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{x}$