HOMEWORK 3 STAT 588

①
$$f(x) = \begin{cases} \frac{x}{4} & 0 < x < 2 \\ 1 - \frac{x}{4} & 2 \leq x < 4 \\ 0 & \text{otherwise} \end{cases}$$

a) $$\mathbb{E}[X] = \int_0^2 \frac{x^2}{4} dx + \int_2^4 x\left(1 - \frac{x}{4}\right) dx$$

$$= \frac{1}{4} \cdot \frac{x^3}{3}\Big|_0^2 + \left(\frac{x^2}{2} - \frac{1}{4} \cdot \frac{x^3}{3}\right)\Big|_2^4$$

$$= \frac{1}{4} \cdot \frac{8}{3} + \frac{4^2}{2} - \frac{1}{4} \cdot \frac{4^3}{3} - \frac{2^2}{2} + \frac{1}{4} \cdot \frac{2^3}{3}$$

$$= 2$$

b) $$\mathbb{E}[X^2] = \int_0^2 x^2 \cdot \frac{x}{4} dx + \int_2^4 x^2\left(1 - \frac{x}{4}\right) dx$$

$$= \frac{1}{4} \cdot \frac{x^4}{4}\Big|_0^2 + \left(\frac{x^3}{3} - \frac{1}{4} \cdot \frac{x^4}{4}\right)\Big|_2^4$$

$$= \frac{1}{16} \cdot 2^4 + \frac{4^3}{3} - \frac{1}{4} \cdot \frac{4^4}{4} - \frac{2^3}{3} + \frac{1}{4} \cdot \frac{2^4}{4}$$

$$= 4.667$$

c) $$\mathbb{E}[3X^2 - 8X] = 3\,\mathbb{E}[X^2] - 8\,\mathbb{E}[X]$$

$$= 3 \cdot (4.667) - 8(2) = -2$$

(2) a) $$f(x) = \frac{|x-1|}{9} \qquad x \in \{-2, -1, 0, 2, 3\}$$

x	-2	-1	0	2	3
$f(x)$	$\frac{1}{3}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{2}{9}$

$$\mathbb{E}[X] = -2 \cdot \frac{1}{3} + (-1)\frac{2}{9} + 0 \cdot \frac{1}{9} + 2 \cdot \left(\frac{1}{9}\right) + 3\left(\frac{2}{9}\right)$$

$$= \frac{-2}{3} - \frac{2}{9} + \frac{2}{9} + \frac{2}{3} = 0.$$

b) $$\mathbb{E}[2X - 5] = 2\mathbb{E}[X] - 5 = -5$$

c) $$\mathbb{E}[X] = \mu = 0.$$

$$\mathbb{E}[X^2] = \mu_2' = (-2)^2 \frac{1}{3} + (-1)^2 \frac{2}{9} + 0\left(\frac{1}{9}\right) + 2^2 \cdot \left(\frac{1}{9}\right) + 3^2\left(\frac{2}{9}\right)$$

$$= \frac{4}{3} + \frac{2}{9} + \frac{4}{9} + 2 = 4.$$

$$\sigma^2 = \mu_2' - \mu^2 = 4 - 0^2 = 4.$$

(3) $f(x) = \begin{cases} \frac{2x}{9} & 0 < x < 3 \\ 0 & \text{otherwise} \end{cases}$

a) $\mu_1' = \mathbb{E}[X] = \int_0^3 x \cdot \frac{2x}{9} \, dx = \frac{2}{9} \cdot \frac{x^3}{3} \Big|_0^3 = 2$

b) $\mu_2' = \mathbb{E}[X^2] = \int_0^3 x^2 \cdot \frac{2x}{9} \, dx = \frac{2}{9} \cdot \frac{x^4}{4} \Big|_0^3 = 4.5$

c) $\sigma^2 = \mu_2' - (\mu_1')^2 = 4.5 - 2^2 = 0.5$

d) $\text{var}(3X + 7) = 9 \, \text{Var}(X) = 9 \cdot (0.5) = 4.5$

(4) $f(x) = \begin{cases} \frac{1}{2} & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$

Moment generating function:

$M_X(t) = \mathbb{E}[e^{tX}] = \int_0^2 e^{tx} \cdot \frac{1}{2} \, dx$

$= \frac{1}{2} \int_0^2 e^{tx} \, dx = \frac{1}{2} \cdot \frac{e^{tx}}{t} \Big|_0^2 = \frac{1}{2} \left(\frac{e^{2t}}{t} - \frac{1}{t} \right)$

(5) $f(x,y) = \begin{cases} 24xy & 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le x+y \le 1 \\ 0 & \text{otherwise} \end{cases}$

$$\mathbb{E}[XY] = \int_{y=0}^{1}\int_{x=0}^{1-y} xy\, 24xy\, dx\, dy$$

$$= \int_{y=0}^{1}\int_{x=0}^{1-y} (24y^2)x^2\, dx\, dy$$

$$= \int_{y=0}^{1} \left((24y^2)\frac{x^3}{3}\Big|_0^{1-y} \right) dy$$

$$= \int_0^1 24y^2 \frac{(1-y)^3}{3}\, dy$$

$$= \int_0^1 8y^2(1-3y+3y^2-y^3)\, dy$$

$$= 8\int_0^1 (y^2 - 3y^3 + 3y^4 - y^5)\, dy$$

$$= 8\left(\frac{y^3}{3} - 3.\frac{y^4}{4} + 3.\frac{y^5}{5} - \frac{y^6}{6}\right)\Big|_0^1$$

$$= 8\left(\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6}\right) = \frac{2}{15}.$$

(6)

		y			
		0	100	200	
x	100	0.2	0.1	0.2	0.5
	250	0.05	0.15	0.3	0.5
		0.25	0.25	0.5	

a) $\mathbb{E}[XY] = 0 \cdot 100 \cdot (0 \cdot 2) + 0 \cdot 250\,(0.05) +$
$+ 100 \cdot 100\,(0.1) + 250 \cdot 100\,(0.15) +$
$+ 100 \cdot 200\,(0.2) + 250 \cdot 200\,(0 \cdot 3)$
$= 23750.$

$\mathbb{E}[X] = 100\,(0.5) + 250\,(0.5) = 175.$

$\mathbb{E}[Y] = 0\,(0.25) + 100\,(0.25) + 200\,(0.5)$
$= 125.$

$Cov(X,Y) = 23\,750 - 175 \times 125 = 1875.$

b) $\mathbb{E}[\frac{3}{4}X + \frac{1}{7}Y] = \frac{3}{4}\mathbb{E}[X] + \frac{1}{7}\mathbb{E}[Y] = 149.1071$

c) $\mathbb{E}[X^2] = 100^2 \cdot (0.5) + 250^2 \cdot (0.5) = 36\,250.$
$Var(X) = 36250 - 175^2 = 5625.$
$\mathbb{E}(Y^2] = 0^2\,(0.25) + 100^2\,(0.25) + 200^2\,(0.5)$
$= 22500$
$var(Y) = 22500 - 125^2 = 6875.$

$Var(\frac{5}{2}X + \frac{2}{3}Y) = \frac{25}{4}\,var(X) + \frac{4}{9}\,var(Y) + \frac{10}{3}\,(cov(X,Y))$

$= 44461.80556$

d) $\operatorname{Cov}\left(\frac{3}{4}X + \frac{1}{7}Y, \frac{5}{2}X + \frac{2}{3}Y\right)$

$$= \frac{3}{4}\cdot\frac{5}{2}\operatorname{Var}(X) + \frac{1}{7}\cdot\frac{2}{3}\operatorname{Var}(Y)$$

$$+ \left(\frac{3}{4}\cdot\frac{2}{3} + \frac{1}{7}\cdot\frac{5}{2}\right)\operatorname{Cov}(X, Y)$$

$$= \frac{15}{8}\cdot 5625 + \frac{2}{21}\cdot 6875 + \frac{6}{7}\cdot 1875$$

$$= 12808.77976$$

⑦ $f(x,y,z) = \frac{1}{54} xyz$

for $x = 1,2,3$; $y = 1,2$; $z = 1,2$

$$\mathbb{E}[X+Y+Z] = \sum_{x=1}^{3}\sum_{y=1}^{2}\sum_{z=1}^{2} \frac{1}{54} xyz(x+y+z)$$

$$= \frac{1}{54}\Big[1.1.1(1+1+1) + 1.1.2(1+1+2) + 1.2.1(1+2+1) + 1.2.2(1+2+2) + 2.1.1(2+1+1) + 2.1.2(2+1+2) + 2.2.1(2+2+1) + 2.2.2(2+2+2) + 3.1.1(3+1+1) + 3.1.2(3+1+2) + 3.2.1(3+2+1) + 3.2.2(3+2+2)\Big]$$

$$= \frac{1}{54}(3+8+8+20+8+20+20+48+15+36+36+84)$$

$$= 5.667 = \frac{17}{3}$$

$$\mathbb{E}[(X+Y+Z)^2] = \sum_{x=1}^{3}\sum_{y=1}^{2}\sum_{z=1}^{2} \frac{1}{54} xyz(x+y+z)^2$$

$$= \frac{1}{54}\Big[1.1.1(1+1+1)^2 + 1.1.2(1+1+2)^2 + 1.2.1(1+2+1)^2 + 1.2.2(1+2+2)^2 + 2.1.1(2+1+1)^2 + 2.1.2(2+1+2)^2 + 2.2.1(2+2+1)^2 + 2.2.2(2+2+2)^2 + 3.1.1(3+1+1)^2 + 3.1.2(3+1+2)^2 + 3.2.1(3+2+1)^2 + 3.2.2(3+2+2)^2\Big]$$

$$= 33.111 = \frac{298}{9}$$

$$\mathrm{Var}(X+Y+Z) = \mathbb{E}[(X+Y+Z)^2] - (\mathbb{E}[X+Y+Z])^2 = 1$$

(8) $$f(y \mid x = \tfrac{1}{4}) = \begin{cases} \frac{1}{30}(6y+1) & 0 < y < 3 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}[Y \mid X = \tfrac{1}{4}] = \int_0^3 \frac{y}{30}(6y+1)\,dy$$

$$= \int_0^3 \left(\frac{1}{30} \cdot 6y^2 + \frac{y}{30}\right) dy$$

$$= \int_0^3 \left(\frac{1}{5} y^2 + \frac{1}{30} y\right) dy$$

$$= \left(\frac{1}{5} \cdot \frac{y^3}{3} + \frac{1}{30} \cdot \frac{y^2}{2}\right)\Bigg|_0^3$$

$$= \frac{1}{5} \cdot \frac{3^3}{3} + \frac{1}{30} \cdot \frac{3^2}{2} = \frac{39}{20}.$$

$$\mathbb{E}[Y^2 \mid X = \tfrac{1}{4}] = \int_0^3 \frac{y^2}{30}(6y+1)\,dy$$

$$= \int_0^3 \left(\frac{1}{5} y^3 + \frac{1}{30} y^2\right) dy$$

$$= \left(\frac{1}{5} \cdot \frac{y^4}{4} + \frac{1}{30} \cdot \frac{y^3}{3}\right)\Bigg|_0^3$$

$$= \frac{1}{20} \cdot 3^4 + \frac{1}{90} \cdot 3^3 = \frac{87}{20}.$$

$$\mathrm{Var}[Y \mid X = \tfrac{1}{4}] = \frac{87}{20} - \left(\frac{39}{20}\right)^2 = \frac{219}{400} = 0.5475$$