2.

(a)divide a, b into a, a+h, a+2h, a+3h, and apply Lagrange polynomial of 3rd order, we get

$$
\begin{aligned}
f(x) = &\frac{(x-(a+h))(x-(a+2h))(x-(a+3h))}{(-h)(-2h)(-3h)}f_0 \\
&+\frac{(x-a)(x-(a+2h))(x-(a+3h))}{(h)(-h)(-2h)}f_1 \\
&+\frac{(x-a)(x-(a+h))(x-(a+3h))}{(2h)(h)(-h)}f_2 \\
&+\frac{(x-a)(x-(a+h))(x-(a+2h))}{(3h)(2h)(3h)}f_3 \\
&+\frac{f^{(4)}(\mu)(x-a)(x-(a+h))(x-(a+2h))(x-(a+3h))}{4!}
\end{aligned}
$$

Where μ is some value between a and b

Integrate both sides

Example: The way we integrate $(x-(a+h))(x-(a+2h))(x-(a+3h))$

Let $x=\frac{3h}{2}(t+1)+a$, then integral becomes

$$\int_{-1}^{1}\frac{3}{2}h\left(\frac{1}{2}h\right)^{3}(3t+1)(3t-1)(3t-3)dt$$

$$=\frac{3}{2}h\left(\frac{1}{2}h\right)^{3}3=-\frac{9}{4}h^{4}$$

$$\text{divide by }(-h)(-2h)(-3h),\text{we get coefficient}\frac{1}{8}$$

integrate all 4 terms we derive

$$\int_{a}^{b}f(x)dx=\frac{3h}{8}(f_0+3f_1+3f_2+f_3)-\frac{3}{80}h^{5}f^{(4)}(c)$$

Note: in the first equation, $f^{(4)}(\mu)$ should depend on x you choose, but in the integral we just treat it as a constant

(b)we use integral $\int_0^1\sqrt{1+x}\,dx$

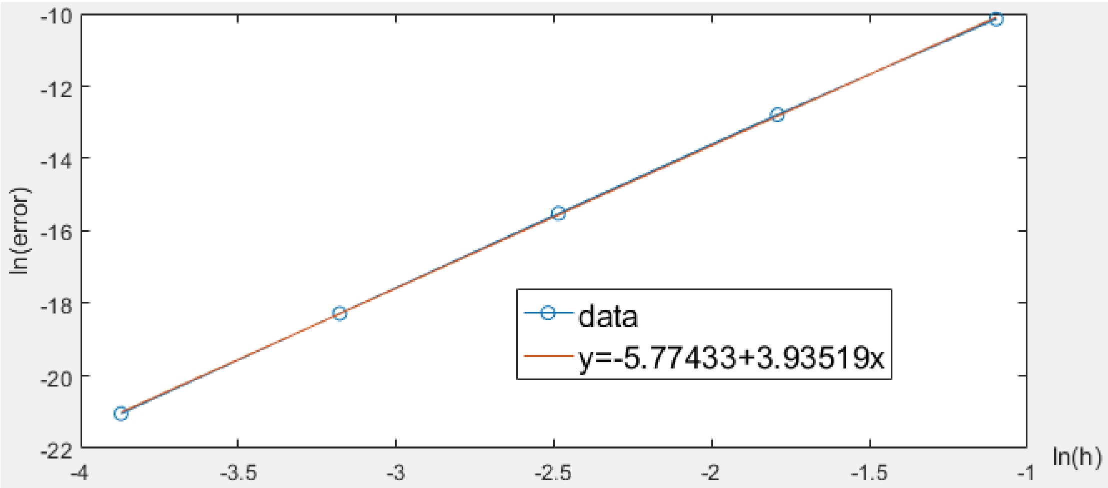
The exact value of the integral is

$$\frac{2}{3}\left(2^{\frac{3}{2}}-1\right)=1.218951416497460$$

Then we apply Simpson's 3/8 rule where N=1,2,4,8,16, we get

Exact value	1.218951416497460		
N		error	Relative error (e-04)
1	1.218912315464783	3.910103267634746e-05	0.320775973079391
2	1.218948613626460	2.802870999962792e-06	0.022994115778761
4	1.218951233411024	1.830864355678585e-07	0.001501999448788
8	1.218951404914901	1.158255891198223e-08	0.000095020677241
16	1.218951415771289	7.261711232331436e-10	0.000005957342626

We plot ln(error) against ln(h)



Which shows error is proportional to h^4

Code

Simpson3_8

```
function I=Simpson3_8(f,N,a,b)
% ***here 3N intervals totally***
%integrate using simpson method
%error proportional to h^5
%f is a handle of function   eg:f=@sin, f=@(x) sin(x)
%a,b:integral range
%n: how many segments [must be even]
x=linspace(a,b,3*N+1);
h=x(2)-x(1);
y=f(x);
w=9*h/8*ones(1,3*N+1);
w(1:3:end)=6*h/8;
w(1)=3*h/8;
w(end)=3*h/8;
I=sum(y.*w);
```

script:

```
ee=[3.910103267634746e-05,2.802870999962792e-06,1.830864355678585e-07
,1.158255891198223e-08,7.261711232331436e-10];
N=[1,2,4,8,16];
```

```
h=1./N/3;
ee=log(ee);
h=log(h);
[a0,a1]=Linearregression(h,ee,ones(1,5));
a0
a1
x=linspace(h(1),h(end),100);
y=a0+a1*x;
plot(h,ee,'-o');
hold on
plot(x,y);
```

linear regression

```
function [a0,a1]=Linearregression(X,Y,U)
%Linear regression, round-off error susceptible
S=sum(1./U./U);
Sx=sum(X./U./U);
Sy=sum(Y./U./U);
t=(X-Sx/S)./U;
Stt=sum(t.*t);
a1=sum(t.*Y./U)/Stt;
a0=(Sy-Sx*a1)/S;
u2a0=(1+Sx*Sx/S/Stt)/S;
u2a1=1/Stt;
Cov=-Sx/S/Stt;
r=Cov/sqrt(u2a0)/sqrt(u2a1);
Y1=a0+a1*X;
Chi2=sum((Y-Y1).*(Y-Y1)./U./U);
[a0,sqrt(u2a0),a1,sqrt(u2a1),Chi2,Cov,r];
```

(c) from the results of (b), we can see the required number N (when relative error is less than 5e-09) is somewhere between 8 and 16

And we just calculate all relative errors from N=8 to N=16, we find when N>=10, relative error begins to be lower than 5e-09

N	Relative error
8	9.502067724129323e-09
9	5.937251615421968e-09
10	3.897859676847526e-09

Script

```
exact=2/3*(power(2,3/2)-1);
f=@(x) sqrt(1+x);
for N=8:16
I=Simpson3_8(f,N,0,1);
(I-exact)/exact
end
```

3.

(a)we get the four roots are

$$x_3 = \sqrt{\frac{15 - 2\sqrt{30}}{35}}, x_2 = -\sqrt{\frac{15 - 2\sqrt{30}}{35}}$$

$$x_4 = \sqrt{\frac{15 + 2\sqrt{30}}{35}}, x_1 = -\sqrt{\frac{15 + 2\sqrt{30}}{35}}$$

And weights are

$$w_1 = w_4 = \frac{3\sqrt{30} - 5}{6\sqrt{30}}$$

$$w_2 = w_3 = \frac{3\sqrt{30} + 5}{6\sqrt{30}}$$

(b)using the substitution $x = \frac{b-a}{2}(t+1) + a$, we could convert the integral to

$$\int_a^b f(x)dx = \int_{-1}^1 f\left(\frac{b-a}{2}(t+1) + a\right)\frac{b-a}{2} \ dt$$

And just sum up the function values of 4 abscissas multiplied by respected weights

Code

```
function I=Gauss_quad_4(f,a,b)
% using 4 points Gassian Legrande method
% exact when polynomial's order <=7
% f: handle
% [a,b] interval
x=zeros(1,4);
w=zeros(1,4);
x(1)=-sqrt((15+2*sqrt(30))/35);
x(2)=-sqrt((15-2*sqrt(30))/35);
x(3)=-x(2);
x(4)=-x(1);
w(1)=(3*sqrt(30)-5)/6/sqrt(30);
w(4)=w(1);
w(2)=(3*sqrt(30)+5)/6/sqrt(30);
w(3)=w(2);
I=sum(f((b-a)/2*(x+1)+a)*(b-a)/2.*w);
```

(c)

```
>> format long
>> I=Gauss_quad_4(f,0,1)

I =

   1.218951433509519
```

Exact value: 1.218951416497460

It has attained the precision of N=8 in (b)’s 3/8 Simpson’s rule, which use 25 points, here only 4 point’s is required.