7.2 – Finance Applications of Fourier Transforms

7.2.1 Review of the Fourier transform

- The Fourier transform defined as follows has many application across many field, including Finance

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x} \; dx$$

- Here $i^2 = -1$ and is am imaginary number. With Fourier analysis, we will be working with complex numbers and complex arithmetric. In C++, we will use the complex<double> structure on a CPU or complex<float> for single precision.

- There are restriction on the functions $f(x)$ for which one can apply Fourier analysis, namely that f is integrable: $\int_{-\infty}^{\infty} |f(x)| dx < \infty$. For example, probability density functions satisfy this property for f.

- Recall the characteristic function.

$$\phi(u) = E\left(e^{iux}\right) = \int_{-\infty}^{\infty} f(x) e^{iux} dx$$

7.2.1 – Fourier Transforms (cont.)

- Or a more general representation for the characteristic function in Probability Theory

$$\phi(u) = E\left(e^{iux}\right) = \int_{\mathbb{R}} e^{iux} dF(x)$$

- One can apply the inverse transform to recover the density function from the characteristic function, if the density function exists.

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(u) e^{-iux} du$$

- There are models for which we can solve and get an analytic solution for the characteristic function even though there are no analytic solutions for the density function and the distribution function. And there are Fourier inversion formulas for calculating probability functions from the characteristic function that we can use in option pricing. We will come back to these later.

7.2.1 – Fourier Transforms (cont.)

- There is also a discrete version of the Fourier transform that is used for time series analysis

$$f_x(k) = \sum_{n=0}^{N-1} x(n)e^{-2\pi i \quad /N}$$

- This is normally calculated for $k = 0, 1, 2, \ldots.., {}^{N}/_{2}$. Note that $f_x(-k) = \overline{f_x(k)}$, which is the complex conjugate and the calculations for $k \geq {}^{N}/_{2}$ are repetitive

- For long time series, one can use the Cooley-Tukey fast Fourier transform to speed up the calculation of the Fourier transform. The fast Fourier transform is a recursive calculation that is not well suited for parallel processing.
- The discrete Fourier transform can be calculated directly for relatively short time series and the calculation is very fast. This is the case for most economic and financial time series. For time series with *N* much greater than 10,000, you should consider the fast Fourier transform. When running the **fast Fourier transform**, you do need to **set *N* to a power of 2**. This is done by padding the end of the time series with zero's to get *N* up to the next power of 2. This does alter the frequencies at which the Fourier transform is calculates. The discrete Fourier transform can be calculated in parallel on the GPU, but the calculation is still not as fast calculating the fast Fourier transform on a CPU. Nvidia has fast Fourier transform functions in the **cufft** library for Cuda C, but I do not recommend using it unless you already have your data in GPU device memory and you want to keep the calculations and results on the GPU. On the GPU, my preference is to calculate the discrete Fourier transform in parallel, and not use the FFT functions in the cufft library.

7.2.1 – Fourier Transforms (cont.)

- I have a sample of time of day observations for the S&P 500 index (SPX) taken at 15 minute intervals. The series runs from 2008 to 2016, and there are over 50,000 observations for open, high, low, and close. I pulled the data from Bloomberg, but it was tricky. Most Bloomberg subscriptions do not include access to time of day pricing.

- First, I looked at the change in the log of the SPX from one close to the next. This includes the close from the end of the day to the close of the first trading time interval from 9:30 to 9:45.

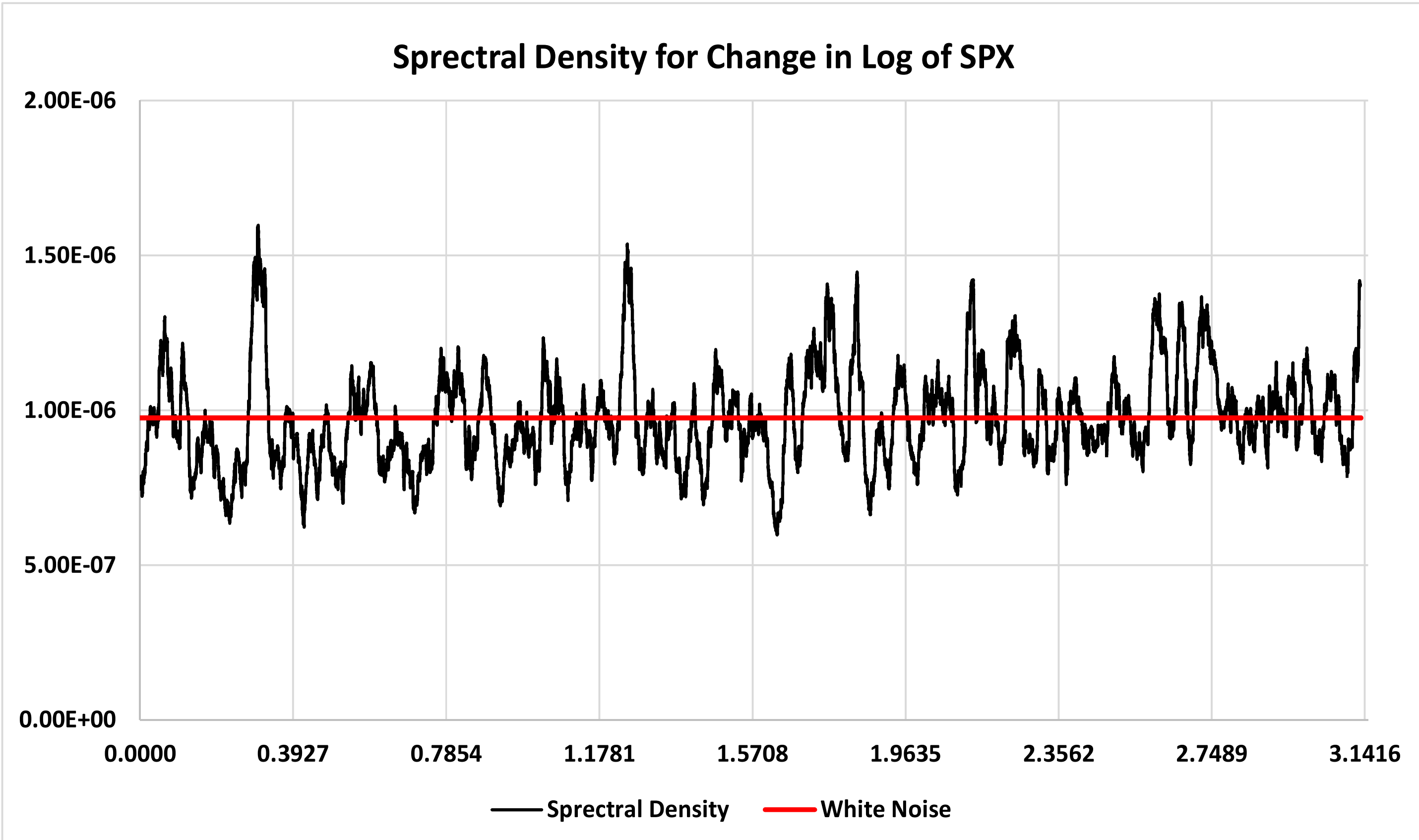
- Describe the estimator for the spectral density function. Fourier transform of the data → calculate the periodogram → then calculate an estimator for the spectral density function (simple averaging of the periodogram).

$$I(\lambda_k) = \frac{1}{2\pi N} f_x((\lambda_k)\, \overline{f_x((\lambda_k)}$$

- Include the spectral density function for white noise, which is flat across all frequencies

$$\frac{\sigma^2}{2\pi}$$

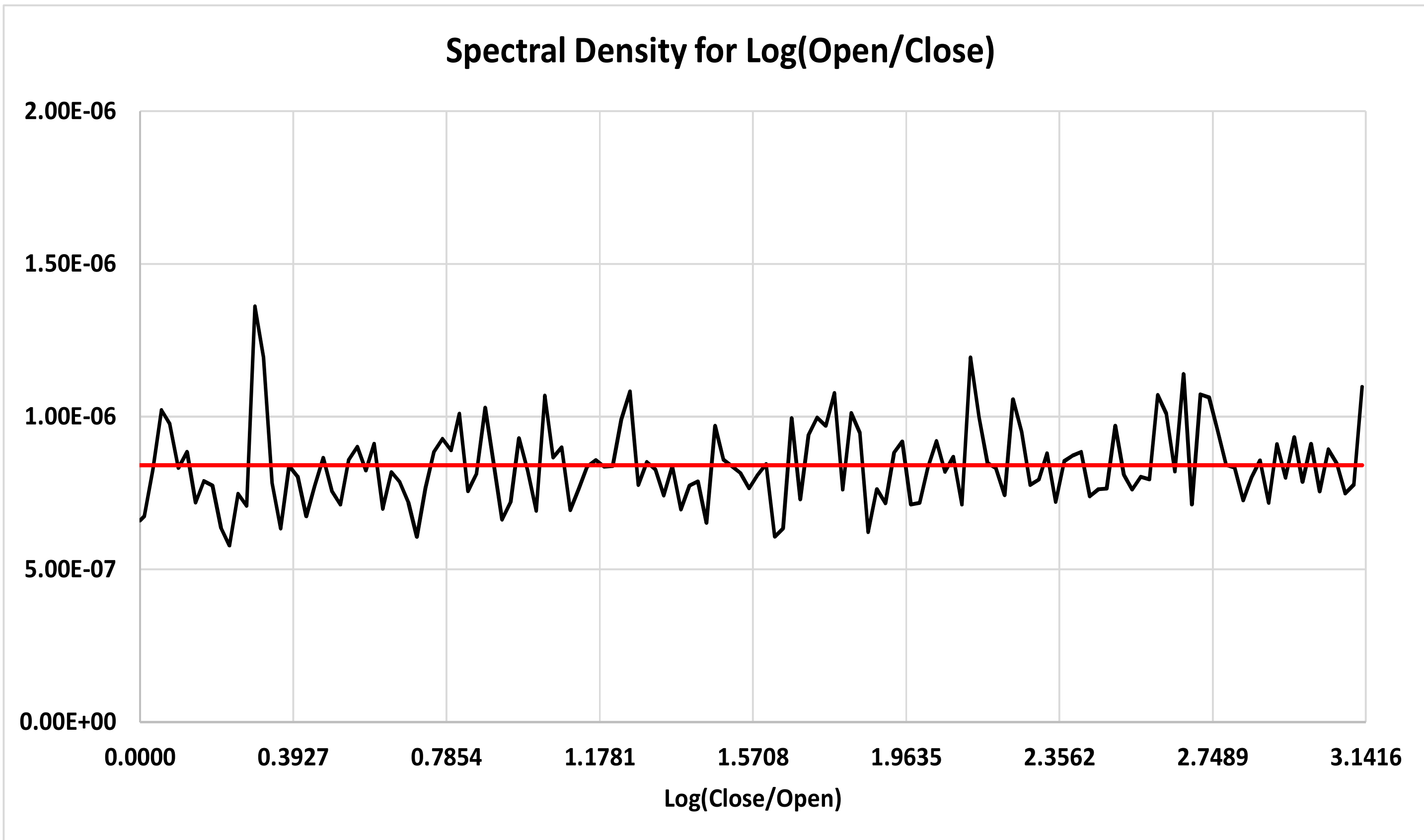
- Demonstrate your code to calculate Fourier transforms.


Sprectral Density for Change in Log of SPX
2.00E-06
1.50E-06
1.00E-06
5.00E-07
0.00E+00
0.0000
0.3927
0.7854
1.1781
1.5708
1.9635
2.3562
2.7489
3.1416
Sprectral Density
White Noise

7.2.1 – Fourier Transforms (cont.)

- Next, I looked at the change in the log of the price from open to close for each 15 minute time interval.

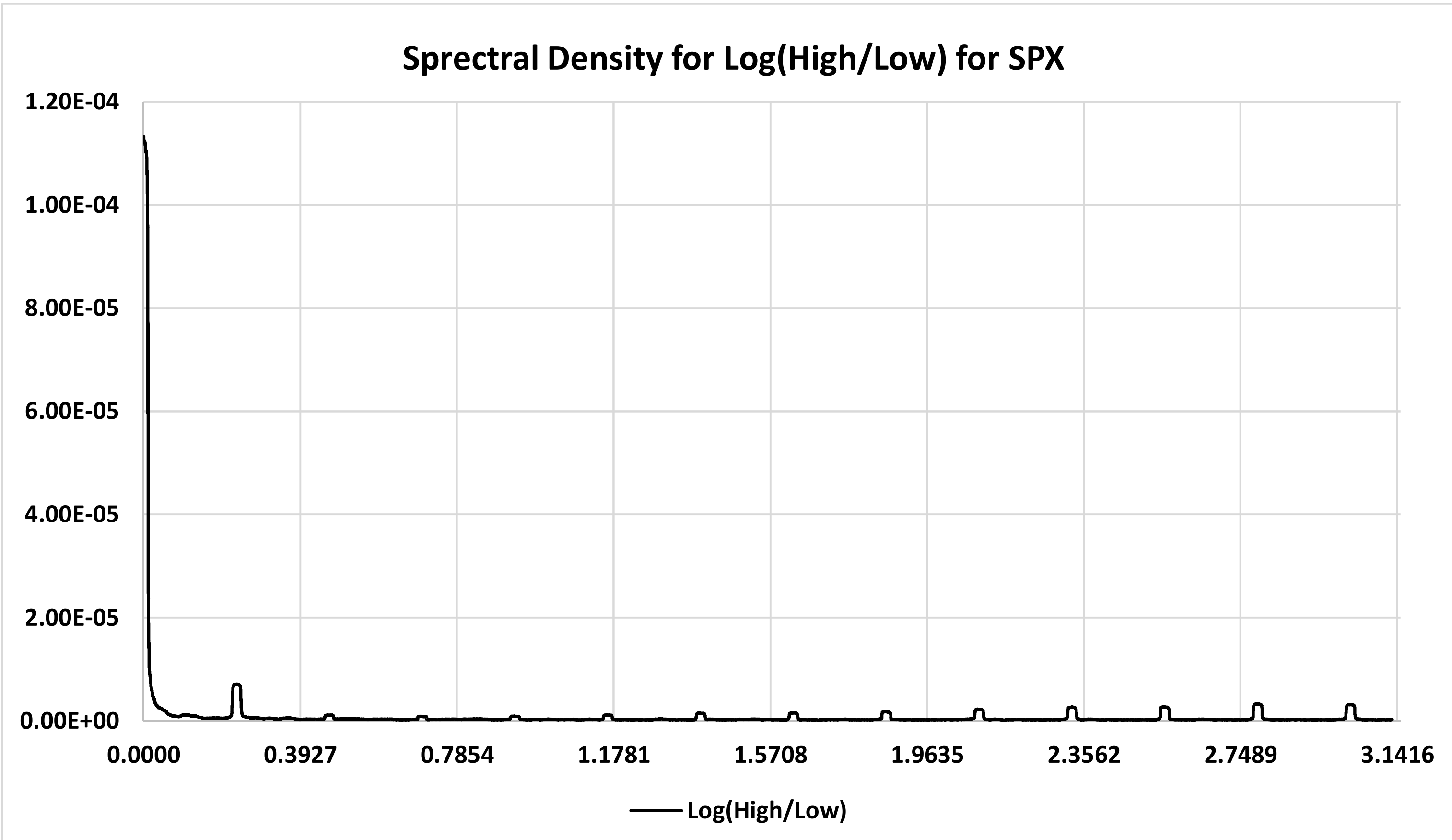
- The results are similar.

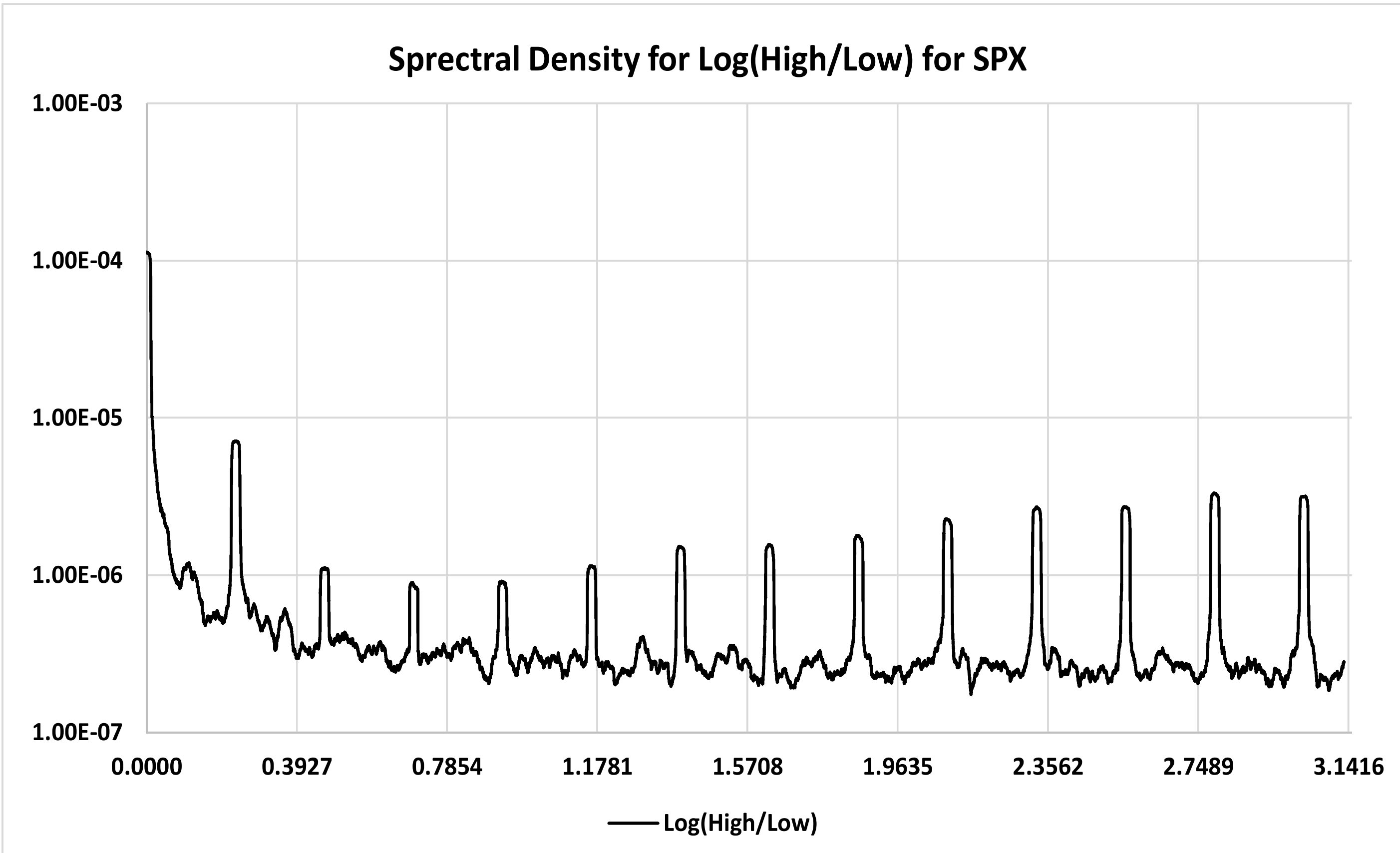

Spectral Density for Log(Open/Close)
2.00E-06
1.50E-06
1.00E-06
5.00E-07
0.00E+00
0.0000
0.3927
0.7854
1.1781
1.5708
1.9635
2.3562
2.7489
3.1416
Log(Close/Open)

7.2.1 – Fourier Transforms (cont.)

- I have also have data for the high and the low. It is well known in Finance that one can get a more efficient estimator of the volatility or the variance by using the high and low. Log (High/Low) is a function of the variance. There is an adjustment factor.

- I applied the Fourier analysis to Log (High/Close) to see if there any interesting patterns in the volatility.


Sprectral Density for Log(High/Low) for SPX
1.20E-04
1.00E-04
8.00E-05
6.00E-05
4.00E-05
2.00E-05
0.00E+00
0.0000
0.3927
0.7854
1.1781
1.5708
1.9635
2.3562
2.7489
3.1416
Log(High/Low)


Sprectral Density for Log(High/Low) for SPX
1.00E-03
1.00E-04
1.00E-05
1.00E-06
1.00E-07
0.0000
0.3927
0.7854
1.1781
1.5708
1.9635
2.3562
2.7489
3.1416
Log(High/Low)

7.2.1 – Fourier Transforms (cont.)

- There is a periodicity based on the observation that there are 27 time periods within each day at which the prices are sampled. The large drop as we move away from the zero frequency is consistent with a low order autoregressive process (1^{st} or 2^{nd} order autoregressive process).

- There are a variety of methods that one might use to model this volatility time series.

- The frequency domain is useful for seeing periodicities, and there are some statistical models that can be estimated using the data in the frequency domain.

- Of course, one can analyze the data in the time domain.

Table 3. Variance of the Change in the Log of SPX by Time of Day, 2008-2015

Variance of the Change in the Log of SPX by Time of Day					
Time Period	**Sum of Squares**	**Count**	**MSE**	**Root MSE**	**Percent**
09:30 - 09:45	0.1252891366	2,013	6.22400E-05	0.7889%	36.88%
09:45 - 10:00	0.0124720633	2,013	6.19576E-06	0.2489%	3.67%
10:00 - 10:15	0.0143330346	2,013	7.12024E-06	0.2668%	4.22%
10:15 - 10:30	0.0090921841	2,013	4.51673E-06	0.2125%	2.68%
10:30 - 10:45	0.0093405952	2,013	4.64014E-06	0.2154%	2.75%
10:45 - 11:00	0.0070829717	2,013	3.51861E-06	0.1876%	2.09%
11:00 - 11:15	0.0066341211	2,013	3.29564E-06	0.1815%	1.95%
11:15 - 11:30	0.0060102294	2,013	2.98571E-06	0.1728%	1.77%
11:30 - 11:45	0.0059695824	2,013	2.96552E-06	0.1722%	1.76%
11:45 - 12:00	0.0052354260	2,013	2.60081E-06	0.1613%	1.54%
12:00 - 12:15	0.0045948618	2,013	2.28259E-06	0.1511%	1.35%
12:15 - 12:30	0.0053584293	2,013	2.66191E-06	0.1632%	1.58%
12:30 - 12:45	0.0046327674	2,013	2.30142E-06	0.1517%	1.36%
12:45 - 13:00	0.0048344970	2,013	2.40164E-06	0.1550%	1.42%
13:00 - 13:15	0.0052992992	2,013	2.63254E-06	0.1623%	1.56%
13:15 - 13:30	0.0051887358	1,995	2.60087E-06	0.1613%	1.53%
13:30 - 13:45	0.0065054747	1,995	3.26089E-06	0.1806%	1.92%
13:45 - 14:00	0.0054008551	1,995	2.70720E-06	0.1645%	1.59%
14:00 - 14:15	0.0069076158	1,995	3.46246E-06	0.1861%	2.03%
14:15 - 14:30	0.0075389555	1,995	3.77893E-06	0.1944%	2.22%
14:30 - 14:45	0.0094437734	1,995	4.73372E-06	0.2176%	2.78%
14:45 - 15:00	0.0084236276	1,995	4.22237E-06	0.2055%	2.48%
15:00 - 15:15	0.0125847864	1,995	6.30816E-06	0.2512%	3.70%
15:15 - 15:30	0.0102452268	1,995	5.13545E-06	0.2266%	3.02%
15:30 - 15:45	0.0123103551	1,995	6.17060E-06	0.2484%	3.62%
15:45 - 16:00	0.0198292824	1,995	9.93949E-06	0.3153%	5.84%
16:00 - 16:15	0.0091184758	1,995	4.57066E-06	0.2138%	2.68%

Note: The analysis in this table is based on changes from close to close, the change for the 9:30 to 9:45 time period is from the close of the previous day to the price at 9:45. On 18 days, there was an early close at 13:15 (1:15 PM). Source: Bloomberg for the time period January 2008 to December 2015.

Table 4. A Decomposition of the Variance across the Opening Period, 9:30-9:45

	Sum of Squares	Count	MSE	Root MSE	Annualized
Previous close to 9:30 open	0.014188068	2013	7.0482E-06	0.2655%	4.21%
Change, 9:30 to 9:45	0.083693076	2013	4.1576E-05	0.6448%	10.24%
Previous close to 9:45	0.125289137	2013	6.2240E-05	0.7889%	12.52%

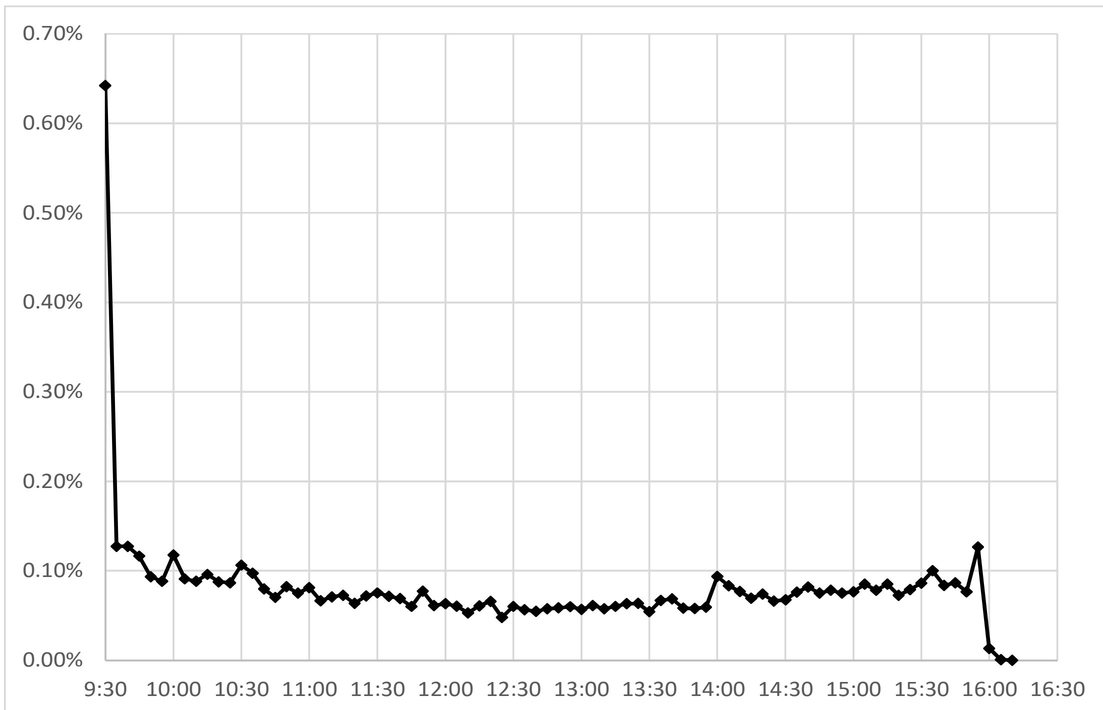
Table 5. SPX Variance across Trading Days, Weekends, and Holidays

All Days	Sum of Squares	Count	MSE	Root MSE	Annualized	% of Total Variance
Close to Open	0.006458369	6,549	9.86161E-07	0.0993%	1.58%	0.80%
Open to Close	0.802815237	6,549	0.000122586	1.1072%	17.58%	99.20%

Close to Close Variance Across	Sum of Squares	Count	MSE	Root MSE	Annualized	Ratio to Trading Day Variance
trading days	0.622386539	5,131	0.000121299	1.1014%	17.48%	
weekends	0.186700454	1,192	0.000156628	1.2515%	19.87%	1.291
holidays	0.007517008	61	0.000123230	1.1101%	17.62%	1.015
holiday-weekend	0.026525115	162	0.000163735	1.2796%	20.31%	1.349

7.2.1 – Fourier Transforms (cont.)

Standard Deviation of SPX over 5 minute intervals



7.2 – Finance Applications of Fourier Transforms (cont.)

7.2.2 Fourier Inversion Formulas for Probability Functions and Option Pricing

- Fourier inversion formulas for distribution functions from Feller, Volume II, Chapter XV. These are used in Finance to calculate probability functions for option pricing models.

$$\frac{F(x+h)-F(x)}{h} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(u)\frac{1-e^{-iuh}}{iuh}e^{-iux}du$$

- Note that if we take the limit as $h \to 0$, we have the probability density function as the inverse Fourier transform of the characteristic function. For a random variable that is nonnegative, and $F(0) = 0$, we can use this formula to calculate the distribution function. Let $x = 0$ and h be the point at which we want to evaluate the probability function. And replace h with x.

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \varphi(u)\frac{1-e^{-iux}}{iu}du$$

7.2.2 – Fourier Inversion Formulas (cont.)

- For random variables that can range from $-\infty$ to ∞, we need to use a modified formula.

$$F(x) \equiv \Pr(X \leq x) = \frac{1}{2} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\varphi(u)e^{-iux}}{iu} du$$

- For calculation purposes, we will use the following

$$F(x) = \frac{1}{2} + \frac{1}{2\pi} \int_{0}^{\infty} \frac{\varphi(-u)e^{iux} - \varphi(u)e^{-iux}}{iu} du = \frac{1}{2} - \frac{1}{\pi} \int_{0}^{\infty} \frac{\operatorname{Im}\left(\varphi(u)e^{-iux}\right)}{u} du$$

- References for the Fourier inversion formulas

Feller, William,, *An Introduction to Probability Theory and Its Applications*, Volume II, 1971.

Davies, R., "Numerical inversion of a characteristic function," *Biometrika* , 1973

Shephard, N.G., "Numerical Integration Rules for Multivariate Inversions," *Journal of Statistical Computation and Simulation*," 1991.

7.2.2 – Fourier Inversion Formulas (cont.)

- For the last formula, there is a requirement that the mean of the distribution exist. At 0, you will need a limit which is the mean of the distribution.

- How are these formulas useful in option pricing? With deterministic interest rates, the valuation for a European call options can be represented as follows

$$C(S(t),t;T,K) = e^{-r(T-t)}\hat{E}(\max[0,S(T)-K]) = e^{-r(T-t)}\int_{-\infty}^{\infty}\max[0,e^{x}-K]\,f(x)dx$$

- Here x is the log of the terminal stock price $S(T)$. We can derive the following representation.

$$C(S(t),t;T,K) = e^{-r(T-t)}\int_{\log K}^{\infty} e^{x}f(x)dx - e^{-r(T-t)}\int_{\log K}^{\infty} Kf(x)dx$$

- Using $S(t) = e^{-r(T-t)}\hat{E}\big(S(T)\big) = e^{-r(T-t)}\hat{E}(e^{x})$, we have

$$C(S(t),t;T,K) = S(t)\int_{\log K}^{\infty}\frac{e^{-r(T-t)}e^{x}f(x)}{S(t)}dx - e^{-r(T-t)}K\int_{\log K}^{\infty} f(x)dx$$

7.2.2 – Fourier Inversion Formulas (cont.)

- Note that

$$f^*(x) = \frac{e^{-r(T-t)} e^{x} f(x)}{S(t)}$$

- is also a probability density function, just a modified version of the risk neutral density function. And note that the structure of this model is similar to the Black-Scholes model. All European call (put) option pricing functions can be expressed in this form, and solving the model comes down to being able to solve and calculate the two distribution functions. Note that the characteristic function for $f^*(x)$ can be derived as follows

$$\varphi^*(u) = \int_{-\infty}^{\infty} e^{iux} f^*(x) dx = \int_{-\infty}^{\infty} \frac{e^{-r(T-t)} e^{iux+x} f(x)}{S(t)} dx = \frac{e^{-r(T-t)} \varphi(u-i)}{S(t)}$$

- where $\varphi(u)$ is the characteristic function for the risk neutral distribution. Finally, we can rewrite the solution as follows

$$C(S(t),t;T,K) = S(t)\,(1 - F^*(\log K)) - e^{-r(T-t)} K\,(1 - F(\log K))$$

$$C(S(t),t;T,K) = S(t)\left(\frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \frac{\operatorname{Im}\left(\varphi^*(u) e^{-iu\log K}\right)}{u} du\right) - e^{-r(T-t)} K\left(\frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \frac{\operatorname{Im}\left(\varphi(u) e^{-iu\,\mathrm{lo}}\right)}{u} du\right)$$

7.2.2 – Fourier Inversion Formulas (cont.)

- Let's apply this Fourier method for European options in the Heston model as an example and follow it through the programming of the calculations. Recall the PDE for the Heston model.

$$0 = \frac{\partial V}{\partial t} + \tfrac{1}{2}v\frac{\partial^2 V}{\partial (\log S)^2} + \tfrac{1}{2}\sigma^2 v\frac{\partial^2 V}{\partial v^2} + \rho\sigma v\frac{\partial^2 V}{\partial(\log S)\partial v} + \left[r - \tfrac{1}{2}v\right]\frac{\partial V}{\partial \log S} + \kappa[\theta - v]\frac{\partial V}{\partial v} - rV$$

- The density function, the distribution function, and the characteristic function each must satisfy a similar PDE, but with different boundary conditions.

$$0 = \frac{\partial \varphi}{\partial t} + \tfrac{1}{2}v\frac{\partial^2 \varphi}{\partial (\log S)^2} + \tfrac{1}{2}\sigma^2 v\frac{\partial^2 \varphi}{\partial v^2} + \rho\sigma v\frac{\partial^2 \varphi}{\partial(\log S)\partial v} + \left[r - \tfrac{1}{2}v\right]\frac{\partial \varphi}{\partial \log S} + [\kappa\theta - (\kappa + \lambda)v]\frac{\partial \varphi}{\partial v}$$

with the boundary condition that as $t \to T$, $\varphi(u) = \exp(iu\log S(t))$

7.2.2 – Fourier Inversion Formulas (cont.)

- To solve for the characteristic function, let's try a guess that it is exponential affine and see if it works

$$\varphi(u) = \exp\{a(u,t) + b(u,t)\log S(t) + c(u,t)v(t)\}$$

- The boundary conditions are $a(u,T) = c(u,T)$ = 0 and $(u,T) = iu$. Apply partial derivatives using the proposed exponential affine solution.

$$0 = \varphi\left(\frac{\partial a}{\partial t} + \frac{\partial b}{\partial t}\log S + \frac{\partial c}{\partial t}v \; + \tfrac{1}{2}vb^2 \; + \tfrac{1}{2}\sigma^2 vc^2 + \rho\sigma vbc + \left[r - \tfrac{1}{2}v\right]b + \; [\kappa\theta - (\kappa + \lambda)v]c\right)$$

- We now have a system of ODE's to solve

$$\log S: \quad 0 = \frac{\partial b}{\partial t}$$

$$v: \quad 0 = \frac{\partial c}{\partial t} + \tfrac{1}{2}b^2 \; + \tfrac{1}{2}\sigma^2 c^2 + \rho\sigma bc - \tfrac{1}{2}b - \; (\kappa + \lambda)c$$

$$\text{Constant:} \qquad 0 = \frac{\partial a}{\partial t} + rb + \; \kappa\theta c$$

7.2.2 – Fourier Inversion Formulas (cont.)

- The solutions to the coefficients in the exponential affine solution are as follows.

$$b(u,t) = iu$$

$$c(u,t) = \frac{(\kappa+\lambda) - \rho\sigma iu + d}{\sigma^2}\left(\frac{1-e^{d(T-t)}}{1-ge^{d(T-t)}}\right)$$

$$a(u,t) = riu(T-t) + \frac{\kappa\theta}{\sigma^2}\left[\big((\kappa+\lambda) - \rho\sigma iu + d\big)(T-t) - 2\log\left(\frac{1-ge^{d(T-t)}}{1-g}\right)\right]$$

$$d = \sqrt{(\rho\sigma iu - (\kappa+\lambda))^2 + \sigma^2(iu+u^2)} \text{ and}$$

$$g = \frac{(\kappa+\lambda) - \rho\sigma iu + d}{(\kappa+\lambda) - \rho\sigma iu - d}$$

7.2.2 – Fourier Inversion Formulas (cont.)

- To summarize, the solution for the European call option pricing function in the Heston model is calculated as follows.

$$C(S(t), t; T, K) = S(t)\,(1 - F^*(\log K)) - e^{-r(T-t)} K\,(1 - F(\log K))$$

$$C(S(t), t; T, K) = S(t)\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\left(\varphi^*(u) e^{-iu\,\mathrm{lo}}\quad\right)}{u}\,du\right)$$
$$- e^{-r(T-t)} K \left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \frac{\mathrm{Im}\left(\varphi(u) e^{-iu\,\mathrm{lo}}\quad\right)}{u}\,du\right)$$

- where $\varphi(u) = \exp\{a(u,t) + b(u,t)\log S(t) + c(u,t)v(t)\}$, with the formulas for the coefficients in this exponential affine solution for the characteristic function given in previous 2 slides.

$$\varphi^*(u) = \frac{e^{-r(T-t)}\varphi(u - i)}{S(t)}$$

7.2.2 – Fourier Inversion Formulas (cont.)

- This gives us the characteristic function for the risk neutral distribution for $\log S(T)$. Just plug into the solution above for the call option pricing function. Now, how does one calculate the integrals for the Fourier inversion formulas? There is an old literature in Mathematical Statistics that provides the numerical methods with precise controls over the approximation error. This is covered in the papers by Davies and Shephard.

- We approximate the following integral with a numerical integration (z = 0 or ½)

$$F(x) + \sum_{j=1}^{\infty} \cos(2\pi z j)\left[\Pr\left(X < x - \tfrac{2\pi j}{h}\right) - \Pr\left(X > x + \tfrac{2\pi j}{h}\right)\right] = \frac{1}{2} - \frac{1}{\pi}\sum_{j=0}^{\infty} \frac{\operatorname{Im}\left(\varphi(h(z+j))e^{-ih(z+j)x}\right)}{(z+j)}$$

- The error is bounded by setting the grid size h so that the probabilities in the bracketed term are probabilities from the tails of the distribution. Let x_L be the extreme point in the left tail of the distribution so that $\Pr(X < x_L) < \varepsilon$, and let x_U be the extreme point in the right tail of the distribution so that $\Pr(X > x_U) < \varepsilon$, where ϵ is the approximation error.

$$h = \max\left(\frac{2\pi}{x - x_L}, \frac{2\pi}{x_U - x}\right)$$

- It is surprising that the grid size is not very small for many of the numerical integrations. Finally, we need to set a limit, N, on the number of terms to calculate in the infinite summation.

7.2.2 – Fourier Inversion Formulas (cont.)

- To set a limit, N, on the number of terms to calculate in the infinite summation.

$$\frac{1}{2} - \frac{1}{\pi}\sum_{j=0}^{N} \frac{\mathrm{Im}\left(\varphi(h(z+j))e^{-ih(z+j)x}\right)}{(z+j)}$$

- Set N so that $|\varphi(h(z+N))|$ is less than ε, the desired approximation error. I typically set this at 10^{-1} with double precision. This is univariate numerical integration and really does not require a GPU. Of course, one could calculate each term in the summation in parallel and then sum.

- Finish with am example using the Duffie, Pan, and Singleton model. Duffie, Pan, and Singleton took the Heston model and added a simultaneous jump to the stock price and the volatility. Example is a calibration during the financial crisis in 2008.

- And note the sign change in the option pricing solution to go from $F(x)$ to $\left(1 - F(x)\right)$.

7..2.3 References for Lecture 7.2 – Finance Applications of Fourier Transforms

Carr, P., and D. Madan, "Option valuation using the fast Fourier transform," *Journal of Computational Finance* 3 (1999), 463-520.

Duffie, D., J. Pan, and K. Singleton, "Transform Analysis and Asset Pricing for Affine Jump-Diffusions," *Econometrica* 68(Nov. 2000), 1343-76.

Lee, R.W., "Option Pricing by Transform Methods: Extensions, Unification, and Error Control," *Journal of Computational Finance* 7 (Spring 2004), 51-86.

Scott, L. "Pricing Stock Options in a Jump-Diffusion Model with Stochastic Volatility and Interest Rates: Application of Fourier Inversion Methods," *Mathematical Finance* 7 (1997), 345-358.