General Topology: final

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5. Bases -I

(a) **(Def: bases)** Let X be a set. A collection $\mathcal{B}$ of subsets of X is a basis if the following holds:

1. For any $x \in X$, there is $B \in \mathcal{B}$ such that $x \in \mathcal{B}$.
2. If $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$, then there is $B_3 \in \mathcal{B}$ such that $x \in B_3 \subset B_1 \cap B_2$.

(b) **(Lemma)** For a basis $\mathcal{B}$, let $\mathcal{T} = \{U \subset X \mid \text{ for any } x \in U, \text{ there is } B_x \in \mathcal{B} \text{ s.t. } x \in B_x \subset U\}$.

1. $\mathcal{T}$ is a topology.
2. $U \in \mathcal{T}$ iff $U = \bigcup_{\alpha \in A} B_\alpha$ for some $\{B_\alpha\}_{\alpha \in A} \subset B$.

6. Bases -II

(a) **(Lemma)** Let X be a topological space, and $\mathcal{B}$ be a collection of open sets of X. Suppose, for any open set U of X and for any $x \in U$, there is $B \in \mathcal{B}$ s.t. $x \in B \subset U$. Then $\mathcal{B}$ is a basis, which generates the topology of X.

(b) **(Def: subbase)** A subbase for a topology on a set X is a collection $\mathcal{S}$ of subsets of X s.t. $\bigcup_{S \in \mathcal{S}} S = X$.

- Let $\mathcal{B} = \{\text{all finite intersections of elements of } \mathcal{S}\} = \{S_1 \cap \cdots \cap S_n \mid S_i \in \mathcal{S}\}$. Then $\mathcal{B}$ is a basis.

7. Continuous functions

(a) **(Thm)** Suppose $\mathcal{B}$ is a basis for the topology of Y. Then a function $f : X \to Y$ is continuous iff $f^{-1}(B)$ is open for all $B \in \mathcal{B}$.

8. Order topology

(a) **(Def)** Suppose X has a simple order, and let $\mathcal{B} = \{(a,b) \mid a < b\} \cup \{[a_0, b) \mid a_0 \text{ is the smallest elt in } X\} \cup \{(a, b_0] \mid b_0 \text{ is the largest elt in } X\}$. Then $\mathcal{B}$ is a basis, and we call the topology generated by $\mathcal{B}$ as the order topology on X.

9. Product topology

(a) **(Def)** Suppose X and Y are topological spaces. The product topology on the set $X \times Y$ is the topology generated by the basis $\mathcal{B} = \{U \times V \mid U \in X \text{ is open}, V \in Y \text{is open}\}$.

(b) **(Thm)** Suppose $\mathcal{B}$, $\mathcal{C}$ are bases for X, Y respectively. Then $\mathcal{D} = \{B \times C \mid B \in \mathcal{B}, C \in \mathcal{C}\}$ is a basis for the product topology on $X \times Y$.

(c) **(Thm)** $\mathcal{S} = \{\pi_1^{-1}(U) \mid U \subset X \text{ is open}\} \cup \{\pi_2^{-1}(V) \mid V \subset Y \text{ is open}\}$ is a subbase for the product topology.

(d) **(Thm)**

1. The product topology is the coarsest(minimal) topology s.t. π_1, π_2 are continuous.
2. $f : Z \to X \times Y$ is continuous $\Leftrightarrow$ $\pi_1 \circ f : Z \to X$ and $\pi_2 \circ f : Z \to Y$ are continuous.

10. Subspace topology

(a) **(Def)** If $X, \mathcal{T}$ is a topological space and Y is a subset of X, then $\mathcal{T}_Y = \{U \cap Y \mid U \in \mathcal{T}\}$ is a topology on Y, called the subspace topology.

(b) **(Lemma)** If $\mathcal{B}$ is a basis for X, $\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$ is a basis for (the subspace topology of Y).

(c) **(Lemma)** Suppose $Y \subset X$ is an open set. Then $U \subset Y$ is open in Y iff U is open in X.

(d) **(Thm)** Suppose $A \subset X$, $B \subset Y$ are subspaces of topological spaces X, Y. Then [the product topology on $A \times B$]=[the subspace topology on the subset $A \times B$ of the product space $X \times Y$].

11. Closedness, closure and interior

(a) **(Thm)** Suppose $A \subset Y \subset X$, Y is closed in X. Then A is closed in Y iff A is closed in X.

(b) **(Def)** Suppose $A \subset X$.

1 int A :=union of all open sets of X contained in A= $\{x \in X \mid^{\exists} \text{nbhd } U \text{ of } x \text{ s.t. } U \subset A\}$.

2 $\overline{A}$ = intersection of all closed sets of X containing A= $\{x \in X \mid^{\forall} \text{nbhd } U \text{ of } x \text{ in } X, U \cap A \neq \emptyset, i.e.\ U \not\subset X - A\}$.

(c) $X - \overline{A} = \text{int}(X - A)$.

12. Limits and Hausdorff Spaces

(a) **(Def)** Suppose $A \subset X$. $x \in X$ is a **limit point** of A if any neighborhood of x contains a point in A which is not equal to x. i.e. $x \in \overline{A - \{x\}}$. For a sequence $\{x_n\}$ in X and $x \in X$, we say that $\{x_n\}$ **converges** to x if for any nbhd U of x, there exists N s.t. $x_n \in U$ whenever $n \geq N$.

(b) **(Thm)** If $x_n \in A$ for all n and $x_n \to x$, then $x \in \overline{A}$. (The converse holds in a metrizable space.)

(c) **(Thm)** X is Hausdorff $\Rightarrow$ any sequence $\{x_n\}$ in X converges at most one point in X.

13. Continuity and closedness

(a) **(Thm)**: TFAE

1 $f : X \to Y$ is continuous.

2 $f(\overline{A}) \subset \overline{f(A)}$ for every subset $A \subset X$.

3 $f^{-1}(B)$ is closed for every closed subset $B \subset Y$.

4 For each $x \in X$ and for each nbhd V of $f(x)$ in Y, there exists a nbhd U of x in X s.t. $f(U) \subset V$.

(b) ****(Thm: Pasting/glueing lemma)**

1 Suppose $X = A_1 \cup A_2 \cup \cdots \cup A_n$ each A_i closed in X and $f_i : A_i \to Y$ is continuous for every $i = 1, 2, \ldots, n$. If $f_i(x) = f_j(x)$ whenever $x \in A_i \cap A_j$, then $f : X \to Y$ given by $f(x) = f_i(x)$ for $x \in A_i$ is well-defined and continuous.

2 Suppose $X = \bigcup_\alpha A_\alpha$ each A_i closed in X and $f_\alpha : A_\alpha \to Y$ is continuous for each α. If $f_\alpha(x) = f_\beta(x)$ whenever $x \in A_\alpha \cap A_\beta$, then $f : X \to Y$ given by $f(x) = f_\alpha(x)$ for $x \in A_\alpha$ is well-defined and continuous.

14. Infinite product spaces -I

(a) **(Def: Box topology on X)**:=the topology generated by the basis $\{\prod_\alpha U_\alpha \mid$ each U_α is open set in $X_\alpha\}$.

(b) **(Def: Product topology on X)**:=the topology generated by the subbasis $\mathcal{S} = \{\pi_\alpha^{-1} \mid \alpha \in J, U_\alpha \subset X_\alpha$ open$\}$ where $\pi_\beta : \prod_\alpha X_\alpha \to X_\beta$ is the projection defined by $\pi_\beta((x_\alpha)_\alpha) = x_\beta$. Then $\mathcal{B} = \{\prod_{\alpha \in J} U_\alpha \mid U_\alpha \subset X_\alpha$ open for all $\alpha, U_\alpha = X_\alpha$ for all α but finitely many$\}$.

(c) **(Thm holds for both)**

1 Suppose $A_\alpha \subset X_\alpha$ is a subspace for each α. Then $\prod_\alpha A_\alpha$ is a subspace of $\prod_\alpha X_\alpha$.

2 If each X_α is Hausdorff, then $\prod_\alpha X_\alpha$ is Hausdorff.

3 Suppose $A_\alpha \subset X_\alpha$ is a subset for each α. Then $\overline{\prod_\alpha A_\alpha} = \prod_\alpha \overline{A_\alpha}$ in $\prod_\alpha X_\alpha$.

15. Infinite product spaces -II

(a) **Thm for Product topology**

1 The product topology on $\prod_\alpha X_\alpha$ is the smallest topology for which the projection $\pi_\beta : \prod_\alpha X_\alpha \to X_\beta$ is continuous for all β.

2 A function $f : Y \to \prod_\alpha X_\alpha$ is continuous iff $\pi_\alpha \circ f : Y \to X_\alpha$ is continuous.

16. Metrizable Spaces -I

(a) **(Def)** A space X is metrizable if there is a metric d on X which induces the topology of X.

(b) **(Thm)** Suppose d and d' are metrices on X inducing topology $\mathcal{T}, \mathcal{T}'$ respectively. Then $\mathcal{T} \subset \mathcal{T}' \Leftrightarrow {}^{\forall}x \in X$ and $\forall \epsilon > 0$, ${}^{\exists}\delta > 0$ such that $B_{d'}(X, \delta) \subset B_d(X, \epsilon)$.

$\therefore$ ${}^{\exists}c$ s.t. $d'(x, y) \geq c \cdot d(x, y)$, ${}^{\forall}x, y \to \mathcal{T} \subset \mathcal{T}'$.

(c) **(Thm)** Let d : Euclidean metric, ρ : square metric ($\rho(x, y) = \max\{d_i(x_i, y_i)\}$). Both d, ρ induce the product topology (on finite product spaces).

(d) **(Def: bounded metric)** A metric d on X is bdd if ${}^{\exists}M > 0$ s.t. $d(x, y) \leq M$ for ${}^{\forall}x, y \in X$. For a metric d on X, define $\overline{d} : X \times X \to \mathbb{R}$ by $\overline{d}(x, y) = \min\{d(x, y), 1\}$. $\overline{d}$ is a metric, which is called the **standard bounded metric** (associated with d).

(e) **(Thm)** $\overline{d}$ and d induce the same topology.

(f) **(Def: Uniform metric)** Suppose $X = \prod_{\alpha \in J} X_\alpha$, (X_α, d_α) a metric space $d_\alpha(x_\alpha, y_\alpha) \leq M$. ${}^{\forall}\alpha$, $x_\alpha \in X_\alpha$, $y_\alpha \in X_\alpha$. Define $\overline{\rho}(x, y) = \sup_{\alpha \in J}\{d_\alpha(x_\alpha, y_\alpha)\}$. Then $\overline{\rho}$ is a metric on X, called the uniform metric. The metric topology induced by $\overline{\rho}$ is called the uniform topology.

(g) **(Thm)** (product topology) $\subset$ (uniform topology) $\subset$ (box topology).

17. Metrizable Spaces -II

(a) **(Def: for Countable infinite product case)** Define $D(x, y) = \sup_i\{\frac{\overline{d_i}(x_i, y_i)}{i}\}$.

(b) **(Thm)** D induces the product topology on X.

(c) **(prop)**

1 A product of countable many metrizable spaces is metrizable.

2 A subspace of metrizable space is metrizable.

3 A metrizable space is Hausdorff.

(d) **(Thm)** $f : X \to Y$ is continuous $\Rightarrow$ Whenever $x_n \to x$ in X, $f(x_n) \to f(x)$ in Y. The converse holds if X is metrizable.

(e) **(Def: Uniformly continuous)** Suppose X is a space and (Y, d) is a metric space. A sequence $\{f_n\}$ of functions $f_n : X \to Y$ uniformly converges to $f : X \to Y$ if $\epsilon > 0$, ${}^{\exists}N$ s.t. $d(f_n(x), f(x))$ for all $x \in X$ and all $n \geq N$.

(f) $f_n \to f$ uniformly and each f_n is continuous $\Rightarrow$ f is continuous.

18. Quotient spaces -I

(a) **(Def)** A function $p : X \to Y$ between spaces X and Y is a quotient map if p is surjective, and for every $U \subset Y$, $p^{-1}(U)$ is open iff U is open (which implies p is continuous).

(b) **(Def)** Suppose $\sim$ is an equivalence relation on a space X. For $x \in X$, let $[x] = \{y \in X \mid x \sim y\}$, the equivalence class of x. Let $X^* = \{[x] \mid x \in X\}$, the set of equivalence classes. $p : X \to X^*$, $p(x) = [x]$. Then X^* equipped with the quotient topology is a quotient space of X.

19. Quotient spaces -II

(a) **(Thm)** Suppose $p : X \to Y$ is a quotient map, $g : X \to Z$ is a function, $g(x) = g(x')$, whehever $p(x) = p(x')$($i.e.$ on each $p^{-1}(y)$, g is constant). Then

1 there is a unique function $f : Y \to Z$ such that $f \circ p = g$.

2 f is continuous iff g is continuous.

3 f is a quotient map iff g is a quotient map.

(b) **(Coro)** Suppose $g : X \to Z$ is surjective and continuous. Let $X^* = \{g^{-1}(z) \mid z \in Z\}$, as a quotient space of X.

1 g induces a bijective continuous map $f : X^* \to Z$. Morever, f is a homeomorphism iff g is a quotient map.

2 If Z is a Hausdorff, then X^* is Hausdorff.

20. Connectedness

(a) **(Def)** A separation of a given space X is a pair U, V of nonempty open subsets of X s.t. $U \cap V = \emptyset$, $U \cup V = X$. X is connected if there is no separation of X. $i.e.$ whenever $U \subset X$ is open and closed, either $U = \emptyset$ or X.

(b) **(Lemma)** C, D forms a separation of X, $Y \subset X$ is connected. $\Rightarrow Y \subset C$ or $Y \subset D$.

(c) **(Thm)**

1 $A_\alpha \subset X$, $\bigcup_\alpha A_\alpha \neq \emptyset$, each A_α is connected $\Rightarrow \bigcup_\alpha A_\alpha$ is connected.

2 $A \subset X$, A is connected, $A \subset B \subset \overline{A}$. $\Rightarrow B$ is connected.

3 $f : X \to Y$ continuous, X is connected $\Rightarrow f(X)$ is connected.

4 X, Y are continuous. $\Rightarrow X \times Y$ is connected.

(d) **(Intermediate Value thm)** If $f : X \to \mathbb{R}$ is continuous, X is connected, and $y \in \mathbb{R}$ is between $f(a)$ and $f(b)$ for some $a, b \in X$, then $y = f(c)$ for some $c \in X$.

21. Path connectedness and components

(a) **(Thm)**

1 A path connected space is connected.

2 $f : X \to Y$ continuous, X path continuous $\Rightarrow f(X)$ path connected.

3 Any convex subset of $\mathbb{R}^n$ is path connected.

(b) **(example)** Topologist's sine curve is connected but not path connected.

(c) **(Def: (path) Component)** Define $\sim$ on X by $x \sim y$ iff $x, y \in A$ for some (path) connected subspace A of X. An equivalence class of $\sim$ is called a (path) component of X.

(d) **(Thm)**

1 Each (path) component of X is connected, disjoint (path) components are disjoint, and X is the union of its (path) components.

2 A nonempty (path) connected subset of X is contained in exactly one (path) component.

22. Local connectedness

(a) **(Def)** X is locally connected at $x \in X$ if for any nbhd V of x in X, there exists a connected nbhd U of x s.t. $x \in U \subset V$.

1 X is **locally connected** if it is locally connected at x for every $x \in X$.

2 X is **locally path connected** at $x \in X$ if for any nbhd V of x in X, there exists a path connected neighborhood U of x s.t. $x \in U \subset V$.

3 X is locally path connected if X is locally path connected at every $x \in X$.

(b) **(prop)** X is locally path connected $\Rightarrow$ X is locally connected.

(c) **(Thm)** X is locally (path) connected iff for any open set U in X, each (path) component of U is open (and closed).

(d) **(Coro)** If X is locally (path) connected, every (path) component of X is open.

(e) **(Thm)** If X is locally path connected, then path components of X and components of X are the same. Especially, if X is locally path connected, X is connected $\Leftrightarrow$ X is path connected.

23. Compactness -I

(a) **(Thm)**

1 X is compact, $Y \subset X$ is closed $\Rightarrow$ Y is compact.

2 X is Hausdorff, $Y \subset X$ is compact $\Rightarrow$ Y is closed.

3 X is compact, $f : X \to Y$ is continuous. $\Rightarrow$ $f(X)$ is compact.

4 $f : X \to Y$ is a continuous bijection , X is compact, Y is Hausdorff. $\Rightarrow$ f is a homeomorphism.

5 X, Y are compact $\Rightarrow$ $X \times Y$ is compact.

6 $[0, 1]$ is compact.

(b) ****(Lemma)** X is Hausdorff, Y is compact. $\Rightarrow$ $^\exists$ disjoint nbhds U, V in X s.t. $x \in U$, $Y \subset V$.

(c) **(Tube lemma)** Suppose $x_0 \times Y \subset N \subset X \times Y$, N is open. If Y is compact, then $^\exists$nbhd U of x_0 in X s.t. $x_0 \times Y \subset U \times Y \subset N$.

(d) **(Extreme value theorem)** $f : X \to Y$ continuous, X compact, Y a simply ordered set (with order topology) $\Rightarrow$ $^\exists a, b \in X$ s.t. $f(a) \le f(x) \le f(b)$ $^\forall x \in X$.

24. Compactness -II

(a) **(Def)** A collection $\mathcal{C}$ of subsets of X has finite intersection property (FIP) if for every finite subcollection $\{C_1, C_2, \cdots, C_n\} \subset \mathcal{C}$, $\bigcap_{i=1}^n C_i \neq \emptyset$.

(b) **(Thm)** X is compact $\Leftrightarrow$ $^\forall$ collection $\mathcal{C}$ of closed subsets of X with FIP, $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

(c) **(Def)** $x \in X$ is an isolation point if $\{x\}$ is open in X.

(d) **(Thm)** X is nonempty, compact, Hausdorff and has no isolation point. $\Rightarrow$ X is uncountable.

(e) ****(Heine-Borel Thm)** $A \subset \mathbb{R}^n$ is compact iff A is closed and bounded w.r.t. the standard metric.

(f) ****(Lebesgue lemma)** Supppose $\mathcal{A}$ is an open cover a metric space (X, d) and X is compact. Then $^\exists \epsilon > 0$ s.t. for every $x \in X$, $B(x, \epsilon)$ is contained in some element of $\mathcal{A}$.

(g) **(Thm)** X is compact $\Rightarrow$ every infinite subset of X has a limit point. (X satisfying this property is called limit point compact.)

(h) **(Def)** X is sequetially compact if every sequence in X has a convergent sequence.

(i) **(Thm)** Suppose X is metrizable. Then TFAE:

1 X is compact.

2 X is limit point compact.

3 X is sequentially compact.

25. Local compactness

(a) **(Def)** X is locally compact if for every $x \in X$, there is a compact set $C \subset X$ containing a nbhd of x.

(b) **(prop)** If X is Hausdorff, then X is locally compact $\Leftrightarrow$ for every $x \in X$ and for every nbhd U of x, there is a nbhd of V of x s.t. $\overline{V}$ is compact and $\overline{V} \subset U$(x has compact nbhd).

(c) ****(Thm)** X is locally compact Hausdorff iff there is a space Y satisfying

1 X is a subspace of Y.

2 $Y \setminus X$ consists a single point. *i.e.* $Y = X \cup \{\infty\}$.

3 Y is compact Hausdorff.

- Furthermore, if Y' is another such space, then there exists a homeomorphism $h : Y \to Y'$ s.t. $h|_X = id_X$. Such a space Y is called a **one-point compactification** of X. (Define $\mathcal{T}_Y = \{U \mid U \subset X \text{ is open in } X\} \cup \{Y - C \mid C \subset X \text{ is compact}\}$.)

26. Tychonoff Thm

(a) An arbitrary product $\prod_\alpha X_\alpha$ of compact spaces is compact.

27. Countability

(a) **(Def)** A space X is called first countable (or X satisfies the 1st countability axiom) if for each $x \in X$, there is a sequence of nbhds $U_1, U_2, \cdots$ of x for any nbhd V of x, $U_i \subset V$ for some i.

(b) **(Thm)**

1 Suppose $A \subset X$. Then there is a sequence $\{x_n\}$ in A s.t. $x_n \to x \in X \Leftrightarrow x \in \overline{A}$.

2 Suppose $f : X \to Y$. Then f is continuous. $\Leftrightarrow f(x_n) \to f(x)$ whenever $x_n \to x$.

(c) **(Def)** X is second countable if X has a countable basis.

(d) **(Thm)** If X is 2nd countable, then there exists a countable dense subset, and every open cover of X has a finite subcover.

28. Separation Axioms

(a) **(Def)**

1 $\mathbf{T_1}$: X is a space each of whose one-point subset is closed. *i.e.* for all $x \neq y$ in X, $\exists$ a nbhd U of x s.t. $y \notin U$. $(X - \{x\} = \bigcup_{y \neq x} U_y)$ **Note: Hausdorff** (T_2) **implies** T_1**.**

- Suppose X below is T_1.

2 **regular**: X is regular if for any closed subset $C \subset X$ and for any $x \in X - C$, $^{\exists}$disjoint open sets U, V s.t. $x \in U$ and $C \subset V$ (satisfies T_3 axioms). ($\Leftrightarrow$ for any nbhd U of $x \in X$, $^{\exists}$ nbhd V of x s.t. $x \in V \subset \overline{V} \subset U$.)

3 **normal**: X is normal if for any closed subset $C, D \subset X$, $^{\exists}$disjoint open sets U, V s.t. $C \subset U$ and $D \subset V$ (satisfies T_4 axioms). ($\Leftrightarrow$ for any closed $C \subset X$ and open $U \subset X$ s.t. $C \subset U$, $^{\exists}$ nbhd V of x s.t. $C \subset V \subset \overline{V} \subset U$.)

- regular $= T_1$+T_3, normal$= T_1$+T_4. **Note: normal** $\Rightarrow$ **regular** $\Rightarrow$ **Hausdorff** $\Rightarrow$ $\mathbf{T_1}$.

(b) **(Thm)** A subspace of Hausdorff is Hausdorff. Any product of Hausdorff spaces is Hausdorff.

(c) **(Thm)** A subspace of regular is regular. Any product of regular spaces is regular.

(d) **(Thm)**

1 X is compact and Hausdorff $\Rightarrow$ X is normal.

2 X is metrizable $\Rightarrow$ X is normal.

3 X is regular and 2nd countable. $\Rightarrow$ X is normal.

29. Urysohn lemma and Tietze extension theorem

(a) ****(Urysohn Lemma)** Suppose X is normal and $C, D \subset X$ are disjoint closed subsets. Then there is a continuous map $f : X \to [0,1]$ s.t. $f(C) = \{0\}$, $f(D) = \{1\}$.

(b) ****(Tietze Extension Theorem)** If X is normal and $A \subset X$ closed, then any continuous map $f : A \to [a, b]$ extends to X. *i.e.* $^{\exists}$continuous $g : X \to [a, b]$ s.t. $g(x) = f(x)$. $^{\forall}x \in A$.

30. Metrization theorem

(a) **(Def)** X embeds into Y if there is a continuous map $f : X \to Y$ s.t. $f : X \to f(X)$ is a homoeomorphism. (f is called an embedding.) *i.e.* X is homeomorphic to a subspace of Y.

(b) **(Thm)** If X is regular and 2nd countable, then X embeds in $\mathbb{R}^\omega = \prod_{n=1}^\infty \mathbb{R}$.

(c) ****(Urysohn Metrization theorem)** X is regular and 2nd countable $\Rightarrow$ X is metrizable.

31. Fundamental group -I

(a) **(Def: Homotopy)** A path $f : I \to X$ with $f(0) = x_0$ and $f(1) = x_1$ is homotopic relatively $\{0, 1\}$ to another path $g : I \to X$ with $g(0) = x_0$ and $g(1) = x_1$. If there is a continuous map $F : I \times I \to X$ s.t. $F(s, 0) = f(s)$, $F(s, 1) = g(s)$, $F(0, t) = x_0$, $F(1, t) = x_1$. $^\forall s, t \in I$. **Note:** $F : f \simeq g$ **ret** $\{0, 1\}$.

(b) **(Def)** $[f]$:=the homotopy class of a path $f : I \to X$.

(c) **(Def)** For path $f : I \to X$ from x_0 to x_1 and $g : I \to X$ from x_1 to x_2, define the product $(f * g)(s) = \begin{cases} f(2s) & \text{if } 0 \le s \le \frac{1}{2} \\ g(2s-1) & \textit{if } \frac{1}{2} \le s \le 1 \end{cases}$, which is well-defined, continuous by the pasting lemma.

(d) **(Lemma)** $f \simeq_p f'$, $g \simeq_p g'$, $f(1) = g(0) \Rightarrow f * g \simeq_p f' * g'$.

32. Fundamental group -II

(a) **(Def: loop)** A path $f : I \to X$ is a loop based at $x_0 \in X$ if $f(0) = x_0 = f(1)$.

$$\pi_1(X, x_0) := \{\text{loops in } X \text{ based at } x_0\}/\simeq_p = \{[f] \mid f : I \to X \text{ is a loop based at } x_0\}.$$

(b) **(Thm)** $\pi_1(X, x_0)$ is a group under the product operation.

33. Simple connectedness and induced homomorphisms

(a) **(Def)** Suppose $x_0, x \in X$, and $\alpha : I \to X$ is a path from x_0 to x_1. Define $\hat{\alpha} : \pi_1(X_1, x_0) \to \pi_1(X_1, x_1)$ where $\hat{\alpha}([f]) = [\overline{\alpha} * f * \alpha]$. $\hat{\alpha}$ is well-defined.

(b) **(Thm)** $\hat{\alpha}$ is group homomorphism.

(c) **(Coro)** If X is path connected, $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$ for all $x_0, x_1 \in X$.

(d) **(Def)** X is simply connected if X is path connected and $\pi_1(X, x_0)$ is the trivial group.

(e) **(Lemma)** Suppose X is simply connected. Then for any paths $f, g : I \to X$ s.t. $f(0) = g(0)$ and $f(1) = g(1)$, we have $f \simeq_p g$.

(f) **(Def: Induced homeomorphism)** Suppose $h : X \to Y$ is a continuous map, and $h(x_0) = y_0$. Define $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ where $h_*([f]) = [h \circ f]$. h_* is well-defined and h_* is a group homomorphism.

(g) ****(Thm: Naturality, Functioriality)**

1 $id_X : X \to X$ induces $(id_X)_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ where $(id_X)_*([f]) = [id_X \circ f] = [f]$.

2 If $h : X \to Y$ and $k : Y \to Z$, $(k \circ h)_* = k_* \circ h_*$ where $h_* : \pi_1(X, x_0) \to \pi_1(Y, h(x_0))$ and $k_* : \pi_1(Y, h(x_0)) \to \pi_1(Z, k \circ h(x_0))$.

(h) **(Coro)** If $h : X \to Y$ is a homeomorphism and $h(x_0) = y_0$, then $h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism.

34. Covering Spaces -I

(a) **(Def) Suppose $p : E \to B$ is continuous and surjective.**

1 An open set $U \subset B$ is evenly covered if $p^{-1}(U) = \bigcup_\alpha V_\alpha$. V_α are disjoint open sets in E, and $p|_{V_\alpha} : V_\alpha \to U$ is a homeomorphism $^\forall \alpha$. Each V_α is called a sheet.

2 p is a covering map if every $b \in B$ has an evenly covered nbhd. When $p : E \to B$ is a covering map, E is called a covering space of B.

(b) **(Thm)** $p : \mathbb{R} \to S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is a covering map.

(c) **(prop)**

1 For $b \in B$, $p^{-1}(b)$ is a discrete subspace of E.

2 An open subset of evenly covered open set is evenly covered.

3 p is an open map.

(d) **(Def: lifting)** Suppose $p : E \to B$ is a covering map, and $f : X \to B$ is a continuous map. $\tilde{f} : X \to E$ is called a lifting of f if $p \circ \tilde{f} = f$. (Here, we regard a lifting as a continuous map.)

(e) **(Thm)** Suppose X is connected, and suppose $\tilde{f}$, $\tilde{f}' : X \to E$ are liftings of $f : X \to E$ s.t. $\tilde{f}(x_0) = \tilde{f}'(x_0)$ for some $x_0 \in X$. Then $\tilde{f} = \tilde{f}'$.

35. Covering spaces -II

(a) **(Thm: Path lifting)** Suppose $f : I \to B$ and $p : E \to B$ a covering map, and $f(0) = b_0 \in B$. Then for any $e_0 \in p^{-1}(b_0)$, there is a lift $\tilde{f} : I \to E$ s.t. $\tilde{f}(0) = e_0$.

(b) **(Thm: Homotopy lifting)** Suppose $F : I \times I \to B$ and $p : E \to B$ a covering map, and $F(0,0) = b_0 \in B$. Then for any $e_0 \in p^{-1}(b_0)$, there is a lift $\tilde{F} : I \times I \to E$ s.t. $\tilde{F}(0,0) = e_0$.

(c) **(Addendum Thm)** If $F : f \simeq_p g$ is a path homotopy, then, $\tilde{F} : \tilde{f} \simeq_p \tilde{g}$ where $\tilde{f}$ =the lift of f with $\tilde{0} = e_0$, $\tilde{g}$ =the lift of g with $\tilde{g}(0) = e_0$. Consequently, $\tilde{f}(1) = \tilde{g}(1)$.

36. Computation of $\pi_1(S^1)$

(a) **(Thm)** Suppose $p : E \to B$ is a covering map, $b_0 \in B$, $e_0 \in p^{-1}(b_0) \subset E$. Define $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ where $\phi([f]) = \tilde{f}(1)$. ϕ is well-defined since $[f] = [g]$ implies $\tilde{f}(1) = \tilde{g}(1)$ by the above.

(b) **(Thm)** $\phi : \pi_1(B, b_0) \to p^{-1}(b_0)$ is a bijection if E is simply connected.

(c) **(Thm)** $\pi_1(S^1, 1)$ is isomorphic to infinite cyclie $\mathbb{Z}$.

37. Retractions and Brouwer's fixed point theorem

(a) **(Thm)** For any $D \subset \mathbb{C} - \{0\}$ containing $S_r' := \{z \in \mathbb{C} \mid |z| = r\}$, there is no logarithm.

(b) **(Def)** Suppose $A \subset X$. A continuous map $r : X \to A$ is retraction if $r(a) = a$ for all $a \in A$. *i.e.* $r \circ i = id_A$ when $i : A \to X$ is an inclusion. If there exists a retraction $r : A \to X$, we say that A is a retract of X.

(c) **(Lemma)** A is a retract of $X \Rightarrow i_X : \pi_1(A, x_0) \to \pi_1(X, x_0)$ is 1-1 for any $x_0 \in X$.

(d) **(Thm)** $S^1 \subset B^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$ is not a retract of B^2.

(e) **(Brouwer's Fixed Point Theorem for B^2)** For any continuous map $f : B^2 \to B^2$, there is $x \in B^2$ s.t. $f(x) = x$, which is called a "fixed point"