Applied Linear Algebra: Chapter 4,5

MATH

19th November 2020

1. determinant

(1) (definition)

(i) A unique function from $A_{n\times n}$ to $\mathbb{R}$(if A is real matrix) satisfying three properties:

(a) (multilinear)

$$\det\begin{pmatrix} kv_1+w_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \det\begin{pmatrix} kv_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \det\begin{pmatrix} w_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}.$$

(b) (alternating)

$$\det\begin{pmatrix} v_i \\ v_j \end{pmatrix} = -\det\begin{pmatrix} v_j \\ v_i \end{pmatrix}$$

where v_k is the k-th row vector.

(c) ($\det I = 1$)

(2) (properties)

(i) $\det A = 0 \Leftrightarrow A$ is singular .

(ii) For triangular matrix A, $\det A = \Pi_{i=1}^n a_{ii}$.

(iii) $\det AB = \det A \det B$.

(iv) $\det A = \det A^T$.

(3) (method)

(i) $\det A = \sum_{\{j_1,j_2,\cdots,j_n\}}(-1)^{\mathrm{sgn}\sigma} a_{1j_1}a_{2j_2}\cdots a_{nj_n}$ where $\mathrm{sgn}\sigma = \begin{cases} 1 & \text{odd(num of transpositions)} \\ 0 & \text{even} \end{cases}$

and $\{j_1, j_2, \cdots, j_n\}$ is every permutation of $\{1, 2, \cdots, n\}$.

(ii) (cofactor) $\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$ Collecting j_1s such that $j_1 = k$ on the above, we get $C_{1k} = \sum_{\{k,j_2,\cdots,j_n\}}(-1)^{\mathrm{sgn}\sigma} a_{2j_2}\cdots a_{nj_n} = (-1)^{k-1}\det M_{1k}$($M_{ij}$ is a submatrix which removes i-th row and j-th column). In general, $C_{ij} = (-1)^{i+j}\det M_{ij}$ so that $\det A = \sum_{i=1}^n a_{ij}C_{ij} = \sum_{j=1}^n (-1)^{i+j} a_{ij}\det M_{ij}$.

(4) (Theorem) $A^{-1} = \frac{1}{\det A}C^T$

pf) We'll show that $\det A \cdot I = AC^T$. Let $B = AC^T$. For diagonal entry, $b_{ii} = \sum_{k=1}^n a_{ik}(C^T)_{ki} = \sum_{k=1}^n a_{ik}C_{ik} = \det A$ by definition. For the nondiagonal entry, $b_{ij} = \sum_{k=1}^n a_{ik}(C^T)_{kj} = \sum_{k=1}^n a_{ik}C_{jk} = 0$ since it's the same as obtaining determinant of $\begin{pmatrix} -v_i- \\ -v_i- \end{pmatrix}$ where v_i is i-th row vector.

2. eigenvalue and eigenvector

(1) (definition)

(i) (eigenvalue) $\lambda \in \mathbb{C}$ is an eigenvalue if $\det(A - \lambda I) = 0$.

(ii) (eigenvector) For fixed λ, v is an eigenvector if $v \in N(A - \lambda I)$. (Then $(A - \lambda I)v = 0$, which eventually implies $Av = \lambda v$)

(iii) (characteristic polynomial of A) $P_A(\lambda) = \det(A - \lambda I)$.

(iv) A is diagonalizable if there exists invertible Q s.t. $A = QDQ^{-1}$ for diagonal matrix D.

(a) (diagonalizable and nonsingular) No eigenvalues are 0. The nonzero eigenvalues maybe distinct or not. If eigenvalues are all distinct, then eigenvectors are independent, However, if eigenvalues are the same, we must check that eigenvectors corresponding to the same eigenvalue are independent.

(b) (diagonalizable and singular) There is eigenvalue of 0. Then $\det A = 0$, which means A is singular. Although 0 is an eigenvalue, the eigenvectors of A might be independent. example: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

(c) (non-diagonalizable and nonsingular) The eigenvalues are all nonzero, but there exists the same eigenvalue such that its eigenvalue are dependent. example: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: We'll use Jordan form for this later.

(d) (non-diagonalizable and singular)

(2) (properties)

(i) $P_A(\lambda)$ is n-th order polynomial.

pf) The diagonal entry of $A - \lambda I = a_{ii} - \lambda$. By definition of det, $P_A(\lambda) = \det(A - \lambda I) = \Pi_{i=1}^{n}(a_{ii} - \lambda) + \cdots$.

(ii) $\mathrm{Tr}(A) = \sum_{i=1}^{n} \lambda_i$.

pf) $P_A(\lambda) = (-1)^n \lambda^n + (-1)^{n-1}(\mathrm{Tr}\, A)\lambda^{n-1} + \cdots = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$. (We only need to regard the multiple of diagonals)

(iii) (counting dupplicated ones) number of eigevalue $0's$ =nullity.

pf) For $\lambda = 0$, $Av = \lambda v = 0$, $v \in N(A)$. Therefore, number of eigenvalue 0's$\leq$ nullity. Consider $v \in N(A)$, $Av = 0$. Then v can be interpreted as an eigenvector whose eigenvalue is 0. Then number of eigenvalue 0's$\geq$ nullity.

(iv) $\det A = \Pi_{i=1}^{n} \lambda_i$.

pf) $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda)$. Let $\lambda = 0$.

(v) If $\lambda_1 \neq \lambda_2$, then v_1 and v_2 are linearly independent. (However, $\lambda_1 = \lambda_2$ generaaly doesn't implies linearly dependence.)

pf) Suppose $c_1 v_1 + c_2 v_2 = 0$ for v_1, v_2. Then $c_1 A v_1 + c_2 A v_2 = c_1 \lambda_1 v + c_2 \lambda_2 v = 0$ and $c_1 \lambda_1 v_1 + c_2 \lambda_1 v_2 = 0$. Then $(\lambda_2 - \lambda_1) c_2 v_2 = 0$, which implies $c_2 = 0$, and consequently $c_1 = 0$.

$\Rightarrow$ More generally, If eigenvalue $\lambda_1, \lambda_2, \cdots, \lambda_n$ are all distinct each other, all corresponding eigenvectors $\{v_1, v_2, \cdots, v_n\}$ are linearly independent. Then, the matrix $Q = \begin{bmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{bmatrix}$ which collects all the independent eigenvectors is invertible. Using $AQ = QD$(collection of $Av = \lambda v$), we get $A = QDQ^{-1}$.

(vi) (Commutativity) Diagonalizable matrices which share same eigenvectors iff $AB = BA$.

pf) Let $A = Q_1 D_1 Q_1^{-1}$ and $B = Q_2 D_2 Q_2^{-1}$.

($\Rightarrow$) Assume A and B share the same eigenvectors. Then $Q_1 = Q_2 = Q$. $AB = (QD_1Q^{-1})(QD_2Q^{-1}) = QD_2D_1Q^{-1} = (QD_2Q^{-1})(QD_1Q^{-1}) = BA$.

($\Leftarrow$) (Best case such that eigenvalues are all distinct) Assume $AB = BA$ and $Av = \lambda v$. $A(Bv) = ABv = BAv = \lambda(Bv)$. Since $\langle$eigenvector of $\lambda\rangle$ is one dimensional, $v//Bv$. $Bv = \eta v$(η is eigenvalue and v is its eigenvector.) Therefore, A and B share the same eigenvectors.

(3) (example) $e^{A+B} = e^A e^B$ does not hold when $AB \neq BA$. (In fact, it holds when $AB = BA$.

pf) $e^{A+B} = \sum_{n=0}^{\infty} \frac{(A+B)^n}{n!}$. $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ and $e^B = \sum_{n=0}^{\infty} \frac{B^n}{n!}$. Consider the term $a_2 = A^k B^{2-k}$. Then for e^{A+B}, the $a_2 = \frac{A^2+AB+BA+B^2}{2}$ where a_2 for $e^A e^B$ is $\frac{A^2}{2} + AB + \frac{B^2}{2}$.

(4) (2016-2 final Problem 2-c) **Vandermonde matrix** Prove that for any $n \geq 1$,

$$\det \begin{bmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^n \\ 1 & a_1 & a_1^2 & \cdots & a_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^n \end{bmatrix} = \Pi_{0 \leq i < j \leq n}(a_j - a_i).$$

pf) It holds for 2×2 matrix(Check). Suppose the equation holds for $n \times n$ matrix. Then the row elimination

$$\det \begin{bmatrix} 1 & a_0 & a_0^2 & \cdots & a_0^n \\ 0 & a_1 - a_0 & a_1^2 - a_0^2 & \cdots & a_1^n - a_0^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_n - a_0 & a_n^2 - a_0^2 & \cdots & a_n^n - a_0^n \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & a_1 - a_0 & (a_1 - a_0)a_1 & \cdots & (a_1 - a_0)a_1^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & a_n - a_0 & (a_n - a_0)a_n & \cdots & (a_n - a_0)a_n^{n-1} \end{bmatrix}$$

$$= \Pi_{i=1}^n (a_i - a_0) \det \begin{bmatrix} 1 & a_1 & \cdots & a_1^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & \cdots & a_n^{n-1} \end{bmatrix} = \Pi_{i=1}^n (a_i - a_0) \Pi_{1 \leq j < l \leq n}(a_l - a_j) = \Pi_{0 \leq i < j \leq n}(a_j - a_i).$$

3. Hermitian

(1) (def) A is called **orthogonally diagonalizable** if A is similar to a diagonal matrix D with an orthogonal matrix Q, *i.e.* $A = QDQ^T$.

(2) (def) A is called **unitarily diagonalizable** if A is similar to a diagonal matrix D with an orthogonal matrix Q, *i.e.* $A = QDQ^*$.

(3) (theorem) If A is Hermitian(in real case, real symmetric), then the following properties hold.

(i) Every eigenvalue of A is real.

(ii) Eigenvectors correspond to distinct eigenvalues are orthogonal. (Then we can make them to orthonormal.)

(iii) A is unitarily diagonalizable. (In real case, A is orthogonally diagonalizable.) It is ensured by **Schur's lemma**.

(4) (def) A is similar to B when there is a matrix M such that $A = M^{-1}BM$.

(5) (theorem) If A, B are similar,

(i) They share the same eigenvalues(counting multiplicity).

(ii) $\det(A - \lambda I) = \det(B - \lambda I)$.

- The converse is not true. For the first case, let $A = I_{2\times 2}$ and $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

(6) **(Schur's lemma)** For $A_{n\times n}$, there always exists unitary M such that $A = MTM^*$ where T is upper triangular. Then since Hermitian matrix makes $A = MTM^* = MT^*M^* = A^*$, $T = T^*$, T is a diagonal matrix.

(7) (theorem) Every normal matrix $NN^* = N^*N$(which includes Hermitian) is unitarily diagonalizable.

(8) **(Jordan (canonical) form)** If $A_{n\times n}$ has $s(\leq n)$ independent eigenvectors, it is similar to a matrix with s blocks so that $A = MJM^{-1}$. Such J always exists for every non-diagonalizable A!!

(9) (method) If $\lambda_1, \lambda_2, \lambda_2$ are eigenvalues of $A_{3\times 3}$ and $\dim(N(A - \lambda_2 I)) = 1$, then we we can pick v_3 satisfying $(A - \lambda_2 I)^2 v_3 = 0$ and not parallel to v_2 *i.e.* $v_3 \in N((A - \lambda_2 I)^2)$ and $v_3 \notin N(A - \lambda_2 I)$. (Since we use '1' on the Jordan block, $Av_3 = \lambda_2 v_3 + v_2$ holds, and cleary $v_3 \in N((A - \lambda_2 I)^2)$ holds.) Note that v_1 and v_3 are independent.

(10) ****(Basis change)** Consider two bases $\{v_1, v_2, v_3\}$ and $\{e_1, e_2, e_3\}$(standard basis) on $\mathbb{R}^3$. For a linear transformation $A = [T]_{e\to e}$, $B = [T]_{v\to v}$ and a basis change matrix $[I]_{e\to v} = M$. Then $A = M^{-1}BM$ holds.

Q. How can we find such matrix M?

- Clearly, $Mv_{1e} = [1, 0, 0]_v^T$, $Mv_{2e} = [0, 1, 0]_v^T$ and $Mv_{3e} = [0, 0, 1]_v^T$ hold. Note that v_{ie} is a column vector of "coefficients". *i.e.* If $v_{1e} = [1, 1, 0]$, then it actually $e_1 + e_2$ on e. Also, $[1, 0, 0]_v^T$ is v_1 on v. In other words, M is a linear transformation which sends a vector to the same vector, although its representation changes depending on basis. Therefore, for $V = \begin{bmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{bmatrix}$, $MV = I$. Thus $M = V^{-1}$. Also, $[I]_{v\to e} = M^{-1} = V$. \$