Applied Linear Algebra: Chapter 1,2,3

MATH

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Almost all concepts covered in Applied Linear Algebra Chapter 1-3. Contact hhyy0401@postech.ac.kr if there is any error or question.

1. Matrix and Gaussian Eliminations

(1) For permutation matrix P, PA makes row exchange of A and AP makes column exchange of A.

(2) (matrix multiplication) Let a_i^* be i-th row vector of A and a_i be i-th column vector of A. The same rule is applied to B. Calculate $C_{m\times p} = A_{m\times n}B_{n\times p}$

  (i) $(c)_{ij} = \sum_{k=1}^n a_{ik}b_{kj} = a_i^* \cdot b_j$

  (ii) $\begin{bmatrix} c_1 & c_2 & \cdots & c_p \end{bmatrix} = A\begin{bmatrix} b_1 & b_2 & \cdots & b_p \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}$

  (iii) $C = a_1b_1^* + a_2b_2^* + \cdots + a_nb_n^* = \sum_{k=1}^n a_kb_k^*$ ($\because (c)_{ij} = (\sum_{k=1}^n a_kb_k^*)_{ij} = \sum_{k=1}^n (a_kb_k^*)_{ij} = \sum_{k=1}^n a_{ik}b_{kj}$.) Here, consider (matrix $\times$ vector) multiplication to compute $C(A)$.

(3) Every matrix has PLU (or $PLDU$) decomposition.

(4) If A is nonsigular matrix and 0 appears in pivot position, then there exists a permutation matrix P such that $PA = LU$ (or $PA = LDU$).

(5) If nonsingular A has LDU decomposition, then its decomposition is unique. *i.e. If* $A = L_1D_1U_1 = L_2D_2U_2$, *then* $L_1 = L_2$, $D_1 = D_2$ *and* $U_1 = U_2$.

(6) TFAE(The followings are equivalent):

For a given square matrix matrix A,

  (i) For all b, there exists a unique solution of $Ax = b$.

  (ii) A is nonsingular.

  (iii) For $PA = LDU$, the pivots(which is on D) are all nonzero.

(7) Assume $A = LDU$. If A is symmetric, than $A = LDL^T$ where L^T is the transpose matrix of L.

(8) When A is nonsingular, using Gauss-Jordan method, we can derive the inverse matrix B of A s.t. $AB = BA = I$.

2. Vector space (over a field $\mathbb{R}$)

(1) For a given vector space V, we need to check "closed under operation$(+,\cdot)$" to determine $W \leq V$(W is a subspace of V). (Then what about $0 \in V$?)

- $\dim V \geq \dim W$.

(2) When solving a system $A_{m\times n}x = b$, $x = x_p + x_N$ where x_p is a particular solution satisfying $Ax_p = b$ and $x_N \in N(A) = \{x \mid Ax = 0\}$ (The particular solution is not unique.)

(3) A basis $S = \{v_1, v_2, \cdots, v_k\}$ of a vector space V is an maximal independent (uniqueness) set and a minimal spanning (existence) set.

(i) Thus, we can express every element of V as a linear combination of S uniquely. i.e. for all $v \in V$, there exists unique c_is s.t. $v = \sum_{i=1}^{k} c_i v_i$.

(ii) The dimension of V = the number of elements in a basis S.

(iii) Usually, a basis is not unique, but its size is unique.

ex) The dimension of $\mathbb{R}^n = n$

(4) **[Four fundamental subspaces of $A_{m\times n}$]**

(i) $N(A) = \{x \in \mathbb{R}^n \mid A_{m\times n}x_{n\times 1} = 0\} \leq \mathbb{R}^n$

- A is nonsingular $\Longleftrightarrow$ $Ax = 0$ has one solution $\Longleftrightarrow$ $N(A) = \{0\}$
- After making A as the reduced echelon form U, positions of columns without pivot are positions of free variables. Let the number of pivots be r. Then [rank of A] $= r$ and [number of free variables]$= n - r$.
- dimension of $N(A) = n - r$
- For a real matrix A, $N(A^TA) = N(A)$

(ii) $N(A^T) = \{y \mid y_{m\times 1}^T A_{m\times n} = 0\} \leq \mathbb{R}^m$

- Using the same argument as above, dimension of $N(A^T) = m - r$.

(iii) $C(A) = \{\text{linear combinations of}A\text{'s columns}\} = \{Ac \mid c \in \mathbb{R}^n\} = \langle v_{i_1}, \cdots v_{i_r}\rangle \leq \mathbb{R}^m$ where $v_{i_k}(k = 1, 2, \cdots, r)$ is a basis of $C(A)$.

- (How to find a basis of $C(A)$) After we get the row echelon form U of A, [the basis of A]= [the positions of pivots in U]. However, $C(A) \neq C(U)$.
- dimension of $C(A) = r$.
- For $Ax = b$, if $b \in C(A)$, $Ax = b$ has a solution. Otherwise, it doesn't have a solution. Then what if dim$C(A) = m$(in this case, $m \leq n$ and we say that A has a full rank m)?

(iv) $C(A^T) = \{\text{linear combinations of A's rows}\} = \langle v_{i_1}^*, \cdots v_{i_r}^*\rangle \leq \mathbb{R}^n$ where $v_{i_k}(k = 1, 2, \cdots, r)$ is a basis of $C(A^T)$.

- $C(A^T) = C(U^T)$
- dimension of $C(A^T) = r$

ex) **(2020-1 midterm-Problem 2)** Let

$$A := \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{1}{3} & \frac{1}{4} & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 0 & \frac{8}{3} & \frac{4}{3} \\ 0 & 0 & 0 \end{bmatrix} = LU.$$

(a) $C(A^T) = C(U^T)$ so $C(A^T) = \left\langle \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{8}{3} \\ \frac{4}{3} \end{bmatrix} \right\rangle$.

(b) Since $N(A) = N(U)$, we need to find x s.t. $Ux = 0$. Let the third column of U be a column of free variable, so $N(A) = \left\langle \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \right\rangle$

(c) For $A = LU$, $L^{-1}A = U$. After calculating

$$L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{3} & 1 & 0 \\ -\frac{1}{4} & -\frac{1}{4} & 1 \end{bmatrix},$$

the third row of L^{-1}(Let y^T) makes $y^T A = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$.(the third row of U) Therefore, $N(A^T) = \left\langle \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \\ 1 \end{bmatrix} \right\rangle$

(d) For any $x = [a, b, c]^T$, $y = Ux = [s, t, 0]^T$.(s, t are any real numbers.) Therefore, the corresponding first and second columns of L are basis elements. $\therefore\ C(A) = \left\langle \begin{bmatrix} 1 \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{1}{4} \end{bmatrix} \right\rangle$

(5) Left/Right inverse

(i) (Existence) Assume $A_{m\times n}$ has full rank $m(m \le n)$, then $Ax = b$ has a solution for all $b \in \mathbb{R}^m$. Morever, A has a right inverse C s.t. $AC = I_m$.

(ii) (Uniqueness) If $A_{m\times n}$ has full rank $n(m \ge n)$, then $Ax = b$ has at most one solution for every b (we don't guarantee existence). Morever, A has left inverse B s.t. $BA = I_n$.

- Left/right inverse is not unique, but we have a formula to get a specific one.
- (left inverse) $B = (A^TA)^{-1}A^T$. (right inverse) $C = A^T(AA^T)^{-1}$.

(iii) (rough ideas): Multiply full rank preserve full rank.(Multiplying doesn't ever increase its rank.)

(6) Linear transformation (**very important ****)

(i) (Definition) A map $T : V \to W$(for two vector spaces V and W) is called linear transformation when $T(cx + y) = cT(x) + T(y)(c \in \mathbb{R}, x, y \in V)$(preserves scalar multiplication and addition).

(ii) (Theorem) For linear transformation $T : V \to W$ with V's basis $S = \{v_1, v_2, \cdots, v_n\}$, T is uniquely defined by $\{Tv_1, Tv_2, \cdots, Tv_n\}$.

(iii) (Theorem) A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ can be expressed with a unique matrix $A_{m\times n}(T(v) = Av$ for $v \in V)$. Reversely, a matrix $A_{m\times n}$ is a linear transformation (i.e.$A(cx + y) = cAx + Ay$).

- (proof) Let a basis of V as $\{v_1, v_2, \cdots, v_n\}$ and a basis of W as $\{w_1, w_2, \cdots, w_m\}$. Note that for every $v \in V$, $w \in W$, $v = (c_1, c_2, \cdots, c_n) = \sum_{i=1}^{n} c_i v_i$ and $w = (d_1, d_2, \cdots, d_n) = \sum_{i=1}^{m} d_i w_i$. Then consider $T(v_i) = \sum_{j=1}^{m} a_{ji} w_j$. For example,

$$\begin{pmatrix} a_{11} & \cdots\cdots \\ a_{21} & \cdots\cdots \\ a_{31} & \cdots\cdots \\ \vdots & \cdots\cdots \\ a_{m1} & \cdots\cdots \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{n\times 1} = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{pmatrix}.$$

Then we get a matrix A of T.

ex) **(2018-2 midterm-Problem 7)** Let V be the vector space of 2×2 matrices with real entries equipped with standard matrix addition and scalar multiplication, i.e.,

$$V := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}.$$

Let a basis of $V = \{v_1, v_2, v_3, v_4\}$ where

$$v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, v_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

. Find the matrix representation of the linear transformation

$$T : V \to V, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a+d & c \\ b & a-d \end{pmatrix}$$

with respect to the basis (with fixed order) $\{v_1, v_2, v_3, v_4\}$.

sol) Since $T(v_1) = v_1 + v_2$, $T(v_2) = v_1 - v_2$, $T(v_3) = v_3$ and $T(V_4) = -v_4$,

$$T = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

ex) **(2019-2 midterm-Problem 11-(b))** Let $\mathbb{P}_4$ be the vector space of polynomials of degrees at most 4 in variable t over $\mathbb{R}$. The map $T : \mathbb{P}_4 \to \mathbb{P}_4$ defined by

$$T(p(t)) = p(t+1)$$

is linear. Find the inverse matrix of the matrix A with the standard basis $\{1, t+1, t^2+1, t^3+1, t^4+1\}$.

sol) Consider T^{-1} instead of calculating A^{-1}. Then we can avoid any complex calculation and get an inverse of A directly. $T^{-1}(p(t)) = p(t-1)$. Then

$$\begin{aligned} T^{-1}(1) &= 1 \\ T^{-1}(t+1) &= t - 1 + 1 = (t+1) - 1 \\ T^{-1}(t^2+1) &= (t-1)^2 + 1 = t^2 - 2t + 2 = (t^2+1) - 2(t+1) + 3 \\ T^{-1}(t^3+1) &= (t-1)^3 + 1 = t^3 - 3t^2 + 3t = (t^3+1) - 3(t^2+1) + 3(t+1) - 1 \\ T^{-1}(t^4+1) &= (t-1)^4 + 1 = t^4 - 4t^3 + 6t^2 - 4t + 2 \\ &= (t^4+1) - 4(t^3+1) + 6(t^2+1) - 4(t+1) + 3. \end{aligned}$$

Then A^{-1}, the matrix representation of T^{-1} becomes

$$A^{-1} = \begin{pmatrix} 1 & -1 & 3 & -1 & 3 \\ 0 & 1 & -2 & 3 & -4 \\ 0 & 0 & 1 & -3 & 6 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

3. Orthogonality

(1) orthogonal and orthogonal complement

(i) (orthogonal) $V, W \le \mathbb{R}^n$ are orthogonal if $v^T w = \langle v, w\rangle = 0$ for all $v \in V$, $w \in W$.

(ii) (orthogonal complement) For a given $V \le \mathbb{R}^n$, the set of "all" vectors orthogonal to v is called orthogonal complement of V and it is denoted by $V^\perp = V_{\text{perp}}$.

(2) $N(A) = (C(A^T))^\perp$ and $N(A^T) = C(A)^\perp$

- (Reason) $N(A) \le C(A^T)^\perp$ and $\dim N(A) + \dim C(A^T) = n$.

(3) $V \cap V^\perp = \{0\}$

(4) Projection matrix P

(i) (definition) A **projection matrix P** is a square matrix that satisfies $P^2 = P$. **P** is an orthogonal projection matrix if $P = P^T$.

(a) If P is projection matrix, then $I - P$ is also a projection matrix.

(b) Let $\{q_1, \cdots, q_n\}$ be any set of n orthogonal vectors in $\mathbb{R}^m$, $n \le m$. Let $Q = \begin{bmatrix} q_1 & q_2 & \cdots & q_n \end{bmatrix}$ be the corresponding $m \times n$ matrix. Then $P = QQ^T$ is orthogonal projection matrix. Here, P is a projection onto $C(P) = C(Q)$ since $Pv = QQ^T v = \sum_{i=1}^n (q_i^T v) q_i$ (linear combination of q_is).

(ii) We want to find $\hat{x}$ s.t. $A\hat{x}$ becomes a projection of b on $C(A)$. Then $A^T(b - A\hat{x}) = 0 \Rightarrow A\hat{x} = A(A^T A)^{-1} A^T b = Pb$. $\therefore P = A(A^T A)^{-1} A^T$. (If it exists.) Such $\hat{x}$ is a least square solution.

(iii) (Don't be confused!) If A is not of full rank, we can make P by removing all dependent columns of A(note as A') and $P = A'(A'^T A')^{-1} A'^T$. Thus, we can always find a least square solution of $A\hat{x} = Pb$. Morever, the solution is not unique generally.

ex) **(2018-2 midterm-Problem 5)** Let $A = \begin{pmatrix} 1 & 1 \\ 4 & 4 \\ 4 & 4 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. What is the vector b_c which is orthogonal projection of b onto the column space $C(A)$ of A?

sol) Solve $A^T A\hat{x} = A^T b$,

$$\begin{pmatrix} 33 & 33 \\ 33 & 33 \end{pmatrix} \hat{x} = \begin{pmatrix} 9 \\ 9 \end{pmatrix}.$$

Therefore, $\hat{x} = \begin{pmatrix} a \\ b \end{pmatrix}$ where $a + b = 9/33$. Then $A\hat{x} = b_c$ is what we want,

$$b_c = \begin{pmatrix} a+b \\ 4(a+b) \\ 4(a+b) \end{pmatrix} = \begin{pmatrix} 3/11 \\ 12/11 \\ 12/11 \end{pmatrix}.$$

sol) Let $A' = \begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}$. Then $Pb = A'(A'^T A')^{-1} A'^T b = \begin{pmatrix} 3/11 \\ 12/11 \\ 12/11 \end{pmatrix}$.

(iv) (Hauseholder reflection) Choose a unit vector $u \in \mathbb{R}^n$, $\|u\| = 1$. Define $H_n = I_n - 2uu^T$. Then H_n is symmetric and $H_n^2 = I_n$. Therefore, H_n is orthogonal matrix.

(a) $H_n u = -u$

(b) $H_n w = w$ for any $w \perp u$.

Denote the hyperplate perpendicular to u by Ω. $H_n x$ is a reflection of x w.r.t. Ω.

(5) Gram-Schmidt Process

(i) a process making basis $S = \{v_1, v_2, \cdots, v_k\} \Rightarrow$ orthonomal basis $\{q_1, q_2, \cdots, q_k\}$ s.t. $\langle v_1, v_2, \cdots, v_k\rangle = \langle q_1, q_2, \cdots, q_k\rangle$, $\|q_j\| = 1$ and $\langle q_i, q_j\rangle = 0 (i \neq j)$.

(a) For an independent set $\{v_1, v_2, \cdots, v_k\}$ we can construct orthogonal set $\{q_1, q_2, \cdots, q_k\}$,

$$\begin{aligned} A_j &= v_j - (q_1^T v_j)q_1 - (q_2^T v_2)q_2 - \cdots (q_{j-1}^T v_{j-1})q_{j-1} \\ q_j &:= \frac{A_j}{\|A_j\|}. \end{aligned}$$

(ii) Orthogonal matrix $Q^T Q = I$

(a) $Q_{n\times n} = \begin{pmatrix} | & | & \vdots & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & \vdots & | \end{pmatrix}$ is an orthogonal matrix.

(b) For $A = QR$ defined above, $C(A) = C(Q)$.

- For $C(A) = \langle v_1, v_2, \cdots, v_k\rangle$ and $C(Q) = \langle q_1, q_2, \cdots, q_k\rangle$(we get q_j by Gram-Schmidt process). Then $v = c_1 q_1 + c_2 q_2 + \cdots + c_k q_k + w$ where $w \in C(Q)^\perp (= C(A)^\perp)$ and $v \in \mathbb{R}^n$. Taking an inner product $\langle q_i, v\rangle$ for every i, we get $c_i = q_i^T v$. Then $v = q_1 q_1^T v + q_2 q_2^T v + \cdots + q_k q_k^T v + w = (q_1 q_1^T + \cdots + q_k q_k^T)v + w = Pv + w$. $\therefore$ Projection onto $C(Q) = \sum_{i=1}^k q_i q_i^T$. (**ref. 2020-1 midterm-Problem 17**)

(c) Q preserves a length($\|Qx\| = \|x\|$) and an angle($\langle Qx, Qy\rangle = \langle x, y\rangle$).

(d) In this case, $A = QR$ where (for example $A_{3\times 3}$)

$$A = \begin{pmatrix} | & | & | \\ a & b & c \\ | & | & | \end{pmatrix}, \ Q = \begin{pmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{pmatrix}, \ R = \begin{pmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T c \end{pmatrix}.$$