Outline :

1. Introduce entanglement entropies to pedestrian (especially me)

   by Witten's "mini-lecture"

Not talked 2. General viewpt. of TQFTs , especially in 2d top. string

   by 1004.2307

3. See an example of entanglement entropy calculation in TQFT.

   by 1809.04574

Maybe it's good to review information theory & entropies first.

By Witten's "mini-introduction"

Classical Information Theory: some entropies & inequalities

1. String: abbaabbbaa... w/ $p(a)=p$, $p(b)=1-p$,

(Shannon) Large N string: pN of a's & $(1-p)N$ of b's, the total # of such strings are

$$\frac{N!}{(pN)!((1-p)N)!} \sim \frac{N^N}{(pN)^{pN}((1-p)N)^{(1-p)N}} = \frac{1}{p^{pN}(1-p)^{(1-p)N}} = 2^{NS}$$

Rmk: All log are based by 2

$S := -p\log_2 p - (1-p)\log_2(1-p)$ the Shannon entropy per letter.

So the # of bits of information that one gains from a string is NS.
(large S)

In the case of k letters $a_1 \cdots a_k$ w/ probability $p_1 \cdots p_k$, denoted A, similarly
distribution

$$S_A = -\sum_{i=1}^{k} p_i \log p_i$$

# of bits in information carried by N symbols: NS_A.

Prop. $2^{NS_A} \geq 1$, $S_A \geq 0$ for any distribution, $S_A = 0$ for $A = \{a, p(a)=1\}$ one letter. (no uncertainty at all)

S_A maximal for A evenly distributed: $A = \{a_1 \cdots a_k, P(a_i) = \frac{1}{k}\}$, $S_A = \log k$.

2. Alice $\Rightarrow$ Bob, $P_{X,Y}(x_i, y_j)$ the probability that y_j is recieved from message x_i,
$X = (x_1 \cdots x_k)$, $Y = (y_1 \cdots y_k)$
(cdtional)

$P_Y(y_j) = \sum_i P_{X,Y}(x_i, y_j)$ prob. that recieving y_i,

<u>Conditional probability</u> $P_{X|Y}(x_i|y_j) = \frac{P_{X,Y}(x_i, y_j)}{P_Y(y_j)}$ of sending x_i when y_j is recieved.

Corresp. entropy : $S_{X|Y=y_j} = -\sum_i P_{X|Y}(x_i|y_j) \log P_{X|Y}(x_i|y_j)$

And the average remaining entropy of X after Y is recived (<u>conditional entropy</u>)

$$S_{X|Y} = \sum_j P_Y(y_j) S_{X|Y=y_j} = -\sum_j P_Y(y_j) \sum_i \frac{P_{X,Y}(x_i,y_j)}{P_Y(y_j)} \log\left(\frac{P_{X,Y}(x_i,y_j)}{P_Y(y_j)}\right)$$

$$= -\sum_{ij} P_{X,Y}(x_i,y_j) \log P_{X,Y}(x_i,y_j) + \sum_{ij} P_{X,Y}(x_i,y_j) \log P_Y(y_j)$$

$$= S_{XY} - S_Y \qquad \left(= \sum_j P_Y(y_j) \log P_Y(y_j)\right)$$

S_{XY} : entropy for the joint distribution, S_Y : entropy for $P_Y(y_j)$ distribution.

Prop. $S_{XY} - S_Y \geq 0$ (which is not true in QM!)

The <u>mutual information</u> $I(X;Y) = S_X + S_Y - S_{XY}$ measures how much one learns about X from recieving Y. $I(X;Y) \geq 0$.

3. Suppose two distributions Q_X, P_X s.t. $Q_X(x_i) = q_i$, $P_X(x_i) = p_i$, $x_1 \cdots x_s$.

(rel.) If only one of them is correct, one can judge the probability of rightness from N-string of "how possible it close to P_X if we assume Q_X is right":

$$\mathcal{P} = \underbrace{\prod_{i=1}^{s} q_i^{p_i N}}_{\text{probability this seq. happens in } Q_X} \underbrace{\frac{N!}{\prod_{j=1}^{s} (p_j N)!}}_{\text{\# of seq. w/ } p_i N \text{ outcome (i.e. is } P_X)} \sim 2^{-N \sum_i p_i (\log p_i - \log q_i)} = 2^{-N S_{P\|Q}}$$

$S_{P_X \| Q_X} = \sum_i p_i (\log p_i - \log q_i)$ the <u>relative entropy</u>.

Prop. $S_{P_X \| Q_X} \geq 0$ and equal only if $P_X = Q_X$.

Rmk: Mutual information $I(X;Y) = S_X + S_Y - S_{XY} = S_{P_{X,Y} \| Q_{X,Y}}$, for $Q_{X,Y} = P_X \cdot P_Y$.

Thus it's positive definite and is zero only if $P_X = P_Y$.

$S_X + S_Y - S_{XY} \geq 0$ is called <u>subadditivity</u>.

Prop. (Monotonicity)

Let X, Y be two random variables, $P_{X,Y}(x_i, y_j)$, $Q_{X,Y}(x_i, y_j)$ are two distributions, $Q_X = \sum_j Q_{X,Y}(x_i, y_j)$, P_X etc., then

$$S(P_{X,Y} \| Q_{X,Y}) \geq S(P_X \| Q_X)$$

"It's hard to disprove hypothesis from observation of X only"

Cor. (Strong subadd.)

Let X, Y, Z, $P_{X,Y,Z}$ joint distribution, then consider

$$Q_{X,Y,Z} = \left(\sum_{Y,Z} P_{X,Y,Z}\right) \cdot \left(\sum_X P_{X,Y,Z}\right) = P_X \cdot P_{Y,Z} \quad \text{(forget correlation of } X \text{ \& } YZ\text{)}$$

$$P_{X,Y} = \sum_Z P_{X,Y,Z}, \quad Q_{X,Y} = \sum_Z Q_{X,Y,Z} = P_X \cdot P_Y \quad \text{(only observe system } XY\text{)}$$

By monotonicity $S_{P_{XYZ} \| Q_{XYZ}} \geq S_{P_{XY} \| Q_{XY}}$

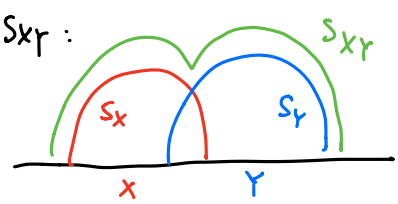
$$\Rightarrow \quad S_X - S_{XYZ} + S_{YZ} \geq S_X - S_{XY} + S_Y$$

$$\Rightarrow \quad S_{XY} + S_{YZ} \geq S_Y + S_{XYZ} \quad \underline{\text{(strong subadditivity)}},$$

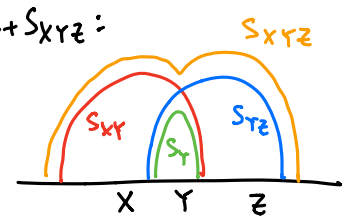
which is true in QM.

Rmk. Aspects from holography / external surfaces

$S_X + S_Y \geq S_{XY}$:

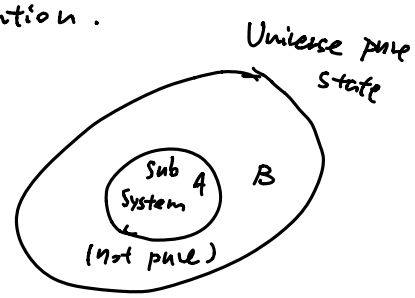


$S_{XY} + S_{YZ} \geq S_Y + S_{XYZ}$:



Quantum Information Theory

1. Density matrix : QM analogy of classical prob. distribution.



Let A be a subsystem w/ $\mathcal{H}_A$ while B be everything else.

Combined Hilbert sp. $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$, & $\psi_{AB} = \psi_A \otimes \psi_B$ and

this system can be regarded as forgetting the B side. Any operator $\mathcal{O}_A$ in $\mathcal{H}_A$,

one has $\mathcal{O}_{AB} = \mathcal{O}_A \otimes id_B$ in $\mathcal{H}_{AB}$ and expectation value factorizes as well.

$$\langle \psi_{AB} | \mathcal{O}_{AB} | \psi_{AB} \rangle = \langle \psi_A | \mathcal{O}_A | \psi_A \rangle \cdot \langle \psi_B | \psi_B \rangle$$

However, a generic $\psi_{AB} \in \mathcal{H}_{AB}$ is not a product but "entangled".

Let $\mathcal{H}_A$ & $\mathcal{H}_B$ are of dim. N, M, then $\mathcal{H}_{AB}$ is of dim. $N \times M$. For instance,

$N=2$, $M=3$, $\psi_{AB} = \begin{pmatrix} \times & \times & \times \\ \times & \times & \times \end{pmatrix}$.

(Schmidt decomp.): A unitary transf. on N & M can make

$$\psi_{AB} \mapsto U \psi_{AB} V = \begin{pmatrix} \sqrt{p_1} & 0 & 0 \\ 0 & \sqrt{p_2} & 0 \end{pmatrix} = \sum_i \sqrt{p_i}\, \psi_A^i \otimes \psi_B^i ,$$

$\psi_{A/B}^i$ orth., $p_i > 0$. Note that they may not be a basis of whole Hil. sp.

take $\psi_{A/B}^i$ unitary vec., then $\sum_i p_i = 1$ is a probability

Then $\langle \psi_{AB} | \mathcal{O}_A \otimes id_B | \psi_{AB} \rangle = \sum_{i,j} \sqrt{p_i p_j} \langle \psi_A^i | \mathcal{O}_A | \psi_A^j \rangle \langle \psi_B^i | \psi_B^j \rangle = \sum_i p_i \langle \psi_A^i | \mathcal{O}_A | \psi_A^i \rangle$

$\langle \mathcal{O}_A \rangle_{\mathcal{H}_{AB}}$ $\qquad$ $\langle \mathcal{O}_A \rangle_{\mathcal{H}_A}$

$$= \mathrm{Tr}_{\mathcal{H}_A} \rho_A \mathcal{O}_A ,$$

for <u>density matrix</u> $\rho_A = \sum_i p_i |\psi_A^i\rangle\langle\psi_A^i|$.

Prop. ρ_A is hermitian positive semi-def of trace 1: $\mathrm{Tr}_{\mathcal{H}_A} \rho_A = \sum_i p_i = 1$.

Any matrix has this property can be "purified" into pure state of partitioned systems AB:

1. hermitian ρ_A can be diagonalized s.t. $\rho_A = \sum_i p_i |\psi_A^i\rangle\langle\psi_A^i|$

2. $\rho_A \geq 0 \Rightarrow \forall\, p_i > 0$

3. ρ_A is density matrix for pure states $\psi_{AB} = \sum_i \sqrt{p_i}\, \psi_A^i \otimes \psi_B^i \in \mathcal{H}_A \otimes \mathcal{H}_B$, some $\mathcal{H}_B$.

Rmk. 1. Purification ψ_{AB} of ρ_A is not unique but up to unitary transf.

2. Any non-neg. self-adj. matrix ρ (density matrix) of tr. 1 has purification, and any two of density matrices ρ_1, ρ_2, $\rho_t = t\rho_1 + (1-t)\rho_2$ for $0 \leq t \leq 1$ is also a density matrix. (convexity)

3. $\mathrm{rk}\, \rho_A = 1 \Rightarrow \psi_{AB} = \psi_A \otimes \psi_B$ is unentangled, system A is pure;

$\mathrm{rk}\, \rho_A > 1 \Rightarrow$ system A is in a mixed state;

$\rho_A = \frac{1}{\mathrm{rk}} \mathrm{Id} \Rightarrow$ System A is maximally mixed.

Similarly, given partition AB, one has ρ_B for instance.

In beginning one already assume ψ_{AB} itself is pure ($\rho_{AB} = 1$), however, we can generally assume the system AB has density matrix ρ_{AB}:

$$\rho_{AB} = \sum_{i,i',\alpha,\alpha'} c_{ii'\alpha\alpha'} |i\rangle_A \otimes |\alpha\rangle_B \cdot {}_A\langle i'| \otimes {}_B\langle \alpha'|$$

Then $\rho_A = \sum_{i,i',\alpha} c_{ii'\alpha\alpha'} |i\rangle_A {}_A\langle i'| = \mathrm{Tr}_{\mathcal{H}_B} \rho_{AB}$ (partial trace)

2. Quantum entropy

Given ρ_A, $S(\rho_A) = -\mathrm{Tr}\, \rho_A \log \rho_A$ (<u>Von-Neumann entropy</u>), which is invar. for unitary transf. $\rho_A \to U \rho_A U^{-1}$.

So it's diagonalized as $\rho_A \log \rho_A = \begin{pmatrix} p_1 \log p_1 & & \\ & p_2 \log p_2 & \\ & & \ddots \end{pmatrix}$, $S(\rho_A) = -\sum_i p_i \log p_i$

the same as Shannon entropy.

Rmk. A generalization: Rényi entropy $S_\alpha(\rho_A) = \frac{1}{1-\alpha} \log \mathrm{Tr}\, \rho_A^\alpha$.

Prop. $0 \le S(\rho_A) \le \log(\dim \rho_A)$, w/ upper-bd. $\rho_A = \frac{1}{\dim}\begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots \end{pmatrix}$ max. mixed.

Prop. (Concavity) ρ_1, ρ_2 are density matrices, $\rho(t) = t\rho_1 + (1-t)\rho_2$ is also density matrix for $0 \le t \le 1$. Then $\frac{d^2}{dt^2} S(\rho(t)) \le 0$, i.e. $S(\rho(t))$ concave.

Then $t S(\rho_1) + (1-t) S(\rho_2) \le S(t\rho_1 + (1-t)\rho_2) = S(\rho(t))$ (line is below S graph),

Rmk. Entropy only decreases under mixing.

3. Other entropies analogying classical

<u>quantum cdt'nl entropy</u>: $S(A|B) = S_{AB} - S_B$ for bipartite system AB,

unlike the classical case, $S(A|B)$ can be negative: $S_{AB} = 0$, $S_B > 0$.

<u>quantum mutual information</u>: $I(A:B) = S_A - S_{AB} + S_B \ge 0$, and $= 0$ iff density matrix factorizes: $\rho_{AB} = \rho_A \otimes \rho_B$.

<u>quantum relative entropy</u> ρ, σ two density matrix on $\mathcal{H}$, then

$S(\rho \| \sigma) = \mathrm{Tr}\, \rho (\log \rho - \log \sigma)$

just like the classical one: if assuming a system is described by σ but actually it's ρ, then to learn that is wrong one needs to observe N copies of systems that $NS(\rho\|\sigma) \gg 1$.

Prop. (Q-monotonicity) Similarly, let ρ_{AB} & σ_{AB} two density matrix of system AB, respectively $\rho_A = \mathrm{Tr}_B \rho_{AB}$, $\sigma_A = \mathrm{Tr}_B \sigma_{AB}$ of A by partial traces. Then

$$S(\rho_{AB}\|\sigma_{AB}) \geq S(\rho_A\|\sigma_A).$$

Cor. (Strong subadditivity)

Consider ABC tripartite w/ ρ_{ABC}, then $\rho_A = \mathrm{Tr}_{BC}\rho_{ABC}$, $\rho_{BC} = \mathrm{Tr}_A \rho_{ABC}$, $\sigma_{ABC} = \rho_A \otimes \rho_{BC}$. Then $\rho_{AB} = \mathrm{Tr}_C \rho_{ABC}$, $\sigma_{AB} = \mathrm{Tr}_C \sigma_{ABC} = \rho_A \otimes \rho_B$

$$S(\rho_{ABC}\|\sigma_{ABC}) \geq S(\rho_{AB}\|\sigma_{AB})$$

$$= S(\rho_{ABC}\|\rho_A \otimes \rho_{BC}) \qquad = S(\rho_{AB}\|\rho_A \otimes \rho_B)$$

$$= S_A + S_{BC} - S_{ABC} \qquad = S_A + S_B - S_{AB}$$

$$\Rightarrow S_{AB} + S_{BC} \geq S_B + S_{ABC}$$

Rmk. For $ABCD$ is pure, $S_{AB} = S_{CD}$, $S_{ABC} = S_D$, then

$$S_{CD} + S_{BC} \geq S_B + S_D$$

i.e. $S(C|D) = S_{CD} - S_D$ or $S(C|B) = S_{BC} - S_B$ can be negative, but $S(C|D) + S(C|B) \geq 0$

The "monogamy of entanglement": given qubit in C can be entangled w/ D and reduce S_{CD} or w/ B and S_{BC}, but not both.

Some remarks

Entanglement is easy to be defined on QM / lattice, but not in general QFT.

In lattice we may just like above, to introduce $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{A^c}$.

If region A has size of $L \gg \epsilon$ (lattice cutoff), consisting N_A sites,

then entropy is as large as $S_A \sim N_A \sim (\frac{L}{\epsilon})^{D-1}$ (vol. law growth)

However if consider short range entanglement across entangling surface,

$S_A \sim (\frac{L}{\epsilon})^{D-2}$ (area law)

given by the bonds being cut by entangling surface.

All of them diverge under continuous limit, thus entanglement entropy in QFT has "essential UV problems": the ultraviolet divergence.

Ref. 1907.08126

What is TQFT?

T(Q)FT is a theory only care about topological structure, for instance,

- CS action: $\int_{M_3} AdA + \frac{2}{3}A^3$ which has no metric dependence.
- Topologically twisted B-model: $Q_B \sim \bar{\partial}$ on target, essential data are Dolbeault cohomology of target sp., worldsheet metric (EMt) is Q-exact.

One sees that it only care about "shape" but not geometric details, so it has strong restriction on the alg. of field theory. (see 2d example)

Fancy language: it's a functor from geometry category to alg. category, by Atiyah's axiomatic approach of TQFTs.

What's a (1-) category? $\begin{cases} \text{obj. (mostly a set w/ extra structure) } (X, Y) \\ \text{Hom. (sets of maps between objs preserving structure)} \end{cases}$

$\mathrm{Hom}(X,Y)$ s.t. $\begin{cases} \text{id. map} \\ \text{composition} \\ \text{associativity} \end{cases}$

(n-) category? 2-Hom. (sets of maps of a Hom set)

Functors between cats.: maps that preserves cat. structures:

$F: \mathcal{C} \to \mathcal{D}$, $\forall X \xrightarrow{\varphi} Y$ in $\mathcal{C}$, then $F(X) \xrightarrow{F(\varphi)} F(Y)$ in $\mathcal{D}$ s.t.

$\exists$ canonical maps s.t.

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "\eta_X"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "\eta_Y"] \\ F(X) \arrow[r, "F(\varphi)"'] & F(Y) \end{tikzcd}$$

Example: Sets, Grps, etc.; Forgetful functor: Grps $\to$ Sets,

Example: Cat. of n-bordisms: $\begin{cases} \text{obj. (closed oriented } (n-1)\text{-mfd)} \ \phi, \Sigma_1, \Sigma_2, \cdots \\ \text{Hom (n-mfd) } M, \ \partial M = \# \Sigma_i \end{cases}$

Rmk: $(\mathrm{Bord}_n, \#, \phi)$ is a monoidal category.

Bord_2: $X =$, $Y =$, $\mathrm{Hom}(X,Y) =$

$\Leftrightarrow$ worldsheet of closed string.

TQFT$_2$ functor: $Z[\bigcirc] = \mathcal{H}$, $Z[\bigcirc \cdots \bigcirc] \in \mathcal{H} \otimes \cdots \otimes \mathcal{H}$, $Z[\bigcirc] \in \mathcal{H}^\vee$

$Z[$ $] = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3^\vee = \mathrm{Hom}(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{H}_3)$

$Z[\phi] \in \mathbb{C}$, since $\Sigma \# \phi = \Sigma \Rightarrow \mathcal{H} \otimes \mathbb{C} = \mathcal{H}$

$Z[$ $] = \mathrm{Hom}(\mathbb{C}, \mathbb{C}) = \mathbb{C}$ (partition function on closed worldsheet)

$Z[$ $] = \mathrm{Hom}(\mathcal{H}, \mathbb{C}) = |\psi\rangle$ a state in $\mathcal{H}$ (s.t. pairing maps $\mathcal{H} \to \mathbb{C}$)

Rmk: $(\mathrm{Vect}, \otimes, \mathbb{C})$ is also a monoidal category. $Z[$ $] = \langle \psi | \phi \rangle = Z[$ $] \in \mathbb{C}$.

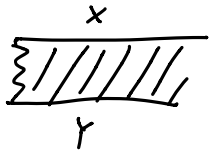
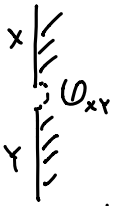
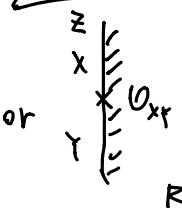
Rmk: All Bord_2 can be regarded as closed RS w/ punctures inside, and has pants decomposition (3-pt. function) , closed string states are inner-punctures of closed RS.

In conclusion, a TQFT functor is a such assignment of Hilbert sp. on punctures.

<u>D-brane view point of category</u>

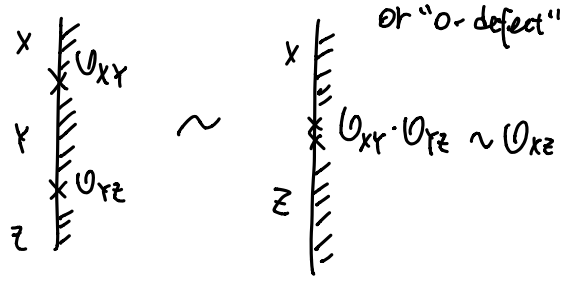
Consider what TQFT functor will assign to open RS, i.e. open string worldsheet

one assign $\mathcal{H}$ on string, but need bdry cdt. X, Y, Z on edges.

For instance, state-operator corresp.: $\mathcal{O}_{XY}$  $\sim$  , or 

thus open string state $\mathcal{O}_{XY}$ is called "bdry changing operator", as edge-punctures of open RS.

Fusion of open strings:



, and it's associative & bilinear.

$\Rightarrow$ Since $TQFT_2$ assign bdry to obj. of some cat., thus open string states are morphism between these objs. satisfying category axioms (one can always assing a trivial puncture as the id map in $\mathrm{Hom}(X,X)$.)

$\Rightarrow$ $TQFT_2$ functor:

closed 2-mfd $\to$ $\mathbb{C}$-#

closed 1-mfd $\to$ $\mathbb{C}$-v.s.

closed 0-mfd $\to$ $\mathbb{C}$-cat. (brave category)

Rmk. For higher TQFT functor, fusion of higher defects gives properties of higher maps in a weak higher category (i.e. pentagon identity, etc.)

$TQFT_n$: closed n-mfd $\to$ $\mathbb{C}$-#, also called extended TQFT.

$(n-1)$-mfd $\to$ $\mathbb{C}$-v.s.

$(n-2)$-mfd $\to$ $\mathbb{C}$-cat.

$(n-3)$-mfd $\to$ $\mathbb{C}$-2-cat.

$\vdots$

One recovers alg. of closed sector from open string category by "center of category".

philosophy: bdry-pictures can not commute, but inner-pictures can.

Algebra of 2d open-closed TQFT (Missed part)

In particular, $TQFT_2$ carries structure that restrict Hilbert sp. into Frobenius alg.

Entanglement branes, modular flow & ETQFT (1811.10785)

Entanglement v.s. factorization of worldsheet w/ bdry

Open-closed TQFT

1. Closed TQFT $\mathcal{C}$: carries Hilbert space on S^1:

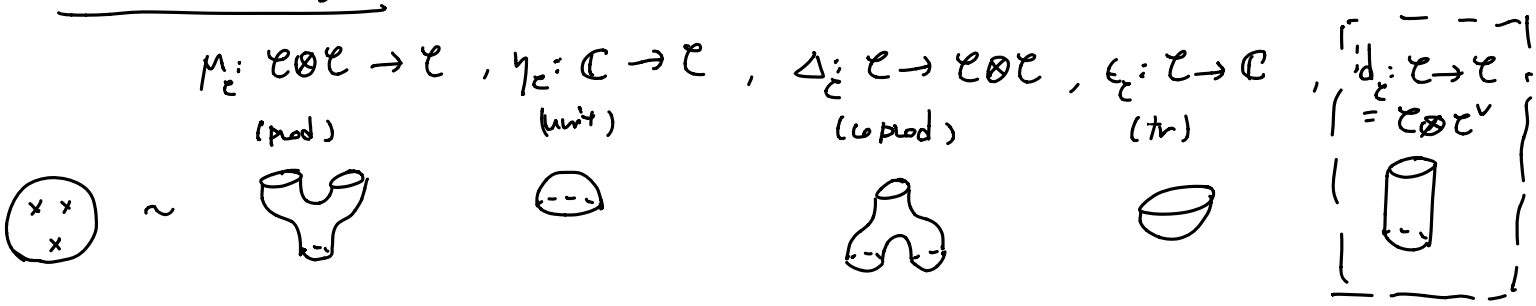
$$\coprod_n S^1 \mapsto \bigotimes_n \mathcal{C}, \quad \text{Cobordism} \mapsto \text{linear maps}.$$

orientation + gluing + pants $\Rightarrow$ Frobenius alg.

dual, composition, multiplication

Frobenius alg. is an alg. $\mathcal{C}$ w/ additional operation:

$\mu_{\mathcal{C}}: \mathcal{C}\otimes\mathcal{C} \to \mathcal{C}$ (prod), $\eta_{\mathcal{C}}: \mathbb{C} \to \mathcal{C}$ (unit), $\Delta_{\mathcal{C}}: \mathcal{C} \to \mathcal{C}\otimes\mathcal{C}$ (coprod), $\epsilon_{\mathcal{C}}: \mathcal{C} \to \mathbb{C}$ (tr), $\mathrm{id}_{\mathcal{C}}: \mathcal{C} \to \mathcal{C} = \mathcal{C}\otimes\mathcal{C}^\vee$

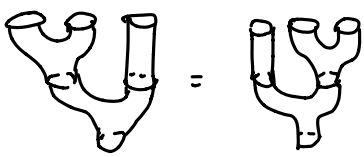


$\tau_{\mathcal{C}}: \mathcal{C}\otimes\mathcal{C} \to \mathcal{C}\otimes\mathcal{C}$ (braiding), $\pi_{\mathcal{C}} = \epsilon\circ\mu: \mathcal{C}\otimes\mathcal{C} \to \mathbb{C}$ (pairing)

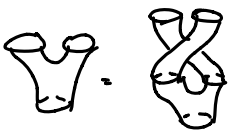
$\Rightarrow$ Closed TQFT$_2$ is a Frobenius alg.

All cpt. oriented 2-mfd. can be obtained from gluing above subjecting to:

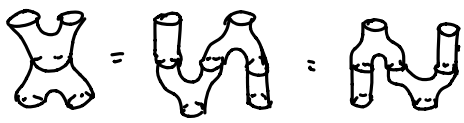
associativity: $\mu\circ(\mathrm{id}\times\mu) = \mu\circ(\mu\times\mathrm{id})$



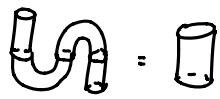
& commutativity: $\mu = \mu\circ\tau$



Frobenius cdt.: $\Delta\circ\mu = (\mu\times\mathrm{id})\circ(\mathrm{id}\times\Delta) = (\mathrm{id}\times\mu)\circ(\Delta\times\mathrm{id})$



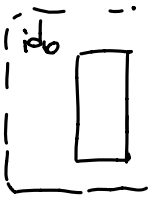
$\Rightarrow$ Zigzag identity: $(\pi\times\mathrm{id})\circ(\mathrm{id}\times\tilde{\pi}) = \mathrm{id}$

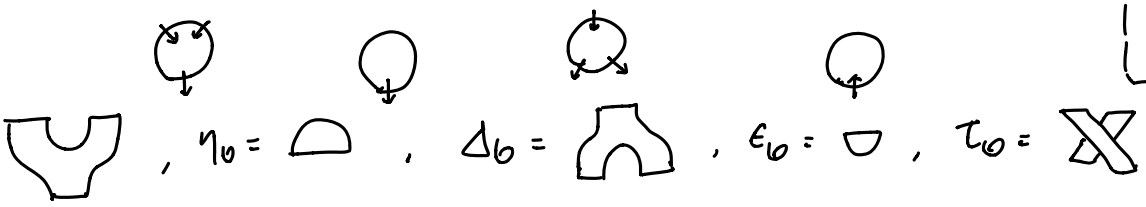


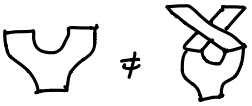
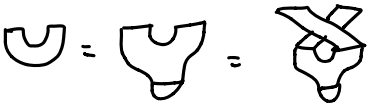
"π is non-degenerate"

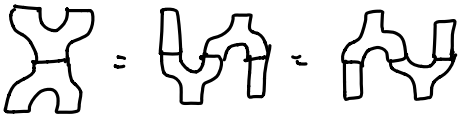
Rmk: sufficiently open 2-mfd w/ oriented punctures invariance: $\Pi(ab,c) = \Pi(a,bc)$:

2. Open TQFT $\mathcal{O}$: carries Hilbert sp. on I:

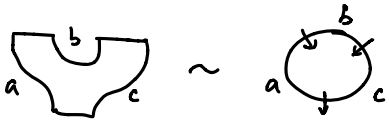


$\mu_0 =$  , $\eta_0 =$, $\Delta_0 =$, $\epsilon_0 =$, $\tau_0 =$

However μ_0 is associative but NOT commutative:  $\neq$, instead one wants alg. to be symmetric ($\tau_0 = \epsilon\circ\mu = \epsilon\circ\mu\circ\tau$): 

Frobenius cdt.: 

$\Rightarrow$ Open TQFT is a symmetric Frobenius alg.

D-brane data: (assigning bdy cdt.) 

, open string stretching branes are $\mathcal{H}_{ab}, \mathcal{H}_{bc}, \mathcal{H}_{ac}^{\vee}$

$M_{\mathcal{O},a,b,c} : \mathcal{H}_{ab} \times \mathcal{H}_{bc} \to \mathcal{H}_{ac}$. And gluing open worldsheets should subject to compatible bdy cdts.

Rmk: Thus open TQFT w/ D-branes actually forms a $\mathbb{C}$-category than algebra.

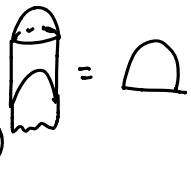
3. Open-closed TQFT

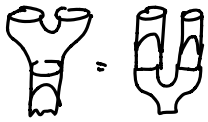
$\Rightarrow$ Knowledgable Frobenius alg. of $\mathcal{C}$ & $\mathcal{O}$, including operations:

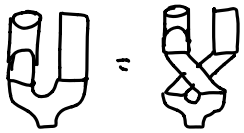
$i : \mathcal{C} \to \mathcal{O}$ (zipper)

$i^* : \mathcal{O} \to \mathcal{C}$ (closed zipper)

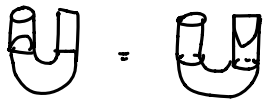
Rmk: I think these guys are disk w/ throat: i.e. inner punctuation

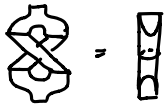
Subjecting to: 1. zipper preserves unit: 
$M_{\mathcal{O}}\circ(i\times id) = M_{\mathcal{O}}\circ\tau_{\mathcal{O}}\circ(i\times id_{\mathcal{O}})$

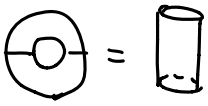
& product: 

2. Knowledge: 

i.e. $i : \mathcal{C} \to \mathcal{O}$ maps to center of open sector

3. Duality :  i.e. cozipper is dual of zipper

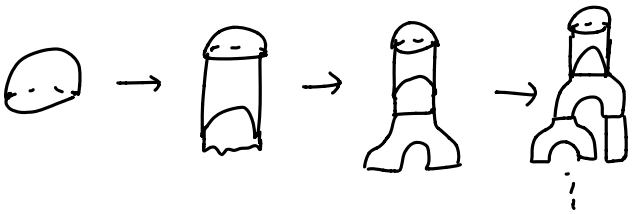
4. Candy :  $\mu_0 \circ \tau \circ \Delta_0 = i \circ i^*$. A familiar version is taking trace:

 , i.e. $\mathrm{Tr}_{\mathcal{H}_0^{\alpha\alpha'}} = \langle \mathcal{O}_\alpha \mathcal{O}_{\alpha'} \rangle$

X Entanglement Brane

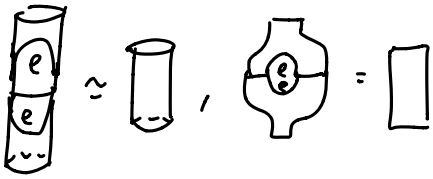
Describe entanglement between regions of space.

Example : Hartle-Hawking state $|HH\rangle =$ (unit) , as entanglement state of intervals :

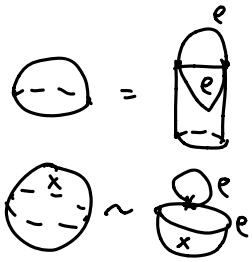
 (just like factorizing Hilbert state)

also, the one w/ bdry cdt.

Reduced density matrix : Let A & B be partition of system, w/ local operator $\mathcal{O}_A$ on A part $\Rightarrow$ $\mathrm{tr}_{\mathcal{H}_A}(\mathrm{tr}_{\mathcal{H}_B}(\rho)\mathcal{O}_A) = \mathrm{tr}(\rho(\mathcal{O}_A \otimes \mathbb{1}))$

$\Rightarrow$ to satisfy above, there exist a bdry cdt. s.t. 

$\Rightarrow$ entanglement brane.

Entanglement brane axiom : 

$\Rightarrow$ any hole in worldsheet can be closed up, any diagram w/ only closed input & output is equivalent to a diagram in closed theory.

From topological to quantum entanglement (1809.04574)

Entangled states in TQFT & von Neumann entropy

Entanglement : non-locality

Topological theory : non-locality (indep. of geometry) d.o.f.

Entanglement in TQFT : two subsystems are entangled if they're cobordant.

TQFT

(E) TQFT is a functor Z between cats. of top. sp. & vect. sp. :

M: (oriented) closed D-mfd $\xrightarrow{Z}$ $\mathbb{C}$-number (partition function is top. invar.)

Σ: closed $(D-1)$-mfd (i.e. bdry of D-mfd) $\xrightarrow{Z}$ $\mathbb{C}$-vector sp. (D-brane on bdry)

Π: closed $(D-2)$-mfd $\xrightarrow{Z}$ $\mathbb{C}$-linear cat. ("open string" vertex, or generally "defects")

$\vdots$

Functoriality : $Z(\Sigma_1 \amalg \Sigma_2 \amalg \cdots \Sigma_n) = Z(\Sigma_1) \otimes Z(\Sigma_2) \otimes \cdots \otimes Z(\Sigma_n)$

Fix $D=2$ (top. string theory), M: Riemann surfaces (w/ punctures),

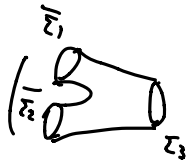
Σ : S^1, Π : pts. (not needed here).

$Z(\Sigma =$ $) = \mathcal{H}$ a v.s. ;

$Z(M =$ s.t. $\partial M = \Sigma) = |\psi\rangle \in \mathcal{H}$ a state in v.s. ;

anti-orientation $\Leftrightarrow$ conjugation

$Z(M =$ $=$ $\amalg$ $) = Z($ $) \otimes Z($ $) = \langle\chi|\psi\rangle \in \mathbb{C}$.

$Z\Big($  $\Big) = \bar{Z}(\Sigma_1) \otimes \bar{Z}(\Sigma_2) \otimes Z(\Sigma_3) = \mathrm{Hom}(\mathcal{H}_1 \otimes \mathcal{H}_2, \mathcal{H}_3)$

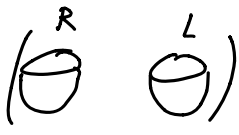
Ex. $D=3$. $Z = \int \mathcal{D}A\, e^{-S_{CS}[A]}$: $\{3\text{-mfd.}\} \to \{\text{Chern-Simons wave functions}\}$

Rmk: Cobordism $\Leftrightarrow$ worldsheet of string evolution. And all string data (v.s.) does not depend on local geometry of worldsheets.

TQFT is like QM. So we may consider entanglement here just like in QM.

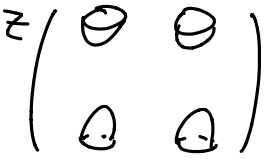
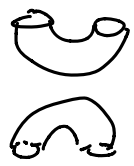
von Neumann entropy ($D=2$)

Hilbert sp. $\mathcal{H}$: $\mathcal{H} = Z\Big(\Sigma_R \amalg \Sigma_L = \big(\bigcirc\ \bigcirc\big)\Big)$

Quantum states $|\psi\rangle \in \mathcal{H}$: $|\psi_1\rangle = Z\Big($  $\Big)$, $|\psi_2\rangle = Z\Big($ $\Big)$

not-entangled (seperated) entangled

Dencity matrix: $\rho = |\psi\rangle\langle\psi| \in \mathrm{End}(\mathcal{H})$

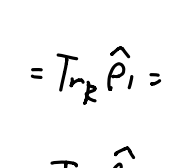
$\hat{\rho}_1 = Z\Big($  $\Big)$, $\hat{\rho}_2 = Z\Big($  $\Big)$

Normalization: $\mathrm{Tr}\,\hat{\rho}_1 = Z\Big($ $\Big)^2$, $\mathrm{Tr}\,\hat{\rho}_2 = Z\Big($ $\Big)$

$\mathrm{Tr} : \mathrm{End}(\mathcal{H}) \to \mathbb{C}$

$\parallel$

$\mathrm{ev} \circ a : \mathrm{End} \simeq \mathcal{H} \otimes \bar{\mathcal{H}} \xrightarrow{a} \bar{\mathcal{H}} \otimes \mathcal{H} \xrightarrow{\mathrm{ev}} \mathbb{C}$

Reduced density matrix: $\hat{\rho}_{L_1} = \mathrm{Tr}_R\, \hat{\rho}_1 = Z\Big($ $\Big)^{-2} \cdot Z\Big($  $\Big) = Z\Big($ $\Big)^{-1} \cdot Z\Big($ $\Big)$,

(partial trace ?)

$\hat{\rho}_{L_2} = \mathrm{Tr}_R\, \hat{\rho}_2 = Z\Big($ $\Big)^{-1} \cdot Z\Big($ $\Big) = Z\Big($ $\Big)^{-1} Z\Big($ $\Big)$

von Neumann formula: $S_L = -\mathrm{Tr}_L\, \rho_L \log \rho_L$ (don't know ...)

Replica trick: $S_L = -\frac{d}{dn} \mathrm{Tr}_L\, \rho_L^n \Big|_{n=1}$.

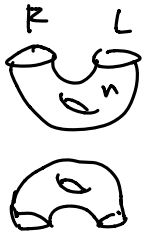
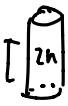
$\mathrm{Tr}_L \rho_{L_1}^n = Z(\bigcirc)^{-n} \cdot \mathrm{Tr}_L \cdot Z\Big($ $\Big\}_{n-1} = Z(\bigcirc)^{-n} \cdot Z(\bigcirc)^{n} = 1,$

$S_{L_1} = 0$

$\mathrm{Tr}_L \rho_{L_2}^n = Z(\bigcirc)^{-n} \cdot \mathrm{Tr}_L\, Z\Big($ $\Big) = Z(\bigcirc)^{1-n},$

$S_{L_2} = -\frac{d}{dn} Z(\bigcirc)^{1-n}\Big|_{n=1} \overset{(\text{r.t.})}{=} \lim_{n\to 0} \frac{d}{dn} Z^n = \log\left[Z(\bigcirc)\right] = \log(\dim \mathcal{H})$

The seperable system has 0 entropy. Entangling operator: $Z\Big($ $\Big)$.

$\rho = Z\Big[$  $\Big]$, $\mathrm{Tr}\rho = Z[g=2n+1]$, $\rho_L = Z[2n+1]^{-1} \cdot Z[$  $]$

Entanglement of links (D=3) (Missed)

$\Sigma = T^2$ which can be associated to a family of Hilbert sp $\mathcal{H}_N^k$, $N \geq 2$, $k \geq 0$,

$\dim \mathcal{H}_N^k = \binom{k+N-1}{N-1}$

$\overset{1:1}{\Leftrightarrow}$ fusion alg. of $SU(N)_k$ WZNW theory, also partition function of $SU(N)_k$ CS.

Entangled pair is or $= |\psi\rangle$ hollowed tube of solid T^2

Similar to the S_{L_2}, $Z(T^2) = \mathcal{H}_N^k$, $S_L = \log Z(T^2 \times S^1) = \log \dim \mathcal{H}_N^k$.

Compared w/ other ref. $|\psi\rangle$ is associated to a Hopf link & max. entangled.