Knot in a Box

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Abstract

Tame knots, which are equivalent to a polygonal knot with a finite number of segments, are well-studied; conversely, wild knots exhibit infinite and pathological behavior, and are difficult to study and classify. Knot mosaics are an example of a representation of tame knots, and our project aims to expand these mosaics to represent some wild knots. We introduce a formal system of infinite repeated insertions generating a wild mosaic, which characterizes a wild knot. This gives insight to many wild knots explored in existing literature and provides methods to generate and classify new examples.

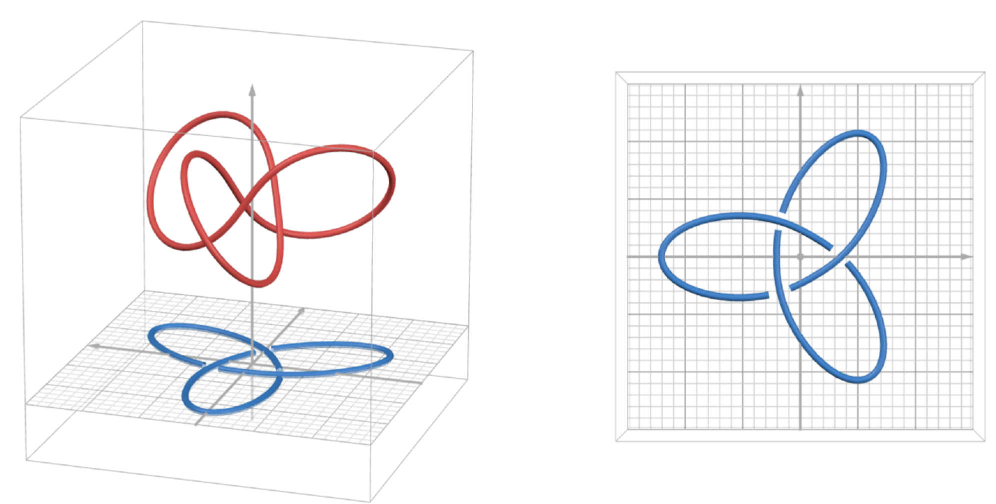
Background and Motivation

Definition. A **knot** is an embedding of the circle S^1 into $\mathbb{R}^3$, creating a simple (non-self-intersecting) closed curve in space.

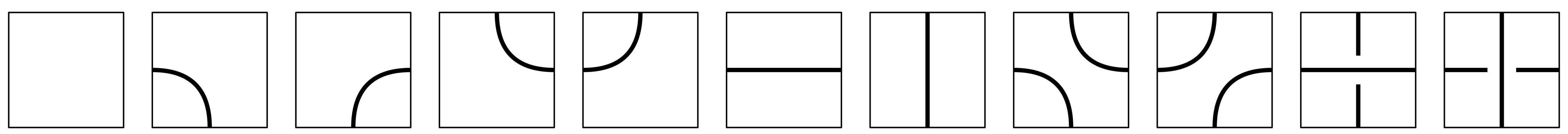
Knots are classified up to a distortion of space which does not cut and re-glue the knot or pass any strands through themselves. These distortions include stretching and deforming. Knots that are equivalent in this way are called **ambient isotopic**.

Definition. A **knot diagram** is a projection of a knot onto $\mathbb{R}^2$, drawn in such a way that crossings involve two transverse strands.

All knots can be drawn as knot diagrams.



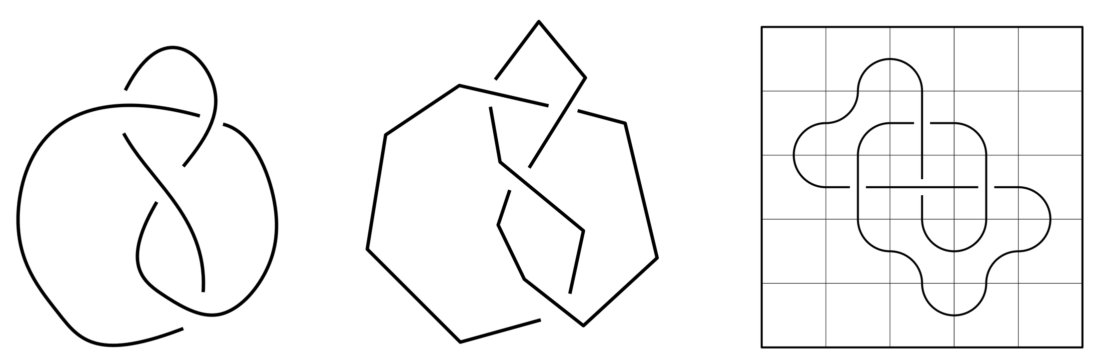
Definition. An **n-mosaic** is an $n \times n$ grid composed from the below set of 11 tiles, denoted $\mathbb{T}$. We say that it is a **knot n-mosaic** if the tiles are suitably connected. This comprises one method of representing and studying knots.



T_0 T_1 T_2 T_3 T_4 T_5 T_6 T_7 T_8 T_9 T_{10}

Definition. A knot is **tame** if it can be represented as the union of a finite set of line segments.

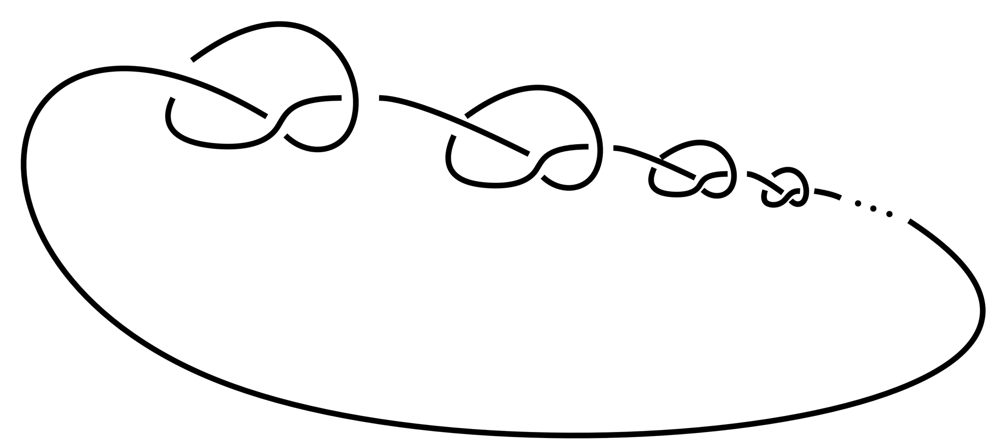
We say that knot mosaics form a *complete invariant* for tame knots: every valid knot mosaic represents a tame knot, and a tame knot can always be represented by a knot mosaic [3].



Definition. A **wild knot** is a knot that is not tame.

Wild knots are sometimes described as *pathological* because of their strange properties; for instance, their strands cannot be thickened to exist in the physical world. The methods used to study tame knots are invalid for wild knots, given their infinite nature.

Therefore, we aim to expand the knot mosaic framework to classify and represent wild knots.

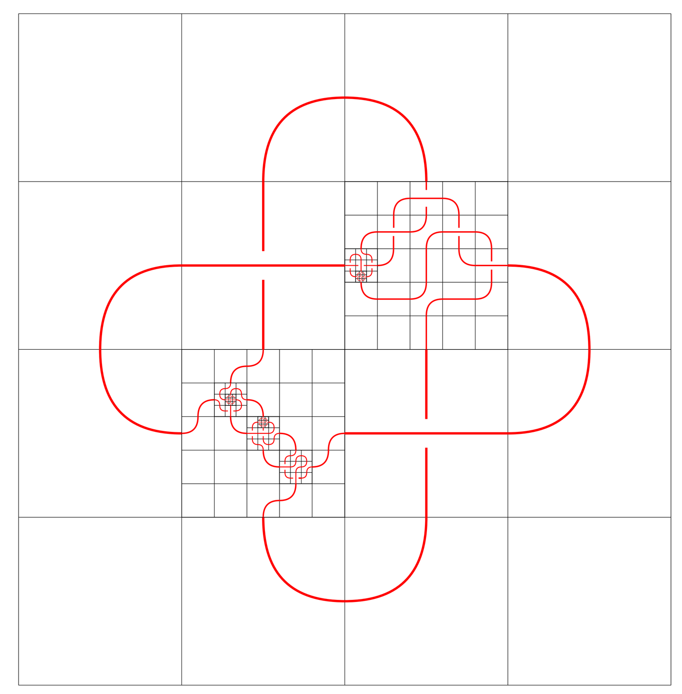


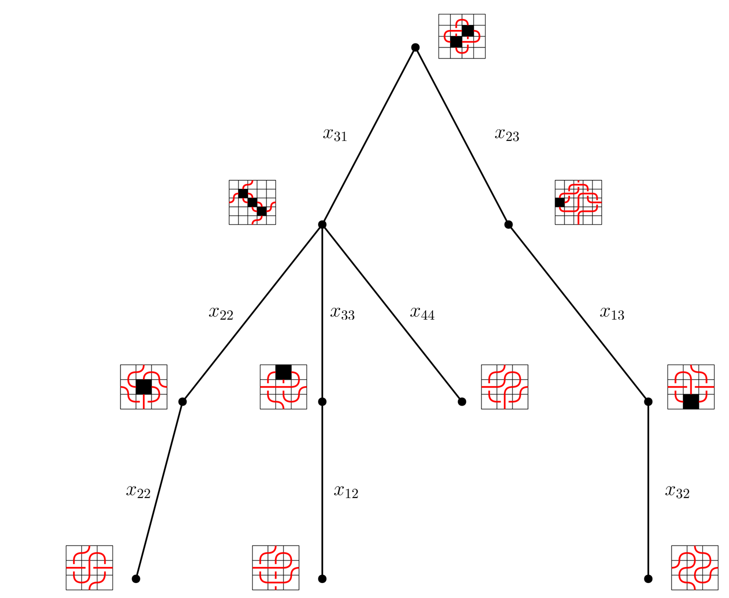
Recursive Mosaics

Definition. A **recursive knot mosaic** is a triple (T, F, i) where T is a (possibly infinite) rooted tree, F associates a (singular) knot mosaic with tile entries to each vertex, and i associates an embedding of mosaics to each edge.

Intuitively, this means dividing the knot diagram into tame sections, then creating mosaic representatives for tame sections of the knot. A tree structure tracks how these tame regions fit together, i.e. where each smaller mosaic is inserted.

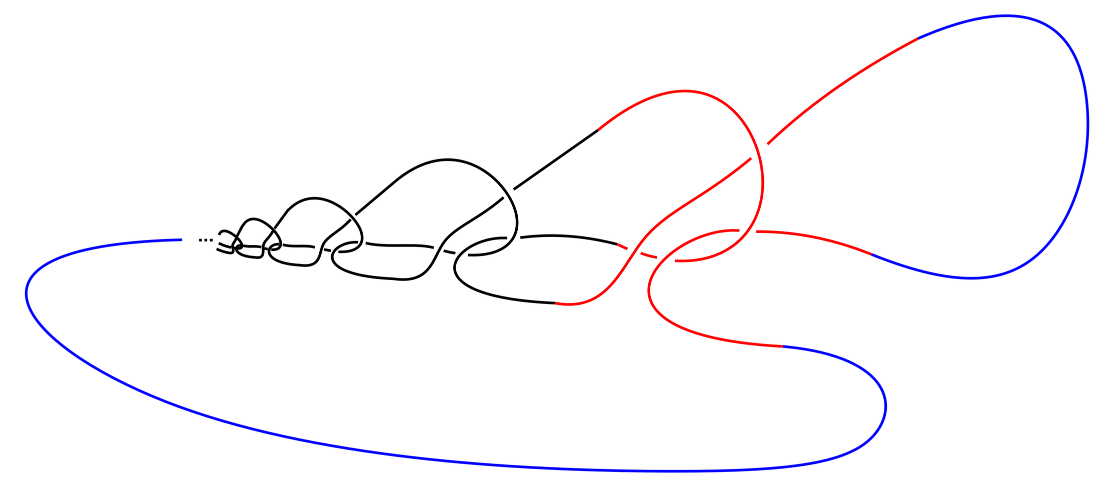
Recursive knot mosaics with *finite* trees can only represent *tame* knots. However, by allowing the tree to be infinite, we can represent certain wild knots.

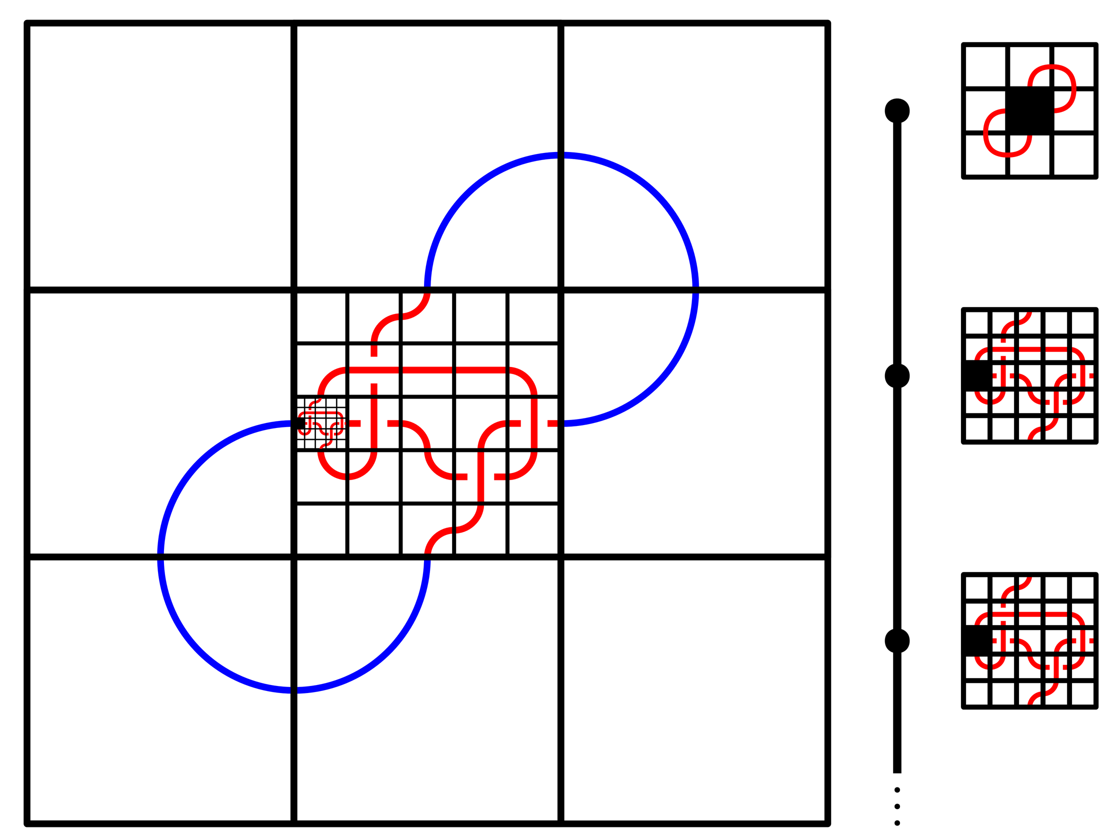




This wild knot, made of infinitely many slip knots, can never actually be unknotted completely [1]. No matter the number of loops we untangle, there are still infinitely many left to go.

This example illustrates the infinite nature of wild knots, which is captured concisely by its recursive knot mosaic.





Acknowledgments

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References

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