
Lecture3

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Laplace equation

- Laplacian operator

- A second-order differentiation

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

- Laplace equation

- For a function, $\phi(\mathbf{r})$, the Laplace equation reads

$$\nabla^2 \phi(\mathbf{r}) = 0$$

- Of course, we need boundary conditions.

- Its application?

- In this course, it is closely related to the Poisson equation.

Analytic solution

- Consider the 1D structure, $[0, a]$.
 - The boundary conditions for $\phi(x)$ are given by $\phi(0) = 0$ and $\phi(a) = 1$.
- It is easy to solve it.
 - In the 1D structure, $\frac{d^2}{dx^2} \phi = 0$.
 - The solution has the form of $\phi(x) = C_1 x + C_0$.
 - Using the boundary conditions, it is found that $\phi(x) = \frac{x}{a}$.

Solution vector

- Let us construct a matrix equation corresponding to the Laplace equation.
 - We assign N points. They are uniformly distributed. For the i -th point, it has the coordinate of
$$x_i = \frac{i - 1}{N - 1} a$$
 - The solution function, $\phi(x)$, can be approximated by
$$[\phi(x_1) \quad \phi(x_2) \quad \cdots \quad \phi(x_{N-1}) \quad \phi(x_N)]^T$$
 - Therefore, N equations are needed to determine the solution vector.
 - Remember that the Laplacian operator is involved.

Discretization

- How to assign N equations
 - For $2 \leq i \leq N - 1$, the discretized Laplacian operator is used.

$$\left. \frac{d^2 \phi}{dx^2} \right|_{x=x_i} \approx \frac{\phi(x_{i+1}) - 2\phi(x_i) + \phi(x_{i-1}))}{\Delta x^2} = 0$$

- The Laplace equation at $x = x_i$ can be written as

$$\frac{1}{(\Delta x)^2} \begin{bmatrix} 0 & \dots & 1 & -2 & 1 & \dots & 0 \end{bmatrix} \begin{bmatrix} \phi(x_1) \\ \vdots \\ \phi(x_{i-1}) \\ \phi(x_i) \\ \phi(x_{i+1}) \\ \vdots \\ \phi(x_N) \end{bmatrix} = [0]$$

Boundary condition

- How to assign N equations (continued)
 - For $i = 1$ or $i = N$, the boundary condition is applied.
 - For example, for $i = N$, the boundary condition can be written in a matrix form:

$$\begin{bmatrix} 0 & \dots & 0 & 0 & 0 & \dots & \color{red}{1} \end{bmatrix} \begin{bmatrix} \phi(x_1) \\ \vdots \\ \phi(x_{i-1}) \\ \phi(x_i) \\ \phi(x_{i+1}) \\ \vdots \\ \phi(x_N) \end{bmatrix} = \begin{bmatrix} \color{red}{1} \end{bmatrix}$$

$N = 5$ example

- Let us collect the previous results explicitly.
 - It is simply given by

$$Ax = \begin{bmatrix} \color{red}{1} & 0 & 0 & 0 & 0 \\ \color{red}{1} & \color{red}{-2} & \color{red}{1} & 0 & 0 \\ 0 & \color{red}{1} & \color{red}{-2} & \color{red}{1} & 0 \\ 0 & 0 & \color{red}{1} & \color{red}{-2} & \color{red}{1} \\ 0 & 0 & 0 & 0 & \color{red}{1} \end{bmatrix} \begin{bmatrix} \phi(x_1) \\ \phi(x_2) \\ \phi(x_3) \\ \phi(x_4) \\ \phi(x_5) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \color{red}{1} \end{bmatrix} = b$$

- How can we solve it numerically?

MATLAB example

- Step-by-step procedure

- First, set the matrix, A .

```
A = zeros(5,5);
```

```
A(1,1) = 1.0;
```

```
A(2,1) = 1.0; A(2,2) = -2.0; A(2,3) = 1.0;
```

```
A(3,2) = 1.0; A(3,3) = -2.0; A(3,4) = 1.0;
```

```
A(4,3) = 1.0; A(4,4) = -2.0; A(4,5) = 1.0;
```

```
A(5,5) = 1.0;
```

- Next , set the vector, b .

```
b = zeros(5,1); b(5,1) = 1.0;
```

- Finally, get the solution vector, x .

```
x = A \ b
```


Generalized Poisson equation?

- Laplace equation

$$\nabla^2 f(\mathbf{r}) = 0$$

- Poisson equation

$$\nabla^2 f(\mathbf{r}) = s(\mathbf{r})$$

- Here, $s(\mathbf{r})$ is the source function.
 - Therefore, the source-free Poisson equation is the Laplace equation.
- Then, why do we consider the “generalized” Poisson equations?
 - What is the exact meaning of the generalization?

Derivation

- One of the four Maxwell equations

$$\nabla \cdot \mathbf{D} = \rho(\mathbf{r})$$

$\mathbf{D}(\mathbf{r})$: Displacement vector, which is given by $\epsilon(\mathbf{r})\mathbf{E}(\mathbf{r})$

$\mathbf{E}(\mathbf{r})$: Electric field

$\epsilon(\mathbf{r})$: Permittivity (assumed as a scalar)

$\rho(\mathbf{r})$: Net charge density

- Under the electrostatic approximation, $\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r})$.
 - Using the above expression, it is found that
$$\nabla \cdot [\epsilon(\mathbf{r})\nabla\phi(\mathbf{r})] = -\rho(\mathbf{r})$$
 - The above equation is (incorrectly) called as the “Poisson equation.”

Homework#3

- Due: AM08:00, September 14 (Next Monday)
- Upload your code and report in your subfolder.
- Problem#1
 - Write a simple code for solving the Laplace equation.
 - In this case, apply $\phi(0) = 1$ and $\phi(a) = -1$.
- Problem#2
 - Write a simple code for solving $\frac{d^2}{dx^2} \phi = \delta(x - \frac{a}{2})$.
 - In this case, apply $\phi(0) = 0$ and $\phi(a) = 0$.