HW3 Laplace equation

20202041 Nuri Park

Problem 1.

Laplace equation : $\nabla^2\phi(x) = 0$,

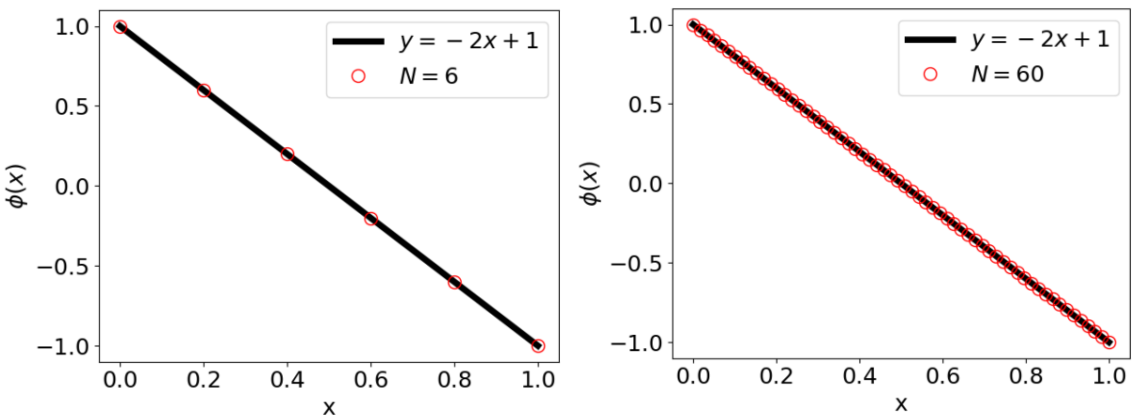
The boundary condition of $\phi(x)$ are given by $\phi(0) = 1$, $\phi(a) = -1$ we set $a = 1$.

Obviously the analytic solution is,

$$Analytic\ solution : \phi(x) = -2x + 1$$

As we did in the previous homework, we could represent Laplacian operator numerically as

$$Discretization\ (N = 3) : \nabla^2\phi(x) \rightarrow \quad \nabla^2 = \frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$



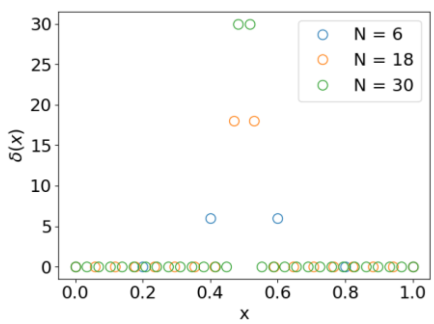
We found that numerical result is similar with analytical result.

Problem 2.

Poisson equation : $\nabla^2\phi(x) = \delta\left(x - \frac{a}{2}\right)$

The Dirac delta function is located at the center $\frac{a}{2}$. Since we should represent it numerically, we need to make it finitely exists. First, the integral of delta function from $-\infty$ to $+\infty$ is one : $\int \delta(x)dx = 1$. Second, we have N points in the x range. Thus we could represent Dirac delta function as,

$$\delta(x) \simeq N, for\ x = \frac{a}{N}, \frac{a+1}{N}$$
$$\simeq 0,\ else$$



Now we consider analytical solution. The difference of the derivative at x=a and x=0 is 1. That is because … :

$$\int dx\nabla^2\phi(x) = \int dx\delta\left(x - \frac{a}{2}\right) = 1$$

$$= \frac{d\phi}{dx}_{x=a} - \frac{d\phi}{dx}_{x=0}$$

By the way, if the range of the integral is out of delta function, the difference of the derivative must be same…

$$\frac{d\phi}{dx}_{x<a/2} = \frac{d\phi}{dx}_{x'<a/2}$$

Therefore, analytical solution is :

$$\phi(x) = -\frac{1}{2}x\,,\ \ (0 < x < 1/2)$$

$$= \frac{1}{2}x - \frac{1}{2}, (1/2 < x < 1)$$