On the formality of nilpotent Lie algebra

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Abstract

An overview of nilpotent Lie algebras, their cohomology and degree of formality. An interesting proof regarding the 'upper bound' of the degree of 2-step nilpotent Lie algebras is also given.

1 Overview of nilpotent Lie algebra

Definition 1.1. *Let G be a group. For all elements g and h in G, the* ***commutator*** *of g and h is defined as*

$$[g, h] = ghg^{-1}h^{-1}.$$

Note that $[g, h] = e_G$ if and only if $gh = hg$, that is, if and only if g and h commute; thus, G is abelian.

Definition 1.2. *Let G be a group. The* ***commutator subgroup*** *of G is*

$$[G, G] = \{[g, h] \mid g, h \in G\}.$$

Proposition 1.1. *The commutator subgroup $[G, G]$ of a group G is normal.*

Proof. Let $g \in G$ and $[x, y] \in [G, G]$. Then,

$$\begin{aligned}
g[x, y]g^{-1} &= g(xyx^{-1}y^{-1})g^{-1} \\
&= gx \cdot (g^{-1}g) \cdot y \cdot (g^{-1}g) \cdot x^{-1} \cdot (g^{-1}g) \cdot y^{-1}g^{-1} \\
&= gxg^{-1}gyg^{-1}gx^{-1}g^{-1}gy^{-1}g^{-1} \\
&= (gxg^{-1})(gyg^{-1})(gx^{-1}g^{-1})(gy^{-1}g^{-1}) \\
&= (gxg^{-1})(gyg^{-1})(gxg^{-1})^{-1}(gyg^{-1})^{-1},
\end{aligned}$$

which is an element of $[G, G]$. Thus, $[G, G]$ is normal. ∎

Recall that, if N is a normal subgroup of a group G, then

$$G/N = \{gN \mid g \in G\}$$

forms a group. In the case of commutator subgroup, $G/_{[G,G]}$ is said to be the ***abelianization*** of the group G.

Inductively, define a series of normal subgroups of G

$$G = G_0 \trianglerighteq G_1 \trianglerighteq ... \trianglerighteq G_n \trianglerighteq ...$$

such that $G_{i+1} = [G, G_i]$. This series is called the ***lower central series*** of G, and it is clear that G_i's are fully invariant subgroups of G.

Definition 1.3. *A group G is* ***nilpotent****, or* ***n-step nilpotent****, if $G_n = \{1\}$ for some n.*

Definition 1.4. *A* ***Lie algebra*** *$\mathfrak{g}$ is a vector space with a bilinear operation $[\cdot,\cdot] : \mathfrak{g} \ x \ \mathfrak{g} \to \mathfrak{g}$, or* ***the Lie bracket****, satisfying:*

- *Skew-symmetry: $[x, y] = -[y, x]$ (thus $[x, x] = 0$), and*
- *Jacobi identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.*

A **Lie subalgebra** $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is a vector space closed under the Lie bracket, i.e, $[h_1, h_2] \in \mathfrak{h}$ for all $h_1, h_2 \in \mathfrak{h}$. An **ideal** $\mathfrak{i}$ of $\mathfrak{g}$ is a subalgebra such that $[x, y] \in \mathfrak{i}$ for all $x \in \mathfrak{i}$ and $y \in \mathfrak{g}$. In other words, $[\mathfrak{g}, \mathfrak{i}] \subseteq \mathfrak{i}$ where $[\mathfrak{g}, \mathfrak{i}]$ is the subalgebra generated by the elements $[x, y]$, with $x \in \mathfrak{i}$ and $y \in \mathfrak{g}$.

In analogy with group theory, one can define the **lower central series of a Lie algebra** $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{g}_0 \supseteq \mathfrak{g}_1 \supseteq ... \supseteq \mathfrak{g}_n \supseteq ...$$

where $\mathfrak{g}_{i+1} = [\mathfrak{g}, \mathfrak{g}_i]$. One thing to observe here is that $\mathfrak{g}_i$ are ideals in $\mathfrak{g}$ for all $i > 0$.

Definition 1.5. *A Lie algebra $\mathfrak{g}$ is* ***nilpotent*** *if $\mathfrak{g}_n = 0$ for some n.*

2 Overview of cohomology and formality

Definition 2.1. *Let V be a vector space over a field k. The* ***dual space*** *V^* is defined as*

$$V^* = Hom(V, k) = \{\varphi : V \to k \mid \varphi \text{ is linear}\}.$$

The dual space V^* is also a vector space, and its dimension equals to the vector space V. If $\mathcal{B} = \{v_1, ..., v_n\}$ is a basis for V, define elements $\epsilon_1, ..., \epsilon_n$ by the formula

$$\epsilon_i(v_j) = \delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Each ϵ_i define an element of $V*$ if one extends by linearity, i.e, if $v = c_1 v_1 + ... + c_n v_n$, then

$$\begin{aligned}\epsilon_i(v) &= \epsilon_i(c_1v_1 + ... + c_nv_n)\\ &= c_1\epsilon_i(v_1) + ... + c_n\epsilon_i(v_n)\\ &= c_i\epsilon_i(v_i)\\ &= c_i.\end{aligned}$$

In fact, $\{\epsilon_i\}$ form a basis for V^*, called **dual basis**.

Definition 2.2. *A **differential k-form** is a tensor field of rank k that is skew-symmetric under exchange of any pair of indices.*

Definition 2.3. *The **exterior algebra** $\bigwedge(V)$ of a vector space V is the direct sum over k in the natural numbers of the vector spaces of alternating differential k-forms on V.*

Definition 2.4. *The **wedge product** $\wedge$, or **exterior product**, of differential forms is a product in an exterior algebra.*

Definition 2.5. *(Properties of wedge products)*
Let $\alpha, \alpha_1, \alpha_2, \beta, \beta_1, \beta_2, \omega$ be differential forms.

- *If α is a p-form and β is a q-form, then $\alpha \wedge \beta = (-1)^{pq}\beta \wedge \alpha$ is a $(p+q)$-form,*
- *Association: $(\alpha \wedge \beta) \wedge \omega = \alpha \wedge (\beta \wedge \omega)$,*
- *Bilinearity:*

$$\begin{aligned}(c_1\alpha_1 + c_2\alpha_2) \wedge \beta = c_1(\alpha_1 \wedge \beta) + c_2(\alpha_2 \wedge \beta)\\ \alpha \wedge (c_1\beta_1 + c_2\beta_2) = c_1(\alpha \wedge \beta_1) + c_2(\alpha \wedge \beta_2),\end{aligned}$$

  where c_1 and c_2 are constants.

Definition 2.6. *The **k-th exterior power** $\bigwedge^k(V)$ is defined as the quotient vector space*

$$\bigwedge^k(V) = \bigotimes^p(V) \big/ W_p,$$

where W_p is the subspace of p-tensors generated by transpositions.

Then, one can form a *chain complex*, which is a sequence of abelian groups (or *chain groups*) connected by homomorphisms:

$$... \xrightarrow{\partial_{n+1}} \bigwedge^n \mathfrak{g} \xrightarrow{\partial_n} \bigwedge^{n-1} \mathfrak{g} \xrightarrow{\partial_{n-1}} \bigwedge^{n-2} \mathfrak{g} \xrightarrow{\partial_{n-2}} ...$$

Homomorphism ∂_n are called *boundary maps* with the property $\partial_n \circ \partial_{n+1} = 0$ for all n.

Definition 2.7. *Let* $\mathfrak{g}$ *be a Lie algebra. The* ***p-th homology groups of*** $\mathfrak{g}$ *is defined as*

$$H_p(\mathfrak{g}) = \frac{Ker(\partial : \wedge^p \mathfrak{g} \to \wedge^{p-1}\mathfrak{g})}{Im(\partial : \wedge^{p+1}\mathfrak{g} \to \wedge^p \mathfrak{g})}$$

Definition 2.8. *Let* $\mathfrak{g}$ *be a Lie algebra and* V *be a vector space. Then, a* ***representation of*** $\mathfrak{g}$ ***on*** V *is a homomorphism:*

$$\rho : \mathfrak{g} \to End(V)$$

where $End(V)$ *is the space of endomorphisms of* V *equipped with the Lie bracket* $[\alpha, \beta] = \alpha\beta - \beta\alpha$ *for all* α, β *in End(V).*

This means that ρ should be a linear map satisfying

$$\rho([X, Y]) = \rho(X)\rho(Y) - \rho(Y)\rho(X)$$

for all X, Y in $\mathfrak{g}$. The vector space V and the map ρ are called **$\mathfrak{g}$-modules**.

The cohomology groups are defined in a similar way as a dual object of homology groups. The *cochain groups* C^n are defined as the dual of the chain group $C_n = \wedge^n \mathfrak{g}$.

Definition 2.9. *The* ***space of*** p***-forms on*** $\mathfrak{g}$ ***with values in*** V*, or the* ***cochain groups****, is defined as*

$$C^p(\mathfrak{g}, V) = Hom(\bigwedge^p \mathfrak{g}, V) \cong \bigwedge^p \mathfrak{g}^* \otimes V.$$

For computing purposes, all Lie algebras are finite-dimensional and V is a trivial $\mathfrak{g}$-module. Then,

$$C^p(\mathfrak{g}, V) \cong \bigwedge^p \mathfrak{g}^*.$$

Lastly, define the *coboundary map* $d : C^p(\mathfrak{g}, V) \to C^{p+1}(\mathfrak{g}, V)$ as follows:

- for every $v \in V$ and $X \in \mathfrak{g}$, $dv(X) = 0.$;
- for every $\alpha \in \mathfrak{g}^*$ and $X_1, X_2 \in g$, $d\alpha(X_1, X_2) = -\alpha([X_1, X_2])$;
- extend to $\wedge^* \mathfrak{g}^*$ by

  $$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta$$

  where $|\alpha|$ is the degree of α in $\mathfrak{g}$,
- and extend to $\wedge^* \mathfrak{g}^* \otimes V$ by

  $$d(\omega \otimes v) = d\omega \otimes v.$$

One important thing to note is that $d_n \circ d_{n-1} = 0$. Then, one acquires the **Chevalley-Eilenberg cochain complex of $\mathfrak{g}$**

$$... \xrightarrow{d_{n-2}} \wedge^{n-1}\mathfrak{g}^* \xrightarrow{d_{n-1}} \wedge^n \mathfrak{g}^* \xrightarrow{d_n} \wedge^{n+1}\mathfrak{g}^* \xrightarrow{d_{n+1}} ...$$

Definition 2.10. *Let* $\mathfrak{g}$ *be a Lie algebra. The* ***p-th cohomology groups of*** $\mathfrak{g}$ *is defined as*

$$H^p(\mathfrak{g}) = \frac{Ker(d : \wedge^p \mathfrak{g}^* \to \wedge^{p+1}\mathfrak{g}^*)}{Im(d : \wedge^{p-1}\mathfrak{g}^* \to \wedge^p \mathfrak{g}^*)}$$

Definition 2.11. *Let* $\mathfrak{g}$ *be a 2-step nilpotent Lie algebra. Then,* $\mathfrak{g}$ *is* ***k-formal*** *if and only if the map* $\Psi_{k+1} : H_{k+1}(\mathfrak{g}) \to H_{k+1}(\mathfrak{a})$ *is injective, where* $\mathfrak{a}$ *is the abelianization of* $\mathfrak{g}$.

A simple observation shows that every 2-step nilpotent Lie algebra $\mathfrak{g}$ is always at least 0-formal:

$$H_1(\mathfrak{g}) = \frac{Ker(\partial_1 : \mathfrak{g} \to 0)}{Im(\partial_2 : \bigwedge^2 \mathfrak{g} \to \mathfrak{g})} \cong \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}], \text{ and}$$

$$H_1(\mathfrak{a}) = \frac{Ker(\partial_1 : \mathfrak{a} \to 0)}{Im(\partial_2 : \bigwedge^2 \mathfrak{a} \to \mathfrak{a})} = \frac{\mathfrak{a}}{0} = \mathfrak{a} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}].$$

It is also worth pointing out that

$$H_k(\mathfrak{a}) = \frac{Ker(\partial_k : \bigwedge^k \mathfrak{a} \to \bigwedge^{k-1} \mathfrak{a})}{Im(\partial_{k+1} : \bigwedge^{k+1} \mathfrak{a} \to \bigwedge^k \mathfrak{a})} = \frac{\bigwedge^k \mathfrak{a}}{0} = \bigwedge^k \mathfrak{a}.$$

Proposition 2.1. *Let* V *be a simplicial complex and* k *be a field. Then,*

$$H^n(K, k) = (H_n(K, k))^*.$$

Remark. *Let* V *and* W *be finite dimensional vector spaces over a field* k*. If the linear transformation* $L : V \to W$ *is injective, then the dual map* $L^* : W^* \to V^*$ *is surjective.*

Using the remark above and Proposition 2.1, one can define formality in terms of cohomology:

Definition 2.12. *Let* $\mathfrak{g}$ *be a 2-step nilpotent Lie algebra. Then,* $\mathfrak{g}$ *is* ***k-formal*** *if and only if the map* $\Psi_{k+1} : H^{k+1}(\mathfrak{a}) \to H^{k+1}(\mathfrak{g})$ *is surjective, where* $\mathfrak{a}$ *is the abelianization of* $\mathfrak{g}$.

Proposition 2.2. *Let* $\mathfrak{g}$ *be a 2-step nilpotent Lie algebra and* $\mathfrak{a} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ *be the abelianization of* $\mathfrak{g}$*. Suppose* $dim(\mathfrak{a}) = r$*. Then,* $\mathfrak{g}$ *is at most* $\lfloor \frac{r}{2} \rfloor$*-formal.*

Proof. Let $\{X_1, X_2, ..., X_r, Y_1, Y_2, ..., Y_s\}$ be a basis of $\mathfrak{g}$, where $\{X_1, ..., X_r\}$ generate $\mathfrak{a}$, $\{Y_1, ..., Y_s\}$ is a basis of $[\mathfrak{g}, \mathfrak{g}]$, and $dim(\mathfrak{g}) = n = r + s$. Suppose $\mathfrak{g}$ is k-formal. By definition 2.12., the map $\Psi_{k+1} : H_{k+1}(\mathfrak{g}) \to H_{k+1}(\mathfrak{a})$ is injective, where $H_{k+1}(\mathfrak{a}) \cong \bigwedge^{k+1}(\mathfrak{a}) = \bigwedge^{k+1}\langle X_1, ...X_r\rangle$. On the other hand, $H_n(\mathfrak{g}) = \langle X_1 \wedge ... \wedge X_r \wedge Y_1 \wedge ... \wedge Y_s\rangle \cong \mathbf{R}$.

Let $x \in H_{k+1}(\mathfrak{g})$ be nonzero. Thus, $x = \sum_I a_I X_I$ with the index set $I = \{i_m \mid 1 \leq i_1 < i_2 < ... < i_{k+1} \leq r\}$ and $a_I \neq 0$. By Poincaré duality, there exists $y \in H_{n-(k+1)}$ such that

$$x \wedge y = X_1 \wedge ... \wedge X_r \wedge Y_1 \wedge ... \wedge Y_s,$$

where $y = \sum_J b_J Z_J$ and an index set $J = (\{1, 2, ..., r\} - \{i_1, i_2, ..., i_k\} \cup \{1, 2, ..., s\})$. Then, by the rank-nullity theorem, $r+s-(k+1) = n-(k+1) > r-(k+1) > k+1$. By working out the inequality, it follows that $k + 1 < r/2$. Therefore, $\mathfrak{g}$ is at most $\lfloor r/2 \rfloor$-formal.

□

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