$= \underline{\underline{0}}$

2). Find the 1^{st} order partial derivatives or rate of chan

(i) $f(x,y) = x^y$

(ii) $f(r,s) = r\ln(r^2+s^2)$

(iii) $f(x,y) = \dfrac{x-y}{x+y}$

(iv) $f(x,y) = x^3 + x^2y^3 - 2y^2$ at $(2,1)$ in x-direction.

(v) $f(x,y) = 4 - x^2 - 2y^2$ at $(1,1)$ in y-direction.

(iii) Verify Clairaut's theorem for $u = xye^y$

$\Rightarrow$ $u = xye^y$

$$\frac{\partial u}{\partial x} = ye^y$$

$$\frac{\partial u}{\partial y} = x\left[y.e^y + e^y\right]$$

$$\frac{\partial^2 u}{\partial x.\partial y} = y.e^y + e^y = u_{xy}$$

$$\frac{\partial^2 u}{\partial y.\partial x} = y.e^y + e^y = u_{yx}$$

$$\therefore u_{xy} = u_{yx}$$

If $u = f\left(\frac{y-x}{xy}, \frac{z-x}{xz}\right)$ then find the value of

$$x^2\frac{\partial u}{\partial x} + y^2\cdot\frac{\partial u}{\partial y} + z^2\cdot\frac{\partial u}{\partial z}$$

$\Rightarrow$ Let $R = \frac{1}{x} - \frac{1}{y}$ & $S = \frac{1}{x} - \frac{1}{z}$

$$u \longrightarrow (R, S) \longrightarrow (x, y, z)$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial R}\cdot\frac{\partial R}{\partial x} + \frac{\partial u}{\partial S}\cdot\frac{\partial S}{\partial x}$$

$$= \frac{\partial u}{\partial R}\cdot\left(\frac{-1}{x^2}\right) + \frac{\partial u}{\partial S}\cdot\left(\frac{-1}{x^2}\right)$$

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$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial R} \cdot \frac{\partial R}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y}$$

$$= \frac{\partial u}{\partial R} \cdot \left(\frac{1}{y^2}\right) + \frac{\partial u}{\partial s}(0)$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial R} \cdot \frac{\partial R}{\partial z} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial z}$$

$$= \frac{\partial u}{\partial R}(0) + \frac{\partial u}{\partial s} \cdot \left(\frac{1}{z^2}\right)$$

$$x^2 \frac{\partial u}{\partial x} + y^2 \cdot \frac{\partial u}{\partial y} + z^2 \cdot \frac{\partial u}{\partial z}$$

$$\cancel{x^2} \cdot \left(-\frac{1}{\cancel{x^2}}\right)\left(\frac{\partial u}{\partial R} + \frac{\partial u}{\partial B}\right) + \cancel{y^2} \cdot \frac{1}{\cancel{y^2}} \cdot \frac{\partial u}{\partial R} + \cancel{z^2} \cdot \frac{1}{\cancel{z^2}} \cdot \frac{\partial u}{\partial s}$$

$$\cancel{-\frac{\partial u}{\partial R}} - \cancel{\frac{\partial u}{\partial s}} + \cancel{\frac{\partial u}{\partial R}} + \cancel{\frac{\partial u}{\partial s}}$$

$$= \underline{\underline{0}}$$

2). Find the 1st order partial derivatives or rate of cha

(i) $f(x, y) = x^y$

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(iii) $f(x, y) = \dfrac{x - y}{x + y}$

(iv) $f(x, y) = x^3 + x^2 y^3 - 2y^2$ at $(2, 1)$ in x-direction.

(v) $f(x, y) = 4 - x^2 - 2y^2$ at $(1, 1)$ in y-direction.

(i) $f(x,y) = x^y$

$$\frac{\partial f}{\partial x} = y \cdot x^{y-1}$$

$$\frac{\partial f}{\partial y} = x^y \cdot \log_e x$$

(ii) $f(r, s) = r^2 \ln(r^2+s^2)$

$$\frac{\partial f}{\partial r} = r^2 \cdot \frac{1}{r^2+s^2} \cdot 2r + \ln(r^2+s^2) \cdot 2r$$

$$\frac{\partial f}{\partial r} = \frac{2r^3}{r^2+s^2} + \ln(r^2+s^2) = \frac{2r^2}{r^2+s^2} + \ln(r^2+s^2)$$

$$\frac{\partial f}{\partial s} = r \cdot \frac{1}{r^2+s^2} \cdot 2s$$

$$= \frac{2rs}{r^2+s^2}$$

(iii) $f(x, y) = \frac{x-y}{x+y}$

$$\frac{\partial f}{\partial x} = \frac{(x+y)(1) - (x-y).(1)}{(x+y)^2}$$

$$= \frac{\cancel{x}+y-\cancel{x}+y}{(x+y)^2} = \frac{2y}{(x+y)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(x+y).(-1) - (x-y).(1)}{(x+y)^2}$$

$$= \frac{-x - xy - x + xy}{(x+y)^2}$$

$$\therefore \frac{\partial f}{\partial y} = \frac{-2x}{(x+y)^2}$$

$f(x,y) = x^3 + x^2y^3 - 2y^2$ at $(2,1)$ in x-direction

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy^3$$

$$\left.\frac{\partial f}{\partial x}\right|_{(2,1)} = 3(2)^2 + 2\times2\times1 = 12 + 4 = \underline{\underline{16}}$$

) $f(x,y) = 4 - x^2 - 2y^2$ at $(1,1)$ in y-direction

$$\frac{\partial f}{\partial y} = (\cancel{3x^2y^2 - 4y)^x} = -4y$$

$$\left.\frac{\partial f}{\partial y}\right|_{(1,1)} = -4(1) = -\underline{\underline{4}}$$

. Find all second order partial derivatives of

) $f(x,y) = x^4 - 3x^2y^3$

) $z = \dfrac{x}{x+y}$ (iii) $z = \sqrt{x^2+y^2}$ (iv) $f(x,y) = \ln(3x +$

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(i) $f(x,y) = x^4 - 3x^2y^3$

$\frac{\partial f}{\partial x} = 4x^3 - 6xy^3$ $\qquad$ $\frac{\partial f}{\partial y} = -9x^2y^2$

$\frac{\partial^2 f}{\partial x^2} = 12x^2 - 6y^3$ $\qquad$ $\frac{\partial^2 f}{\partial y^2} = -18x^2y$

$\frac{\partial^2 f}{\partial x . \partial y} = -18xy^2$ $\qquad$ $\frac{\partial^2 f}{\partial y . \partial x} = -18xy^2$

(ii) $z = \frac{x}{x+y}$

$\frac{\partial z}{\partial x} = \frac{(x+y)(1) - x(1)}{(x+y)^2}$ $\qquad$ $\frac{\partial z}{\partial y} = x\left(\frac{-1}{(x+y)^2}\right)$

$\frac{\partial z}{\partial x} = \frac{y}{(x+y)^2}$ $\qquad$ $\frac{\partial z}{\partial y} = \frac{-x}{(x+y)^2}$

$\frac{\partial^2 z}{\partial x^2} = \frac{-2y}{(x+y)^3}$ $\qquad$ $\frac{\partial^2 z}{\partial y^2} = \frac{2x}{(x+y)^3}$

$$\frac{\partial^2 z}{\partial x . \partial y} = \frac{(x+y)^2 . (1) - y\,2(x+y)}{(x+y)^4}$$

$$= \frac{(x+y)^2 - 2y(x+y)}{(x+y)^4}$$

$$= \frac{x^2+y^2+\cancel{2xy}-\cancel{2xy}-2y^2}{(x+y)^4}$$

$$\frac{\partial^2 z}{\partial x . \partial y} = \frac{(x^2-y^2)}{(x+y)^4} = \frac{x-y}{(x+y)^3}$$

$$\frac{\partial^2 z}{\partial y.\partial x} = \frac{(x+y)^2.(-1) - (-x)2(x+y)}{(x+y)^4}$$

$$= \frac{-x^2-y^2-\cancel{2xy}+2x^2+\cancel{2xy}}{(x+y)^4}$$

$$= \frac{x^2-y^2}{(x+y)^4}$$

$$\therefore \frac{\partial^2 z}{\partial y.\partial x} = \frac{x-y}{(x+y)^3}$$

(iii) $z = \sqrt{x^2+y^2}$

$$\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x^2+y^2}}.\cancel{2}x \qquad \frac{\partial z}{\partial y} = \frac{1}{2\sqrt{x^2+y^2}}.\cancel{2}y$$

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} \qquad \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\sqrt{x^2+y^2}.(1) - x.\frac{1}{\cancel{2}\sqrt{x^2+y^2}}.\cancel{2}x}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{\sqrt{x^2+y^2}} - \frac{x^2}{(x^2+y^2)^{3/2}} = \frac{x^2+y^2-x^2}{(x^2+y^2)^{3/2}} = \frac{y^2}{(x^2+y^2)^3}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\sqrt{x^2+y^2}.(1) - y.\frac{1}{\cancel{2}\sqrt{x^2+y^2}}.\cancel{2}y}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{\sqrt{x^2+y^2}} - \frac{y^2}{(x^2+y^2)^{3/2}} = \frac{x^2+y^2-y^2}{(x^2+y^2)^{3/2}} = \frac{x^2}{(x^2+y^2)^3}$$

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$$\frac{\partial^2 z}{\partial x.\partial y} = x.\left(\frac{-1}{2}\right)(x^2+y^2)^{-3/2} . 2y$$

$$= \frac{-x}{2(x^2+y^2)^{3/2}} . 2y = \frac{-xy}{(x^2+y^2)^{3/2}}$$

$$\frac{\partial^2 z}{\partial y.\partial x} = y.\left(\frac{-1}{\cancel{2}}\right)(x^2+y^2)^{-3/2} . \cancel{2}x$$

$$= \frac{-xy}{(x^2+y^2)^{3/2}}$$

iv) $f(x,y) = \ln(3x+5y)$

$$\frac{\partial f}{\partial x} = \frac{1}{3x+5y} . 3$$

$$\frac{\partial f}{\partial y} = \frac{5}{3x+5y}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{-3.3}{(3x+5y)^2}$$

$$= \frac{-9}{(3x+5y)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-5.5}{(3x+5y)^2}$$

$$\therefore \frac{\partial^2 f}{\partial y^2} = \frac{-25}{(3x+5y)^2}$$

$$\frac{\partial^2 f}{\partial x.\partial y} = \frac{-3.5}{(3x+5y)^2}$$

$$= \frac{-15}{(3x+5y)^2}$$

$$\frac{\partial^2 f}{\partial y.\partial x} = \frac{-5.3}{(3x+5y)^2}$$

$$= \frac{-15}{(3x+5y)^2}$$

4. Verify the conclusion of Clairaut's theorem:

(i) $u = x\sin(x+2y)$

(ii) $u = x^4y^2 - 2xy^5$

5. If $u = \sin x \sin at$ then show that $u_{tt} = a^2 u_{xx}$

$\Rightarrow$ $\frac{\partial u}{\partial x} = \sin at . \cos x$ $\qquad$ $\frac{\partial u}{\partial t} = \sin x \cos at . (a)$

$\frac{\partial^2 u}{\partial x^2} = -\sin x \sin at = u_{xx}$ $\qquad$ $\frac{\partial^2 u}{\partial t^2} = -a^2 \sin at . \sin x = u_{t}$

$u_{tt} = -a^2 \sin at . \sin x$

$= a^2(-\sin x \sin at)$

$\therefore u_{tt} = a^2 u_{xx}$

(ii) If $u = \frac{t}{a^2t^2 - x^2}$ S.T. $u_{tt} = a^2 u_{xx}$

$u_x = \frac{-t}{(a^2t^2-x^2)^2} . (-2x) = \frac{2xt}{(a^2t^2-x^2)^2}$

$u_{xx} = \frac{(a^2t^2-x^2)^2 . 2t - 2xt\, 2(a^2t^2-x^2).(-2x)}{(a^2t^2-x^2)^4}$

$= \frac{(a^2t^2-x^2)^2 . 2t + 8x^2t(a^2t^2-x^2)}{(a^2t^2-x^2)^4}$

$= \frac{\cancel{a^2t^2-x^2}\,[2t^3a^2 - 2tx^2 + 8x^2t]}{(a^2t^2-x^2)^4}$

$\therefore u_{xx} = \dfrac{2a^2t^3 + 6x^2t}{(a^2t^2 - x^2)^3}$

$$u_t = \frac{a^2t^2 - x^2 .(1) - t(+a^2(2t)}{(a^2t^2 - x^2)^2}$$

$$= \frac{a^2t^2 - x^2 - 2a^2t^2}{(a^2t^2 - x^2)^2}$$

$$= \frac{-a^2t^2 - x^2}{(a^2t^2 - x^2)^2}$$

$$u_{tt} = \frac{(a^2t^2 - x^2)^2 (-2a^2t) - (-a^2t^2 - x^2) 2(a^2t^2 - x^2).(}{(a^2t^2 - x^2)^4}$$

$$= \cancel{a^2t^2 - x^2} \left[\frac{[a^2t^2 - x^2][-2a^2t] + [a^2t^2 + x^2] 4a^2t}{(a^2t^2 - x^2)^4} \right]$$

$$= \frac{2a^2t[-a^2t^2 + x^2 + 2a^2t^2 + 2x^2]}{(a^2t^2 - x^2)^3}$$

$$= \cancel{\left(\frac{2a^2t[\cancel{a^2t^2 + 3x^2})}{(a^2t^2 - x^2)^3} \right)} = \frac{2a^2t[a^2t^2 + 3x^2]}{(a^2t^2 - x^2)^3}$$

$$= \frac{2a^2t(a^2t^2 + 3x^2)}{(a^2t^2 - x^2)^3} = \frac{a^2(2a^2t^3 + 6x^2t)}{(a^2t^2 - x^2)^3}$$

$$u_{tt} = \frac{a^2(2a^2t^3 + 6x^2t)}{(a^2t^2 - x^2)^3}$$

$$= a^2 \left[\frac{2a^2t^3 + 6x^2t}{(a^2t^2 - x^2)^3} \right]$$

$$= a^2 u_{xx}$$

$$\therefore u_{tt} = a^2 u_{xx}$$

Use the chain rule to find $\frac{dz}{dt}$ and $\frac{dw}{dt}$ for

(i) $z = x^2y + 3xy^4$, $x = \sin 2t$, $y = \cos t$

(ii) $z = \sqrt{x^2+y^2}$, $x = e^{2t}$, $y = e^{-2t}$

(iii) $z = \sin x . \cos y$. $x = \pi t$ $y = \sqrt{t}$

(iv) $w = xe^{y/z}$, $x = t^2$, $y = 1-t$, $z = 1+2t$

(i) $z = x^2y + 3xy^4$

$x = \sin 2t$. $y = \cos t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2xy + 3y^4) . 2\cos 2t + (x^2 + 12xy^3) . (-\sin t)$$

$$= (2\sin 2t . \cos t + 3\cos^4 t) . 2\cos 2t + (\sin^2 2t + 12 \sin 2t \cos^3 t) . (-\sin t)$$

$$\frac{dz}{dt} = 4\sin 2t \cos 2t \cos t + 6\cos^4 t . \cos 2t - \sin t \sin^2 2t - 12 \sin 2t \sin t \cos^3 t$$

(ii) $z = \sqrt{x^2+y^2}$, $x = e^{2t}$, $y = e^{-2t}$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} . \frac{dx}{dt} + \frac{\partial z}{\partial y} . \frac{dy}{dt}$$

$$= \frac{1}{\cancel{2}\sqrt{x^2+y^2}} . \cancel{2}x . 2 . e^{2t} + \frac{1}{\cancel{2}\sqrt{x^2+y^2}} . \cancel{2}y . -2 . e^{-2t}$$

$$= \frac{2xe^{2t} - 2ye^{-2t}}{\sqrt{x^2+y^2}} = \frac{2e^{2t}x - 2e^{-2t}y}{\sqrt{e^{4t}+e^{-4t}}} .$$

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$$= \frac{2.e^{2t}.e^{2t} - 2.e^{-2t}.e^{-2t}}{\sqrt{e^{4t}+e^{-4t}}}$$

$$\frac{dz}{dt} = \frac{2.e^{4t} - 2e^{-4t}}{\sqrt{e^{4t}+e^{-4t}}}$$

(iii) $z = \sin x.\cos y$, $x = \pi t$, $y = \sqrt{t}$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}.\frac{dx}{dt} + \frac{\partial z}{\partial y}.\frac{dy}{dt}$$

$$= \cos x.\cos y.(\pi) - \sin x \sin y . \frac{1}{2\sqrt{t}}$$

$$\frac{dz}{dt} = \cos(\pi t).\cos(\sqrt{t}).(\pi) - \sin(\pi t).\sin(\sqrt{t}).\frac{1}{2\sqrt{t}}$$

(iv) $w = x.e^{y/z}$, $x = t^2$, $y = 1-t$, $z = 1+2t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}.\frac{dx}{dt} + \frac{\partial w}{\partial y}.\frac{dy}{dt} + \frac{\partial w}{\partial z}.\frac{dw}{dt}$$

$$= e^{y/z}.2t + x.e^{y/z}.\frac{1}{z}.(-1) + x.e^{y/z}.y.2($$

$$= e^{y/z}\left[2t - \frac{x}{z} \frac{\cancel{t-1}\,x y 2}{z^2}\right]$$

$$= e^{1-t/1+2t}.\left[2t - \frac{t^2}{1+2t} \frac{\cancel{t-1}\,2(t^2-t^3)}{(1+2t)^2}\right]$$

$$= e^{\frac{1-t}{1+2t}}\left[\frac{2t(1+2t) - t^2(1+2t) - 2(t^2-t^3)}{(1+2t)^2}\right]$$

$$= e^{\frac{1-t}{1+2t}}\left[\frac{2t+4t^2-t^2-\cancel{2t^3}-2t^2+\cancel{2t^3}}{(1+2t)^2}\right]$$

$$\therefore \frac{dw}{dt} = e^{\frac{1-t}{1+2t}}\left[\frac{t^2+2t}{(1+2t)^2}\right]$$

Use chain rule to find $\frac{\partial z}{\partial s}$ & $\frac{\partial z}{\partial t}$

(i) $z = \frac{x}{y}$, $x = se^t$, $y = 1+se^{-t}$

(ii) $z = e^{xy}\tan y$; $x = s+2t$, $y = \frac{s}{t}$

(iii) $z = e^r\cos\theta$, $r = st$, $\theta = \sqrt{s^2+t^2}$

$\Rightarrow$ (i) $z = \frac{x}{y}$, $x = se^t$, $y = 1+se^{-t}$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$$

$$= \frac{1}{y}\cdot e^t + x\cdot\left(\frac{-1}{y^2}\right)\cdot(+e^{-t})$$

$$= \frac{e^t}{y} - \frac{xe^{-t}}{y^2}$$

$$= \frac{e^t}{(1+se^{-t})} - \frac{se^t\cdot e^{-t}}{(1+se^{-t})^2}$$

$$= \frac{e^t(1+se^{-t}) - s\theta}{(1+se^{-t})^2} = \frac{e^t+\cancel{s}-\cancel{s}}{(1+se^{-t})^2} = \frac{e^t}{(1+se^{-t})^2}$$

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$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= \frac{1}{y} \cdot se^{t} + \left(\frac{-1}{y^2}\right) \cdot x . s . (-1)(e^{-t})$$

$$= \frac{se^{t}}{y} + \frac{xse^{-t}}{y^2}$$

$$= \frac{se^{t}}{(1+se^{-t})} + \frac{se^{t} . s . e^{-t}}{(1+se^{-t})^2}$$

$$= \frac{se^{t} . (1+se^{-t}) + s^2}{(1+se^{-t})^2}$$

$$\frac{\partial z}{\partial t} = \frac{se^{t}+s^2+s^2}{(1+se^{-t})^2} = \frac{se^{t}+2s^2}{(1+se^{-t})^2}$$

(ii) $z = e^{xy} \tan y$, $x = s + 2t$, $y = \frac{s}{t}$

$$\frac{\partial z}{\partial s}$$