Chapter: 2. Functions and Graphs

Function:

A function f is a rule that assigns to each element x in a set A exactly one element, called $f(x)$, in a set B.

We usually consider functions for which the sets A and B are sets of real numbers. The set A is called the domain of the function. The number $f(x)$ is the value of f at x .The range of f is the set of all possible values of $f(x)$ as x varies through-out the domain.

A symbol that represents an arbitrary number in the domain of a function is called an independent variable. A symbol that represents a number in the range of is called a dependent variable.
Function behaves as a machine

If x is in the domain of the function then when x enters the machine, it's accepted as an input and the machine produces an output $f(x)$ according to the rule of the function. Thus, we can think of the
Domain: as the set of all permissible inputs and
Range: as the set of all possible outputs.

Construction of a function:

<u>Find a formula for the described</u>: A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30ft, express the area of the window as a function of the width x of the window.

Soln:

Perimeter of the window =30ft (given)

$2h + x + \pi\frac{x}{2} = 30$ (h-height, x-width, $x/2$—radius)

$4h + 2x + \pi x = 60$

$$\boxed{h = \frac{60-(2+\pi)x}{4}}$$

Area=$\boldsymbol{xh + \frac{1}{2}\pi\frac{x^2}{4}}$

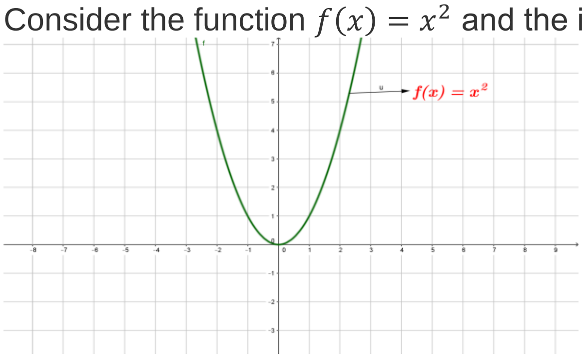
$=x\frac{60-(2+\pi)x}{4} + \frac{\pi x^2}{8}$=$15x - \frac{x^2}{8}(4+\pi)$

Transformations:

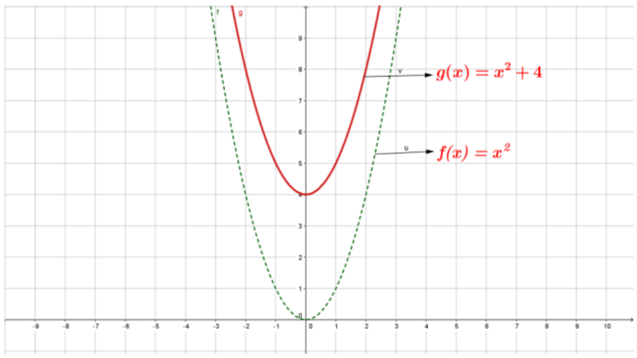
Definition: It is a method of obtaining a new function from the old function **OR** When the graph of a function changes appearance or location is called transformation.

a) Vertical transformation: It is transformation that shifts the graph of a function **up** or **down** relative to the original graph. This happens when constant is added to a y-coordinate of the function. If we add **positive** constant graph will shift up and if we add **negative** constant graph will shift down.

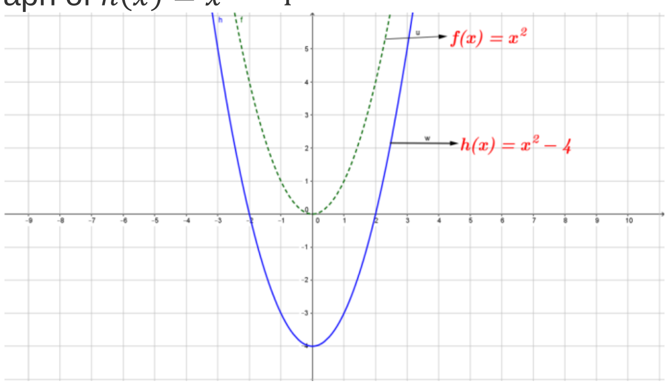
Consider the function $f(x) = x^2$ and the it's graph



Now new functions are defined as $g(x) = x^2 + 4$ and

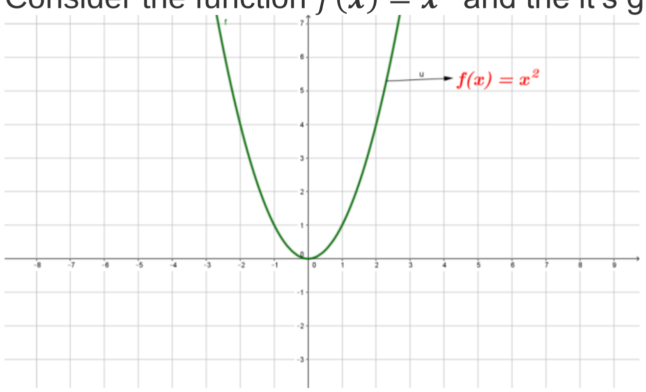


And graph of $h(x) = x^2 - 4$

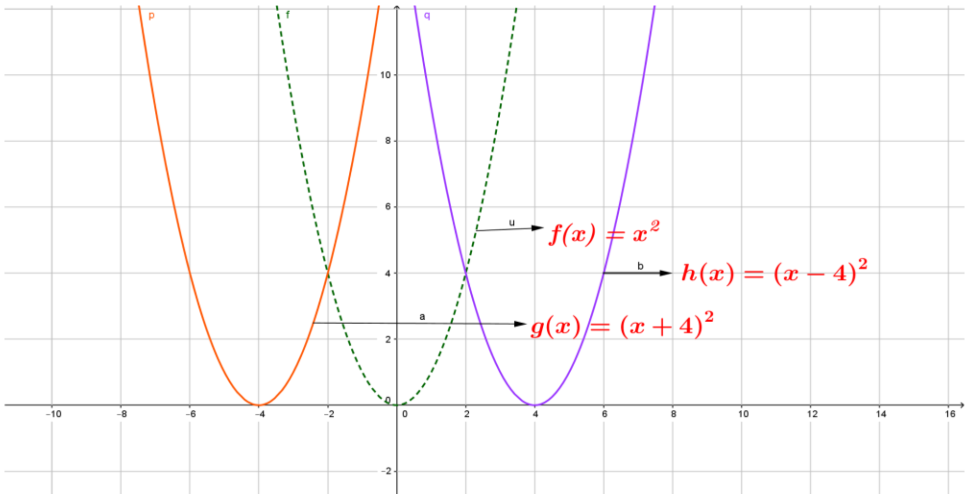


b) Horizontal Transformation: It is transformation that shifts the graph **left** or **right** relative to the original graph. This occurs when we add or subtract constant from $x - coordinate$

Consider the function $f(\boldsymbol{x}) = \boldsymbol{x^2}$ and the it's graph



New functions are defined as $g(x) = (x+4)^2$ and $h(x) = (x-4)^2$

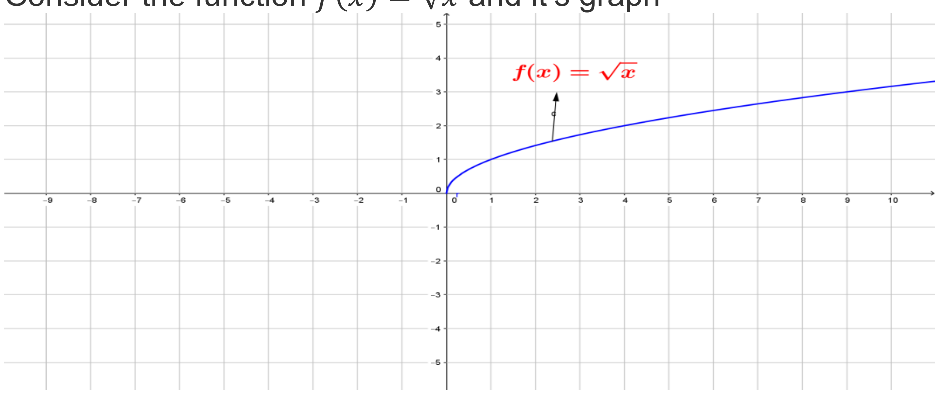


c) Reflection: It is a transformation in which mirror image of the graph is produced about an axis. Here we consider the reflection about $x - axis$ and $y - axis$.

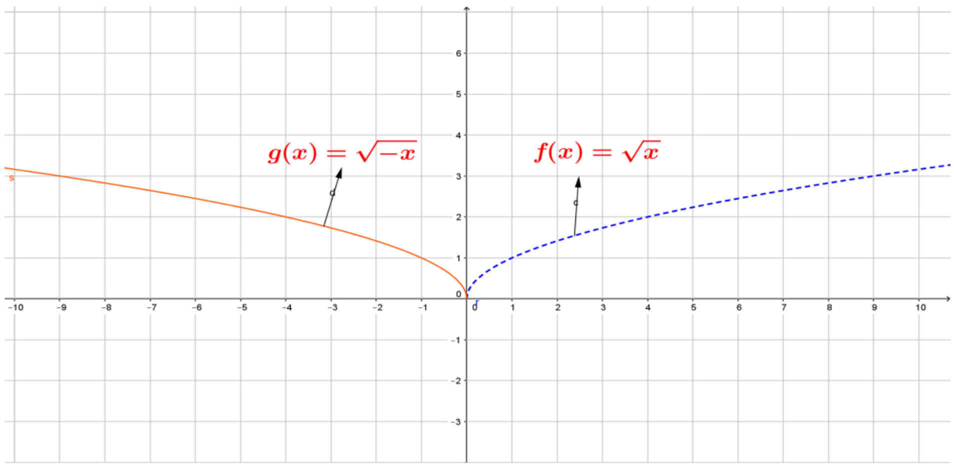
The graph of function is reflected about the $x - axis$ if each Y-coordinate is multiplied by -1.

And the graph of a function is reflected about $y - axis$ if x-coordinate is multiplied by -1.

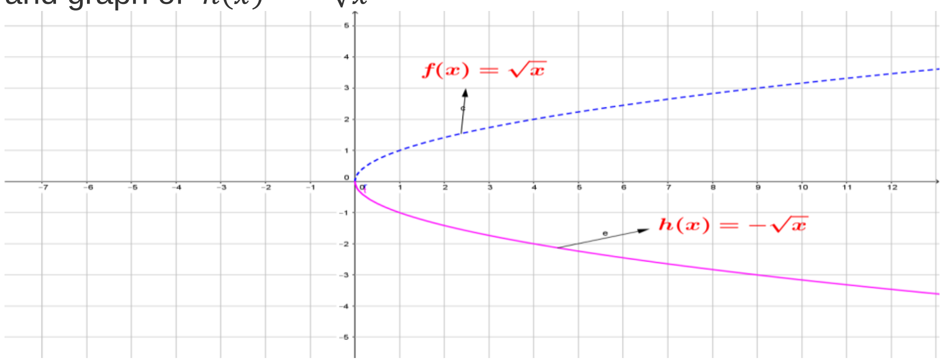
Consider the function $f(x) = \sqrt{x}$ and it's graph



New functions are by $g(x) = \sqrt{-x}$



and graph of $h(x) = -\sqrt{x}$

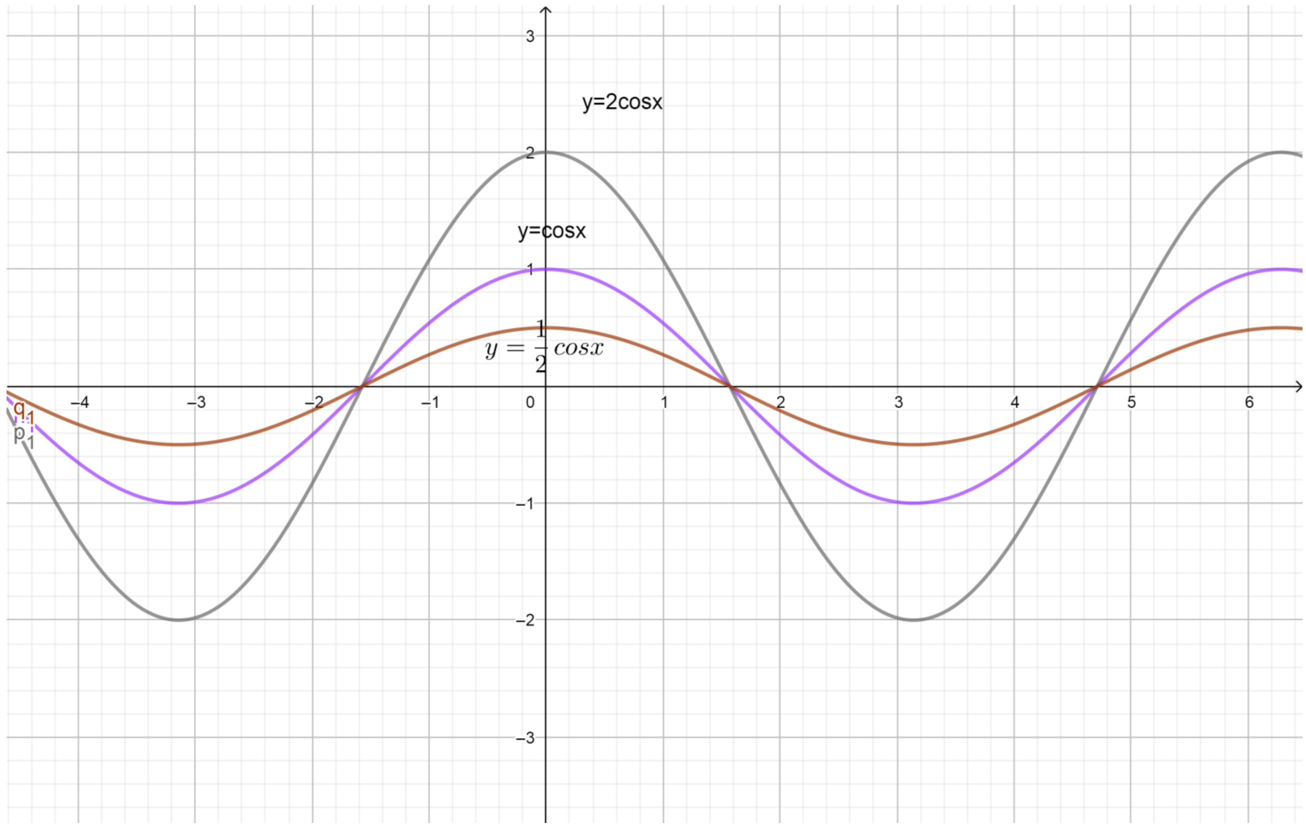


In general,

Vertical shift up by $k = 4$ units	$g(x) = f(x) + 4 = x^2 + 4$
Vertical shift down by $k = 4$ units	$h(x) = f(x) - 4 = x^2 - 4$
Horizontal shift left by $k = 4$ units	$g(x) = f(x + k) = (x + 4)^2$
Horizontal shift right by $k = 4$ units	$h(x) = f(x - k) = (x - 4)^2$
Reflection about $y - axis$	$g(x) = f(-x) = \sqrt{-x}$
Reflection about $x - axis$	$h(x) = -f(x) = -\sqrt{x}$

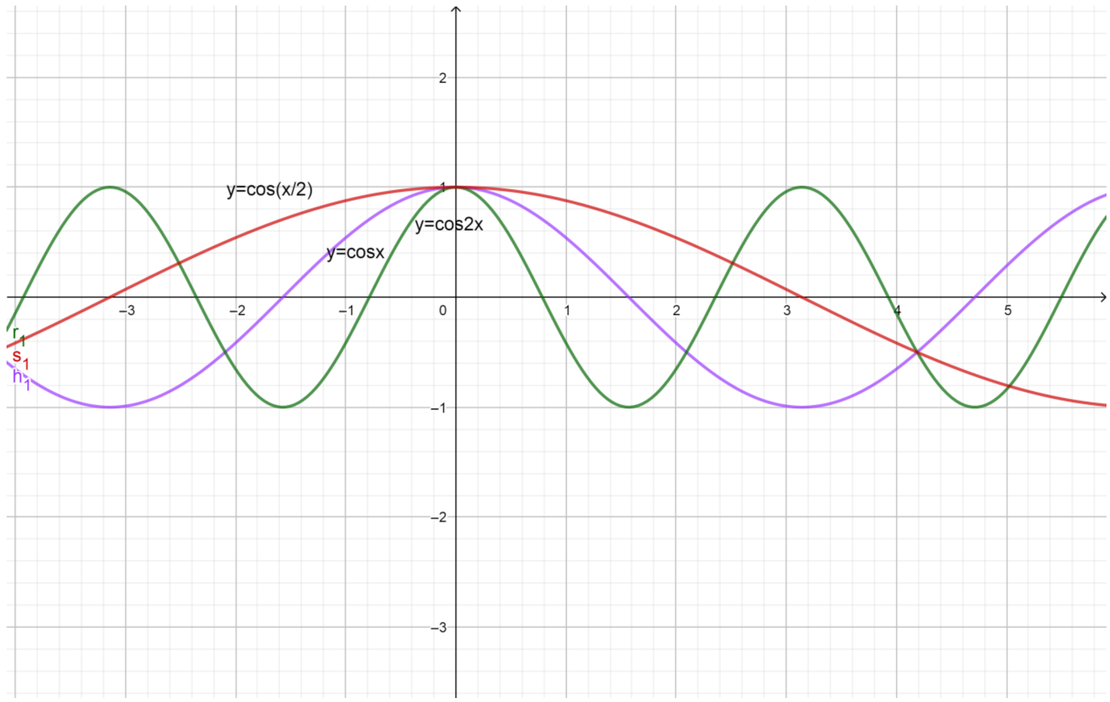
d) Vertical Stretching and Compressing:

It is transformation that Stretch or compress vertically relative to the original graph. This happens when constant is multiplied to the **function.** If we multiply **constant k graph will stretch vertically by a k factor** if we multiply **constant $\frac{1}{k}$ graph will compress vertically by a k factor**



e) **Horizontal Stretching and Compressing**:

f) It is transformation that Stretch or compress horizontally relative to the original graph. This happens when constant is multiplied to the **variable**. If we multiply **constant k graph will compress horizontally by a k factor** if we multiply **constant $\frac{1}{k}$ graph will stretch horizontally by a k factor**



In general,

Vertical stretch by $k = 2$ units	$g(x) = 2f(x) = 2cosx$
Vertical compress by $k = 2$ units	$h(x) = 1/2f(x) = \frac{1}{2}cosx$
Horizontal stretch by $k = 2$ units	$g(x) = f\left(\frac{1}{2}x\right) = cos(\frac{x}{2})$
Horizontal compress by $k = 2$ units	$h(x) = f(2x) = cos(2x)$

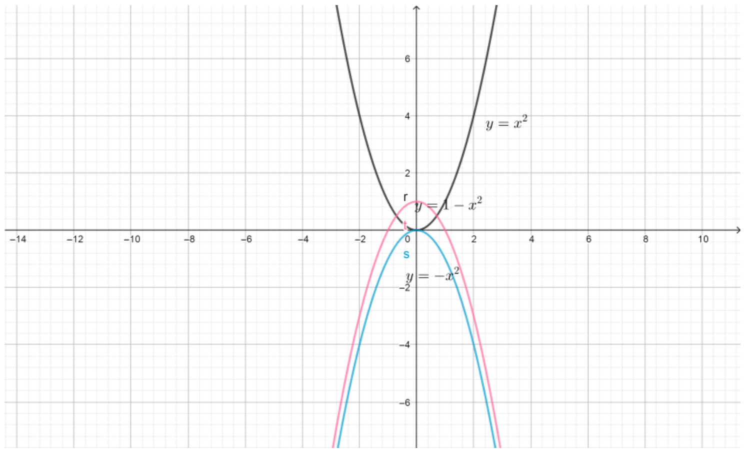
g) Obtain the following graphs by applying appropriate transformations and explain. Hence determine the domain and range.

(i) $y = 1 - x^2$ (ii) $y = (x+1)^2$ (iii) $y = x^2 - 4x + 3$ (iv) $y = \sqrt{x+3}$ (v) $y = -2^{-x}$

(i) $y = 1 - x^2$

Soln:

Basic function is $y = x^2$ Reflection on X-axis gives $y = -x^2$ shift this graph one unit up gives $y = 1 - x^2$



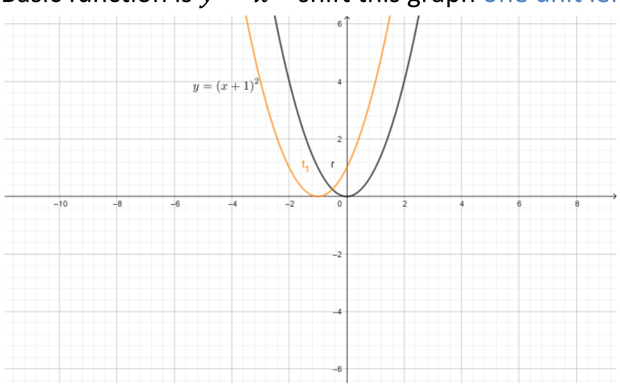
Domin::$(-\infty, \infty)$

Range:$[1, -\infty)$

(ii) $\mathrm{y} = (\mathrm{x}+1)^2$

Soln:

Basic function is $y = x^2$ shift this graph one unit left gives $y = (x+1)^2$



Domin::$(-\infty, \infty)$

Range:$[0, \infty)$

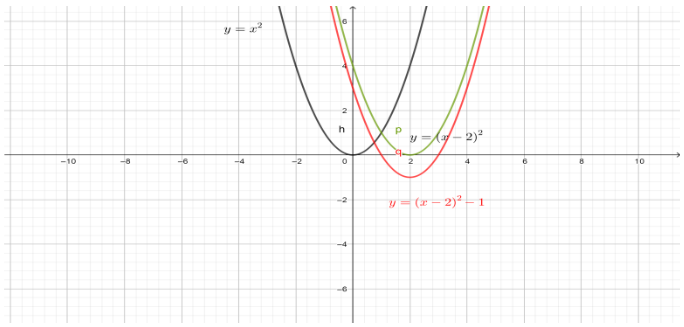
(iii) $y = x^2 - 4x + 3$

Soln:

$y = x^2 - 4x + 3 = (x-2)^2 - 1$

Basic function is $y = x^2$ shift this graph two units right gives $y = (x-2)^2$

shift this graph one unit down gives $y = (x-2)^2 - 1$



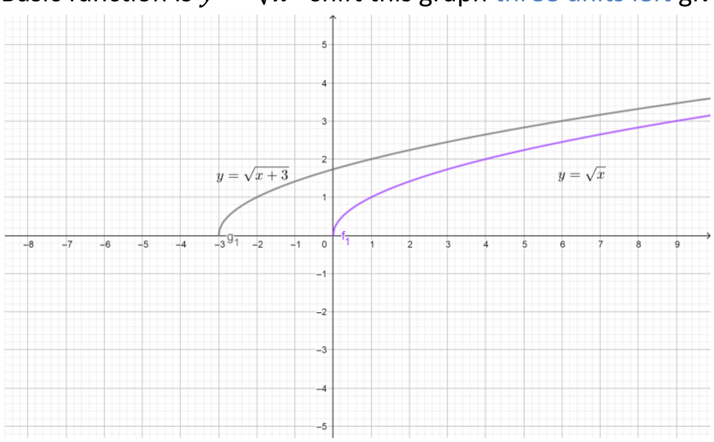
Domin: $(-\infty, \infty)$

Range: $[-1, \infty)$

(iv) $y = \sqrt{x+3}$

Soln:

Basic function is $y = \sqrt{x}$ shift this graph three units left gives $y = \sqrt{x+3}$



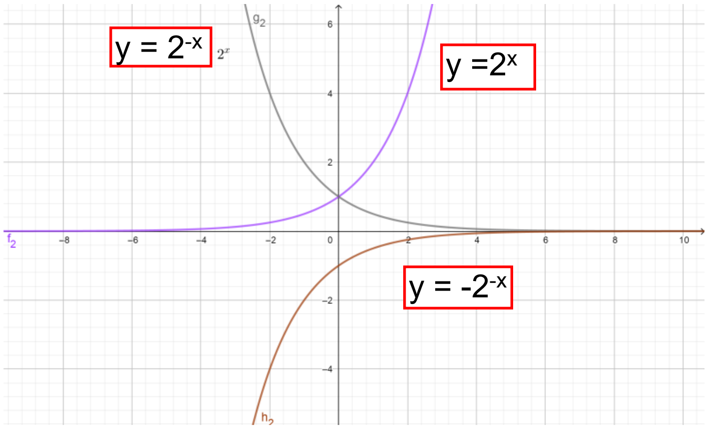
Domin: $[-3, \infty)$

Range: $[0, \infty)$

v) $y = -2^{-x}$

Soln:

Basic function is $y = 2^x$ Reflection on Y-axis gives $y = 2^{-x}$ now Reflecting this graph on X-axis gives $y = -2^{-x}$



Domin:$(-\infty, \infty)$

Range:$[0, -\infty)$

linear model:

Q1. Biologists have noticed that the chirping rate of crickets of a certain species is related to temperature and the relationship appears to be nearly linear. A cricket produces 113 chirps per minute at 70^0F and 173 chirps per minute at 80^0F. (i) Find linear equation that models the temperature T as a function of the no. of chirps per minute N. (ii) what is the slope of the graph? What does it represent? (iii) If the crickets are chirping at 150 chirps per minute, estimate the temperature.

soln :

T-temperature, N-no of chirps, by seeing the question (i) we can judge which is dependent and independent variables

Here T is dependent on N

Linear equation is T= m N + C

By putting m and C value we get linear model for estimating temperature for given number of chirps.

given data is when N=113 T =70, when N=173 T=80

so linear equations are 70=113m+C and 80=173m+C

by solving these two equations we get m=0.166 and C=51.16

then linear model is (i) T=(0.166)N+51.6

ii) slope is m=0.166 means increase of 0.166 cricket chirps per minute corresponds to an increase of 1^0F.

(iii) by using the model $T = (0.166)N + 51.6$

when $N = 150$ $T = 760F$

Q2. A town has a population 1000 people at time$t = 0$. In each of the following cases; write a formula for the population P, of the town as a function of year$t$. (i)The population increases by 50 people a year. (ii)The population increases by 5% a year.

Soln :

i)Initial population is 1000 ie at t=0 P=1000 and population is function of t

Every year population increases by 50 (given) then it is slope of the model

Formula or model for the population P is $P = 50t + 1000$

ii) population increases by 5% a year, ie increasing factor is 5% then it is a exponential function

$P = P_0 a^t$ -----1

at t=0 P=1000 ie $P_0 = 1000$

at t=1 P=1050 substitute in eqn 1 we get

$1050 = 1000(a)^1 \rightarrow a = 1.05$

Formula for the population at any time t increases by 5% a year is $P = 1000(1.05)^t$

Exponential model:

Q3. An isotope of sodium, ^{24}Na, has a half life of 15 hours. A sample of this isotope has mass 2g. (i) Find the amount remaining after 60 hours. (ii) Find the amount remaining after t hours. (iii) Estimate the amount remaining after 4 days.

Soln :

^{24}Na, has a half life of 15 hours ie Substance is decaying so function is exponential function having decay factor. $P = P_0 a^t$ -----1

At $t = 0$ $P = 2g \rightarrow$ $P_0 = 2$

At t=15 hours $P = \frac{P_0}{2} = 1$ substituting in eqn 1 we get

$1 = 2(a)^{15} \rightarrow a = (0.5)^{1/15}$

(ii) So the amount remaining after t hours is $P = 2(0.5)^{t/15}$

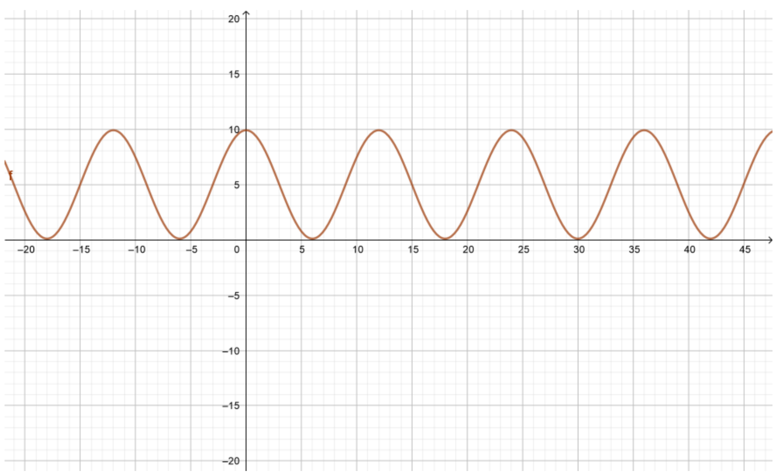
(i) the amount remaining after 60 hours is $P = 2(0.5)^{60/15} = 2(0.5)^4 = 0.125$

(iii) the amount remaining after 4 days=96 hours is $P = 2(0.5)^{96/15} = 2(0.5)^{6.4} = 0.0236$

Trigonometric model:

Q4. On Feb 10, 1990, high tide in Boston was at midnight. The water level at high tide was 9.9 feet; at low tide, it was 0.1 feet. Assuming the next high tide is at exactly 12 noon and that the height of the water is given by a sine or cosine curve, find a formula for the water level in Boston as a function of time.

Soln :



Let y be the water level in feet and let t be the time measured in hours from midnight.

The oscillations have amplitude= $\frac{high\ tide - low\ tide}{2}$ =$\frac{9.9-0.1}{2} = 4.9\ feet$

high tide in Boston was at midnight and next high tide is at exactly 12 noon

therefore period t =12 hours $t = \frac{2\pi}{B} \rightarrow 12B = 2\pi \rightarrow B = \frac{\pi}{6}$

since water is highest at midnight, when t=0. The oscillations are best represented by cosine function.

We can say height above average=$4.9\cos\left(\frac{\pi}{6}\right)t$

Since the average water level was = $\frac{9.9+0.1}{2} = 5\ feet$

We shift the cosine up by adding 5 so $y = 5 + 4.9\cos\left(\frac{\pi}{6}\right)t$