Infinite series

? Determine whether the series converges or diverges

(ii) $\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2+1}$

$\rightarrow$ $a_n = \frac{\cos^2 n}{n^2+1}$

wkt = $\cos^2 n < 1$, $\forall n$

$\Rightarrow \frac{\cos^2 n}{n^2+1} < \frac{1}{n^2+1}$ $(\because n^2+1>0)$ — (1)

Also $n^2+1 > n^2$

$\Rightarrow \frac{1}{n^2+1} < \frac{1}{n^2}$ — (2)

from (1) and (2)

$\frac{\cos^2 n}{n^2+1} < \frac{1}{n^2}$

But $\sum \frac{1}{n^2}$ is cgt by P-series test $(\because p=2>1)$

By Comparison test

$\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2+1}$ is cgt

iii) $\Sigma \frac{n^2-5}{n^3+n+2}$

$\rightarrow$ $a_n = \frac{n^2-5}{n^3+n+2}$ $\quad b_n = \frac{n^2}{n^3} = \frac{1}{n}$

$\Rightarrow$ $\Sigma b_n = \Sigma 1/n$ is dgt ($\because P=1$) by P-series test

$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{n^2-5}{n^3+n+2}\right)}{(1/n)} = \lim_{n\to\infty}\left(\frac{n^2-5}{n^3+n+2}\right)n$

$= \lim_{n\to\infty} \frac{\cancel{n^3}\left(1-\frac{5}{n^2}\right)}{\cancel{n^3}\left(1+\frac{1}{n^2}+\frac{2}{n^3}\right)} = \lim_{n\to\infty} \frac{\left(1-\frac{5}{n^2}\right)}{1+\frac{1}{n^2}+\frac{2}{n^3}}$

$= \frac{1}{1} = 1\ (\neq 0)$ $\therefore$ By limit Comparison test Σa_n is also dgt

v) $1+\frac{2^2}{2!}+\frac{3^2}{3!}+\frac{4^2}{4!}+ \cdots$

$\rightarrow$ Given Series $= \sum_{n=1}^{\infty} \frac{n^2}{n!}$

$\therefore a_n = \frac{n^2}{n!}$ $\quad a_{n+1} = \frac{(n+1)^2}{(n+1)!} = \frac{(n+1)^2}{(n+1)n!}$

$= \frac{n+1}{n!}$

$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{\left(\frac{n+1}{n!}\right)}{\left(\frac{n^2}{n!}\right)} = \lim_{n\to\infty}\left(\frac{n+1}{n^2}\right)$

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{n}{n^2}+\frac{1}{n^2}\right) = \lim_{n\to\infty}\left(\frac{1}{n}+\frac{2}{n^2}\right)$

$= 0\ (<1)$

$\therefore$ By Ratio test $\Sigma \frac{n^2}{n!}$ is cgt//

11) Expand $\tan x$ in powers of $f\left(x-\frac{\pi}{4}\right)$. Hence find the value of $\tan 46^\circ$ considering first 4 terms.

$f(x) = \tan x$ $\qquad f\left(\frac{\pi}{4}\right) = 1$

$f'(x) = \sec^2 x$ $\qquad f'\left(\frac{\pi}{4}\right) = 2$

$f''(x) = 2(\sec^2 x \tan x)$ $\qquad f''\left(\frac{\pi}{4}\right) = 4$

$f'''(x) = 6\sec^4 x - 4\sec^2 x$ $\qquad f'''\left(\frac{\pi}{4}\right) = 16$

Taylor's series at $x = \frac{\pi}{4}$ is

$\tan\frac{\pi}{4} = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x-\frac{\pi}{4}\right)^2 +$

$\frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(x-\frac{\pi}{4}\right)^3 + \cdots$

$$\tan\frac{\pi}{4} = f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{4}\right)\left(x-\frac{\pi}{4}\right) + \frac{f''\left(\frac{\pi}{4}\right)}{2!}\left(x-\frac{\pi}{4}\right)^2$$
$$+ \frac{f'''\left(\frac{\pi}{4}\right)}{3!}\left(x-\frac{\pi}{4}\right)^3 + \cdots$$

$$= 1 + 2\left(x-\frac{\pi}{4}\right) + 2\left(x-\frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x-\frac{\pi}{4}\right)^3$$

12) Compute the value of $\sin 62^\circ$ using Taylor Series Considering first 4-terms

→ Let $f(x) = \sin x$; $a = \frac{\pi}{3}$ (because 60° is near to 62°

$f'(x) = \cos x$ $\quad f(\pi/3) = \sqrt{3}/2 \quad f^{iv}(\pi/3) = \sqrt{3}/2$

$f''(x) = -\sin x$ $\quad f'(\pi/3) = 1/2$

$f'''(x) = -\cos x$ $\quad f''(\pi/3) = -\sqrt{3}/2$

$f^{iv}(x) = \sin x$ $\quad f'''(\pi/3) = -1/2$

Taylor Series is:

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a)$$
$$+ \frac{(x-a)^4}{4!}f^{iv}(a)$$

$$\sin x = f(\pi/3) + (x-\pi/3)f'(\pi/3) + \frac{(x-\pi/3)^2}{2!}f''(\pi/3) +$$
$$\frac{(x-\pi/3)^3}{3!}f'''(\pi/3) + \frac{(x-\pi/3)^4}{4!}f^{iv}(\pi/3) + \cdots$$

$$\sin x = \sqrt{3}/2 + (x-\pi/3)\left[\frac{1}{2}\right] + \frac{(x-\pi/3)^2}{2!}\left[\frac{-\sqrt{3}}{2}\right] + \frac{(x-\pi/3)^3}{6}$$

$62^\circ = 60^\circ + 2^\circ = \pi/3 + \pi/90$ | $\because \pi = 180^\circ$, $\pi/180^\circ = 1^\circ$ | $\frac{\pi}{90} = 2^\circ$

put $x = 62^\circ = \pi/3 + \pi/90$

$$\sin 62^\circ = \frac{\sqrt{3}}{2} + \left(\frac{\pi}{3} + \frac{\pi}{90} - \frac{\pi}{3}\right)\left[\frac{1}{2}\right] + \left(\frac{\pi}{3} + \frac{\pi}{90} + \frac{\pi}{3}\right)^2\left[\frac{-\sqrt{3}}{4}\right]$$
$$+ \left(\frac{\pi}{3} + \frac{\pi}{90} - \frac{\pi}{3}\right)^3\left(-\frac{1}{12}\right) + \cdots$$

$$\sin 62^\circ = 0.8666 + \left(\frac{\pi}{90}\right)(0.5) + \left(\frac{\pi}{90}\right)^2[-0.9333] +$$
$$[\pi/90]^3[-0.0833] + \cdots$$

$$\sin 62^\circ = 0.866$$

Integral Calculas

i) Trace the Curve

i) $x^{2/3} + y^{2/3} = a^{2/3}$

$\rightarrow$ The given equation is $x = t$, $y = a\sin^3 t$

1) symmetry

The Curve is Symmetrical about x axis

2) Tangents

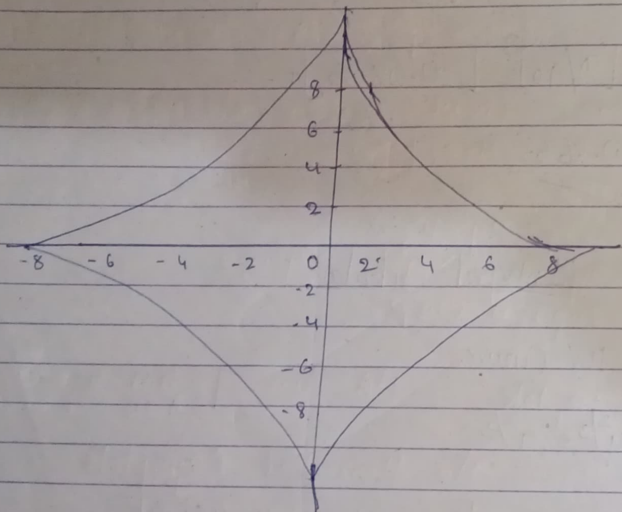
$\frac{dx}{dt} = -3t\sin t$, $\frac{dy}{dt} = 3t\cos t \rightarrow \frac{dy}{dx} = \tan t$

3) origin
The curve does not pass through the origin

4.) largest value of r is a

5) Table

t	0	$\pi/2$	π	$3\pi/2$	2π
x	a	0	$-a$	0	a
y	0	a	0	$-a$	0
$\frac{dy}{dx}$	0	$-\infty$	0	∞	0



ii) $y^2(2a-x) = x^3$

$\rightarrow y^2 = \frac{x^3}{2a-x} \Rightarrow y^2 = x^2 \cdot \frac{x}{2a-x}$

$\Rightarrow y = x\sqrt{\frac{x}{2a-x}}$

① Domain
$\Rightarrow x \geq 0$ & $2a-x > 0 \Rightarrow x \geq 0$ & $2a > x$
$\Rightarrow x \geq 0$ & $x < 2a$
$\Rightarrow 0 \leq x \leq 2a$

② intercepts
x-intercept: put $y=0 \Rightarrow [x=0]$
y intercept: put $x=0 \Rightarrow y=0$

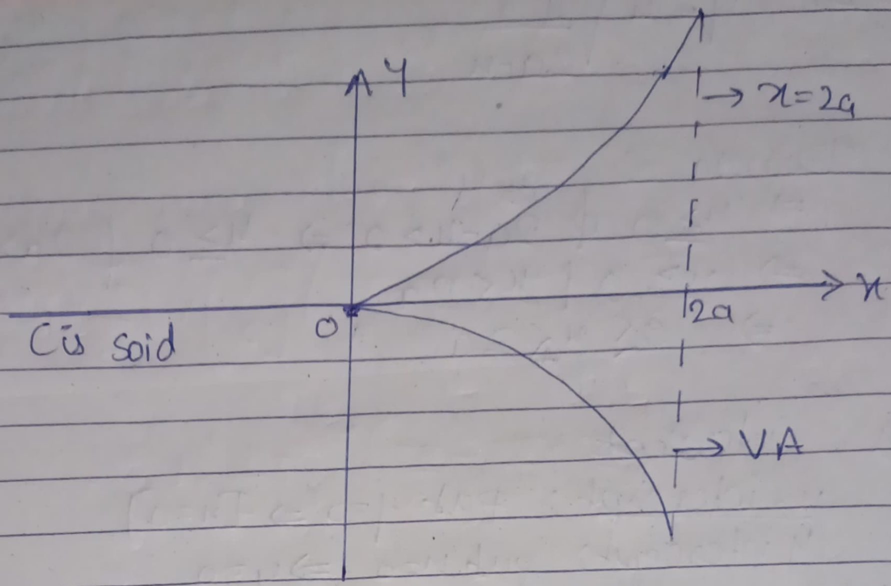
3) Symmetric: About x-axis
($\because$ only y power is even)

④ origin: Curve passes through origin
($\because x=0 \Rightarrow y=0$)
from ①
$2ay^2 - xy^2 = x^3$
$\Rightarrow 2ay^2 = 0 \Rightarrow y^2 = 0 \Rightarrow y = 0, 0$
$\therefore$ origin is cusp

⑤ Asymptote: HA: $\lim y = \lim_{x\to\infty} x\sqrt{\frac{x}{2a-x}}$
No Horizontal Asymptote

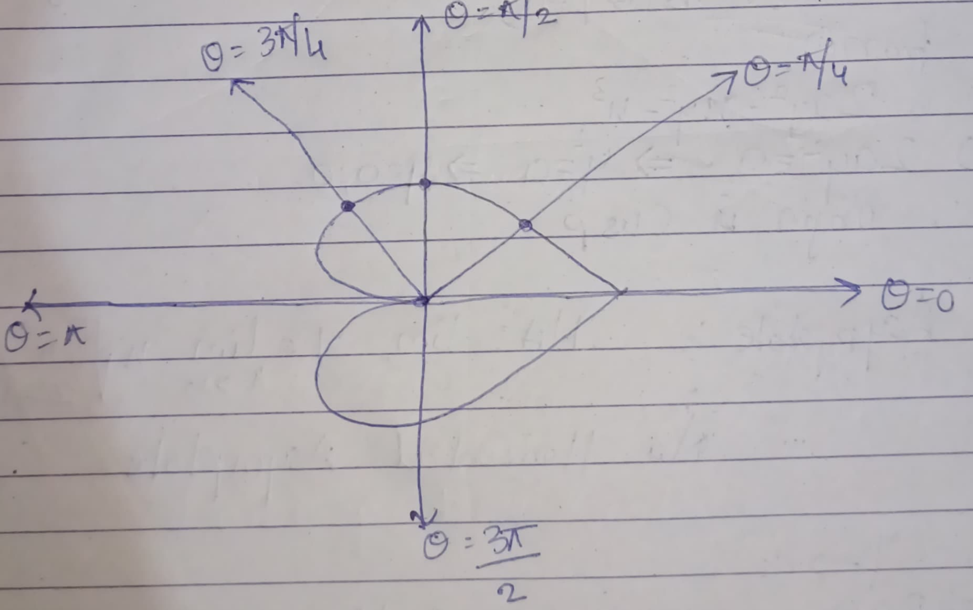
VA: $y \to \infty$
$\Rightarrow 2a - x = 0 \Rightarrow x = 2a$ is VA

Graph



4) Find the perimeter of the Curve

(ii) $r = a(1+\cos\theta)$ $a>0$,



Cardioide:

③ Evalute using beta & Gamma function

a) $\int_0^{\pi/2} \sin^6\theta \cos^7\theta \, d\theta$

soln Here $p=6, q=7$

wkt $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\left(\frac{p+1}{2}\right)+\left(\frac{q+1}{2}\right)\right)}$

$$\therefore \int_0^{\pi/2} \sin^6\theta \cos^7\theta \, d\theta = \frac{\Gamma\left(\frac{6+1}{2}\right)\Gamma\left(\frac{7+1}{2}\right)}{2\,\Gamma\left(\left(\frac{6+1}{2}\right)+\left(\frac{7+1}{2}\right)\right)} = \frac{\Gamma\left(7/2\right)\Gamma\left(8/2\right)}{2\,\Gamma\left(\frac{7}{2}+\frac{8}{2}\right)}$$

$$= \frac{\Gamma\left(7/2\right)\Gamma(4)}{2\,\Gamma\left(15/2\right)} = \frac{\cancel{\Gamma\left(7/2\right)}\,(3!)}{2\cdot\frac{13}{2}\cdot\frac{11}{2}\cdot\frac{9}{2}\cdot\frac{7}{2}\,\cancel{\Gamma\left(7/2\right)}}$$

$$= \frac{6}{\frac{13.11.9.7}{8}} = \frac{\cancel{2}\,6\times 8}{13.11.\cancel{9}_{3}.7} = \frac{16}{13.11.3.7}$$

$$= \frac{16}{3003} = 0.0053$$

~~b)~~ $\cancel{\int_0^{\pi/2} \sqrt{\tan x}\, dx}$

b) $\int_0^1 \log\left(\frac{1}{y}\right) dy$

sol put $\log\left(\frac{1}{y}\right)=t$

$$\frac{1}{y}=e^{t}$$

$$\frac{1}{e^{t}}=y \; ; \; y=e^{-t}$$

$dy=-e^{-t}dt$ as $y=0,\ t=\infty$

$y=1 \quad t=0$

$$\int_{\infty}^{0} t(-e^{-t})dt=\int_{0}^{\infty} te^{-t}=\underline{\underline{1}}$$