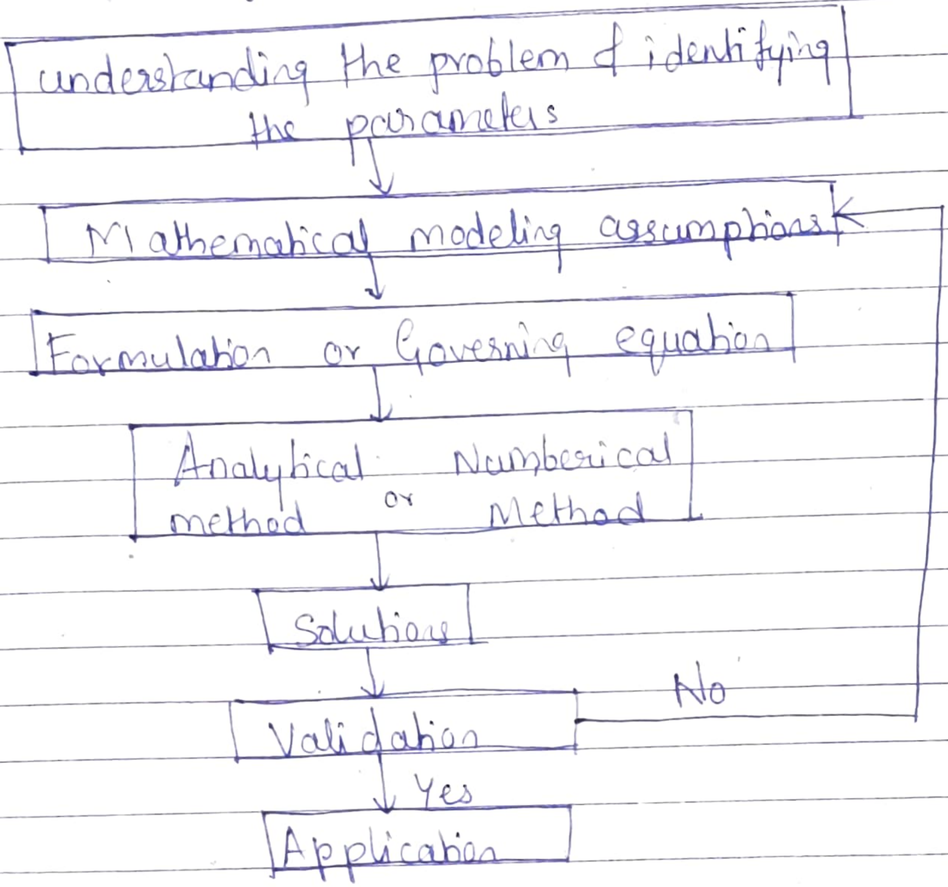
Assignment

Chapterwise plan

1. Introduction to Mathematical Modeling

1) What is mathematical modeling? And explain the process of mathematical modeling.

→ mathematical Modeling is the process that uses mathematics to represent, Analyse, Make Prediction Provide insight into real world phenomenon



→ The first step towards mathematical modeling is about understanding the problem & identifying

→ the parameters (In this step we analysis the problem and see which parameters have major influence on the solution to the problem

→ The next step is to Construct the basic frame work of the model by making Certain assumption (in this step we state those parameters which are not essential and can be neglected.

→ If the assumptions are sufficiently precise they may lead directly to the formulation or governing equations" In Some Cases the formulation itself is the solution" if the formulation is not the Solution then we apply analytical method to solve the equation when analytical methods are uproductable we can use numberical are upproductable we can use numberical methods to optain Solution

→ After obtaining the Solution we start testing the Validity of the model by Comparing the theoretical and practical result if the model is valid then we go more towards applications if not we recheck our assumptions and repeat the steps until we get a valid model.

2. Construct a mathematical model for the velocity prior to opening the parachute when a parachutist of mass m kg jumps out of a stationary hot air balloon where drag co-efficient is c kg/s

→

Understanding the problem: Find the velocity prior to opening the parachute.

identifying the parameters: Forces acting on the body and mass

Mathematical Modeling Assumptions: No horizontal force is acting on the body and mass of the parachute is negligible

Formulation or Governing equations:-

By the Newton's Second law; $F = ma$

Net force acting on the body is $F = F_d + F_u$ — ①

Substituting in equation 1 we get

$$ma = mg - cv$$

$$m\frac{dv}{dt} = mg - cv$$

$$\boxed{\frac{dv}{dt} = g - \left(\frac{c}{m}\right)v}$$

is a model for acceleration

$$\frac{dv}{dt} + \left(\frac{c}{m}\right)v = g$$

Solving by analytical method

$$I.F = e^{\int \frac{c}{m} dt} = e^{\left(\frac{c}{m}\right)t}$$

$$v \times I.F = \int I.F \times g \, dt$$

$$v \times e^{\left(\frac{c}{m}\right)t} = g \frac{e^{\left(\frac{c}{m}\right)t}}{\frac{c}{m}} + k \quad \text{--- (2)}$$

At $t=0, v=0 \rightarrow k = -\frac{gm}{c}$ subs in equation in (2)

$$v(t) = \frac{gm}{c}\left(1-e^{-(c/m)t}\right)$$

$\therefore$ velocity of parachutist is the solution of the model

Validation :-

Consider parachutist of mass - $m = 68.1$ kg jumps out of a stationary hot air balloon Find the velocity prior to opening the chute the drag Co-efficient is $c = 12.5$ kg/s

Soln:-

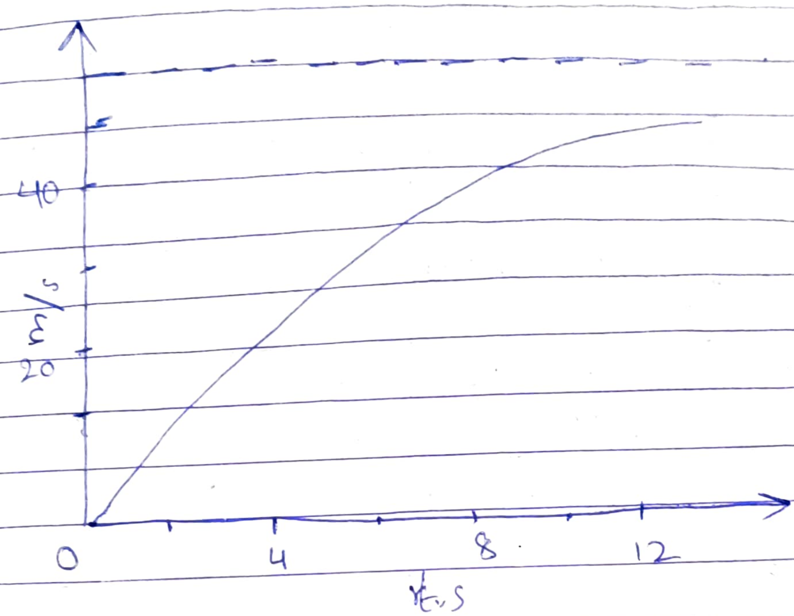
velocity is $v(t) = \frac{gm}{c}\left(1-e^{-(c/m)t}\right)$

By substituting $g = 9.8\ m/s^2$ $c = 12.5$ kg/s. $m = 68.1$ kg

velocity at any time:- $v(t) = 53.39\left(1-e^{-0.18355t}\right)$

For different values

t(sec)	0	2	4	8
v(m/s)	0	16.40	27.77	41.10



conclusion : Table and graph tells us that after long time velocity remains Constant that is 53.39 m/s is the terminal velocity

2 Functions and Graphs

7.
→ (i) Given: $T_1 = 70°F$, $N_1 = 113$
$T_2 = 80°F$ $N_2 = 173$
$T = f(N)$ is linear
we have $\boxed{T = mN + C}$
where, m = slope c = y - intercept
$70 = m(113) + C$ — ①
& $80 = m(173) + C$ — ②

solving ② - ① we get

$80 = m(173) + \cancel{C}$
$\underset{(-)}{70} = \underset{(-)}{m}(\cancel{1}13) + \underset{(-)}{\cancel{C}}$; & $C = 51.1667$

$10 = 60m$
$m = \frac{1}{6} = 0.1667$

∴ $T = \frac{1}{6} N + 51.1667$ — ③

(ii) slope = 1/6
Represent = For every 6 chirps produced the temperature is 1 degree higher

(iii) Put $N = 150$ in eqn ③
$T = \frac{1}{6}(150) + 51.1667 = 76.1667°F$

(8)

→ Given $t=0$ $P_0 = 1000$

(i) Population increases by 50 people a year

$t=1$, $P_1 = P_0 + 50 = 1000+50$
$t=2$, $P_2 = P_1 + 50 = (1000+50) = 1000+2(50)$
$t=3$, $P_3 = P_3 + 50 = 1000 + 2(50) + 50$
$= 1000+350$

$\therefore$ $P_t = 1000 + t(50)$

(ii) Population increases by 5% a year
$P_1 = P_0 + 5\% P_0 = P_0 + 0.05 P_0 = P_0(1+0.05)$
$= P_0(1.05)$
$P_2 = P_1 + 5\% P_1 = P_1 + 0.05 P_1 = P_1(1.05)$
$= (1.05)(1.05)P_0 = (1.05)^2 P_0$
$P_3 = (1.05)^3 P_0$
$P_t = (1.05)^t P_0 \Rightarrow \boxed{P_t = (1.05)^t 1000}$

(9)

→ Given ; $t=0$; $Q_0 = 2g$
$t_{1/2} = 15$ hrs ; $Q = Q_0/2 = \frac{2}{2} = 1g$

i) Find the amount remaining after 60 hour
ii) Find the amount remaining after t hours
iii) Estimate the amount remaining after 4 dc

To find ; i) $t = 60$ hrs, $Q = ?$
ii) $t = t$, $Q = ?$ iii) $t = 4$ days. &
wkt ; $Q = Q_0 a^t$, $Q = 2a^t$ —— (1)
Put $t = 15$ & $Q = 1$

$1 = 2a^{15} \Rightarrow 1/2 = a^{15}$

$a = \left(\frac{1}{2}\right)^{1/15}$

Eq (1) reduces to

$Q = 2\left(\frac{1}{2}\right)^{t/15}$ — (2)

i) Put $t = 60$ in Eqn (2)

$Q = 2\left(\frac{1}{2}\right)^{60/15} = 0.125\,g$

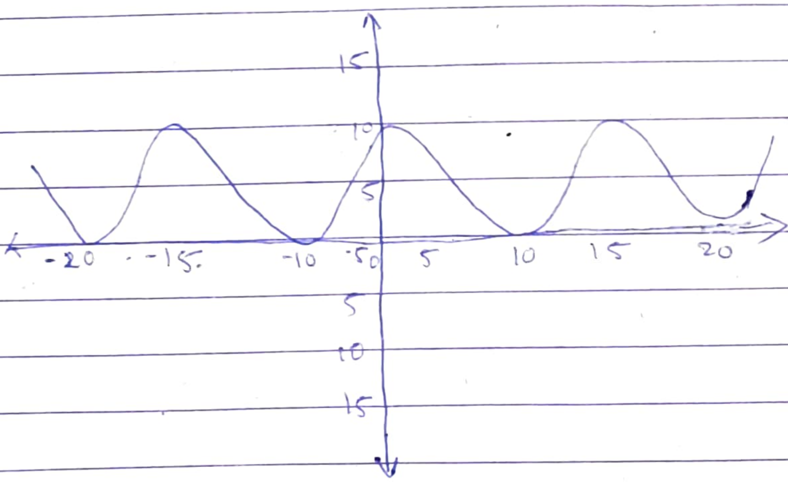
iii) $t = 4$ days $= 4 \times 24 = 96$ hrs

Put $t = 96$ in Eqn (2)

$Q = 2\left(\frac{1}{2}\right)^{96/15} = 0.0237\,g$

10)

Soln



The oscillations have amplitude $= \frac{\text{high tide} - \text{low tide}}{2}$

$= \frac{9.9 - 0.1}{2} = 4.9$ feet

~~3. Calculus of functions and models~~

Since water is highest at mid night, when $t=0$
The Oscillations are best represented by Cosine functions

We Can Say height above average $= 4.9\cos\left(\frac{\pi}{6}\right)t$

Since the average water level was $= \frac{9.9+0.1}{2} = 5\text{ ft}$

we shift the Cosine up by adding 5 so
$y = 5 + 4.9\cos\left(\frac{\pi}{6}\right)t$

~~graphs~~

3. Calculus of function and models

2. Define horizontal and vertical asymptote and hence find the horizontal and vertical asymptotes of the curves

i) $y = \frac{x^2}{x^2-1}$

→ *) The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if

$\lim_{x \to a} f(x) = \infty$

*) The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if

$\lim_{x \to \infty} f(x) = L$

i) $y = \frac{x^2}{x^2-1}$

→ Vertical asymptotes

Here $f(x) = \frac{x^2}{x^2-1}$

$f(x) = \frac{x^2}{x^2-1}$ becomes ∞ if $x^2 - 1 = 0$

$\therefore (x-1)(x+1) = 0$

$x = 1, -1$

$\therefore x = 1, -1$ are vertical asymptotes.

Horizontal asymptotes;

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{x^2}{x^2-1}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{x^2\left(1-\frac{1}{x^2}\right)}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{\left(1-\frac{1}{x^2}\right)}$$

$$\lim_{x\to\infty} f(x) = 1$$

$\therefore$ $y=1$ is the horizontal asymptotes

ii) $y = \dfrac{x^3}{x^2+3x-10}$

→ Vertical asymptotes :-

Here $f(x) = \dfrac{x^3}{x^2+3x-10}$

$f(x) = \dfrac{x^3}{x^2+3x-10}$ becomes ∞ if $x^2+3x-10=0$

$\therefore$ $(x+5)(x-2)=0$

$x=-5, 2$

$\therefore$ $x=-5, 2$ are vertical asymptotes,

Horizontal asymptotes :

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3}{x^2+3x-10}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3}{x^3\left(\frac{1}{x}+\frac{3}{x^2}-\frac{10}{x^3}\right)}$$

$\therefore$ No horizontal asymptotes

(22)

using L - Hospital's rule

i) $\lim_{x\to 0} \left(\frac{1}{\sin x} - \frac{1}{x}\right)$ $(\infty - \infty)$ form

$= \lim_{x\to 0} \left(\frac{x - \sin x}{x\sin x}\right) \left(\frac{0}{0}\right)$ form, applying L Hospitals rule

$= \lim_{x\to 0} \frac{1 - \cos x}{x\cos x + \sin x} \left(\frac{0}{0}\right)$ form

$= \lim_{x\to 0} \frac{\sin x}{-x\sin x + \cos x + \cos x} = 0$

ii) $\lim_{x\to 0} (\cot x)^{\frac{1}{\log x}}$ (∞^0) form

$L = \lim_{x\to 0} (\cot x)^{\frac{1}{\log x}}$

$\log L = \lim_{x\to 0} \frac{1}{\log x} \cdot \log(\cot x)$ $(0.\infty)$ form

$\log L = \lim_{x\to 0} \frac{\log(\cot x)}{\log x} \left(\frac{\infty}{\infty}\right)$ form. applying L Hospital rule

$\log L = \lim_{x\to 0} \frac{1}{\cot x}\left(\frac{-\operatorname{cosec}^2 x}{1/x}\right)$

$\log L = \lim_{x\to 0} \frac{x}{\sin x} \cdot \lim_{x\to 0} \left(\frac{1}{\cos x}\right)$

$\log L = -1$

$L = e^{-1} = \frac{1}{e}$

iii) $\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

$L = \lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{1/x^2}$ 1^∞ form

$\log L = \lim_{x\to 0}\frac{1}{x^2}\cdot\log\left(\frac{\sin x}{x}\right)$ $(\infty.0)$ form

$\log L = \lim_{x\to 0}\frac{\left(\log\left(\frac{\sin x}{x}\right)\right)}{x^2}$ $\frac{0}{0}$ form, applying L'Hospital rule

$\text{Log} L = \lim_{x\to 0}\left[\frac{\left(\frac{1}{\left(\frac{\sin x}{x}\right)}\cdot\frac{x\cos x-\sin x}{x^2}\right)}{2x}\right]$

$\log L = \lim_{x\to 0}\left(\frac{x}{\sin x}\right)\left(\frac{x\cos x-\sin x}{2x^3}\right)$

$\text{Log} L = \lim_{x\to 0}\left(\frac{x\cos x-\sin x}{2x^3}\right)$ $\frac{0}{0}$ form applying L'Hospital rule

$\log L = \lim_{x\to 0}\frac{-x\sin x+\cos x-\cos x}{6x^2} = \lim_{x\to 0}\left(\frac{-1}{6}\right)\left(\frac{\sin x}{x}\right)$

$= \frac{-1}{6}$

$L = e^{\left(\frac{1}{6}\right)}$