Triple Integrals

(1) i) $\iiint_B xyz^2\,dV$ $\quad B = \{(x,y,z) \mid 0 \le x \le 1,\ -1 \le y \le 2,\ 0 \le z \le 3\}$

Sol^n: $\displaystyle\int_0^1\int_{-1}^2\int_0^3 xyz^2\,dz\,dy\,dx = \int_0^1\int_{-1}^2 xy\left[\frac{z^3}{3}\right]_0^3 dy\,dx$

$\displaystyle I = \int_0^1\int_{-1}^2 xy \times 9\,dy\,dx = 9\int_0^1 x\left[\frac{y^2}{2}\right]_{-1}^2 dx = 9\int_0^1 x\left[\frac{4}{2} - \frac{1}{2}\right] dx$

$\displaystyle I = \frac{27}{2}\left[\frac{x^2}{2}\right]_0^1 = \frac{27}{4}$ //

ii) $\iiint_E (xz - y^3)\,dV$ $\quad E = \{(x,y,z) \mid -1 \le x \le 1;\ 0 \le y \le 2,\ 0 \le z \le 1\}$

$\displaystyle I = \int_{-1}^1\int_0^2\int_0^1 (xz - y^3)\,dz\,dy\,dx = \int_{-1}^1\int_0^2 \left[\frac{xz^2}{2} - y^3z\right]_0^1 dy\,dx$

$\displaystyle I = \int_{-1}^1\int_0^2 \frac{x}{2} - y^3\,dy\,dx = \int_{-1}^1 \left[\frac{xy}{2} - \frac{y^4}{4}\right]_0^2 dx$

$\displaystyle I = \int_{-1}^1 \frac{2x}{2} - \frac{16}{4}\,dx = \int_{-1}^1 x - 4\,dx = \left[\frac{x^2}{2} - 4x\right]_{-1}^1$

$\displaystyle I = \left(\left(\cancel{\tfrac{1}{2}} - 4\right) - \left(\cancel{\tfrac{1}{2}} + 4\right)\right) = -8$ //

(iii) $\displaystyle\iiint_E \frac{z}{x^2+z^2}\,dV$ $\quad E = \{(x,y,z) \mid 1 \le y \le 4,\ y \le z \le 4,\ 0 \le x \le 2\}$

$\displaystyle I = \int_1^4\int_y^4\int_0^2 \frac{z}{x^2+z^2}\,dx\,dz\,dy =$

$\displaystyle I = \int_1^4\int_y^4 z\,\frac{1}{z}\left[\tan^{-1}\left(\frac{x}{z}\right)\right]_0^2 dz\,dy = \int_1^4\int_y^4 \tan^{-1} 1 - \tan^{-1} 0\ \,dz\,dy$

$$I = \int_1^4 \int_y^4 \frac{\pi}{4}\, dz\, dy = \frac{\pi}{4}\int_1^4 [4-y]\, dy$$

$$I = \frac{\pi}{4}\left[4y - \frac{y^2}{2}\right]_1^4 = \frac{\pi}{4}\left[(16-8)-\left(4-\frac{1}{2}\right)\right]$$

$$I = \frac{\pi}{4}\left(8-\frac{7}{2}\right) = \frac{\pi}{4}\left(\frac{16-7}{2}\right)$$

$$\boxed{I = \frac{9\pi}{8}}$$

$$\cancel{②\ i)\ \int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dz\,dy\,dx}$$

$$\cancel{I=\int_{-c}^{c}\int_{-b}^{b}\left(x^2z+y^2z+\frac{z^3}{3}\right)_{-a}^{a}dy\,dx = \int_{-c}^{c}\int_{-b}^{b}\left(ax^2+ay^2+\frac{a^3}{3}\right)dy\,dx}$$

$$\cancel{I=\int_{-c}^{c}\left[ax^2y+\frac{ay^3}{3}+\frac{a^3y}{3}\right]_{-b}^{b}dx = \int_{-c}^{c}\left(abx^2+\frac{ab^3}{3}+\frac{a^3b}{3}\right)dx}$$

$$\cancel{I=\left[ab\frac{x^3}{3}+\frac{ab^3}{3}x+\frac{a^3bx}{3}\right]_{-c}^{c} = ab\frac{c^3}{3}+\frac{ab^3}{3}c+\frac{a^3bc}{3}}$$

② $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}(x^2+y^2+z^2)\,dz\,dy\,dx$

$$I=\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}\left(x^2z+y^2z+\frac{z^3}{3}\right)_{-a}^{a}dy\,dx = \int_{-c}^{c}\int_{-b}^{b}\left[\left(x^2a+ay^2+\frac{a^3}{3}\right)-\left(-ax^2-ay^2-\frac{a^3}{3}\right)\right]dy\,dx$$

$$I=\int_{-c}^{c}\int_{-b}^{b} x^2a+ay^2+\frac{a^3}{3}+ax^2+ay^2+\frac{a^3}{3}\,dy\,dx = 2\int_{-c}^{c}\int_{-b}^{b} x^2a+ay^2+\frac{a^3}{3}\,dy\,dx$$

$$I=2\int_{-c}^{c}\left[ax^2y+\frac{ay^3}{3}+\frac{a^3}{3}y\right]_{-b}^{b}dx = 2\int_{-c}^{c}\left[\left(ax^2b+\frac{ab^3}{3}+\frac{a^3b}{3}\right)-\left(-ax^2b-\frac{ab^3}{3}-\frac{a^3b}{3}\right)\right]dx$$

$$I=4\int_{-c}^{c} abx^2+\frac{ab^3}{3}+\frac{a^3b}{3}\,dx = 4\left[\left(ab\frac{x^3}{3}+\frac{ab^3}{3}x+\frac{a^3b}{3}x\right)\right]_{-c}^{c}$$

$$I = 4\left[\left(\frac{abc^3}{3}+\frac{ab^3c}{3}+\frac{a^3bc}{3}\right)\cdot\left(-\frac{abc^3}{3}-\frac{ab^3c}{3}-\frac{a^3bc}{3}\right)\right]$$

$$I = 8\left[\frac{abc^3}{3}+\frac{ab^3c}{3}+\frac{a^3bc}{3}\right]$$

$$I = \frac{8abc\left(a^2+b^2+c^2\right)}{3} \quad ॥$$

(ii) $$\int_0^{2\pi}\int_0^{\pi/4}\int_0^{a} r^2\sin\theta \, dr\, d\theta\, d\phi$$

$$I = \int_0^{2\pi}\int_0^{\pi/4}\frac{\sin\theta}{3}\left[r^3\right]_0^a d\theta\, d\phi = \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4}\sin\theta\, a^3\, d\theta\, d\phi$$

$$I = -\frac{a^3}{3}\int_0^{2\pi}\left[\cos\theta\right]_0^{\pi/4} d\phi = -\frac{a^3}{3}\int_0^{2\pi}\frac{1}{\sqrt{2}}-1\, d\phi$$

$$I = \frac{a^3}{3}\left(1-\frac{1}{\sqrt{2}}\right)\left[2\pi\right] = \frac{2\pi a^3}{3}\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)$$

$$\boxed{I = \frac{2\pi a^3(\sqrt{2}-1)}{3\sqrt{2}} \quad ॥}$$

(iii) $$\int_0^1\int_x^{2x}\int_0^{y} 2xyz\, dz\, dy\, dx$$

$$I = 2\int_0^1\int_x^{2x} xy\left[\frac{z^2}{2}\right]_0^y dy\, dx = \int_0^1\int_x^{2x} xy^3\, dy\, dx = \int_0^1\frac{x}{4}\left[y^4\right]_x^{2x} dx$$

$$I = \frac{1}{4}\int_0^1 x\left[16x^4-x^4\right]dx = \frac{1}{4}\int_0^1 x\left[15x^4\right]dx = \frac{15}{4}\int_0^1 x^5\, dx$$

$$I = \frac{15}{4}\left[\frac{x^6}{6}\right]_0^1 = \frac{15}{24}(1) = 5/8 \quad ॥$$

$$\int_0^2\int_0^{z^2}\int_0^{y-z}(2x-y)\,dx\,dy\,dz$$

$$\int_0^2\int_0^{z^2}\left[x^2-yx\right]_0^{y-z} = \int_0^2\int_0^{z^2} \cancel{y^2}+z^2-2yz-\cancel{y^2}+yz\,dy\,dz$$

$$\int_0^2\int_0^{z^2} z^2-yz\,dy\,dz = \int_0^2\left[z^2y-\frac{y^2z}{2}\right]_0^{z^2}dz = \int_0^2 z^4-\frac{z^5}{2}\,dz$$

$$=\left[\frac{z^5}{5}-\frac{z^6}{12}\right]_0^2 = \frac{32}{5}-\frac{\cancel{64}^{16}}{\cancel{12}_3} = \frac{32}{5}-\frac{16}{3}$$

$$=\frac{96-80}{15} = \frac{16}{15}$$

$$\int_0^{\pi/2}\int_0^{y}\int_0^{x}\cos(x+y+z)\,dz\,dx\,dy.$$

Put $t=x+y+z$
$dt=1\,dz$
Put $z=0\Rightarrow t=x+y$
$z=x\Rightarrow t=2x+y$

$$\int_0^{\pi/2}\int_0^{y}\int_{x+z}^{2x+y}\cos t\,dt\,dx\,dy$$

$$\int_0^{\pi/2}\int_0^{y}\left[\sin t\right]_{x+z}^{2x+y}dx\,dy = \int_0^{\pi/2}\int_0^{y}\left[\sin(2x+y)-\sin(x+y)\right]dx\,dy$$

$t_1=2x+y$ $\quad t_2=x+y$
$dt_1=2dx$ $\quad dt_2=dx$
$x=0\Rightarrow t_1=y$ $\quad x=0\Rightarrow t_2=y$
$x=y\Rightarrow t_1=2y+y$ $\quad x=y\Rightarrow t_2=2y$
$t_1=3y$

$$=\int_0^{\pi/2}\left[\int_y^{3y}\sin t_1\left(\frac{dt_1}{2}\right)-\int_y^{2y}\sin t_2\,(dt_2)\right]dy$$

$$=\int_0^{\pi/2}\left[-\frac{1}{2}\left[\cos t_1\right]_y^{3y}+\left[\cos t_2\right]_y^{2y}\right]dy$$

$$=\int_0^{\pi/2}\left[-\frac{1}{2}\left[\cos 3y-\cos y\right]+\left[\cos(2y)-\cos y\right]\right]dy$$

$$I=-\frac{1}{2}\left\{\left[\frac{\sin 3y}{3}\right]_0^{\pi/2}-\left[\sin y\right]_0^{\pi/2}\right\}+\left[\frac{\sin 2y}{2}-\sin y\right]_0^{\pi/2}$$

$$=-\frac{1}{2}\left[\frac{\sin 270}{3}-\frac{\sin 90}{1}\right]+\left[\frac{\sin 180}{2}-\frac{\sin 90}{1}\right]$$

$$=-\frac{1}{2}\left(-\frac{1}{3}-1\right)+\left(0-1\right) = -\frac{1}{2}\left(\frac{-1-3}{3}\right)-1 = \frac{-1}{2}\left[\frac{-4}{3}\right]-1$$

$= \frac{2}{3} - 1 \quad = \quad -\frac{1}{3}$,,

③ Use triple Integral to find the volume of.

i) Solid Enclosed by Cylinder $y = x^2$ & plane $z = 0$ & $y + z = 1$

Sol^n:

$\boxed{z = 0 \quad \text{to} \quad z = 1 - y}$

$\boxed{y = x^2 \quad \text{to} \quad y = 1 \ [\because z = 0]}$

$y + z = 1$

Put $y = x^2$ & $z = 0$

$x^2 = 1$

$x = \pm 1$

$\boxed{x = -1 \quad \text{to} \quad x = 1}$

$$V = \int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} dz\, dy\, dx = \int_{-1}^{1} \int_{x^2}^{1} 1 - y \, dy\, dx$$

$$V = \int_{-1}^{1} \left(y - \frac{y^2}{2}\right)_{x^2}^{1} dx = \int_{-1}^{1} \left(1 - \frac{1}{2}\right) - \left(x^2 - \frac{x^4}{2}\right) \cdot dx$$

$$V = \int_{-1}^{1} \frac{1}{2} - x^2 + \frac{x^4}{2}\, dx = \left[\frac{1}{2}x - \frac{x^3}{3} + \frac{x^5}{10}\right]_{-1}^{1}$$

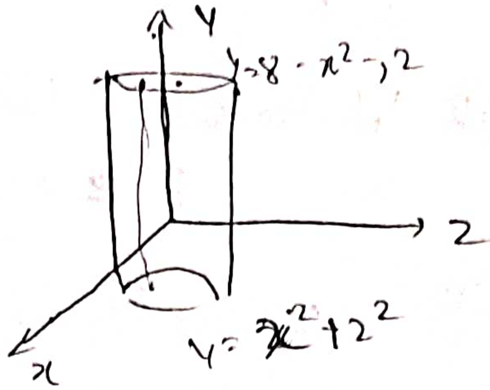
$$V = \left[\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{10}\right) - \left(\frac{-1}{2} + \frac{1}{3} - \frac{1}{10}\right)\right] = \frac{1}{2} - \frac{1}{3} + \frac{1}{10} + \frac{1}{2} - \frac{1}{3} + \frac{1}{10}$$

$$V = \frac{2}{2} - \frac{2}{3} + \frac{2}{10} = 1 - \frac{2}{3} + \frac{1}{5} = \frac{15 - 10 + 3}{15}$$

$$\boxed{V = \frac{8}{15}}$$,,

(ii) solid enclosed by cylinders $y = x^2 + z^2$ & $y = 8 - x^2 - z^2$

In Cylindrical Coord.



$x = r\cos\theta \quad z = r\sin\theta \quad y = y \rightarrow (i)$

$dx\,dy\,dz = r\,dr\,d\theta\,dy$

Eq (i) ⇒ $x^2 + z^2 = 8 - x^2 - z^2$

$r^2 = 8 - r^2$

$2r^2 = 8$

$r^2 = 4$

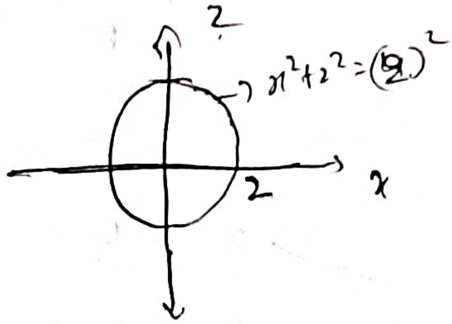
$\boxed{r = 2}$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$x^2 + z^2 \le y \le 8 - x^2 - z^2$

$r^2 < y < 8 - r^2$



$$V = \int_0^{2\pi}\int_0^2\int_{r^2}^{8-r^2} r\,dy\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r\,[8 - r^2 - r^2]\,dr\,d\theta$$

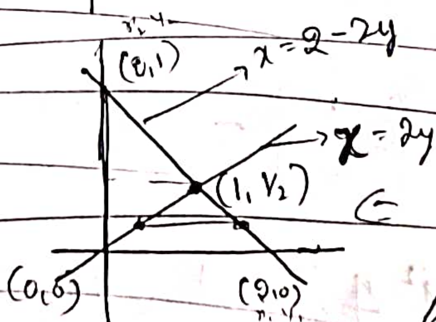
$$V = \int_0^{2\pi}\int_0^2 (8r - 2r^3)\,dr\,d\theta = \int_0^{2\pi}\left[4r^2 - \frac{r^4}{2}\right]_0^2 d\theta$$

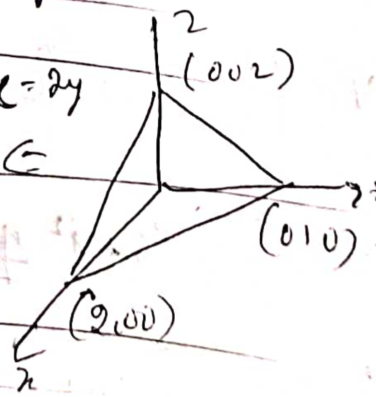
$$V = \int_0^{2\pi} 16 - \frac{16}{2}\,d\theta = 8\,[2\pi] = 16\pi$$

(iii) Tetrahedron T bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$ & $z = 0$

$x + 2y + z = 2$

$\frac{x}{2} + \frac{y}{1} + \frac{z}{2} = 1$





$0 \le z \le 2 - x - 2y$

$2y \le x \le 2 - 2y$

$0 \le y \le 1/2$

$y - 0 = \frac{1 - 0}{0 - 2}(x - 2)$

$-2y = x - 2$

$x = 2 - 2y$

$2y = 2 - 2y$

$4y = 2 \Rightarrow 2y = 1 \quad \boxed{y = 1/2} \quad \boxed{x = 1}$

(iii) Tetrahedron T bounded by the planes $x+2y+z=2$, $x=2y$, $x=0$ & $z=0$

Sol:

$$x+2y+z=2$$

$$\frac{x}{2}+\frac{y}{1}+\frac{z}{2}=1$$

P.I of l.i

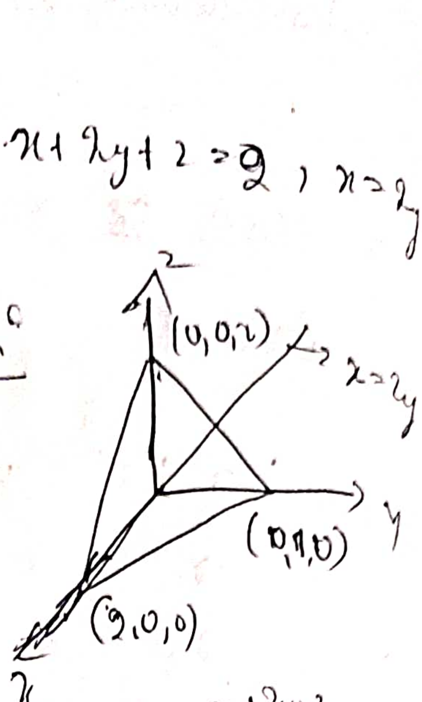
$$\frac{x}{2}=\frac{2-x}{2}$$

$$2x-2=0$$

$$2(x-1)=0$$

$$\boxed{x=1}$$

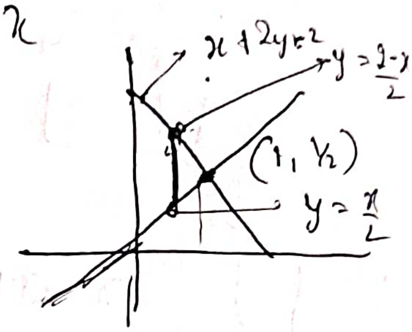
$$\boxed{y=\frac{1}{2}}$$



$$0 \le z \le 2-x-2y$$

$$\frac{x}{2} \le y \le \frac{2-x}{2}$$

$$0 \le x \le 1$$



$$V=\int_0^1\int_{x/2}^{2-x/2}\int_0^{2-x-2y} dz\,dy\,dx$$

$$V=\int_0^1\int_{x/2}^{\frac{(2-x)}{2}}(2-x-2y)\,dy\,dx=\int_0^1\Big[(2-x)y-y^2\Big]_{x/2}^{\frac{(2-x)}{2}}dx$$

$$V=\int_0^1\left\{\left[\frac{(2-x)(2-x)}{2}-\frac{(2-x)^2}{4}\right]-\left[(2-x)\frac{x}{2}-\frac{x^2}{4}\right]\right\}dx$$

$$V=\int_0^1\left\{(2-x)^2\left[\frac{1}{2}-\frac{1}{4}\right]-\left[\frac{2x-x^2}{2}-\frac{x^2}{4}\right]\right\}dx$$

$$V=\int_0^1\left\{(2-x)^2\left[\frac{1}{4}\right]-\left(\frac{4x-2x^2-x^2}{4}\right)\right\}dx$$

$$V=\int_0^1\frac{1}{4}(4+x^2-4x)-\frac{1}{4}(4x-3x^2)\}\,dx$$

$$V=\frac{1}{4}\int_0^1 4+x^2-4x-4x+3x^2\,dx$$

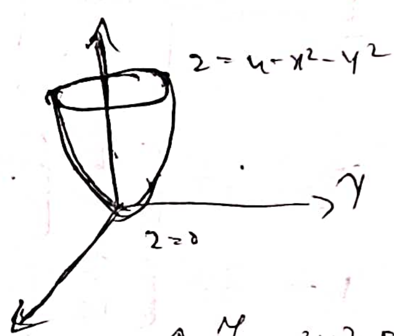
$$V = \frac{1}{4}\int_0^1 4 + 4x^2 - 8x \, dx = \int_0^1 1 + x^2 - 2x \, dx$$

$$V = \left[x + \frac{x^3}{3} - x^2\right]_0^1 = \left[\cancel{1} + \frac{1}{3} - \cancel{1}\right]$$

$$\boxed{V = \frac{1}{3}}$$

(4) Problems on changing to Cylindrical Co-ordinates

i) Evaluate $\iiint_B (x+y+z)\, dV$ E: the Solid in first Octant under Paraboloid $z = 4 - x^2 - y^2$

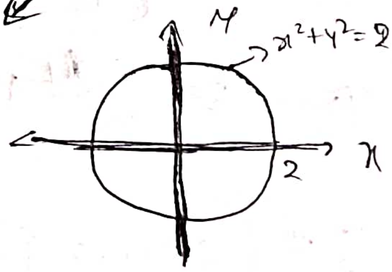


Sol:

$0 \le z \le 4 - x^2 - y^2 \Rightarrow \underline{0 \le z \le 4 - r^2}$

$0 \le r \le 2$

$0 \le \theta \le \frac{\pi}{2}$



$$I = \int_0^{\pi/2}\int_0^2\int_0^{4-r^2} (x+y+z)\, r\, dz\, dr\, d\theta$$

$$I = \int_0^{\pi/2}\int_0^2 \left[xz + yz + \frac{z^2}{2}\right]_0^{4-r^2} r\, dr\, d\theta$$

$$I = \int_0^{\pi/2}\int_0^2 \left[x(4-r^2) + y(4-r^2) + \frac{(4-r^2)^2}{2}\right] r\, dr\, d\theta$$

$$I = \int_0^{\pi/2}\int_0^2 r^2\cos\theta\,(4-r^2) + r^2\sin\theta\,(4-r^2) + \frac{r(4-r^2)^2}{2}\, dr\, d\theta$$

$$I = \int_0^{\pi/2}\int_0^2 \cos\theta\,(4r^2 - r^4) + \sin\theta\,(4r^2 - r^4) + \frac{16r + r^5 - 8r^3}{2}\, dr\, d\theta$$

$$I = \int_0^{\pi/2} \cos\theta \left[\frac{4r^3}{3}-\frac{r^5}{5}\right]_0^2 + \sin\theta\left[\frac{4r^3}{3}-\frac{r^5}{5}\right]_0^2 + \frac{1}{2}\left\{8\left[r^2\right]_0^2 + \left[\frac{r^6}{6}\right]_0^2 - 2\left[r^4\right]_0^2\right\} d\theta$$

$$I = \int_0^{\pi/2} \cos\theta\left[\frac{32}{3}-\frac{32}{5}\right] + \sin\theta\left(\frac{32}{3}-\frac{32}{5}\right) + 32 + \frac{64}{6} - 32 \; d\theta$$

$$I = \int_0^{\pi/2} 32\cos\theta\left[\frac{5-3}{15}\right] + 32\sin\theta\left[\frac{5-3}{15}\right] + \frac{1}{2}\left[32\left(\cancel{1}+\frac{1}{3}-\cancel{1}\right)\right] d\theta$$

$$\cancel{I = 32\int_0^{\pi/2} \cos\theta \cdot \frac{2}{15} + \sin\theta\, \frac{2}{15} + \left[\frac{1}{6}\right] d\theta}$$

$$\cancel{I = 32\left[\frac{2}{15}\int_0^{\pi/2}\cos\theta+\sin\theta\; d\theta + \frac{1}{6}\int_0^{\pi/2} d\theta\right]}$$

$$\cancel{I = 32\left[\frac{2}{15}\left[\sin\theta-\cos\theta\right]_0^{\pi/2} + \frac{1}{6}\left(2\pi\right)\right]}$$

$$\cancel{I = 32\left[\frac{2}{15}\left[(1-0)-(0-1)\right] + \frac{2\pi}{3}\right]}$$

$$\cancel{I = 32\left[\frac{4}{15}+\frac{2\pi}{3}\right] = \frac{128}{15}+\frac{64\pi}{3}}$$

$$I = \int_0^{\pi/2} 32\cos\theta\left[\frac{2}{15}\right] \times 32\sin\theta\left(\frac{2}{15}\right) + \frac{32}{6}\; d\theta$$

$$I = 32\int_0^{\pi/2}\left[\frac{2}{15}\cos\theta+\frac{2}{15}\sin\theta\right] + \int_0^{\pi/2}\frac{1}{6}\, d\theta$$

$$I = 32\left[\frac{2}{15}\left[\sin\theta-\cos\theta\right]_0^{\pi/2} + \frac{1}{6}\times\frac{\pi}{2}\right]$$

$$I = 32\left[\frac{2}{15}\times 2+\frac{\pi}{12}\right] = \frac{128}{15}+\frac{\cancel{32}^{\,8}\pi}{\cancel{12}_{\,3}}$$

$$\boxed{I = \frac{128}{15}+\frac{8\pi}{3}}$$

//

Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\, dz\, dy\, dx$

$\dfrac{\sqrt{4-r^2\cos^2\theta}}{\sqrt{r^2-(r\cos\theta)^2}}$

~~In Cylindrical Co ordinates,~~

$\cancel{y = r\cos\theta \quad z = r\sin\theta \quad dy\,dz = r\,dr\,d\theta}$

$\cancel{x = x}$

$\cancel{I = \int_{-2}^{2}\int^{-\sqrt{4-}}}$

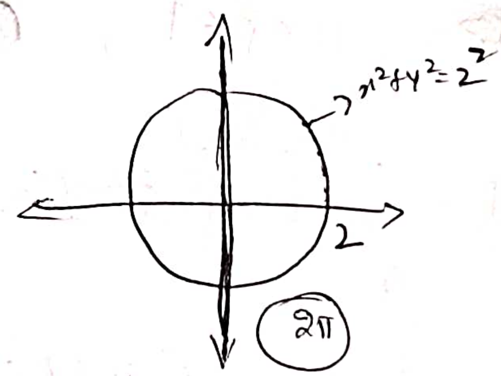
Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\, dz\, dy\, dx$

Sol:- $y = -\sqrt{4-x^2}$

Sq. on b.s

$y^2 = 4 - x^2$

$x^2+y^2 = 4$



$r \le z \le 2$, $\quad 0 \le r \le 2$, $\quad 0 \le \theta \le 2\pi$

$$I = \int_0^{2\pi}\int_0^2\int_r^2 r^2 \cdot r\, dz\, dr\, d\theta$$

$$I = \int_0^{2\pi}\int_0^2 [r^3]\,[2-r]\, dr\, d\theta = \int_0^{2\pi}\left[\frac{2r^4}{4} - \frac{r^5}{5}\right]_0^2 d\theta$$

$$I = \int_0^{2\pi}\left[\frac{16}{2} - \frac{32}{5}\right] d\theta = \int_0^{2\pi} \frac{80-64}{10}\, d\theta = \frac{16}{10}[2\pi]$$

$$I = \frac{16\pi}{5}$$ //

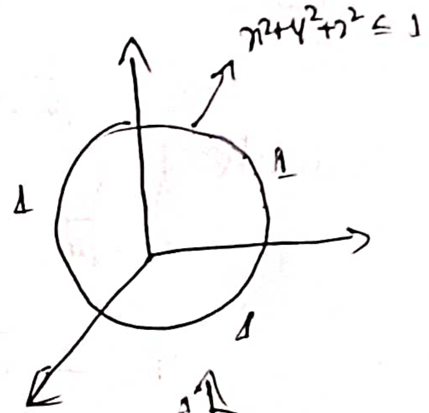
⑤ Problems on changing to Spherical Co-ordinates

i) Evaluate $\iiint_B x e^{x^2+y^2+z^2} \, dV$ B: the portion of the unit ball $x^2+y^2+z^2 \leq 1$ lies in the first Octant

Sol: In Spherical Coordinates,

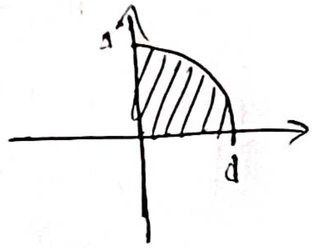
$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi$.

$dz\,dy\,dx = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$



$x^2+y^2+z^2 = \rho^2$

$0 \leq \rho \leq 1 , \quad 0 \leq \phi \leq \frac{\pi}{2} , \quad 0 \leq \theta \leq \pi/2$



$$I = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho \sin\phi \cos\theta \, e^{\rho^2} \cdot \underline{\rho^2 \sin\phi} \, d\rho \, d\phi \, d\theta$$

$$I = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 \rho^3 \sin^2\phi \cos\theta \, e^{\rho^2} \, d\rho \, d\phi \, d\theta$$

$$I = \int_0^{\pi/2} \cos\theta \int_0^{\pi/2} \sin^2\phi \int_0^1 \rho^2 \cdot \rho \cdot e^{\rho^2} \, d\rho \, d\phi \, d\theta$$

$$I = \int_0^{\pi/2} \cos\theta \int_0^{\pi/2} \sin^2\phi \int_0^1 t \, e^t \left(\frac{dt}{2}\right) d\phi \, d\theta$$

$$I = \frac{1}{2} \int_0^{\pi/2} \cos\theta \int_0^{\pi/2} \sin^2\phi \left[t e^t - e^t\right]_0^1 d\phi \, d\theta$$

$$I = \frac{1}{2} \int_0^{\pi/2} \cos\theta \int_0^{\pi/2} \sin^2\phi \left[(e - e) - (0 - 1)\right] d\phi \, d\theta$$

$$I = \frac{1}{2} \int_0^{\pi/2} \cos\theta \int_0^{\pi/2} \sin^2\phi \, d\phi \, d\theta$$

$t = \rho^2$

$dt = 2\rho \, d\rho$

$\frac{dt}{2} = \rho \, d\rho$

$\rho = 0 \Rightarrow t = 0$

$\rho = 1 \Rightarrow t = 1$

$t e^t - \int 1 e^t \, dt$

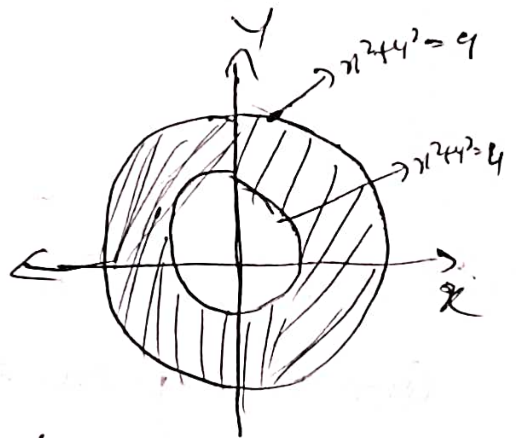
$t e^t - e^t$

$$I = \frac{1}{2}\int_0^{\pi/2} \cos\theta \times \frac{1}{2} B\left(\frac{3}{2}, \frac{1}{2}\right) d\theta = \frac{1}{4}\int_0^{\pi/2} \cos\theta \, \frac{\Gamma 3/2 \, \Gamma 1/2}{\Gamma 2}$$

$$I = \frac{1}{4}\int_0^{\pi/2} \cos\theta \, \frac{\frac{1}{2}\sqrt{\pi}\,\sqrt{\pi}}{1} \, d\theta = \frac{1}{4}\times\frac{1}{2}\pi \int_0^{\pi/2} \cos\theta \, d\theta$$

$$I = \frac{\pi}{8}\left[\sin\theta\right]_0^{\pi/2} = \frac{\pi}{8}(1) = \frac{\pi}{8} \quad //$$

) Evaluate $\iiint_E (x^2+y^2)\, dv$ where E: lies betⁿ the spheres.
$x^2+y^2+z^2=4$ & $x^2+y^2+z^2=9$.



In Spherical Co-ordinates,

$x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$. $z = \rho\cos\phi$

$dx\,dy\,dz = \rho^2\sin\phi \, d\rho\, d\theta\, d\phi$

$2 \le \rho \le 3$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$

$$I = \int_0^{2\pi}\int_0^{\pi}\int_2^{3} \left(\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta\right) \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$\left[\int_0^{\pi} = 2\int_0^{\pi/2}\right]$$

$$I = \int_0^{2\pi}\int_0^{\pi}\int_2^{3} \rho^2\sin^2\phi \cdot \rho^2\sin\phi \cdot d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_2^{3} \rho^4 \sin^3\phi \, d\rho\, d\phi\, d\theta = [2\pi] \cdot \cancel{2}\frac{1}{\cancel{2}} B\left(2, \frac{1}{2}\right) \frac{1}{5}\left[(3)^5 - (2)^5\right]$$

$$= (2\pi)\, \frac{\Gamma 2 \, \Gamma 1/2}{\Gamma 2+1/2} \times \frac{1}{5}\left[243 - 32\right] = \frac{2\pi}{5}\, \frac{\sqrt{\pi}}{\frac{3}{2}\times\frac{1}{2}\times\sqrt{\pi}} \times 211 = \frac{8\pi}{15}\times 211$$

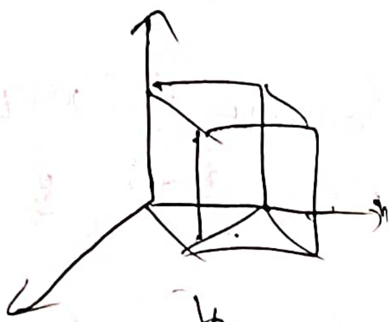
$$\boxed{I = \frac{1688\pi}{15}}$$

⑦ Find the total mass of the region in the Cube $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$ with the density at any point given by xyz

Sol^n: Mass $= \iiint_V \text{density } dv$

$$\text{Mass} = \int_0^1\int_0^1\int_0^1 xyz \, dz\, dy\, dx = \left[\frac{x^2}{2}\right]_0^1 \left[\frac{y^2}{2}\right]_0^1 \left[\frac{z^2}{2}\right]_0^1$$

$$\boxed{\text{Mass} = \frac{1}{8} \text{ Kg}}$$



⑧ Find the mass of a plate which is formed by the Co-ordinate planes & the planes & the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ the density is given by $\rho = kxyz$.

Sol^n:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$z = c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$$

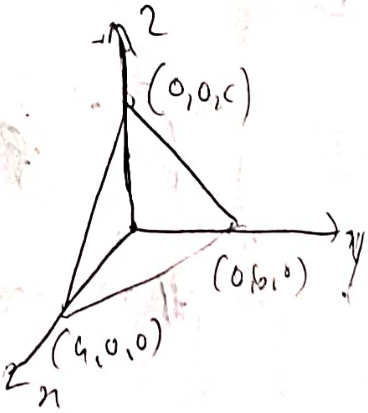
$$y = b\left(1 - \frac{x}{a}\right)$$

$$x = 0 \text{ to } a$$

$$0 \le z \le c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$$

$$0 \le y \le b\left(1 - \frac{x}{a}\right)$$

$$0 \le x \le a$$



$(x-y-z)^2 =$
$x^2+y^2+z^2-2xy-2xz+yz+zy$

$M = \iiint_V \text{density } dv$

$$M = \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} k\,xyz \; dz\,dy\,dx$$

$$M = \int_0^a \int_0^{b(1-\frac{x}{a})} k\,xy \left[\frac{z^2}{2}\right]_0^{c(1-\frac{x}{a}-\frac{y}{b})} dy\,dx = \frac{k}{2}\int_0^a\int_0^b xy\left[c\left(1-\frac{x}{a}-\frac{y}{b}\right)\right]^2 dy\,dx$$

$$M = \frac{kc^2}{2}\int_0^a\int_0^{b(1-\frac{x}{a})} xy\left[\left(1-\frac{x}{a}\right)^2+\frac{y^2}{b^2}-2\left(1-\frac{x}{a}\right)\left(\frac{y}{b}\right)\right] dy\,dx$$

$$M = \frac{kc^2}{2}\int_0^a\int_0^{b(1-\frac{x}{a})} xy\left[1+\frac{x^2}{a^2}-2\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2y}{b}\left(1-\frac{x}{a}\right)\right] dy\,dx$$

$$M = \frac{kc^2}{2}\int_0^a\int_0^{b(1-\frac{x}{a})} xy\left[1-\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2y}{b}+\frac{2xy}{ab}\right) dy\,dx$$

$$M = \frac{kc^2}{2}\int_0^a\int_0^{b(1-\frac{x}{a})} xy-\frac{x^3y}{a^2}+\frac{xy^3}{b^2}-\frac{2xy^2}{b}+\frac{2x^2y^2}{ab}\; dy\,dx.$$

$$M = \frac{kc^2}{2}\int_0^a \frac{xb^2}{2}-\frac{x^3b^2}{2a^2}+\frac{xb^4}{4b^2}-\frac{2xb^3}{3b}+\frac{2x^2b^3}{3ab}\; dx.$$

$$M = \frac{kc^2}{2}\left[\frac{b^2a^2}{4}-\frac{b^2a^4}{8a^2}+\frac{b^4a^2}{8b^2}-\frac{2b^3a^2}{6b}+\frac{2b^3a^3}{9ab}\right]$$

$$M = \frac{kc^2}{2}\left[\frac{b^2a^2}{4}-\frac{a^2b^2}{8}+\frac{a^2b^2}{8}-\frac{a^2b^2}{3}+\frac{2a^2b^2}{9}\right]$$

$$M = \frac{kc^2}{2}\times a^2b^2\left[\frac{1}{4}-\frac{1}{3}+\frac{1}{9}\right] = \frac{ka^2b^2c^2}{2}\left[\frac{9-12+4}{36}\right]$$

$$M = \frac{ka^2b^2c^2}{72}$$

$$M = \frac{kc^2}{2}\int_0^a \left[\frac{xy^2}{2} - \frac{x^3y^2}{2a^2} + \frac{xy^4}{4b^2} - \frac{2xy^3}{3b} + \frac{2x^2y^3}{3ab}\right]_0^{b(1-\frac{x}{a})}$$

$$M = \frac{kc^2}{2}\int_0^a \left[\frac{xb^2}{2}\left(1-\frac{x}{a}\right)^2 - \frac{x^3b^2}{2a^2}\left(1-\frac{x}{a}\right)^2 + \frac{xb^4}{4b^2}\left(1-\frac{x}{a}\right)^4 - \frac{2xb^3}{3b}\left(1-\frac{x}{a}\right)^3 + \frac{2x^2b^3}{3ab}\left(1-\frac{x}{a}\right)^3\right.$$

$$M = \frac{kc^2}{2}\int_0^a \frac{xb^2}{2}\left(1-\frac{x}{a}\right)^2 - \frac{x^3b^2}{2a^2}\left(1-\frac{x}{a}\right)^2 + \frac{xb^2}{4}\left(1-\frac{x}{a}\right)^4 - \frac{2xb^2}{3}\left(1-\frac{x}{a}\right)^3 + \frac{2x^2b^2}{3a}\left(1-\frac{x}{a}\right)^3$$

$\boxed{x = a(1-t)} \Leftarrow t = 1 - \frac{x}{a}$ Put $x = 0 \Rightarrow t = 1$

$dt = -\frac{1}{a}\,dx$ $x = a \Rightarrow t = 0$

$-a\,dt = dx$

$$M = \frac{kc^2}{2}\int_1^0 \left[\frac{a(1-t)}{2}t^2 - \frac{a^3(1-t)^3b^2}{2a^2}t^2 + \frac{a(1-t)b^2}{4}t^4 - \frac{2a(1-t)b^2}{3}t^3 + \frac{2a^2(1-t)^2b^2}{3a}t^3\right](-a\,dt)$$

$$M = \frac{kc^2}{2}\int_0^1 \frac{a^2(1-t)}{2}t^2 - \frac{a^4(1-t)^3b^2t^2}{2a^2} + \frac{a^2(1-t)b^2t^4}{4} - \frac{2a^2(1-t)b^2}{3}t^3 + \frac{2a^3(1-t)^2b^2}{3a}t^3$$

$$M = \frac{kc^2}{2}\int_0^1 \frac{a^2(1-t)}{2}t^2 - \frac{a^2(1-t)^3b^2}{2}t^2 + \frac{a^2b^2(1-t)}{4}t^4 - \frac{2a^2b^2(1-t)}{3}t^3 + \frac{2a^2b^2(1-t)^2t^3}{3}\,dt$$

$$M = \frac{kc^2}{2}\int_0^1 \frac{a^2}{2}(t^2-t^3) - \frac{a^2b^2}{2}(t^2-3t^3+3t^4-t^5) + \frac{a^2b^2}{4}(t^4-t^5) - \frac{2a^2b^2}{3}(t^3-t^4)$$

$$+ \frac{2a^2b^2}{3}(t^3+t^5-2t^4)\Big]\,dt$$

$$M = \frac{kc^2}{2}\left[\frac{a^2}{2}\left[\frac{t^3}{3}-\frac{t^4}{4}\right]_0^1 - \frac{a^2b^2}{2}\left[\frac{t^3}{3} - \frac{3}{4}t^4 + \frac{3}{5}t^5 - t^6\right]_0^1 + \frac{a^2b^2}{4}\left[\frac{t^5}{5}-\frac{t^6}{6}\right]_0^1 - \frac{2a^2b^2}{3}\left[\frac{t^4}{4}-\frac{t^5}{5}\right]_0^1\right. +$$

$$\left.\frac{2a^2b^2}{3}\left[\frac{t^4}{4}+\frac{t^6}{6}-\frac{2t^5}{5}\right]_0^1\right]$$

$$M = \frac{Kc^2}{2}\left[\frac{a^2}{2}\left(\frac{1}{12}\right) - \frac{a^2b^2}{2}\left(\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - 1\right) + \frac{a^2b^2}{4}\left(\frac{1}{5} - \frac{1}{6}\right) - \frac{2a^2b^2}{3}\left(\frac{1}{4} - \frac{1}{5}\right) + \right.$$

$$\left. \frac{2a^2b^2}{3}\left(\frac{1}{4} + \frac{1}{6} - \frac{2}{5}\right)\right]$$

$$M = \frac{Kc^2}{2}\left[\frac{a^2}{24} + \frac{49a^2b^2}{120} + \frac{a^2b^2}{120} - \frac{2a^2b^2}{60} + \frac{2a^2b^2}{180}\right]$$

$$M = \frac{Kc^2}{2}\left[\frac{15a^2 + 147a^2b^2 + 3a^2b^2 - 12a^2b^2 + 4a^2b^2}{360}\right]$$