1. Define limit of a function and Explain why graphically

(i) $\lim_{x\to 0} \frac{1}{x^2}$(ii) $\lim_{x\to 0} \frac{|x|}{x}$ does not exists.

Solution:

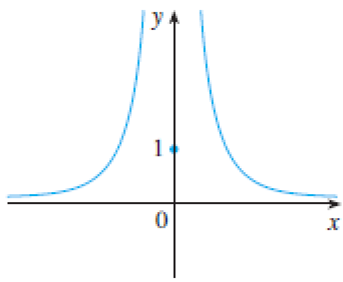
Definition :

Let f be a function defined on some open interval that contains the number a , except possibly at a itself. Then we say that the limit of $f(x)$ as x approaches a is L and we write

$\lim_{x\to a} f(x) = L$

Explain graphically : i)$\lim_{x\to 0} \frac{1}{x^2}$ does not exist

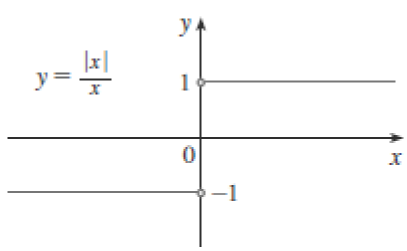
Solution :



In the above graph ,as x approaches to zero from either sides , the values of $\frac{1}{x^2}$ are positive and are arbitrary large (i.e. it approaches to infinity as x approaches to zero) so we can say that $\lim_{x\to 0} \frac{1}{x^2}$ does not exist

ii) $\lim_{x\to 0} \frac{|x|}{x}$ does not exists.

Solution:



$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x}$=$\lim_{x\to 0^+} 1 = 1$

$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{x}{-x}$=$\lim_{x\to 0^-} -1 = -1$

Since the right- and left-hand limits are different, it follows that $\lim_{x\to 0}\frac{|x|}{x}$ does not exists.

From The graph of the function $f(x) = \frac{|x|}{x}$ is shown in figure, the right hand and left hand limits are different

2. Define horizontal and vertical asymptote and hence find the horizontal and vertical asymptotes of the curves

(i)$y = \frac{x^2}{x^2-1}$ (ii) $y = \frac{x^3}{x^2+3x-10}$

Solutions:

Definitions

The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if $\lim_{x\to a} f(x) = \infty$

The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if $\lim_{x\to\infty} f(x) = L$

Vertical and horizontal asymptotes :

(i)$y = \frac{x^2}{x^2-1}$

Solution :

Vertical asymptotes

Here $f(x) = \frac{x^2}{x^2-1}$

$f(x) = \frac{x^2}{x^2-1}$ becomes ∞ if $x^2 - 1 = 0$

$\therefore (x-1)(x+1) = 0$

$x = 1, -1$

$\therefore x = 1, -1$ are vertical asymptotes.

Horizontal asymptotes:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{x^2-1}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{x^2(1-\frac{1}{x^2})}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{(1-\frac{1}{x^2})}$$

$$\lim_{x\to\infty} f(x) = 1$$

$\therefore y = 1$ is the horizontal asymptotes

(ii) $y = \frac{x^3}{x^2+3x-10}$

Solution:

Vertical asymptotes

Here $f(x) = \frac{x^3}{x^2+3x-10}$

$f(x) = \frac{x^3}{x^2+3x-10}$ becomes ∞ if $x^2 + 3x - 10 = 0$

$\therefore (x+5)(x-2) = 0$

$x = -5,2$

$\therefore x = -5,2$ are vertical asymptotes.

Horizontal asymptotes:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3}{x^2+3x-10}$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^3}{x^3(\frac{1}{x}+\frac{3}{x^2}-\frac{10}{x^3})}$$

No horizontal asymptotes

3) Define continuity. For what values of the constant c is the function

$f(x) = \begin{cases} x^2 - c^2, & x < 4 \\ cx + 20, & x \geq 4 \end{cases}$ continuous on $(-\infty, \infty)$

Solution:

A function f is **continuous at a number *a*** if

$$\lim_{x\to a} f(x) = f(a)$$

(i.e) f is **continuous at a number *a*** if

i) $f(a)$ is defined

ii) $\lim_{x\to a} f(x)$

iii)) $\lim_{x\to a} f(x) = f(a)$

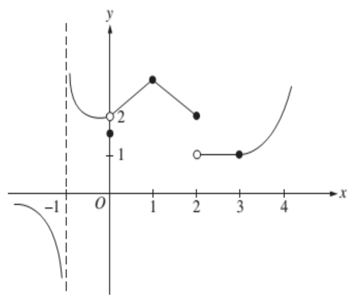
b) Since f is continuous function

$$\lim_{x\to 4^+} f(x) = \lim_{x\to 4^-} f(x)$$

$$\lim_{x\to 4^+} x^2 - c^2 = \lim_{x\to 4^-} cx + 20$$
$$16 - c^2 = 4c + 20$$
$$c^2 + 4c + 4 = 0$$
$$\therefore c = -2, -2$$

4. The graph of a function f is shown below. If $\lim_{x\to b} f(x)$ exists and f is not continuous at b, then for what value of b given below the statement is true or false. Give reason

(a) -1 (b) 0 (c) 1 (d) 2 (e) 3

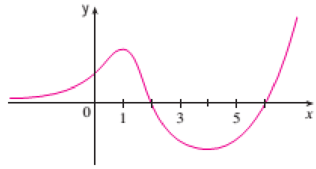


Solution:

b		Reason
$b = -1$	False	Limit does not exists. (vertical asymptote)
$b = 0$	True	Limit exist and f is not continuous
$b = 1$	False	Limit exists but f is continuous
$b = 2$	False	Limit of f does not exist
$b = 3$	False	Limit exists but f is continuous.

5. For each initial approximation, determine graphically what happens if Newton's

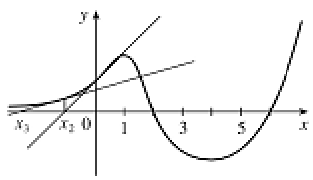
method is used for the function whose graph is shown.



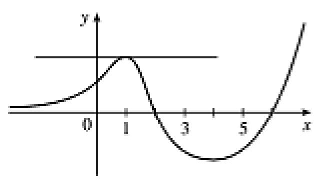
(a)$x_1 = 0$ (b) $x_1 = 1$ (c) $x_1 = 3$ (d) $x_1 = 4$ (e) $x_1 = 4$

Solution:

(a) At $x_1 = 0$, then x_2 is negative, and x_3 is more negative The sequence of approximations do not converge, that is, Newton's method fails.

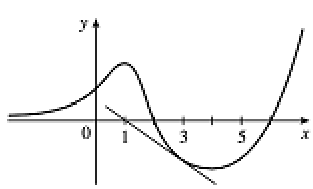


(b) At $x_1 = 1$, the tangent line is horizontal and Newton's method fails.

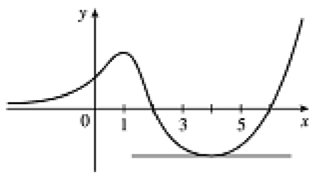


(c) At $x_1 = 3$, $x_2 = 1$ then and we have the same situation as in part (b).

Newton's method fails again.



(d) At $x_1 = 4$, the tangent line is horizontal and Newton's method fails.



(e) At $x_1 = 5$, then Newton's method will lead us to the root.

6. State Intermediate value theorem .check that there is a root in a given interval
   i)$x^3 - 6x + 1 = 0$ between 2 and 3 ii)$\ln(x) = e^{-x}$ in (1,2)

   Solution:

   Statement :

   Suppose that f is continuous on the closed interval $[a, b]$ and let $'N'$ be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$ then there exist a number c in (a, b) such that $f(c) = N$

check that there is a root in a given interval i)$x^3 - 6x + 1 = 0$ between 2 and 3

Solution:

Here $f(x) = x^3 - 6x + 1$

$f(x) = x^3 - 6x + 1$

$f(2) = 2^3 - 12 + 1 \quad = -3 < 0$

$f(3) = 3^3 - 18 + 1 \quad = 10 \ > 0$

Therefore by intermediate value theorem there exist a value 'c' between 2 and 3 such that $f(c) = 0$.

Therefore root exist between 2 and 3

7. Find the real root of the following equation by Bisection method up to five iterations.

$x + log_{10}x = 3.375$

Solution:

Here $f(x) = x + \log_{10} x - 3.375$

First we find the location of the root

$f(1) = -2.375$

$f(2) = -1.0740$

$f(3) = 0.1021$

Therefore by intermediate value theorem, root of the given equation lies in the interval (2,3)

1st iteration

$$x_1 = \frac{2+3}{2} = 2.5$$

$f(x_1) = -0.4771$

Therefore the root lies between the interval (2.5,3)

2nd iteration:

$$x_2 = \frac{2.5+3}{2} = 2.75$$

$f(x_2) = -0.1857$

Therefore the roots lies between the interval (2.75,3)

3rd iteration:

$$x_3 = \frac{2.75+3}{2} = 2.875$$

$f(x_3) = -0.0414$

Therefore the root lies between the interval (2.875,3)

4th iteration:

$$x_4 = \frac{2.875+3}{2} = 2.9375$$

$f(x_4) = 0.0305$

Therefore the roots lies between the interval (2.875,2.9375)

5th iteration:

$$x_5 = \frac{2.875+2.9375}{2} = 2.9063$$

After the 5th iteration the approximate root of the given equation is

$x = 2.9063$

8. Find the real root of the following equation by Newton Raphson method correct to four decimal places.

$f(x) = x\sin(x) + \cos(x)$ near $x = \pi$

Solution:

Given

$f(x) = xsin(x) + \cos(x)$

$f'(x) = x\cos(x)$

Newton formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Given the root is near $x = \pi$

Therefore the initial approximation is $x_0 = \pi$

1st iteration:

$$x_1 = \pi - \frac{f(\pi)}{f'(\pi)} = \pi - \frac{-1}{-3.1416}$$

$x_1 = 2.8233$

2nd iteration:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.8233 - \frac{f(2.8233)}{f'(2.8233)} = 2.8233 - \frac{-0.0662}{-2.6815} = 2.7986$$

$x_2 = 2.7986$

3rd iteration:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.7986 - \frac{f(2.7986)}{f'(2.7986)} = 2.7986 - \frac{-0.0006}{-2.6356}$$

$x_3 = 2.7984$

4th iteration:

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 2.7984 - \frac{f(2.7984)}{f'(2.7984)} = 2.7984 - \frac{-0.00004}{-2.6352} = 2.7984$$

$x_4 = 2.7984$

Therefore the root of the given equation is

$x_4 = 2.7984$

9. Using Newton's method, find the x-coordinate of the point where the curve $y = cos(x) - xe^x$ crosses the x-axis.

Solution:

Here given $f(x) = cos(x) - xe^x$

$f'(x) = -\sin(x) - e^x - xe^x$

$f(0) = 1$

$f(0.5) = 0.0532$

$f(1) = -2.1780$
Newton formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Let us take

$x_0 = 0.5$

1st iteration:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.5 + \frac{0.0532}{2.9525} = 0.5180$$

2nd iteration:

$$x_2 = 0.5180 - \frac{f(0.5180)}{f'(0.5180)} = 0.5180 - \frac{0.0007}{3.0434} = 0.5178$$

3rd iteration:

$$x_3 = 0.5178 - \frac{f(0.5178)}{f'(0.5178)} = 0.5178 - \frac{0.0001}{3.0423} = 0.5178$$

Therefore the real root for the given equation is $x = 0.5178$

DERIVATIVES

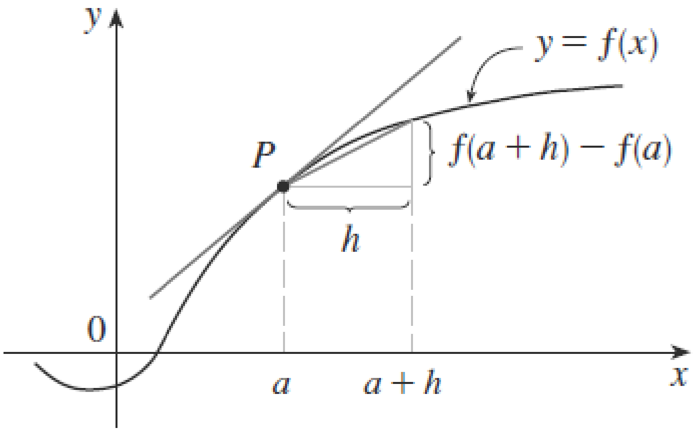
The derivative of a function f at a number a, denoted by $f'(a)$, is $f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ if this limit exists.

Limits of the form $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ arise whenever we calculate a rate of change in any of the sciences or engineering, such as a rate of reaction in chemistry or a marginal cost in economics. Since this type of limit occurs so widely, it is given a special name and notation.

Interpretation of the Derivative as the Slope of a Tangent

The tangent line to $y = f(x)$ at $\left(a, f(a)\right)$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

Thus, the geometric interpretation of a derivative is given below



(a) $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$

= slope of tangent at P

= slope of curve at P

Interpretation of the Derivative as a rate of change

We defined instantaneous rate of change of $y = f(x)$ with respect to x at $x = x_1$ as

Instantaneous rate of change $= \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$

Where $\Delta x = x_2 - x_1$ and $\Delta y = f(x_2) - f(x_1)$

The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when

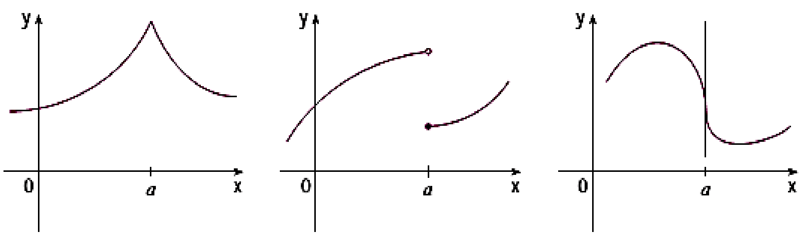
$x = a$.

The derivative of a function $y = f(x)$ is also written as

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

Given any number x for which this limit exists, we assign to x the number $f'(x)$. So we can
regard f' as a new function, called the derivative of f.

Ex: In following graphs of f, explain why f is not differentiable at x = a



(a)

(b) (c)

(a) If the graph of a function has a "corner" or "kink" in it (as in the figure (a)), then the graph of has no tangent at this point and is not differentiable there. In trying to compute $f'(a)$, we find that the left and right limits are different.
(b) If f is not continuous at a, then is not differentiable at a (as in figure (b)). So at any discontinuity (for instance, a jump discontinuity) fails to be differentiable.
(c) If the curve has a **vertical tangent line** when $x = a$ as in figure (c), that is f is continuous at a and $\lim_{x\to a}|f'(x)| = \infty$. This means that the tangent lines become steeper and steeper as $x \to a$. Thus, the function is not differentiable.

Maximum and minimum values of a function

A function $f(x)$ has an absolute maximum at k if $f(k) \geq f(x), \forall x \in D$, where D is the domain of $f(x)$. The number $f(k)$ is called the **absolute maximum** value of f on D.
Similarly, $f(x)$ has an absolute minimum at k if $f(k) \leq f(x), \forall x \in D$, where D is the domain of $f(x)$. The number $f(k)$ is called the **absolute minimum** value of f on D.

The maximum and minimum values of f are called the extreme values of f.

A function $f(x)$ has a local maximum at k if $f(k) \geq f(x)$, when x is near k. The number $f(k)$ is called the **local maximum** value of f near k. Similarly, a function $f(x)$ has a local minimum at k if $f(k) \leq f(x)$, when x is near k. The number $f(k)$ is called the **local minimum** value of f near k.

Critical number:

A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ doesnot exists.

Closed interval method to find absolute maxima and absolute minima:

Step1: Find the values of f at the critical numbers of f .

Step2: Find the values of f at the endpoints of the interval.

Step3: The largest of the values from step1 and step2 is the absolute maximum values and the smallest of these values is the absolute minimum value.

1st Derivative test to find local maxima and local minima:

Suppose that c is a critical number of a continuous function f,

(a) If f' changes from +ve to –ve at c, then f has a local maximum at c.

(b) If f' changes from -ve to +ve at c, then f has a local minimum at c.

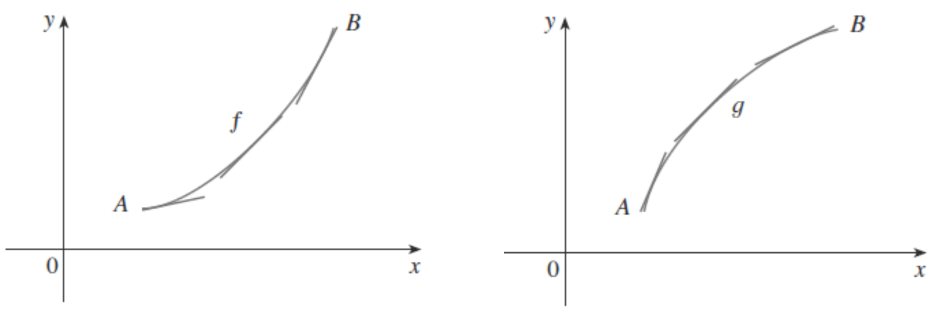
(c) If f' does not change sign at c, then f has no local maximum or minimum at c.

Second derivative of a function:

Since the derivative is itself a function, we can consider its derivative. For a function f, the 2nd derivative is denoted by f'' and is given by $f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

Concavity :

If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on an interval I, then it is called **concave downward** on I.



(a) Concave upward (b) Concave downward

Concavity test:

(a) If $f"(x) > 0, \forall x \in I$, then f is concave upward on I.

(b) If $f"(x) < 0, \forall x \in I$, then f is concave downward on I.

Inflection Point:

A point P on a curve $y = (x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or concave downward to concave upward at P.

A point at which the graph of a function changes its concavity is called an inflection point of f.

The 2nd derivative test:

Suppose f'' is continuous near c,

(a) If $f'(c) = 0$ and $"(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $"(c) < 0$, then f has a local maximum at c.

(c) If $f'(c) = 0$ and $"(c) = 0$, then this test fails.

Examples:

1. Find the critical numbers of the function $f(x) = x^3 + 3x^2 - 24x$.

   Now, $f'(x) = 3x^2 + 6x - 24 = 3(x+4)(x-2)$

   Therefore, $f'(x) = 0$ if $x = -4$, $x = 2$

   Thus, $x = -4, 2$ are the critical numbers.

2. Find the absolute minimum and maximum values of f on the given interval

   $f(x) = 2x^3 - 3x^2 - 12x + 1$ in [-2,3].

   We have $f'(x) = 6x^2 - 6x - 12$

   Here, $f'(x) = 0$ when $x = -1,2$ (critical numbers)

   By Closed interval method, we have

   $f(2) = -19$ (Absolute minimum value)

   $f(-1) = 8$ (Absolute maximum value)

   $f(-2) = -3$

   $f(3) = -8$

3. For the given functions (a) Find the intervals on which f is increasing or decreasing

   (b)Find the local maximum and minimum values of f

   (c)Find the intervals of concavity and the inflection points.

   (i) $f(x) = \cos^2(x) - 2\sin(x)$, $0 \le x \le 2\pi$

   Here $f'(x) = -2\cos x\,(1 + sinx)$

   $f'(x) = 0$ if $x = \frac{\pi}{2}, \frac{3\pi}{2}$ (critical numbers)

Interval	-cosx	1+sinx	Sign of f'	Nature of f
$\left(0, \frac{\pi}{2}\right)$	-	+	-	Decreasing
$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$	+	+	+	Increasing

$\left(\frac{3\pi}{2}, 2\pi\right)$	-	+	-	Decreasing

Therefore, by 1st derivative test, local maximum exists at $x = \frac{\pi}{2}$ and local minimum exists at

$$x = \frac{3\pi}{2}$$

Now, $f''(x) = 2sinx(sinx + 1) - 2\cos^2 x$

$f''(x) = 4\sin^2 x\ + 2sinx - 2$

$f''(x) = \left(sinx - \frac{1}{2}\right)(sinx + 1)$

Here $f''(x) = 0$ if $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$

Interval	$sinx - \frac{1}{2}$	$sinx + 1$	Sign of f''	Concavity
$\left(0, \frac{\pi}{6}\right)$	-	+	-	Concave down
$\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$	+	+	+	Concave up
$\left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$	-	+	-	Concave down
$\left(\frac{3\pi}{2}, 2\pi\right)$	-	+	-	Concave down

(ii) $f(x) = \frac{x^2}{x^2+3}$

Here $f'(x) = \frac{6x}{(x^2+3)^2}$

Then, $f'(x) = 0$ if $x = 0$ (critical number)

Interval	Sign of f'	Nature of f
$(-\infty, 0)$	-	Decreasing
$(0, \infty)$	+	Increasing

Therefore, by 1st derivative test, local minimum exists at $x = 0$ and there is no local maximum for the given function.

Now, $f''(x) = \frac{18(1-x^2)}{(x^2+3)^3} = \frac{18(1+x)(1-x)}{(x^2+3)^3}$

Here, $f''(x) = 0$ if $x = 1, -1$

Interval	$(1+x)$	$(1-x)$	Sign of f''	Concavity
$(-\infty, -1)$	-	+	-	Concave down
$(-1,1)$	+	+	+	Concave up
$(1, \infty)$	+	-	-	Concave down

Note that concavity changes at $x = -1$ and $x = 1$

Therefore, $x = -1,1$ are the inflection points.

4. Using second derivative test classify the critical points of the given function as local maxima or local minima $f(x) = \frac{x}{x^2+4}$

Soln: We have $f'(x) = \frac{4-x^2}{(x^2+4)^2} = \frac{(2-x)(2+x)}{(x^2+4)^2}$

Here $f'(x) = 0$ if $x = 2, -2$ (Critical numbers)

Now, $f''(x) = \frac{-2x(12-x^2)}{(x^2+4)^3}$

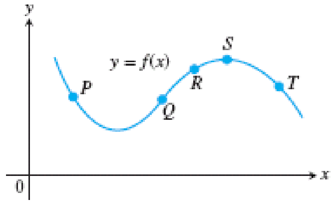
We find values of f'' at critical numbers, we get

$f''(-2) = \frac{1}{16} > 0$ and $f''(2) = -\frac{1}{16} < 0$

Thus, by 2nd derivative test, we get

Local maximum exists at $x = 2$ and local minimum exists at $x = -2$.

5. For the given graph, classify y' and y"as positive , negative or zero at labeled points P,Q,R,S,T

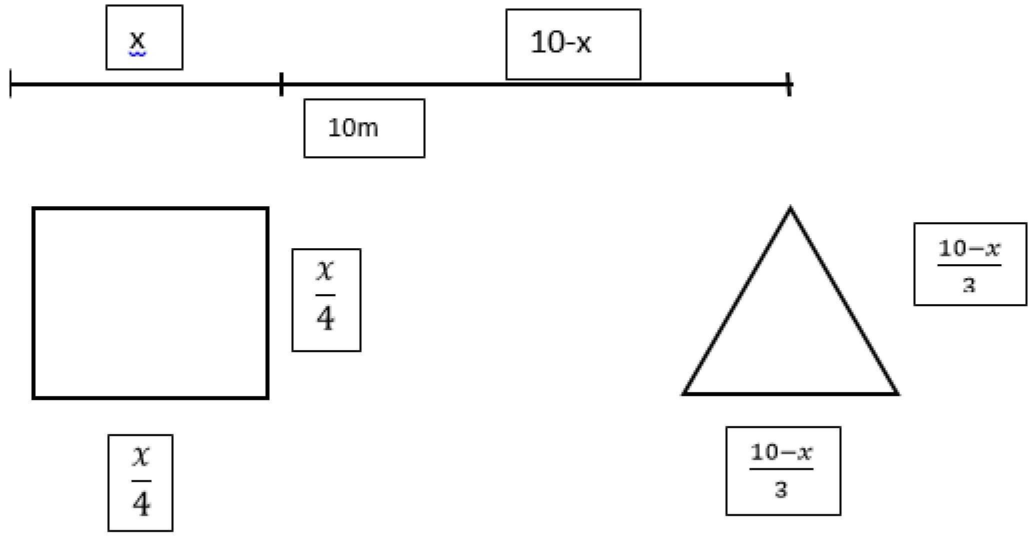


Soln:

Point	y'	y''
P	-	+
Q	+	0
R	+	-
S	0	-
T	-	-

6. A piece of wire 10m long is cut into two pieces. One piece is bent into a square and the other is bent into an equilateral triangle. How should the wire be cut so that the total area enclosed is (a) A maximum? (b) A minimum?

Soln:



The total area enclosed by a square and a equilateral triangle is calculated as

$$A = \left(\frac{x}{4}\right)^2 + \frac{\sqrt{3}}{4}\left(\frac{10-x}{3}\right)^2 = \frac{x^2}{16} + \frac{(10-x)^2}{12\sqrt{3}} \rightarrow (1)$$

Here $x \in [0,10]$

Thus, we can apply Closed interval method to find the maximum and minimum values of our Area function (1). For that, we first find

$$A'(x) = \frac{6\sqrt{3}x - 8(10-x)}{48\sqrt{3}}$$

Here $A'(x) = 0$ if $x = 4.349$ (critical number)

Now, we apply the Closed interval method,

$A(4.349) = 2.718$

$A(0) = 4.81$

$A(10) = 6.25$

Thus, (a) maximum area occurs when x=10m, that is all the wire is used for the square.

(b) minimum area occurs when x=4.349m, that is the wire of length x=4.349m is used for square and (10-x)m wire is used for triangle.

Sketching of curves:

Ex: Sketch the graph of a function that satisfies all of the given conditions:

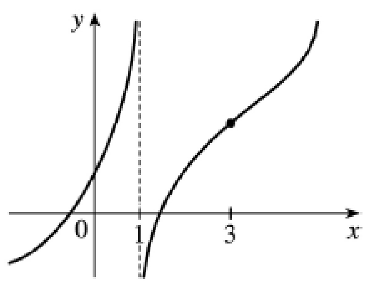
1. (i) $f'(x) > 0$ for all $x \neq 1$, vertical asymptote x=1
   (ii) $f''(x) > 0$ if $x < 1$ or $x > 3$
   (iii) $f''(x) < 0$ if $1 < x < 3$

Soln: (i) $f(x)$ is increasing in $(-\infty, 1)$ and $(1, \infty)$

(ii) $f(x)$ is cocave up on $(-\infty, 1)$ and $(3, \infty)$

(iii) $f(x)$ is concave down on $(1,3)$

Here x=3 is an inflection point.



2. (i) $f'(x) >\ 0$ if $|x| < 2$
   (ii) $f'(x) <\ 0$ if $|x| > 2$
   (iii) $f'(2) = 0$
   (iv) $\lim_{x\to\infty} f(x) = 1$
   (v) $f(-x) = f(x)$
   (vi) $f''(x) <\ 0$ if $x < 0 < 3$, $f''(x) >\ 0$ if $x > 3$

Soln: (i) $f(x)$ is increasing in $(-2,2)$

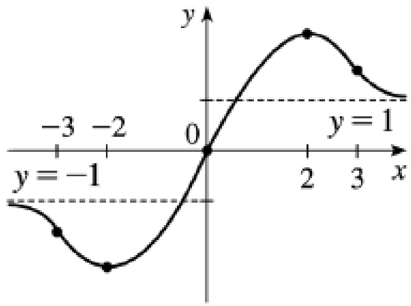
(ii) $f(x)$ is decreasing in $(-\infty, -2)$ and $(2, \infty)$

(iii) Horizontal tangent at x=2

(iv) Horizontal asymptote at y=1

(v) Odd function (Symmetry about origin)

(vi) Concave down on $(0,3)$ and concave up on $(3, \infty)$



Indeterminate forms

Suppose $\lim_{x\to a}\frac{f(x)}{g(x)}$ and if $\lim_{x\to a} f(x) = 0 \; or \; \infty$ and $\lim_{x\to a} g(x) = 0 \; or \; \infty$. Then, the given limit is of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form, which is called an Indeterminate form.

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \infty$ then $\lim_{x\to a} f(x)\, g(x) = 0.\infty$ form. Then, $fg = \frac{f}{1/g} = \frac{0}{0}$ form.

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ then $\lim_{x\to a}\big(f(x) - g(x)\big) = \infty - \infty$ form. Using a common denominator or rationalization or factoring out a common factor, we have to reduce it to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

Indeterminate powers:

Suppose $L = \lim_{x\to a}[f(x)]^{g(x)}$ then,

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ the L is of the form 0^0.

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ the L is of the form ∞^0.

If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \pm\infty$ the L is of the form 1^∞.

Then, $\log_e L = \lim_{x\to a} g(x).\log_e[f(x)]$ which will reduces to $0.\infty$ form.

L'Hospital's Rule:

Suppose $f(x)$ and $g(x)$ are differentiable and $f(a) = 0$ and $g(a) = 0$ then $\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$, if $g'(a) \neq 0$.

Examples:

1. Identify the following indeterminate forms and evaluate using L-Hospital's rule.

(i) $\lim_{x\to 0}\left(\frac{1}{sinx} - \frac{1}{x}\right)$ $(\infty - \infty)$ form

$= \lim_{x\to 0}\left(\frac{x - sinx}{x\, sinx}\right)$ $\left(\frac{0}{0}\right)$ form , applying L'Hospital rule

$$= \lim_{x\to 0} \frac{1-cosx}{x\,cosx+sinx} \quad \left(\frac{0}{0}\right) \text{ form}$$

$$= \lim_{x\to 0} \frac{sinx}{-x\,sinx+cosx+cosx}$$

$$= 0$$

(ii) $\lim_{x\to 0}(cotx)^{\frac{1}{logx}}$ (∞^0) form

$$L = \lim_{x\to 0}(cotx)^{\frac{1}{logx}}$$

$$logL = \lim_{x\to 0} \frac{1}{logx}\,.\log(cotx) \quad (0.\infty) \text{ form}$$

$$logL = \lim_{x\to 0} \frac{\log(cotx)}{logx} \quad \left(\frac{\infty}{\infty}\right) \text{ form, applying L'Hospital rule}$$

$$logL = \lim_{x\to 0} \frac{1}{cotx}.\left(\frac{-cosec^2x}{\frac{1}{x}}\right)$$

$$logL = \lim_{x\to 0} \frac{x}{sinx}.\lim_{x\to 0}\left(\frac{1}{cosx}\right)$$

$$logL = -1$$

$$L = e^{-1} = \frac{1}{e}$$

(iii) $\lim_{x\to 0}\left(\frac{sinx}{x}\right)^{\frac{1}{x^2}}$

$$L = \lim_{x\to 0}\left(\frac{sinx}{x}\right)^{\frac{1}{x^2}} \quad 1^\infty \text{ form}$$

$$logL = \lim_{x\to 0} \frac{1}{x^2}\,.\log\left(\frac{sinx}{x}\right) \quad (\infty.0) \text{ form}$$

$$logL = \lim_{x\to 0} \frac{\left(\log\left(\frac{sinx}{x}\right)\right)}{x^2} \quad \frac{0}{0} \text{ form, applying L'Hospital rule}$$

$$logL = \lim_{x\to 0}\left[\frac{\left(\frac{1}{\left(\frac{sinx}{x}\right)}\frac{x\,cosx - sinx}{x^2}\right)}{2x}\right]$$

$$logL = \lim_{x\to 0}\left(\frac{x}{sinx}\right)\left(\frac{x\,cosx - sinx}{2x^3}\right)$$

$logL = \lim_{x\to 0}\left(\frac{x\,cosx - sinx}{2x^3}\right)$ $\frac{0}{0}$ form, applying L'Hospital rule

$$logL = \lim_{x\to 0}\frac{-x\,sinx + cosx - cosx}{6x^2} = \lim_{x\to 0}\left(-\frac{1}{6}\right)\left(\frac{sinx}{x}\right) = -\frac{1}{6}$$

$$L = e^{-\left(\frac{1}{6}\right)}$$