LINEAR DIFFERENTIAL EQUATION.

Suppose the eqn to be solved is $\frac{dy}{dx} + Py = Q$

where P & Q are function of x. Such an equation is called linear differential equation. We find Integrating factor.

$$I.F = e^{\int P \cdot dx}.$$

& Sol^n is $y \times I.F = \int (I.F) \times Q\,dx + C_1$

Ex) Solve $\frac{dy}{dx} + y(\sec x) = \tan x$.

→ Given eqn is of the form $\frac{dy}{dx} + Py = Q$.

So here $P = \sec x$ $\quad Q = \tan x$.

$$I.F = e^{\int p \cdot dx}$$
$$= e^{\int \sec x\, dx}.$$
$$= e^{\log(\tan x + \sec x)}$$
$$= \sec x + \tan x.$$

Its sol^n is

$$y(\sec x + \tan x) = \int (\sec x + \tan x) \times \tan x\, dx + C_1$$

$$y(\sec x + \tan x) = \int (\tan^2 x + \sec x \tan x)\, dx.$$

$$y(\sec x + \tan x) = \int (\sec^2 x - 1)\, dx + \int \sec x \tan x\, dx$$

$$y(\sec x + \tan x) = \tan x - x + \sec x\, dx.$$

$$\frac{dy}{dx} + \frac{y}{x\log x} = \frac{2\log x}{x\,\cancel{x\log x}}$$

$$\frac{dy}{dx} + \frac{y}{x\log x} = 2.$$

$$P = \frac{1}{x\log x} \quad Q = \frac{2}{x}$$

$$\text{I.F.} = e^{\int \frac{dx}{x\log x}}$$
$$= \log x.$$

Sol^n is $y(\log x) = \int (\log x)\frac{2}{x}\,dx + c$

$$\log x = t$$
$$\frac{1}{x} = dt$$

$$= \int 2t\,dt + c$$

$$= t^2 + c$$

$$y(\log x) = (\log x)^2 + c.$$

3) $\sqrt{1-y^2}\,dx = (\sin^{-1}y - x)\,dy$

$$\frac{dx}{dy} = \frac{\sin^{-1}y - x}{\sqrt{1-y^2}}$$

$$\frac{dx}{dy} + \frac{x}{\sqrt{1-y^2}} = \frac{\sin^{-1}y}{\sqrt{1-y^2}}$$

$$e^{\int \frac{1}{\sqrt{1-y^2}}dy} = e^{\sin^{-1}y}.$$

Sol^n

$$x\left(e^{\sin^{-1}y}\right) = \int e^{\sin^{-1}y} \times \frac{\sin^{-1}y}{\sqrt{1-y^2}}\,dy + c$$

$$\sin^{-1}y = t.$$
$$\frac{1}{\sqrt{1-y^2}} = dt$$

$$= \int e^t\, t\, dt.$$

$$= te^t - e^t + c.$$

$$= e^t[t-1] + c$$

$$\cancel{x e^{\sin^{-1}y}} = e^{\sin^{-1}y}[\sin^{-1}y - 1] + c$$

$$x = \sin^{-1}y - 1 + c.$$

Linear Bernoulli's Equation.

$$\frac{dy}{dx} + Py = Qy^n.$$

where P & Q are functions of x which is non linear eqn known as Bernoulli's equation.

To reduce this equation to linear we divide throughout by y^n.

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q.$$

Put $z = y^{1-n}$ in above eqn then the given eqn becomes a linear one then solve as usual.

Ex: Solve $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y.$

$\rightarrow$ $\div$ by $\cos^2 y$.

$$\sec^2 y\frac{dy}{dx} + \frac{x\sin 2y}{\cos^2 y} = x^3\cos^2 y$$

$$\sec^2 y\frac{dy}{dx} + \frac{2\sin y\cos x}{\cos^2 y} = x^3\cos^2 y \quad \text{—①}$$

$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$ —— ①.

Put $z = \tan y$.

$\frac{dz}{dx} = \sec^2 y \frac{dy}{dx}$.

Now eqn ① becomes

$\frac{dz}{dx} + 2xz = x^3$

$P = 2x \quad Q = x^3$

$I.F. = e^{\int 2x\,dx}$

$= e^{x^2}$

$Sol^n = z(e^{x^2}) = \int e^{x^2} \times x^3\, dx + C$

$\tan y \times e^{x^2} = \int e^{x^2} \times x^3\, dx$

Put $t = x^2$

$\cancel{2x = \frac{dt}{dx}}$ $2x\,dx = dt$

$= \frac{1}{2}\int t e^t\, dt$

$= \frac{1}{2}[te^t - e^t] + C$

$= \frac{1}{2} e^t [t - 1] + C$

$\tan y \cdot e^{x^2} = \frac{1}{2} e^{x^2}[x^2 - 1] + C$

Solve $xy(1 + xy^2)\frac{dy}{dx} = 1$

$(xy + x^2y^3)\frac{dy}{dx} = 1$

$\frac{dx}{dy} = xy + x^2y^3$.

$\frac{dx}{dy} - xy = x^2y^3$

$\div x^2$

$\frac{1}{x^2}\frac{dx}{dy} - \frac{x^{-1}y}{} = y^3$ —— ①.

Put $z = x^{-1}$

$-1 x^{-2}\frac{dx}{dy} = \frac{dz}{dy}$

$\frac{dz}{dy} + zy = y^3$

$P = y \quad Q = y^3$

$I.F. = e^{\int y\,dy} = e^{\frac{y^2}{2}}$

$z e^{\frac{y^2}{2}} = \int (e^{\frac{y^2}{2}}) y^3\, dy + C$.

Put $\frac{y^2}{2} = t$.

$y\,dy = dt$

$dy = \frac{dt}{y}$

$ze^t = \int e^t\, 2t\, dt + C$.

$= 2\int e^t\, t\, dt$

$= 2(te^t - e^t)$

$\frac{1}{x} e^{y^2/2} = 2(te^t - e^t) + C$

$= \int (\sec x - \cancel{\sec x \sin x \tan x} + \cancel{\tan x} - \tan x \sin x) dx$

$= \int (\sec x - \tan x \sin x)$

$= \int \sec x (1 - \sin^2 x)$

$= \sec x \cos^2 x$

$= \int \cos x$

$= \sin x \, dx + C$

Exact Differential Equation

The D.E of the form $M dx + N dy = 0$ where both M & N are functions of x & y are said to be exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

And its soln is $\int\limits_{y\ \text{constant}} M dx + \int\limits_{x\ \text{const}} (\text{terms of } N \text{ free from } x) dy = C$

Ex) Solve $3x(xy-2)dx + (x^3 + 2y)dy = 0$.

$\Rightarrow$ The given d.E is of form

$x^3y - \frac{3}{2}x^2 + y^2 = C$

Solve $(y\sin 2x)dx - (1+y^2+\cos^2 x)dx = 0$

$\Rightarrow M = y\sin 2x \qquad N = -(1+y^2+\cos^2 x)$

$\frac{\partial M}{\partial y} = \sin 2x \qquad \frac{\partial N}{\partial x} = \sin 2x$

The given D.E is exact

Sol^n is

$\int y\sin 2x + \int (1+y^2)\,dy = C$

$-y\frac{\cos 2x}{2} - y - \frac{y^3}{3} = C$

Reducable to Exact

Many D.E of the form $Mdx + Ndy = 0$ that are not exact can be made exact by multiplying the terms by appropriate functions called as I.F.

Rules 1) If the eqn is homogeneous & $Mx + Ny \neq 0$ then $\frac{1}{Mx+Ny}$ is I.F

2) If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is function of x alone say $f(x)$ then $e^{\int f(x)dx}$ is I.F

3) $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is function of y alone say $g(y)$ then $e^{\int g(y)dy}$ is I.F.

4) If the function is of form

$f_1(x,y)\,y\,dx + f_2(x,y)\,x\,dy = 0$

$M = y f_1(x,y)$; $N = f_2(x,y)\,x\,dy$

then $Mx - Ny \neq 0$ then $\frac{1}{Mx-Ny}$ is I.F

Ex) Solve $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

$(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$

$\frac{\partial M}{\partial y} = x^2 - 4x \qquad \frac{\partial N}{\partial x} = 3x^2 - 6x$

By rule ①

$x^3y - 2x^2y^2 - x^3y + 3x^2y^2 = 0$

$x^2y^2 \neq 0$

$I.F = \frac{1}{x^2y^2}$

Multiply throughout by $\frac{1}{x^2y^2}$

$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$

$\frac{\partial M}{\partial y} = -y^{-2} \qquad \frac{\partial N}{\partial x} = -y^{-2}$

Sol^n is

$\left(\frac{x}{y} - 2\log x\right) + 3\log x = C$

$M \left(\frac{\partial x}{} - \frac{}{\partial y} \right)$

$$\frac{1}{y^4 - 2y}(y^3 - y^4) = \frac{\cancel{y^3 - y^4} - 3(y^3+2)}{y^4+2y} \qquad \frac{\cancel{y^4\left(\frac{1}{y} - 1\right)}}{\cancel{y^4\left(1 - \frac{2}{y^3}\right)}}$$

$$= \frac{-3(y^3+2)}{y^4+2y} \qquad \cancel{\frac{\frac{1}{y} - 1}{1 - \frac{2}{y^3}}}$$

$$= \frac{-3(y^3+2)}{y(\cancel{y^3+2})} \qquad = \cancel{\frac{\frac{1-y}{y}}{\frac{y^3-2}{y^3}}}$$

$$= -\frac{3}{y} \qquad = \cancel{\frac{1-y}{y} \times \frac{y^3}{y^3-2}}$$

$$I.F = e^{\int -\frac{3}{y} dy} \qquad = \cancel{\frac{y^2(1-y)}{y^3-2}}$$

$$= e^{-\log y^3}$$

$$= \frac{1}{y^3}$$

$$\left(-y - \frac{2}{y^2}\right)dx + \left(x + 2y + \frac{4x}{y^3}\right)dy = 0.$$

$$-\int y + \frac{2}{y^2} + \int 2y = 0.$$

Solve: Ex) $(x^2y^2 + xy + 1)\,y\,dx + (x^2y^2 - xy + 1)\,x\,dy = 0$

$M = (x^2y^2 + xy + 1)\,y$ $\quad N = (x^2y^2 - xy + 1)\,x$

$\frac{\partial M}{\partial y} = 3x^2y^2 + 2xy + 1$ $\quad = x^3y^2 - x^2y + x$

$\frac{\partial N}{\partial x} = 3y^2x^2 - 2xy + 1$

$\cancel{x^2y^3} + x^2y^2 + \cancel{xy} - \cancel{x^2y^3} + x^2y^2 + \cancel{xy} = 0$

$2x^2y^2$

$I.F = \frac{1}{2x^2y^2}$

$\cancel{x^2y^2 + xy + y}$

$(x^2y^3 + xy^2 + y) \times \frac{1}{2x^2y^2}\,dx + (x^3y^2 - x^2y + x) \times \frac{1}{2x^2y^2}\,dy = 0$

$\left(\frac{y}{2} + \frac{1}{2x} + \frac{1}{2x^2y}\right)dx +$

APPLICATION OF DIFFERENTIAL EQUATIONS

Orthogonal Trajectories

Consider 2 plane curves (family curves) say F_1 & F_2 suppose each member of a family F_1 cuts every member of family F_2 at right angles & then we say that the members of family F_2 are orthogonal trajectories of family F_1 & vice versa.

Ex:-) Let F_1 be the family of concentric circles with centres at origin & F_2 be the family of straight lines passing through the origin then every member of F_2 cut every member of F_1 at right angles. Thus straight lines through the origin are orthogonal trajectories of concentric circles centered at the origin & vice versa.

Step 1: Obtain the D.E of given family

Step 2: Change $\frac{dy}{dx}$ to $-\frac{dx}{dy}$ in the D.E obtained in step 1

The resulting equation is D.E of orthogonal trajectories.

Step 3: Solve the D.E obtained in step 2

The general soln of cartesian eqn is orthogonal trajectories.

Step 4:

Find O.T of families of curves $x^2 + y^2 = c^2$

Differentiate given eqn w.r.t x.

$\cancel{2}x + \cancel{2}y\frac{dy}{dx} = 0$

$x + y\frac{dy}{dx} = 0$ | D.E of given family

Change $\frac{dy}{dx}$ to $-\frac{dx}{dy}$

$x = -y\frac{dy}{dx}$

$\frac{dx}{dy} = \cancel{\frac{y}{x}}$ $\quad x - y\frac{dx}{dy} = 0$ | D.E of O.T

$$2x + 2y\frac{dy}{dx} + 2g = 0$$

$$g = -\left(x + y\frac{dy}{dx}\right)$$

$$x^2 + y^2 + 2\left(-x + y\frac{dy}{dx}\right) + c = 0$$

$$x^2 + y^2 - 2x - 2y\frac{dy}{dx} + c = 0$$

$$-2xy\frac{dy}{dx} - x^2 + y^2 + c = 0 \quad \text{--- (2)} \quad \Big|\ \text{D.E of given family}$$

Change $\frac{dy}{dx}$ to $-\frac{dx}{dy}$

$$2xy\frac{dx}{dy} - x^2 + y^2 + c = 0 \quad \Big|\ \text{D.E of O.T}$$

$$\Rightarrow \quad 2x\frac{dx}{dy} - \frac{x^2}{y} + y + \frac{c}{y} = 0$$

$$2x\frac{dx}{dy} - \frac{x^2}{y} = -y - \frac{c}{y}$$

Substitute $u = x^2$

$$2x\frac{dx}{dy} = \frac{du}{dy}$$

$$\frac{du}{dy} - \frac{2}{y} = -y + \frac{c}{y} \quad \text{--- (4)}$$

[Linear in u]

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$P = -\frac{1}{y}$ $Q = -y - \frac{c}{y}$

$I.F = e^{\int -\frac{dy}{y}}$

$= e^{\log y^{-1}}$

$= \frac{1}{y}$

$u. I.F = \int$

$\frac{u}{y} = \int \frac{1}{y}\left[-y - \frac{c}{y}\right] dy + C$

$= -\int 1 + \frac{c}{y^2} dy + c$

$= -y + \frac{c}{y} + K.$

$\frac{u}{y} = \frac{c}{y} - y + K$

$\frac{x^2}{y} = \frac{c}{y} - y + K$

$x^2 = c - y^2 + Ky$

$x^2 + y^2 = Ky + C$

$x^2 + y^2 - Ky + C = 0$

Find O.T of family of curves $\frac{x^2}{a^2} + \frac{y^2}{b^2 + \lambda} = 1$ where λ is parameter.

Diff w.r.t x, we get

$\frac{x}{a^2} + \frac{y}{b^2+\lambda}\frac{dy}{dx} = 0$ — ②

From ① & ② we get

$b^2 + \lambda = \frac{a^2 y^2}{a^2 + y x^2}$ $b^2 + \lambda = -\frac{a^2 y}{x}\frac{dy}{dx}$

$\frac{x}{a^2} + \frac{y(a^2 + x^2)}{a^2 y^2}\frac{dy}{dx} = 0$

$\frac{a^2 y^2}{a^2 - x^2} = -\frac{a^2 y}{x}\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{-xy}{a^2 - x^2}$ | D.E of given family of curves.

Change $\frac{dy}{dx}$ to $-\frac{dx}{dy}$

$\frac{dx}{dy} = \frac{a^2 - x^2}{xy}$ | D.E of O.T

$\frac{dy}{dx} = \frac{xy}{a^2 - x^2}$

$\frac{dy}{xy} = \frac{dx}{a^2 - x^2}$

$dy(a^2 - x^2) = (dx)\, xy$

$\int \left(\frac{a^2 - x^2}{x}\right) dy = y\, dy.$

$\frac{a^2}{2} = a^2 \log x - \frac{x^2}{2} + C$

$x^2 + y^2 = 2a^2 \log x + K$

Show that family of parabolas $x^2 = 4a(y + a)$

→

Given

$2x = 4a\frac{dy}{dx}$

$a = \frac{x}{2\left(\frac{dy}{dx}\right)}$

Substituting

$$x^2 = 4\left(\frac{x}{2\left(\frac{dy}{dx}\right)}\right)\left(y + \frac{x}{2\left(\frac{dy}{dx}\right)}\right)$$

$$x\frac{dy}{dx} = 2\frac{\left(y\frac{dy}{dx} + \frac{x}{2}\right)}{\left(\frac{dy}{dx}\right)}$$

$$x\left(\frac{dy}{dx}\right)^2 - 2y\frac{dy}{dx} - x = 0 \quad \text{--- (3)}$$

which is given D.E

$$x\left(-\frac{dx}{dy}\right)^2 + 2y\frac{dx}{dy} - x = 0$$

$$x\left(\frac{dx}{dy}\right)^2 + 2y\frac{dx}{dy} - x = 0$$

$$\div \left(\frac{dx}{dy}\right)^2$$

$$x + 2y\frac{dy}{dx} - x\left(\frac{dy}{dx}\right)^2 = 0$$

$$x\left(\frac{dy}{dx}\right) + 2y\frac{dy}{dx} - x = 0 \quad \text{--- (4)}$$

Since D.E (3) & (4) are same it follows that the given family of parabolas are self orthogonal.

<u>To find O.T for Polar Curves.</u>

Given $f(r\theta) = 0$

* Diff. w.r.t θ.
* Eliminate the parameter to obtain D.E of
* Replace $\frac{dr}{d\theta}$ by $\to -r^2\frac{d\theta}{dr}$
* We get D.E of O.T
* Solve the new D.E
* We get the D.E of orthogonal system

Ex: Find the O.T of curves $r^2 = a^2\cos 2\theta$

$$\frac{r^2}{\cos 2\theta} = a^2$$

Diff w.r.t θ

$$\frac{\cos 2\theta \, 2r\frac{dr}{d\theta} - r^2(-\sin 2\theta)2}{(\cos 2\theta)^2}$$

$$\cos 2\theta \frac{dr}{d\theta} + r\sin 2\theta = 0 \quad \text{--- (1) D.E of given family.}$$

Replace $\frac{dr}{d\theta} \to -r^2\frac{d\theta}{dr}$ in eqn (1)

$$\cos 2\theta\left(-r^2\frac{d\theta}{dr}\right) + \cancel{r}\sin 2\theta = 0$$

$$-\cos 2\theta \, r\frac{d\theta}{dr} + \sin 2\theta = 0 \quad \text{--- (2) D.E of O.T}$$

$$\frac{dr}{r} = \frac{\cos 2\theta \, d\theta}{\sin 2\theta}$$

$$\int\frac{dr}{r} = \int \cot 2\theta \, d\theta$$

$$\log r = \frac{\log \sin 2\theta}{2} + \log c$$

$$2\log r = \log \sin 2\theta + 2\log c$$

$$r^2 = c^2 \sin 2\theta$$

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Find the O.T of family of cardioides $r = a(1+\cos\theta)$

$\rightarrow \frac{r}{1+\cos\theta} = a.$

$$\frac{(1+\cos\theta)\frac{dr}{d\theta} + r\sin\theta}{(1+\cos\theta)^2} = 0$$

$(1+\cos\theta)\frac{dr}{d\theta} + r\sin\theta = 0$ — ① | D.E of given family

Replace $\frac{dr}{d\theta} \rightarrow -r^2\frac{d\theta}{dr}$

$-(1+\cos\theta)r^{\cancel{2}}\frac{d\theta}{dr} + \cancel{r}\sin\theta = 0$

$+r(1+\cos\theta)\frac{d\theta}{dr} = \sin\theta$

$\frac{dr}{r} = \frac{d\theta}{\sin\theta}(1+\cos\theta)$

$\log r = \int \frac{(1+\cos\theta)}{\sin\theta} d\theta$

$= \frac{\cancel{2}\cos^2\frac{\theta}{2}}{\cancel{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$= \int \cot\frac{\theta}{2} d\theta$

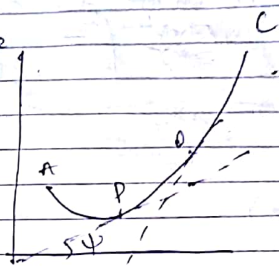
$\log r = \frac{\log\sin\frac{\theta}{2}}{\frac{1}{2}} + \log c$

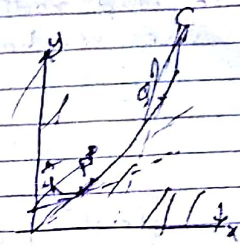
$\frac{1}{2}\log r = \log\sin\frac{\theta}{2} + 2\log c$

$r^{\frac{1}{2}} = \sin\frac{\theta}{2} c^{\frac{1}{2}}$

RADIUS OF CURVATURE

Curvature





Let C be a cartesian curve with fixed point A Let P be any point on C such that arc $AP = x$. the mean curvature of arc AQ is defined by $\frac{\delta\psi}{\delta s}$

$\frac{\delta\psi}{\delta s}$ & $\lim_{\delta \to 0}\frac{\delta\psi}{\delta s} - \frac{\delta\psi}{\delta s}$

(Amount of bending of curve at P is called curvature) is defined as the curvature of curve at P & is denoted by K

$\therefore$ Curvature $K = \frac{d\psi}{ds}$

Radius of curvature

The reciprocal of curvature is called as the radius of curvature is called as ρ

$\therefore \rho = \frac{1}{K} = \frac{1}{\frac{d\psi}{ds}} = \frac{ds}{d\psi}$