I) If V is the tetrahedron bounded by planes $x=0, y=0, z=0$ and $x+y+z=4$ then express $\int_V f\,dv$ (where f is function of x, y, z) as a triple integral & hence evaluate $\int_V x\,dv$

Solⁿ: Given : $x=0 \rightarrow$ Eqⁿ of xy-plane.

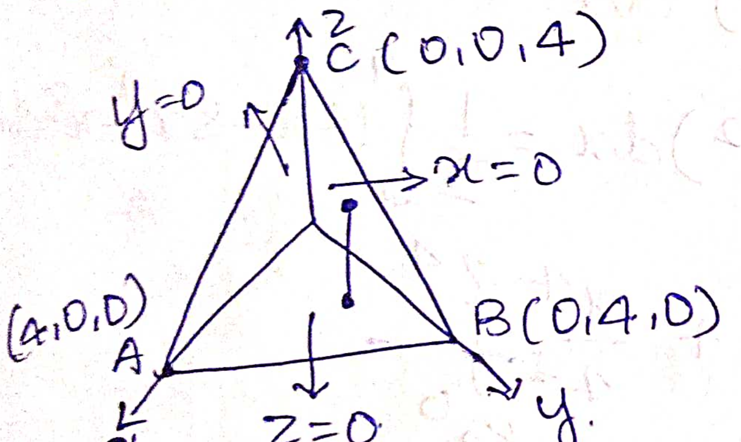
$y=0 \rightarrow$ Eqⁿ of xz-plane.

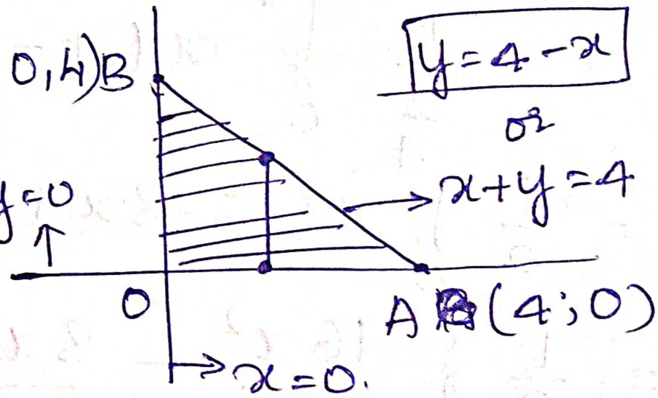
$z=0 \rightarrow$ Eqⁿ of xy-plane.

$x+y+z=4$ — ① (Also $z=4-x-y$).

To find : (i) $\int_V f\,dv$ (ii) $\int_V x\,dv$

Eqⁿ ① is a plane that cuts x-axis, y-axis and z-axis at 4.





(i) $$\int_V f\,dv = \int_0^4 \int_0^{4-x} \int_0^{4-x-y} f\,dz\,dy\,dx$$

(ii) $$\int_V x\,dv = \int_0^4 \int_0^{4-x} \int_0^{4-x-y} x\,dz\,dy\,dx.$$

$$= \int_0^4 \int_0^{4-x} x\left[\int_0^{4-x-y} dz\right] dy\,dx.$$

$$= \int_0^4 \int_0^{4-x} x\,[z]_0^{4-x-y}\,dy\,dx$$

$$= \int_0^4 \int_0^{4-x} x[4-x-y]\,dy\,dx$$

$$= \int_0^4 x \int_0^{4-x} [(4-x)-y]\,dy\,dx$$

$$= \int_0^4 x \left[(4-x)y - \frac{y^2}{2}\right]_0^{4-x} dx$$

$$= \int_0^4 x\left[(4-x)^2 - \frac{(4-x)^2}{2}\right] dx.$$

$$= \int_0^4 x(4-x)^2\left[1-\frac{1}{2}\right] dx$$

$$= \frac{1}{2}\int_0^4 x(4-x)^2\,dx.$$

$$= \frac{1}{2}\int_0^4 x(16-8x+x^2)\,dx = \frac{1}{2}\int_0^4 16x-8x^2+x^3\,dx$$

$$= \frac{1}{2}\left\{\frac{16x^2}{2} - \frac{8x^3}{3} + \frac{x^4}{4}\right\}_0^4$$

$$= \frac{1}{2}\left\{8(16) - \frac{8}{3}(64) + \frac{\cancel{4}^{43}}{\cancel{4}}\right\}$$

$$= \frac{16}{2}\left\{8 - \frac{8}{3}(4) + 4\right\} = 4\left\{8 - \frac{32}{3} + 4\right\}$$

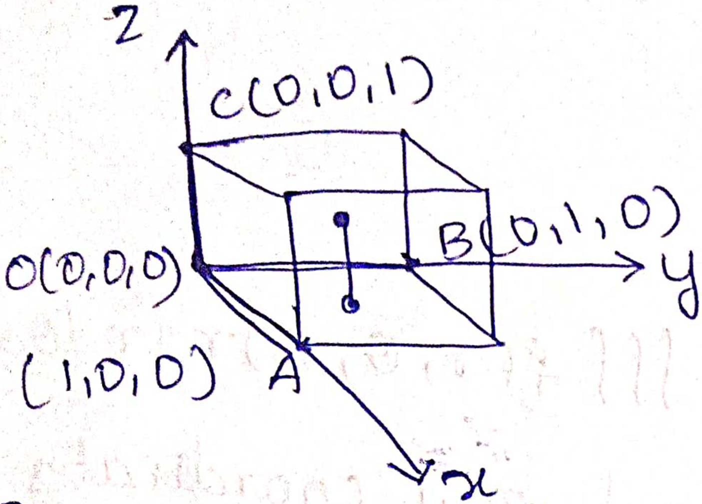
$$= 4(12-10.6667) = 4(2.6667)$$

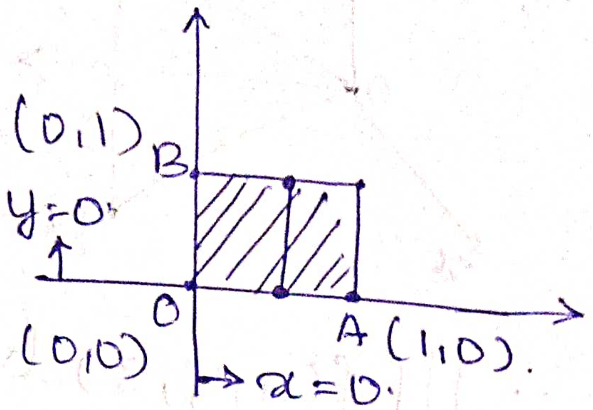
$$= 10.6667$$

6) consider the cube of side 1, then express the integral $\int_V f\,dV$ (where f is a function of x, y, z) as a triple integral and hence evaluate $\int_V (y^2+z^2)\,dV$

Sol^n: Given : cube of side = 1

To find : (i) $\int_V f\,dv$ (ii) $\int_V (y^2+z^2)\,dv$





(i) $\int_V f\,dv = \int_0^1\int_0^1\int_0^1 f\,dz\,dy\,dx$

(ii) $\int_V (y^2+z^2)\,dv = \int_0^1\int_0^1\int_0^1 (y^2+z^2)\,dz\,dy\,dx$

$= \int_0^1\int_0^1 \left[y^2 z + \frac{z^3}{3}\right]_0^1 dy\,dx = \int_0^1\int_0^1 \left[y^2+\frac{1}{3}\right]dy\,dx$

$= \int_0^1 dx \int_0^1 y^2 + \tfrac{1}{3}\,dy = [x]_0^1 \left[\frac{y^3}{3}+\frac{y}{3}\right]_0^1$

$= [1-0]\left[\tfrac{1}{3}+\tfrac{1}{3}\right] = \tfrac{2}{3}$ //.

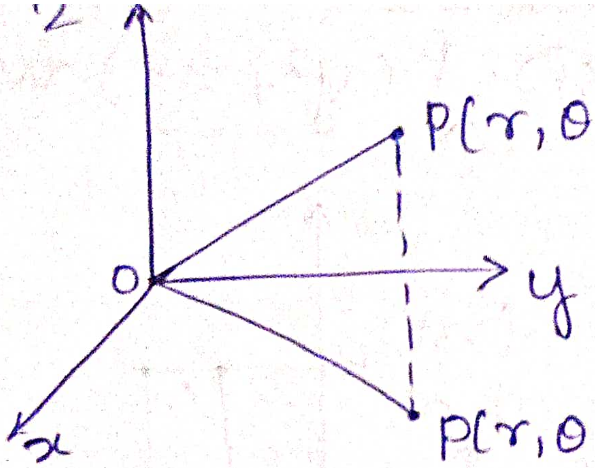
Triple Integrals in Cylindrical co-ordinates :

In cylindrical co-ordinates ,

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$



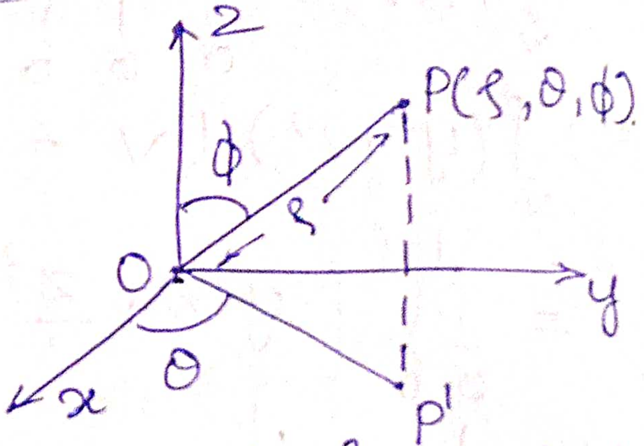
$$\therefore \iiint_V f(x,y,z)\,dv = \iiint f(r,\theta,z)\,r\,dr\,d\theta\,dz$$

Triple Integrals in spherical coordinates

$x = \rho\,\text{sen}\,\phi\cos\theta$

$y = \rho\,\text{sen}\,\phi\,\text{sen}\,\theta$

$z = \rho\cos\phi$



$$\iiint_V f(x,y,z)\,dv = \iiint f(\rho,\theta,\phi)\,\rho^2\,\text{sen}\,\phi\,d\rho\,d\theta\,d\phi$$

where $\rho \geq 0$, $0 \leq \theta \leq 2\pi$,

$0 \leq \phi \leq \pi$