# The Length of an Arc

## 1 Introduction

Many times when solving problems it is useful to know the length of a curve. By the end of this lesson we will be able to find the lengths of curves using integrals and derivatives!

## 2 Derivation for functions

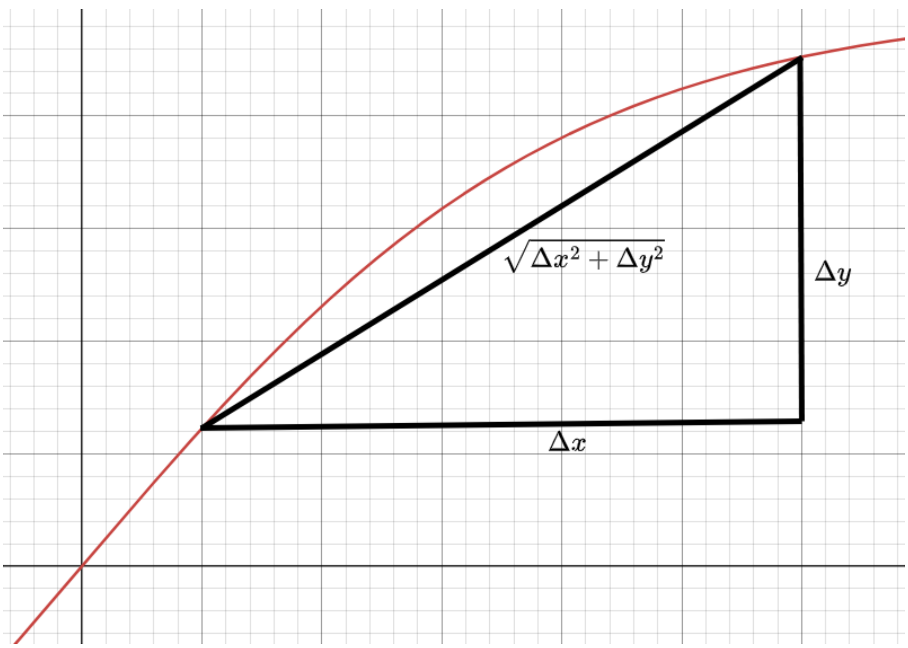
We already know how to calculate the length of a straight line using the Pythagorean theorem

$$\text{Length of a straight line} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1}$$

We can first start by splitting our curve into many small straight line segments segments of width $\Delta x$

The sum of all these line segments (as $\Delta x$ goes to zero) approaches the length of the line

$$\sum \sqrt{\Delta x^2 + \Delta y^2} \tag{2}$$



So now we have to find a way to write this where we can evaluate a riemann sum (and thus an integral). We can factor out $\Delta x^2$ from both deltas inside the equation

$$\sum \sqrt{\Delta x^2 (1 + (\frac{\Delta y}{\Delta x})^2)} \tag{3}$$

$$\sum \sqrt{(1 + (\frac{\Delta y}{\Delta x})^2)} \cdot \Delta x \tag{4}$$

As $\Delta x$ approaches 0, we realize this is a riemann sum and can use the fundamental theorem of calculus to evalute it We also see that $\frac{\Delta x}{\Delta y}$ is the derivative of the function, so we can replace it in our formula

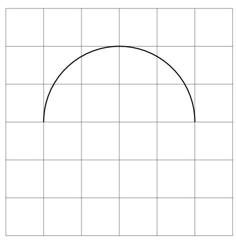
$$\text{Length of a curve} = \int_a^b \sqrt{1 + f'(x)^2} \cdot dx \tag{5}$$

# 3 Practice

**Example 1.** Find the arc length of a unit semicircle (since we know the formula for the perimeter of a circle we know our answer should be $\frac{2\pi(1)}{2} = \pi$)

$$f(x) = \sqrt{1 - x^2} \tag{6}$$

$$\text{Arc length} = \int_{-1}^{1} \sqrt{1 + f'(x)^2} dx \tag{7}$$



$f'(x) = \frac{-2x}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}}$

$[f'(x)]^2 = \frac{x^2}{1-x^2}$ $\quad \sqrt{1+[f'(x)]^2} = \sqrt{\frac{x^2+1-x^2}{1-x^2}} =$

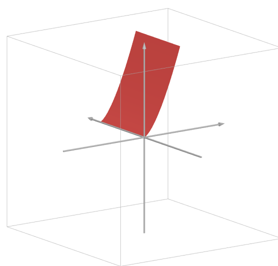
$\frac{1}{\sqrt{1-x^2}}$

$\int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = \arcsin(x)\Big|_{-1}^{1} = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi$

**Example 2.** Find the arc length of $\log(\sec(x))$ from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$

$$f(x) = \log(\sec(x))$$
$$f'(x) = \frac{1}{\sec x} \sec x \tan x = \tan x$$
$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1+\tan^2 x}\, dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec x\, dx = \ln|\sec x + \tan x| \Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$
$$= \ln|\sqrt{2}+1| - \ln|\sqrt{2}-1|$$

**Example 3.** Find the surface area length of strip $y = x^{3/2}$ from $x = 0$ to $x = 4$, when stretched across the z-axis by 4 units



$$f(x) = x^{\frac{3}{2}} \quad f'(x)\ \frac{3}{2}\sqrt{x}$$
$$\int_0^4 \sqrt{1+\frac{9}{4}x}\, dx \qquad u = 1+\frac{9}{4}x \quad du = \frac{9}{4}$$
$$\frac{4}{9}\int \sqrt{u}\, du = \frac{4}{9}\frac{2}{3} u^{\frac{3}{2}} \Rightarrow \frac{8}{27}\left(1+\frac{9}{4}x\right)^{\frac{3}{2}} \Big|_0^4 =$$
$$\frac{8}{27}\sqrt{1000} - \frac{8}{27} = \frac{8}{27}(10\sqrt{10}-1)$$
$$\text{Arc Length} \cdot \text{Width} = \left[\frac{8}{27}(10\sqrt{10}-1)\right]4 = \frac{32}{27}(10\sqrt{10}-1)$$

# 4 Arc Length of Parametric Functions

For the derivation of the arc length of parametric functions, we will go a slightly different route
We will be using the derivative of both the $x$ component and the $y$ component
We know the our change in $y$ can be found by using the parametric equation for $y$ based on $t$, and the same thing can be said for the change in $x$

$$\text{Change in y} = \Delta y = \frac{\Delta y}{\Delta t} \cdot \Delta t \tag{8}$$

$$\text{Change in x} = \Delta x = \frac{\Delta x}{\Delta t} \cdot \Delta t \tag{9}$$

Now lets use our initial formula again to find the arc length of a parametric function

$$= \sum \sqrt{(\frac{\Delta x}{\Delta t} \cdot \Delta x)^2 + (\frac{\Delta y}{\Delta t} \cdot \Delta t)^2} \tag{10}$$

$$= \sum \sqrt{(\Delta t)^2((\frac{\Delta x}{\Delta t})^2 + (\frac{\Delta y}{\Delta t})^2)} \tag{11}$$

$$= \sum \sqrt{(\frac{\Delta x}{\Delta t})^2 + (\frac{\Delta y}{\Delta t})^2} \cdot \Delta t \tag{12}$$

$$\tag{13}$$

And treating this as a riemann sum we can then use integral calculus to evaluate as $\Delta t$ goes to zero and this leaves us with the following formula for the arc length of a parametric function:

$$\text{Length of a parametric curve} = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \cdot dt \tag{14}$$

# 5 Finding lengths of parametric functions

**Example 1.** Lets revisit our first example, finding the length of half an arc of a unit circle (from 0 to $\pi$), since we can now rewrite it using the following parametric functions

$$x = \sin(t) \tag{15}$$

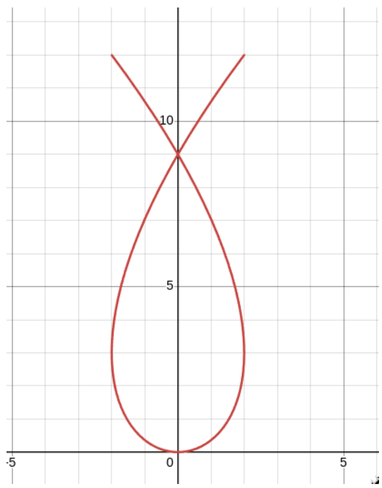
$$y = \cos(t) \tag{16}$$

$x' = \cos t \quad y' = -\sin t \quad x'^2 = \cos^2 t \quad y'^2 = \sin^2 t$

$$\int_0^\pi \underbrace{\sqrt{\cos^2 t + \sin^2 t}}_{1}\, dt = t\Big|_0^\pi = \pi$$

**Example 2.** Find the length of the following parametric curve from $t = -2$ to $t = 2$

$$x = t^3 - 3t \tag{17}$$
$$y = 3t^2 \tag{18}$$



$$x' = 3t^2 - 3 \quad y' = 6t$$

$$x'^2 = 9t^4 - 18t^2 + 9 \quad y'^2 = 36t^2$$

$$\int_{-2}^{2} \sqrt{9t^4 - 18t^2 + 9 + 36t^2}\, dt = \int_{-2}^{2} \sqrt{(3t^2+3)^2}\, dt$$

$$= t^3 + 3t \Big|_{-2}^{2} = (8+6) - (-8-6) = 28$$