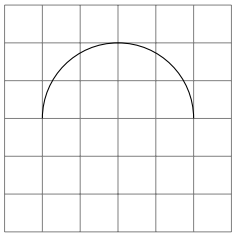
The Length of an Arc

1 Practice

Example 1. Find the arc length of a unit semicircle (since we know the formula for the perimeter of a circle we know our answer should be $\frac{2\pi(1)}{2} = \pi$)

$$f(x) = \sqrt{1-x^2} \tag{1}$$

$$\text{Arc length} = \int_{-1}^{1} \sqrt{1+f'(x)^2}dx \tag{2}$$



SOLUTION:

$$f'(x) = \frac{1}{2\sqrt{1-x^2}} \cdot \frac{-2x}{1} = \frac{-x}{\sqrt{1-x^2}}dx \tag{3}$$

$$L = \int_{-1}^{1} \sqrt{1+(\frac{-x}{\sqrt{1-x^2}})^2}dx \tag{4}$$

$$L = \int_{-1}^{1} \sqrt{1+\frac{x^2}{1-x^2}}dx \tag{5}$$

$$L = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx \tag{6}$$

$$L = \sin^{-1}(1) - \sin^{-1}(-1) \tag{7}$$

$$L = \pi \tag{8}$$

Example 2. Find the arc length of $\ln(\sec(x))$ from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$

$$f(x) = \ln(\sec(x)) \tag{9}$$
$$f'(x) = \tan(x) \tag{10}$$
$$\text{Arc Length} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{1 + (tan(x))^2}dx \tag{11}$$
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sqrt{(\sec(x))^2}dx \tag{12}$$
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec(x)dx \tag{13}$$
$$= \ln(|\tan(\frac{\pi}{4}) + \sec(\frac{\pi}{4})|) - \ln(|\tan(-\frac{\pi}{4}) + \sec(-\frac{\pi}{4})|) \tag{14}$$
$$= \ln(\sqrt{2} + 1) - \ln(\sqrt{2} - 1) \tag{15}$$

Example 3. Find the arc length of $y = x^{3/2}$ from $x = 0$ to $x = 4$

2 Finding lengths of parametric functions

Example 1. Lets revisit our first example, finding the length of half an arc of a unit circle (from 0 to π), since we can now rewrite it using the following parametric functions

$$x = sin(t) \tag{16}$$
$$y = cos(t) \tag{17}$$

SOLUTION:

$$\frac{dx}{dt} = \cos(x) \tag{18}$$
$$\frac{dy}{dt} = -\sin(x) \tag{19}$$
$$L = \int_0^{\pi} \sqrt{(\cos(x))^2 + (-\sin(x))^2}dx \tag{20}$$
$$L = \int_0^{\pi} 1 \cdot dx \tag{21}$$
$$L = \pi - 0 \tag{22}$$
$$L = \pi \tag{23}$$

Example 2. Find the length of the following parametric curve from $t = -2$ to $t = 2$

$$x = t^3 - 3t \tag{24}$$
$$y = 3t^2 \tag{25}$$

$$\frac{dx}{dt} = t^3 - 3t \tag{26}$$
$$\frac{dy}{dt} = 3t^2 \tag{27}$$

3 "Difficulties"

3. Find the arc length of an ellipse

$$f(x) = \sqrt{1 - \frac{x^2}{9}} \tag{28}$$
$$f'(x) = \frac{-\frac{2x}{9}}{2\sqrt{1 - \frac{x^2}{9}}} \tag{29}$$