The Length of an Arc

1 Introduction

Many times when solving problems it is useful to know the length of a curve. By the end of this lesson we will be able to find the lengths of curves using integrals and derivatives!

2 Derivation for functions

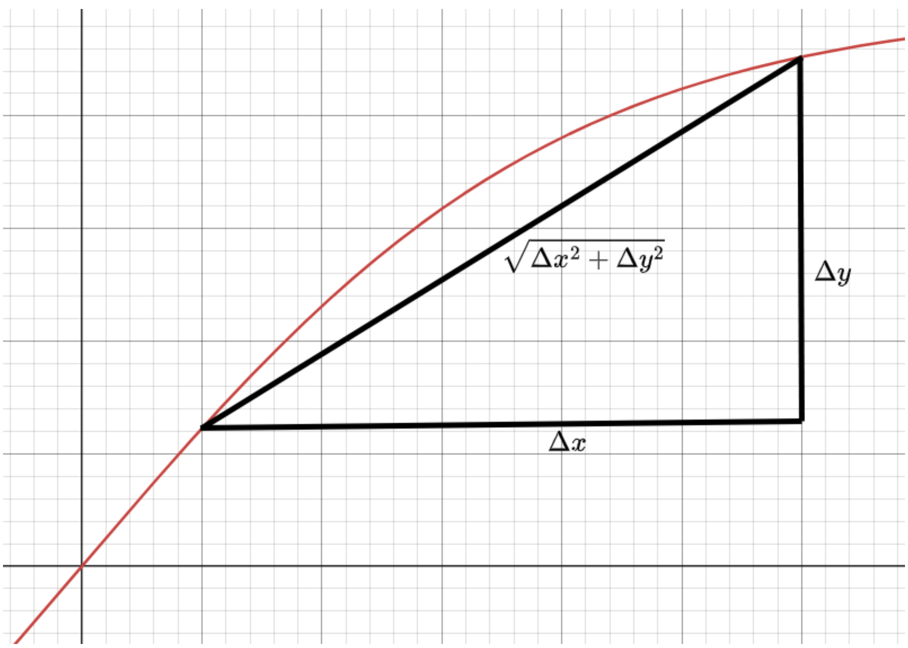
We already know how to calculate the length of a straight line using the Pythagorean theorem

$$\text{Length of a straight line} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1}$$

We can first start by splitting our curve into many small straight line segments segments of width Δx

The sum of all these line segments (as Δx goes to zero) approaches the length of the line

$$\sum \sqrt{\Delta x^2 + \Delta y^2} \tag{2}$$



So now we have to find a way to write this where we can evaluate a riemann sum (and thus an integral). We can factor out Δx^2 from both deltas inside the equation

$$\sum \sqrt{\Delta x^2 (1 + (\frac{\Delta y}{\Delta x})^2)} \tag{3}$$

$$\sum \sqrt{(1 + (\frac{\Delta y}{\Delta x})^2)} \cdot \Delta x \tag{4}$$

As Δx approaches 0, we realize this is a riemann sum and can use the fundamental theorem of calculus to evalute it We also see that $\frac{\Delta x}{\Delta y}$ is the derivative of the function, so we can replace it in our formula

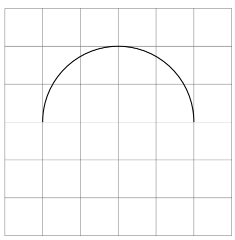
$$\text{Length of a curve} = \int_a^b \sqrt{1 + f'(x)^2} \cdot dx \tag{5}$$

3 Practice

Example 1. Find the arc length of a unit semicircle (since we know the formula for the perimeter of a circle we know our answer should be $\frac{2\pi(1)}{2} = \pi$)

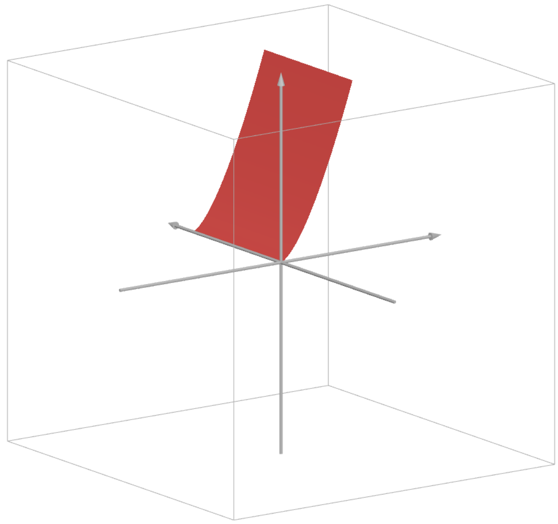
$$f(x) = \sqrt{1 - x^2} \tag{6}$$

$$\text{Arc length} = \int_{-1}^{1} \sqrt{1 + f'(x)^2} dx \tag{7}$$



Example 2. Find the arc length of $\log(\sec(x))$ from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$

Example 3. Find the surface area length of strip $y = x^{3/2}$ from $x = 0$ to $x = 4$, when stretched across the z-axis by 4 units



4 Arc Length of Parametric Functions

For the derivation of the arc length of parametric functions, we will go a slightly different route
We will be using the derivative of both the x component and the y component
We know the our change in y can be found by using the parametric equation for y based on t

$$\text{Change in y} = \Delta y = \frac{\Delta y}{\Delta t} \cdot \Delta t \tag{8}$$

$$\text{Change in x} = \Delta x = \frac{\Delta x}{\Delta t} \cdot \Delta t \tag{9}$$

Now lets use our initial formula again to find the arc length of a parametric function

$$= \sum \sqrt{(\frac{\Delta x}{\Delta t} \cdot \Delta x)^2 + (\frac{\Delta y}{\Delta t} \cdot \Delta t)^2} \tag{10}$$

$$= \sum \sqrt{(\Delta t)^2((\frac{\Delta x}{\Delta t})^2 + (\frac{\Delta y}{\Delta t})^2)} \tag{11}$$

$$= \sum \sqrt{(\frac{\Delta x}{\Delta t})^2 + (\frac{\Delta y}{\Delta t})^2} \cdot \Delta t \tag{12}$$

$$\tag{13}$$

And treating this as a riemann sum we can then use integral calculus to evaluate as Δt goes to zero and this leaves us with the following formula for the arc length of a parametric function:

$$\text{Length of a parametric curve} = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \cdot dt \tag{14}$$

5 Finding lengths of parametric functions

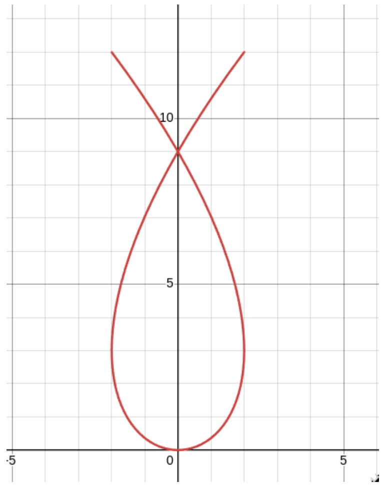
Example 1. Lets revisit our first example, finding the length of half an arc of a unit circle (from 0 to π), since we can now rewrite it using the following parametric functions

$$x = sin(t) \tag{15}$$

$$y = cos(t) \tag{16}$$

Example 2. Find the length of the following parametric curve from $t = -2$ to $t = 2$

$$x = t^3 - 3t \tag{17}$$
$$y = 3t^2 \tag{18}$$



6 "Difficulties"

3. Find the arc length of an ellipse

$$f(x) = \sqrt{1 - \frac{x^2}{9}} \tag{19}$$

$$f'(x) = \frac{-\frac{2x}{9}}{2\sqrt{1 - \frac{x^2}{9}}} \tag{20}$$