The Length of an Arc

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1 Introduction

"Whenever you're trying to calculate something that you can approximate as a sum of a bunch of tiny things, always try and use an integral." - Grant Sanderson

Many times when solving problems it is useful to know the length of a curve.

2 Derivation

We already know how to calculate the length of a straight line using the Pythagorean theorem

$$\text{Length of a straight line} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1}$$

We can first start by splitting our surve into small segments of width Δx Since we also know the derivative of the function we can now approximate a small portion of the curve with a straight line

$$\text{Slope of line at x} = \frac{dy}{dx} = f'(x) \tag{2}$$

$$\text{Change in y at x} = \frac{dy}{dx} \cdot dx = f'(x)dx \tag{3}$$

$$\tag{4}$$

Since we now know the change in y and the change in x, we can now use the pythagorean theorem to calculate the length of small portion of the line

$$\text{Length of a segment} = \sqrt{dx^2 + (f'(x) \cdot dx)^2} \tag{5}$$

We want to find the sum of all these lengths, but before we can factor out the Δx we need to simplify the equation

$$\begin{aligned} \text{Length of a segment} &= \sqrt{dx^2 + f'(x)^2 \cdot dx^2} && (6) \\ &= \sqrt{1 + f'(x)^2} \cdot dx && (7) \end{aligned}$$

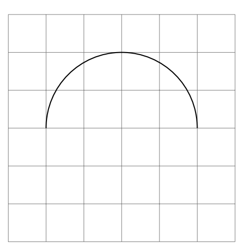
Now we can integrate this function over an interval $[a, b]$ and this gives us the formula for the length of a curve

$$\text{Length of a curve} = \int_a^b \sqrt{1 + f'(x)^2} \cdot dx \tag{8}$$

3 Practice

1. Find the arc length of a unit semicircle (since we know the formula for the perimeter of a circle we know our answer should be $\frac{2\pi(1)}{2} = \pi$)

$$\begin{aligned} f(x) &= \sqrt{1 - x^2} && (9) \\ \text{Arc length} &= \int_{-1}^{1} \sqrt{1 + f'(x)^2} dx && (10) \end{aligned}$$



2. Find the arc length of $\log(\sec(x))$ from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$

4 "Difficulties"

3. Find the arc length of an ellipse

$$f(x) = \sqrt{1 - \frac{x^2}{9}} \tag{11}$$

$$f'(x) = \frac{-\frac{2x}{9}}{2\sqrt{1 - \frac{x^2}{9}}} \tag{12}$$