Assignment 2 Solutions

March 18, 2019

Question 1 to 4

1) 21R, 8B, 4G.

$$\frac{21C_2 \cdot 8C_2 \cdot 4C_2}{33C_6} \rightarrow 0.0319$$

2) a $$1 \cdot \frac{25}{26} \cdot \frac{24}{26} \cdot \frac{23}{26} \cdot \frac{22}{26} = 0.664$$

- INDEPENDENT → with replacement
- 1st letter can be anything ∴ probability : 1.

b) $$\frac{26C5}{26^5} =$$ There is exactly one arrangement which will be in increasing order and distinct

3) a $7C_3$.

b $$\frac{1C_1 \cdot 6C_2}{7C_3} = 0.429$$

c) $$\frac{1C_1 \cdot 1C_1 \cdot 5C_1}{7C_3} = 0.143$$

4) $$p_{x,y}(x,y) = \left[\left(\frac{5}{6}\right)^{x-1}\left(\frac{1}{6}\right)\right]\left[\left(\frac{5}{6}\right)^{y-1}\left(\frac{1}{6}\right)\right] = p_X(x)\, p_Y(y)$$

hence independent

Figure 1: Q1 -4 solutions

5 a) We can take P(A) to be the probability of A finding a butterfly. It is represented by **1** in the **A** column below. The 0 represents P(A'), i.e, A not finding a butterfly. The same is the case for B and C as well, such that

$$P(A) = 0.17, P(A') = 0.83, P(B) = 0.25, P(B') = 0.75, P(C) = 0.45, P(C') = 0.55$$

By using these probabilities, we are able to find X and Y, and subsequently, the $P_{x,y}(x, y)$ for each combination below.

A	B	C	X	Y	$P_{x,y}(x, y)$
0	0	0	0	3	0.342375
0	0	1	1	2	0.280125
0	1	0	1	2	0.114125
0	1	1	2	1	0.093375
1	0	0	1	2	0.070125
1	0	1	2	1	0.057375
1	1	0	2	1	0.023375
1	1	1	3	0	0.019125

Table 1: Table showing the Joint PMF

Adding all of the $P_{x,y}(x, y)$ in the table gives us 1, which is the case for joint PMF.

b)

$$
\begin{aligned}
P(Z_1 = 1) &= 0.17 \\
P(Z_2 = 1) &= 0.25 \\
P(Z_3 = 1) &= 0.45 \\
E(Z_1 = 1) &= 1 * 0.17 \\
E(Z_2 = 1) &= 1 * 0.25 \\
E(Z_3 = 1) &= 1 * 0.45 \\
\therefore E(Z) = E(Z_1 = 1) + E(Z_2 = 1) + E(Z_3 = 1) &= 0.17 + 0.25 + 0.45 = 0.87
\end{aligned}
$$

6 a)

A	B	X	Y	$P_{x,y}(x, y)$
0	0	0	0	$\frac{2}{5} * \frac{1}{4} = \frac{2}{20}$
0	1	0	1	$\frac{2}{5} * \frac{3}{4} = \frac{6}{20}$
1	0	1	0	$\frac{3}{5} * \frac{2}{4} = \frac{6}{20}$
1	1	1	1	$\frac{3}{5} * \frac{2}{4} = \frac{6}{20}$

Table 2: Joint PMF of X and Y

b) If Y = 1, this means that there is now only 2 chocolate cookies and 2 non-chocolate cookies for A to choose from, therefore,

$$
\begin{aligned}
P_{x,y}(0|1) &= \frac{2}{4} \\
P_{x,y}(1|1) &= \frac{2}{4}
\end{aligned}
$$

c) If Y = 0, this means that there are 3 chocolate cookies and 1 non-chocolate cookies for A to choose from, therefore,

$$P_{x,y}(0|0) = \frac{1}{4}$$
$$P_{x,y}(1|0) = \frac{3}{4}$$

d) 2 Random Variables are said to be independent if

$$P_{x,y}(x, y) = P_x(x).P_y(y)$$

For $P_x(x)$,

$$P(X = 0) = \frac{2}{20} + \frac{6}{20} = \frac{8}{20}$$
$$P(X = 1) = \frac{6}{20} + \frac{6}{20} = \frac{12}{20}$$

For $P_y(y)$,

$$P(Y = 0) = \frac{2}{20} + \frac{6}{20} = \frac{8}{20}$$
$$P(Y = 1) = \frac{6}{20} + \frac{6}{20} = \frac{12}{20}$$

We check if each of $P_{x,y}(x, y) = P_x(x).P_y(y)$, i.e

$$P_{x,y}(0, 0) = \frac{2}{20} \neq P_x(0).P_y(0) = \frac{8}{20}.\frac{8}{20}$$

As they aren't equal, we know that they are dependent, and do not need to check for $P_{x,y}(1, 1)$ and $P_{x,y}(2, 2)$

7 This question is a binomial distribution problem. And is similar to tossing a biased coin 3 times. With the probability of getting a head (a call from telemarketer). The four possible outcomes are getting 0 calls, 1 call, 2 calls or 3 calls. The following are the 4 probabilities for each outcomes,

$$P(X = 0) =^3 C_0(\frac{7}{8})^3 = \frac{343}{512} \tag{1}$$
$$P(X = 1) =^3 C_1(\frac{1}{8})(\frac{7}{8})^2 = \frac{147}{512} \tag{2}$$
$$P(X = 2) =^3 C_2(\frac{1}{8})^2(\frac{7}{8}) = \frac{21}{512} \tag{3}$$
$$P(X = 3) =^3 C_3(\frac{1}{8})^3 = \frac{1}{512} \tag{4}$$

The proof that it going fine is that all the probabilities add up to one and hence the sample space is valid and abides by the axioms.

The plot is as follows,

8 As it says n is until the telemarketer calls. It is a geometric random variable. Therefore, there will be n calls before the $(n+1)^{th}$ call which will be by the telemarketer. Answers to most questions are hidden in their extremes therefore, let's start with 0 callers and 1 caller before the telemarketer's call,

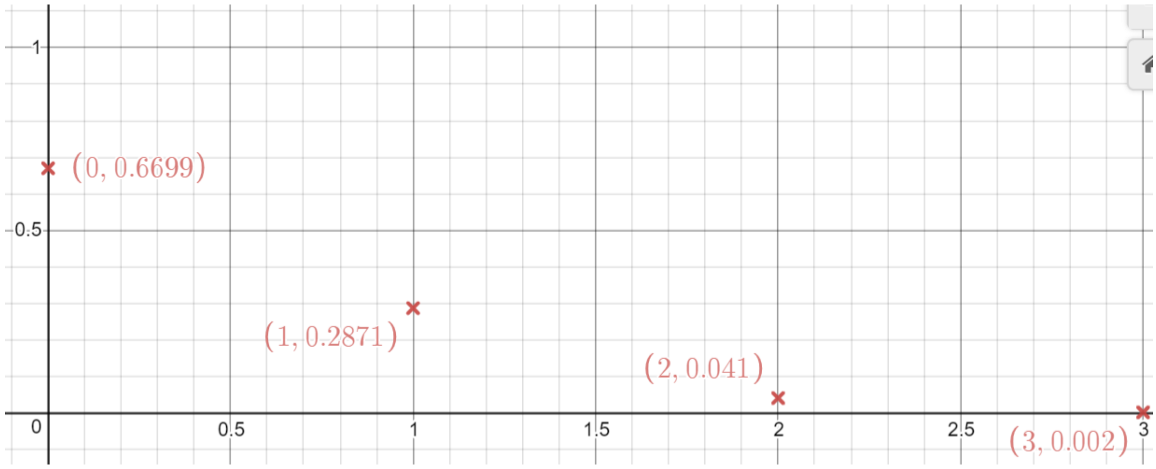

Figure 2: PMF of X

$$P(X=0) = \frac{1}{8} \tag{5}$$

$$P(X=1) = \frac{7}{8}\frac{1}{8} \tag{6}$$

$$P(X=2) = (\frac{7}{8})^2\frac{1}{8} \tag{7}$$

$$\vdots \tag{8}$$

$$P(X=n-1) = (\frac{7}{8})^{n-1}\frac{1}{8} \tag{9}$$

$$P(X<n) = \sum_{i=0}^{n}((\frac{7}{8})^{i-1}\frac{1}{8}) \tag{10}$$

$$P(X \geq n) = 1 - P(X<n) = 1 - \sum_{i=0}^{n}((\frac{7}{8})^{i-1}\frac{1}{8}) \tag{11}$$

9 A very binary approach would be to see if the sets X and Y are niether disjoint nor anyone of them is a subset of the other. Another approach would be to observe the joint PMF table for X and Y. The following is the joint PMF for X and Y,

Y \ **X**	**0**	**1**	**2**
2	**4/52 x 3/51**	**0**	**0**
1	**4/52 x 36/51**	**12/52 x 4/51**	**0**
0	**36/52 x 35/51**	**12/52 x 36/51**	**12/52 x 11/51**

Figure 3: Joint PMF for X and Y

We can see the zero enteries in the table, which show that the occurence of one directly influences the other to the extent that it can't even happen therefore the two events are dependent. The zeroes are due to the fact that we can only draw two cards.

10 The following is the definition of the indicator random variable,

$$\begin{aligned} \text{if } X_i &: X_i = i - 1 \\ \text{else } &: X_i = 0 \end{aligned}$$

Now,

$$P(X = X_i) = ^5 C_{i-1}(\frac{4}{52})^{i-1}(\frac{48}{52})^{5-i+1} \tag{12}$$

Now calculating the expectation,

$$E[X] = \sum_{i=1}^{5} X_i P(X = X_i) \tag{13}$$

$$E[X] = 0(^5C_0(\frac{4}{52})^0(\frac{48}{52})^5) + 1(^5C_1(\frac{4}{52})^1(\frac{48}{52})^4) + \ldots + 4(^5C_4(\frac{4}{52})^4(\frac{48}{52})^1) \tag{14}$$

Now for the without replacement case,

$$P(X = X_i) = \frac{(^4C_{i-1})(^{48}C_{5-i+1})}{(^{52}C_5)} \tag{15}$$

$$E[X] = \sum_{i=1}^{5} X_i P(X = X_i) \tag{16}$$

$$E[X] = 0\frac{(^4C_0)(^{48}C_5)}{(^{52}C_5)} + \ldots + 4\frac{(^4C_4)(^{48}C_1)}{(^{52}C_5)} \tag{17}$$

11 The following maps all the possible outcomes,

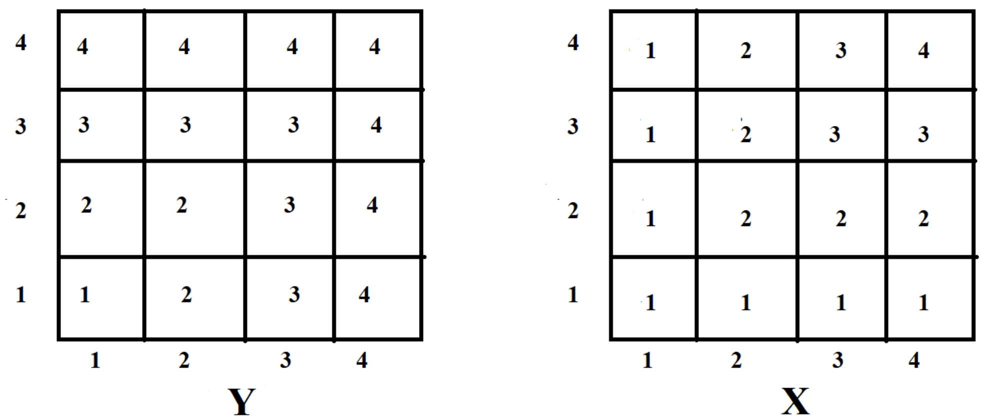

Figure 4: Mapping of all possible outcomes.

Since the events are independent, $p_{X,Y}(x, y) = p_X(x)p_Y(y)$,

$$p_X(X = i) = \frac{9 - 2i}{16} \tag{18}$$

$$p_Y(Y = i) = \frac{2i - 1}{16} \tag{19}$$

$$p_{X,Y}(X = i, Y = j) = \frac{(2i - 1)(9 - 2j)}{256} \tag{20}$$

12 (a) We can see that if 3 telemarketers are calling on average in 7-days, then for a single day, the average will be:

$$mean[single\ day] = \frac{3}{7} \tag{21}$$

Realize that we don't know the number of telemarketers we are dealing with. Let's denote the number by n. Now, observe that this is basically a binomial process in which, for a given day, a telemarketer will either call or not call. Since we know that the mean for a binomial process is given by $n \times p$, we can use that to calculate p as follows:

$$p = \frac{mean[single\ day]}{n} \tag{22}$$

This p is the probability for any telemarketer to call on any given day.

(b) Since we've already identified the number of telemarketers calling on a given day as a binomial random variable, we can see that:

$$P(no\ telemarketers) = \binom{n}{0} \times p^0 \times (1 - p)^{n-0} \tag{23}$$

(c) This can again be simply found using:

$$P(exactly\ two\ telemarketers) = \binom{n}{2} \times p^2 \times (1 - p)^{n-2} \tag{24}$$

13 (a) **A** can get either 0, 1 or 2 red marbles. Let **Y** be a random variable denoting the number of marbles **A** gets. Now, **Y** can take the following values: 0, 1, 2. The probability for each one of them can be calculated as follows:

$$P(Y = 0) = \frac{6 \times 5}{8 \times 7} \tag{25}$$

$$P(Y = 1) = \frac{2 \times 6}{8 \times 7} + \frac{6 \times 2}{8 \times 7} \tag{26}$$

$$P(Y = 2) = \frac{2 \times 1}{8 \times 7} \tag{27}$$

$$P(Y = y) = \begin{cases} \frac{30}{56}, & y = 0 \\ \frac{24}{56}, & y = 1 \\ \frac{2}{56}, & y = 2 \end{cases} \tag{28}$$

(b) The expectation can be found using the following formula:

$$E[Y] = \sum_{i=1}^{n} y_i \times p(y_i) \tag{29}$$

This will expand out as follows:

$$E[Y] = 0 \times \frac{30}{56} + 1 \times \frac{24}{56} + 2 \times \frac{2}{56} \tag{30}$$

(c) Now, let's think about **B**. Let **X** be a random variable denoting the number of marbles **B** gets. Now, **X** can, like **Y**, take the following values: 0, 1, 2.

It can be zero in the following three cases: **Y** is zero and **X** is zero, **Y** is one and **X** is zero, **Y** is two and **X** is zero. This implies:

$$\begin{aligned} P(X=0) = P(Y=0) \times \frac{4 \times 3}{6 \times 5} &+ \\ P(Y=1) \times \frac{5 \times 4}{6 \times 5} &+ \\ P(Y=2) \times \frac{6 \times 5}{6 \times 5} & \end{aligned} \tag{31}$$

Similarly,

$$\begin{aligned} P(X=1) = P(Y=0) \times \frac{2 \times 4}{6 \times 5} &+ \\ P(Y=0) \times \frac{4 \times 2}{6 \times 5} &+ \\ P(Y=1) \times \frac{1 \times 4}{6 \times 5} &+ \\ P(Y=1) \times \frac{4 \times 1}{6 \times 5} & \end{aligned} \tag{32}$$

$$P(X=2) = P(Y=0) \times \frac{2 \times 1}{6 \times 5} \tag{33}$$

The expectation can be found using the following formula:

$$E[X] = \sum_{i=1}^{n} x_i \times p(x_i) \tag{34}$$

14 (a) Let our indicator variable be denoted as **X** which is defined as follows:

$$X = \begin{cases} 1, & student = oncampus \\ 0, & student = offcampus \end{cases} \tag{35}$$

Now, we can find the expected value of **X** for one student ($E[X_1]$), and since the selections are independent, we will simply multiply that expectation by 6:

$$\begin{aligned} E[on\ campus] = E[X_1] + E[X_2] + E[X_3] &+ \\ E[X_4] + E[X_5] + E[X_6] & \end{aligned} \tag{36}$$

Since, X_1, X_2, . . . X_6 are independent as well as identical in their probability distribution, we can replace all of them with X leading us to:

$$E[on\ campus] = 6 \times E[X] \tag{37}$$

Now,

$$E[X] = 0.4 \times 1 + 0.6 \times 0 \tag{38}$$

(b) Let us define Y as a random variable denoting the presence or absence of a high value in a single die roll. This implies that:

$$Y = \begin{cases} 1, & roll\ outcome = 5\ or\ 6 \\ 0, & otherwise \end{cases} \tag{39}$$

We can also write the PMF for Y as follows:

$$P(Y = y) = \begin{cases} \frac{2}{6}, & y = 1 \\ \frac{4}{6}, & y = 0 \end{cases} \tag{40}$$

This leads us to the following expression for the expectation of Y:

$$E[Y] = 1 \times \frac{2}{6} + 0 \times \frac{4}{6} \tag{41}$$

Now, using the same argument as the one employed in part (a), we can calculate the expecation of X as follows:

$$E[X] = 7 \times E[Y] \tag{42}$$

15 (a)

$$P(X = 0) = \frac{6 \times 5 \times 4 \times 3}{8 \times 7 \times 6 \times 5} \tag{43}$$

(b) Now, $X = 1$ can happen in four ways: Alice's first marble is red, second is not red, and both of Bob's marbles are not red; Alice's first marble is not red, second is red, and both of Bob's marbles are not red; Alice's both marbles are not red, Bob's first marble is red, and second is not red; Alice's both marbles are not red, Bob's first marble is not red, and second is red. This leads us to:

$$\begin{aligned} P(X = 1) = & \frac{2 \times 6 \times 5 \times 4}{8 \times 7 \times 6 \times 5} + \\ & \frac{6 \times 2 \times 5 \times 4}{8 \times 7 \times 6 \times 5} + \\ & \frac{6 \times 5 \times 2 \times 4}{8 \times 7 \times 6 \times 5} + \\ & \frac{6 \times 5 \times 4 \times 2}{8 \times 7 \times 6 \times 5} \end{aligned} \tag{44}$$

(c) $P(X = 2)$ can simply be calculated by subtracting the previous two probabilities from 1:

$$P(X = 2) = 1 - [P(X = 0) + P(X = 1)] \tag{45}$$

16 (a) We shall try to observe a general pattern in order to solve this question. Now, the minimum possible value for the sum is obviously 3 (1, 1, 1), whereas the maximum possible value is 18 (6, 6, 6). There is only one configuration which will make them possible. As we move on, for the sum to be 4, there are three possible configurations: (1, 1, 2), (1, 2, 1), (2, 1, 1)) - similar is the case for the sum to be 17. Moving on; for the sum to be 5, there are six possible configurations: (1, 1, 4), (1, 4, 1), (4, 1, 1), (2, 2, 1), (1, 2, 2), (2, 1, 2) - similar is the case for the sum to be 16. We'll need one more puzzle piece in order to see the pattern appear. Let's look at the case of the sum being 6, there are ten possible configurations: (1, 1, 4), (1, 4, 1), (4, 1, 1), (2, 2, 2), (1, 2, 3), (3, 2, 1), (1, 3, 2), (3, 1, 2), (2, 1, 3), (2, 3, 1) - similar is the case for the sum to be 15. The pattern should be clear by now: 1, 3, 6, 10, . . . , 10, 6, 3, 1. However, this pattern does not work for sums greater than 8 and less than 13 (it works for sums greater than 13 since the cases are symmetric). The reason for this is that for a sum of 9, our method implicitly assumes that outcomes like (7, 1, 1) or (1, 7, 1) can also occur, which is simply not possible. Hence, we need to do these cases separately. For the sum to be 9, our pattern predicts 28 possible configurations; however, we can simply remove the three impossible configurations of (7, 1, 1), (1, 7, 1), and (1, 1, 7), and hence the possible configurations become 25 - similar is the case for the sum to be 12. Similarly, for the sum to be 10, our pattern predicts 36 possible configurations including the following impossible ones: (7, 1, 2), (7, 2, 1), (2, 1, 7), (2, 7, 1), (1, 2, 7), (1, 7, 2), (1, 1, 8), (1, 8, 1), (8, 1, 1); hence, our true value shall be 27 - similar is the case for the sum to be 11.

$$P(X = x) = \begin{cases} \frac{1}{6\times6\times6}, & x = \ 3 \ or \ 18 \\ \frac{3}{6\times6\times6}, & x = \ 4 \ or \ 17 \\ \frac{6}{6\times6\times6}, & x = \ 5 \ or \ 16 \\ \frac{10}{6\times6\times6}, & x = \ 6 \ or \ 15 \\ \frac{15}{6\times6\times6}, & x = \ 7 \ or \ 14 \\ \frac{21}{6\times6\times6}, & x = \ 8 \ or \ 13 \\ \frac{24}{6\times6\times6}, & x = \ 9 \ or \ 12 \\ \frac{27}{6\times6\times6}, & x = \ 10 \ or \ 11 \\ 0, & x = \ otherwise \end{cases} \tag{46}$$

Refer to Figure 5.

(b) We can again try to observe a pattern. Let's start with the maximum being 1, this is possible in only one case: (1, 1, 1). For the maximum to be 2, eight configurations are possible: 3 involving (1, 1, 2), 3 involving (1, 2, 2) and one being (2, 2, 2). For the maximum to be 3, 19 configurations are possible: 3 involving (1, 1, 3), 3 involving (1, 3, 3), 3 involving (2, 2, 3), 3 involving (2, 3, 3), 6 involving (1, 2, 3), and one being (3, 3, 3). While it might not be as obvious as the previous part, there is still a pattern which is being followed. For $Y = 1$ we had one configuration, which is the cube of 1 minus the cube of 0; for $Y = 2$ we had seven configurations, which can be found by subtracting the cube of 2 and the cube of 1; similarly for $Y = 3$ we had 19 configurations, which can be found by subtracting the cube of 3 and the cube of 2. Also, note that for this question, the argument of symmetry can not be applied.

Hence:

$$P(Y = y) = \begin{cases} \frac{1^3 - 0^3}{6\times6\times6}, & y = 1 \\ \frac{2^3 - 1^3}{6\times6\times6}, & y = 2 \\ \frac{3^3 - 2^3}{6\times6\times6}, & y = 3 \\ \frac{4^3 - 3^3}{6\times6\times6}, & y = 4 \\ \frac{5^3 - 4^3}{6\times6\times6}, & y = 5 \\ \frac{6^3 - 5^3}{6\times6\times6}, & y = 6 \\ 0, & y = otherwise \end{cases} \tag{47}$$

Refer to Figure 6.

(c) Since the roll of the first die will not be affected by the other two. This one can simply be given as:

$$P(Z = z) = \begin{cases} \frac{1}{6}, & z = 1, 2, 3, 4, 5, 6 \\ 0, & z = otherwise \end{cases} \tag{48}$$

Refer to Figure 7.

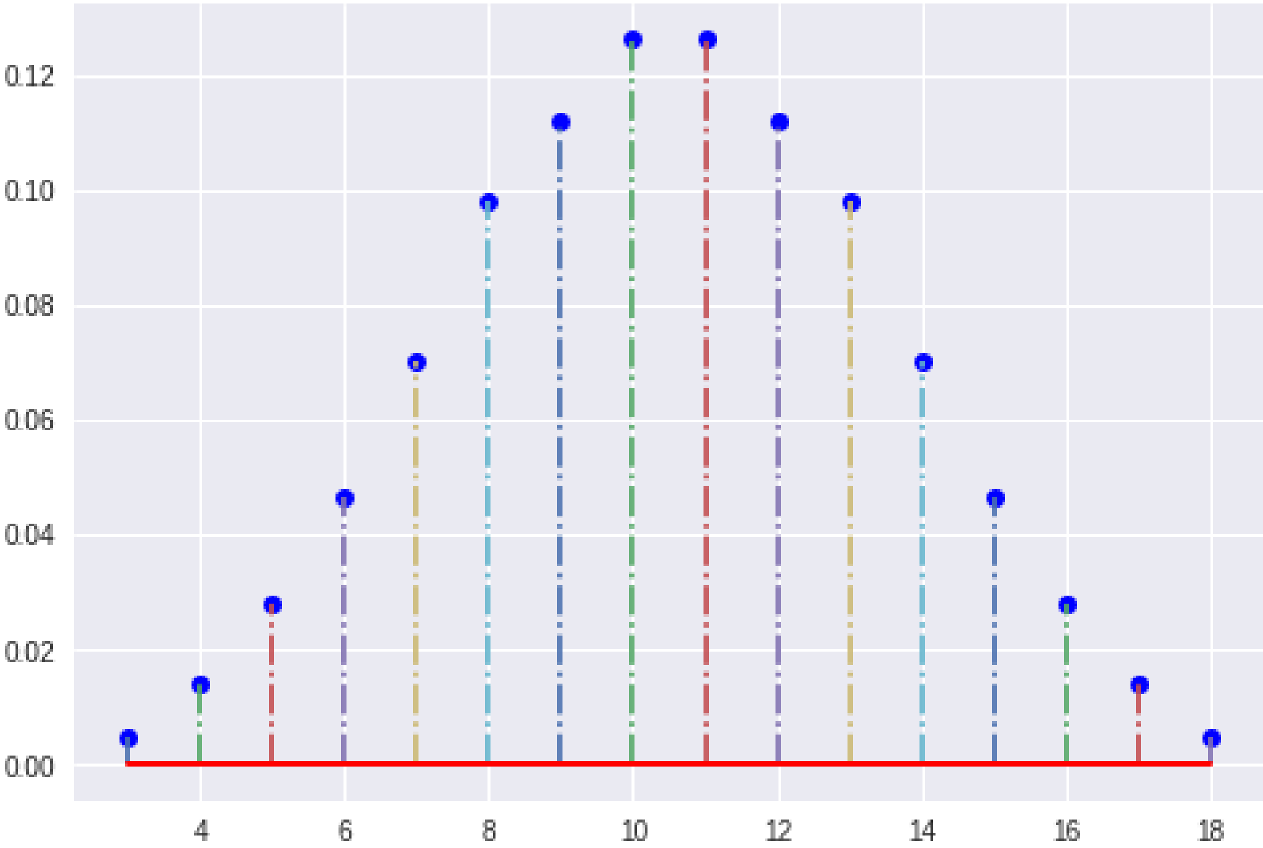

Figure 5: PMF of X

17 The possible outcomes for two four sided dice can be seen in figure 1. X is the outcome of the first die (row-wise) whereas Y is the maximum of the two outcomes (for example: for an outcome of (2, 1), Y will have a value of 2).

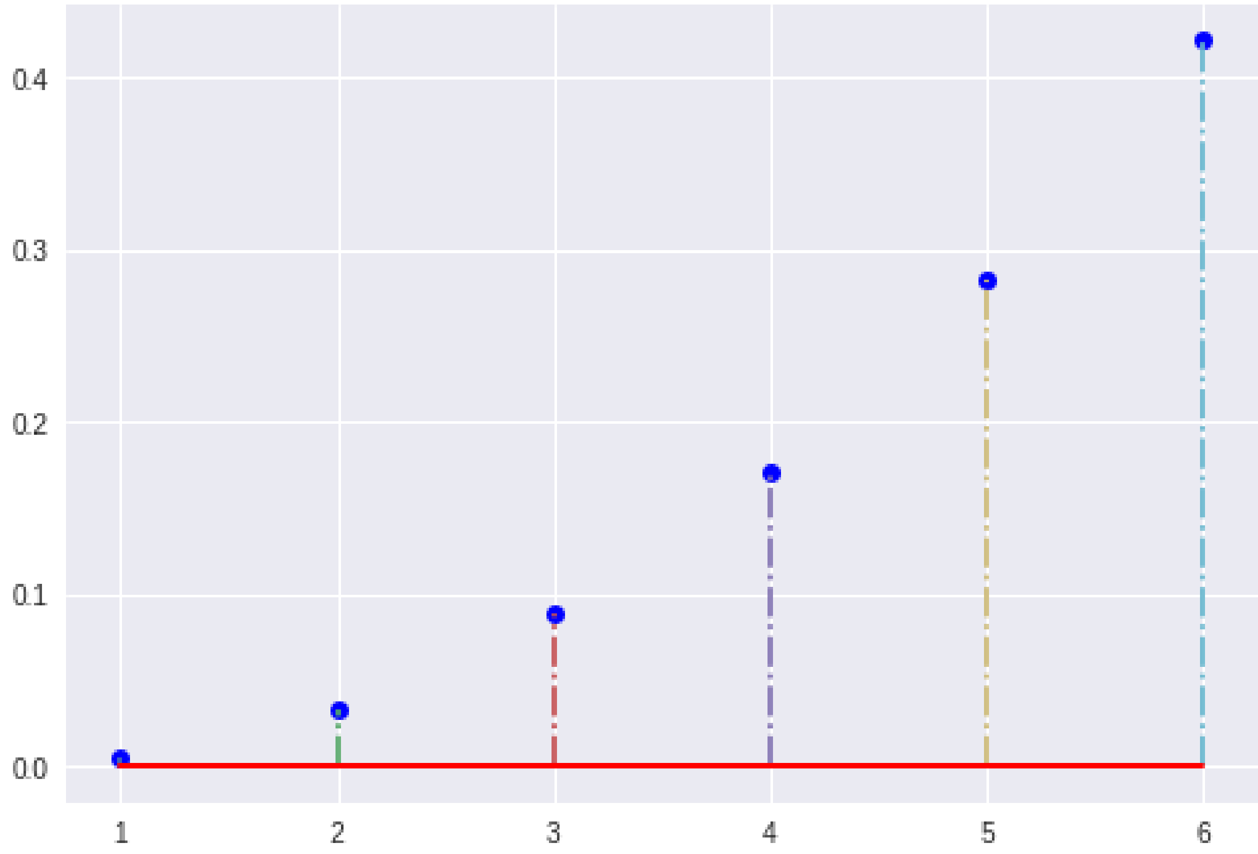

Figure 6: PMF of Y

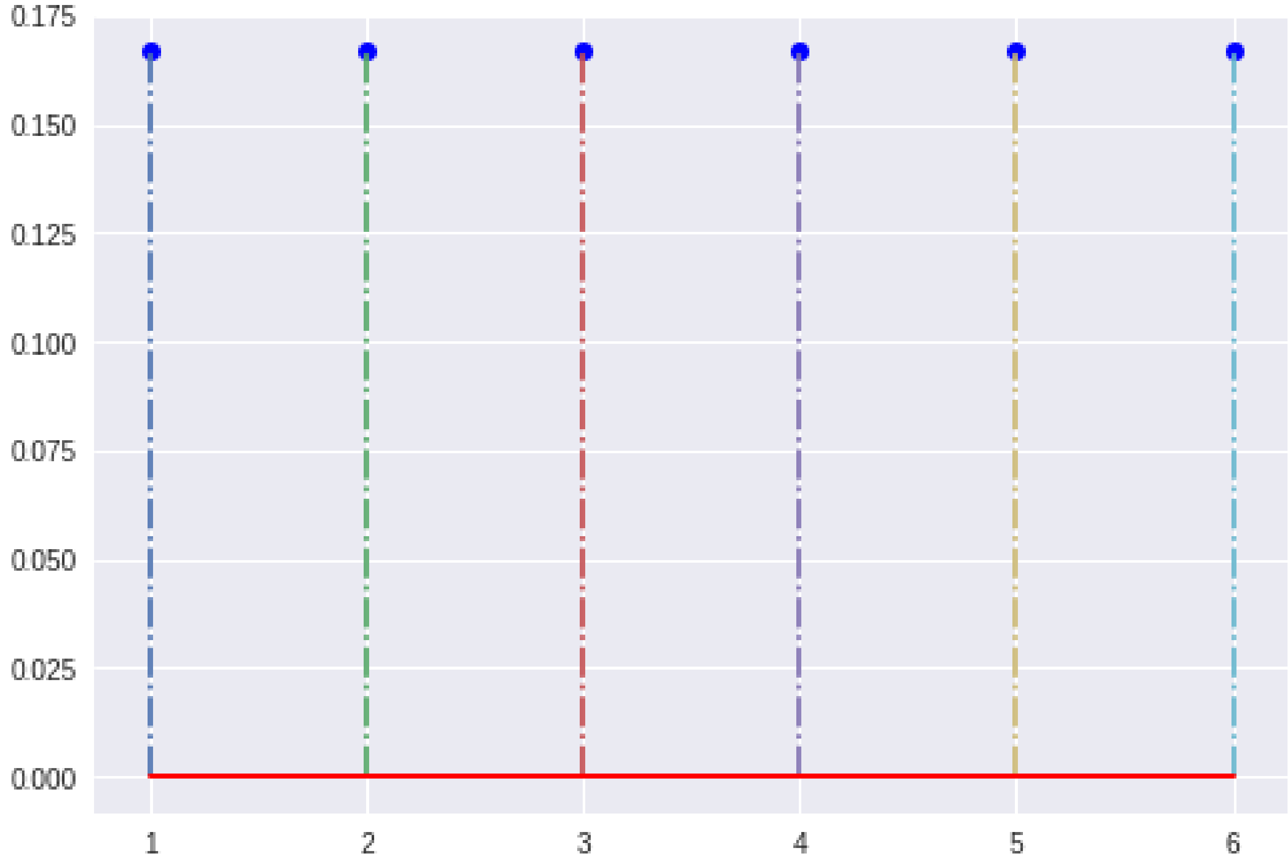

Figure 7: PMF of Z

From the Figure 8, we can easily write down the PMF for Y.

$$P(Y = y) = \begin{cases} \frac{1}{16}, & y = 1 \\ \frac{3}{16}, & y = 2 \\ \frac{5}{16}, & y = 3 \\ \frac{7}{16}, & y = 4 \\ 0, & y = otherwise \end{cases} \tag{49}$$

In order to find the conditional probability of X given Y, we will have to repeat the process for each value of Y.

Now, given $Y = 1$, X can only take one value i.e. $X = 1$. Our universe is simply: (1, 1):

$$P(X = x|Y = 1) = \begin{cases} 1, & x = 1 \\ 0, & x = otherwise \end{cases} \tag{50}$$

Moving on, given $Y = 2$, X can take two values, either 1 or 2. Our universe is now restricted to: (1, 2), (2, 1), (2, 2):

$$P(X = x|Y = 2) = \begin{cases} \frac{1}{3}, & x = 1 \\ \frac{2}{3}, & x = 2 \\ 0, & x = otherwise \end{cases} \tag{51}$$

Moving on, given $Y = 3$, X can take three values, either 1, 2, or 3. Our universe is now restricted to: (1, 3), (2, 3), (3, 3), (3, 2), (3, 1):

$$P(X = x|Y = 3) = \begin{cases} \frac{1}{5}, & x = 1, 2 \\ \frac{3}{5}, & x = 3 \\ 0, & x = otherwise \end{cases} \tag{52}$$

Moving on, given $Y = 4$, X can take all four values, 1, 2, 3, or 4. Our universe is now restricted to: (1, 4), (2, 4), (3, 4), (4, 4), (4, 3), (4, 2), (4, 1):

$$P(X = x|Y = 4) = \begin{cases} \frac{1}{7}, & x = 1, 2, 3 \\ \frac{4}{7}, & x = 4 \\ 0, & x = otherwise \end{cases} \tag{53}$$

	1	2	3	4
1	(1,1)	(1,2)	(1,3)	(1,4)
2	(2,1)	(2, 2)	(2,3)	(2,4)
3	(3,1)	(3,2)	(3,3)	(3,4)
4	(4,1)	(4,2)	(4,3)	(4,4)

Figure 8: Outcomes of rolling 2 four sided dice