Discreet 12.9.5.24

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Problem Statement

If S_1, S_2, S_3 are the sum of the first n natural numbers, their squares, and their cubes, respectively, show that

$$9(S_2)^2 = (S_3)(1+8(S_1))$$

Solution

Equation	Expression	Description
S_1	$\frac{n(n+1)}{2}$	sum of n natural numbers
S_2	$\frac{n(n+1)(2n+1)}{6}$	sum of squares
S_3	$\left(\frac{n(n+1)}{2}\right)^2$	sum of cubes

Table 1: Input Equations

Now, let's substitute these expressions into the given equation $9(S_2)^2 = (S_3)(1+8(S_1))$ and simplify:

$$
\begin{aligned}
9\left(\frac{n(n+1)(2n+1)}{6}\right)^2 &= \left(\frac{n(n+1)}{2}\right)^2\left(1+8\cdot\frac{n(n+1)}{2}\right)\\
\frac{9}{36}\left(n(n+1)(2n+1)\right)^2 &= \frac{1}{4}\left(n(n+1)\right)^2\left(1+4n(n+1)\right)\\
\frac{1}{4}\left(n(n+1)(2n+1)\right)^2 &= \frac{1}{4}\left(n(n+1)\right)^2\left(4n(n+1)+1\right)\\
\left(n(n+1)(2n+1)\right)^2 &= \left(n(n+1)\right)^2\left(4n(n+1)+1\right)
\end{aligned}
$$

The last equation holds true, which verifies that $9(S_2)^2 = (S_3)(1+8(S_1))$ for the given expressions of S_1, S_2, and S_3.