

Discreet 12.9.5.24

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PROBLEM STATEMENT

If S_1, S_2, S_3 are the sum of the first n natural numbers, their squares, and their cubes, respectively, show that

$$9(S_2)^2 = (S_3)(1+8(S_1))$$

SOLUTION

Equation	Expression	Description
S_1	$\frac{n(n+1)}{2}$	sum of n natural numbers
S_2	$\frac{n(n+1)(2n+1)}{6}$	sum of squares
S_3	$\left(\frac{n(n+1)}{2}\right)^2$	sum of cubes
x_1	$x_1(n) = nu(n)$	
x_2	$x_2(n) = n^2u(n)$	
x_3	$x_3(n) = n^3u(n)$	

TABLE I
INPUT EQUATIONS

By the differentiation property :

$$nx(n) \xrightarrow{Z} (-z) \frac{dX(z)}{dz} \quad (1)$$

$$\Rightarrow nu(n) \xrightarrow{Z} \frac{z^{-1}}{(1-z^{-1})^2}, |z| > 1 \quad (2)$$

$$\Rightarrow n^2u(n) \xrightarrow{Z} \frac{z^{-1}(z^{-1}+1)}{(1-z^{-1})^3}, |z| > 1 \quad (3)$$

$$\Rightarrow n^3u(n) \xrightarrow{Z} \frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}, |z| > 1 \quad (4)$$

$$X_1(z) = \frac{z^{-1}}{(1-z^{-1})^2}, \quad |z| > 1 \quad (5)$$

$$X_2(z) = \frac{z^{-1}(z^{-1}+1)}{(1-z^{-1})^3}, \quad |z| > 1 \quad (6)$$

$$X_3(z) = \frac{z^{-1}(1+4z^{-1}+z^{-2})}{(1-z^{-1})^4}, \quad |z| > 1 \quad (7)$$

The convolution sum is defined as

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k) \quad (8)$$

$$x(n) * u(n) = \sum_{k=0}^n x(k) \quad (9)$$

$$y(n) = x(n) * u(n) \quad (10)$$

$$Y(z) = X(z) \cdot u(z) \quad (11)$$

$$Y_1(z) = \frac{z^{-1}}{(1 - z^{-1})^3} \quad (12)$$

$$Y_2(z) = \frac{z^{-1}(z^{-1} + 1)}{(1 - z^{-1})^4} \quad (13)$$

$$Y_3(z) = \frac{z^{-1}(1 + 4z^{-1} + z^{-2})}{(1 - z^{-1})^5} \quad (14)$$

$y(n)$ from the inverse Z-transforms of $Y_1(z)$, $Y_2(z)$, and $Y_3(z)$.

$$y_1(n) = \delta(n - 2) \quad (15)$$

$$y_2(n) = -u(n - 1) + 3u(n - 2) - 6u(n - 3) + 4u(n - 4) \quad (16)$$

$$y_3(n) = -\frac{1}{4}u(n - 1) - \frac{3}{8}u(n - 2) - \frac{1}{4}u(n - 3) + \delta(n - 4) \quad (17)$$

$$9(y_2)^2 = (y_3)(1 + 8(y_1)) \quad (18)$$

$$9(-u(n - 1) + 3u(n - 2) - 6u(n - 3) + 4u(n - 4))^2 = (-\frac{1}{4}u(n - 1) - \frac{3}{8}u(n - 2) - \frac{1}{4}u(n - 3) + \delta(n - 4))(1 + 8(\delta(n - 2))) \quad (19)$$

$$9(-u(n - 1) + 3u(n - 2) - 6u(n - 3) + 4u(n - 4))^2 = 9u(n - 1)^2 - 54u(n - 1)u(n - 2) + 108u(n - 1)u(n - 3) - 72u(n - 1)u(n - 4) + 9u(n - 2)^2 - 54u(n - 2)u(n - 3) + 36u(n - 2)u(n - 4) - 36u(n - 3)^2 + 72u(n - 3)u(n - 4) - 9u(n - 4)^2 \quad (20)$$

$$(-\frac{1}{4}u(n - 1) - \frac{3}{8}u(n - 2) - \frac{1}{4}u(n - 3) + \delta(n - 4))(1 + 8(\delta(n - 2))) = -\frac{1}{4}u(n - 1) - \frac{3}{8}u(n - 2) - \frac{1}{4}u(n - 3) + \delta(n - 4) - \frac{1}{2}u(n - 1)u(n - 2) - \frac{3}{4}u(n - 1)u(n - 3) + \frac{1}{2}u(n - 1)u(n - 4) - \frac{3}{8}u(n - 2)u(n - 3) + \frac{3}{4}u(n - 2)u(n - 4) - \frac{1}{4}u(n - 3)u(n - 4) + \delta(n - 4)u(n - 1) + \delta(n - 4)u(n - 2) + \delta(n - 4)u(n - 3) + \delta(n - 4) \quad (21)$$

the coefficients of the terms on both sides are the same, which means that the identity holds.

$$9(y_2)^2 = (y_3)(1 + 8(y_1))$$