Application Assignment: Filter Design

EE23BTECH11026 - Hiba Muhammed

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1 Introduction

We are supposed to design the equivalent FIR and IIR filter realizations for filter number 114. This is a bandpass filter whose specifications are available below.

2 Filter Specifications

The sampling rate for the filter has been specified as $F_s = 48$ kHz. If the un-normalized discrete-time (natural) frequency is F, the corresponding normalized digital filter (angular) frequency is given by $\omega = 2\pi\left(\frac{F}{F_s}\right)$.

2.1 The Digital Filter

1. *Tolerances:* The passband (δ_1) and stopband (δ_2) tolerances are given to be equal, so we let $\delta_1 = \delta_2 = \delta = 0.15$.

2. *Passband:* The passband of filter is $4 + 0.6j$ kHz to $4 + 0.6(j + 2)$ kHz, where

$$j = (r - 11000)\%\sigma \tag{1}$$
$$r = \text{roll number (last five digits)} \tag{2}$$
$$\sigma = \text{sum of those digits} \tag{3}$$

In this project,

$$r = 11006 \tag{4}$$
$$\text{so } \sigma = 8 \tag{5}$$
$$\text{and } j = 6\%8 = 6 \tag{6}$$

So the passband range for the bandpass filter is from 7.6 kHz to 8.8 KHz. Hence, the un-normalized discrete time filter passband frequencies are

$$F_{p1} = 8.8\text{kHz} \tag{7}$$
$$F_{p2} = 7.6\text{kHz} \tag{8}$$

The corresponding normalized digital filter passband frequencies are

$$\omega_{p1} = 2\pi \frac{F_{p1}}{F_s} = 0.367\pi \tag{9}$$

$$\omega_{p2} = 2\pi \frac{F_{p2}}{F_s} = 0.317\pi \tag{10}$$

The centre frequency is then given by

$$\omega_c = \frac{\omega_{p1} + \omega_{p2}}{2} \tag{11}$$

$$= 0.342\pi \tag{12}$$

3. *Stopband:* The *transition band* for bandpass filters is $\Delta F = 0.3$ kHz on either side of the passband. Hence, the un-normalized *stopband* frequencies are

$$F_{s1} = 8.8 + 0.3 = 9.1\text{kHz} \tag{13}$$

$$F_{s2} = 7.6 - 0.3 = 7.3\text{kHz} \tag{14}$$

The corresponding normalized frequencies are

$$\omega_{s1} = 0.379\pi \tag{15}$$

$$\omega_{s2} = 0.304\pi \tag{16}$$

2.2 The Analog filter

In the bilinear transform, the analog filter frequency (Ω) is related to the corresponding digital filter frequency (ω) as $\Omega = \tan\frac{\omega}{2}$. Using this relation, we obtain the analog passband and stopband frequencies as $\Omega_{p1} = 0.6502$, $\Omega_{p2} = 0.5436$ and $\Omega_{s1} = 0.6773$, $\Omega_{s2} = 0.5175$ respectively.

3 The IIR Filter Design

Filter Type: We are supposed to design filters whose stopband is monotonic and passband equiripple. Hence, we use the *Chebyschev approximation* to design our bandpass IIR filter.

3.1 The Analog Filter

1. *Low Pass Filter Specifications:* If $H_{a,BP}(j\Omega)$ be the desired analog band pass filter, with the specifications provided in Section 2.2, and $H_{a,LP}(j\Omega_L)$ be the equivalent low pass filter, then

$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega} \tag{17}$$

where

$$\Omega_0 = \sqrt{\Omega_{p1}\Omega_{p2}} = 0.5945 \tag{18}$$

$$B = \Omega_{p1} - \Omega_{p2} = 0.1066 \tag{19}$$

$$\tag{20}$$

The low pass filter has the passband edge at $\Omega_{Lp} = 1$ and stopband edges at $\Omega_{Ls_1} = 1.4583$ and $\Omega_{Ls_2} = -1.5525$. We choose the stopband edge of the analog low pass filter as $\Omega_{Ls} = \min(|\Omega_{Ls_1}|, |\Omega_{Ls_2}|) = 1.4583$.

2. *The Low Pass Chebyschev Filter Paramters:* The magnitude squared of the Chebyschev low pass filter is given by

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\Omega_L/\Omega_{Lp})} \tag{21}$$

where $c_N(x)$ is the Chebyshev's polynomial of order N and given as,

$$c_N(x) = \begin{cases} \cosh(N\cosh^{-1}(x)) & \text{for } |x| > 1 \\ \cos(N\cos^{-1}(x)) & \text{for } |x| < 1 \end{cases} \tag{22}$$

$$c_0(x) = 1 \tag{23}$$

$$c_1(x) = x \tag{24}$$

$$c_N(x) = 2xc_{N-1}(x) - c_{N-2}(x) \tag{25}$$

$c_N(x)$ and the integer N, which is the order of the filter, and ϵ are design paramters. Since $\Omega_{Lp} = 1$, (21) may be rewritten as

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\Omega_L)} \tag{26}$$

Also, the design paramters have the following constraints

$$\frac{\sqrt{D_2}}{c_N(\Omega_{Ls})} \le \epsilon \le \sqrt{D_1},$$

$$N \ge \left\lceil \frac{\cosh^{-1}\sqrt{D_2/D_1}}{\cosh^{-1}\Omega_{Ls}} \right\rceil, \tag{27}$$

where

$$D_1 = \frac{1}{(1-\delta)^2} - 1 = 0.3841 \tag{28}$$

$$D_2 = \frac{1}{\delta^2} - 1 = 43.4444 \tag{29}$$

After appropriate substitutions, we obtain

$$N \ge 4 \tag{30}$$

$$0.3268 \le \epsilon \le 0.6197 \tag{31}$$

In Figure 2, we plot $|H(j\Omega_L)|$ for a range of values of ϵ, for $N = 4$. We find that for larger values of ϵ, $|H(j\Omega_L)|$ decreases in the transition band. We choose $\epsilon = 0.4$ for our IIR filter design.

3. *The Low Pass Chebyschev Filter:* Thus, we obtain

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + 0.16c_4^2(\Omega_L)} \tag{32}$$

where

$$c_4(x) = 8x^4 - 8x^2 + 1. \tag{33}$$

The poles of the frequency response in (21) lying in the left half plane are in general obtained as

$$p(k) = -\sinh\phi\sin\phi(k) + j\cosh\phi\cos\phi(k) \tag{34}$$

where

$$\phi = \frac{1}{N}\sinh^{-1}\left(\frac{1}{\epsilon}\right) \tag{35}$$

$$\phi(k) = \frac{(2k+1)}{N}\pi \quad k = 0, \ldots, N-1 \tag{36}$$

The following code generates the poles for $N = 4$, stores it in a .txt file and plots the pole-zero plot in Figure 1,

```
https://github.com/hibamuhd/Filter−Design/blob/main/codes/pole_zero.py
```

And the poles are stored into the following .txt file,

```
https://github.com/hibamuhd/Filter−Design/blob/main/codes/poles.txt
```

Thus, for N even, the low-pass stable Chebyschev filter, with a gain G has the form (Only the poles on the left side of the $j\omega$ axis would be considered to ensure stability of the filter)

$$H_{a,LP}(s_L) = \frac{G_{LP}}{(s_L - p(1))(s_L - p(2))(s_L - p(3))(s_L - p(4))} \tag{37}$$

Substituting $N = 4$, $\epsilon = 0.4$ and $H_{a,LP}(j) = \frac{1}{\sqrt{1+\epsilon^2}}$, we obtain

$$H_{a,LP}(s_L) = \frac{0.3125}{s_L^4 + 1.1068s_L^3 + 1.6125s_L^2 + 0.9140s_L + 0.3366} \tag{38}$$

In Figure 2 we plot $|H(j\Omega)|$ using (32) and (38), thereby verifying that our low-pass Chebyschev filter design meets the specifications.

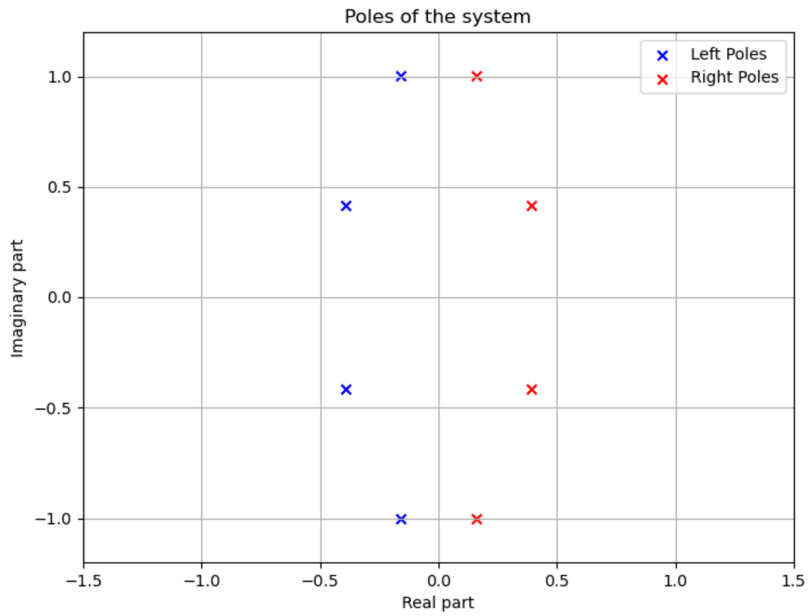

Figure 1: pole-zero plot

4. *The Band Pass Chebyschev Filter:* The analog bandpass filter is obtained from (38) by substituting $s_L = \frac{s^2+\Omega_0^2}{Bs}$. Hence

$$H_{a,BP}(s) = G_{BP} H_{a,LP}(s_L)\Big|_{s_L=\frac{s^2+\Omega_0^2}{Bs}}, \tag{39}$$

where G_{BP} is the gain of the bandpass filter. After appropriate substitutions, and evaluating the gain such that $H_{a,BP}(j\Omega_{p1}) = 1$, we obtain

$$H_{a,BP}(s) = \frac{4.3489 \times 10^{-5} s^4}{s^8 + 0.1179s^7 + 1.4320s^6 + 0.1262s^5 + 0.7625s^4 + 0.0446s^3 + 0.1789s^2 + 0.0052s + 0.0156} \tag{40}$$

Where,

$$G_{BP} = 1.0788 \tag{41}$$

The above substitution is done by the following code,

```
https://github.com/hibamuhd/Filter−Design/blob/main/codes/coeff_analog.
    py
```

And the coefficients are stored into the .txt file,

```
https://github.com/hibamuhd/Filter−Design/blob/main/codes/
    coefficients_analog.txt
```

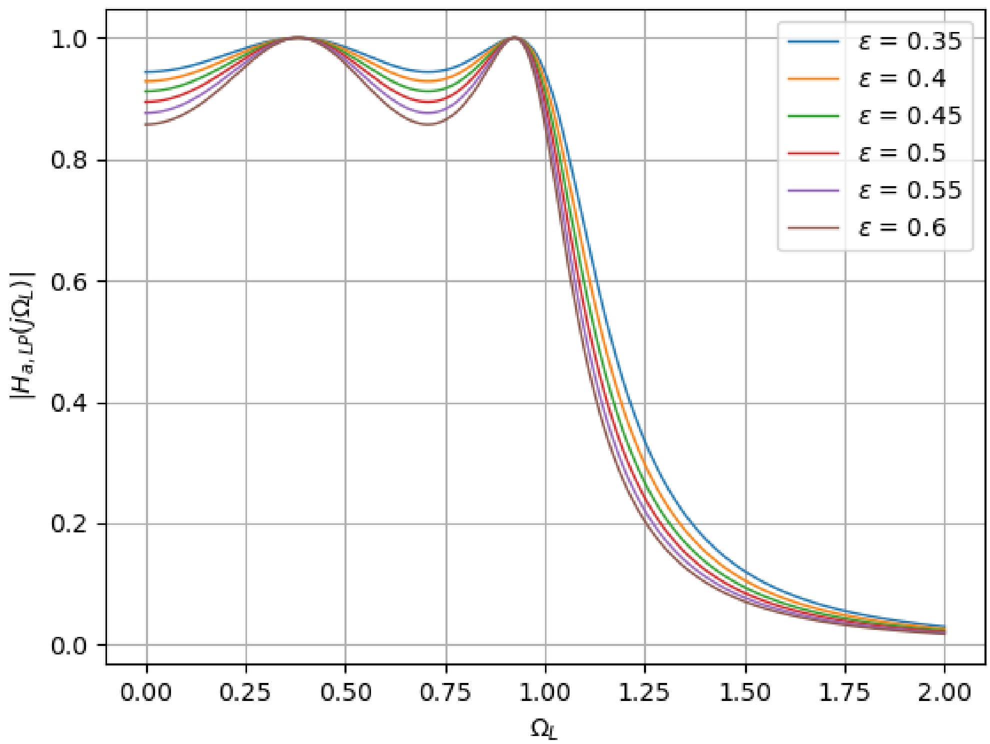

Figure 2: The Analog Low-Pass Frequency Response for $0.35 \leq \epsilon \leq 0.6$

In Figure 3, we plot $|H_{a,BP}(j\Omega)|$ as a function of Ω for both positve as well as negative frequencies. We find that the passband and stopband frequencies in the figure match well with those obtained analytically through the bilinear transformation.

3.2 The Digital Filter

From the bilinear transformation, we obtain the digital bandpass filter from the corresponding analog filter as

$$H_{d,BP}(z) = GH_{a,BP}(s)\big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \tag{42}$$

where G is the gain of the digital filter. From (40) and (42), we obtain

$$H_{d,BP}(z) = G\frac{N(z)}{D(z)} \tag{43}$$

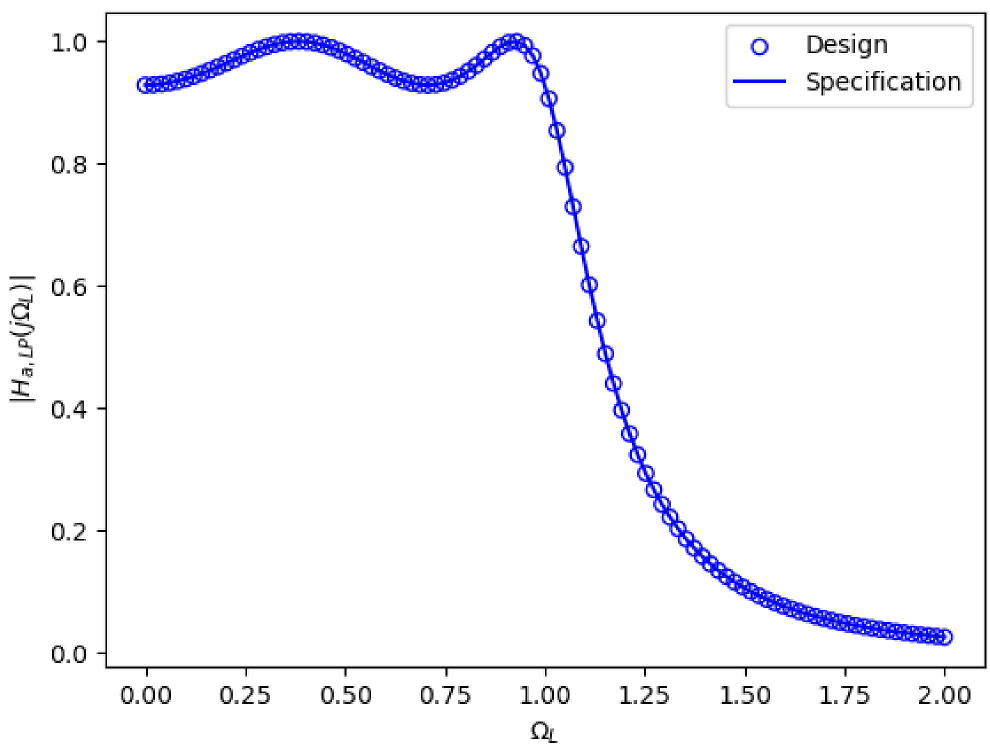

Figure 3: The magnitude response plots from the specifications in Equation 32 and the design in Equation 38

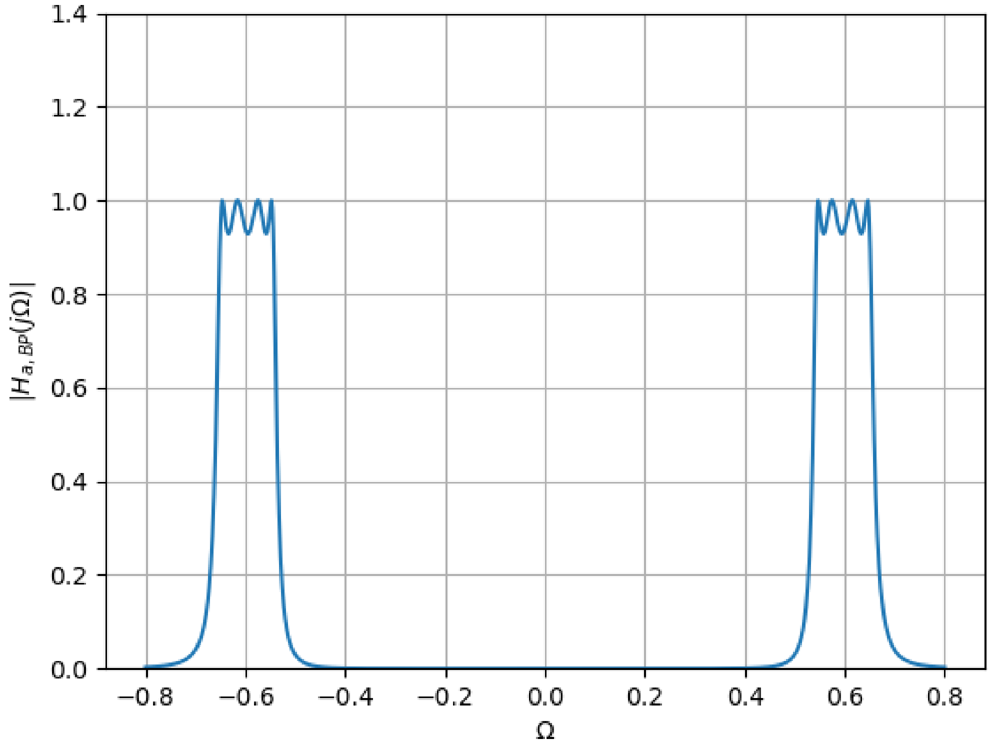

Figure 4: The analog bandpass magnitude response plot from Equation 40

where $G = 4.3489 \times 10^{-5}$,

$$N(z) = 1 - 4z^{-2} + 6z^{-4} - 4z^{-6} + z^{-8} \tag{44}$$

and

$$\begin{aligned} D(z) = 3.6830 - 13.7277z^{-1} + 33.2138z^{-2} - 51.2028z^{-3} + 59.5578z^{-4} \\ -49.0243z^{-5} + 30.4476z^{-6} - 12.0480z^{-7} + 3.0950z^{-8} \end{aligned} \tag{45}$$

Again the substitution is done by the code,

```
https://github.com/hibamuhd/Filter−Design/blob/main/codes/coeff_digital.py
```

And the the coefficients are then stored in this .txt file,

```
https://github.com/hibamuhd/Filter−Design/blob/main/codes/coefficients_digital.
    txt
```

The plot of $|H_{d,BP}(z)|$ with respect to the normalized angular freqency (normalizing factor π) is available in Figure 5. Again we find that the passband and stopband frequencies meet the specifications well enough.

4 The FIR Filter

We design the FIR filter by first obtaining the (non-causal) lowpass equivalent using the Kaiser window and then converting it to a causal bandpass filter.

4.1 The Equivalent Lowpass Filter

The lowpass filter has a passband frequency ω_l and transition band $\Delta\omega = 2\pi\frac{\Delta F}{F_s} = 0.0125\pi$. The stopband tolerance is δ.

1. The *passband frequency* ω_l is defined as $\omega_l = \frac{\omega_{p1}-\omega_{p2}}{2}$. Substituting the values of ω_{p1} and ω_{p2} from section 2.1, we obtain $\omega_l = 0.025\pi$.

2. *The impulse response* $h_{lp}(n)$ of the desired lowpass filter with cutoff frequency ω_l is given by

$$h_l(n) = \frac{\sin(n\omega_l)}{n\pi} w(n), \tag{46}$$

   where $w(n)$ is the Kaiser window obtained from the design specifications.

4.2 The Kaiser Window

The Kaiser window is defined as

$$\begin{aligned} w(n) &= \frac{I_0\left[\beta N\sqrt{1-\left(\frac{n}{N}\right)^2}\right]}{I_0(\beta N)}, \qquad -N \le n \le N, \quad \beta > 0 \\ &= 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise,} \end{aligned} \tag{47}$$

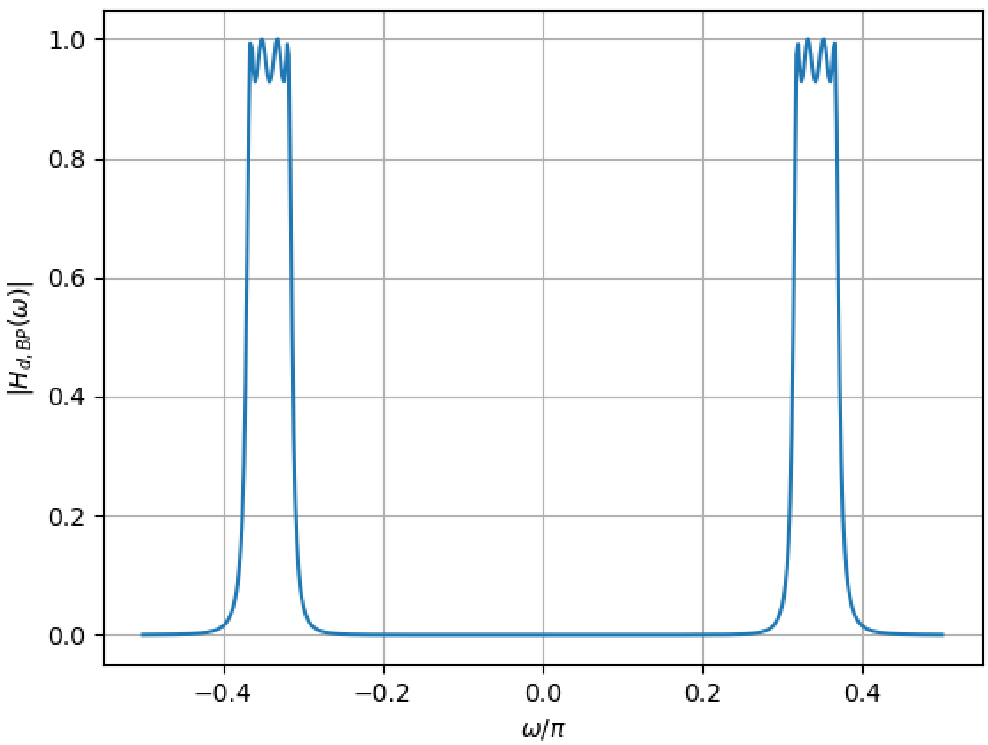

Figure 5: The magnitude response of the bandpass digital filter designed to meet the given specifications

where $I_0(x)$ is the modified Bessel function of the first kind of order zero in x and β and N are the window shaping factors. In the following, we find β and N using the design parameters in section 2.1.

1. N is chosen according to
$$N \geq \frac{A-8}{4.57\Delta\omega}, \tag{48}$$
where $A = -20\log_{10}\delta$. Substituting the appropriate values from the design specifications, we obtain $A = 16.4782$ and $N \geq 48$.

2. β is chosen according to
$$\beta N = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.5849(A-21)^{0.4} + 0.07886(A-21) & 21 \leq A \leq 50 \\ 0 & A < 21 \end{cases} \tag{49}$$
In our design, we have $A = 16.4782 < 21$. Hence, from (49) we obtain $\beta = 0$.

3. We choose $N = 100$, to ensure the desired low pass filter response. Substituting in (47) gives us the rectangular window
$$\begin{aligned} w(n) &= 1, \; -100 \leq n \leq 100 \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{50}$$

From (46) and (50), we obtain the desired lowpass filter impulse response

$$\begin{aligned} h_{lp}(n) &= \frac{\sin(\frac{n\pi}{40})}{n\pi} \qquad -100 \leq n \leq 100 \\ &= 0, \qquad \text{otherwise} \end{aligned} \tag{51}$$

The response of the filter in (51) is shown in Figure 6.

4.3 The FIR Bandpass Filter

The centre of the passband of the desired bandpass filter was found to be $\omega_c = 0.275\pi$ in Section 2.1. The impulse response of the desired bandpass filter is obtained from the impulse response of the corresponding lowpass filter as

$$h_{bp}(n) = 2h_{lp}(n)cos(n\omega_c) \tag{52}$$

Thus, from (51), we obtain

$$\begin{aligned} h_{bp}(n) &= \frac{2\sin(\frac{n\pi}{40})\cos(0.342n\pi)}{n\pi} \qquad -100 \leq n \leq 100 \\ &= 0, \qquad \text{otherwise} \end{aligned} \tag{53}$$

The frequency response of the FIR bandpass filter designed to meet the given specifications is plotted in Figure 7.

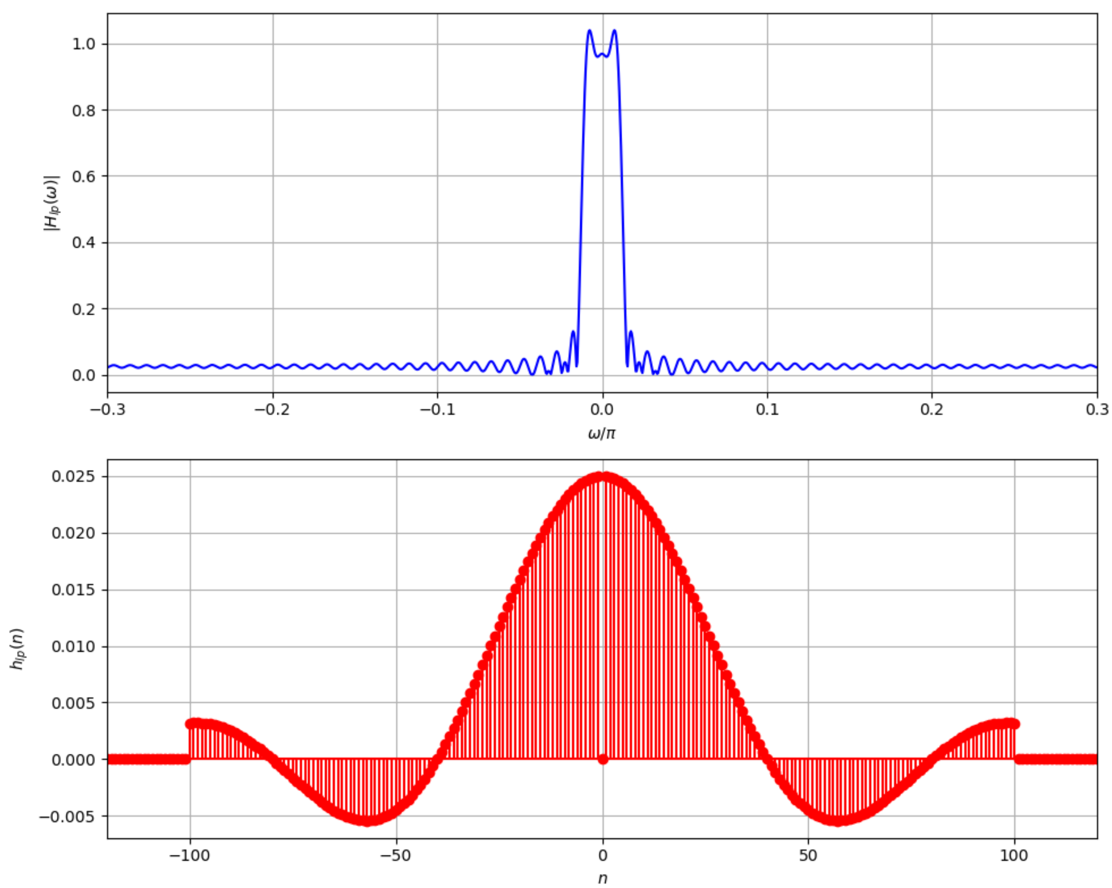

Figure 6: The frequency and the impulse response of the FIR lowpass digital filter designed to meet the given specifications

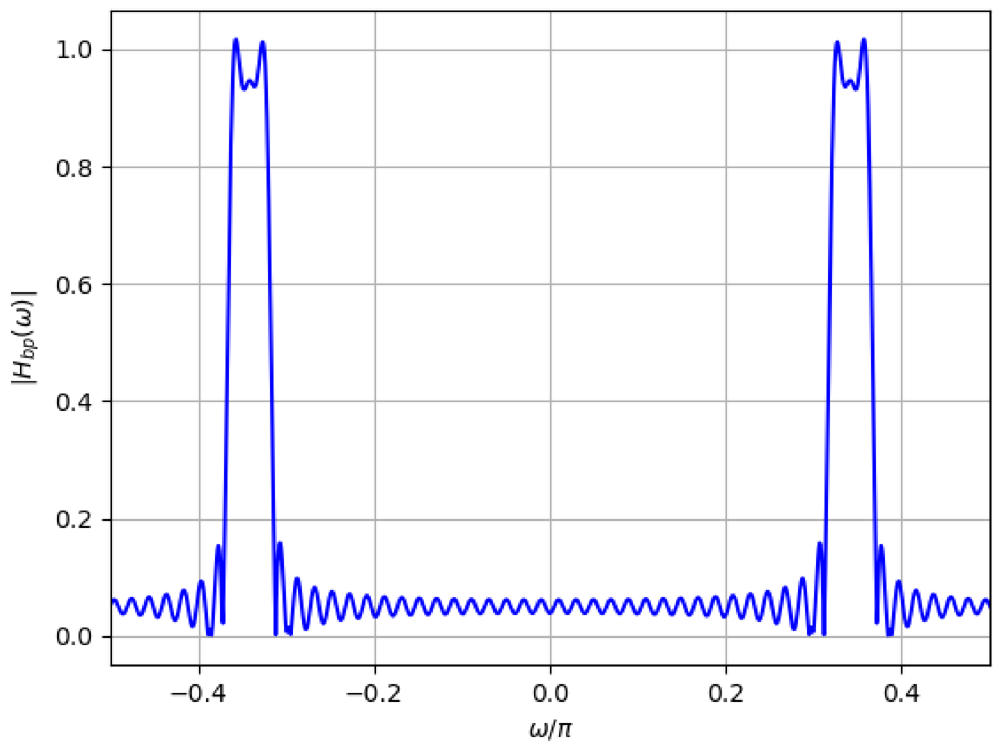

Figure 7: The frequency response of the FIR bandpass digital filter designed to meet the given specifications