Gate 2023 EC 58

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Problem Statement

Let $x_1(t) = u(t+1.5) - u(t-1.5)$ and $x_2(t)$ is shown in the figure below. For $y(t) = x_1(t) * x_2(t)$, the $\int_{-\infty}^{\infty} y(t)\,dt$ is __________.

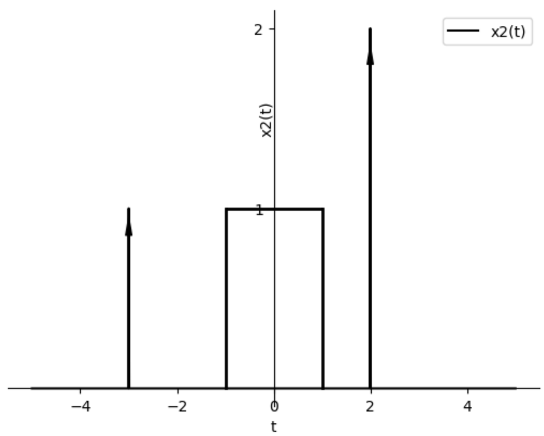

Fig. 1. Figure

Solution

Input Parameters

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Function	**Expression**	**Description**
$x_1(t)$	$u(t+1.5) - u(t-1.5)$	Step function with delay and width parameters.
$X_1(\omega)$		Fourier Transform of $x_1(t)$.
$x_2(t)$	$\delta(t+3) + \text{rect}\left(\frac{t}{2}\right) + 2\delta(t-2)$	Impulse function followed by a rectangle and two impulses.
$X_2(\omega)$		Fourier Transform of $x_2(t)$.

$$x_1(t) = u(t+1.5) - u(t-1.5) \tag{1}$$

$$x_1(t) = \text{rect}\left(\frac{t}{3}\right) \tag{2}$$

$$\text{rect}\left(\frac{t}{a}\right) \xrightarrow{f} f(\omega) = a \times \sin(\omega\frac{a}{2}) \tag{3}$$

$$\text{where } rect\left(\frac{t}{a}\right) = \begin{cases} 1 & \text{if } |t| < \frac{a}{2} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

The Fourier Transform of $rect\left(\frac{t}{a}\right)$:

$$
\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} rect\left(\frac{t}{a}\right) e^{-i\omega t} dt && (5)\\
&= \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-i\omega t} dt && (6)\\
&= \left. \frac{1}{-i\omega} e^{-i\omega t} \right|_{-\frac{a}{2}}^{\frac{a}{2}} && (7)\\
&= \frac{1}{-i\omega}(e^{-i\omega \frac{a}{2}} - e^{i\omega \frac{a}{2}}) && (8)\\
&= \frac{2}{\omega} \sin\left(\frac{\omega a}{2}\right) && (9)\\
&= \boxed{F(\omega) = a \ \text{sinc}\left(\frac{\omega a}{2}\right)} && (10)\\
X_1(\omega) &= 3\text{sinc}(1.5\omega) && (11)\\
x_2(t) &= \delta(t+3) + \text{rect}\left(\frac{t}{2}\right) + 2\delta(t-2) && (12)\\
X_2(\omega) &= e^{3j\omega} + 2\text{sinc}(\omega) + 2e^{-2j\omega} && (13)\\
y(t) &= x_1(t) * x_2(t) && (14)\\
Y(\omega) &= X_1(\omega) \cdot X_2(\omega) && (15)\\
&\text{We know:} && (16)\\
Y(\omega) &= \int_{-\infty}^{\infty} y(t) e^{-j\omega t} dt && (17)\\
\int_{-\infty}^{\infty} y(t) dt &= Y(0) && (18)\\
Y(0) &= X_1(0) \cdot X_2(0) && (19)\\
&= 3(1+2+2) && (20)\\
&= 15 && (21)
\end{aligned}
$$