Gate 2023 EC 58

HIBA MUHAMMED
EE23BTECH11026

Problem Statement

Let $x_1(t) = u(t+1.5) - u(t-1.5)$ and $x_2(t)$ is shown in the figure below. For $y(t) = x_1(t) * x_2(t)$, the $\int_{-\infty}^{\infty} y(t)\,dt$ is ________.

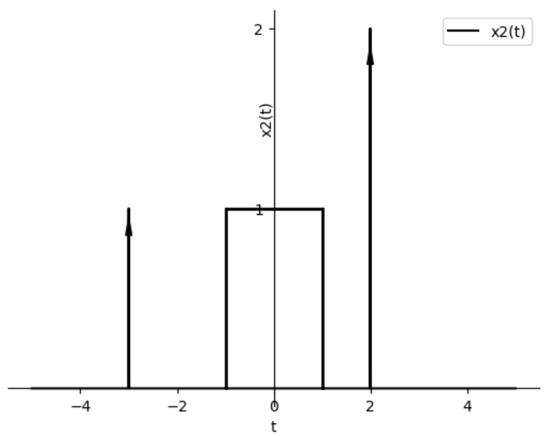

Fig. 1. Figure

Solution

Input Parameters

Input Parameters		
Function	**Expression**	**Description**
$x_1(t)$	$u(t+1.5) - u(t-1.5)$	Step function with delay and width parameters.
$X_1(f)$		Fourier Transform of $x_1(t)$.
$x_2(t)$	$\delta(t+3) + \text{rect}\left(\frac{t}{2}\right) + 2\delta(t-2)$	Impulse function followed by a rectangle and two impulses.
$X_2(f)$		Fourier Transform of $x_2(t)$.

$$x_1(t) = u(t+1.5) - u(t-1.5) \tag{1}$$

$$x_1(t) = \text{rect}\left(\frac{t}{3}\right) \tag{2}$$

$$\text{rect}\left(\frac{t}{a}\right) \overset{\mathcal{F}}{\longleftrightarrow} a \times \sin(2\pi f \frac{a}{2}) \tag{3}$$

$$\text{where rect}\left(\frac{t}{a}\right) = \begin{cases} 1 & \text{if } |t| < \frac{a}{2} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

The Fourier Transform of $rect\left(\frac{t}{a}\right)$:

$$F(f) = \int_{-\infty}^{\infty} \text{rect}\left(\frac{t}{a}\right) e^{-j2\pi f t} dt \tag{5}$$
$$= \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-j2\pi f t} dt \tag{6}$$
$$= \frac{1}{-j2\pi f} e^{-j2\pi f t}\Big|_{-\frac{a}{2}}^{\frac{a}{2}} \tag{7}$$
$$= \frac{1}{-j2\pi f}(e^{-j2\pi f \frac{a}{2}} - e^{j2\pi f \frac{a}{2}}) \tag{8}$$
$$= \frac{2}{2\pi f} \sin\left(\frac{2\pi f a}{2}\right) \tag{9}$$
$$= \boxed{F(f) = a \ \text{sinc}\left(\frac{2\pi f a}{2}\right)} \tag{10}$$
$$X_1(f) = 3\text{sinc}(1.5 \cdot 2\pi f) \tag{11}$$
$$x_2(t) = \delta(t+3) + \text{rect}\left(\frac{t}{2}\right) + 2\delta(t-2) \tag{12}$$
$$X_2(f) = e^{3j \cdot 2\pi f} + 2\text{sinc}(2\pi f) + 2e^{-2j \cdot 2\pi f} \tag{13}$$
$$Y(f) = \int_{-\infty}^{\infty} y(t) e^{-j2\pi f t} dt \tag{14}$$
$$\int_{-\infty}^{\infty} y(t) = Y(0) \tag{15}$$
$$y(t) = x_1(t) * x_2(t) \tag{16}$$
$$y(t) = \int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau)\, d\tau \tag{17}$$
$$= \int_{-\infty}^{\infty} \text{rect}\left(\frac{\tau}{3}\right) \cdot \left(\delta(t+3-\tau) + \text{rect}\left(\frac{t-\tau}{2}\right) + 2\delta(t-\tau-2)\right) d\tau \tag{18}$$
$$= \delta(t+3) + 2\text{rect}\left(\frac{t}{2}\right) + 4\delta(t-2) \tag{19}$$
$$Y(f) = X_1(f) \cdot X_2(f) \tag{20}$$
$$= 3\text{sinc}(1.5 \cdot 2\pi f) \cdot (e^{3j \cdot 2\pi f} + 2\text{sinc}(2\pi f) + 2e^{-2j \cdot 2\pi f}) \tag{21}$$
$$Y(0) = 3\text{sinc}(0) \cdot (e^0 + 2\text{sinc}(0) + 2e^0) \tag{22}$$
$$= 3 \cdot (1 + 2 + 2) \tag{23}$$
$$= 15 \tag{24}$$

Therefore, the value of $\int_{-\infty}^{\infty} y(t)\, dt$ is 15