Gate 2023 EC 58

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Problem Statement

Let $x_1(t) = u(t+1.5) - u(t-1.5)$ and $x_2(t)$ is shown in the figure below. For $y(t) = x_1(t) * x_2(t)$, the $\int_{-\infty}^{\infty} y(t)\,dt$ is __________.

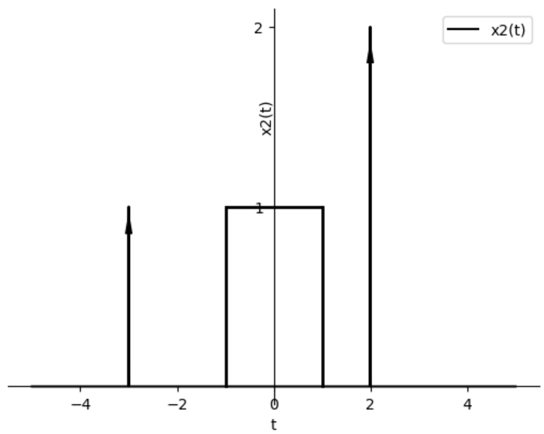

Fig. 1. Figure

Solution

Input Parameters

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Function	**Expression**	**Description**
$x_1(t)$	$u(t+1.5) - u(t-1.5)$	Step function with delay and width parameters.
$X_1(f)$		Fourier Transform of $x_1(t)$.
$x_2(t)$	$\delta(t+3) + \text{rect}\left(\frac{t}{2}\right) + 2\delta(t-2)$	Impulse function followed by a rectangle and two impulses.
$X_2(f)$		Fourier Transform of $x_2(t)$.

$$x_1(t) = u(t+1.5) - u(t-1.5) \tag{1}$$

$$x_1(t) = \text{rect}\left(\frac{t}{3}\right) \tag{2}$$

$$\tag{3}$$

The Fourier Transform of $rect\left(\frac{t}{a}\right)$:

$$
\begin{aligned}
F(f) &= \int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{t}{a}\right) e^{-j2\pi f t} dt && (4)\\
&= \int_{-\frac{a}{2}}^{\frac{a}{2}} e^{-j2\pi f t} dt && (5)\\
&= \frac{1}{-j2\pi f} e^{-j2\pi f t}\Big|_{-\frac{a}{2}}^{\frac{a}{2}} && (6)\\
&= \frac{1}{-j2\pi f}(e^{-j2\pi f \frac{a}{2}} - e^{j2\pi f \frac{a}{2}}) && (7)\\
&= \frac{2}{2\pi f} \sin\left(\frac{2\pi f a}{2}\right) && (8)\\
&= \boxed{F(f) = a \ \mathrm{sinc}\left(\frac{2\pi f a}{2}\right)} && (9)
\end{aligned}
$$

$$
\mathrm{rect}\left(\frac{t}{a}\right) \stackrel{\mathcal{F}}{\leftrightarrow} a \times \sin(2\pi f \frac{a}{2}) \quad (10)
$$

$$
\text{where } \mathrm{rect}\left(\frac{t}{a}\right) = \begin{cases} 1 & \text{if } |t| < \frac{a}{2} \\ 0 & \text{otherwise} \end{cases} \quad (11)
$$

$$
\begin{aligned}
X_1(f) &= 3\mathrm{sinc}(1.5 \cdot 2\pi f) && (12)\\
x_2(t) &= \delta(t+3) + \mathrm{rect}\left(\frac{t}{2}\right) + 2\delta(t-2) && (13)\\
X_2(f) &= e^{3j \cdot 2\pi f} + 2\mathrm{sinc}(2\pi f) + 2e^{-2j \cdot 2\pi f} && (14)\\
y(t) &= x_1(t) * x_2(t) && (15)\\
y(t) &= \int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau)\, d\tau && (16)\\
&= \int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{\tau}{3}\right) \cdot \left(\delta(t+3-\tau) + \mathrm{rect}\left(\frac{t-\tau}{2}\right) + 2\delta(t-\tau-2)\right) d\tau && (17)\\
&= \delta(t+3) + 2\mathrm{rect}\left(\frac{t}{2}\right) + 4\delta(t-2) && (18)\\
Y(f) &= \int_{-\infty}^{\infty} y(t) e^{-j2\pi f t} dt && (19)\\
\int_{-\infty}^{\infty} y(t) &= Y(0) && (20)\\
Y(f) &= X_1(f) \cdot X_2(f) && (21)\\
&= 3\mathrm{sinc}(1.5 \cdot 2\pi f) \cdot (e^{3j \cdot 2\pi f} + 2\mathrm{sinc}(2\pi f) + 2e^{-2j \cdot 2\pi f}) && (22)\\
Y(0) &= 3\mathrm{sinc}(0) \cdot (e^0 + 2\mathrm{sinc}(0) + 2e^0) && (23)\\
&= 3 \cdot (1+2+2) && (24)\\
&= 15 && (25)
\end{aligned}
$$

Therefore, the value of $\int_{-\infty}^{\infty} y(t)\, dt$ is 15