Hiba Nejjari

Optimization and operations research

E3: Application of the Karush-Kuhn-Tucker Conditions

- $f(x,y) = xy$ and $x+y \leq 100$

$x \leq 40$; $x, y \geq 0$

→ Note that the feasible region is bounded, so a global maximum must exist: we write the constraints as $g_1(x,y) = 100 - x - y \geq 0$, $g_2(x,y) = x - 40 \leq 0$ and $g_3(x,y) = -y \leq 0$. So the KKT conditions can be written as:

- $y - \lambda_1 + \lambda_2 = 0$
- $x - \lambda_1 = 0$
- $\lambda_1 (100 - x - y) = 0$
- $\lambda_2 (x - 40) = 0$

with the primal feasibility conditions:

- $x + y \leq 100$
- $x \leq 40$
- $x, y \geq 0$

and the dual feasibility conditions:

$\lambda_1, \lambda_2 \geq 0$

In each of the complementary slackness equations $\lambda_i (b_i - g_i(x,y)) = 0$ at least 1 of the 2 factors must be 0

Case 1: Suppose $\lambda_1 = 0$. The first KKT condition says $y - \lambda_1 + \lambda_2 = y + \lambda_2 = 0$ and the second says $x - \lambda_1 = x = 0$. Since x and y are nonnegative due to the primal feasibility condition $x = y = \lambda_2 = 0$ however since $f(0,0) = 0$ while $f(x,y) > 0$ within the interior of the feasible region, that is not a local maximum.

Case 2: Suppose $\lambda_1 > 0 \iff 100 \geq x+y$ is active.

The KKT condition becomes $x+y=100$. At least x or y must be positive.

Case 2a: Suppose $x>0$. Then $\lambda_2 = 0$ since the second constraint is not active (we have $x \leq 40$, not $x=40$). The first condition simplifies to $y=1$ and the second to $x=1$. But to satisfy $x+y=100$, $x=y=50$ which is outside of our bounds for x. This case does not provide a valid candidate for the local maximum.

Case 2b: Suppose $x=40$. This activates the second constraint. The KKT conditions give us $\lambda_1 = 40$ and $y=60$. Since y must be non negative and $x+y \leq 100$, this is a feasible solution. This case gives us the point $(40, 60)$ which is within the bounds of the feasible region.

Case 3: Considering $x=40$ and $y=0$. This point satisfies all the KKT conditions but would yield an objective ~~value~~ function value of 0, which is not the maximum because $60>0$

Conclusion: The global maximum of the function $f(x,y)=xy$ satisfying all the constraints is at ~~is~~ the point $(40,60)$. (2400)