MPC Cheat Sheet - 2020Fall - Yujie He

Mat.Derivation: vec $\mathbf{a}$; mat $\mathbf{A}$; taking derivative of $\mathbf{x}$ 1) mat & vec product: $\nabla \mathbf{A}\mathbf{x} = \mathbf{A}$; 2) inner product: $\nabla(\mathbf{a}^T\mathbf{x}) = \mathbf{a}$; 3) $\nabla\|\mathbf{x}\|_2^2 = \nabla(\mathbf{x}^T\mathbf{x}) = 2\mathbf{x}$; $\nabla \mathbf{x}^T\mathbf{A}\mathbf{x} = (\mathbf{A}+\mathbf{A}^T)\mathbf{x}$ (symmetry: $\nabla \mathbf{x}^T\mathbf{A}\mathbf{x} = 2\mathbf{A}\mathbf{x}$)

$G(s) = \frac{w^2}{s^2+2\zeta w s + w^2} \to \dot{x} = \begin{bmatrix} -2\zeta\omega & -\omega^2 \\ 1 & 0 \end{bmatrix} x$

Distabce $\sqrt[p]{d^p}$: $p=0$-number of >0; $p=1$: sum of axes ; $p=2$-Euclidean; $p=\infty$-max value (convex $p\ge 1$)
Comp a system is stable (exist Lyapunov function) → eigenvalues in the unit ball → given $A/A+BK$ → $|\lambda I - A| = 0$, get $\{\lambda_i\}$ → $(\lambda I - A)x = 0$, get $\{e_i\}$

System Theory Basics

Models: Continuous-time system: $\dot{x} = A^c x + B^c u$
Solution: $x(t) = e^{A^c(t-t_0)}x_0 + \int_{t_0}^t e^{A^c(t-\tau)}Bu(\tau)d\tau$

Discrete-time system: $x(k+1) = g(x(k), u(k))$.
Discr-time linear system: $x(k+1) = Ax(k) + Bu(k)$.
Analysis: Open-loop $x_{k+1} = Ax_k + Bu_k$, $y = Cx_k$
1) **Stability:** An LTI system is globally asympt. stable $\lim_{k\to\infty} x(k) = 0\ \forall x(0)\in\mathbb{R}^n$ iff $|\lambda_j| < 1\ \forall j = 1,..,n$;
2) **Controllable** $\iff \text{rank}([B\ AB\ \cdots\ A^{n-1}B]) = n$;
$\exists 0$ in control matrix B $\iff$ not ctrl.
3) **Stabilizable** $\Longleftarrow$ iff all of its uncontrollable modes are stable; Controllability implies stabilizability
4) **Observable** iff $\text{rank}([C\ (CA)\ \cdots\ (CA^{n-1})]^T) = n$;
$\exists 0$ in output matrix C $\iff$ not obs.

Unconstrained Optimal Control

Stage cost $l(x,u) = x^TQx + u^TRu$, Q, R pos.definite;
$J^*(x(0)) := \min_u \sum_{i=0}^{N-1} l(x_i,u_i) + x_N^T P x_N$; s.t. $x_{i+1} = Ax_i + Bu_i, x_0 = x$; Set $Q = C^T$ and $R = \rho I$ → $\sum_{i=0}^N \|y_i\|^2 + u_i\rho\|u_i\|^2$ (Large ρ leads to small input energy and weakly controlled)

Bellman recursion/Parametric

Cost function: $V^\star(x_0) := \min_{\mathbf{u}} \sum_{i=0}^N l(x_i,u_i)$ s.t. $x_{i+1} = Ax_i + Bu_i$; Comp DP: 1) Assume PSD $V_{i+1}(x_{i+1}) = x_{i+1}^T H_{i+1} x_{i+1}$; 2) iterate $V(x_i)$ backwards for $N-1$ to 0 given constraint; 3) Setting $\nabla_{u_i} V = 0$; $2(Au_i + Bx_i)^T H_{i+1} B + 2\mathcal{R}u_i = 0$; 4) obtain optimal input $u_i^* = K_i x_i$ until optimal $u_0^*(x_0)$

Conclusion: $V_i^*(x_i)$ is quadratic and positive definite ; Optimizer $u_0^*(x)$ is linear function of current state

Transformed into matrix representation
$V^\star(x_0) := \min_u \mathbf{x}^T\mathcal{Q}\mathbf{x} + \mathbf{u}^\top\mathcal{R}\mathbf{u}$, s.t. $\mathcal{A}\mathbf{x} + \mathcal{B}\mathbf{u} = \mathcal{C}x_0$

$$\begin{bmatrix} -1 & 0 & \cdots & \cdots & 0\\ A & -1 & 0 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots & \vdots \\ \cdots & \cdots & \cdots & A & -1\end{bmatrix}\mathbf{x} + \mathcal{B}\mathbf{u} = \begin{bmatrix}-A\\0\\\vdots\\0\end{bmatrix}x_0$$

where $\mathcal{B} = \text{diag}\{B\}$; $\mathbf{x} = [x_1^T,\dots,x_N^T]$; $\mathbf{u} = [u_0^T,\dots,u_{N-1}^T]$ $\mathcal{Q} = \text{diag}(Q)$; $\mathcal{R} = \text{diag}(R)$
Least-squares solver to obtain $\mathbf{u} = \mathcal{K}x_0$, where $\mathcal{K} = -(\mathcal{R}+F^T\mathcal{Q}F)^{-1}F^T\mathcal{Q}G$ (different from DP)

Infin.Horizon.Ctrl/Stability of LQR

Infinite-horizon will be stable → optimal cost is Lyapunov
Bellman equation: Can find a function V such that $V^*(x) = \min_u l(x,u) + V^*(Ax+Bu)$ so that $V^*(x) = V_\infty^*(x)$; → DT Riccati Equation (DRDE) for Infin.LQR $u = Kx$, $K = (R+B^TPB)^{-1}B^TPA$:
$P = Q + A^TPA - A^TPB(R+B^TPB)^{-1}B^TPA$
Optimal **cost-to-go**: $J_i^\star(x_i) = x_i^\top P_i x_i$.

Lyapunov function: 1) positive$-\{0\}$: $V(0) = 0$, $V(x)\ge 0$; 2) mono.decre. $V(f(x)) - V(x) \le -\alpha(x)$ $\exists V(x) \implies$ asymptotically stable in Ω.
LTI system, then $\iff$ DT Lyapunov equation: $V(x) = x^\top Px$ with $P>0$: $A^\top PA - P = -Q$, $Q>0$
Pro $V^*(x_1) = V^*(x_0) - x_0^T(Q + K^TRK)x_0 < V^*(x_0)$

Optimization

Feasibility and Stability

Feasible set: set of feasible variables z (satisfies the constraints); Optimizer: achieves min.cost $z\in\mathcal{C}$ and $p^* = f(z^*)$; **infeasible** if $\mathcal{C}$ is empty
In order to ensure feasibility and stability, we must introduce $l_f(\cdot)$ and $\mathcal{X}_f$ to mimic an infinite horizon.

Theorem: The closed-loop system under MPC $x(k+1) = Ax(k) + Bu_0^\star(x(k))$ is *recursively feasible* and *asymptotically stable* if:
1. Stage cost is positive definite;
2. Terminal set is invariant under control $\kappa_f(x_i)$ $x_{i+1} = Ax_i + B\kappa_f(x_i)\in\mathcal{X}_f \quad \forall x_i\in\mathcal{X}_f$ All state and input constraints are satisfied in $\mathcal{X}_f$: $\mathcal{X}_f\subset\mathcal{X} \quad \kappa_f(x_i)\in\mathcal{U} \quad \forall x_i\in\mathcal{X}_f$
3. Terminal cost is a Lyapunov function in $\mathcal{X}_f$: $l_f(x_{i+1}) - l_f(x_i) \le -l(x_i,\kappa_f(x_i)) \quad \forall x_i\in\mathcal{X}_f$

Option 1: Choose $P = P_\infty$ solution of LQR control and $\mathcal{X}_f$ maximum invariant set of $x_{i+1} = (A+BF_\infty)x_i$ $\mathcal{X}_f\subseteq\mathcal{X}$; $\quad F_\infty x_i\in\mathcal{U} \quad \forall x_i\in\mathcal{X}_f$
Option 2: $P = A^\top PA + Q$ assuming no control input after horizon (only possible if A is stable).
Under these assumptions, $J^\star(x)$ is Lyapunov function: $J^\star(x(k+1)) - J^\star(x(k)) \le -l(x(k), u^\star(k))$

Convex Sets

$\lambda z_1 + (1-\lambda)z_2 \in S$ for all $z_1,z_2\in S, \lambda\in[0,1]$-convex combination of points inside S are also inside S
Intersection of convex sets is convex. **Union** is not
Hyperplane: $\{x\in\mathbb{R}^n \mid a^\top x = b\}$ (affine and convex)
Halfspace: $\{x\in\mathbb{R}^n\mid a^\top x\le b\}$ open if $<$, closed if $\le$.
Polyhedron: finite intersection of closed halfspaces: $P = \{x\in\mathbb{R}^n \mid a_i^\top x\le b_i\}$
Polytope is **bounded** polyhedron → convex set

Convex Functions & Problems

$f: \text{dom}(f)\to\mathbb{R}$ is **convex** $\iff$ $\text{dom}(f)$ is convex and $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$
$\iff f(y)\ge f(x) + \nabla f(x)^\top(y-x)$ if differentiable
$\iff \nabla^2 f(x)\succeq 0\ \forall x\in\text{dom}(f)$ if twice-differentiable.
A function f is **concave** if $-f$ is convex.

Some operations preserve convexity: non-negative weighted sum, composition with affine function, pointwise maximum/supremum, partial minimization.
Convex Problems: An optimization problem is **convex** if $\min_{x\in\text{dom}(f)} f(x)$; [subj. to $g_i(x)\le 0 \quad i=1,\cdots,m$ $h_i(x) = 0 \quad i = 1,\cdots,p$]; $\text{dom}(f), f, g_i$ are convex, $h_i(x) = a_i^\top x - b$ are affine.

For convex problems, local optima are global optima. A **stricly** convex problem has also a **unique** minimizer. min convex or min concave → convex problem

Linear programs: $f(x) = c^\top x$. The solution can be:
1. unbounded $\implies p^\star = -\infty$
2. bounded and unique $\implies p^\star\in\mathbb{R}$, x_{opt} is a point
3. bounded and multiple $\implies x_{opt}\subseteq\mathbb{R}^s$

QP: $f(x) = \frac12 x^\top Hx + q^\top x(+r)$. If solution exists, it can lie inside the feasible space or on its boundary.

Constrained minimization

Turn constrained problem into unconstrained problem with **Barrier method**: barrier function $\phi(z)$ with **indicator function** I $\phi(z) = \sum_{i=1}^m I_-(g_i(z))$ (keep g(z) neg, if outside feasible set → ∞; but Underivable)

Augmented via log: $\phi(z) = -\sum_{i=1}^m \log(-g_i(z))$;
Comp **gradient**: $\nabla\phi = \sum_{i=1}^m \frac{1}{-g_i(z)}\nabla g_i(z)$; **hessian**: $\nabla^2\phi = \sum_{i=1}^m \frac{1}{g_i(z)^2}\nabla g_i(z)\nabla g_i(z)^T + \frac{1}{-g_i(z)}\nabla^2 g_i(z)$;
Path-following Method-start from analytical center, $\arg\min_z \phi(z)$ decrease during optimization as $\kappa\to 0$, f(z) dominates & reaches opt

Unconstrained minimization

1) **Necessary** condition $f(\cdot)$ differentiable at z^*, a local minimizer → $\nabla f(z^*) = 0$; 2) **Sufficient** condition: $f(\cdot)$ twice differentiable at z^*, Hessian $\nabla^2 f(z^*) > 0$ is positive definite → local minimizer; **Theo.**: with (1,2), if f convex, z^* is global optimizer iff $\nabla f(z^*) = 0$

Descent Methods $z^{(k+1)} = z^{(k)} + t^{(k)}\Delta z^{(k)}$;
1) Descent direction Δz; Gradient descent: $\Delta z := -\nabla f(z)$; Newton method (invert the Hessian): $\Delta z = -\nabla^2 f(z)^{-1}\nabla f(z)$ 2) step size t (Line-search): $t^* = \text{argmin}_{t>0} f(z + t\Delta z)$
Descent direction $\delta z \iff$ overall.cost $f(z^{k+1}) < f(z^k)$ → Pro $\nabla f(z)^T\delta z < 0 \iff \nabla f + \kappa\nabla\phi)^T\delta_{nt} < 0$ (NT)

Barrier Interior-point 1) **Centering step** using Newton's Method: Compute $z^*(\kappa)$ by minimizing $f(z) + \kappa\phi(z)$ starting from z 2) Update $z := z^*(\kappa)$ (repeat) 3) Stopping criterion: Stop if $m_\kappa < \epsilon$
4) Decrease barrier parameter: $\kappa := \mu\kappa$
Comp Centering step Δz_{nt} min 2nd-order approx.: $(\nabla^2 f(z) + \kappa\nabla^2\phi(z))\Delta z_{nt} = -\nabla f(z) - \kappa\nabla\phi(z)$

Unconstrained Control

Invariance

Invariance: Region in which an autonomous system will satisfy the constraints **for all time**; **Controlled invariance**: Region for which there exists a controller so that the system satisfies the constraints for all time
A set $\mathcal{O}$ is **positive invariant** for the autonomous system $x^+ = f(x)$ if $x_i\in\mathcal{O} \implies x_{i+1}\in\mathcal{O}\ \forall k = \{0,1,\dots\}$
The **max.invar.** $\mathcal{O}_\infty\subset\mathbb{X}$ contains all invar. sets $\mathcal{O}$
Pre-Set: $\text{pre}(S) = \{x \mid g(x)\in S\}$ (states evolve in S).
Positive invariant $\mathcal{O} \iff \mathcal{O}\subseteq\text{pre}(\mathcal{O})$ $\iff \text{pre}(\mathcal{O})\cap\mathcal{O} = \mathcal{O}$

Control Invariant Set $\mathcal{C}\subseteq\mathbb{X}$ is $x_i\in\mathcal{C}\Rightarrow\exists u_i\in\mathbb{U}$ such that $x^+ = f(x_i,u_i)\in\mathcal{C}$ for all $i\in\mathbb{N}^+$
Maxi.ctrl.invar. $\mathcal{C}_\infty$ is the largest set for any controller
Pro **If no** state constraints X, we can setting $u=0$ so input constraints are met everywhere → $\mathcal{C}_\infty = \mathbb{R}^2$

Algo.-Compute O_∞/C_∞: input g, $\mathcal{X}$; output $\mathcal{O}_\infty$
1. $\Omega_0\leftarrow\mathcal{X}$
2. **while** $\Omega_i\ne\Omega_{i-1}$ **do**
3. $\quad\Omega_{i+1}\leftarrow \text{pre}(\Omega_i)\cap\Omega_i$
4. $\quad$**if** $\Omega_{i+1} = \Omega_i$ **then**
5. $\quad\quad$**return** $\mathcal{O} = \Omega_i$
6. $\quad$**end**
7. **end**

If $\mathcal{X}$ and $\mathcal{U}$ are boxes, and $x(k+1) = (A+BF)x(k)$ is linear, it is sufficient to check all corner points of $\mathcal{X}$ to prove its invariance.

Represent set Ω_i as Polytopes
1) **inequility** form $P := \{x\mid Ax\le b\}$;
2) **convex hull**: $\text{conv}(S)$ is the smallest convex set containing S. given a set of points $\{v_1,\dots,v_k\}\in\mathbb{R}^d$ (weighted sum of points)

1-D case: $[a,b]\oplus[c,d] = [a+c, b+d]$; higher dim → **Minkowski Sum:** $A\oplus B := \{x+y \mid x\in A, y\in B\}$

Conditions using **inequality**: 1) Input saturation: $u_{lb}\le u\le u^{ub} \to [1\ \ -1]^T u \le [u^{ub}\ \ -u_{lb}]^T$;
2) Rate constraints: $\|x_i - x_{i+1}\|_\infty\le\alpha$ → $\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{pmatrix}x_i\\x_{i+1}\end{pmatrix}\le\mathbf{1}\alpha$; 3) overshoot ≤ 0.1 response to a step of size r: $y\le 1.1r$;
Intersec. $I = S\cap T = \left\{x\mid [C\ D]^T x\le [c\ d]^T\right\}$ (stack)
Pre-set: $S = \{x\mid Fx\le f\}$, $\text{pre}(S) = \{x\mid FAx\le f\}$

MPC & Practical MPC

Main idea: Introduce **terminal cost, constraints** to ensure stability, feasibility to **guarantee valid approx. infin.**
$J^\star(x) = \min_{x,u}\sum_{i=0}^{N-1} x_i^TQx_i + u_i^TRu_i + x_N^TPx_N$; s.t. $x_{i+1} = Ax_i + Bu_i$, $x_0 = x$, $Cx_i + Du_i\le b$, $x_N\in\mathcal{X}_f$

1) Up to time k = N, calculate with constraints; 2) For $k>N$, drop the the constraints → Unconstrained LQR starting from state x_N;

Pro How to define terminal?
1) pos.invar. **stage cost** $Q,R >= 0$; 2) **control law**: first set local law $K_f = 0$, satisfy $Ax + K_f u\in X_f$
3) **set**: All states and input constraints are in X_f as $\mathcal{X}_f\subseteq\mathbb{X}, \kappa_f(x)\in\mathbb{U}$ for all $x\in\mathcal{X}_f$ 4) **cost** satisfy stability condition: $V_f(x^+) - V_f(x)\le -l(x,\kappa_f(x))\ \forall x\in\mathcal{X}_f$;
As local κ_f: $-l(x,\kappa_f(x)) = -x^TQx$, so $x^TA^TQ_fAx - x^TQ_fx\le -x^TQx,\ \forall x\in X_f \iff A^TQ_fA - Q_f\le -Q$ [e.g., We can implement cost as $V_f(x_N) = x_N^TPx_N$, where P from DARE]

Feasible set $\mathcal{X}_N$: the set of initial states x for which the MPC problem with horizon N is feasible;
Recursive feasibility: For all feasible initial states, feasibility is guaranteed along the closed-loop trajectory.
Pro $\exists$ a feasible solution $(x_{0\dots u}, u_{0\dots u})$ at all time instance when starting from a feasible initial point $x_{0(1)}$ and (next step) remain in the constraint set X.
Stability: A pos.invar. X for system containing a neighborhood of the origin in its interior.
Pro Asymptotic Stability: 1) Lyapunov stable; 2) Approaching 0: $\lim_{k\to\infty}\|x_k\| = 0$ for all $x(0)\in\mathcal{X}$

Soft-Constrained MPC

Noise & may infeasible → Enlarging set → Relax state constraints by introducing slack variables ϵ_i and penalize them by adding $\rho(\epsilon_i)=\epsilon_i^T S\epsilon_i$ to the cost.
Same solution (if feasible) in the original problem → Quadratic penalty is added for controllability.

$J^\star_{\text{soft}}(x)\le J^\star(x)$ for all feasible $x\in S$ → if standard MPC feasible, soft-const must be feasible
Increasing s reduces size but longer duration of violation.
Increasing v peak violation ↑ but with shorter duration.

Reference Tracking

The reference $r=Cx_s$ is achieved by state x_s which should be a steady-state: $x_s=Ax_s+Bu_s$.

Target conditions: $\begin{bmatrix} I-A & -B\\ H & 0\end{bmatrix}\begin{bmatrix}x_s\\ u_s\end{bmatrix}=\begin{bmatrix}0\\ r\end{bmatrix}$

- If $n_u=n_r \implies$ unique solution
- If $n_u>n_r \implies \infty$ solutions: find $\min u_s^T R_s u_s$
- If $n_u<n_r \implies$ impossible: $\min Q_s(Hx_s-r)^2$

Delta-formulation: $\Delta x=x-x_s$ and $\Delta u=u-u_s$
$\Delta x_{i+1}=A\Delta x+B\Delta u$; $l=\Delta x_i^T Q\Delta x_i+\Delta u_i^T R\Delta u_i$
$H_x x\le h_x \implies H_x\Delta x\le h_x-H_x x_s$
$H_u u\le h_u \implies H_u\Delta u\le h_u-H_u u_s$

If, in addition to the three conditions stated before, $\{x_s\}\oplus\mathcal{X}_f\subseteq\mathcal{X},\quad K\Delta x+u_s\in\mathcal{U}\quad\forall\Delta x\in\mathcal{X}_f$ then the closed-loop system converges to x_s for $k\to\infty$.
Input to apply: $u^\star(k)=\Delta u_0^\star(x(k))+u_s$.

Constant Disturbance Rejection

Remove offset, converge to desired setpoint
Model: $x(k+1)=A(x(k))+Bu(k)+B_d d(k)$
$d(k+1)=d(k)$, $y(k)=Cx(k)+C_d d(k)$
The augmented system is **observable** $\iff (A,C)$ is observable and rank $\left(\begin{bmatrix}A-I & B_d\\ C & C_d\end{bmatrix}\right)=n_x+n_d$

Linear State Estimator:
$\begin{bmatrix}\hat{x}(k+1)\\ \hat{d}(k+1)\end{bmatrix}=\begin{bmatrix}A & B_d\\ 0 & I\end{bmatrix}\begin{bmatrix}\hat{x}(k)\\ \hat{d}(k)\end{bmatrix}+\begin{bmatrix}B\\ 0\end{bmatrix}u(k)+\begin{bmatrix}L_x\\ L_d\end{bmatrix}\left[-y(k)+C\hat{x}(k)+C_d\hat{d}(k)\right]$
Error dyn.: $\hat{e}_x(k)=x(k)-\hat{x}(k), \hat{e}_d(k)=d(k)-\hat{d}(k)$
$\begin{bmatrix}\hat{e}_x(k+1)\\ \hat{e}_d(k+1)\end{bmatrix}=\left(\begin{bmatrix}A & B_d\\ 0 & I\end{bmatrix}+\begin{bmatrix}L_x\\ L_d\end{bmatrix}\begin{bmatrix}C & C_d\end{bmatrix}\right)\begin{bmatrix}\hat{e}_x(k)\\ \hat{e}_d(k)\end{bmatrix}$
L_x, L_d are linear estimator to achieve $d_s=\hat{d}$

New target conditions:
$\begin{bmatrix}I-A & -B\\ C & 0\end{bmatrix}\begin{bmatrix}x_s\\ u_s\end{bmatrix}=\begin{bmatrix}-B_d\hat{d}\\ r-C\hat{d}\end{bmatrix}$

Procedure: 1) Estimate $\hat{x}$ and $\hat{d}$; 2) Obtain (x_s,u_s) from target conditions using $\hat{d}$; 3) Solve MPC problem with (x_s,u_s) and using $\hat{d}$. If $n_y=n_d$ and the target steady-state is strictly feasible, then the target is achieved with zero offset.

Robust MPC

$x(k+1)=Ax(k)+Bu(k)+w(k),\ w\in\mathcal{W}$ bounded.
A set $\mathcal{O}^{\mathcal{W}}$ is **robust positive invariant** if
$x(k)\in\mathcal{O}^{\mathcal{W}} \implies x(k+1)=g(x(k),w)\in\mathcal{O}^{\mathcal{W}}\ \forall w\in\mathcal{W}$

Robust Pre-Set: Pre-set ∀ values of disturbance:
$\text{pre}^{\mathbb{W}}(S)=\{x\,|\,g(x,w)\in S\quad\forall w\in\mathcal{W}\}$
Given $\Omega=\{x|Fx\le f\}$ and $g(x,w)=Ax+w$,
$\text{pre}^{\mathbb{W}}(\Omega)=\{x|FAx+Fw\le f\ \forall w\in\mathcal{W}\}$
$=\{x|F_iAx\le f_i-\max_w F_iw\ \ \forall i\}=A(\Omega\ominus\mathbb{W})$

Comp robust.invar. $\Omega\cap\text{pre}^{\mathbb{W}}\{\Omega\}$; e.g. $X=[-10.10]$, $|w|<=1$, $x^+=1/2x+w$
$\text{pre}^{\mathbb{W}}(\Omega)=\{x|-10\le 1/2x+w\le 10,\forall w\in[-1,1]\}$
$=\{x|-20-2w\le x\le 20-2w,\forall w\in[-1,1]\}$
$=\{x|18\le x\le 18\}\to\Omega\cap pre=[-10,\ 10]$

A set $\mathcal{O}^{\mathcal{W}}$ is robust pos. inv. $\iff \mathcal{O}^{\mathcal{W}}\subseteq\text{pre}^{\mathcal{W}}(\mathcal{O}^{\mathcal{W}})$
In order to compute it, we can use Algorithm 1.

Robust Open-Loop MPC

Nominal system + offset caused by the disturbance:
$J^\star(x(0)):=\min_u l_f(x_N)+\sum_{i=0}^{N-1}l(x_i,u_i)$; s.t.
$x_{i+1}=Ax_i+Bu_i$; $x_i\in\mathcal{E}=\bigoplus_{k=0}^{i-1}A^k\mathbb{W}$; $u_i\in\mathcal{U}$,
$x_0=x(k)$, $x_N\in\mathcal{X}_f\ominus(\mathcal{W}\oplus A\mathcal{W}\oplus\cdots\oplus A^{N-1}\mathcal{W})$
where $\mathcal{X}_f$ is a robust invariant set for the system
$x(k+1)=(A+BF)x(k)$ for some stabilizing F.
This problem has a very small region of attraction.

Tube-MPC

Separate control authority in two parts: v controls the nominal system $z(k+1)=Ax(k)+Bv(k)$ and another that compensates disturbances: $u_i=K(x_i-z_i)+v_i$.
We fix K offline and optimize the nominal trajectory.
Error dynam.: $e_{i+1}=x_{i+1}-z_{i+1}=(A+BK)e_i+w_i$

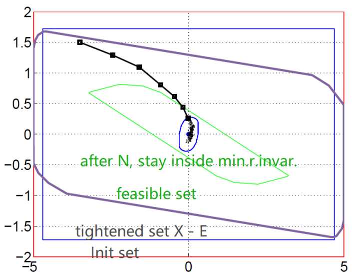
Minimum robust invariant set: $\mathcal{E}=\bigoplus_{k=0}^{i-1}A^k\mathbb{W}$
the smallest set in which the state will remain inside.
Therefore $x_i\in\{z\}_i\oplus\mathcal{E}\subseteq\mathcal{X} \impliedby z_i\in\mathcal{X}\ominus\mathcal{E}$ and $u_i\in K\mathcal{E}\oplus v_i\subset\mathcal{U}\impliedby v_i\in\mathcal{U}\ominus K\mathcal{E}$
Algo.-Compute $\mathcal{E}=F_\infty$: 0) input A, output F_∞; 1) $\Omega_0=\{0\}$; 2) loop and update $\Omega_{i+1}\leftarrow\Omega_i W$ until $\Omega_{i+1}=\Omega_i$ 3) return F_∞

Tube-MPC Formulation:

$$(V^\star(x_0),Z^\star(x_0)):=\arg\min_{V,Z}\ l_f(z_N)+\sum_{i=0}^{N-1}l(z_i,v_i)$$
$$\text{subj. to } z_{i+1}=Az_i+Bv_i\quad\forall i=0,1,\ldots,N-1$$
$$z_i\in\mathcal{X}\ominus\mathcal{E}\qquad v_i\in\mathcal{U}\ominus K\mathcal{E}\quad\forall i$$
$$z_N\in\mathcal{X}_f\subseteq\mathcal{X}\ominus\mathcal{E}\qquad x_0\in z_0\oplus\mathcal{E}$$

And apply $\mu_{\text{tube}}(x):=K(x-z_0^\star(x))+v_0^\star(x)$;
$\lim_{k\to\infty}z_0^\star(x(k))=0$ but $\lim_{k\to\infty}x(k)\in\{0\}\oplus\mathcal{E}$

Assumptions: 1) The stage cost is a positive definite function; 2) The terminal set is invariant for the **nominal** system under the local control law $\kappa_f(z)$:
$Az+B\kappa_f(z)\in\mathcal{X}_f\qquad\forall z\in\mathcal{X}_f$ **Pro** **Comp**
All tightened constraints are satisfied in $\mathcal{X}_f$:
$\mathcal{X}_f\subseteq\mathcal{X}\ominus\mathcal{E};\qquad\kappa_f(z)\in\mathcal{U}\ominus K\mathcal{E}\quad\forall z\in\mathcal{X}_f$
3) Terminal cost is a Lyapunov function in $\mathcal{X}_f$
Pontryagin Diff.: $A\ominus B:=\{x\,|\,x+e\in A,\ \forall e\in B\}$
Property: $A\ominus B\oplus B\subseteq A$ (but not equal)



Offline: 1) Choose **stabilizing K** such that $\|A+BK\|<1$; 2) **Comp** minimal robust invariant set $\mathcal{E}=\bigoplus_{k=0}^{\infty}A^k\mathbb{W}$ for $x^+=(Ax+BK)x$; 3) **Comp** tightened constraints: $\tilde{\mathcal{X}}:=\mathcal{X}\ominus\mathcal{E}$ and $\tilde{\mathcal{U}}:=\mathcal{U}\ominus K\mathcal{E}$; 4) Choose cost and terminal set $\mathcal{X}_f$
Online: 1_ Measure/estimate the state x; 2) Solve the optimization problem; 3) Apply input $u=\mu_{\text{tube}}(x)$

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the system ignoring noise is **Input-to-State Stable**:
Bound that monotonically decreases to $\max\{\|w\||w\in\mathbb{W}\}$ (noise size) → Converges to ≈ 0

Explicit MPC

KKT Conditions for $\min f(z)$. A constraints coeff.; λ introduced variables; conditions for optimality:

1. Stationarity: $\nabla f(z)+A^{\mathsf{T}}\lambda=0$
2. Primal feasibility: $Az\le b$, original constraints
3. Dual feasibility: $\lambda\ge 0$ constraints for λ
4. Complementarity: $\lambda^{\mathsf{T}}(Az-b)=0$

$$f^\star(x)=\min_z\frac{1}{2}z^2+2xz\quad\text{s.t. } z\ge x-1,\quad z\ge 0$$

$\nabla_z\mathcal{L}=z+2x-\lambda-\nu=0$ Stationarity
$x-1-z\le 0,\ z\ge 0$ Primal feasibility
$\lambda,\nu\ge 0$ Dual feasibility
$\lambda(z-x-1)=\nu z=0$ Complementarity

Four complementarity cases:

$\lambda=0\quad z\ge x-1,\ \nu=0\quad z\ge 0\ \to\begin{cases}z^\star(x)&=-2x\\ f^\star(x)&=-2x^2\\ x&\le 0\end{cases}$

$\lambda=0\quad z\ge x-1,\ \nu\ge 0\quad z=0\ \to\begin{cases}z^\star(x)=0\\ f^\star(x)=0\\ 0\le x\le 1\end{cases}$

$\lambda\ge 0\quad z=x-1,\ \nu=0\quad z\ge 0\ \to\begin{cases}z^\star(x)&=x-1\\ f^\star(x)&=\frac{5}{2}x^2-3x+\frac{1}{2}\\ x&\ge 1\end{cases}$

$\lambda\ge 0\quad z=x-1,\ \nu\ge 0\quad z=0\ \to\begin{cases}z^\star(x)&=0\\ f^\star(x)&=0\\ x&=1\end{cases}$

Comp $[v+(-A^T)\lambda-\nabla f(z)=0]$; $[s+A^Tz=b]$
Comp value $f^*(z)$ by setting (v,s), (z,λ), (v,λ) =0
Comp given a q, plot vector q and cheak the cone area from (e_1,e_2) or (e_2,m_1) or (m_1,m_2) or (m_2,e_1) and set others as 0 → obtain by $(M-I)^{-1}q$

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pQP: $J^\star(x):=\min_u\frac{1}{2}u^TQu+(Fx+f)^Tu$, s.t. $Gu\ge Ex+e,\ \ u\ge 0$
→ Parametric Linear Complementarity (**pLCP**):
$Iw-Mz=Qx+q;\ w,z\ge 0, w^Tz=0;$

KKT Conditions:
$Qu+Fx+f-G^T\lambda-\nu=0$ Stationarity
$-s+Gu=Ex+e\ ,\ u\ge 0$ Primary feasibility
$\lambda,\ \nu\ge 0$ Dual feasibility
$\nu^Tu=0\ ,\ \lambda^Ts=0$ Complementarity

Stationarity, Primal feasibility: $\begin{bmatrix}I & 0\\ 0 & I\end{bmatrix}\begin{pmatrix}\nu\\ s\end{pmatrix}-\begin{bmatrix}Q & -G^T\\ G & 0\end{bmatrix}\begin{pmatrix}u\\ \lambda\end{pmatrix}=\begin{bmatrix}F\\ -E\end{bmatrix}x+\begin{bmatrix}f\\ -e\end{bmatrix}$
Primal and dual feasibility $\nu,s,u,\lambda\ge 0$
Complementarity $\nu^Tu=s^T\lambda=0$

Find cone containing q (critical region): define the polyhedral critical region using the solution
$CR(B):=\left\{x\mid A_B^{-1}(q+Qx)\ge 0\right\}$
Convex pLCP → sufficient matrix → 1) unoverlapped cones (unique solution); 2) connected domain (connected neighbour) → Calculate piecewise affine function and online evaluation → Point Location by **sequential** or **logarithmic** search

Nonlinear MPC

Nonlinear system $x_{i+1}=f(x_i,u_i)$ → nonconvex overall cost $\arg\min\sum_{i=0}^{N-1}l(x_i,u_i)+V_f(x_N)$ while same theory and assumption on feasibility and stability.
Challenges: 1) hard to calculate the invariance set → drop the terminal constraints; 2) local minimal

1) Compute nonconvex → descent method; example: Newton's; Gauss-Newton; Sequential QP
2) Discretiziation: $u(t)=u(t_k)\ \forall t\in[t_k,t_{k+1})$
Non-linear: $x(k+1)=x(k)+T_s\cdot g^c(x(k),u(k))$
Naive-Euler: $A=I+T_sA^c$ and $B=T_sB^c$
Exact: $A=e^{A^cT_s}$ and $B=(A^c)^{-1}(A-I)B^c$ **Solution:** lin. comb. of initial state and inputs
$x(k+N)=A^Nx(k)+\sum_{i=0}^{N-1}A^iBu(k+N-1-i)$

Advanced: Direct integration / Collocation
Runge-Kutta (RK)-2 as example $x^+=x(t+h)$
2nd-order Taylor series $x^+=x+h\dot{x}+\frac{h^2}{2}\ddot{x}+\mathcal{O}\left(h^3\right)$
$x^+\approx x+\frac{h}{2}f(x)+\frac{h}{2}f(x+hf(x))=x+h\left(\frac{1}{2}k_1+\frac{1}{2}k_2\right)$
where $k_1=f(x)$ and $k_2=f\left(x+hk_1\right)$

RK4 use higher-order Taylor series → high acc.
$x_{k+1}=x_k+h\left(\frac{k_1}{6}+\frac{k_2}{3}+\frac{k_3}{3}+\frac{k_4}{6}\right)$, where
$k_1=f\left(t_k,x_k\right)$, $k_2=f\left(t_k+\frac{h}{2},x_k+\frac{h}{2}k_1\right)$,
$k_3=f(t_k+\frac{h}{2},x_k+\frac{h}{2}k_2)$, $k_4=f\left(t_k+h,x_k+hk_3\right)$

Remark

- union of a finite set of ellipses **not necessarily** convex
- intersection of an ellipse and a polytope → convex
- **symmetric** $Q=Q^T$ and **nonnegtive eigenvalues** $Q\succeq 0$ (all nonnegtive ×) → guarante optimization prob $\min x^TQx$ with local $x=x^*$ a **global** minimum
- quadratic function $xTPx$ is convex iff P is PSD.

- $J^*\left(x_{k+1}\right)-J^*\left(x_k\right)\le -I\left(x_k,u_0^*\right)$ eusures $J^*(x)$ Lyapunov function → stability
- ∃ subset of pos.invar. ≠ invariant
- **Not** all pos.invar. for system $x^+=f(x)$ can be written as a polyhedron $\{x|Gx\le h\}$
- **intersection** and **union** of two pos.invar. are invar.
- **Convex hull** of pos.invar. is invar. if f **linear**
- max.invar. for system is **union** of given invar.

- ∃ **possible** that N-step sequence $\in\mathbb{X}$ and MPC controller $\in\mathbb{U}$ given x_0 not in max.ctrl.invar. C_∞
- Given x_0 in max.**ctrl**.invar. C_∞ for system $f(x,u)$ with constraints → $\exists u_0\in\mathbb{U}$ that $f(x_0,u_0)\notin\mathbb{X}$
- $\exists x\in\mathbb{X}$ and $\exists u\in\mathbb{U}$ such that $f(x,u)\in C\iff$ **Not** $\exists x\in\mathbb{X}-\{C\}$ and $\exists u\in\mathbb{U}$ such that $f(x,u)\in C$, given same setting before with (X,S)

- x_s always in the interior (boundary ×) of invar.
- **No** slack variable weight ρ ensure soft-constrained $J^\star_{\text{soft}}(\bar{x})=J^\star(\bar{x})$ standard [≈ soft with higher ρ]

- Minimal **robust** invar. $F_\infty>\mathbb{X}\to$ feasible set $\emptyset$
- $S\subseteq\text{pre}(S\ominus W)=pre^{\mathbb{W}}(s)$ is invar. of the **uncertain** system with $w\in\mathbb{W}$
- Bounded disturbance syetem $x^+=Ax+Bu+Ew$, max.**robust** crtl.invar. C_∞ will =↓, =↑, and =↓ after 0.5 $\mathbb{U}$, $\mathbb{W}$, and $\mathbb{X}$

- MPC value function for all x: $J^*(x)$ unconstrained <= constrained infin.hori.ctrl = constrained fin.hori.ctrl w/o terminal <= w.terminal

Credits

Most material was taken from the lecture notes of *Model Predictive Control* given by **Prof. Colin Jones** .
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