EPFL | MGT-418 : Convex Optimization | Tutorial 4

Exercises – Fall 2021

Exercise 1 (**Optimal paper recycling**)

A company produces five types of paper products: (1) newsprint, (2) wall paper, (3) coated paper, (4) uncoated paper, (5) writing paper. For each type j of paper, the demand d_j is given in thousands of tons (see table below). The company meets the demand in two ways. It can either manufacture new paper from fresh wood or it can recycle old paper. A certain percentage m_j of the demand for paper type j must be satisfied with paper produced from fresh wood, but the rest can come from recycling. When paper is recycled, there is some loss of material. In fact, α tons of recycled paper result in $t_j\alpha$ tons of new paper of type j. In the table below, the s_j column indicates the amount of old paper (in thousands of tons) of type j that is available for recycling. Note that a given type of recycled paper can only be used to produce certain types of new paper. This information is given in the last column.

j	d_j	m_j	s_j	t_j	can be used for
1	3475	0%	2000	0.85	1,2
2	1223	47%	1600	0.9	1,2,3,4
3	2260	50%	1000	0.85	2,3,4,5
4	2700	40%	990	0.85	2,3,5
5	2950	30%	2800	0.90	5

Formulate an LP to solve the problem of producing enough of each product to meet the demand while minimizing the use of fresh wood. Implement your LP using YALMIP or CVXPY and solve it with GUROBI. Report the amount of fresh wood used in the optimal solution.

Exercise 2 (**Minimum fuel optimal control**)

Consider a linear dynamical system with state $x_t \in \mathbb{R}^n$, for $t = 0, \ldots, N$, and an actuator or input signal $u_t \in \mathbb{R}$, for $t = 0, \ldots, N-1$. The dynamics of the system is given by the linear recurrence

$$x_{t+1} = Ax_t + bu_t \quad \forall t = 0, \ldots, N-1,$$

with $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ given. We assume that the initial state is zero, *i.e.*, $x_0 = 0$. The *minimum fuel optimal control problem* is to choose inputs $u_0, \ldots, u_{N-1}$ to minimize the total fuel consumed,

$$F(u_0, \ldots, u_{N-1}) = \textstyle\sum_{t=0}^{N-1} f(u_t),$$

subject to the constraint that $x_N = x_{\text{des}}$, where N is the (given) time horizon and $x_{\text{des}} \in \mathbb{R}^n$ is the (given) desired final state. The function $f : \mathbb{R} \to \mathbb{R}$ is the fuel use map for the actuator, and gives the amount of fuel used as a function of the actuator signal amplitude. In this problem we use

$$f(a) = \begin{cases} |a| & \text{if } |a| \le 1 \\ a^2 & \text{if } |a| > 1. \end{cases}$$

This means that fuel usage is proportional to the magnitude of the actuator signal for actuator signals between -1 and 1, while for larger actuator signals the fuel usage grows quadratically in the magnitude of the actuator signal. Formulate the minimum fuel optimal control problem as a QCQP. Implement it using YALMIP or CVXPY and solve it with either GUROBI or MOSEK for the parameter values that are given below. Report the total fuel consumption in the optimal solution. *Hint:* Use epigraphical variables to move nonlinear terms from the objective to the constraints.

$$A = \begin{bmatrix} -1 & 0.4 & 0.8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix}, \qquad x_{\text{des}} = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \qquad N = 10.$$

Exercise 3 (**Minimum time path problem**)

Figure 1 below shows a light particle that moves from the origin to the point $p = (4, 2.5) \in \mathbb{R}^2$ crossing three layers of fluids with different densities. In the first layer, the light particle travels at speed v_1. In the second and in the third layer, the light particle travels at lower speeds $v_2 = v_1/\eta_2$ and $v_3 = v_1/\eta_3$, respectively, where $\eta_2, \eta_3 > 1$. For the sake of this exercise, assume that $v_1 = 1$, $\eta_2 = 1.5$, $\eta_3 = 1.3$. Set up an SOCP to determine the fastest (*i.e.*, minimum time) path from the origin to point p. It turns out that, in reality, light would actually follow this path. Implement your SOCP using YALMIP or CVXPY, and solve it with either GUROBI or MOSEK. *Hint:* Use the break point coordinates p_1, p_2 and the path leg lengths ℓ_1, ℓ_2, ℓ_3 as variables, and argue why, in this problem, equality constraints of the form $\ell_i =$ "something" can be equivalently substituted by inequality constraints $\ell_i \geq$ "something". Report the travel time in the optimal solution.

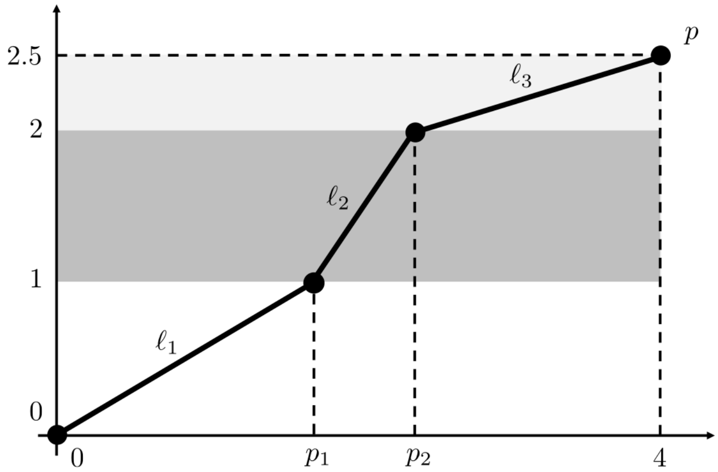

Figure 1: Travel path of a light particle moving across different fluid layers.

Exercise 4 (**Norm approximations via LPs**)

Formulate the following convex optimization problems as LPs. The parameters $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ can be considered as given. Explain in detail the relation between the optimal solution of each original problem and the optimal solution of its equivalent LP.

(a) minimize $\|Ax - b\|_\infty$ (∞-norm approximation)

(b) minimize $\|Ax - b\|_1$ (1-norm approximation)

(c) minimize $\|Ax - b\|_1$ subject to $\|x\|_\infty \leq 1$

(d) minimize $\|x\|_1$ subject to $\|Ax - b\|_\infty \leq 1$

(e) minimize$\|Ax - b\|_1 + \|x\|_\infty$

Exercise 5 (**SOC reformulation of hyperbolic constraints**)

Formulate the *geometric mean maximization* problem given below with variable $x \in \mathbb{R}^n$ and parameters $a_i \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ as an SOCP. *Hint:* Consider first the case $m = 2$. Then, consider general values of m and argue why without loss of generality one can assume that $m = 2^k$ ($k \in \mathbb{N}$). Use epigraphical variables and the second-order cone reformulation of hyperbolic constraints.

$$\begin{array}{ll} \text{maximize} & \left(\prod_{i=1}^m (a_i^\top x - b_i)\right)^{\frac{1}{m}} \\ \text{subject to} & a_i^\top x \geq b_i \quad \forall i = 1, \ldots, m \end{array}$$