EPFL | MGT-418 : Convex Optimization | Tutorial 4

Solutions – Fall 2021

Solution 1 (**Optimal paper recycling**)

We choose our decision variables (with indices $i, j \in \{1, \dots, 5\}$) as described below,

$$\begin{aligned} x_j &= \text{number of tons (in thousands) of freshwood used to produce new paper of type } j, \\ y_{ij} &= \text{number of tons (in thousands) of old paper of type } i \text{ used to produce new paper of type } j. \end{aligned}$$

The objective to be minimized is the total amount of freshwood used, *i.e.*, $\sum_{j=1}^{5} x_j$. There are several requirements that need to be formalized as constraints. First of all, we chose our decision variables to represent nonnegative quantities,

$$x_j \geq 0, \quad y_{ij} \geq 0 \qquad \forall i = 1, \dots, 5 \quad \forall j = 1, \dots, 5.$$

Second, we would like to produce enough paper of each type so as to meet the respective demand,

$$x_j + t_j \textstyle\sum_{i=1}^{5} y_{ij} = d_j \qquad \forall j = 1, \dots, 5.$$

Third, we need to respect the minimum amount of freshwood required in each paper type

$$x_j \geq m_j d_j \qquad \forall j = 1, \dots, 5.$$

Fourth, we can only use as much old paper of each type as is available to us

$$\textstyle\sum_{j=1}^{5} y_{ij} \leq s_i \qquad \forall i = 1, \dots, 5.$$

Fifth, we need to ensure the recycling compatibility of old and new paper. We do this by ensuring that $y_{ij} = 0$ for all incompatible combinations of old paper type i and new paper type j.

$$y_{13} = y_{14} = y_{15} = 0, \quad y_{25} = 0, \quad y_{31} = 0, \quad y_{41} = y_{44} = 0, \quad y_{51} = y_{52} = y_{53} = y_{54} = 0.$$

In summary, we obtain the following LP formulation of the optimal paper recycling problem

$$\begin{array}{lll} \text{minimize} & \sum_{j=1}^{5} x_j & \\ \text{subject to} & x_j \geq 0, \ y_{ij} \geq 0 & \forall i = 1, \dots, 5 \quad \forall j = 1, \dots, 5, \\ & x_j + t_j \sum_{i=1}^{5} y_{ij} = d_j & \forall j = 1, \dots, 5, \\ & x_j \geq m_j d_j & \forall j = 1, \dots, 5, \\ & \sum_{j=1}^{5} y_{ij} \leq s_i & \forall i = 1, \dots, 5, \\ & y_{13} = y_{14} = y_{15} = 0, \ y_{25} = 0, \ y_{31} = 0, & \\ & y_{41} = y_{44} = 0, \ y_{51} = y_{52} = y_{53} = y_{54} = 0. & \end{array}$$

The MATLAB script *sol04_ex1.m* implements this LP using YALMIP and solves it with GUROBI. The solutions needs 5755.5 tons of fresh wood.

Solution 2 (**Minimum fuel optimal control**)

The decision variables of this problem are the system states $x_t \in \mathbb{R}^n$, for $t = 0, \dots, N$, and the input signals $u_t \in \mathbb{R}$, for $t = 0, \dots, N-1$. The objective to be minimized is the total fuel consumption $F(u_0, \dots, u_{N-1}) = \sum_{t=0}^{N-1} f(u_t)$. In terms of constraints, we need to impose the system dynamics

$x_{t+1} = Ax_t + bu_t$ for $t = 0, \dots, N-1$, the initial state $x_0 = 0$ and the desired final state $x_N = x_{\text{des}}$. A direct formulation of the problem might thus look as follows

$$\begin{array}{ll} \text{minimize} & \sum_{t=0}^{N-1} f(u_t) \\ \text{subject to} & x_{t+1} = Ax_t + bu_t \qquad \forall t = 0, \dots, N-1 \\ & x_0 = 0,\ x_N = x_{\text{des}}. \end{array}$$

This is already a convex optimization problem. To see it, note that $f(a)$ is a convex function because it is the pointwise maximum of $|a|$ and a^2, which are both convex functions. The objective is therefore convex because it is a (non-negative weighted) sum of convex functions. Furthermore, the feasible set is a polyhedron and hence convex. Still, the above formulation is not yet a QCQP. Following the hint, we thus introduce epigraphical variables $z_t \in \mathbb{R}$, for $t = 0, \dots, N-1$, so that $z_t \geq f(u_t)$. The latter inequality is equivalent to the following two constraints

$$z_t \geq |u_t| \iff -z_t \leq u_t \leq z_t \qquad \text{and} \qquad z_t \geq u_t^2.$$

With this, the minimum fuel optimal control problem can be cast as the following QCQP,

$$\begin{array}{lll} \text{minimize} & \sum_{t=0}^{N-1} z_t & \\ \text{subject to} & x_{t+1} = Ax_t + bu_t & \forall t = 0, \dots, N-1 \\ & -z_t \leq u_t \leq z_t,\ z_t \geq u_t^2 & \forall t = 0, \dots, N-1 \\ & x_0 = 0,\ x_N = x_{\text{des}}. & \end{array}$$

The MATLAB script *sol04_ex2.m* implements this QCQP using YALMIP and solves it with GUROBI. The total fuel consumption is 70.5.

Solution 3 (**Minimum time path problem**)

Following the hint, we choose decision variables $p_1, p_2 \in \mathbb{R}$ to denote the breakpoint coordinates and decision variables $\ell_1, \ell_2, \ell_3 \in \mathbb{R}$ to denote the respective length of the various path legs. The objective to be minimized is the total travel time of the light particle, which amounts to

$$\frac{\ell_1}{v_1} + \frac{\ell_2}{v_2} + \frac{\ell_3}{v_3} = \frac{1}{v_1}\ell_1 + \frac{\eta_2}{v_1}\ell_2 + \frac{\eta_3}{v_1}\ell_3.$$

In terms of constraints, we need to ensure that ℓ_1, ℓ_2 and ℓ_3 indeed represent the length of the various path legs. One natural way of achieving this would be to impose the equality constraints

$$\ell_1 = \left\| \begin{pmatrix} p_1 \\ 1 \end{pmatrix} \right\|_2, \quad \ell_2 = \left\| \begin{pmatrix} p_2 - p_1 \\ 1 \end{pmatrix} \right\|_2, \quad \ell_3 = \left\| \begin{pmatrix} 4 - p_2 \\ 0.5 \end{pmatrix} \right\|_2.$$

However, these equality constraints are not convex as they do not define a convex feasible set. Luckily, there is a workaround. Because all coefficients in the objective are positive, the minimization will push ℓ_1, ℓ_2 and ℓ_3 to take values as small as possible. Instead of constraining the exact values of ℓ_1, ℓ_2 and ℓ_3, it is therefore sufficient to impose appropriate lower bounds on ℓ_1, ℓ_2 and ℓ_3 and count on the fact that, at optimality, these lower bounds will be binding. Thanks to this reasoning, we can replace the (non-convex) equality constraints with (convex) *in*equality constraints and obtain the SOCP

$$\begin{array}{ll} \text{minimize} & \frac{1}{v_1}\ell_1 + \frac{\eta_2}{v_1}\ell_2 + \frac{\eta_3}{v_1}\ell_3 \\ \text{subject to} & \ell_1 \geq \left\| \begin{pmatrix} p_1 \\ 1 \end{pmatrix} \right\|_2, \quad \ell_2 \geq \left\| \begin{pmatrix} p_2 - p_1 \\ 1 \end{pmatrix} \right\|_2, \quad \ell_3 \geq \left\| \begin{pmatrix} 4 - p_2 \\ 0.5 \end{pmatrix} \right\|_2. \end{array}$$

The MATLAB script *sol04_ex3.m* implements this SOCP using YALMIP and solves it with MOSEK. The travel time is 5.72.

Solution 4 (**Norm approximations via LPs**)

Remark: Throughout this solution, the notation $\mathbf{1}$ denotes the appropriately sized vector of all ones.

(a) Introduce an auxiliary variable $t \in \mathbb{R}$ and reformulate as shown below. At optimality, the bounds on $Ax-b$ are as tight as possible. The rigid structure of $t\mathbf{1}$, however, does not allow the bounds to adapt element-wise to the entries of $Ax-b$. Hence, the solution $(x^\star, t^\star)$ of the reformulation and the solution $x^\star_{\text{orig}}$ of the original problem are related through $x^\star = x^\star_{\text{orig}}$ and $t^\star = \|Ax^\star_{\text{orig}} - b\|_\infty$.

$$\begin{array}{llcll} \underset{x}{\text{minimize}} & \|Ax-b\|_\infty & \iff & \underset{x,\,t}{\text{minimize}} & t \\ & & & \text{subject to} & -t\mathbf{1} \leq Ax-b \leq t\mathbf{1} \end{array}$$

(b) Introduce an auxiliary variable $y \in \mathbb{R}^m$ and proceed as shown below. At optimality, the bounds on $Ax-b$ are again as tight as possible. But now, the flexible structure of y allows the bounds to adapt element-wise to the entries of $Ax-b$. Thus, the solution $(x^\star, y^\star)$ of the reformulation and the solution $x^\star_{\text{orig}}$ of the original problem are related through $x^\star = x^\star_{\text{orig}}$ and $y^\star_i = |(Ax^\star_{\text{orig}} - b)_i|$.

$$\begin{array}{llcll} \underset{x}{\text{minimize}} & \|Ax-b\|_1 & \iff & \underset{x,\,y}{\text{minimize}} & \mathbf{1}^\top y \\ & & & \text{subject to} & -y \leq Ax-b \leq y \end{array}$$

The discussed techniques to reformulate expressions containing $\|\cdot\|_\infty$ and $\|\cdot\|_1$ in a linear fashion are very useful and can be applied in more general settings. The parts below provide examples for this.

(c) Use an auxiliary variable $y \in \mathbb{R}^m$ and proceed as below. The solution $(x^\star, y^\star)$ of the reformulation and the solution $x^\star_{\text{orig}}$ of the original problem are related via $x^\star = x^\star_{\text{orig}}$ and $y^\star_i = |(Ax^\star_{\text{orig}} - b)_i|$.

$$\begin{array}{llcll} \underset{x}{\text{minimize}} & \|Ax-b\|_1 & \iff & \underset{x,\,y}{\text{minimize}} & \mathbf{1}^\top y \\ \text{subject to} & \|x\|_\infty \leq 1 & & \text{subject to} & -y \leq Ax-b \leq y \\ & & & & -\mathbf{1} \leq x \leq \mathbf{1} \end{array}$$

(d) Use an auxiliary variable $y \in \mathbb{R}^m$ and proceed as below. The solution $(x^\star, y^\star)$ of the reformulation and the solution $x^\star_{\text{orig}}$ of the original problem are related through $x^\star = x^\star_{\text{orig}}$ and $y^\star_i = |(x^\star_{\text{orig}})_i|$.

$$\begin{array}{llcll} \underset{x}{\text{minimize}} & \|x\|_1 & \iff & \underset{x,\,y}{\text{minimize}} & \mathbf{1}^\top y \\ \text{subject to} & \|Ax-b\|_\infty \leq 1 & & \text{subject to} & -y \leq x \leq y \\ & & & & -\mathbf{1} \leq Ax-b \leq \mathbf{1} \end{array}$$

(e) Introduce auxiliary variables $y \in \mathbb{R}^m$ and $t \in \mathbb{R}$ to respectively linearize $\|Ax-b\|_1$ and $\|x\|_\infty$. Then, reformulate as below. The solution $(x^\star, y^\star, t^\star)$ of the reformulation and the solution $x^\star_{\text{orig}}$ of the original problem are related through $x^\star = x^\star_{\text{orig}}$, $y^\star_i = |(Ax^\star_{\text{orig}} - b)_i|$ and $t^\star = \|x^\star_{\text{orig}}\|_\infty$.

$$\begin{array}{llcll} \underset{x}{\text{minimize}} & \|Ax-b\|_1 + \|x\|_\infty & \iff & \underset{x,\,y,\,t}{\text{minimize}} & \mathbf{1}^\top y + t \\ & & & \text{subject to} & -y \leq Ax-b \leq y \\ & & & & -t\mathbf{1} \leq x \leq t\mathbf{1} \end{array}$$

Solution 5 (**SOC reformulation of hyperbolic constraints**)

Recall that a hyperbolic constraint in $x \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$ can be cast as a SOC constraint through

$$\|x\|_2^2 \leq st,\ \ s \geq 0,\ \ t \geq 0 \quad \Longleftrightarrow \quad \left\| \begin{pmatrix} 2x \\ s-t \end{pmatrix} \right\|_2 \leq s + t. \tag{$\star$}$$

Introducing auxiliary variables $z_i := a_i^\top x - b_i$, the geometric mean maximization problem becomes

$$\begin{array}{lll} \text{maximize} & \left(\prod_{i=1}^m z_i\right)^{\frac{1}{m}} & \\ \text{subject to} & z_i = a_i^\top x - b_i & \forall i = 1, \ldots, m \\ & z_i \geq 0 & \forall i = 1, \ldots, m. \end{array}$$

For $m = 2$, the objective function reads $\sqrt{z_1 z_2}$. We can thus introduce a *reversed* epigraphical variable $\overline{z} \geq 0$ so that $\overline{z}^2 \leq z_1 z_2$ (this is usually referred to as a *hypographical* variable) to obtain the equivalent reformulation below. Applying the transformation ($\star$) to the last constraint yields the desired SOCP.

$$\begin{array}{lll} \text{maximize} & \overline{z} & \\ \text{subject to} & z_i = a_i^\top x - b_i & \forall i = 1, 2 \\ & z_i \geq 0,\ \overline{z} \geq 0 & \forall i = 1, 2 \\ & \overline{z}^2 \leq z_1 z_2. & \end{array}$$

For general m, one can assume w.l.o.g. that $m = 2^k$ ($k \in \mathbb{N}$). If $2^{k-1} < m < 2^k$, factor constant terms $a_i^\top x - b_i$ with $a_i = 0$, $b_i = -1$ into the objective product so as to reach the next higher power of 2,

$$\prod_{i=1}^m (a_i^\top x - b_i) = \prod_{i=1}^m (a_i^\top x - b_i) \cdot \prod_{i=m+1}^{2^k} (a_i^\top x - b_i) \quad \text{where} \quad a_i = 0,\ b_i = -1\ \forall i = m+1, \ldots, 2^k.$$

This leaves the objective product unchanged but modifies the overall objective from $(\prod_{i=1}^m (a_i^\top x - b_i))^{\frac{1}{m}}$ to $(\prod_{i=1}^{2^k} (a_i^\top x - b_i))^{\frac{1}{2^k}} = (\prod_{i=1}^m (a_i^\top x - b_i))^{\frac{1}{2^k}} = ((\prod_{i=1}^m (a_i^\top x - b_i))^{\frac{1}{m}})^{\frac{m}{2^k}}$. The two objectives are related through a power function with exponent $\frac{m}{2^k} > 0$. As this function is monotonically increasing on $\mathbb{R}_+$, the original and the modified problem share the same maximizer and are thus equivalent. Following the same idea as before, we can then introduce hypographical variables $\overline{z}_j \geq 0$, $j = 1, \ldots, 2^{k-1}$, and gather the terms being multiplied in the objective pairwise with hyperbolic constraints

$$\begin{array}{lll} \text{maximize} & \left(\prod_{j=1}^{2^{k-1}} \overline{z}_j\right)^{\frac{1}{2^{k-1}}} & \\ \text{subject to} & z_i = a_i^\top x - b_i & \forall i = 1, \ldots, 2^k \\ & z_i \geq 0 & \forall i = 1, \ldots, 2^k \\ & \overline{z}_j \geq 0 & \forall j = 1, \ldots, 2^{k-1} \\ & \overline{z}_j^2 \leq z_{2j-1} z_{2j} & \forall j = 1, \ldots, 2^{k-1}. \end{array}$$

Note that this procedure reduced the number of terms being multiplied in the objective by half. Now, introducing new hypographical variables $\overline{\overline{z}}_\ell \geq 0$, $\ell = 1, \ldots, 2^{k-2}$, and repeating the procedure yields

$$\begin{array}{lll} \text{maximize} & \left(\prod_{\ell=1}^{2^{k-2}} \overline{\overline{z}}_\ell\right)^{\frac{1}{2^{k-2}}} & \\ \text{subject to} & z_i = a_i^\top x - b_i & \forall i = 1, \ldots, 2^k \\ & z_i \geq 0 & \forall i = 1, \ldots, 2^k \\ & \overline{z}_j \geq 0 & \forall j = 1, \ldots, 2^{k-1} \\ & \overline{\overline{z}}_\ell \geq 0 & \forall \ell = 1, \ldots, 2^{k-2} \\ & \overline{z}_j^2 \leq z_{2j-1} z_{2j} & \forall j = 1, \ldots, 2^{k-1} \\ & \overline{\overline{z}}_\ell^2 \leq \overline{z}_{2\ell-1} \overline{z}_{2\ell} & \forall \ell = 1, \ldots, 2^{k-2}. \end{array}$$

Iterating this procedure k times results in an equivalent reformulation with only one linear term in the objective and $2^k - 1 \in \mathcal{O}(m)$ hyperbolic constraints (polynomial growth in the problem input m). Applying the transformation ($\star$) to each hyperbolic constraint finally produces the desired SOCP.