Part3-Divide & Conquer Problem

Completed on	@2020/03/15
Key videos	https://www.coursera.org/learn/algorithms-divide-conquer/lecture/IUiUk/o-n-log-n-algorithm-for-counting-inversions-ii https://www.coursera.org/learn/algorithms-divide-conquer/lecture/YXmYz/strassens-subcubic-matrix-multiplication-algorithm https://www.coursera.org/learn/algorithms-divide-conquer/lecture/nf0jk/o-n-log-n-algorithm-for-closest-pair-i-advanced-optional https://www.coursera.org/learn/algorithms-divide-conquer/lecture/cER7y/o-n-log-n-algorithm-for-closest-pair-ii-advanced-optional
Note	In-depth analysis to three typical divide-and-conquer algorithms with lower complexity, i.e., Counting Inversions, Strassen's Matrix Multiplication Algorithm, and Algorithm for Closest Pair.

Note

▼ O(n log n) Algorithm for Counting Inversions I

- 基本问题：计算一个无序数组中逆序对的数量 ⇒ 应用于相似性度量（similarity measure between two ranked lists）

  如：[1, 3, 5, 2, 4, 6]有[3,2], [5,2], [5,4]三对

  Input : array A containing the numbers 1,2,3,..,n in some arbitrary order

  Output : number of inversions = number of pairs (i,j) of array indices with i<j and A[i] > A[j]

- 解法：暴力 or 分治

  暴力算法，双层循环（double loop）

Brute-force : $\theta(n^2)$ time

Can we do better ? Yes!

KEY IDEA # 1 : Divide + Conquer

- 分治算法，按照之前MergeSort的方法，对于N个数，分治算法的复杂度为O(logn)，因此只需保证其中的算法复杂度为O(n)即可低于暴力算法！

```
Count (array A, length n)
    if n=1, return 0
    else
        X = Count (1st half of A, n/2)
        Y = Count (2nd half of A, n/2_
        Z = CountSplitInv(A,n)    CURRENTLY UNIMPLEMENTED
    return x+y+z
```

Goal : implement CountSplitInv in linear (O(n)) time then count will run in O(nlog(n)) time [just like Merge Sort]

O(n log n) Algorithm for Counting Inversions II

- 进一步优化，受到MergeSort启发，进行排序并计数

```
Sort-and-Count (array A, length n)
if n=1, return 0
else
Sorted version of 1st half    (B,X) = Sort-and-Count(1st half of A, n/2)
Sorted version of 2nd half    (C,Y) = Sort-and-Count(2nd half of A, n/2)
Sorted version of A           (D,Z) = CountSplitInv(A,n)    CURRENTLY UNIMPLEMENTED
return X+Y+Z
```

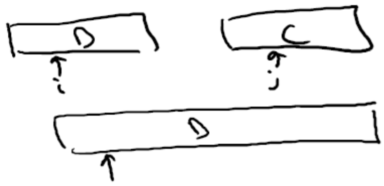
Goal : implement CountSplitInv in linear (O(n)) time

=> then Count will run in O(nlog(n)) time [just like Merge Sort]

- 算法实现：

Pseudocode for Merge:

```
D = output [length = n]
B = 1st sorted array [n/2]
C = 2nd sorted array [n/2]
i = 1
j = 1
```

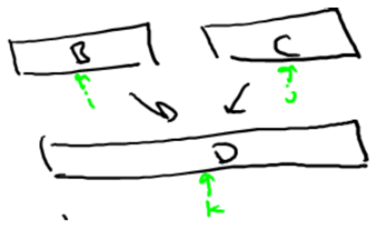



```
for k = 1 to n
        if B(i) < C(j)
                D(k) = B(i)
                i++      count = count
                         相当于符合顺序不增加!
        else [C(j) < B(i)]
                D(k) = C(j)
                j++      count = count+[(n/2)-j]
                         相当于增加了未排序逆序的个数
end
```

(ignores end cases)

-- while merging the two sorted subarrays, keep running total of number of split inversions

-- when element of 2^{nd} array C gets copied to output D, increment total by number of elements remaining in 1^{st} array B



merge running total

Run time of subroutine : O(n) + O(n) = O(n)

=> Sort_and_Count runs in O(nlog(n)) time [just like Merge Sort]

▼ Strassen's Subcubic Matrix Multiplication Algorithm

- 一般乘法：输入两个NxN的矩阵，输出结果

  考虑最简单的2×2矩阵，输出2×2个元素，每个元素有2次乘法得到结果，最终算法复杂度：

$$T(n) = \mathcal{O}(n^{\log_2 8}) = \mathcal{O}(n^3)$$

(all n X n matrices)

Where z_{ij} = (i^{th} row of X) . (j^{th} column of Y)

$$= \sum_{k=1}^{n} X_{ik} \cdot Y_{kj}$$

Note : input size = $\theta(n^2)$

- 分治思想：Strassen's Algorithm相比于一般乘法算法，中间仅需要7次乘法运算！

Strassen’s Algorithm (1969)

Step 1 : recursively compute only 7 (cleverly chosen) products

Step 2 : do the necessary (clever) additions + subtractions (still $\theta(n^2)$ time)

Fact : better than cubic time!

- 算法解析（来源wiki）

将A, B, C分成相等大小的方块矩阵：

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{1,1} & \mathbf{A}_{1,2} \\ \mathbf{A}_{2,1} & \mathbf{A}_{2,2} \end{bmatrix}, \mathbf{B} = \begin{bmatrix} \mathbf{B}_{1,1} & \mathbf{B}_{1,2} \\ \mathbf{B}_{2,1} & \mathbf{B}_{2,2} \end{bmatrix}, \mathbf{C} = \begin{bmatrix} \mathbf{C}_{1,1} & \mathbf{C}_{1,2} \\ \mathbf{C}_{2,1} & \mathbf{C}_{2,2} \end{bmatrix}$$

即

$$\mathbf{A}_{i,j}, \mathbf{B}_{i,j}, \mathbf{C}_{i,j} \in F^{2^{n-1} \times 2^{n-1}}$$

于是

$$\begin{aligned} \mathbf{C}_{1,1} &= \mathbf{A}_{1,1}\mathbf{B}_{1,1} + \mathbf{A}_{1,2}\mathbf{B}_{2,1} \\ \mathbf{C}_{1,2} &= \mathbf{A}_{1,1}\mathbf{B}_{1,2} + \mathbf{A}_{1,2}\mathbf{B}_{2,2} \\ \mathbf{C}_{2,1} &= \mathbf{A}_{2,1}\mathbf{B}_{1,1} + \mathbf{A}_{2,2}\mathbf{B}_{2,1} \\ \mathbf{C}_{2,2} &= \mathbf{A}_{2,1}\mathbf{B}_{1,2} + \mathbf{A}_{2,2}\mathbf{B}_{2,2} \end{aligned}$$

引入新矩阵标记M，是由A、B矩阵元素加减的乘积组合

引入新矩阵

$$
\begin{aligned}
\mathbf{M}_1 &:= (\mathbf{A}_{1,1} + \mathbf{A}_{2,2})(\mathbf{B}_{1,1} + \mathbf{B}_{2,2}) \\
\mathbf{M}_2 &:= (\mathbf{A}_{2,1} + \mathbf{A}_{2,2})\mathbf{B}_{1,1} \\
\mathbf{M}_3 &:= \mathbf{A}_{1,1}(\mathbf{B}_{1,2} - \mathbf{B}_{2,2}) \\
\mathbf{M}_4 &:= \mathbf{A}_{2,2}(\mathbf{B}_{2,1} - \mathbf{B}_{1,1}) \\
\mathbf{M}_5 &:= (\mathbf{A}_{1,1} + \mathbf{A}_{1,2})\mathbf{B}_{2,2} \\
\mathbf{M}_6 &:= (\mathbf{A}_{2,1} - \mathbf{A}_{1,1})(\mathbf{B}_{1,1} + \mathbf{B}_{1,2}) \\
\mathbf{M}_7 &:= (\mathbf{A}_{1,2} - \mathbf{A}_{2,2})(\mathbf{B}_{2,1} + \mathbf{B}_{2,2})
\end{aligned}
$$

可得:

$$
\begin{aligned}
\mathbf{C}_{1,1} &= \mathbf{M}_1 + \mathbf{M}_4 - \mathbf{M}_5 + \mathbf{M}_7 \\
\mathbf{C}_{1,2} &= \mathbf{M}_3 + \mathbf{M}_5 \\
\mathbf{C}_{2,1} &= \mathbf{M}_2 + \mathbf{M}_4 \\
\mathbf{C}_{2,2} &= \mathbf{M}_1 - \mathbf{M}_2 + \mathbf{M}_3 + \mathbf{M}_6
\end{aligned}
$$

最终，输出矩阵的四个元素可以由七个矩阵M加减组合得到！

▼ O(n log n) Algorithm for Closest Pair I [Advanced - Optional]

- 问题描述与解法分析

The Closest Pair Problem

Input : a set $P = \{p_1, \dots, p_n\}$ of n points in the plane R^2.

Notation : $d(p_i, p_j)$ = Euclidean distance

So if $p_i = (x_i, y_i)$ and $p_j = (x_j, y_j)$

$$d(p_i, p_j) = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}$$

Output : a pair $p*, q* \in P$ of distinct points that minimize d(p,q) over p,q in the set P

- 解法分析

Initial Observations

Assumption : (for convenience) all points have distinct x-coordinates, distinct y-coordinates.

Brute-force search : takes $\theta(n^2)$ time.

1-D Version of Closest Pair :

1. Sort points (O(nlog(n)) time)
2. Return closest pair of adjacent points (O(n) time)

暴力解法：双层循环嵌套O(n2)

一维解法：先排序O(nlogn)，然后计算相邻元素距离O(n)

- 二维算法步骤与分析

分别对x与y排序，但是仍然尚不足以解决问题！⇒ 需要额外的合并函数符合O(n)复杂度！

ClosestPair(P_x, P_y)

1. Let Q = left half of P, R = right half of P. Form Q_x, Q_y, R_x, R_y [takes O(n) time]
2. (p_1,q_1) = ClosestPair(Q_x,Q_y)
3. (p_2,q_2) = ClosestPair(R_x,R_y)
4. Let $\delta = min\{d(p_1,q_1), d(p_2,q_2)\}$
5. (p_3,q_3) = ClosestSplitPair(P_x,P_y,δ)
6. Return best of (p_1,q_1), (p_2,q_2), (p_3,q_3)

BASE CASE OMITTED

WILL DESCRIBE NEXT

Requirements
1. O(n) time
2. Correct whenever closest pair of P is a split pair

中间增加了步骤4，提前计算子问题中的最小值

- 伪代码分析：

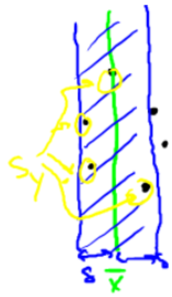
然后找到集合的中心，然后结合之前计算的\delta，可以排除掉中间带区域内的点，并按照y排序的集合strip Sy

然后对于每个点，搜寻y坐标比他大的7个点即可，复杂度控制在O(n)

ClosestSplitPair(P_x, P_y, δ)

Let $\bar{x}$ = biggest x-coordinate in left of P. (O(1) time)

Let S_y = points of P with x-coordinate in
Sorted by y-coordinate (O(n) time)



Initialize best = δ, best pair = NULL
For i = 1 to $|S_y|$ - 7
  For j = 1 to 7
    Let p,q = i^{th}, $(i+j)^{th}$ points of S_y
    If d(p,q) < best
      best pair = (p,q), best = d(p,q)

O(1) time

O(n) time

At end return best pair

▼ O(n log n) Algorithm for Closest Pair II [Advanced - Optional]

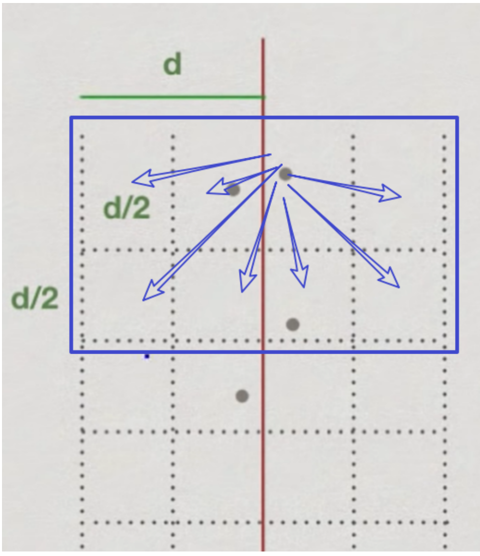
- 两大假设：

Claim : Let $p \in Q, q \in R$ be a split pair with d(p,q) < δ

Then: (A) p and q are members of S_y

(B) p and q are at most 7 positions apart in S_y.

- 尤其是第二点，为什么要计算至多七个位置的点（由于从小到大进行遍历，因此不需要周围除本身的16个方块）



References

- Strassen's Subcubic Matrix Multiplication Algorithm

[施特拉森演算法](https://zh.wikipedia.org/wiki/%E6%96%BD%E7%89%B9%E6%8B%89%E6%A3%AE%E6%BC%94%E7%AE%97%E6%B3%95)
https://zh.wikipedia.org/wiki/%E6%96%BD%E7%89%B9%E6%8B%89%E6%A3%AE%E6%BC%94%E7%AE%97%E6%B3%95

[Strassen algorithm](https://en.wikipedia.org/wiki/Strassen_algorithm)
https://en.wikipedia.org/wiki/Strassen_algorithm

- 乘法算法

[Matrix multiplication algorithm](https://en.wikipedia.org/wiki/Matrix_multiplication_algorithm)
https://en.wikipedia.org/wiki/Matrix_multiplication_algorithm

- closest pair

[Closest Pair of Points | Divide and Conquer | GeeksforGeeks](https://youtu.be/0W_m46Q4qMc)
https://youtu.be/0W_m46Q4qMc

[Closest pair of points](https://www.youtube.com/watch?v=6UBDkbVhJck)
https://www.youtube.com/watch?v=6UBDkbVhJck

- 分治法与暴力法求最近的二维点对

[Closest Pair Implemetation Python](https://stackoverflow.com/questions/28237581/closest-pair-implemetation-python)
https://stackoverflow.com/questions/28237581/closest-pair-implemetation-python

[二维空间最近点对问题 python_Python_荒谬小孩-CSDN博客](https://blog.csdn.net/wangkai0011/article/details/80518314)
https://blog.csdn.net/wangkai0011/article/details/80518314

[空间最小距离点对--python_Python_woshilsh的博客-CSDN博客](https://blog.csdn.net/woshilsh/article/details/89956482)
https://blog.csdn.net/woshilsh/article/details/89956482

[python动态演示分治法解决最近对问题](https://www.cnblogs.com/whitehawk/p/10853875.html)
https://www.cnblogs.com/whitehawk/p/10853875.html