Industrial Computer Vision

- Geometric Camera Models

11th lecture, 2022.11.23

Lecturer: Youngbae Hwang

Contents

- Pinhole camera model vs. Lens camera model
- Camera Projection Matrix
- Geometric Camera Calibration
- Radial Distortion

Let’s say we have a sensor…

digital sensor
(CCD or CMOS)

… and an object we like to photograph

What would an image taken like this look like?

Bare-sensor imaging

real-world
object

digital sensor
(CCD or CMOS)

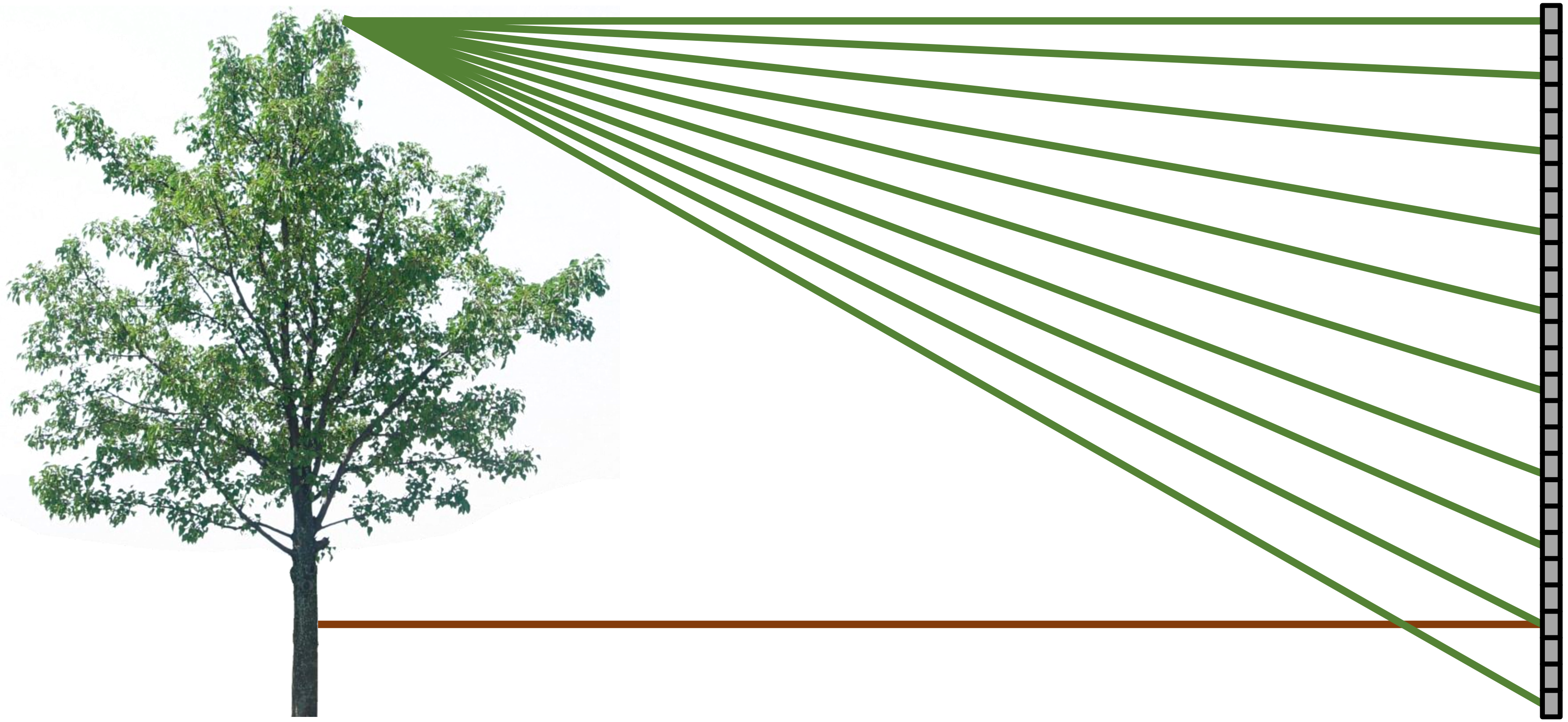
Bare-sensor imaging

real-world
object

digital sensor
(CCD or CMOS)

Bare-sensor imaging

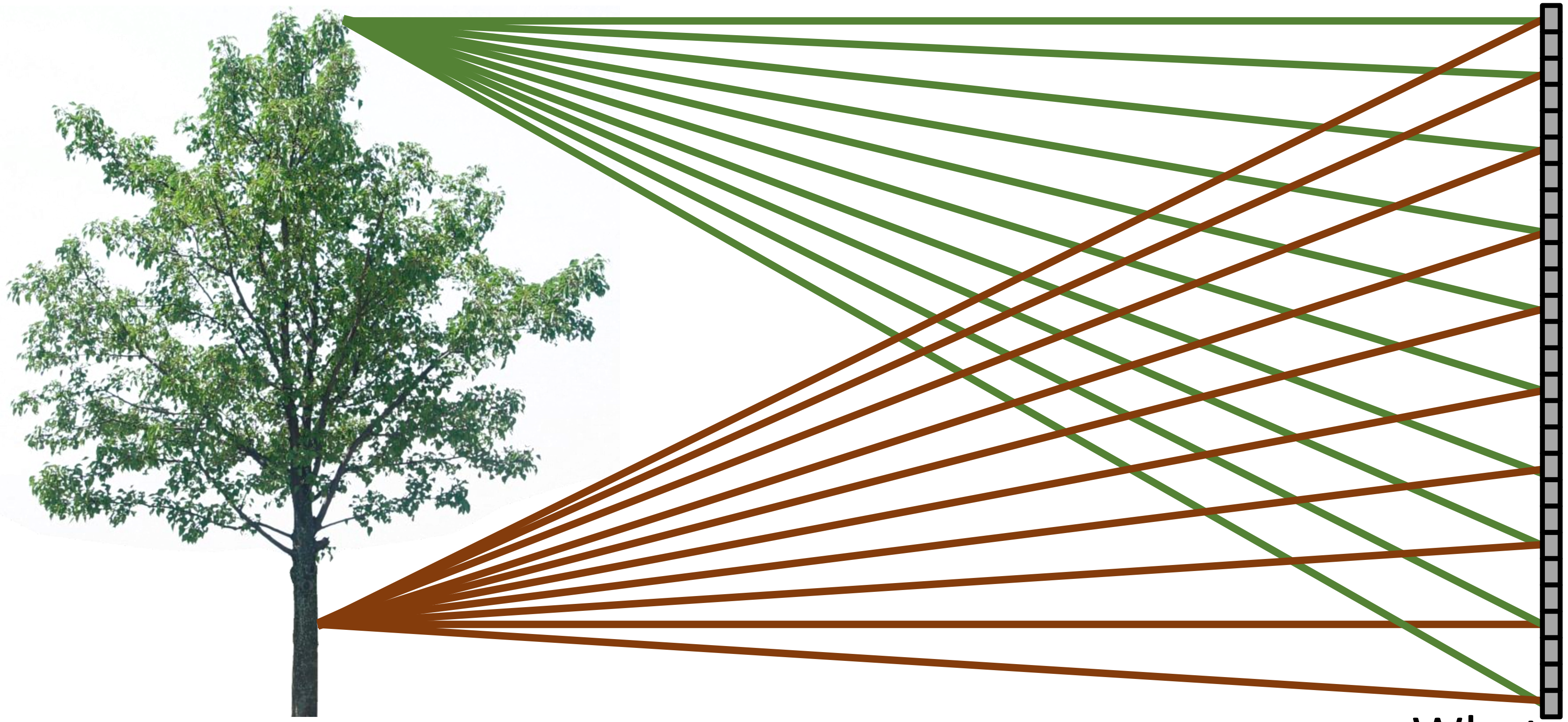
real-world object



digital sensor (CCD or CMOS)

Bare-sensor imaging

real-world object



digital sensor (CCD or CMOS)

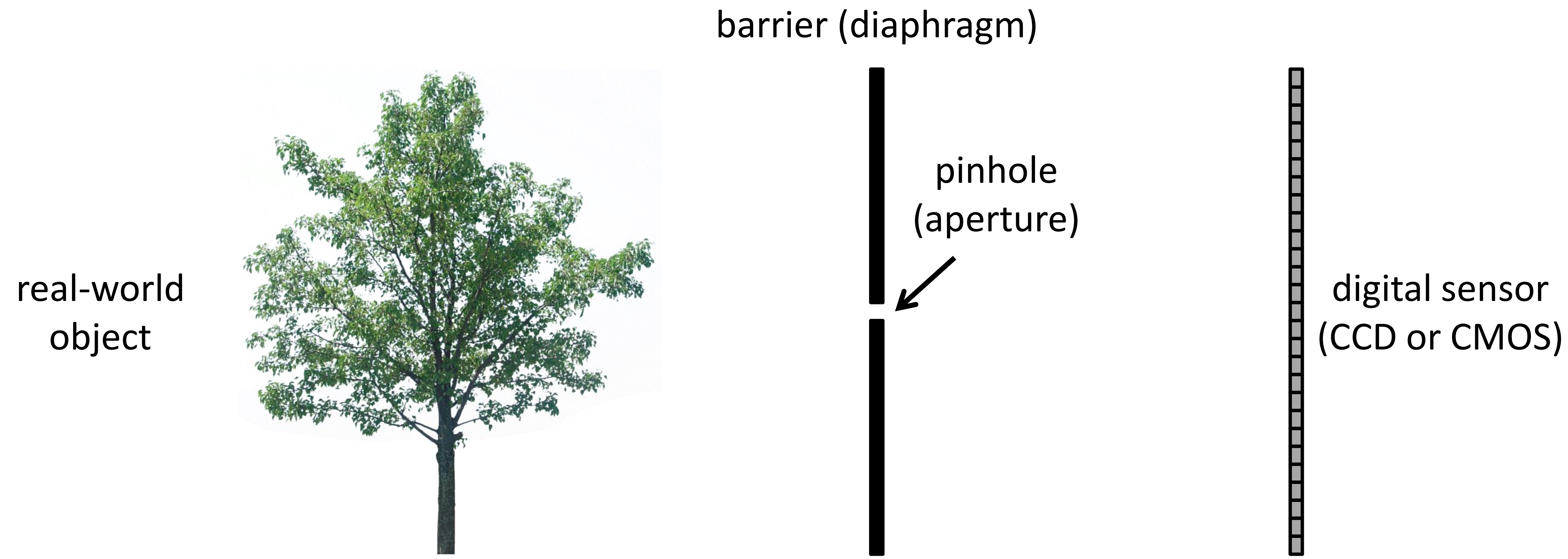
What does the image on the sensor look like?

All scene points contribute to all sensor pixels

Bare-sensor imaging

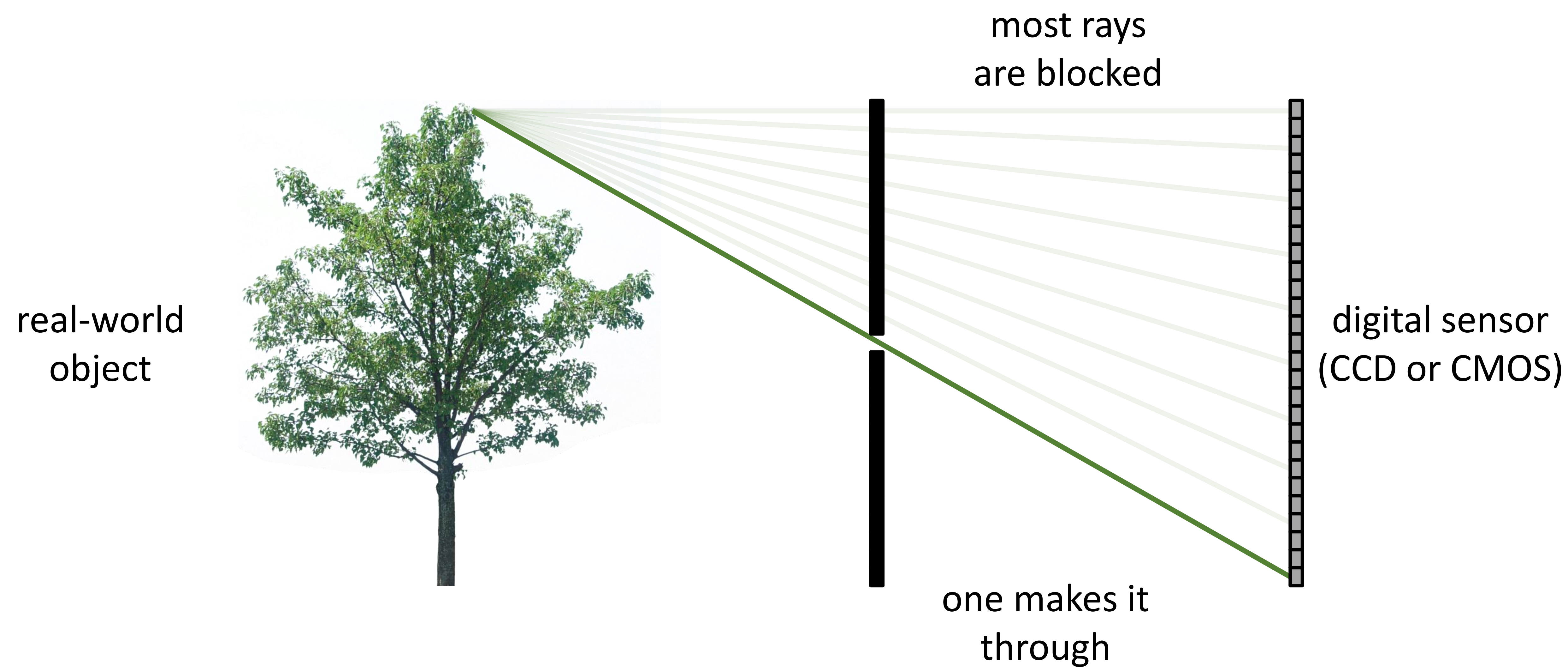
All scene points contribute to all sensor pixels

Let’s add something to this scene

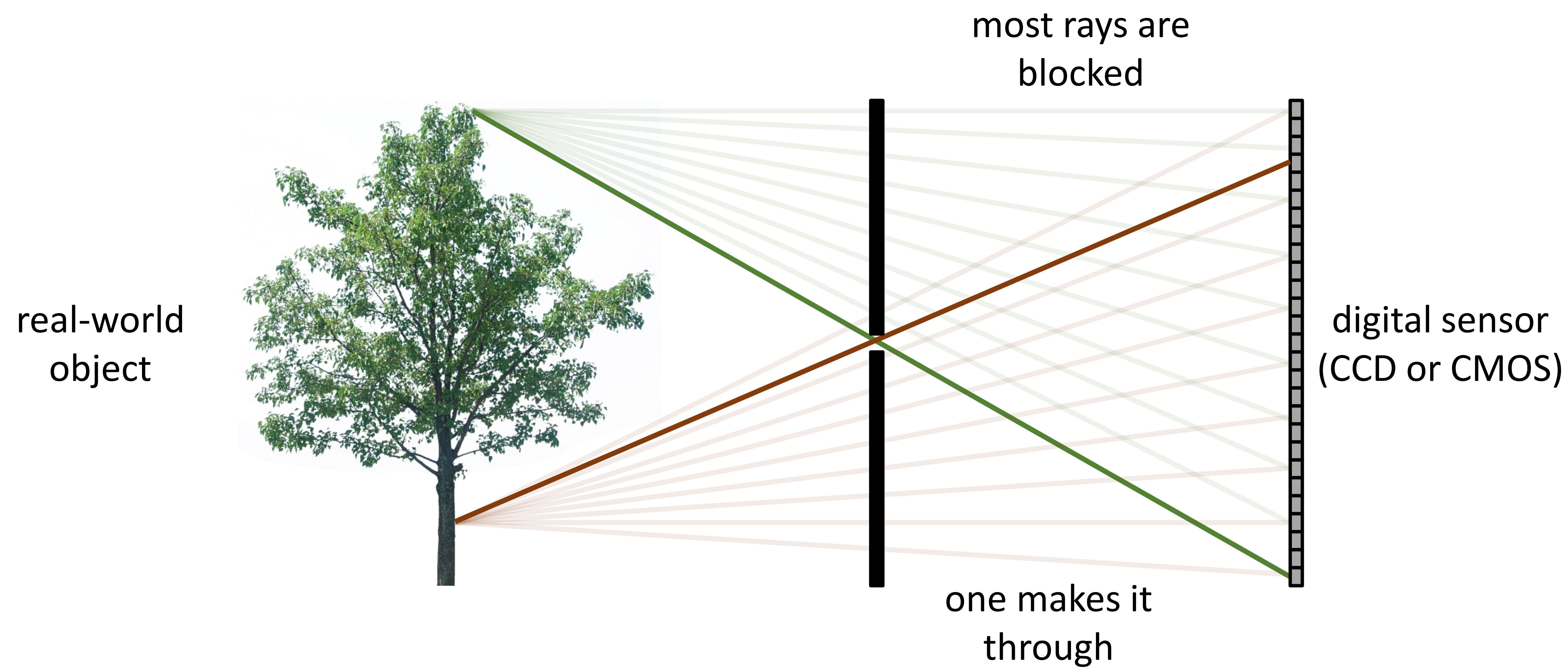


What would an image taken like this look like?

Pinhole imaging

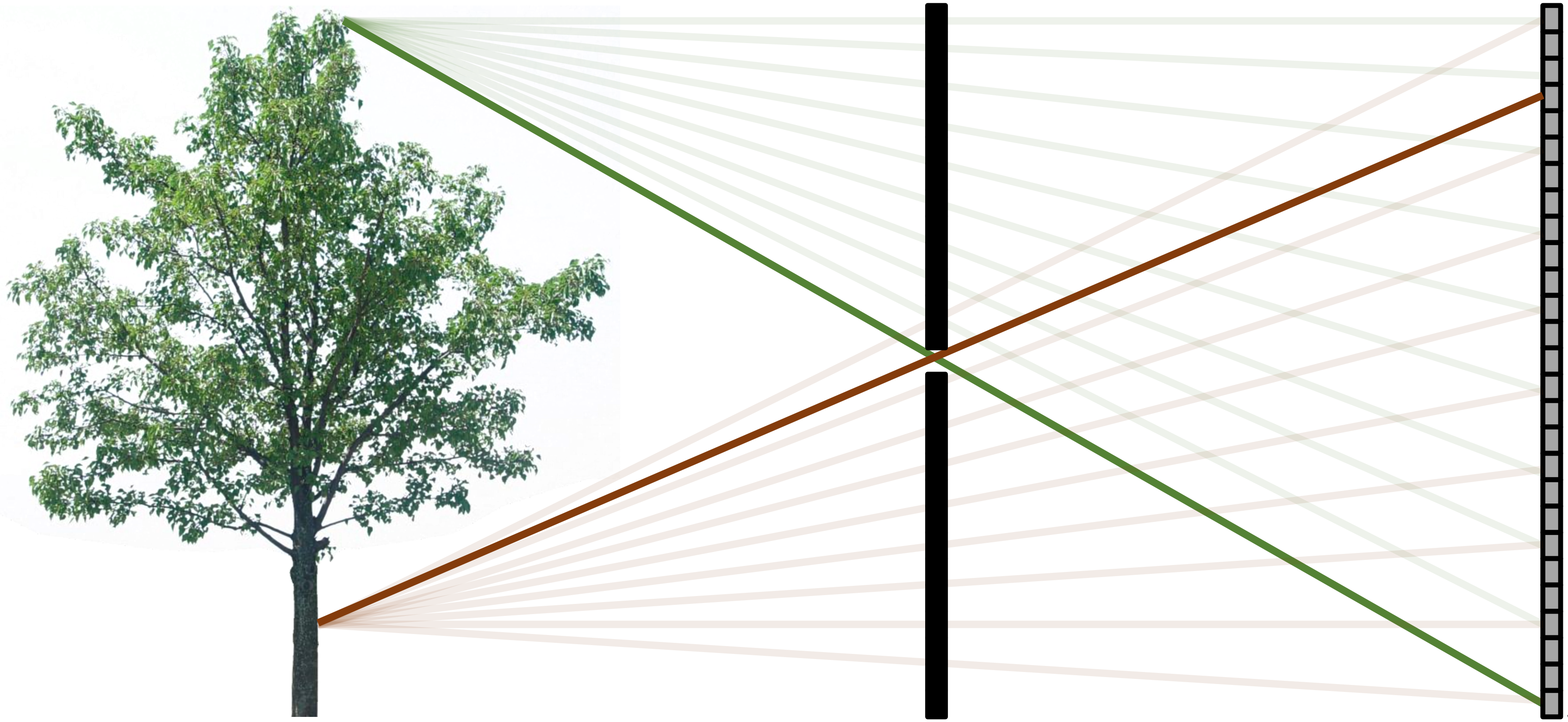


Pinhole imaging



Pinhole imaging

real-world object



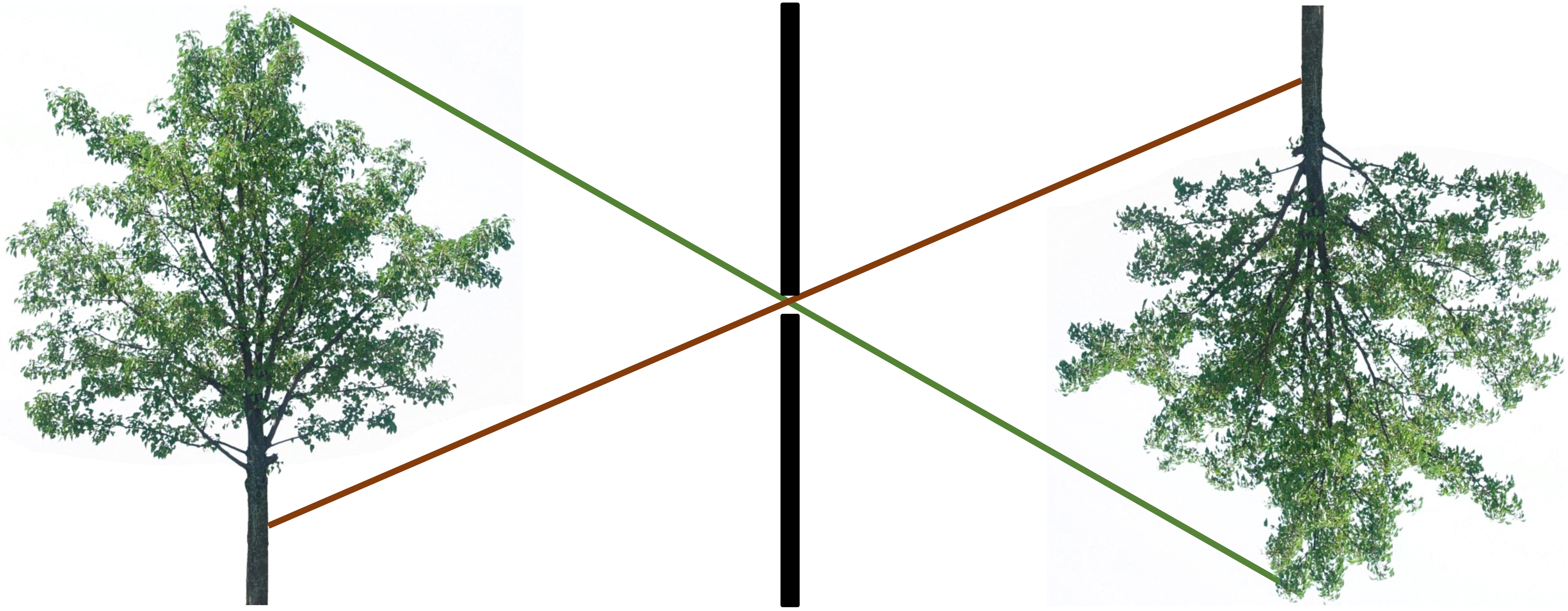
digital sensor (CCD or CMOS)

What does the image on the sensor look like?

Each scene point contributes to only one sensor pixel

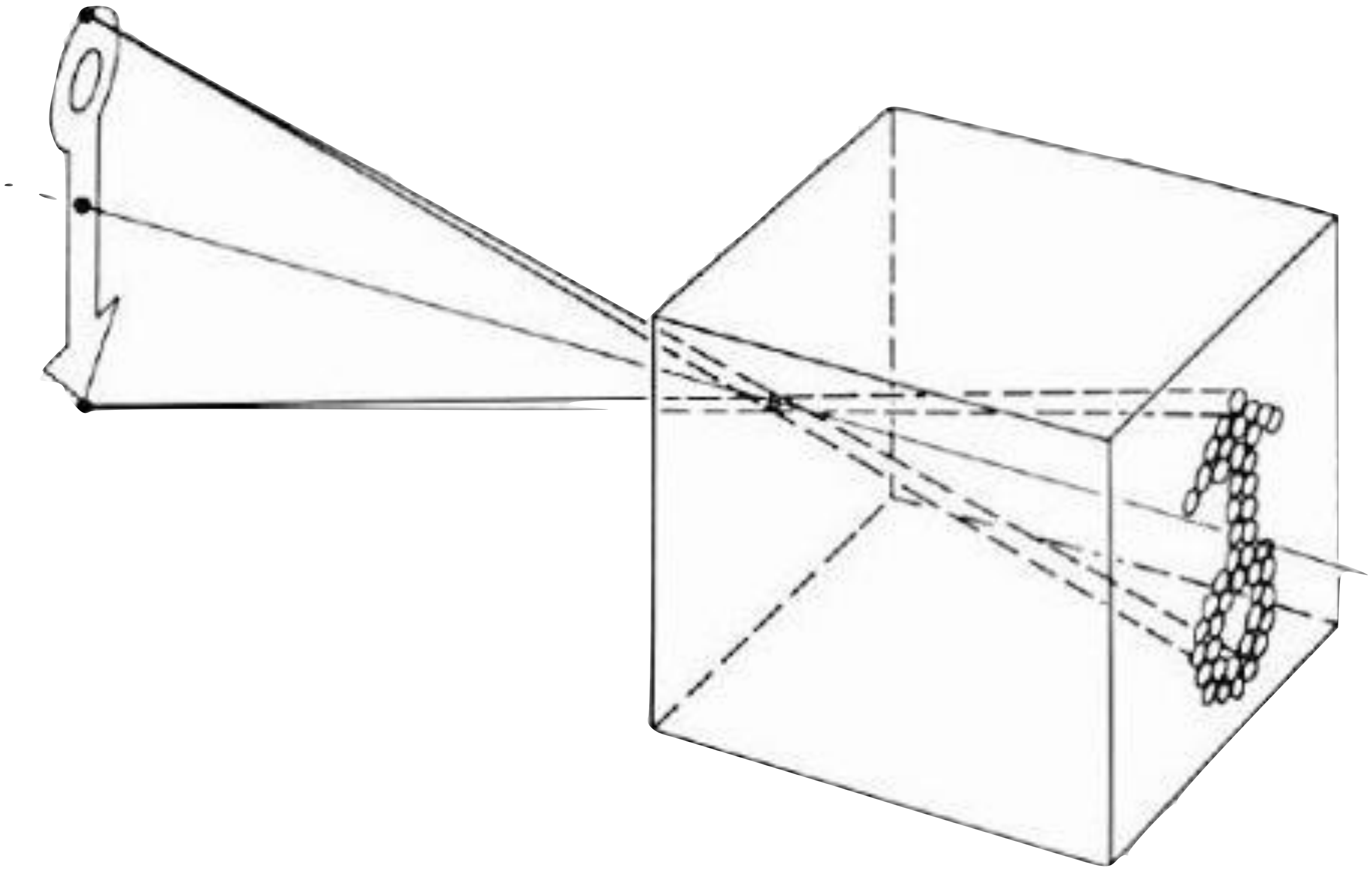
Pinhole imaging

real-world object



copy of real-world object (inverted and scaled)

Pinhole camera a.k.a. camera obscura

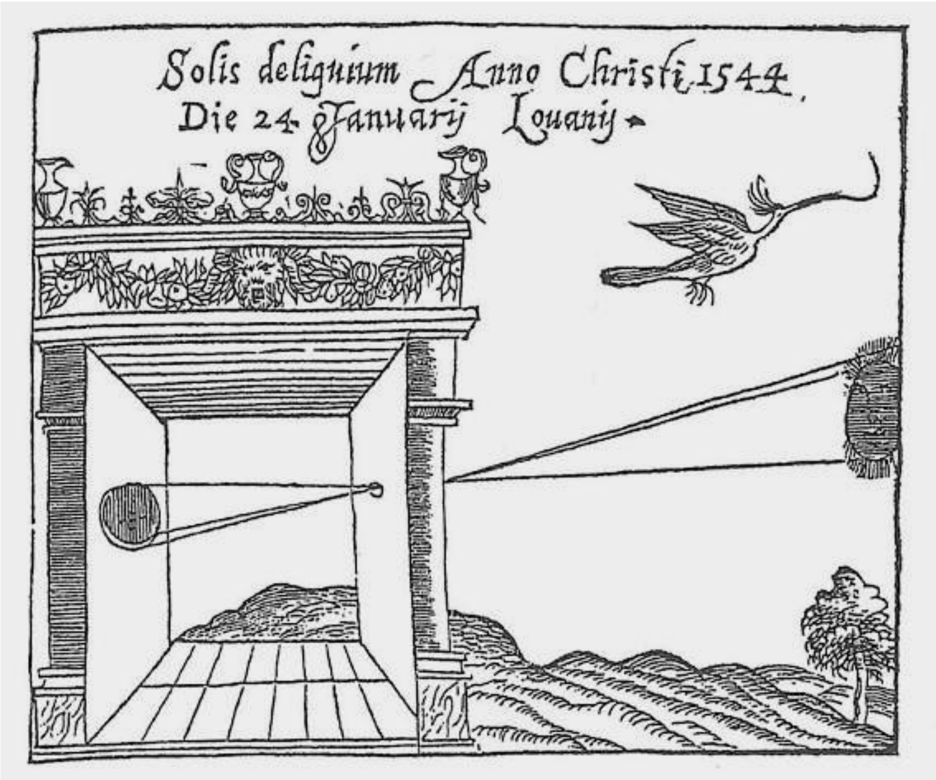


Pinhole camera a.k.a. camera obscura

First mention …

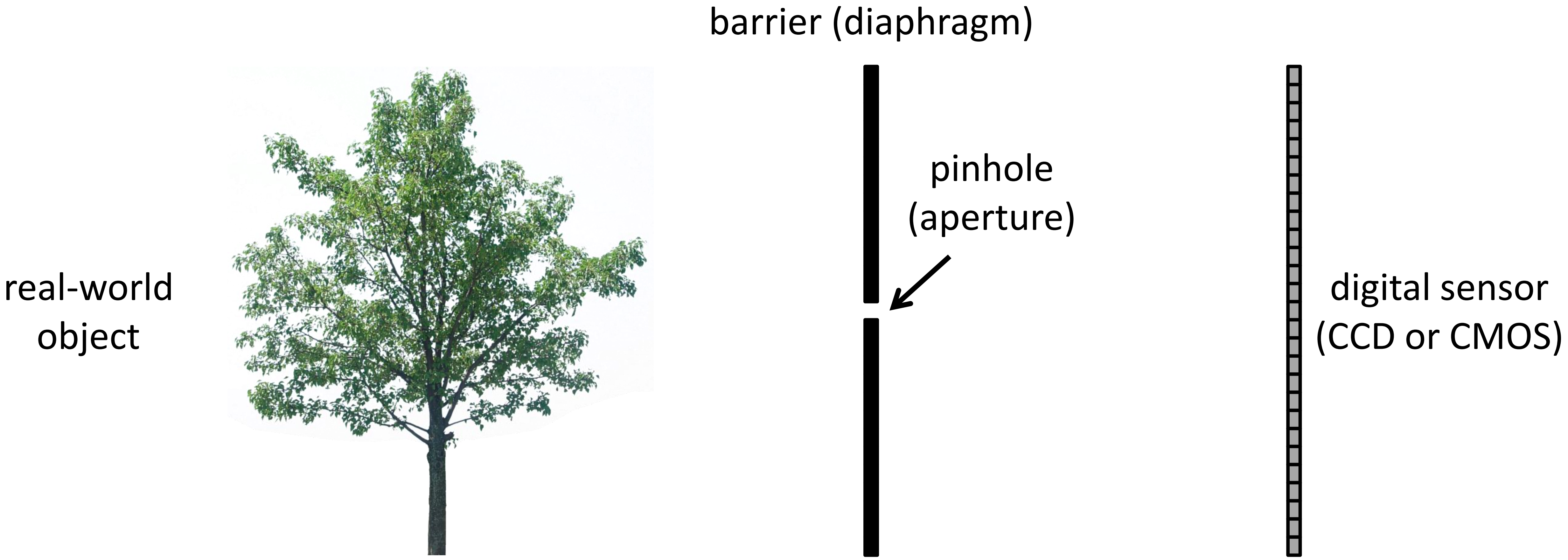
Chinese philosopher Mozi
(470 to 390 BC)

First camera …

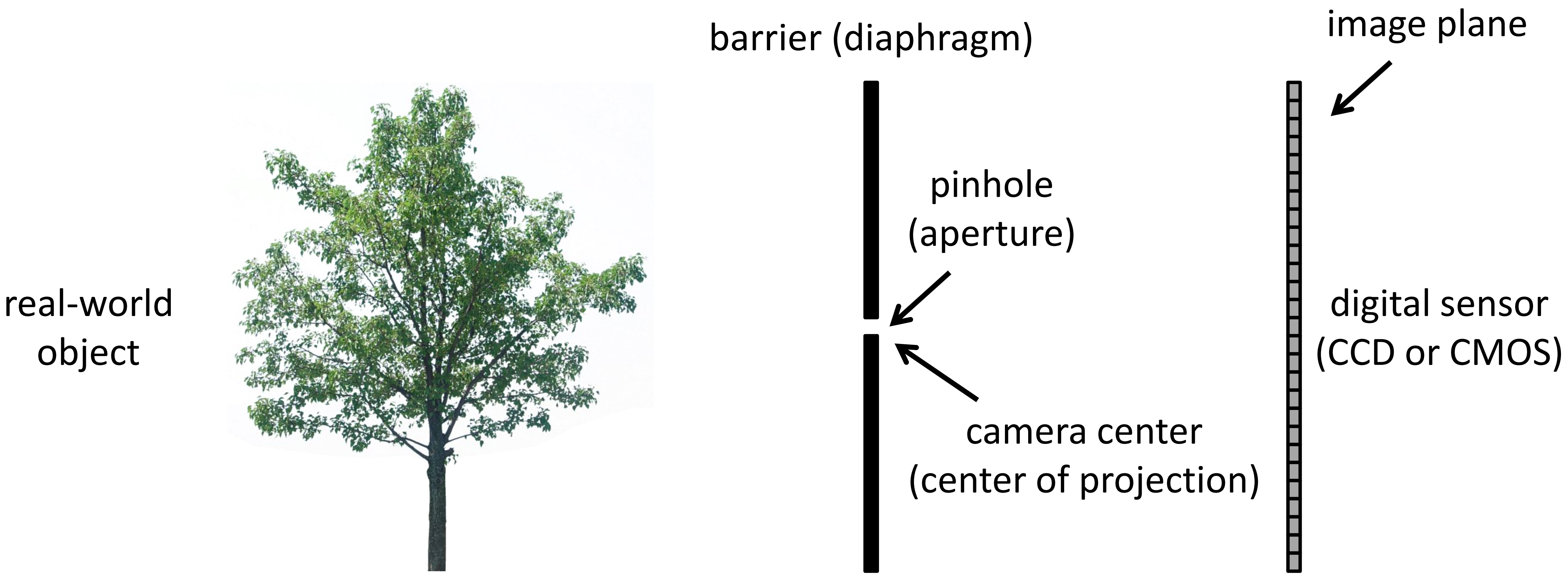


Greek philosopher Aristotle
(384 to 322 BC)

Pinhole camera terms

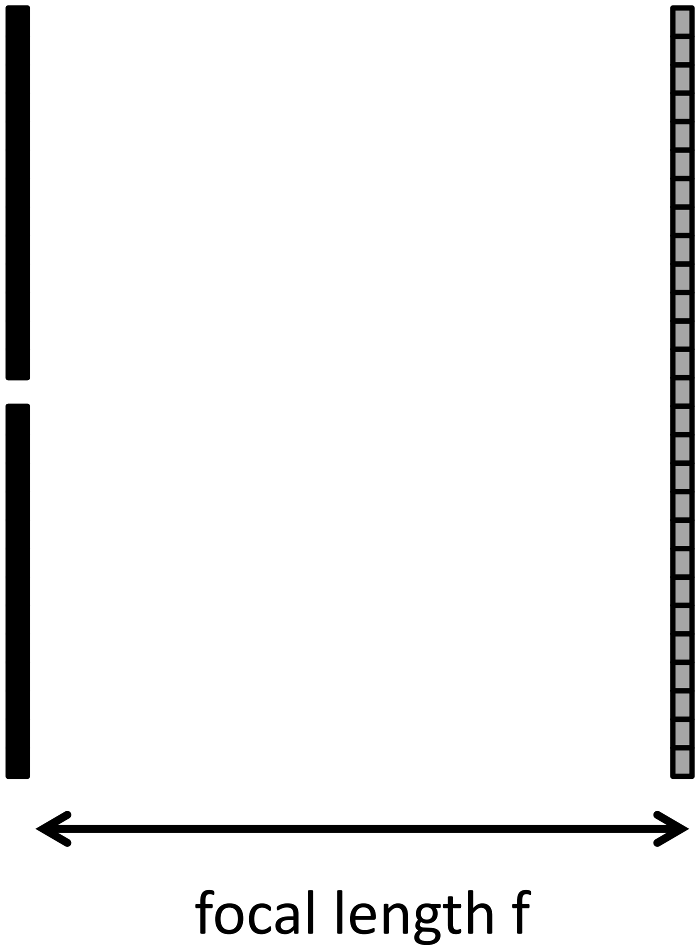


Pinhole camera terms



Focal length

real-world
object



Focal length

- What happens as we change the focal length?

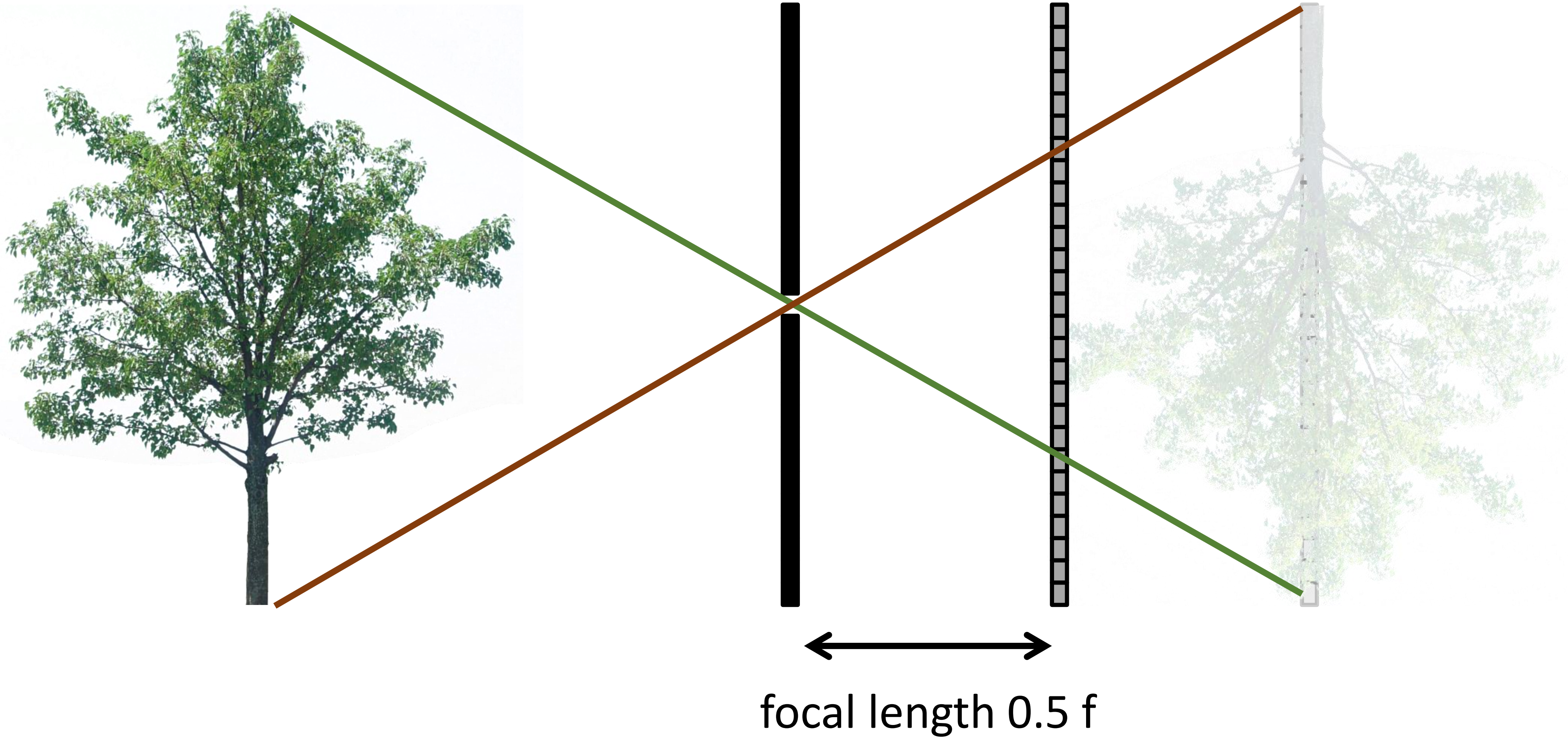
real-world
object

focal length 0.5 f

Focal length

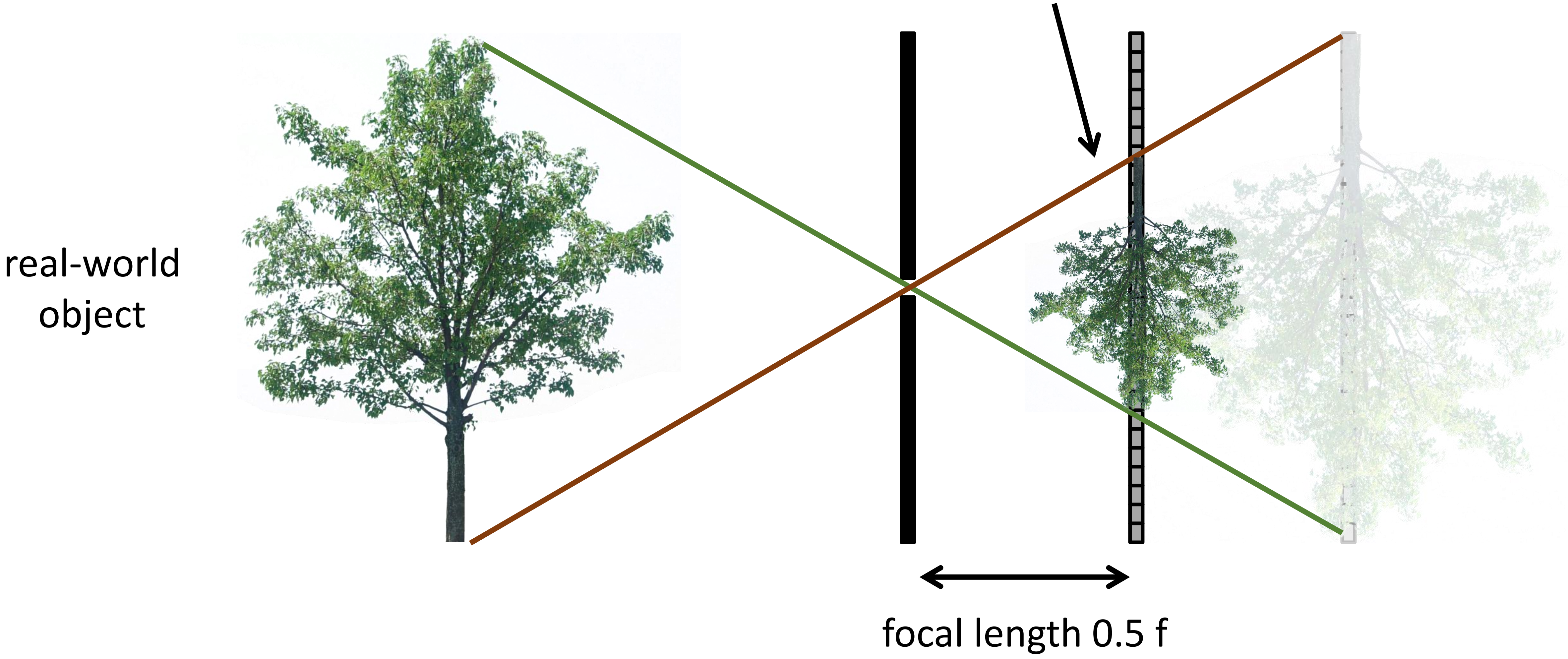
- What happens as we change the focal length?

real-world object

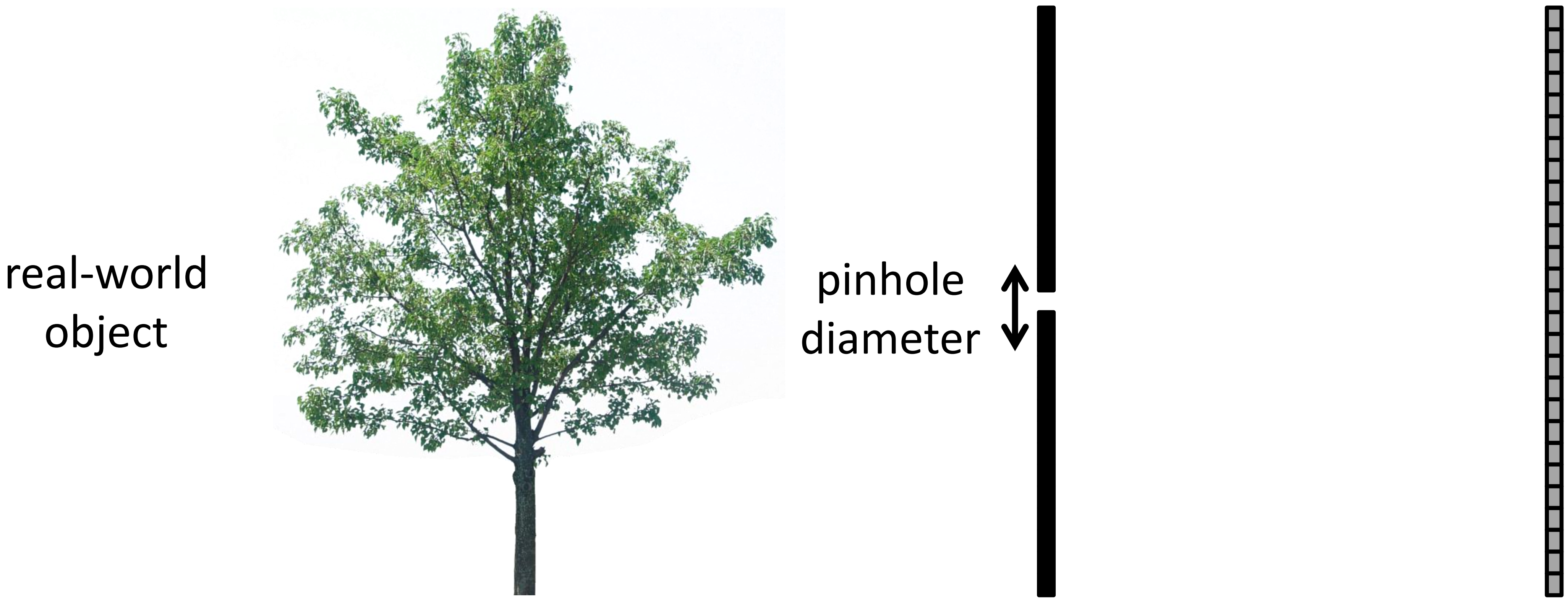


Focal length

- What happens as we change the focal length?



Pinhole size

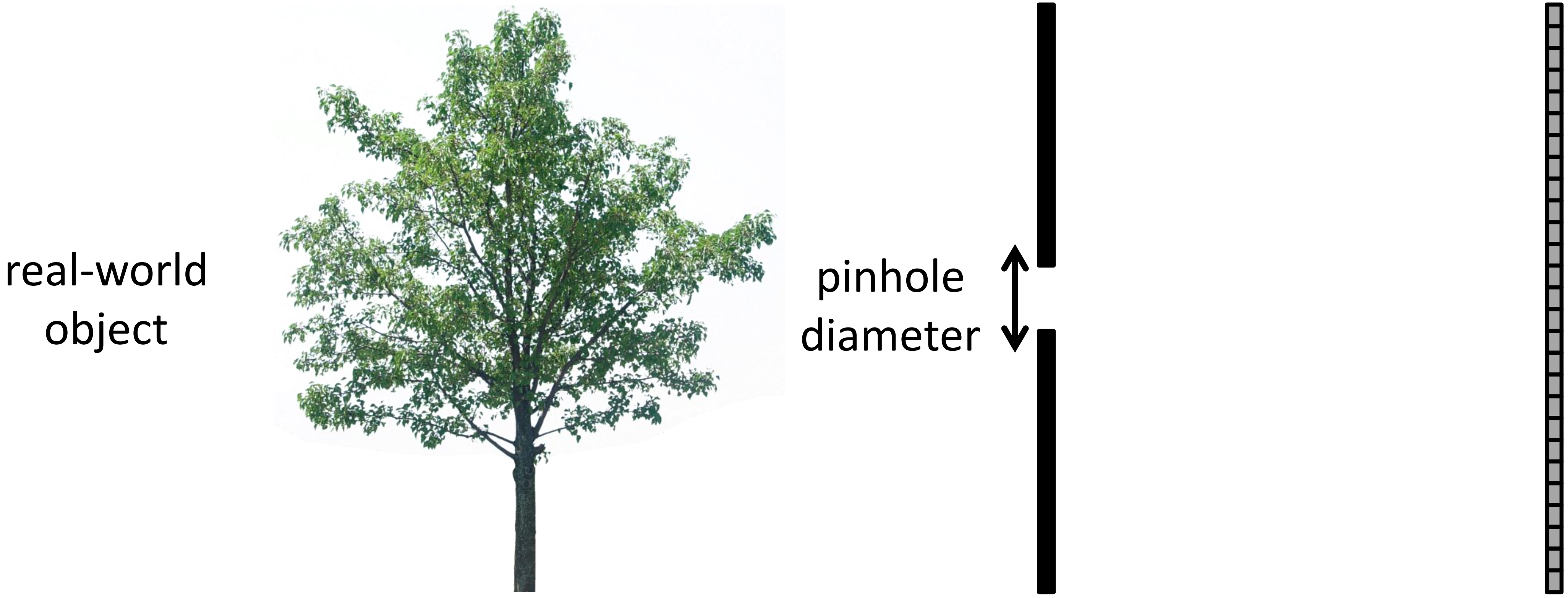


Ideal pinhole has infinitesimally small size

- In practice that is impossible.

Pinhole size

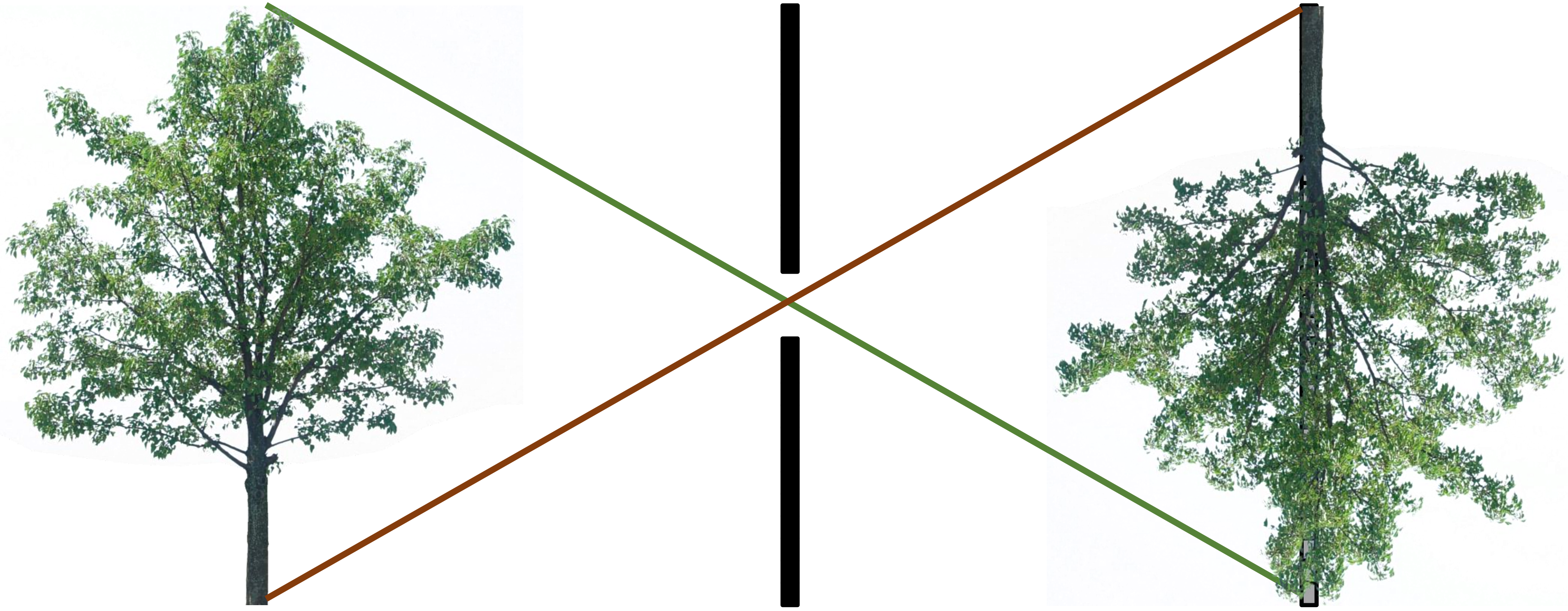
- What happens as we change the pinhole diameter?



Pinhole size

- What happens as we change the pinhole diameter?

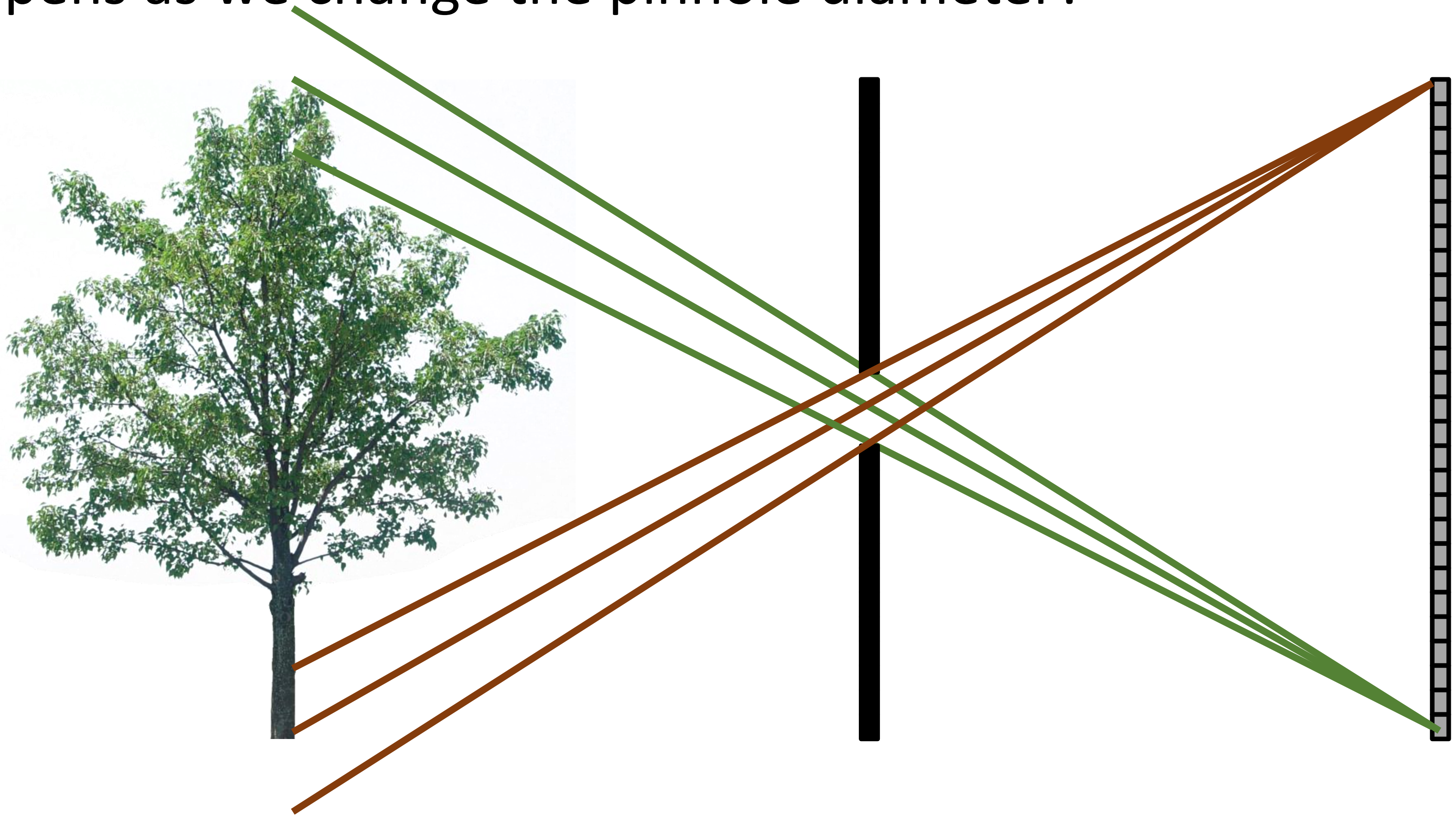
real-world object



Pinhole size

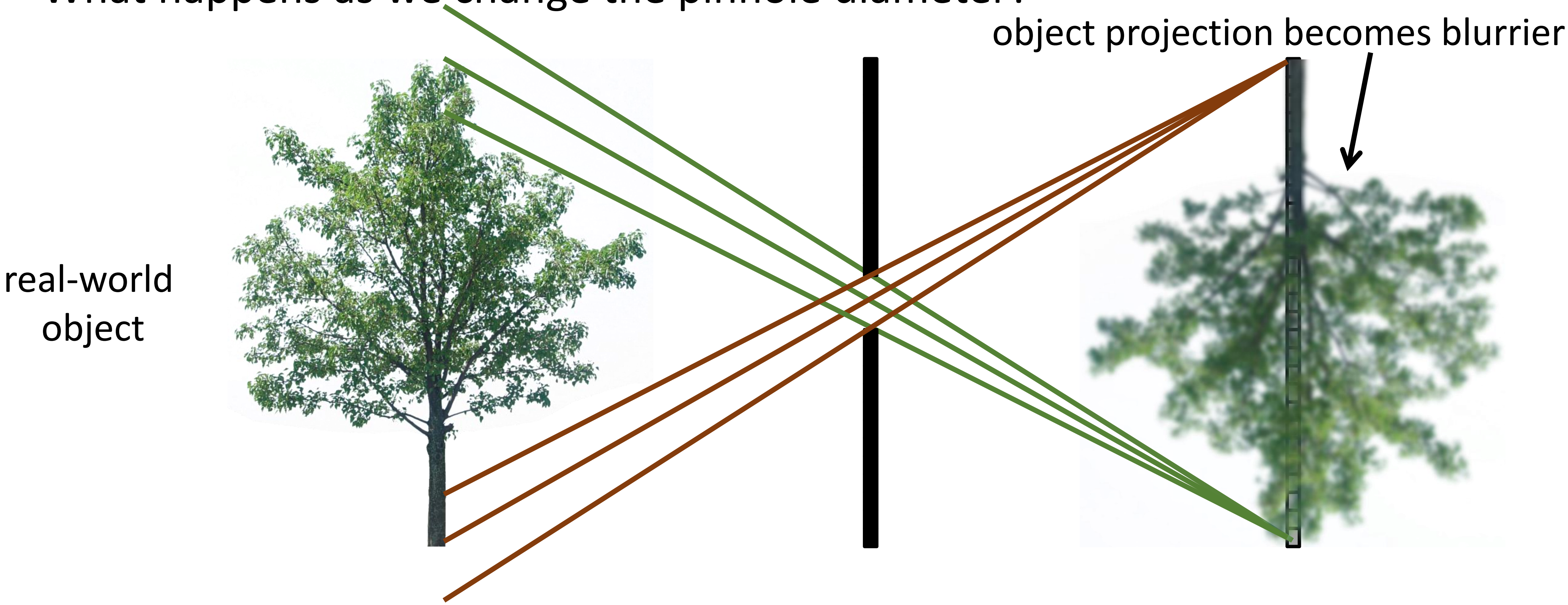
- What happens as we change the pinhole diameter?

real-world
object

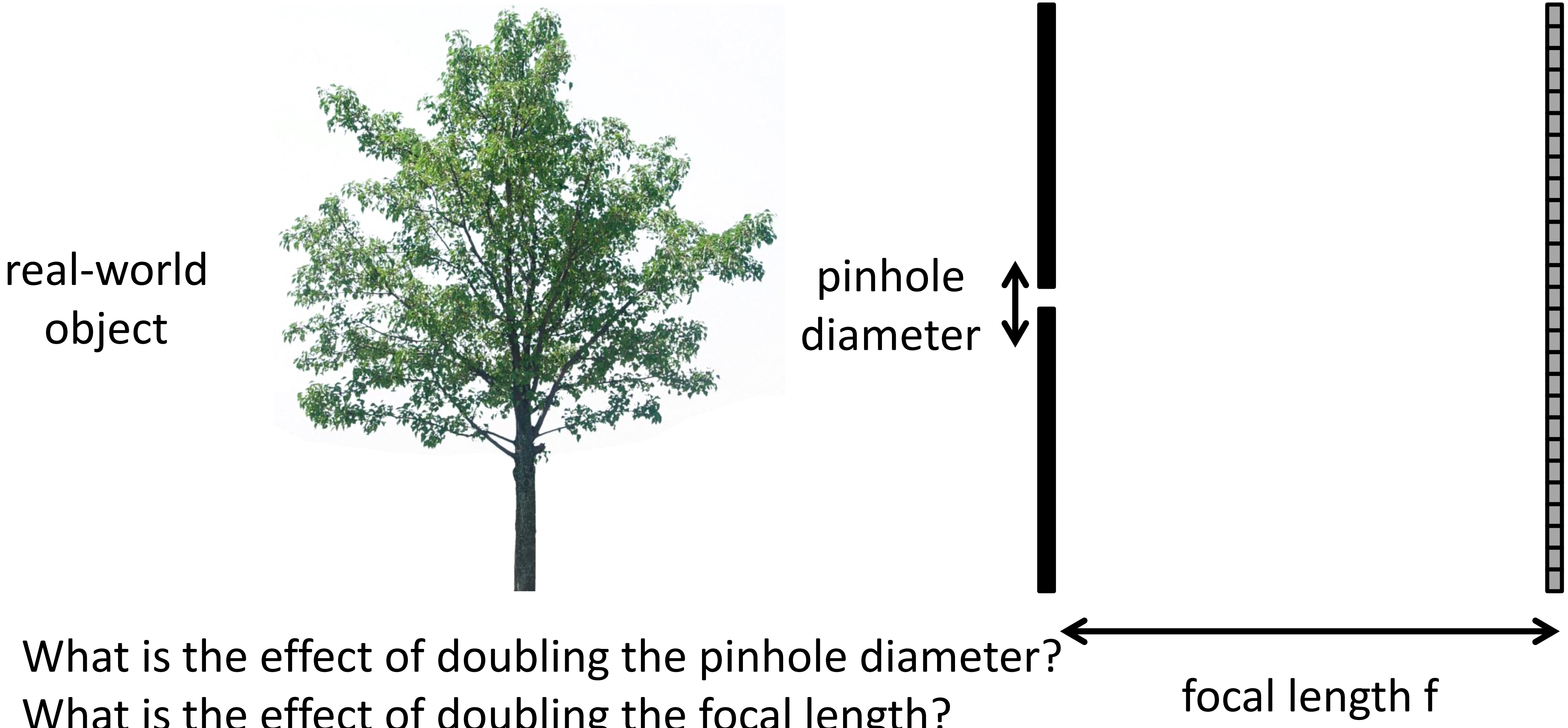


Pinhole size

- What happens as we change the pinhole diameter?

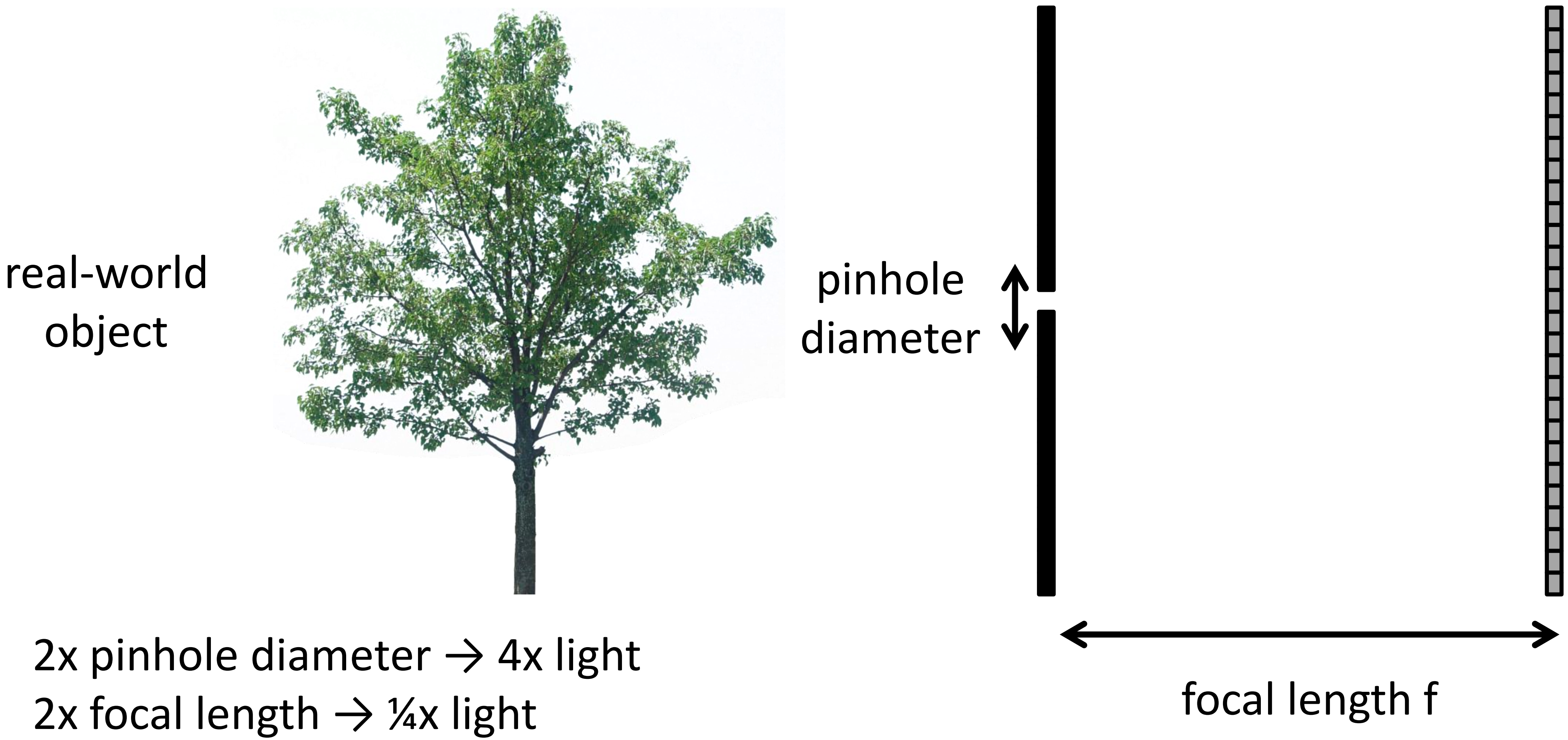


What about light efficiency?



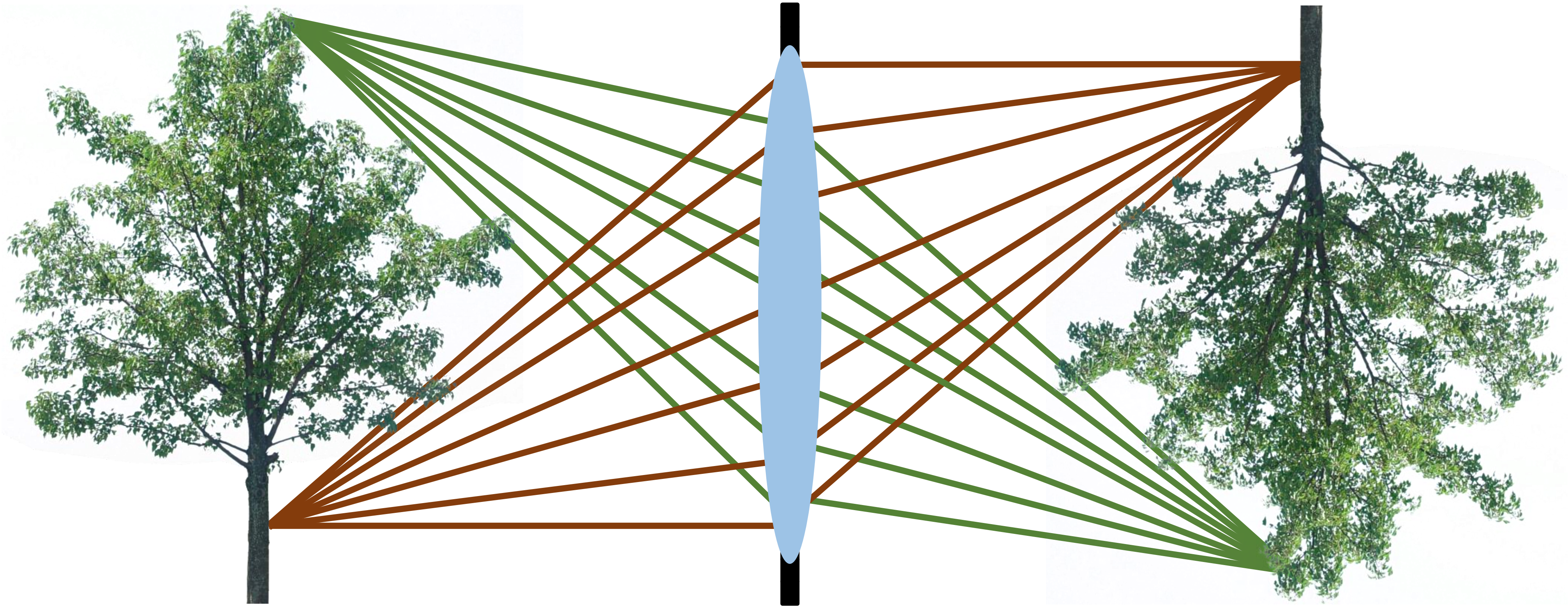
- What is the effect of doubling the pinhole diameter?
- What is the effect of doubling the focal length?

What about light efficiency?



- 2x pinhole diameter → 4x light
- 2x focal length → ¼x light

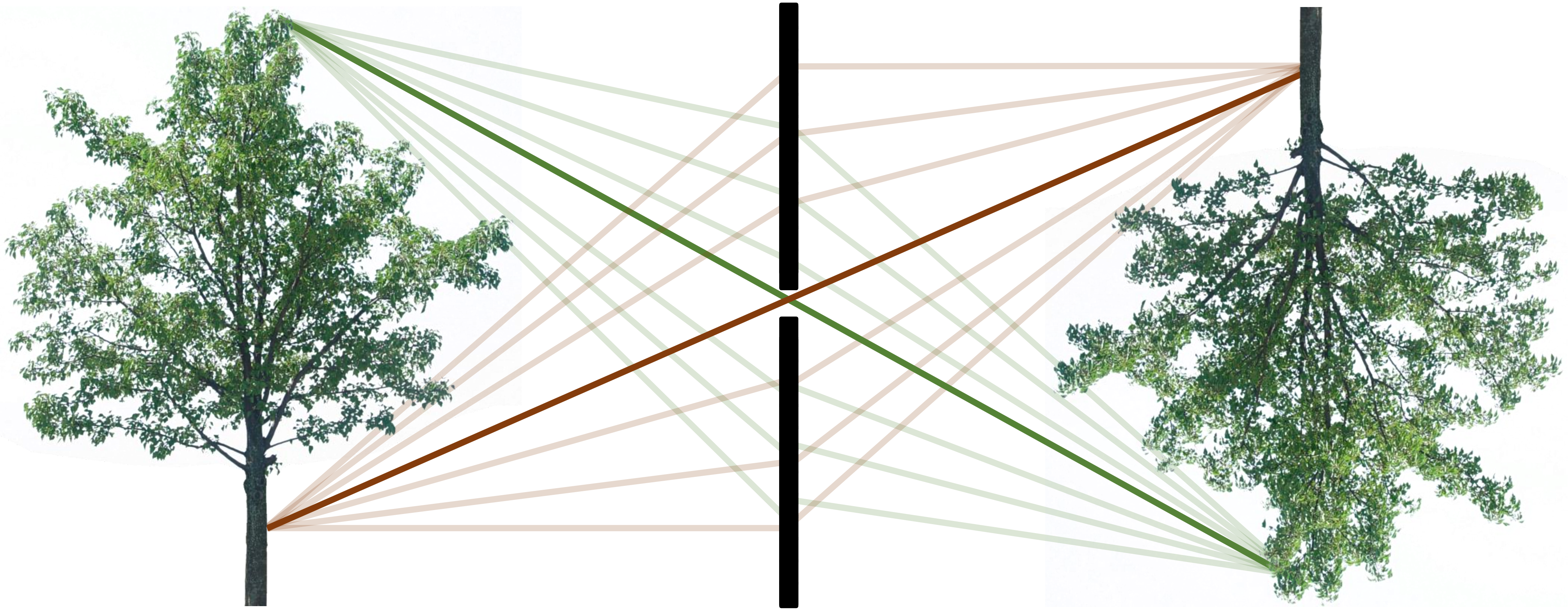
The lens camera



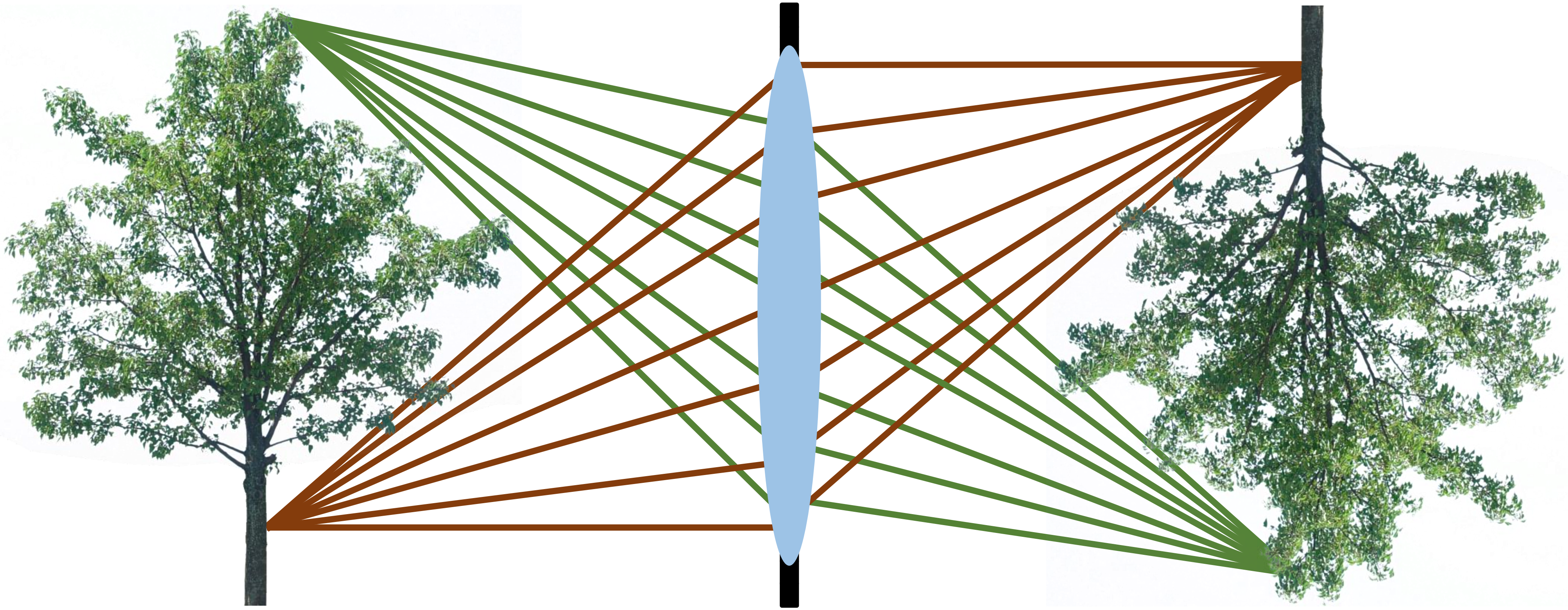
Lenses map “bundles” of rays from poi
nts on the scene to the sensor.

How does this mapping work exactly?

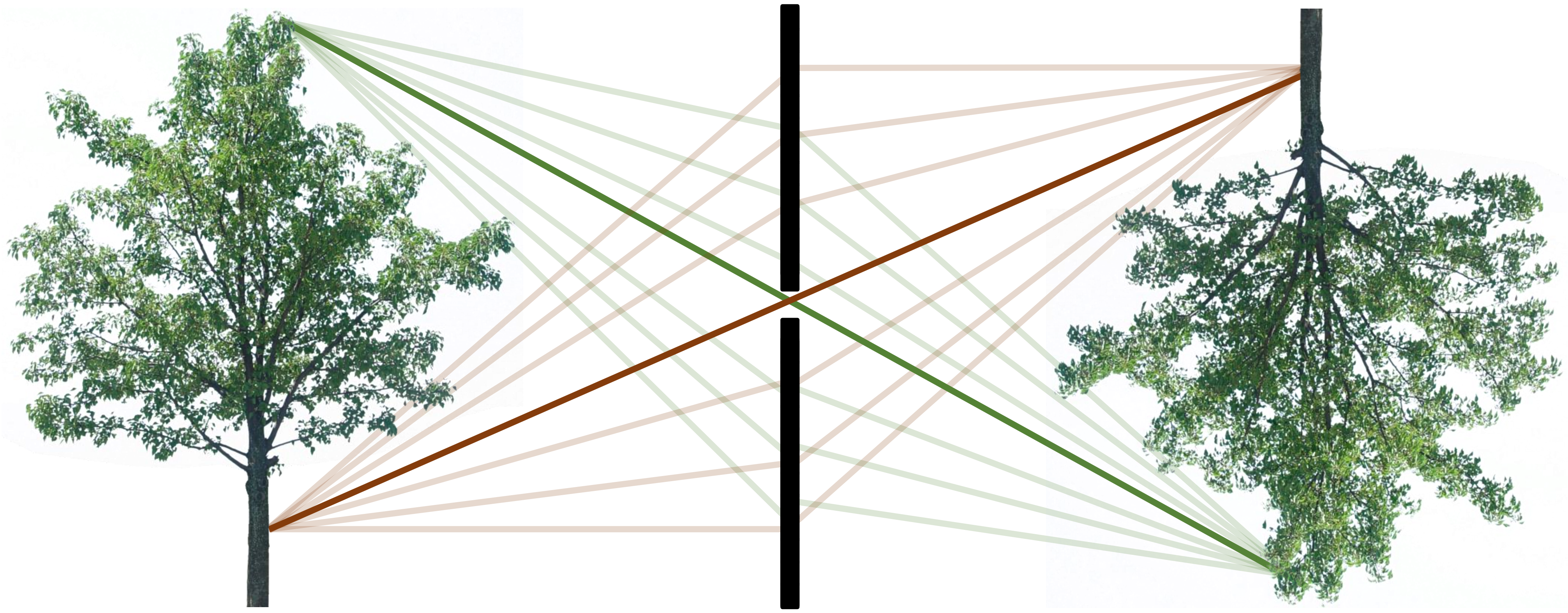
The pinhole camera



The lens camera

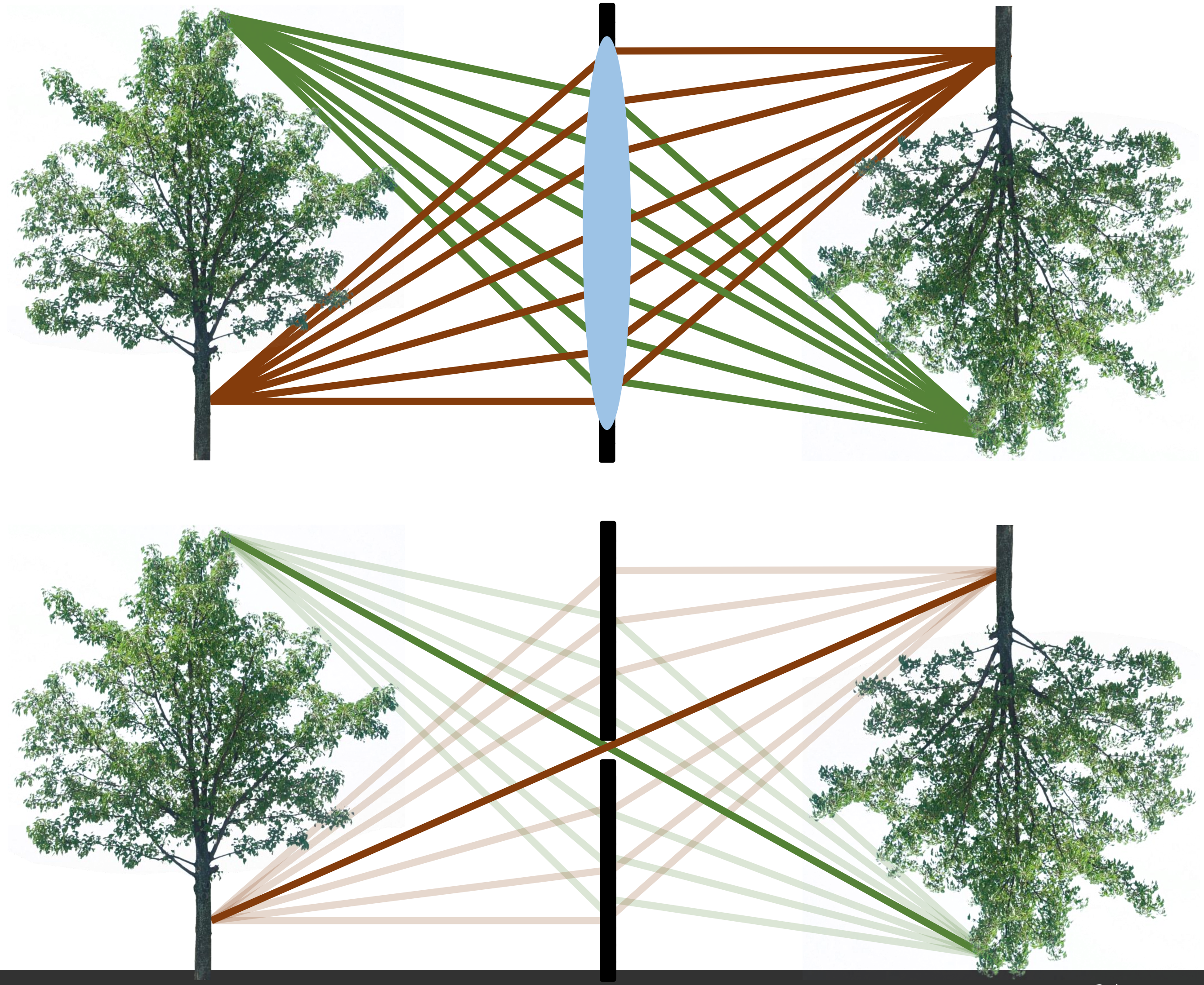


The pinhole camera



Central rays propagate in the same way for both models!

Describing both lens and pinhole cameras



We can derive properties and descriptions that hold for both camera models if:

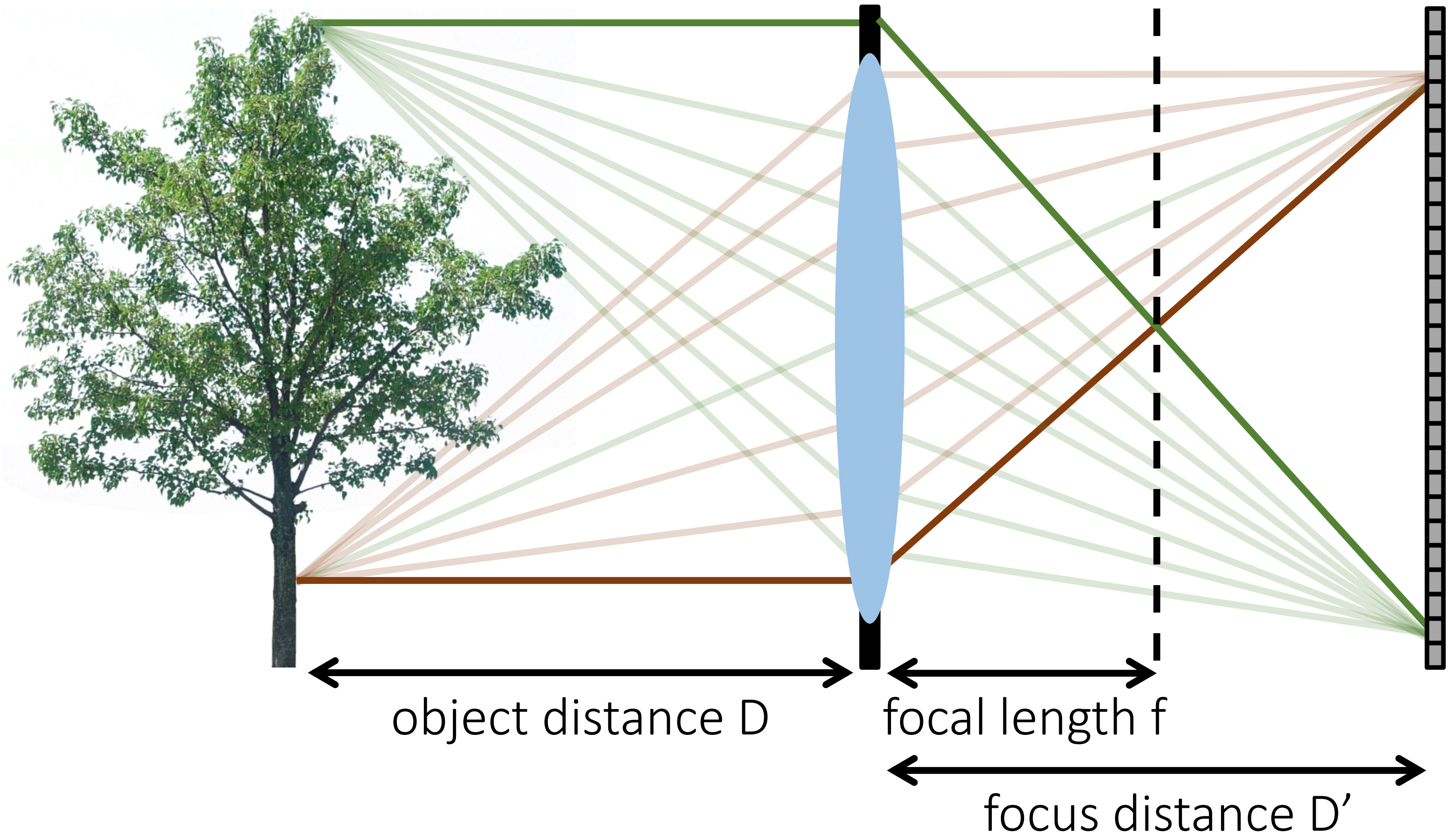
- We use only central rays.
- We assume the lens camera is in focus.

Important difference: focal length

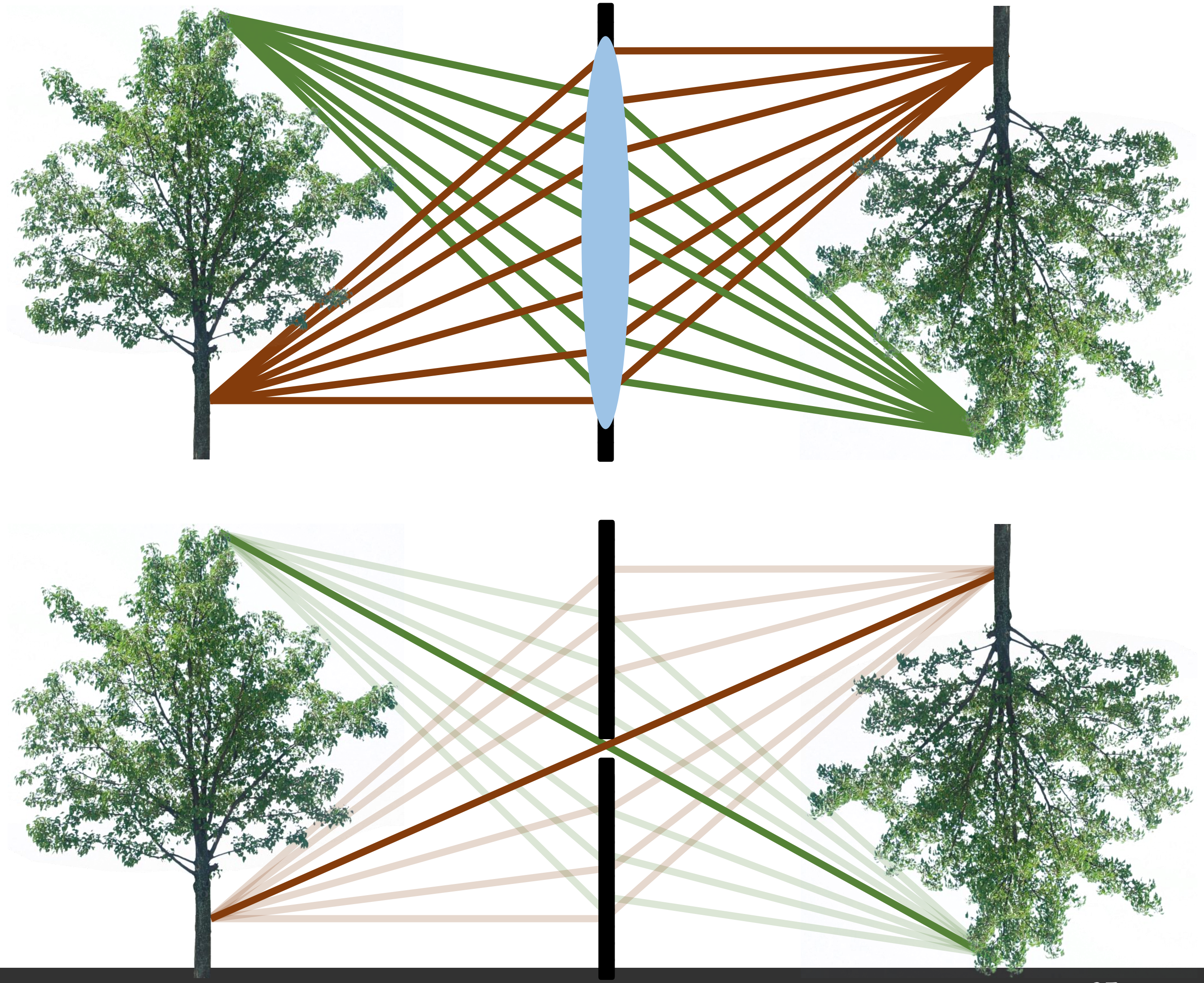
- In a pinhole camera, focal length is distance between aperture and sensor

Important difference: focal length

- In a lens camera, focal length is distance where parallel rays intersect



Describing both lens and pinhole cameras



We can derive properties and descriptions that hold for both camera models if:

- We use only central rays.
- We assume the lens camera is in focus.
- We assume that the focus distance of the lens camera is equal to the focal length of the pinhole camera.

Remember: *focal length* f refers to different things for lens and pinhole cameras.

- In this lecture, we use it to refer to the aperture-sensor distance, as in the pinhole camera case.

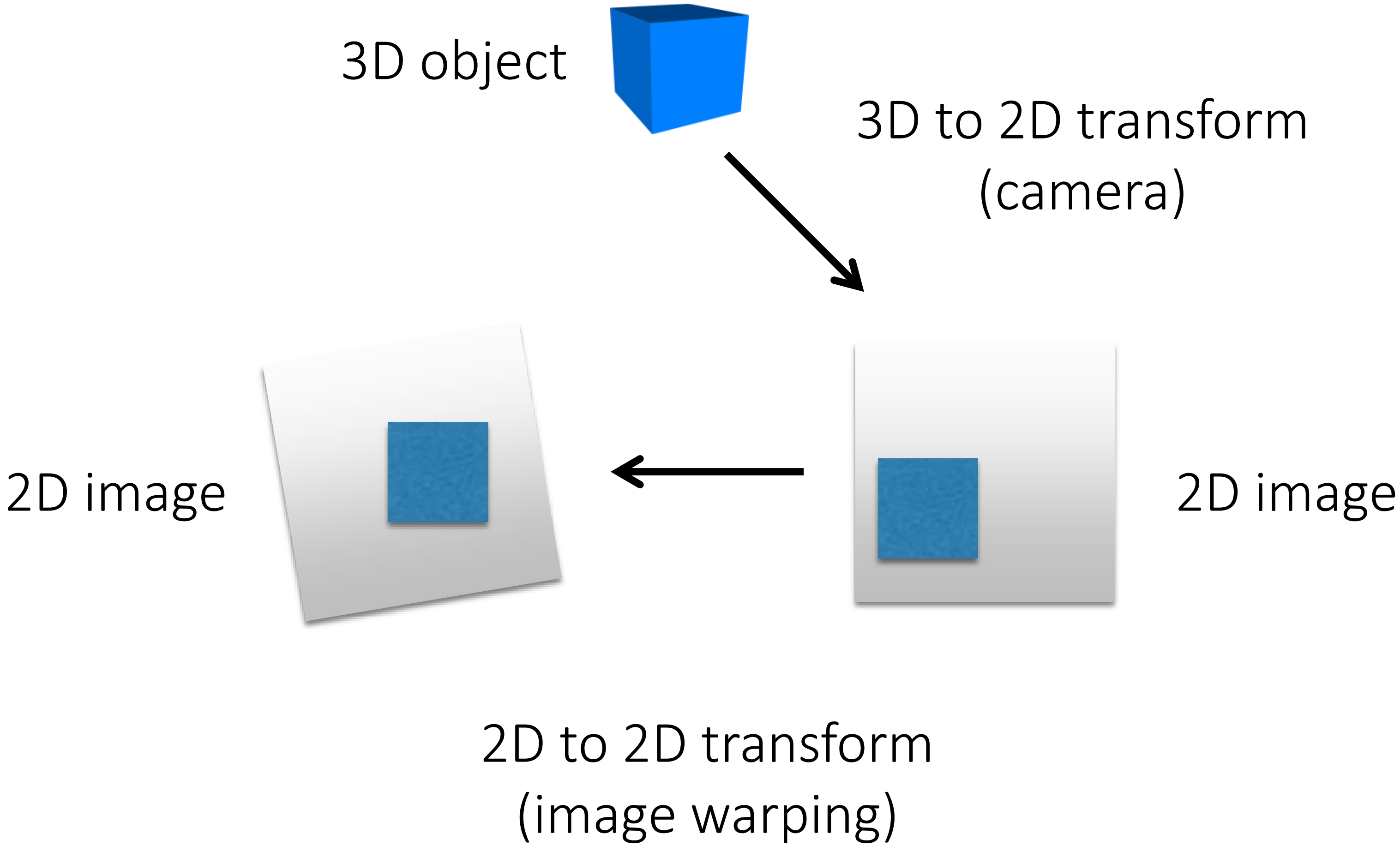
The camera as a coordinate transformation

A camera is a mapping from:

the 3D world

to:

a 2D image



The camera as a coordinate transformation

A camera is a mapping from:

the 3D world

to:

a 2D image

homogeneous coordinates

$$\boldsymbol{x} = \mathbf{P}\mathbf{X}$$

2D image point — camera matrix — 3D world point

What are the dimensions of each variable?

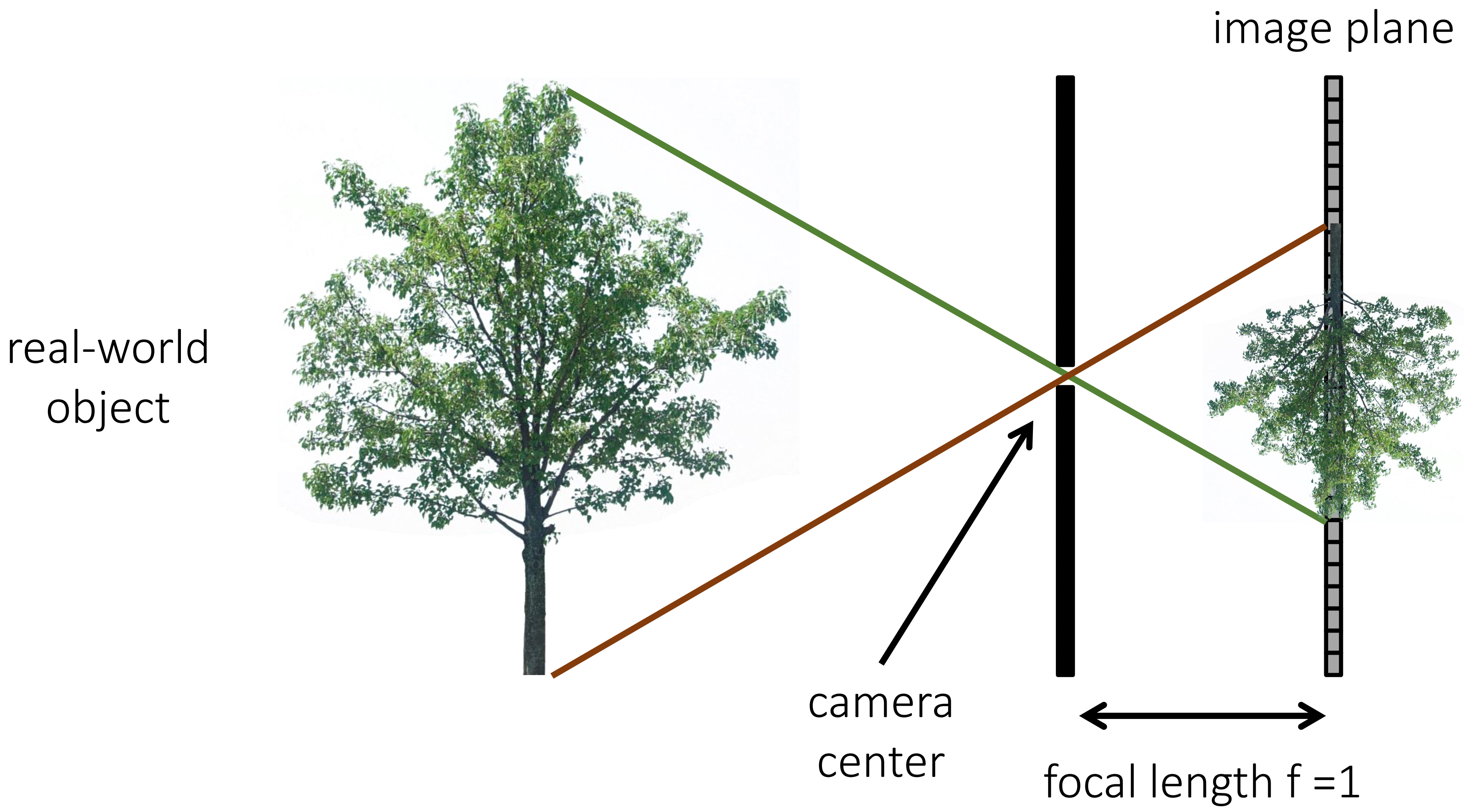
The camera as a coordinate transformation

$$\boldsymbol{x} = \mathbf{PX}$$

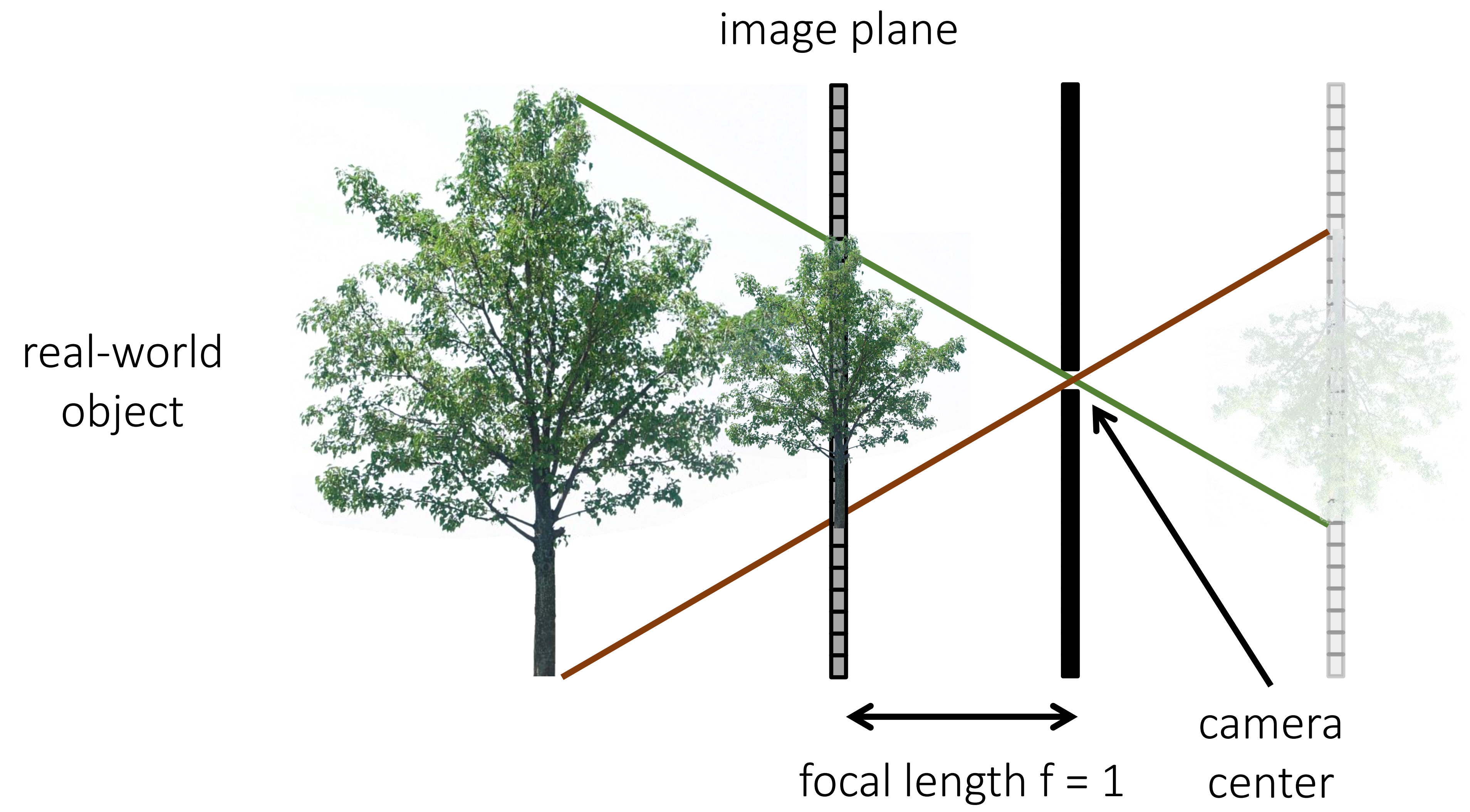
$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

homogeneous image coordinates	camera matrix	homogeneous world coordinates
3 x 1	3 x 4	4 x 1

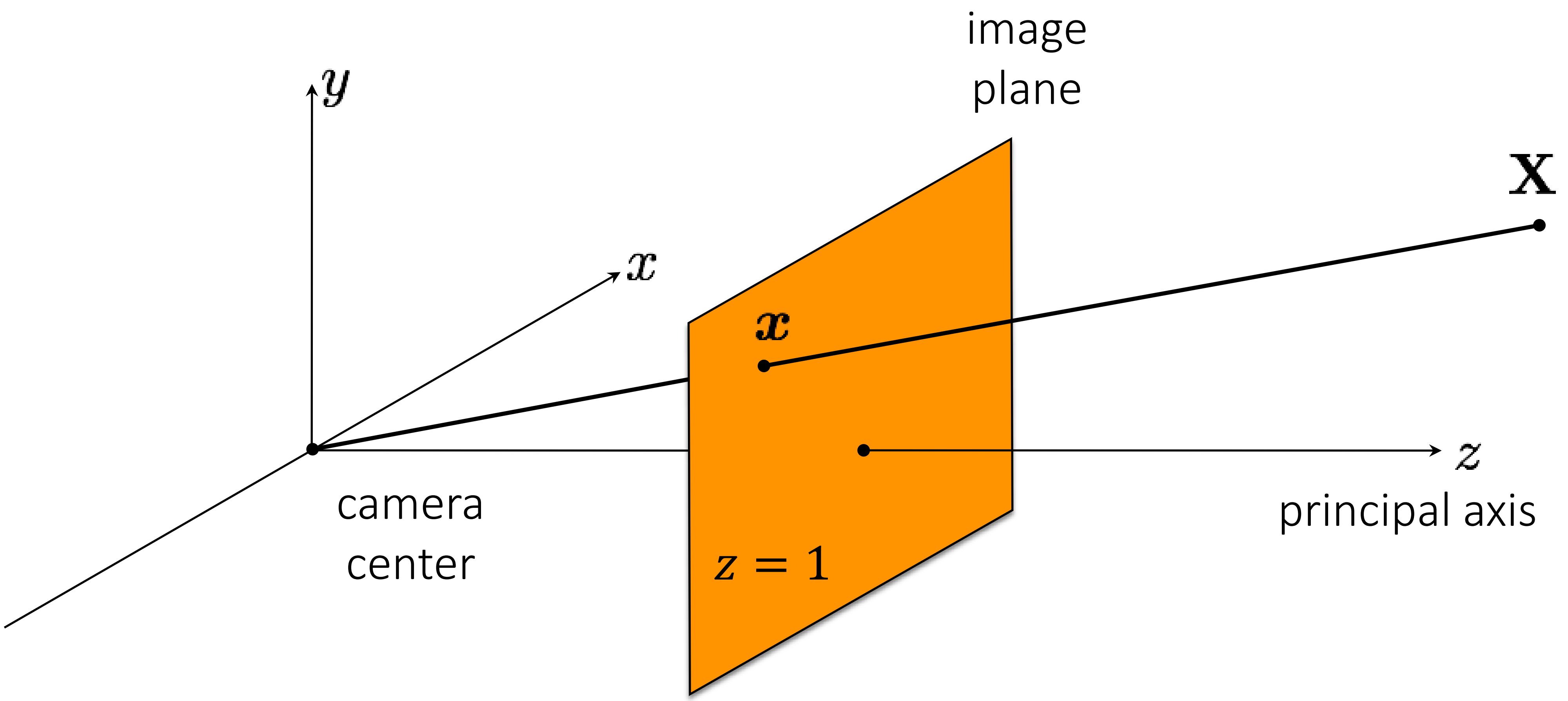
The pinhole camera



The (rearranged) pinhole camera

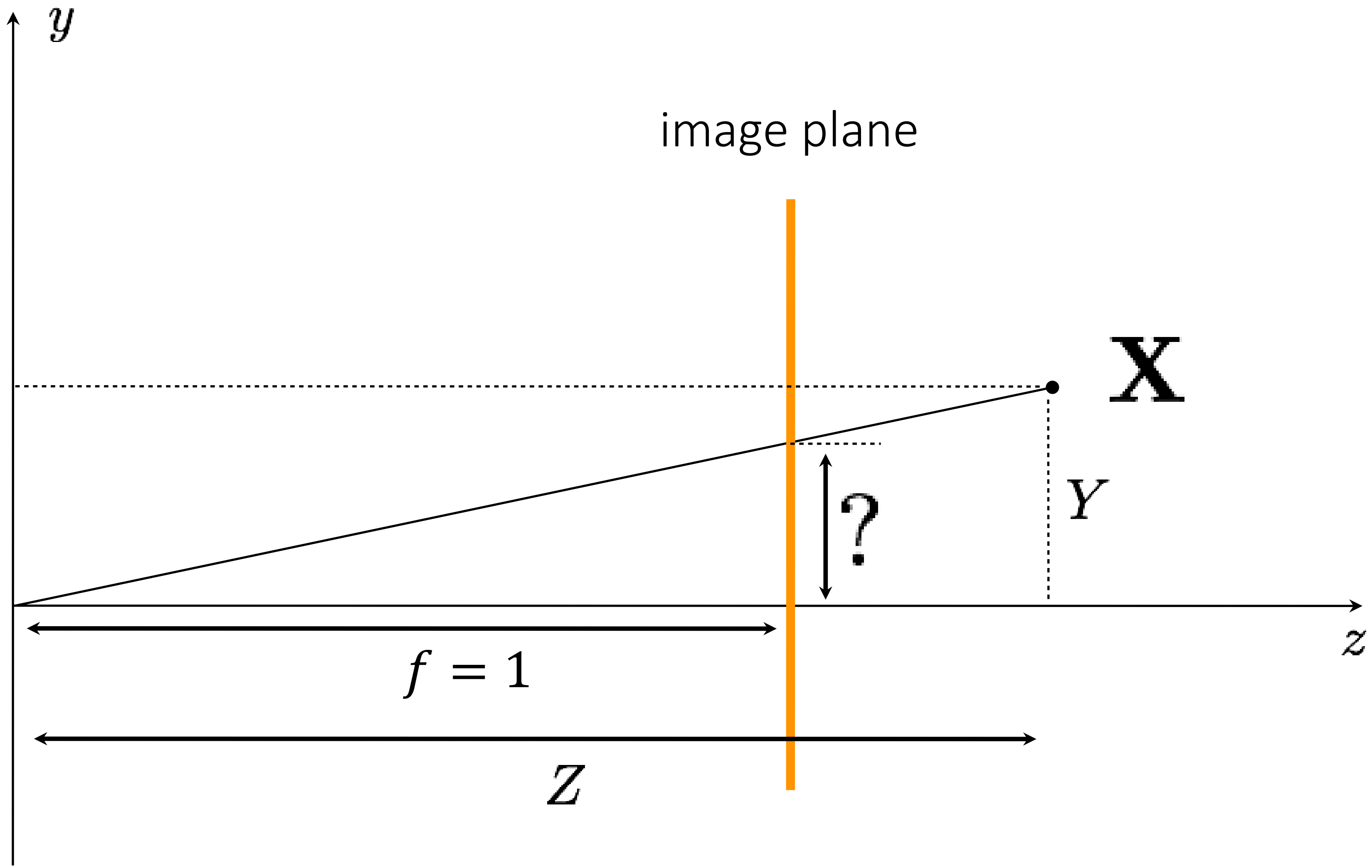


The (rearranged) pinhole camera



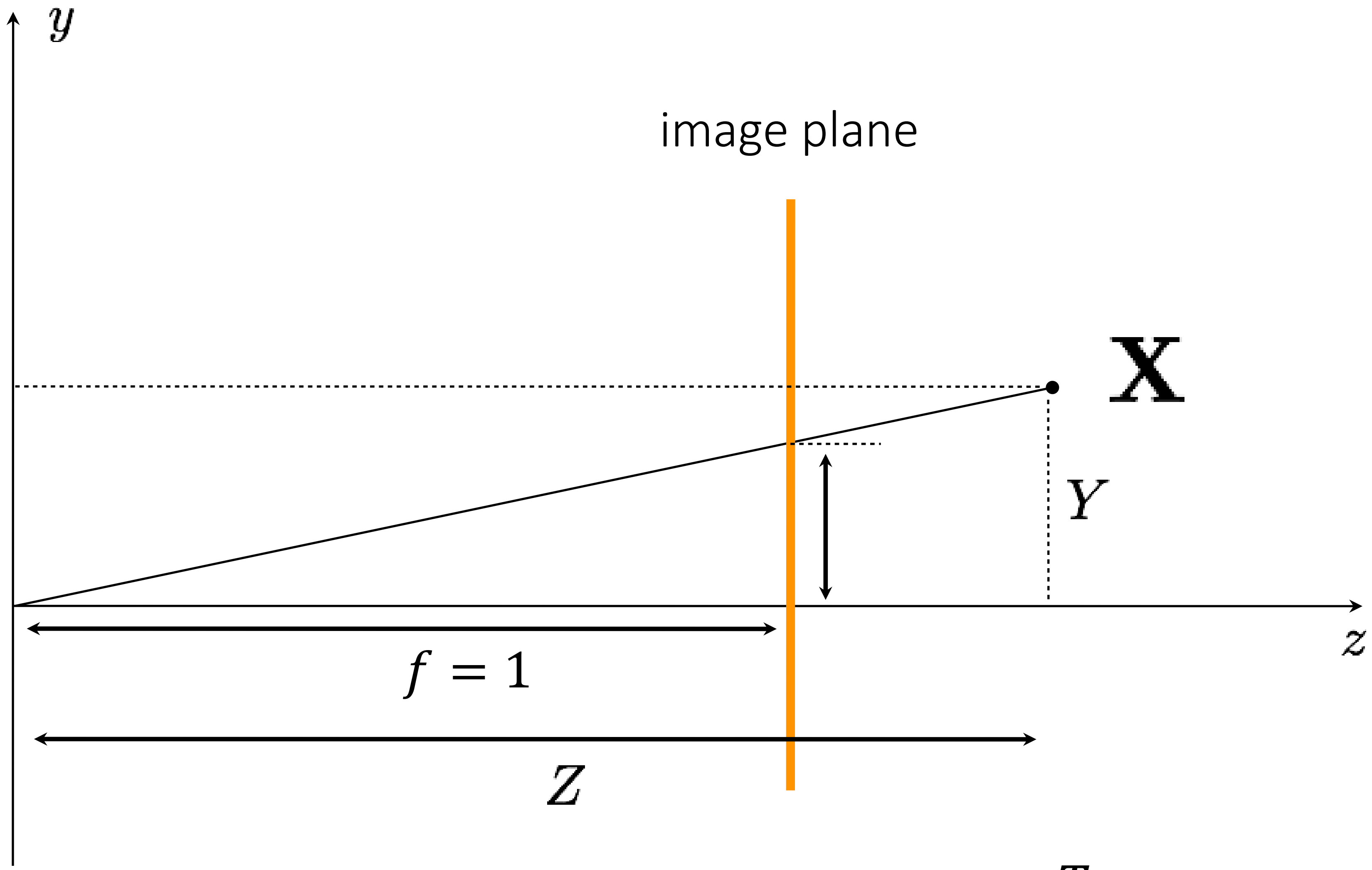
What is the equation for image coordinate **x** in terms of **X**?

The 2D view of the (rearranged) pinhole camera



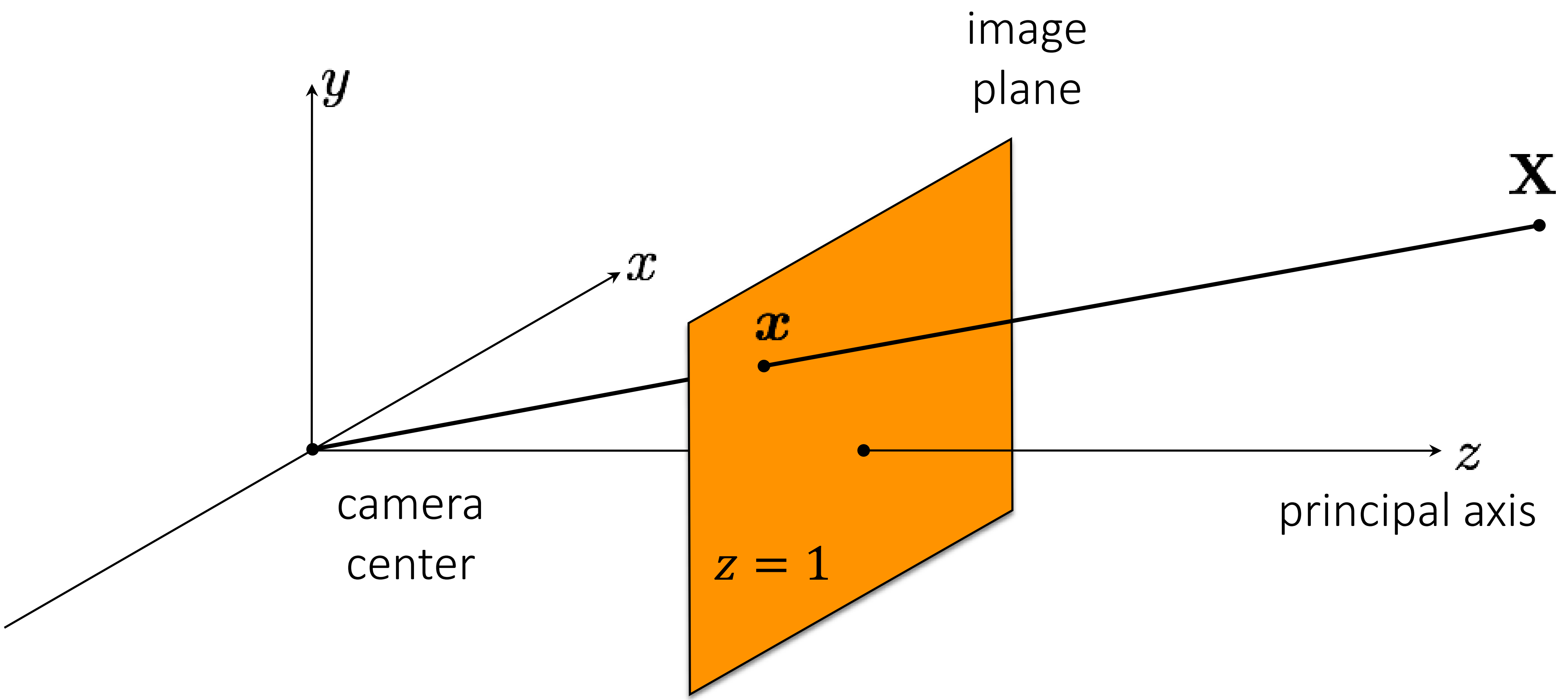
What is the equation for image coordinate **x** in terms of **X**?

The 2D view of the (rearranged) pinhole camera



$$[X \quad Y \quad Z]^T \rightarrow [X/Z \quad Y/Z]$$

The (rearranged) pinhole camera



What is the camera matrix **P** for a pinhole camera? $\boldsymbol{x} = \mathbf{PX}$

The pinhole camera matrix

Relationship from similar triangles:

$$[X \quad Y \quad Z]^T \rightarrow [X/Z \quad Y/Z]$$

General camera model in *homogeneous coordinates*:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

What does the pinhole camera projection look like?

$$\mathbf{P} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix}$$

The pinhole camera matrix

Relationship from similar triangles:

$$[X \quad Y \quad Z]^T \rightarrow [X/Z \quad Y/Z]$$

General camera model *in homogeneous coordinates*:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

What does the pinhole camera projection look like?

The perspective projection matrix

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The pinhole camera matrix

Relationship from similar triangles:

$$[X \quad Y \quad Z]^T \rightarrow [X/Z \quad Y/Z]$$

General camera model *in homogeneous coordinates*:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

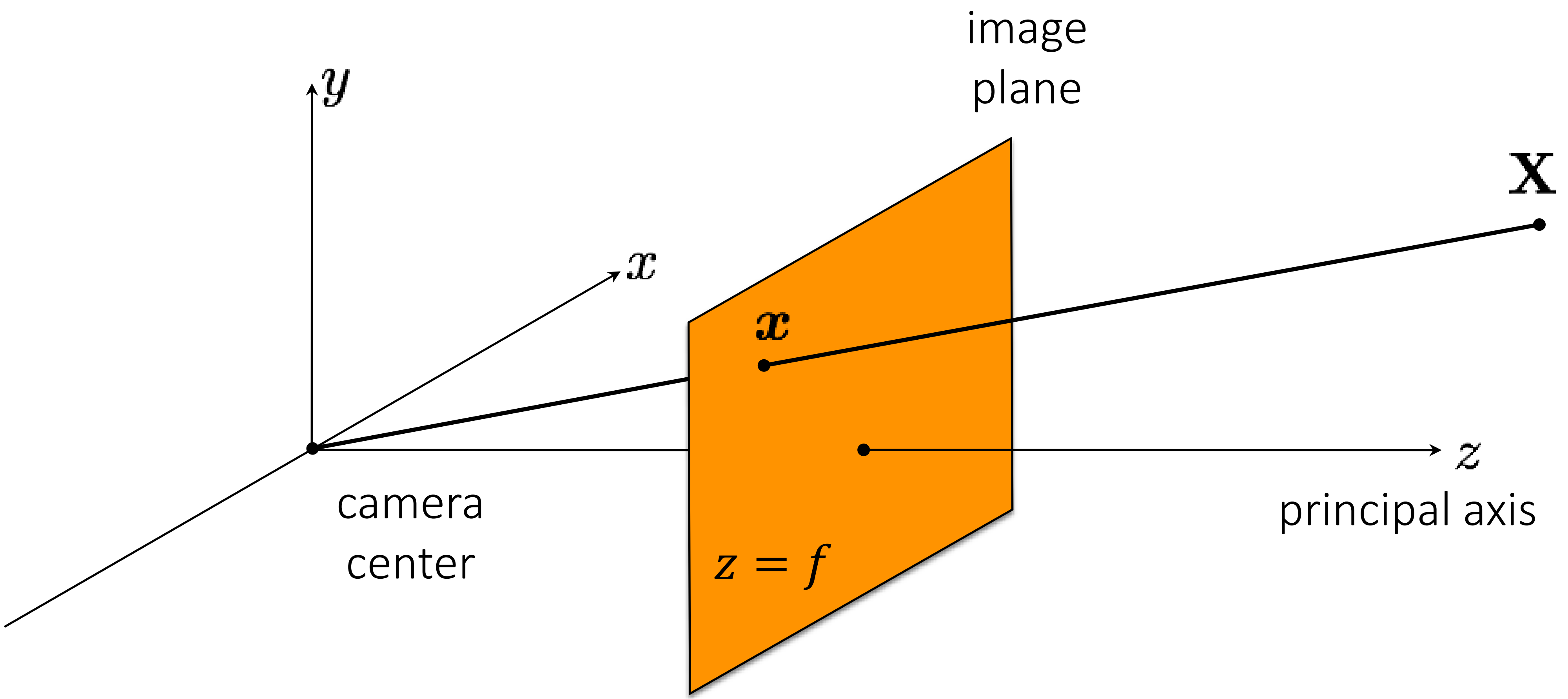
What does the pinhole camera projection look like?

The perspective projection matrix

$$\mathbf{P} = \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] = [\mathbf{I} \quad | \quad \mathbf{0}]$$

alternative way to write the same thing

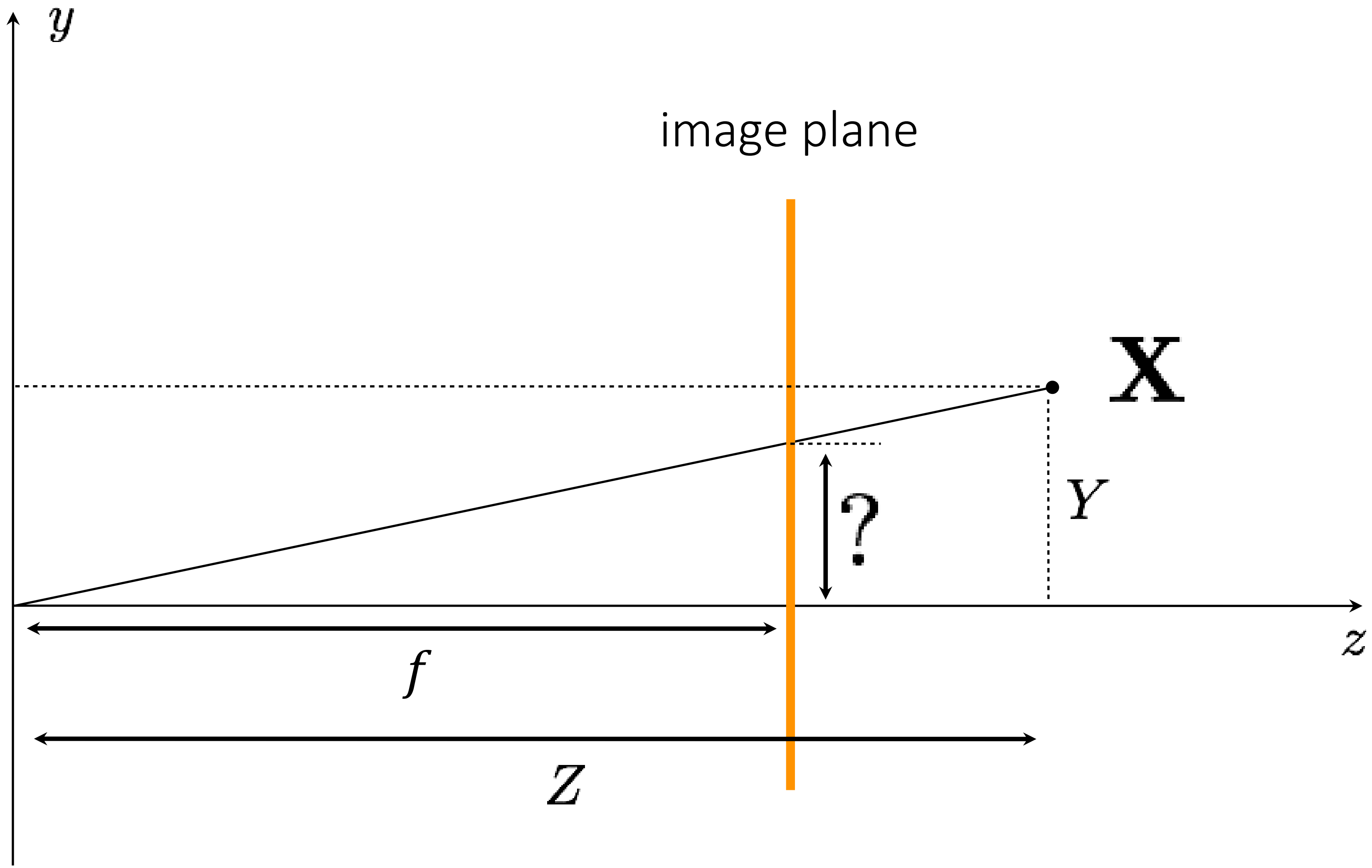
More general case: arbitrary focal length



What is the camera matrix **P** for a pinhole camera?

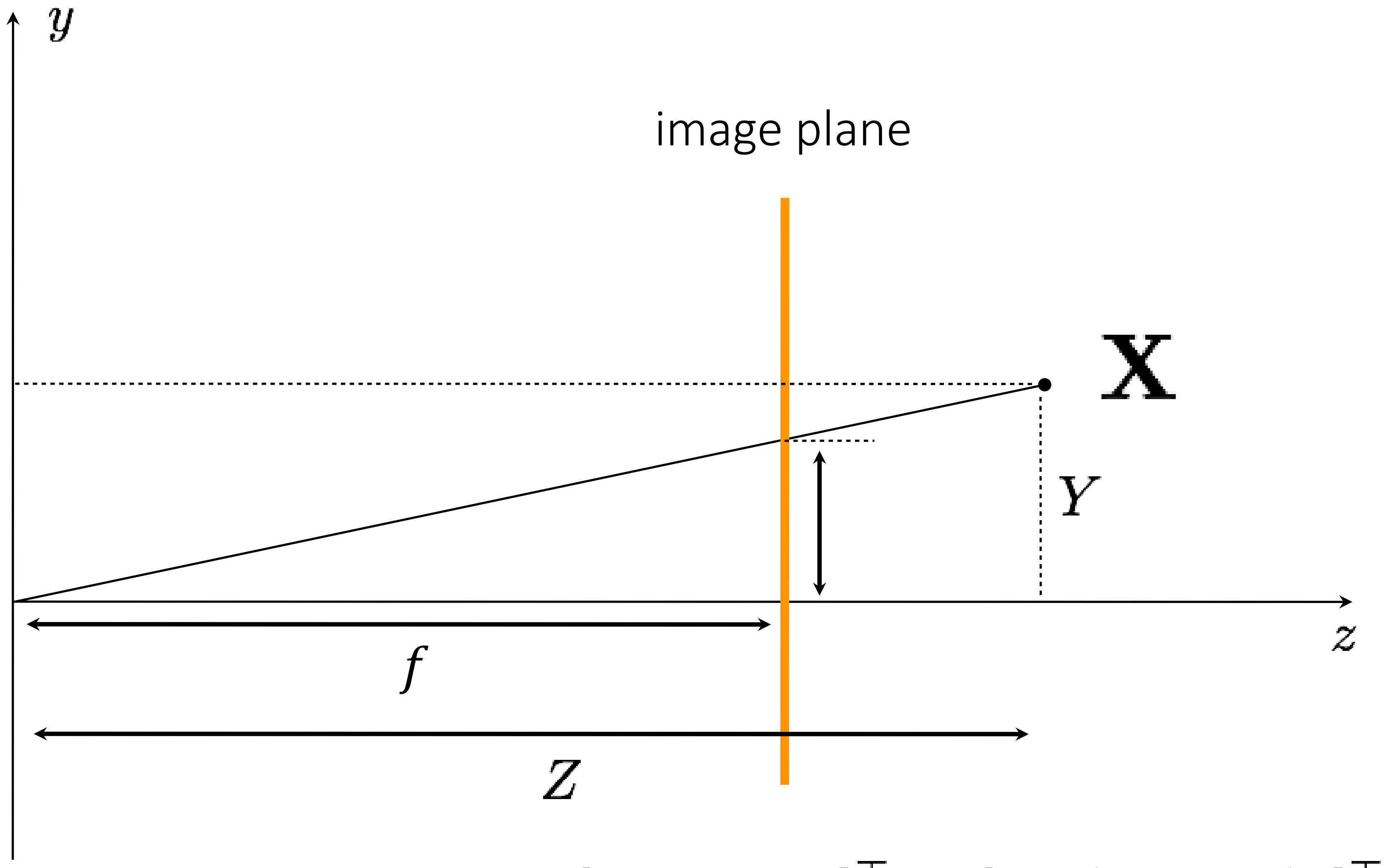
$$\boldsymbol{x} = \mathbf{P}\mathbf{X}$$

More general (2D) case: arbitrary focal length



What is the equation for image coordinate **x** in terms of **X**?

More general (2D) case: arbitrary focal length



$$[X \quad Y \quad Z]^\top \mapsto [fX/Z \quad fY/Z]^\top$$

The pinhole camera matrix for arbitrary focal length

Relationship from similar triangles:

$$[X \quad Y \quad Z]^\top \mapsto [fX/Z \quad fY/Z]^\top$$

General camera model *in homogeneous coordinates*:

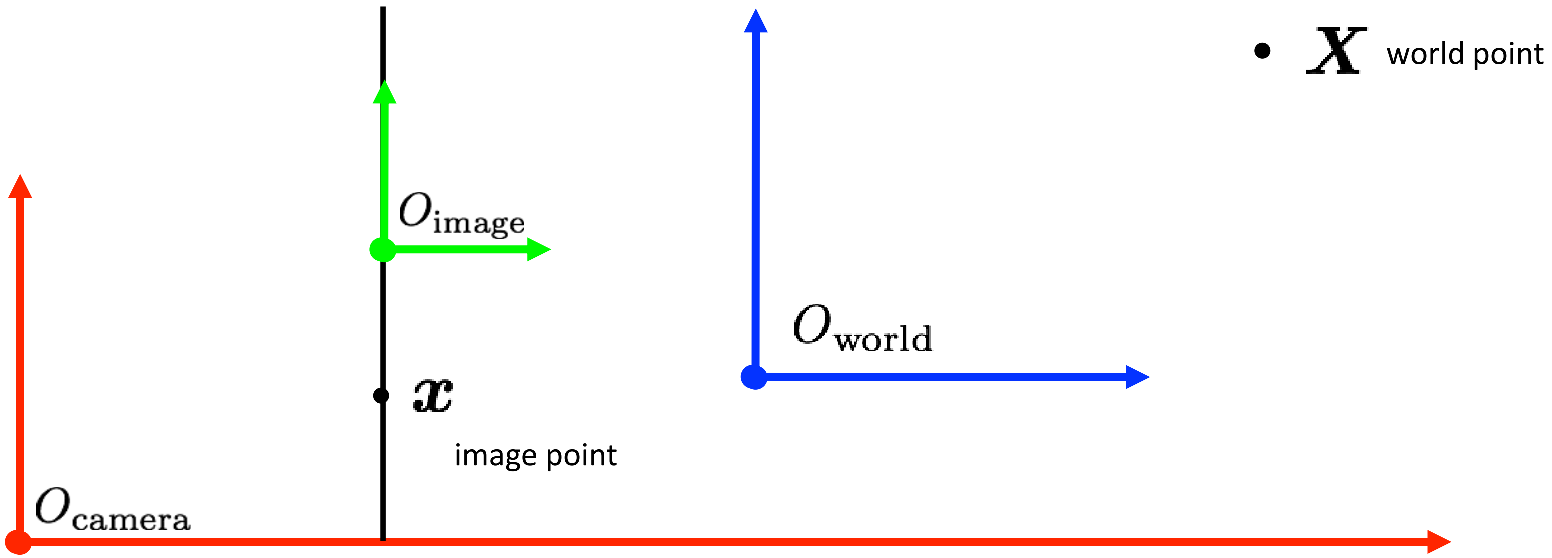
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

What does the pinhole camera projection look like?

$$\mathbf{P} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

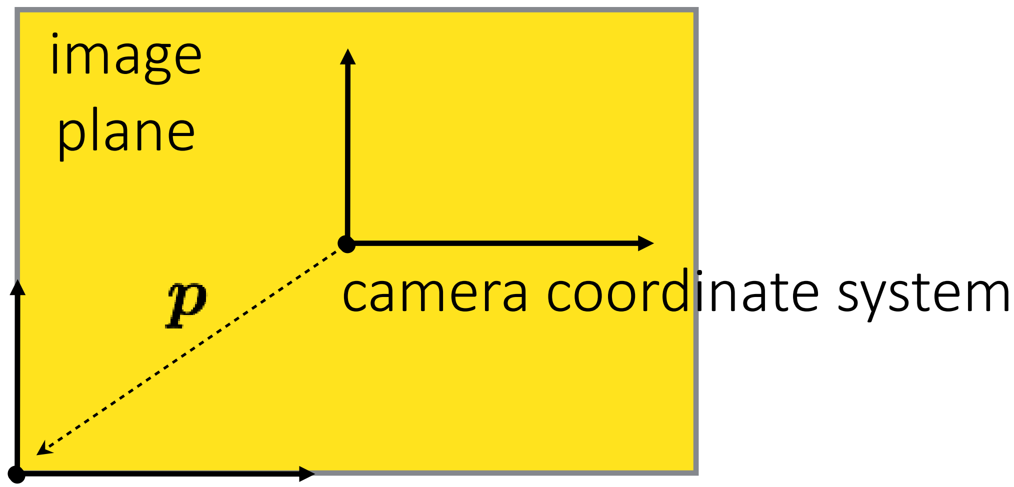
Generalizing the camera matrix

In general, the camera and image have *different* coordinate systems.



Generalizing the camera matrix

In particular, the camera origin and image origin may be different:

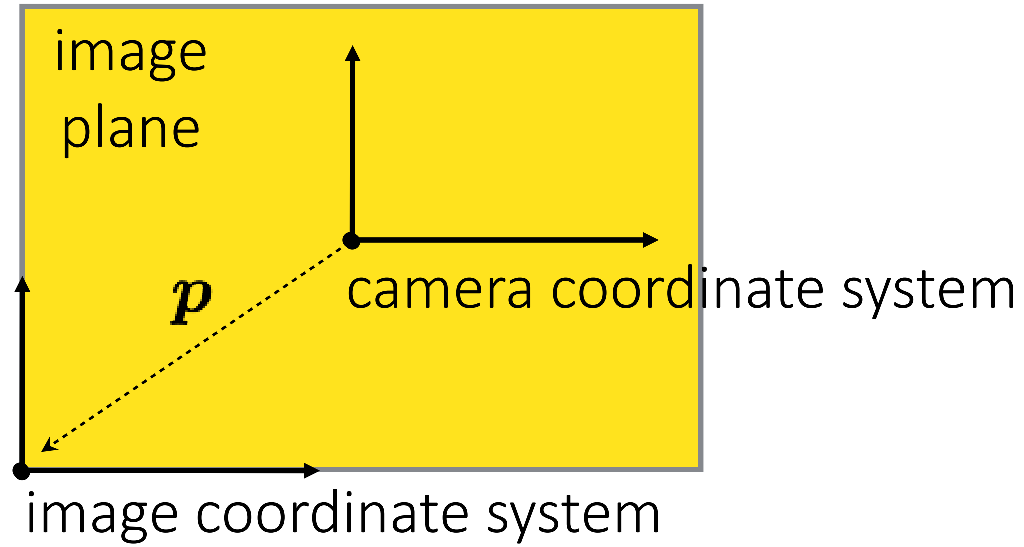


How does the camera matrix change?

$$\mathbf{P} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Generalizing the camera matrix

In particular, the camera origin and image origin may be different:



How does the camera matrix change?

shift vector transforming camera origin to image origin

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x & 0 \\ 0 & f & p_y & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Camera matrix decomposition

We can decompose the camera matrix like this:

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

What does each part of the matrix represent?

Camera matrix decomposition

We can decompose the camera matrix like this:

$$\mathbf{P} = \left[\begin{array}{ccc} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{array}\right] \left[\begin{array}{ccc:c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

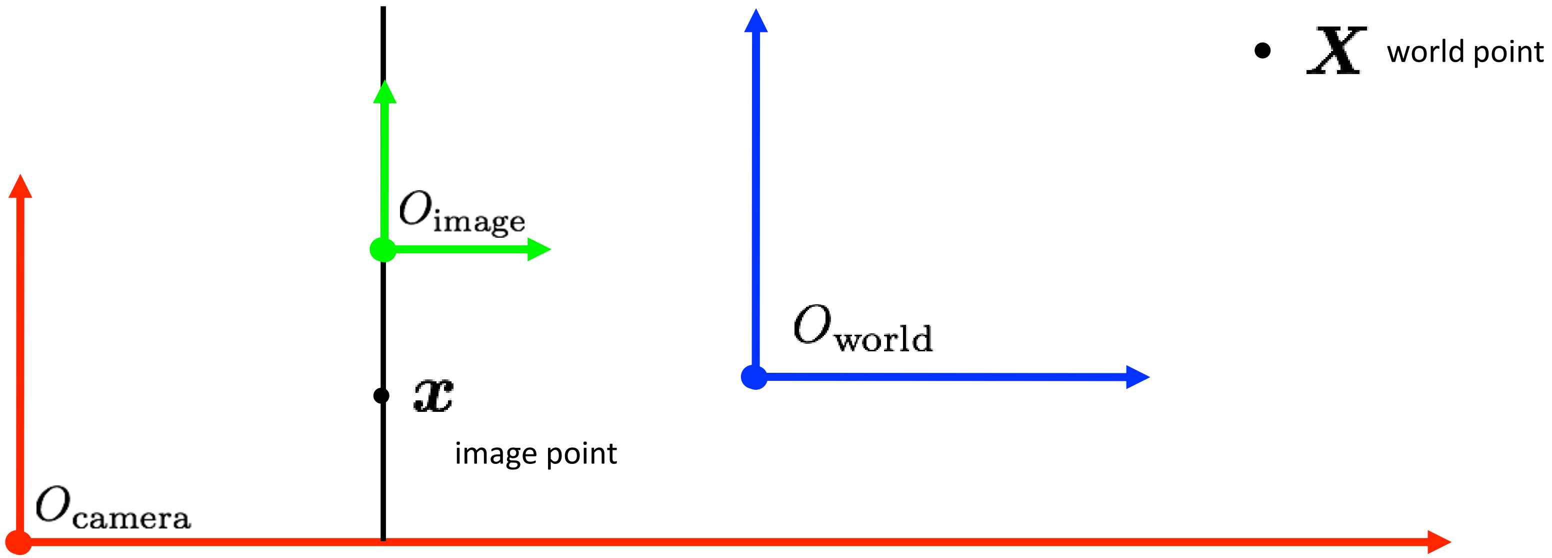
(homogeneous) transformation from 2D to 2D, accounting for not unit focal length and origin shift

(homogeneous) perspective projection from 3D to 2D, assuming image plane at z = 1 and shared camera/image origin

Also written as: $\mathbf{P} = \mathbf{K}[\mathbf{I}|\mathbf{0}]$ where $\mathbf{K} = \left[\begin{array}{ccc} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{array}\right]$

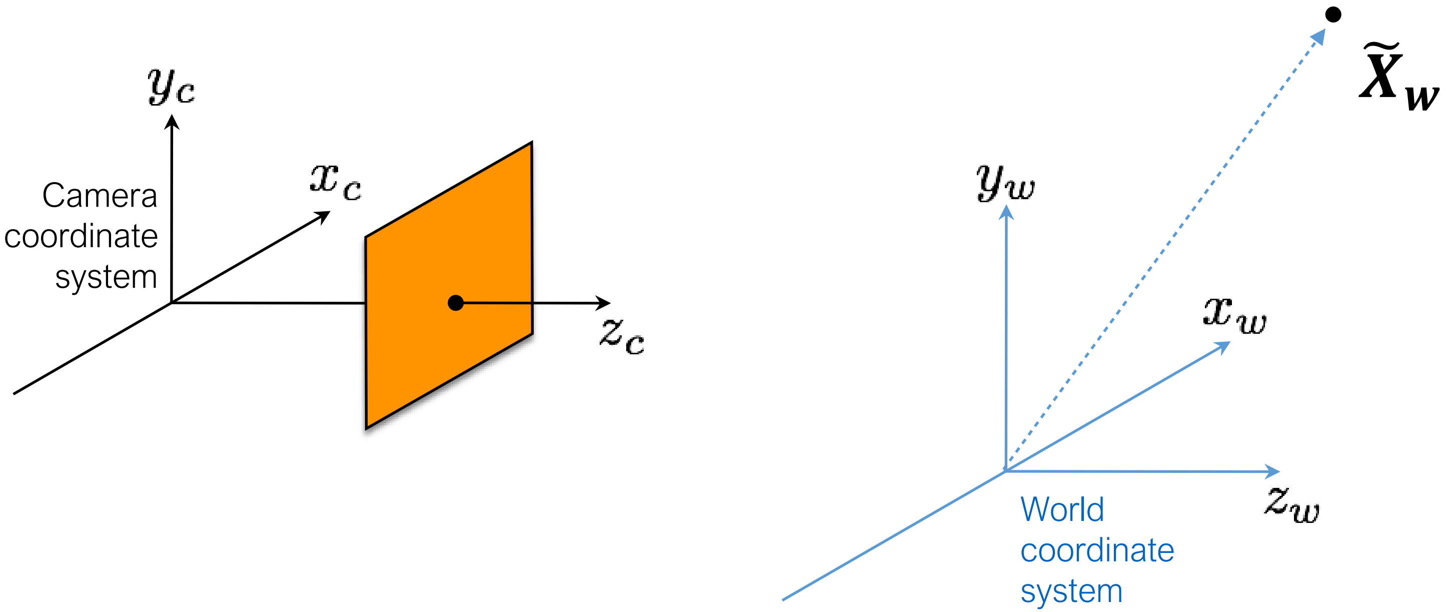
Generalizing the camera matrix

In general, there are *three*, generally different, coordinate systems.



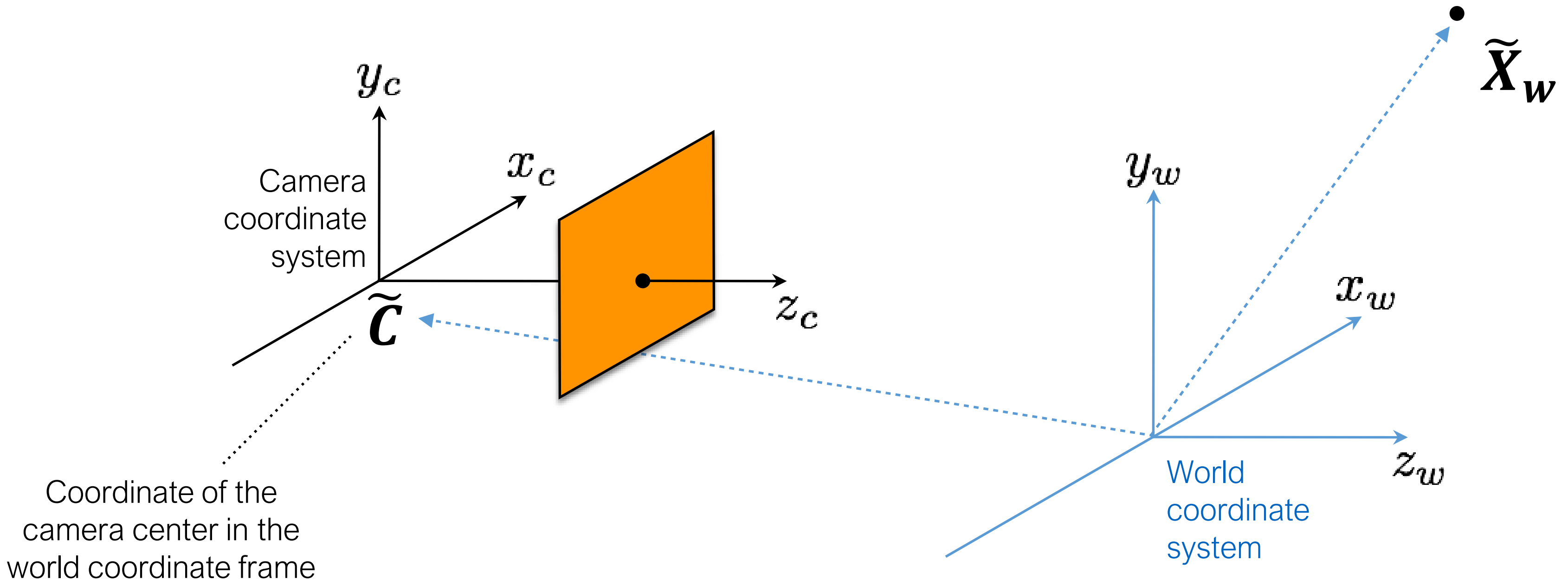
We need to know the transformations between them.

World-to-camera coordinate system transformation

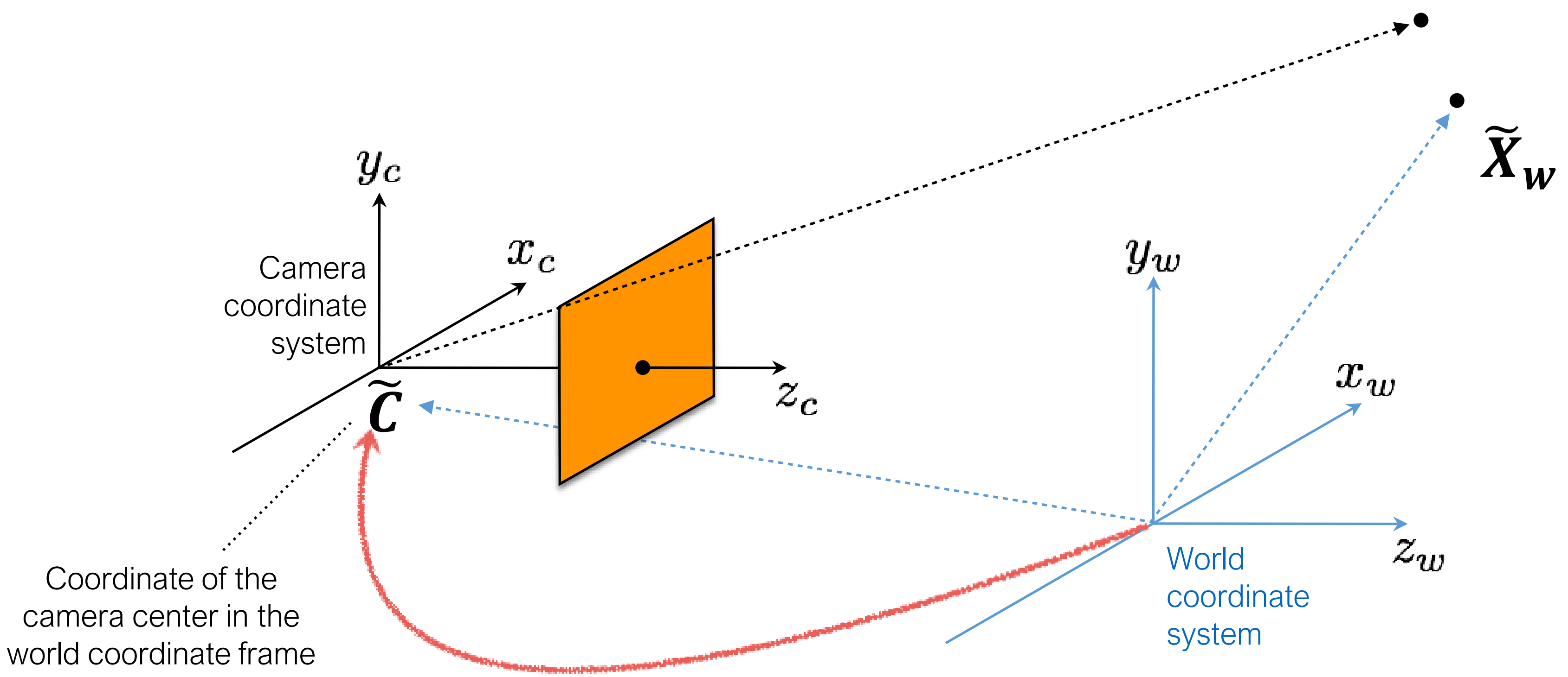


tilde means *heterogeneous* coordinates

World-to-camera coordinate system transformation



World-to-camera coordinate system transformation

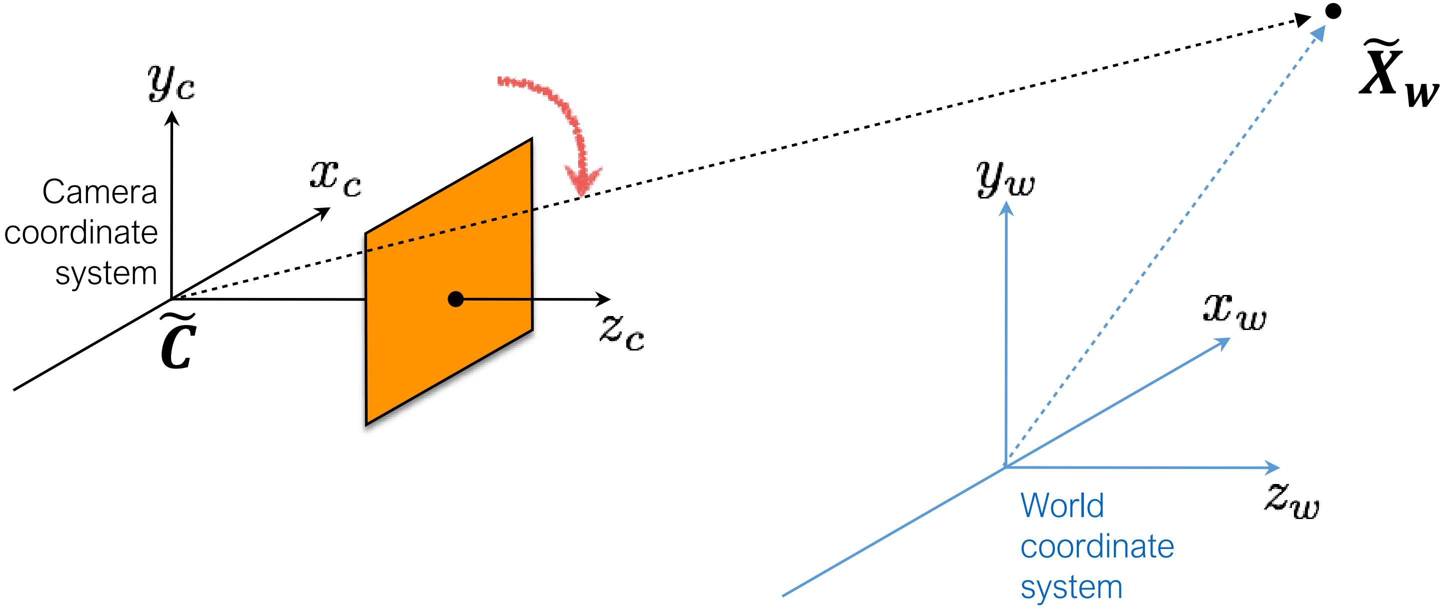


Why aren't the points aligned?

$$(\tilde{X}_w - \tilde{C})$$

translate

World-to-camera coordinate system transformation



points now
coincide

$$\boldsymbol{R} \cdot \left(\tilde{\boldsymbol{X}}_{\boldsymbol{w}} - \tilde{\boldsymbol{C}}\right)$$

rotate translate

Modeling the coordinate system transformation

In heterogeneous coordinates, we have:

$$\widetilde{\mathbf{X}}_{\mathbf{c}} = \mathbf{R} \cdot \left(\widetilde{\mathbf{X}}_{\mathbf{w}} - \tilde{\mathbf{C}}\right)$$

How do we write this transformation in homogeneous coordinates?

Modeling the coordinate system transformation

In heterogeneous coordinates, we have:

$$\widetilde{\mathbf{X}}_{\mathbf{c}} = \mathbf{R} \cdot \left(\widetilde{\mathbf{X}}_{\mathbf{w}} - \tilde{\mathbf{C}}\right)$$

In homogeneous coordinates, we have:

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & -\mathbf{RC} \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

or

$$\mathbf{X_c} = \begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ \mathbf{0} & 1 \end{bmatrix} \mathbf{X_w}$$

Incorporating the transform in the camera matrix

The previous camera matrix is for homogeneous 3D coordinates in camera coordinate system:

$$\mathbf{x} = \mathbf{P}\mathbf{X_c} = \mathbf{K}[\mathbf{I}|\mathbf{0}]\mathbf{X_c}$$

We also just derived:

$$\mathbf{X_c} = \begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ \mathbf{0} & 1 \end{bmatrix} \mathbf{X_w}$$

Putting it all together

We can write everything into a single projection:

$$\mathbf{x} = \mathbf{P}\mathbf{X}_{\mathbf{w}}$$

The camera matrix now looks like:

$$\mathbf{P} = \left[\begin{array}{ccc} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{array}\right] [\mathbf{I} \quad | \quad \mathbf{0}] \begin{bmatrix} \mathbf{R} & -\mathbf{R}\tilde{\mathbf{C}} \\ \mathbf{0} & 1 \end{bmatrix}$$

intrinsic parameters (3 x 3): correspond to camera internals (image-to-image transformation)

perspective projection (3 x 4): maps 3D to 2D points (camera-to-image transformation)

extrinsic parameters (4 x 4): correspond to camera externals (world-to-camera transformation)

Putting it all together

We can write everything into a single projection:

$$\mathbf{x} = \mathbf{P}\mathbf{X}_\mathbf{w}$$

The camera matrix now looks like:

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\mathbf{R} \;\vdots\; -\mathbf{RC} \right]$$

intrinsic parameters (3 x 3):
correspond to camera internals
(sensor not at f = 1 and origin shift)

extrinsic parameters (3 x 4):
correspond to camera externals
(world-to-image transformation)

General pinhole camera matrix

We can decompose the camera matrix like this:

$$\mathbf{P} = \mathbf{K}\mathbf{R}[\mathbf{I} | -\mathbf{C}]$$

(translate first then rotate)

Another way to write the mapping:

$$\mathbf{P} = \mathbf{K}[\mathbf{R}|\mathbf{t}]$$

where $\mathbf{t} = -\mathbf{R}\mathbf{C}$

(rotate first then translate)

General pinhole camera matrix

$$\mathbf{P} = \mathbf{K}[\mathbf{R}|\mathbf{t}]$$

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc:c} r_1 & r_2 & r_3 & t_1 \\ r_4 & r_5 & r_6 & t_2 \\ r_7 & r_8 & r_9 & t_3 \end{array}\right]$$

intrinsic parameters

extrinsic parameters

$$\mathbf{R} = \begin{bmatrix} r_1 & r_2 & r_3 \\ r_4 & r_5 & r_6 \\ r_7 & r_8 & r_9 \end{bmatrix} \qquad \mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \end{bmatrix}$$

3D rotation

3D translation

More general camera matrices

The following is the standard camera matrix we saw.

$$\mathbf{P} = \begin{bmatrix} f & 0 & p_x \\ 0 & f & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\mathbf{R} \;\vdots\; -\mathbf{RC} \right]$$

More general camera matrices

CCD camera: pixels may not be square.

$$\mathbf{P} = \begin{bmatrix} \alpha_x & 0 & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\mathbf{R} \;\vdots\; -\mathbf{RC} \right]$$

How many degrees of freedom?

More general camera matrices

CCD camera: pixels may not be square.

$$\mathbf{P} = \begin{bmatrix} \alpha_x & 0 & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\mathbf{R} \mid -\mathbf{RC} \right]$$

How many degrees of freedom?

10 DOF

More general camera matrices

Finite projective camera: sensor be skewed.

$$\mathbf{P} = \begin{bmatrix} \alpha_x & s & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\mathbf{R} \;\middle|\; -\mathbf{RC} \right]$$

How many degrees of freedom?

More general camera matrices

Finite projective camera: sensor be skewed.

$$\mathbf{P} = \begin{bmatrix} \alpha_x & s & p_x \\ 0 & \alpha_y & p_y \\ 0 & 0 & 1 \end{bmatrix} \left[\mathbf{R} \;\vdots\; -\mathbf{RC} \right]$$

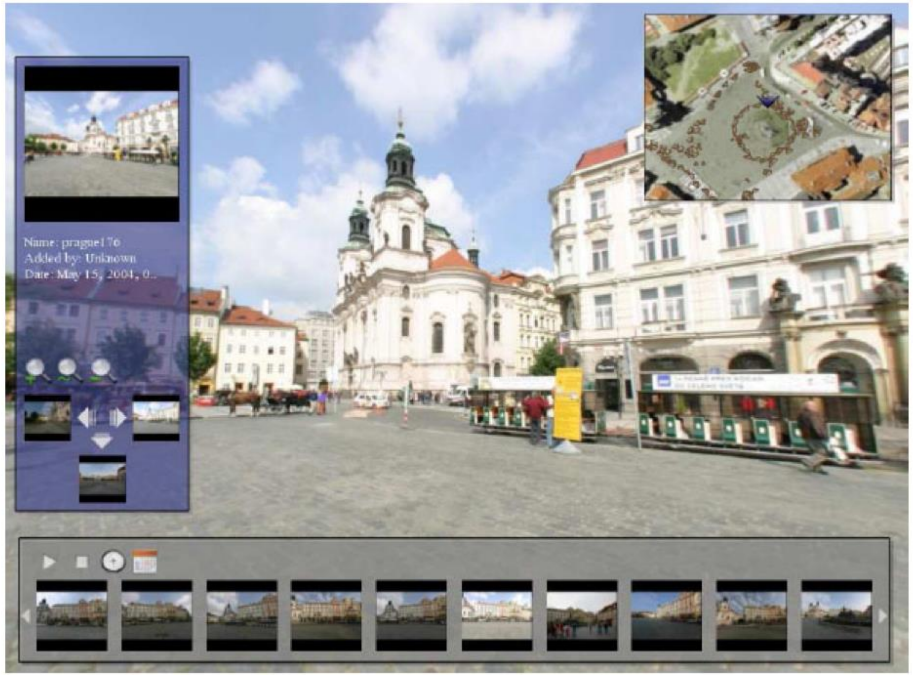
How many degrees of freedom?

11 DOF

Geometric camera calibration

	Structure (scene geometry)	Motion (camera geometry)	**Measurements**
Camera Calibration (a.k.a. Pose Estimation)	known	estimate	3D to 2D correspond ences
Triangulation	estimate	known	2D to 2D coorespond ences
Reconstruction	estimate	estimate	2D to 2D coorespond ences

Pose Estimation



Given a single image,
estimate the exact position of the photographer

Geometric camera calibration

Given a set of matched points

$$\{\mathbf{X}_i, \boldsymbol{x}_i\}$$

point in 3D space

point in the image

and camera model

$$\boldsymbol{x} = \boldsymbol{f}(\mathbf{X}; \boldsymbol{p}) = \mathbf{P}\mathbf{X}$$

projection model

parameters

Camera matrix

Find the (pose) estimate of

$$\mathbf{P}$$

We'll use a **perspective** camera model for pose estimation

Same setup as homography estimation

(slightly different derivation here)

Mapping between 3D point and image points

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

What are the unknowns?

Mapping between 3D point and image points

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} — & \boldsymbol{p}_1^\top & — \\ — & \boldsymbol{p}_2^\top & — \\ — & \boldsymbol{p}_3^\top & — \end{bmatrix} \begin{bmatrix} | \\ \boldsymbol{X} \\ | \end{bmatrix}$$

Heterogeneous coordinates

$$x' = \frac{\boldsymbol{p}_1^\top \boldsymbol{X}}{\boldsymbol{p}_3^\top \boldsymbol{X}} \qquad y' = \frac{\boldsymbol{p}_2^\top \boldsymbol{X}}{\boldsymbol{p}_3^\top \boldsymbol{X}}$$

(non-linear relation between coordinates)

How can we make these relations linear?

Mapping between 3D point and image points

$$x' = \frac{\boldsymbol{p}_1^\top \boldsymbol{X}}{\boldsymbol{p}_3^\top \boldsymbol{X}} \qquad y' = \frac{\boldsymbol{p}_2^\top \boldsymbol{X}}{\boldsymbol{p}_3^\top \boldsymbol{X}}$$

Make them linear with algebraic manipulation…

$$\boldsymbol{p}_2^\top \boldsymbol{X} - \boldsymbol{p}_3^\top \boldsymbol{X} y' = 0$$

$$\boldsymbol{p}_1^\top \boldsymbol{X} - \boldsymbol{p}_3^\top \boldsymbol{X} x' = 0$$

Now we can setup a system of linear equations with multiple point correspondences

Mapping between 3D point and image points

$$\boldsymbol{p}_2^\top \boldsymbol{X} - \boldsymbol{p}_3^\top \boldsymbol{X} y' = 0$$

$$\boldsymbol{p}_1^\top \boldsymbol{X} - \boldsymbol{p}_3^\top \boldsymbol{X} x' = 0$$

How do we proceed?

Mapping between 3D point and image points

$$\boldsymbol{p}_2^\top \boldsymbol{X} - \boldsymbol{p}_3^\top \boldsymbol{X} y' = 0$$

$$\boldsymbol{p}_1^\top \boldsymbol{X} - \boldsymbol{p}_3^\top \boldsymbol{X} x' = 0$$

In matrix form … $$\begin{bmatrix} \boldsymbol{X}^\top & \boldsymbol{0} & -x'\boldsymbol{X}^\top \\ \boldsymbol{0} & \boldsymbol{X}^\top & -y'\boldsymbol{X}^\top \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \end{bmatrix} = \boldsymbol{0}$$

How do we proceed?

Mapping between 3D point and image points

$$\boldsymbol{p}_2^\top \boldsymbol{X} - \boldsymbol{p}_3^\top \boldsymbol{X} y' = 0$$

$$\boldsymbol{p}_1^\top \boldsymbol{X} - \boldsymbol{p}_3^\top \boldsymbol{X} x' = 0$$

In matrix form …

$$\begin{bmatrix} \boldsymbol{X}^\top & \boldsymbol{0} & -x'\boldsymbol{X}^\top \\ \boldsymbol{0} & \boldsymbol{X}^\top & -y'\boldsymbol{X}^\top \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \end{bmatrix} = \boldsymbol{0}$$

For N points …

$$\begin{bmatrix} \boldsymbol{X}_1^\top & \boldsymbol{0} & -x'\boldsymbol{X}_1^\top \\ \boldsymbol{0} & \boldsymbol{X}_1^\top & -y'\boldsymbol{X}_1^\top \\ \vdots & \vdots & \vdots \\ \boldsymbol{X}_N^\top & \boldsymbol{0} & -x'\boldsymbol{X}_N^\top \\ \boldsymbol{0} & \boldsymbol{X}_N^\top & -y'\boldsymbol{X}_N^\top \end{bmatrix} \begin{bmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \end{bmatrix} = \boldsymbol{0}$$

How do we solve this system?

Mapping between 3D point and image points

Solve for camera matrix by

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \|\mathbf{A}\boldsymbol{x}\|^2 \text{ subject to } \|\boldsymbol{x}\|^2 = 1$$

$$\mathbf{A} = \begin{bmatrix} \boldsymbol{X}_1^\top & \mathbf{0} & -x'\boldsymbol{X}_1^\top \\ \mathbf{0} & \boldsymbol{X}_1^\top & -y'\boldsymbol{X}_1^\top \\ \vdots & \vdots & \vdots \\ \boldsymbol{X}_N^\top & \mathbf{0} & -x'\boldsymbol{X}_N^\top \\ \mathbf{0} & \boldsymbol{X}_N^\top & -y'\boldsymbol{X}_N^\top \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \end{bmatrix}$$

SVD!

Mapping between 3D point and image points

Solve for camera matrix by

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \|\mathbf{A}\boldsymbol{x}\|^2 \text{ subject to } \|\boldsymbol{x}\|^2 = 1$$

$$\mathbf{A} = \begin{bmatrix} \boldsymbol{X}_1^\top & \mathbf{0} & -x'\boldsymbol{X}_1^\top \\ \mathbf{0} & \boldsymbol{X}_1^\top & -y'\boldsymbol{X}_1^\top \\ \vdots & \vdots & \vdots \\ \boldsymbol{X}_N^\top & \mathbf{0} & -x'\boldsymbol{X}_N^\top \\ \mathbf{0} & \boldsymbol{X}_N^\top & -y'\boldsymbol{X}_N^\top \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \end{bmatrix}$$

Solution **x** is the column of **V** corresponding to smallest singular value of

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$$

Mapping between 3D point and image points

Solve for camera matrix by

$$\hat{\boldsymbol{x}} = \arg\min_{\boldsymbol{x}} \|\mathbf{A}\boldsymbol{x}\|^2 \text{ subject to } \|\boldsymbol{x}\|^2 = 1$$

$$\mathbf{A} = \begin{bmatrix} \boldsymbol{X}_1^\top & \mathbf{0} & -x'\boldsymbol{X}_1^\top \\ \mathbf{0} & \boldsymbol{X}_1^\top & -y'\boldsymbol{X}_1^\top \\ \vdots & \vdots & \vdots \\ \boldsymbol{X}_N^\top & \mathbf{0} & -x'\boldsymbol{X}_N^\top \\ \mathbf{0} & \boldsymbol{X}_N^\top & -y'\boldsymbol{X}_N^\top \end{bmatrix} \qquad \boldsymbol{x} = \begin{bmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \boldsymbol{p}_3 \end{bmatrix}$$

Equivalently, solution ***x*** is the Eigenvector corresponding to smallest Eigenvalue of $\mathbf{A}^\top\mathbf{A}$

Mapping between 3D point and image points

Now we have:
$$\mathbf{P} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix}$$

Are we done?

Mapping between 3D point and image points

Almost there …

$$\mathbf{P} = \begin{bmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{bmatrix}$$

How do you get the intrinsic and extrinsic parameters from the projection matrix?

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array} \right]$$

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array}\right]$$

$$\mathbf{P} = \mathbf{K}[\mathbf{R}|\mathbf{t}]$$

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array} \right]$$

$$\begin{aligned} \mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R}| - \mathbf{Rc}] \\ &= [\mathbf{M}| - \mathbf{Mc}] \end{aligned}$$

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array}\right]$$

$$\begin{aligned} \mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R}| - \mathbf{Rc}] \\ &= [\mathbf{M}| - \mathbf{Mc}] \end{aligned}$$

Find the camera center **C**

What is the projection of the camera center?

Find intrinsic **K** and rotation **R**

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array}\right]$$

$$\begin{aligned}\mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R}| - \mathbf{Rc}] \\ &= [\mathbf{M}| - \mathbf{Mc}]\end{aligned}$$

Find the camera center **C**

$$\mathbf{Pc} = \mathbf{0}$$

How do we compute the camera center from this?

Find intrinsic **K** and rotation **R**

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array}\right]$$

$$\begin{aligned} \mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R}| - \mathbf{Rc}] \\ &= [\mathbf{M}| - \mathbf{Mc}] \end{aligned}$$

Find the camera center **C**

$$\mathbf{Pc} = \mathbf{0}$$

SVD of P!

***c** is the Eigenvector corresponding to smallest Eigenvalue*

Find intrinsic **K** and rotation **R**

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array}\right]$$

$$\begin{aligned} \mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R}| - \mathbf{Rc}] \\ &= [\mathbf{M}| - \mathbf{Mc}] \end{aligned}$$

Find the camera center **C**

$$\mathbf{Pc} = \mathbf{0}$$

SVD of P!

***c** is the Eigenvector corresponding to smallest Eigenvalue*

Find intrinsic **K** and rotation **R**

$$\mathbf{M} = \mathbf{KR}$$

Any useful properties of K and R we can use?

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array}\right]$$

$$\begin{aligned} \mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R}| - \mathbf{Rc}] \\ &= [\mathbf{M}| - \mathbf{Mc}] \end{aligned}$$

Find the camera center **C**

$$\mathbf{Pc} = \mathbf{0}$$

SVD of P!

***c** is the Eigenvector corresponding to smallest Eigenvalue*

Find intrinsic **K** and rotation **R**

$$\mathbf{M} = \mathbf{KR}$$

right upper triangle

orthogonal

How do we find K and R?

Decomposition of the Camera Matrix

$$\mathbf{P} = \left[\begin{array}{ccc|c} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{array}\right]$$

$$\begin{aligned} \mathbf{P} &= \mathbf{K}[\mathbf{R}|\mathbf{t}] \\ &= \mathbf{K}[\mathbf{R}| - \mathbf{Rc}] \\ &= [\mathbf{M}| - \mathbf{Mc}] \end{aligned}$$

Find the camera center **C**

$$\mathbf{Pc} = \mathbf{0}$$

SVD of P!

***c** is the Eigenvector corresponding to smallest Eigenvalue*

Find intrinsic **K** and rotation **R**

$$\mathbf{M} = \mathbf{KR}$$

QR decomposition

Geometric camera calibration

Given a set of matched points

$$\{\mathbf{X}_i, \boldsymbol{x}_i\}$$

point in 3D space

point in the image

Where do we get these matched points from?

and camera model

$$\boldsymbol{x} = \boldsymbol{f}(\mathbf{X}; \boldsymbol{p}) = \mathbf{P}\mathbf{X}$$

projection model

parameters

Camera matrix

Find the (pose) estimate of

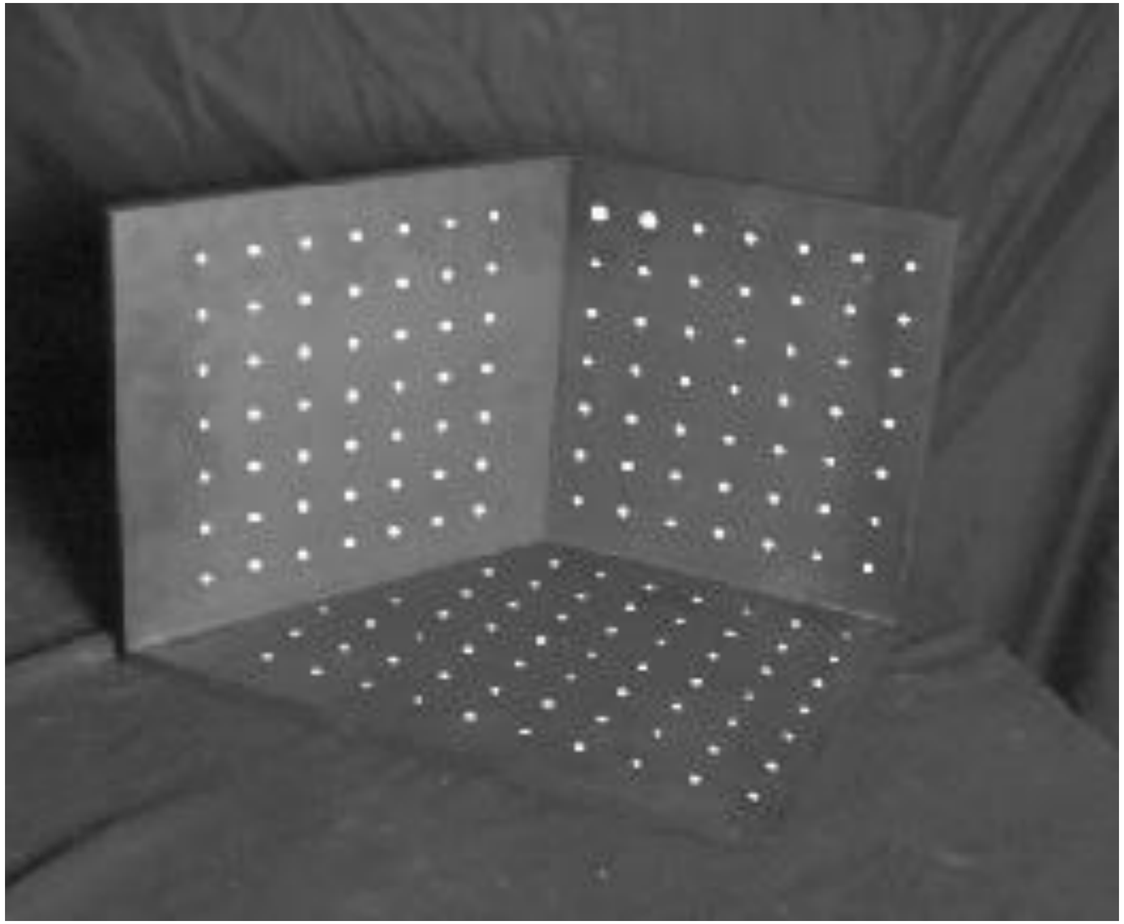
$$\mathbf{P}$$

We'll use a **perspective** camera model for pose estimation

Calibration using a reference object

Place a known object in the scene:

- identify correspondences between image and scene
- compute mapping from scene to image



Issues:

- must know geometry very accurately
- must know 3D->2D correspondence

Geometric camera calibration

Advantages:

- Very simple to formulate.
- Analytical solution.

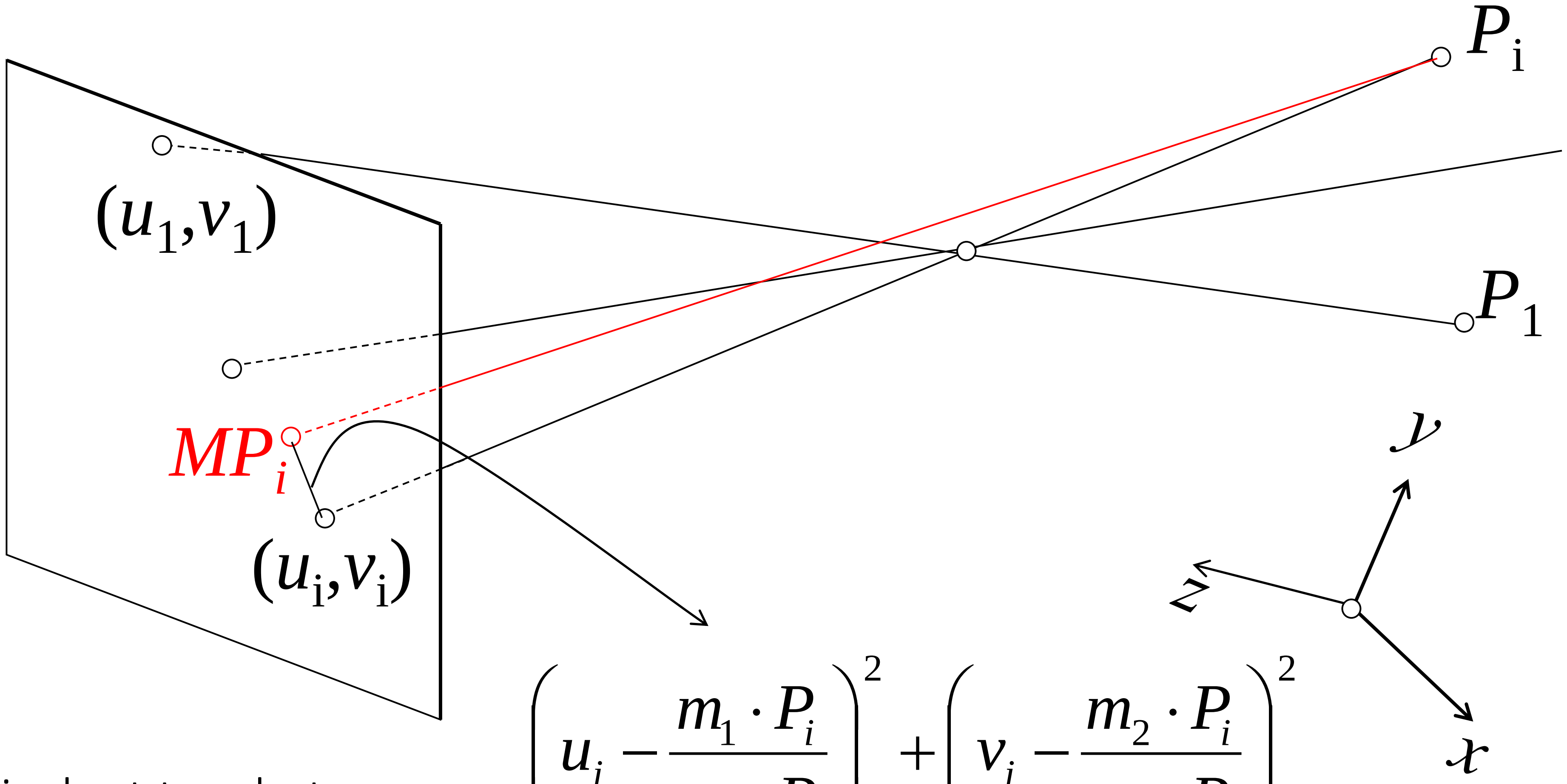
Disadvantages:

- Doesn’t model radial distortion.
- Hard to impose constraints (e.g., known *f*).
- Doesn’t minimize the correct error function.

For these reasons, *nonlinear methods* are preferred

- Define error function E between projected 3D points and image positions
  - E is nonlinear function of intrinsics, extrinsics, radial distortion
- Minimize E using nonlinear optimization techniques

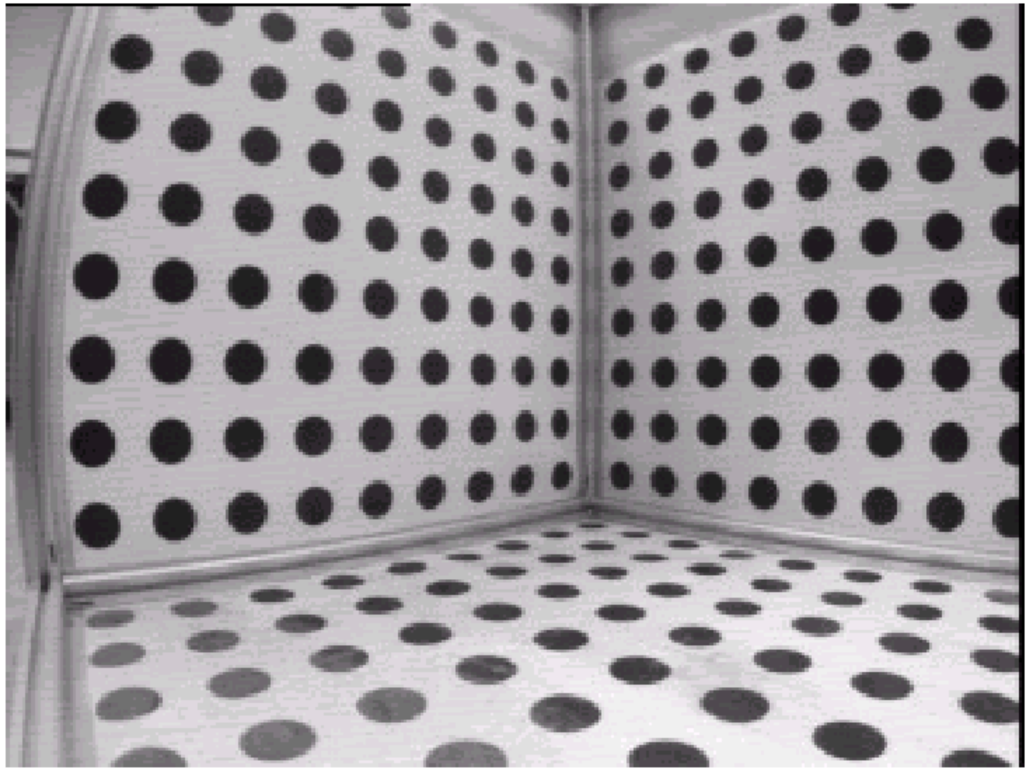
Minimizing reprojection error



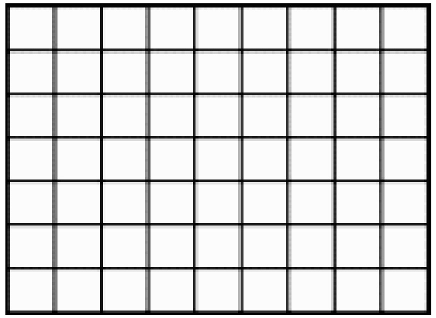
$$\left(u_i - \frac{m_1 \cdot P_i}{m_3 \cdot P_i}\right)^2 + \left(v_i - \frac{m_2 \cdot P_i}{m_3 \cdot P_i}\right)^2$$

Is this equivalent to what we were doing previously?

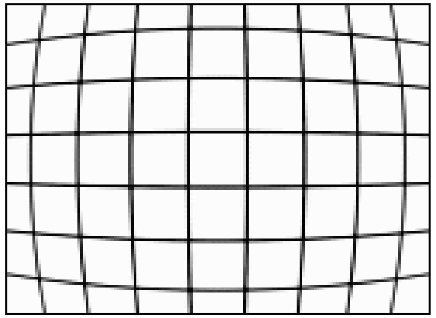
Radial distortion



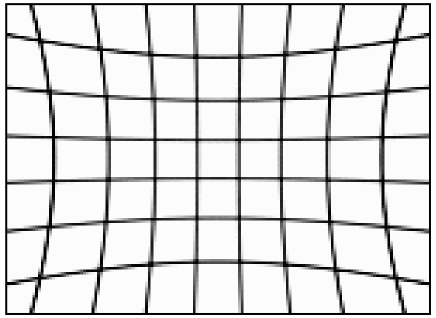
What causes this distortion?



no distortion

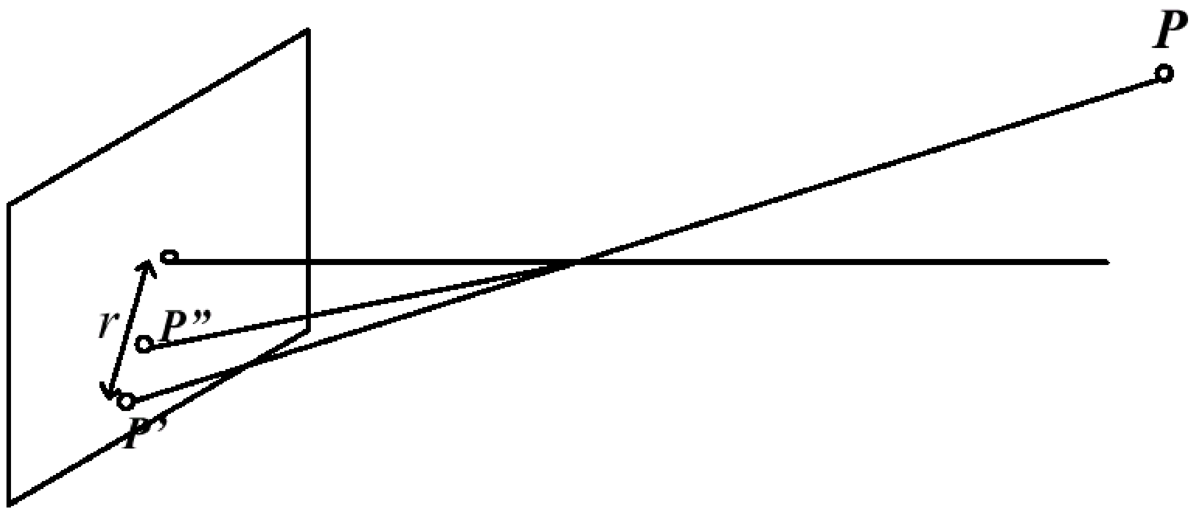


barrel distortion



pincushion distortion

Radial distortion model



Ideal:

$$x' = f\frac{x}{z}$$

$$y' = f\frac{y}{z}$$

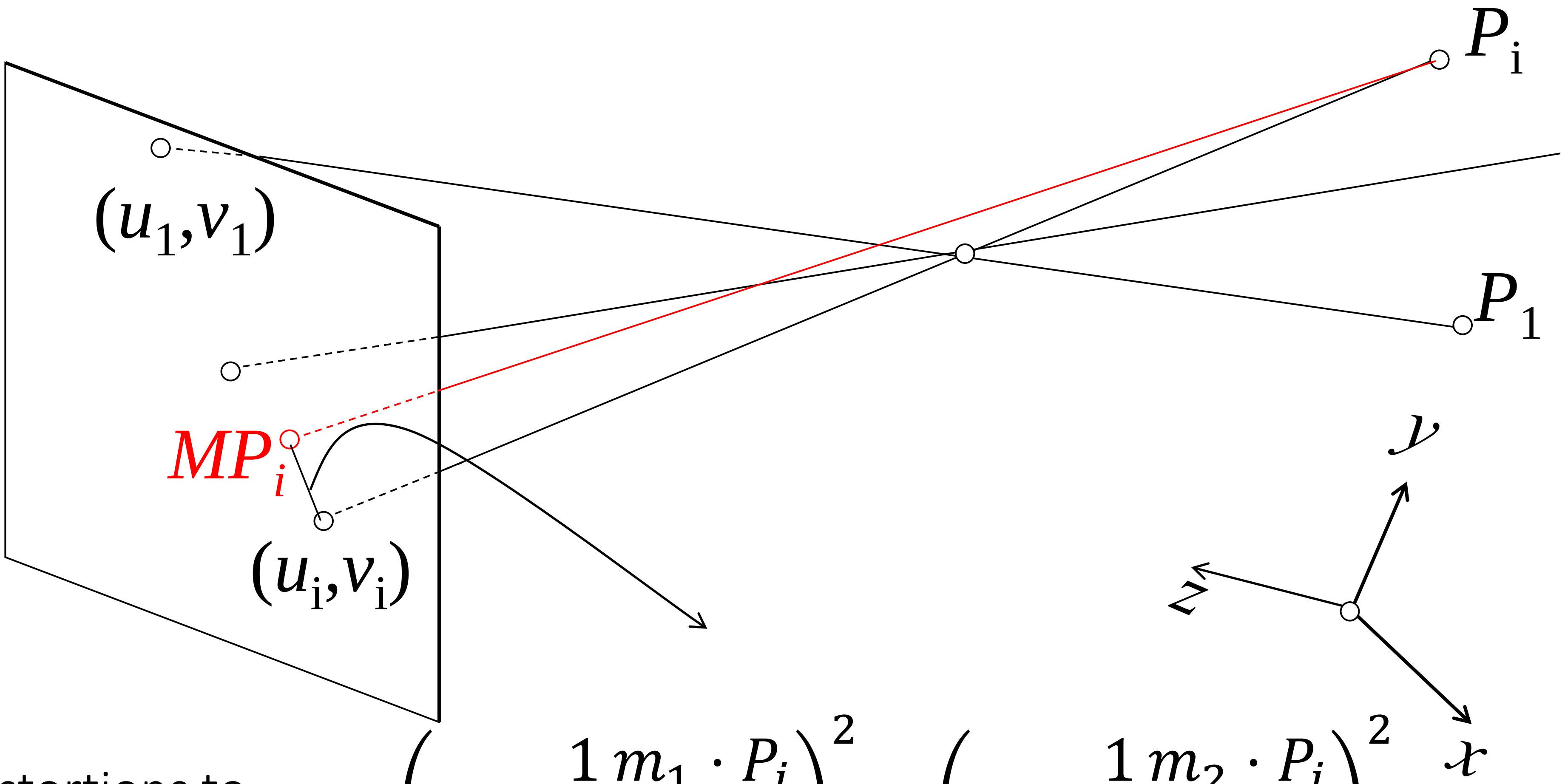
Distorted:

$$x'' = \frac{1}{\lambda}x'$$

$$y'' = \frac{1}{\lambda}y'$$

$$\lambda = 1 + k_1 r^2 + k_2 r^4 + \cdots$$

Minimizing reprojection error with radial distortion



Add distortions to reprojection error:

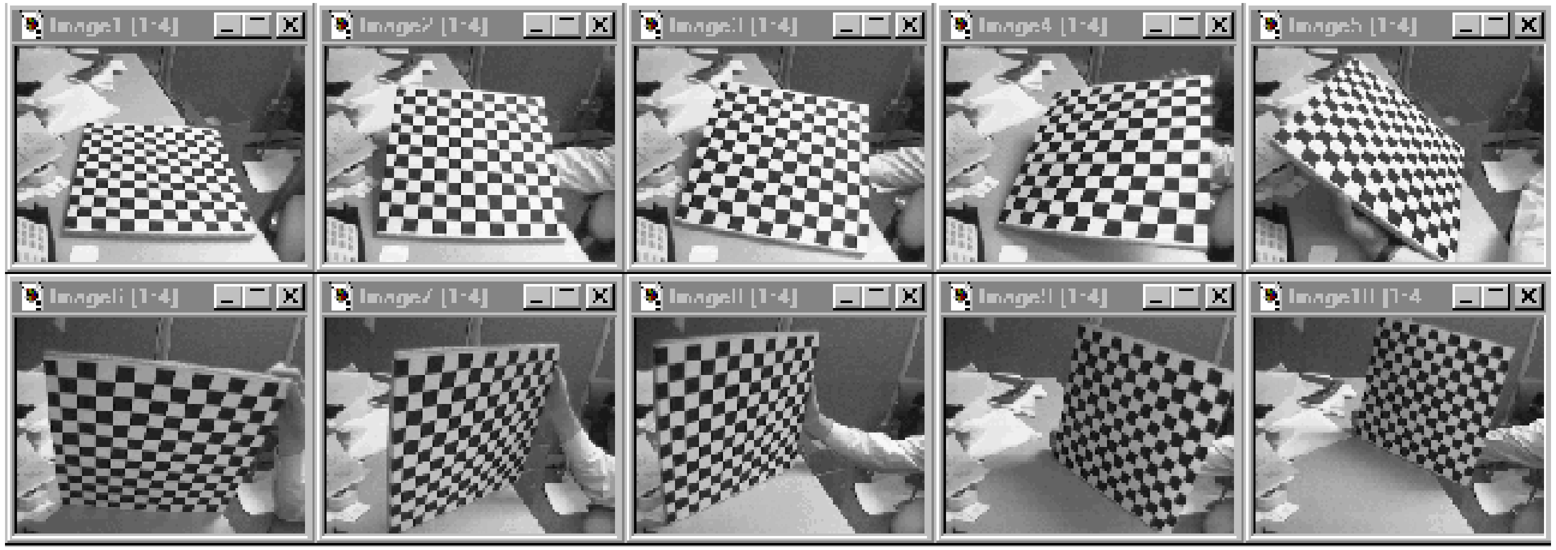
$$\left(u_i - \frac{1}{\lambda}\frac{m_1 \cdot P_i}{m_3 \cdot P_i}\right)^2 + \left(v_i - \frac{1}{\lambda}\frac{m_2 \cdot P_i}{m_3 \cdot P_i}\right)^2$$

Correcting radial distortion

before

after

Alternative: Multi-plane calibration

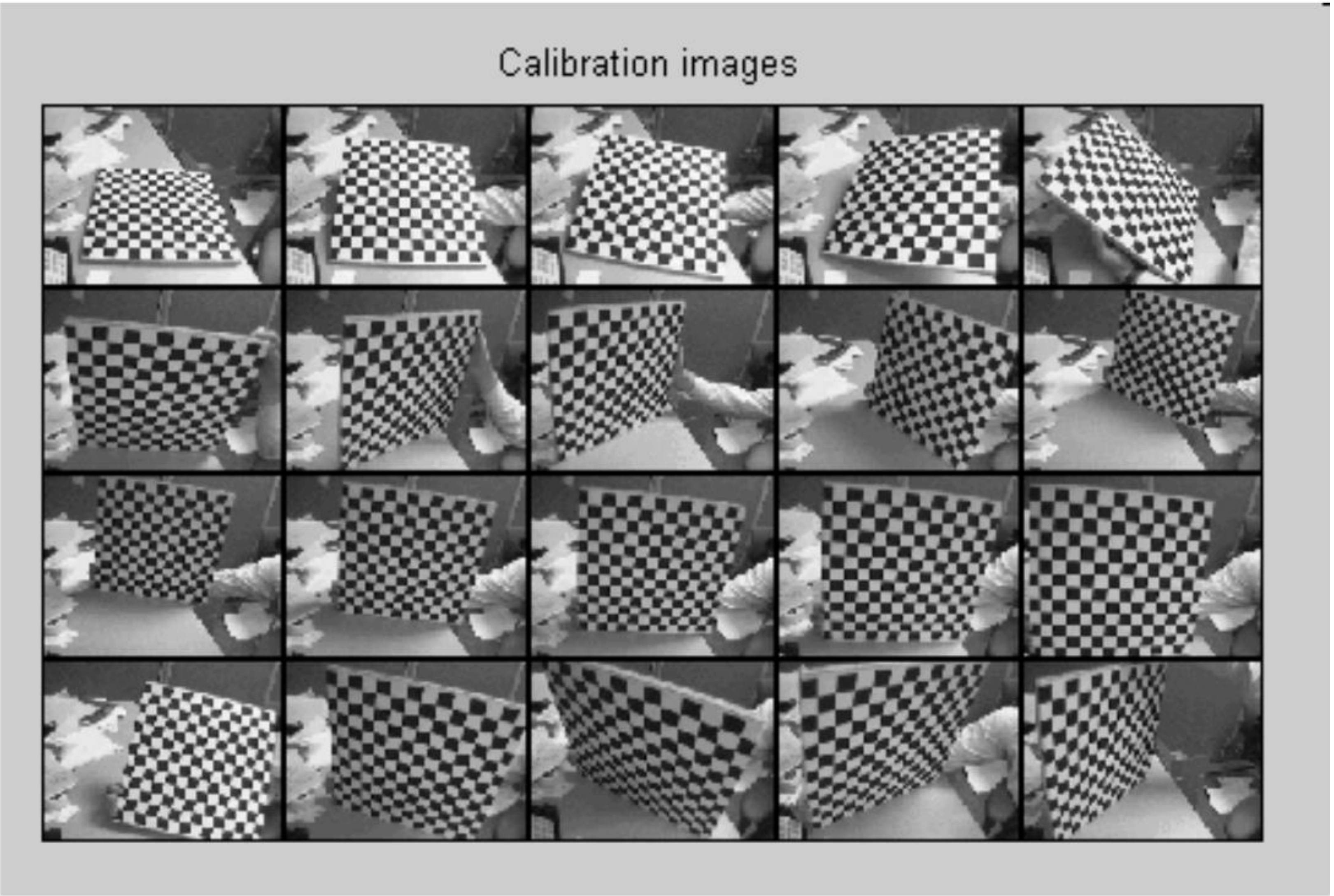


Advantages:

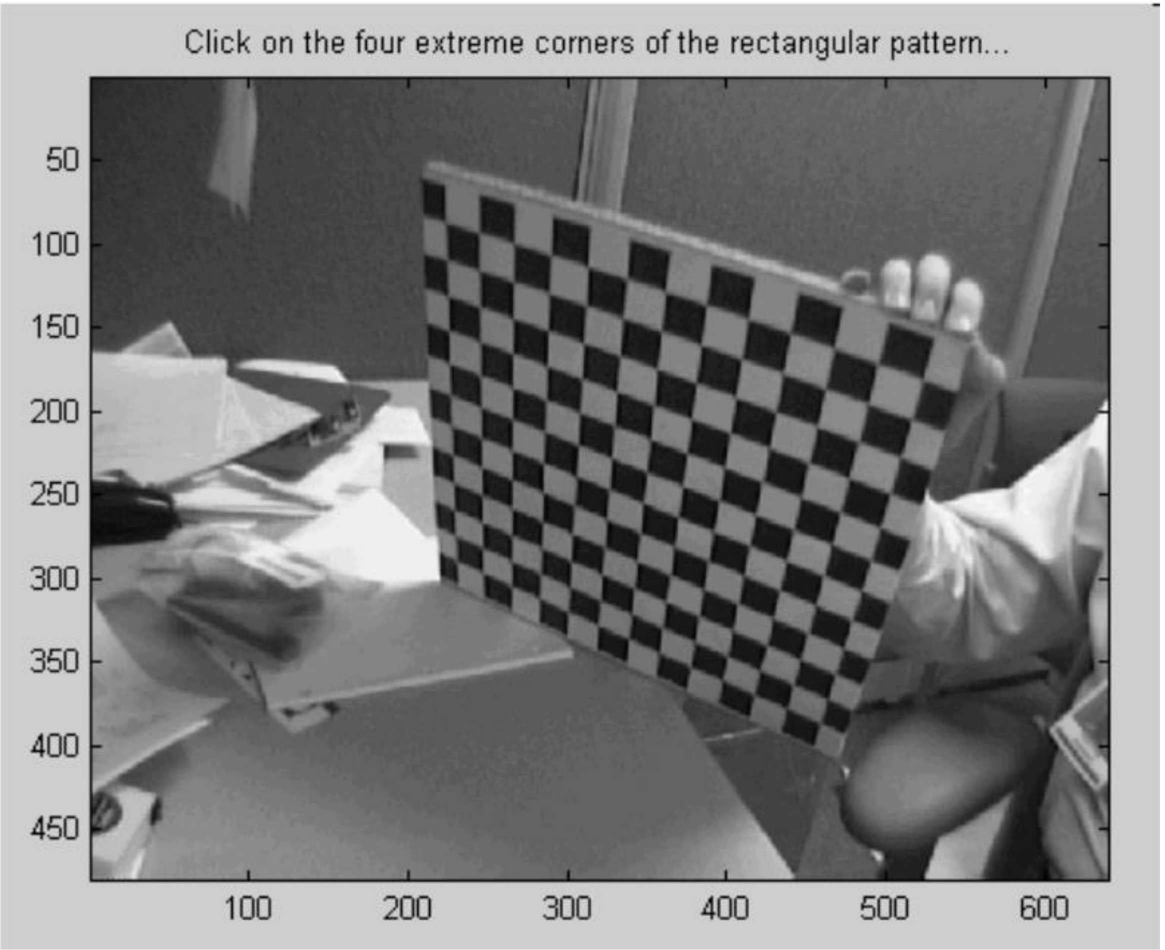
- Only requires a plane
- Don't have to know positions/orientations
- Great code available online!
  - Matlab version: http://www.vision.caltech.edu/bouguetj/calib_doc/index.html
  - Also available on OpenCV.

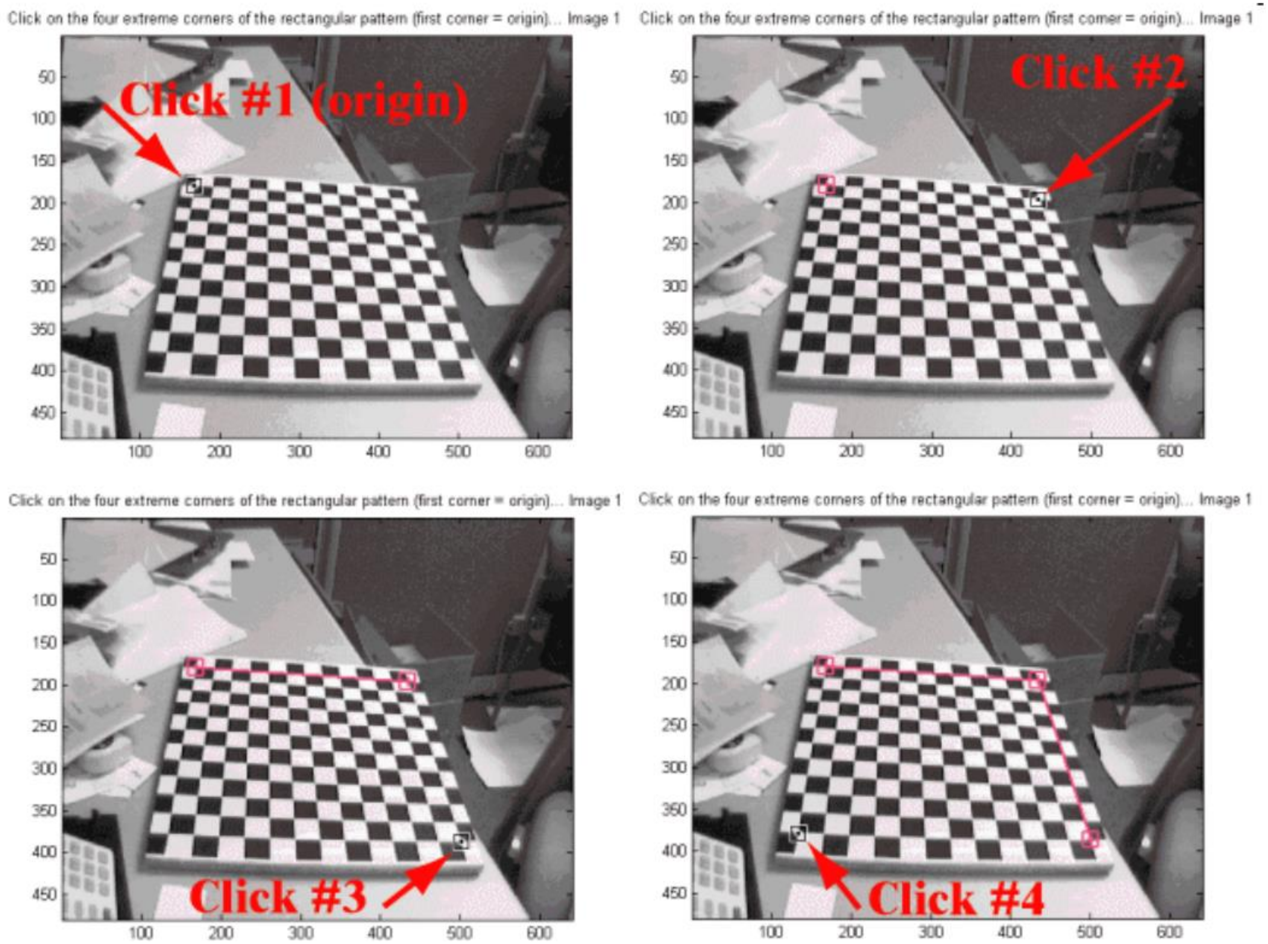
Disadvantage: Need to solve non-linear optimization problem.

Step-by-step demonstration

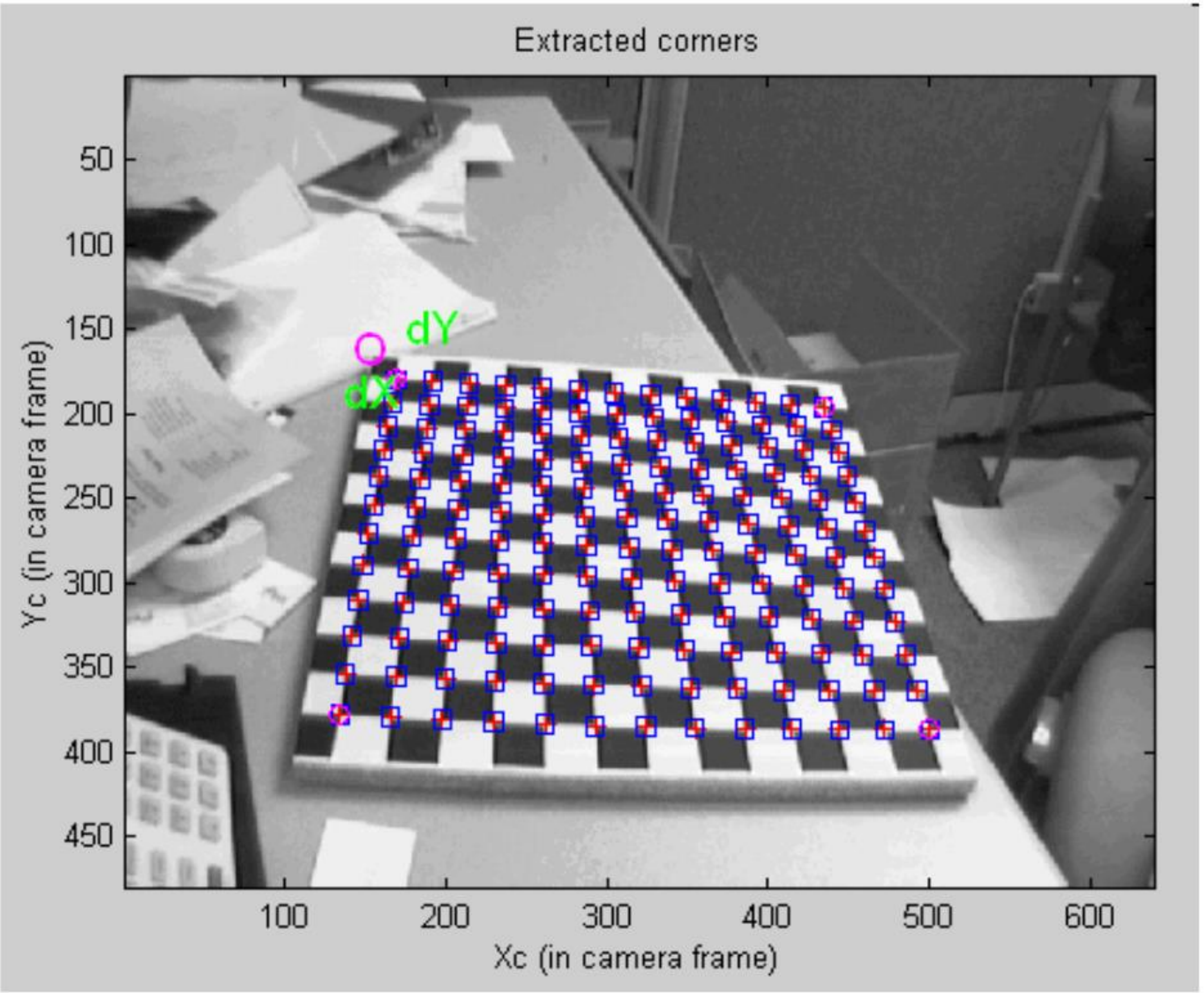


Step-by-step demonstration

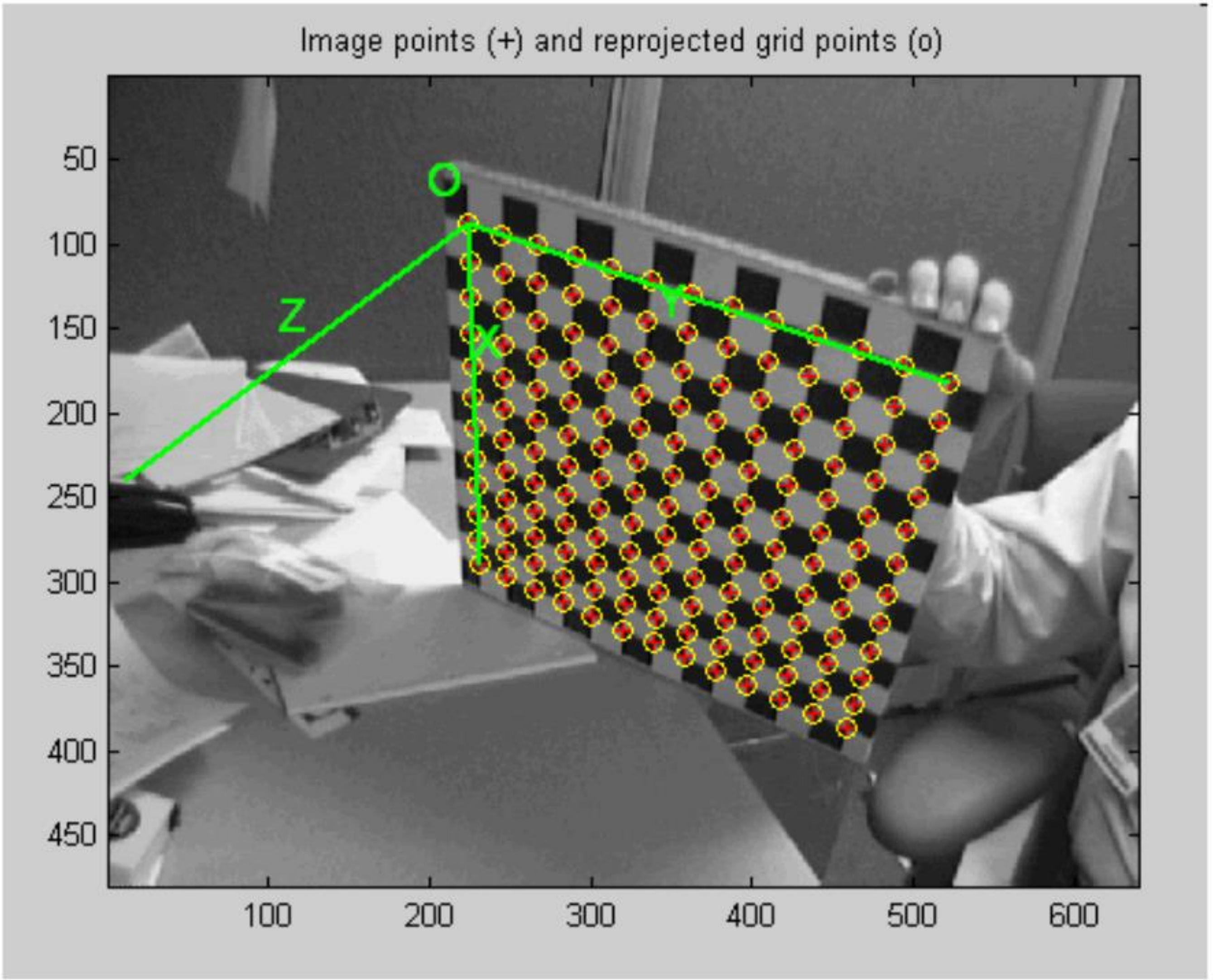




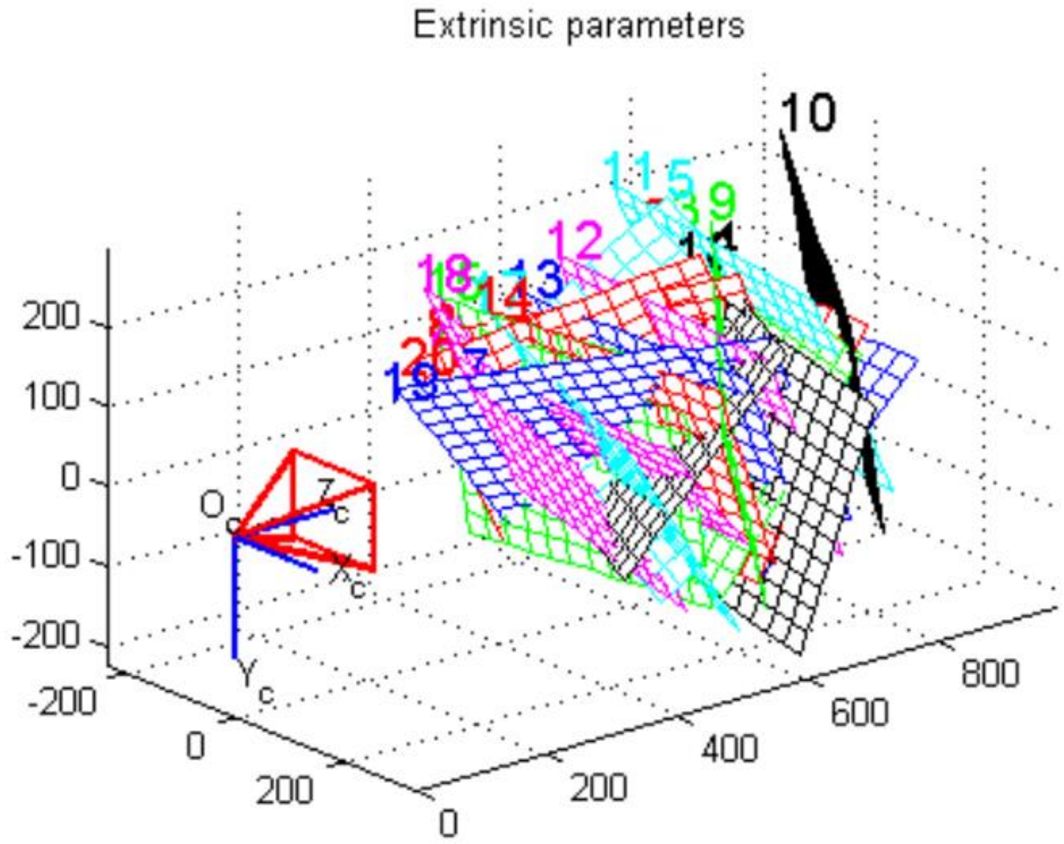
Step-by-step demonstration

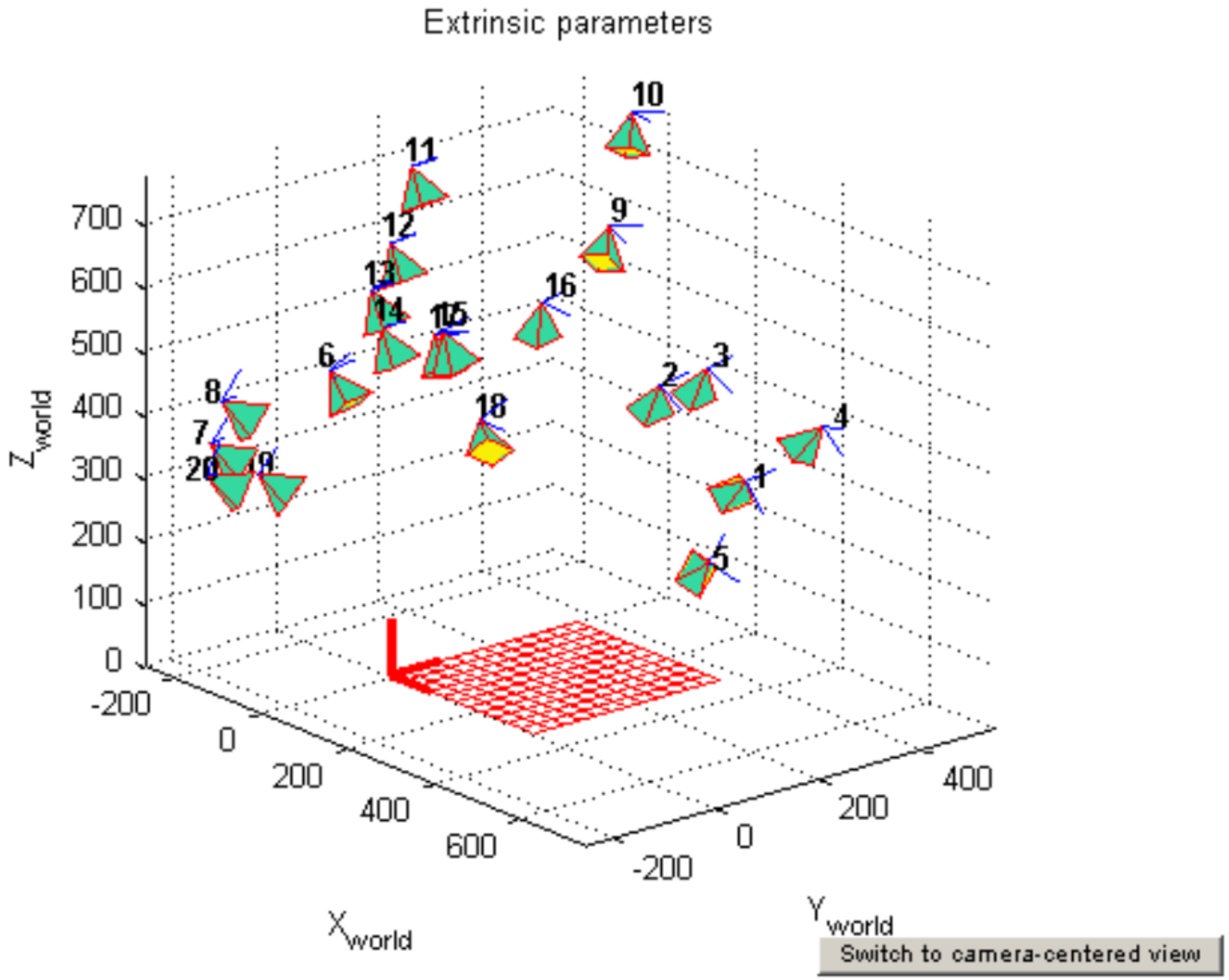


Step-by-step demonstration



Step-by-step demonstration





What does it mean to “calibrate a camera”?

Many different ways to calibrate a camera:

- Radiometric calibration.
- Color calibration.
- Geometric calibration.
- Noise calibration.
- Lens (or aberration) calibration.

Pinhole camera model calibration

```python
import cv2
import numpy as np
import os

pattern_size = (10, 7)
samples = []
file_list = os.listdir('../data/pinhole_calib')
img_file_list = [file for file in file_list if file.startswith('img')]

for filename in img_file_list:

    frame = cv2.imread(os.path.join('../data/pinhole_calib', filename))
    res, corners = cv2.findChessboardCorners(frame, pattern_size)

    img_show = np.copy(frame)
    cv2.drawChessboardCorners(img_show, pattern_size, corners, res)
    cv2.putText(img_show, 'Samples captured: %d' % len(samples), (0, 40),
                cv2.FONT_HERSHEY_SIMPLEX, 1.0, (0, 255, 0), 2)
    cv2.imshow('chessboard', img_show)

    wait_time = 0 if res else 30
    k = cv2.waitKey(wait_time)

    if k == ord('s') and res:
        samples.append((cv2.cvtColor(frame, cv2.COLOR_BGR2GRAY), corners))
    elif k == 27:
        break
```

```python
cv2.destroyAllWindows()

criteria = (cv2.TERM_CRITERIA_EPS + cv2.TERM_CRITERIA_MAX_ITER, 30, 1e-3)

for i in range(len(samples)):
    img, corners = samples[i]
    corners = cv2.cornerSubPix(img, corners, (10, 10), (-1,-1), criteria)

pattern_points = np.zeros((np.prod(pattern_size), 3), np.float32)
pattern_points[:, :2] = np.indices(pattern_size).T.reshape(-1, 2)

images, corners = zip(*samples)

pattern_points = [pattern_points]*len(corners)

rms, camera_matrix, dist_coefs, rvecs, tvecs = cv2.calibrateCamera(
    pattern_points, corners, images[0].shape, None, None)

np.save('camera_mat.npy', camera_matrix)
np.save('dist_coefs.npy', dist_coefs)

print(np.load('camera_mat.npy'))
print(np.load('dist_coefs.npy'))
```

Fisheye camera model calibration

```python
import cv2
import numpy as np
import os

pattern_size = (8, 6)
samples = []

file_list = os.listdir('../data/fisheyes')
img_file_list = [file for file in file_list if file.startswith('Fisheye1_')]

for filename in img_file_list:

    frame = cv2.imread(os.path.join('../data/fisheyes', filename))
    res, corners = cv2.findChessboardCorners(frame, pattern_size)

    res, corners = cv2.findChessboardCorners(frame, pattern_size)

    img_show = np.copy(frame)
    cv2.drawChessboardCorners(img_show, pattern_size, corners, res)
    cv2.putText(img_show, 'Samples captured: %d' % len(samples), (0, 40),
                cv2.FONT_HERSHEY_SIMPLEX, 1.0, (0, 255, 0), 2)
    cv2.imshow('chessboard', img_show)

    wait_time = 0 if res else 30
    k = cv2.waitKey(wait_time)

    if k == ord('s') and res:
        samples.append((cv2.cvtColor(frame, cv2.COLOR_BGR2GRAY), corners))
    elif k == 27:
        break

cv2.destroyAllWindows()
```

```python
criteria = (cv2.TERM_CRITERIA_EPS + cv2.TERM_CRITERIA_MAX_ITER, 30, 1e-3)

for i in range(len(samples)):
    img, corners = samples[i]
    corners = cv2.cornerSubPix(img, corners, (10, 10), (-1,-1), criteria)

pattern_points = np.zeros((1, np.prod(pattern_size), 3), np.float32)
pattern_points[0, :, :2] = np.indices(pattern_size).T.reshape(-1, 2)

images, corners = zip(*samples)

pattern_points = [pattern_points]*len(corners)

print(len(pattern_points), pattern_points[0].shape, pattern_points[0].dtype)
print(len(corners), corners[0].shape, corners[0].dtype)

rms, camera_matrix, dist_coefs, rvecs, tvecs = cv2.fisheye.calibrate(
    pattern_points, corners, images[0].shape, None, None)

np.save('camera_mat.npy', camera_matrix)
np.save('dist_coefs.npy', dist_coefs)

print(np.load('camera_mat.npy'))
print(np.load('dist_coefs.npy'))
```

Stereo rig calibration

```python
import cv2
import glob
import numpy as np

PATTERN_SIZE = (9, 6)
left_imgs = list(sorted(glob.glob('../data/stereo/case1/left*.png')))
right_imgs = list(sorted(glob.glob('../data/stereo/case1/right*.png')))
assert len(left_imgs) == len(right_imgs)

criteria = (cv2.TERM_CRITERIA_EPS + cv2.TERM_CRITERIA_MAX_ITER, 30, 1e-3)
left_pts, right_pts = [], []
img_size = None

for left_img_path, right_img_path in zip(left_imgs, right_imgs):
    left_img = cv2.imread(left_img_path, cv2.IMREAD_GRAYSCALE)
    right_img = cv2.imread(right_img_path, cv2.IMREAD_GRAYSCALE)
    if img_size is None:
        img_size = (left_img.shape[1], left_img.shape[0])

    res_left, corners_left = cv2.findChessboardCorners(left_img, PATTERN_SIZE)
    res_right, corners_right = cv2.findChessboardCorners(right_img, PATTERN_SIZE)

    corners_left = cv2.cornerSubPix(left_img, corners_left, (10, 10), (-1,-1),
                                    criteria)
    corners_right = cv2.cornerSubPix(right_img, corners_right, (10, 10), (-1,-1),
                                     criteria)

    left_pts.append(corners_left)
    right_pts.append(corners_right)
```

```python
pattern_points = np.zeros((np.prod(PATTERN_SIZE), 3), np.float32)
pattern_points[:, :2] = np.indices(PATTERN_SIZE).T.reshape(-1, 2)
pattern_points = [pattern_points] * len(left_imgs)

err, Kl, Dl, Kr, Dr, R, T, E, F = cv2.stereoCalibrate(
    pattern_points, left_pts, right_pts, None, None, None, None, img_size, flags=0)

print('Left camera:')
print(Kl)
print('Left camera distortion:')
print(Dl)
print('Right camera:')
print(Kr)
print('Right camera distortion:')
print(Dr)
print('Rotation matrix:')
print(R)
print('Translation:')
print(T)

np.save('stereo.npy', {'Kl': Kl, 'Dl': Dl, 'Kr': Kr, 'Dr': Dr, 'R': R, 'T': T, 'E': E, 'F': F,
                       'img_size': img_size, 'left_pts': left_pts, 'right_pts': right_pts})
```

Distorting and undistorting points

```python
import cv2
import numpy as np

camera_matrix = np.load('../data/pinhole_calib/camera_mat.npy')
dist_coefs = np.load('../data/pinhole_calib/dist_coefs.npy')

img = cv2.imread('../data/pinhole_calib/img_00.png')
pattern_size = (10, 7)
res, corners = cv2.findChessboardCorners(img, pattern_size)
criteria = (cv2.TERM_CRITERIA_EPS + cv2.TERM_CRITERIA_MAX_ITER, 30, 1e-3)
corners = cv2.cornerSubPix(cv2.cvtColor(img, cv2.COLOR_BGR2GRAY),
                           corners, (10, 10), (-1,-1), criteria)

h_corners = cv2.undistortPoints(corners, camera_matrix, dist_coefs)
h_corners = np.c_[h_corners.squeeze(), np.ones(len(h_corners))]

img_pts, _ = cv2.projectPoints(h_corners, (0, 0, 0), (0, 0, 0), camera_matrix, None)
```

```python
for c in corners:
    cv2.circle(img, tuple(c[0]), 10, (0, 255, 0), 2)

for c in img_pts.squeeze().astype(np.float32):
    cv2.circle(img, tuple(c), 5, (0, 0, 255), 2)

cv2.imshow('undistorted corners', img)
cv2.waitKey()
cv2.destroyAllWindows()

img_pts, _ = cv2.projectPoints(h_corners, (0, 0, 0), (0, 0, 0), camera_matrix, dist_coefs)

for c in img_pts.squeeze().astype(np.float32):
    cv2.circle(img, tuple(c), 2, (255, 255, 0), 2)

cv2.imshow('reprojected corners', img)
cv2.waitKey()
cv2.destroyAllWindows()
```

Removing lens distortion effects

```
import cv2
import numpy as np

camera_matrix = np.load('../data/pinhole_calib/camera_mat.npy')
dist_coefs = np.load('../data/pinhole_calib/dist_coefs.npy')
img = cv2.imread('../data/pinhole_calib/img_00.png')

cv2.imshow('original image', img)

ud_img = cv2.undistort(img, camera_matrix, dist_coefs)
cv2.imshow('undistorted image1', ud_img)

opt_cam_mat, valid_roi = cv2.getOptimalNewCameraMatrix(camera_matrix, dist_coefs, img.shape[:2][::-1], 0)
ud_img = cv2.undistort(img, camera_matrix, dist_coefs, None, opt_cam_mat)
cv2.imshow('undistorted image2', ud_img)

cv2.waitKey(0)
cv2.destroyAllWindows()
```