Image Reconstruction And Poisson's equation

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Parallel Computations for Large-Scale Problems I

Outline

Introduction

Question

What have image processing and the solution of partial differential equations in common?

Digital Images

Definition

A (digital) *image* is an $M \times N$-matrix of pixel values , the *pixmap*.

- We will assume that each pixel is represented by its gray level. Thus, we assume the image to be black/white.
- A colored image consists of a collection of pixmaps.
- We assume that the type of a pixel is `double`. In practice, most often 8-bit values are used (`unsigned char`)

Smoothing, Sharpening, Noise Reduction

Smoothing suppresses large fluctuations in intensity over the image

Sharpening accentuates transitions and enhances the details

Noise reduction suppresses a noise signal present in an image

Smoothing By Local Filtering

Idea: Replace each pixel value u_{mn} by the mean of the surrounding pixels:

$$\begin{aligned}\tilde{u}_{mn} = & \frac{1}{9}\big(u_{m-1,n-1} + u_{m-1,n} + u_{m-1,n+1} + \\ & u_{m,n-1} + u_{mn} + u_{m,n+1} + \\ & u_{m+1,n-1} + u_{m+1,n} + u_{m+1,n+1}\big)\end{aligned}$$

Smoothing: Example

Original

Result

Weighted Masks

The mean value can be conveniently be described by a a 3×3 matrix W,

$$W = \frac{1}{9}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Application:

$$\begin{aligned} \tilde{u}_{mn} =& w_{-1,-1}u_{m-1,n-1} + w_{-1,0}u_{m-1,n} + w_{-1,1}u_{m-1,n+1} \\ &+ w_{0,-1}u_{m.n-1}w_{0,0}u_{mn} + w_{0,1}u_{m,n+1} \\ &+ w_{1,-1}u_{m+1,n-1}w_{1,0}u_{m+1,n} + w_{1,1}u_{m+1,n+1} \end{aligned}$$

Mathematically: Convolution

Noise Reduction

$$W = \frac{1}{16}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 8 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

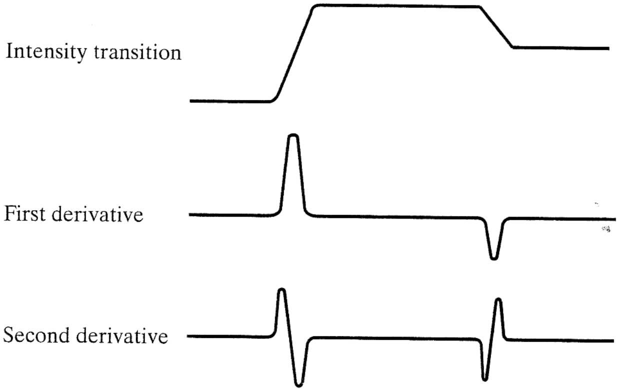
Noisy image

denoised

Edge Detection

- *Edge detection* is the high-lightening of the edges of an object, where an edge is a significant change in the gray-level intensity.
- Basic idea: The rate of change of a quantity can be measured by the *magnitude of its derivative(s)*

Example


Intensity transition
First derivative
Second derivative

The Laplace Operator

Definition

For any function u defined on some two-dimensional domain, the *Laplacian* Δu of u is defined as

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

Approximating The Laplacian

- Approximate the derivatives,

$$\frac{\partial^2 u}{\partial x^2}(x,y) \approx \frac{1}{h^2}(u(x-h,y) - 2u(x,y) + u(x+h,y)), \quad h > 0$$

  and similarly for $\partial^2 u/\partial y^2$.
- We obtain the weight matrix,

$$W = \frac{1}{h^2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

- *This fits exactly in our framework!*

Edge Detection By The Discrete Laplacian

Original

Edges

Using First Order Derivatives: Sobel Operator

Original

Edges

Poisson's Equation

- Ubiquitous equation
  - Fluid flow, electromagnetics, gravitational interaction, ...
- In two dimensions, Poisson's equation reads:
  - Solve $\Delta u = f(x, y)$ for $(x, y) \in \Omega$,
  - subject to the boundary condition $u(x, y) = g(x, y)$ for $(x, y) \in \partial\Omega$
- For simplicity, consider only $\Omega = (0, 1) \times (0, 1)$.
- Generalizations to other dimensions are obvious.

Discrete Approximation

- Define a *mesh* (or *grid*): For a given N, let

$$h = 1/(N-1), \quad x_m = mh, \quad y_n = nh$$

- Let $u_{mn} \approx u(x_m, y_n)$, $f_{mn} = f(x_m, y_n)$.
- Using the Laplace approximation from above, we obtain a system of equations

$$\frac{1}{h^2}(u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1} - 4u_{mn}) = f_{mn},$$
$$0 < m, n < N-1$$

- In the context of pde's, the matrix W is usually called a *stencil*.

Jacobi Iteration

- *Basic idea*: Rewrite the equations as

$$u_{mn} = \frac{1}{4}(u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1} - h^2 f_{mn})$$

- For some starting guess (e.g., $u_{mn} = 0$), iterate this equation,

```
while (not_done)
  for (m,n) in 1:N-2 x 1:N-2
    ut(m,n)=(u(m-1,n)+u(m+1,n)+u(m,n-1)
                 +u(m,n+1)-h^2f(m,n))/4;
  end
  u = ut;
end
```

Accuracy

- How do we know that the answer is "good enough"?
    - When the computed solution has reached a reasonable approximation to the exact solution
    - When we can validate the computed solution in the field
- But often we do not know the exact solution, and must estimate the error, e.g.,
    - Stop when the residual is small enough, $r = Au - f$
    - Stop when the change $u - u'$ in u is small.
    - Both approaches must be designed carefully!

Boundary Conditions

- Evaluating the stencil is not possible near the boundary.
- For Poisson's equation, invoke the boundary condition.
- In image processing, there are two possibilities:
  1. Discard the boundary (the new image is 2 pixels smaller in both dimensions).
  2. Modify the weight matrices such that only existing neighbors are used.

The Common Denominator

Conclusion

The methods considered use a *uniform mesh* for their data.

- Such methods are very common in applications
- They can be easily adapted to problems of any (spatial) dimension

Observations

Observations

- The computations for each point u_{ij} are completely decoupled
- The number of operations per data point is constant
- The new value at each data point depends only on its nearest neighbors

Conclusion

A good parallelization strategy is *data partitioning*.

To keep things as simple as possible, consider only a one-dimensional array (a vector)

$$u = (u_0, ..., u_{M-1})^T$$

Data Distribution

Definition

Assume that we have P processes (enumerated $0, ..., P-1$). A *P-fold data distribution* of the index set $\mathcal{M} = \{0, ..., M-1\}$ is a bijective mapping μ which maps each *global index* $m \in \mathcal{M}$ to a pair of indices (p, i) where p is the process identifier and i the *local index*.

Notes:

- This definition allows for the fact that the number of elements on each process varies with p. Of course, this is necessary if P does not divide M evenly.
- Technically, we assume that the local index set on each process is a set of consecutive integers, often (but not always!) $0 \le i < I_p$.

Example: Linear Data Distribution

Idea

Split the vector into equal chunks and allocate the p-the chunk to process p.

- $p = 0 : u_0, u_1, \ldots, u_{l_0 - 1}$
- $p = 1$: $u_{l_0}, \ldots, u_{l_0 + l_1 - 1}$
- $p : u_{l_{p-1}}, \ldots, u_{l_{p-1} + l_p - 1}$
- If P does not divide M evenly, distribute the remaining R elements to the first few processes.
- The *load-balanced linear data distribution* is:

$$
\begin{aligned}
L &= \left\lfloor \frac{M}{P} \right\rfloor \\
R &= M \mod P \\
\mu(m) &= (p, i) \text{ where } \begin{cases} p = & \max\left(\left\lfloor \frac{m}{L+1} \right\rfloor, \left\lfloor \frac{m-R}{L} \right\rfloor \right) \\ i = & m - pL - \min(p, R) \end{cases} \\
l_p &= \left\lfloor \frac{M + P - p - 1}{P} \right\rfloor \\
\mu^{-1}(p, i) &= pL + \min(p, R) + i
\end{aligned}
$$

Example: Scatter Distribution

Idea

Allocate consecutive vector components to consecutive processes.

- $p = 0$: $u_0, u_P, u_{2P}, \ldots$
- $p = 1$: $u_1, u_{P+1}, \ldots$
- p: $u_p, u_{P+p}, \ldots$

The *load balanced scatter distribution* is:

$$\mu(m) = (p, i) \text{ where } \begin{cases} p = & m \mod P \\ i = & \left\lfloor \frac{m}{P} \right\rfloor \end{cases}$$

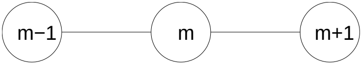
$$I_p = \left\lfloor \frac{M + P - p - 1}{P} \right\rfloor$$

$$\mu^{-1}(p, i) = iP + p$$

A Distributed Vector

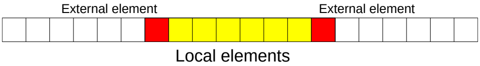
- The one-dimensional version of the convolution formula reads

$$\tilde{u}_m = w_{-1}u_{m-1} + w_0 u_m + w_1 u_{m+1}$$



- Each evaluation needs its neighbors. Consequently, the *linear data distribution* is most appropriate

- Each process needs one element stored on the processes to the "left" and "right"

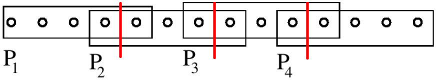


Conclusion

In addition to the local data, introduce *ghost cells*.

Ghost Cells

- Two adjacent processes p and $p+1$ need to share *two* data points. This is called the *overlap* between two processes



  The overlap is $a = 2$ in this case.
- The overlap is dependent on the width of the stencil
- The local array $u_i, 0 \leq i < I_p$ will be surrounded by two cells (with the exception of the first and the last processes)
- This is conveniently done by enlarging the local vector

Linear Distribution With Overlap

- Let $q = a/2$.

$$\begin{aligned}
\tilde{M} =& M + (P-1)a \\
L =& \left\lfloor \frac{\tilde{M}}{P} \right\rfloor \\
R =& \tilde{M} \mod P \\
I_p =& \begin{cases} L+1, & \text{for } p < R, \\ L, & \text{for } p \geq R \end{cases} \\
\mu^{-1}(p,i) =& (L-a)p + \min(R,p) + i
\end{aligned}$$

- Note that the latter formulae is only defined for

$$0 < i < I_p - 1$$

- So $u[0]$ and $u[I_p - 1]$ do not contain valid entries!

Fill The Ghost Cells: Communication

- Before we can start applying the stencil, the ghost cells must be filled
- Attempted erroneous solution (assume $a = 2$ for simplicity)

```
receive(u[0],p-1);
send(u[1],p-1);
receive(u[Ip-1],p+1);
send(u[Ip-2],p+1);
```

Deadlock!

- Mismatch in communication. All processes waiting to receive
- Possible solutions:
  - Rewrite program so that calls to `send` and `receive` are matched
  - non-blocking communication

Communication: A New Attempt

- Exchange send and receive:

```
if p > 0
   send(u[1],p-1);
   receive(u[0],p-1);
end
if p < P-1
   receive(u[Ip-1],p+1);
   send(u[Ip-2],p+1);
end
```

Code works! But very **inefficient**!

- Most processes are idle during communication
- Possible solution: Use different communication pattern

An Efficient But Unreliable Solution

```
send(u[Ip-2],p+1);
receive(u[0],p-1);
send(u[1],p-1);
receive(u[Ip-1],p+1);
```

Properties:

+ communication time is optimal:

$$2(t_{\mathsf{startup}} + 8t_{\mathsf{data}})$$

- Relies on the network to buffer the messages. *This is not guaranteed by MPI!*

The Safe Solution

The idea is a red-black (chequerboard) coloring:

- Even *p*: assign red
- Odd *p*: assign black

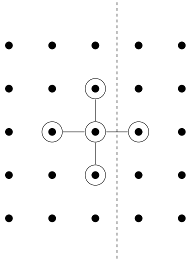
Communication appears in two steps: red/black and black red:

```
if mycolor == red
   send(u[Ip_2],p+1);
   receive(u[Ip-1],p+1);
   send(u[1],p-1);
   receive(u[0],p-1);
else
   receive(u[0],p-1);
   send(u[1],p-1);
   receive(u[Ip-1],p+1);
   send(u[Ip-2],p+1);
end
```

Communication time is only doubled compared to the previous version.

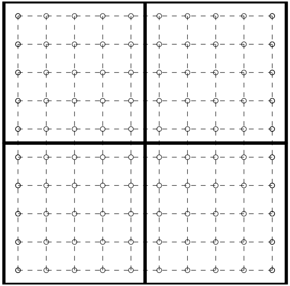
Generalizations To Two Dimensions

- Sample stencil (Poisson):

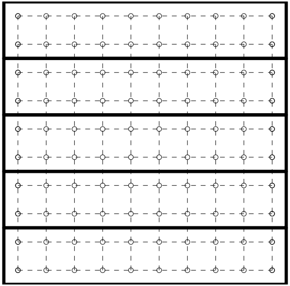


- Use an array of $R = P \times Q$ processes
- Distribute equal chunks of the pixmap/solution onto these processes
- Different partitions are called *process geometry* or *process topology*

process topology



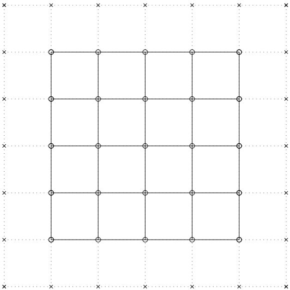
$P = Q = 2$



$P = 1, \ Q = 5$

Ghost Cells

- Each process needs values found on neighboring processes
- Use *ghost cells*,



  - Circles: local grid points
  - Crosses: ghost points

- The memory map is constructed individually for the x and y directions along the lines of the 1D example

Communication of Ghost Points

Question

How should the exchange of the ghost points corresponding to the inter-process boundaries be implemented?

The handling of the outer boundaries depends on the problem at hand (either ignore them or apply physical boundary conditions).

Some notation

For a process with "coordinates" (p, q), the neighbors are defined as follows (if they exist):

neighbor	coordinates
east	$(p+1, q)$
west	$(p-1, q)$
north	$(p, q+1)$
south	$(p, q-1)$

Non-Blocking Implementation

1. Initiate send (MPI_Isend) to east, west, north, and south neighbors (if present)
2. Initiate receive (MPI_Irecv) from west, east, south, and north neighbors
3. Evaluate the stencil away from the boundaries
4. Wait for communication to complete
5. Evaluate stencil near boundaries

Red-Black Communication

- generalizes the red-black communication in 1D
- Associate each process with a color (red or black) in the p and q directions such that no neighbor has the same color
- East-west sweep

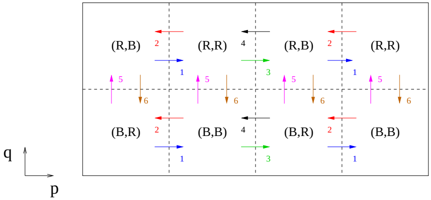
```
if color(1) == black
   send(p+1,q);
   receive(p+1,q);
   send(p-1,q);
   receive(p-1,q);
else
   receive(p-1,q);
   send(p-1,q);
   receive(p+1,q);
   send(p+1,q);
end
```

Red-Black Communication (cont)

- South-north sweep

```
if color(2) == black
   send(p,q+1);
   receive(p,q+1);
   send(p,q-1);
   receive(p,q-1);
else
   receive(p,q-1);
   send(p,q-1);
   receive(p,q+1);
   send(p,q+1);
end
```

Red-Black Communication (cont)



Number and colors show the communication pattern
process color indicated by (q, p) (note the order)

Red-Black Communication Time

- We assume a perfectly load balanced (linear) distribution,

$$I_p \approx \frac{M}{P}, \quad J_q \approx \frac{N}{Q}$$

- East-west sweep:

$$t_{\mathrm{comm},1} = C(P)(t_{\mathrm{startup}} + I_p t_{\mathrm{data}})$$

where

$$C(P) = \begin{cases} 0, & \text{if } P = 1 \\ 2, & \text{if } P = 2 \\ 4, & \text{if } P \geq 3 \end{cases}$$

- Similarly, for the South-north sweep:

$$t_{\mathrm{comm},2} = C(Q)(t_{\mathrm{startup}} + J_q t_{\mathrm{data}})$$

- Total communication time

$$t_{\mathrm{comm}} \approx (C(P) + C(Q))t_{\mathrm{startup}} + \frac{t_{\mathrm{data}}}{PQ}(C(P)QM + C(Q)PN)$$

Computation Time

- Assume a (compact) stencil

$$W = \begin{pmatrix} w_{-1,-1} & w_{0,-1} & w_{1,1} \\ w_{-1,0} & w_{0,0} & w_{0,1} \\ w_{-1,1} & w_{0,1} & w_{1,1} \end{pmatrix}$$

- Let w be the number of nonzero entries in W. Then

$$t_{\mathrm{comp},pq} = \alpha w I_p J_q t_a \approx \alpha w \frac{MN}{PQ} t_a$$

  ($0 < \alpha$ is a small constant)

- Best sequential time

$$T_s^* = \alpha w M N t_a$$

Speedup

$$
\begin{aligned}
S_R = S_{PQ} &= \frac{T_s^*}{T_R} \\
&\geq R \frac{\alpha w M N t_a}{\alpha w M N t_a + 8 R t_{\mathrm{startup}} + 4(QM + PN) t_{\mathrm{data}}} \\
&\geq R \frac{1}{1 + \frac{8 R t_{\mathrm{startup}}}{\alpha w M N t_a} + \frac{4}{\alpha w}\left(\frac{P}{M} + \frac{Q}{N}\right)\frac{t_{\mathrm{data}}}{t_a}}
\end{aligned}
$$

Conclusions

- For constant R, the speedup reaches an optimal value if MN becomes large
- If MN is fixed, the speedup will eventually degrade if R gets larger
- The speedup becomes better if $(P/M + Q/N)$ attains a minimum for a given problem size and a given number of processes

Optimal process Topology

- For a given problem size MN and a given number of processes R, find P and $Q = R/P$ such that

$$\Phi(P) = \left(\frac{P}{M} + \frac{Q}{N}\right)$$

becomes minimal

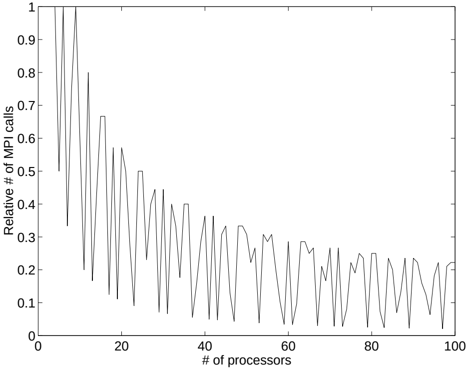
- A simple calculation gives

$$P = \sqrt{\frac{M}{N}R}$$

(provided that these are integers)

- In the case $M = N$ and R being a square, $P = \sqrt{R}$

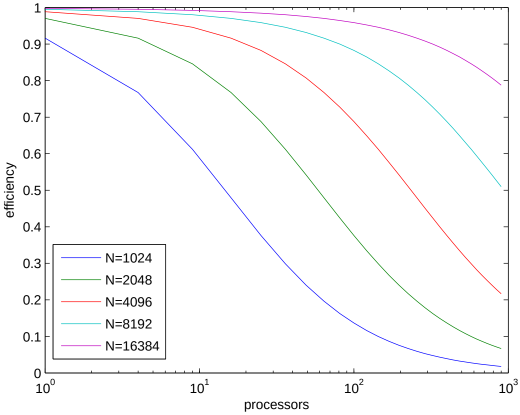
Practical Aspects



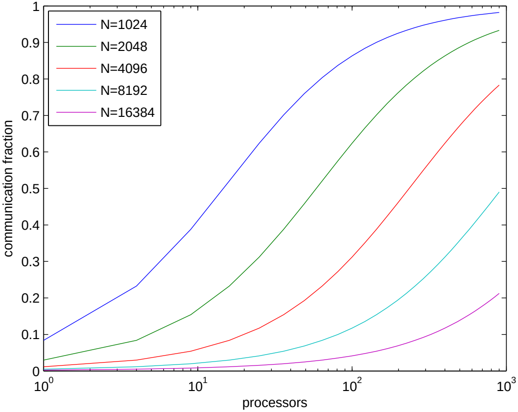
Relative number of MPI calls (compared to naive implementation)

Efficiency

For typical data on lucidor, this is the efficiency $E_R = S_R/R$



Communication Fraction



Surface to Volume Ratio

Observation:

- The *computation time* t_{comp} is proportional to the *area* $I_p \times J_q$ of the data
- The *communication time* t_{comm} is proportional to the *perimeter* $2(I_p + J_q)$

"Area-perimeter law"

The communication time is negligible if the number of data $M \times N$ is large compared to the number of processes.

The Curse of Dimensionality

As we move to higher dimensional spaces, communication becomes relatively more costly,

- in 1D: $2/N$
- in 2D: $4N/N^2 = 4/N$
- in 3D: $6N^2/N^3 = 6/N$

Virtual Topologies

Virtual Topologies

MPI includes a number of standard routines for defining and handling different process topologies. They are called *virtual topologies*. These routines lead to a great simplification of the programming efforts needed.

Jacobi Iteration

- *Basic idea*: Rewrite the equations as

$$u_{mn} = \frac{1}{4}(u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1} - h^2 f_{mn})$$

- For some starting guess (e.g., $u_{mn} = 0$), iterate this equation,

```
while (not_done)
  for (m,n) in 1:N-2 x 1:N-2
    ut(m,n)=(u(m-1,n)+u(m+1,n)+u(m,n-1)
                 +u(m,n+1)-h^2f(m,n))/4;
  end
  u = ut;
end
```

Gauss-Seidel Iteration

Observation: The Jacobi iteration converges very slowly

$$u_{mn}^{k+1} = \frac{1}{4}(u_{m-1,n}^{k} + u_{m+1,n}^{k} + u_{m,n-1}^{k} + u_{m,n+1}^{k} - h^2 f_{mn})$$

Idea

Use the new (better?) values as soon as they are available

Gauss-Seidel Iteration (cont)

```
while (not_done)
  for (m,n) in 1:N-2 x 1:N-2
    u(m,n)=(u(m-1,n)+u(m+1,n)+u(m,n-1)
             +u(m,n+1)-h^2f(m,n))/4;
  end
  % u = ut;
end
```

Observation

This iteration depends on the order of the unknown!

Lexicographic Order

Definition

The lexicographic order of the array u_{mn} is given by

$$u_{11}, u_{21}, u_{31}, \ldots u_{M,1}, u_{21}, u_{22}, \ldots, u_{MN}$$

The lexicographic order corresponds to

```
for n = 1:N
   for m = 1:M
      % u(m,n) = ...
   end
end
```

Pipelined Computations

- Gauss-Seidel iterations are purely sequential
- Assume a $P \times Q$ process grid as before
- *Process (p, q) cannot start computing before the values on processes $(p-1, q)$ and $(p, q-1)$ are available*
- This leads to *pipelined computations*.
- At every moment in time, only the processes along diagonals are active.

How to Parallelize?

Idea

Use red-black ordering!

- Black points: $m+n$ is even
- Red points: $m+n$ is odd
- Gauss-Seidel Iteration

$$u_{mn} = \frac{1}{4}(u_{m-1,n} + u_{m+1,n} + u_{m,n-1} + u_{m,n+1} - h^2 f_{mn})$$

- *If u_{mn} is black, the values on the right hand side are all red and vice versa.*
- The "black sweep" and the "red sweep" can be parallelized independently
- Note: This is a different kind of iteration!

Final Remarks

- More efficient methods for solving Poisson's equation include *multigrid methods*
- For a full 9-point stencil, four colors are needed
- Today, the most complex parallel circuit in a PC is the *GPU* (graphic processing unit)
- Not surprisingly, the GPU is used as a parallel solver unit even for PDEs
- MPI includes the possibility to define *virtual topologies* thus simplifying the design of the communication a lot

What Did We Learn?

- Evaluation of stencils for different purposes (image processing, solutions of partial differential equations)
- Data distributions, ghost points, practical aspects
- Efficient communication strategies
- Performance evaluation of the corresponding algorithms
- Pipelined computations (Gauss-Seidel iterations)
- Reformulation of recursive algorithms