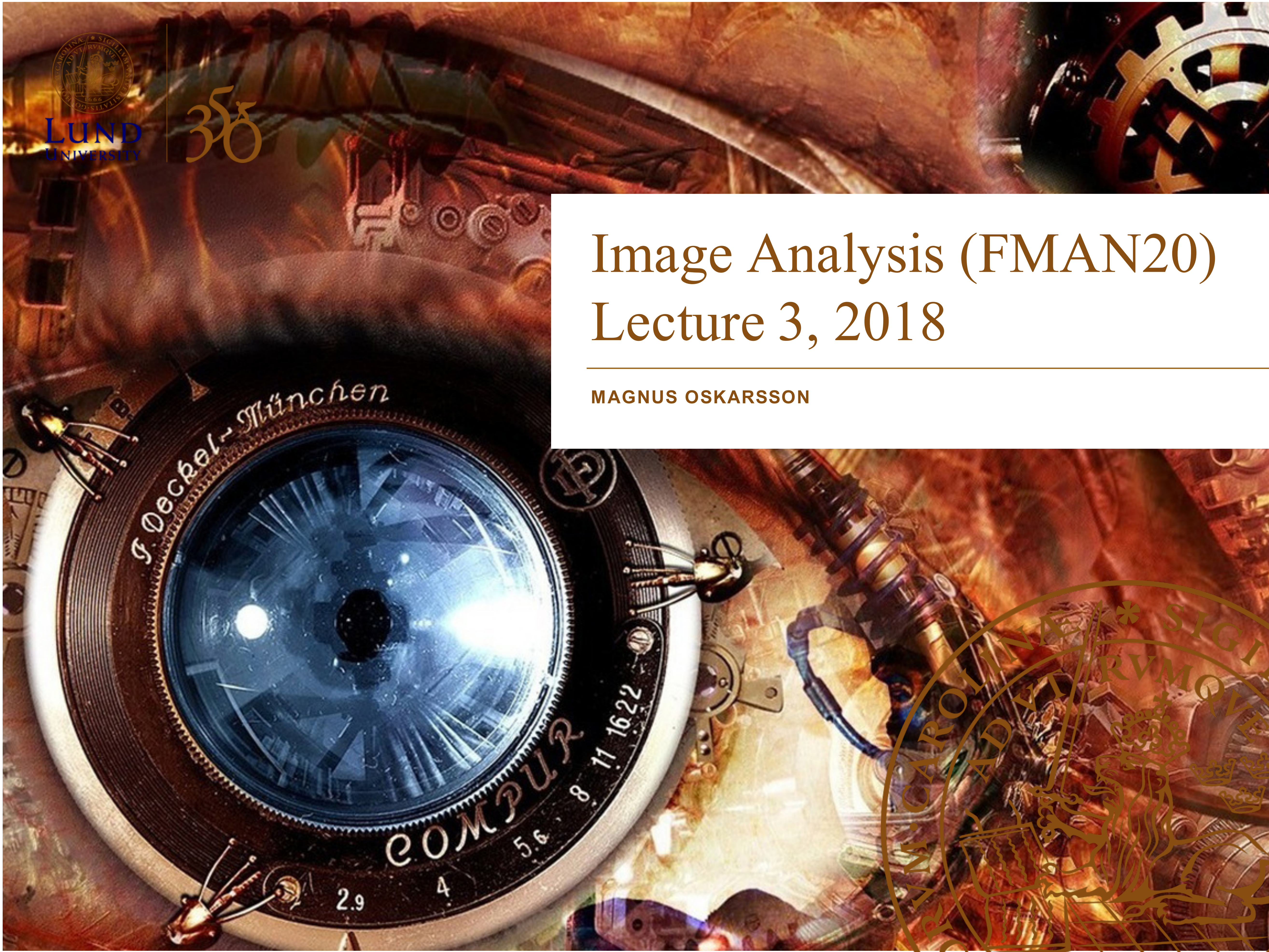
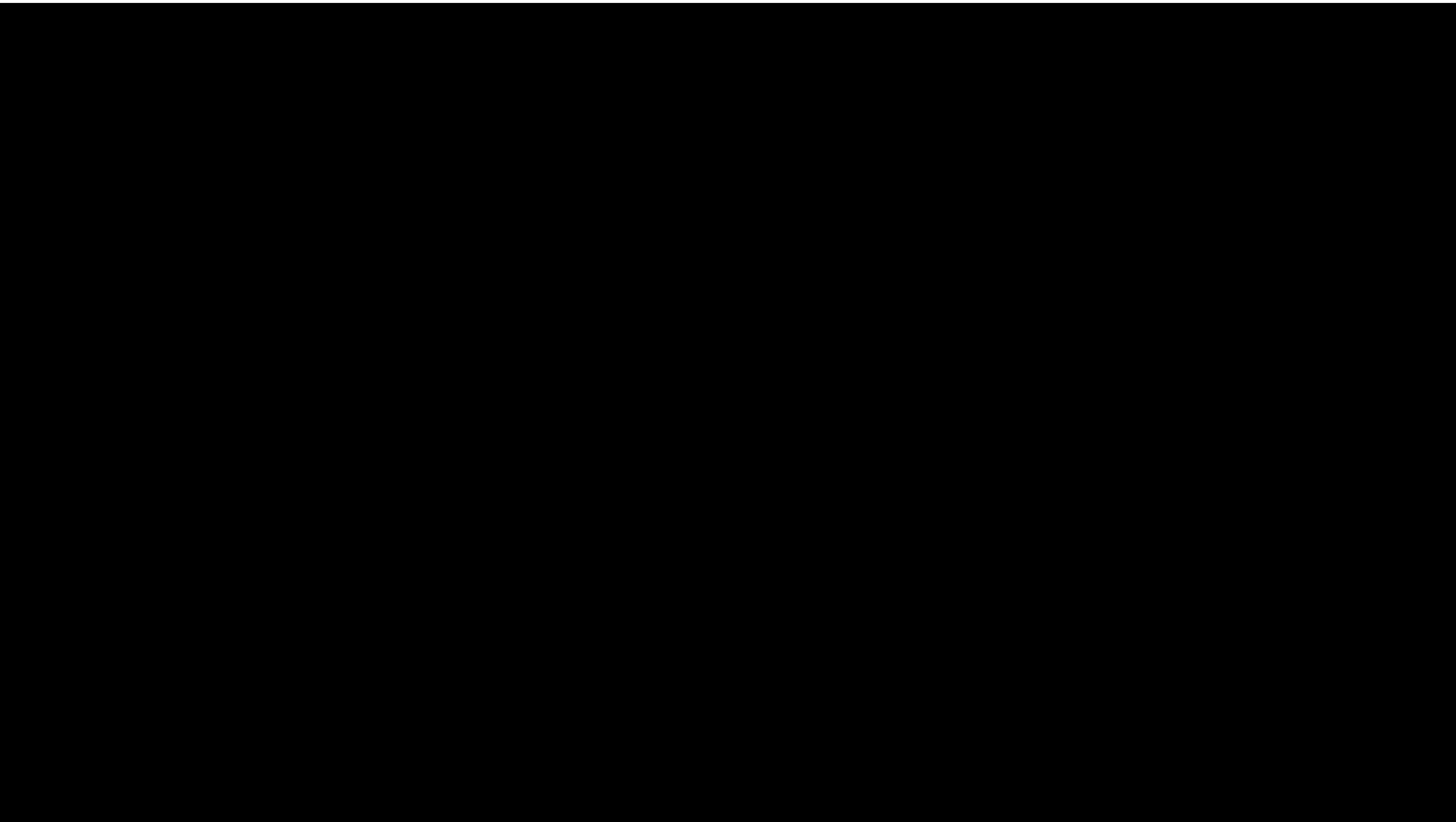


Image Analysis (FMAN20)
Lecture 3, 2018

MAGNUS OSKARSSON

Image Analysis - Motivation



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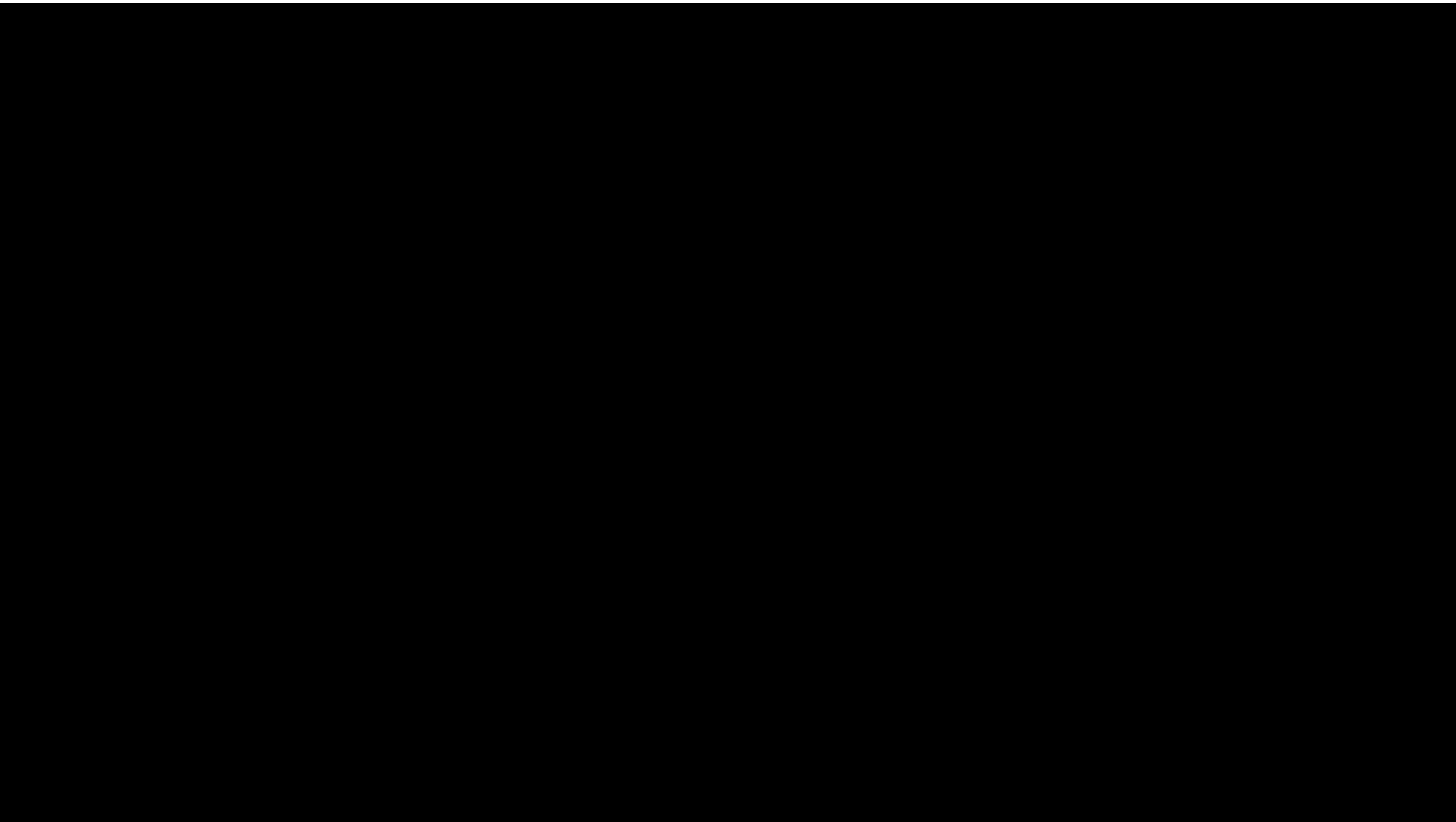

Overview –
Linear Algebra and FFT

1. Linear Algebra
    1. **Vector space – ’A matrix is a vector’ What does this mean?**
    2. Basis, coordinates
    3. Scalar product
    4. Projection onto a subspace
    5. Projection onto an affine ‘subspace’
    6. (Principal Component Analysis – Recipe)
    7. Change of basis
2. Fourier Transform

Vector spaces R^n and C^n

The following linear spaces are well-known:

- $\mathbb{R}^n$: all $n \times 1$-matrices, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad x_i \in \mathbb{R}$
- $\mathbb{C}^n$: all $n \times 1$-matrices, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad x_i \in \mathbb{C}$

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Basis

Definition

$e_1, \dots e_n \in \mathbb{R}^n$ is a **basis** in $\mathbb{R}^n$ if

- they are linearly independent
- they span $\mathbb{R}^n$.

■

Example (3D space)

$e_1, e_2, e_3 \in \mathbb{R}^3$ is a **basis** *in $\mathbb{R}^3$ if they are not located in the same plane.* ■

Canonical basis (normal basis)

Example (canonical basis)

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

is called the **canonical basis** *in* $\mathbb{R}^n$*.*

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 e_1 \dots + x_n e_n \quad .$$

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Coordinates

Let $e_1, e_2, \ldots, e_n$ be a basis. Then for every x there is a unique set of scalars ξ_i such that

$$x = \sum_{i=1}^{n} \xi_i e_i \quad .$$

These scalars are called the **coordinates** for x in the basis $e_1, e_2, \ldots, e_n$.

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Scalar product

Definition

Let A be a (complex) matrix. Introduce

$$A^* = (\bar{A})^T \quad .$$

■

Definition

Let x and y be two vectors in $\mathbb{R}^n$ ($\mathbb{C}^n$). **The scalar product** of x and y is defined as

$$\boxed{x \cdot y = \sum \bar{x}_i y_i = x^* y} \quad .$$

■

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General Vector Space

- A 'General' Vector Space is a collection of objects called **vectors**, which can be added together and also be multiplied by 'numbers' called **scalars**, where the **addition** and **multiplication with scalars** fulfill a set of rules.

(i)	$\bar{\mathbf{u}} + \bar{\mathbf{v}} = \bar{\mathbf{v}} + \bar{\mathbf{u}}$	*(commutativity)*
(ii)	$(\bar{\mathbf{u}} + \bar{\mathbf{v}}) + \bar{\mathbf{w}} = \bar{\mathbf{u}} + (\bar{\mathbf{v}} + \bar{\mathbf{w}})$	*(associativity)*
(iii)	$\bar{\mathbf{v}} + \bar{\mathbf{0}} = \bar{\mathbf{v}}$	*(zero existence)*
(iv)	$\bar{\mathbf{v}} + (-\bar{\mathbf{v}}) = \bar{\mathbf{0}}$	*(negative vector existence)*
(v)	$k(l\bar{\mathbf{v}}) = (kl)\bar{\mathbf{v}}$	*(associativity)*
(vi)	$1\bar{\mathbf{v}} = \bar{\mathbf{v}}$	*(multiplicative one)*
(vii)	$0\bar{\mathbf{v}} = \bar{\mathbf{0}}$	*(multiplicative zero)*
$(viii)$	$k\bar{\mathbf{0}} = \bar{\mathbf{0}}$	*(multiplicative zero vector)*
(ix)	$k(\bar{\mathbf{u}} + \bar{\mathbf{v}}) = k\bar{\mathbf{u}} + k\bar{\mathbf{v}}$	*(distributivity 1)*
(x)	$(k + l)\bar{\mathbf{v}} = k\bar{\mathbf{v}} + l\bar{\mathbf{v}}$	*(distributivity 2)*

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General Vector Space

- A ’General’ Vector Space is a collection of objects called **vectors**, which can be added together and also be multiplied by ’numbers’ called **scalars**, where the **addition** and **multiplication with scalars** fulfill a set of rules.
- There are many examples of such vectors spaces. The vectors can for example be
    - Geometrical vectors in three dimensions
    - N-tuples of real numbers
    - Functions
    - Polynomials
    - Matrices
    - Tensors

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Example - polynomials

- Vectors - Polynomials of degree 2
- Scalars – Real numbers

Example 3.2.1. *Polynomials in one variable of degree 2 is a vector space. One possible basis is*

$$\bar{\mathbf{e}}_1(x) = 1, \quad \bar{\mathbf{e}}_2(x) = x, \quad \bar{\mathbf{e}}_3(x) = x^2.$$

The polynomial $\bar{\mathbf{u}}(x) = 5x^2 + 3x - 2$ *has coordinates* $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \\ 5 \end{pmatrix}$, *since*

$$\bar{\mathbf{u}} = \underbrace{u_1}_{-2}\underbrace{\bar{\mathbf{e}}_1}_{1} + \underbrace{u_2}_{3}\underbrace{\bar{\mathbf{e}}_2}_{x} + \underbrace{u_3}_{5}\underbrace{\bar{\mathbf{e}}_3}_{x^2} = 5x^2 + 3x - 2.$$

The dimension of the vector space is 3*.*

Example - matrices

- Vectors – Matrices of size 2x2
- Scalars – Real numbers

Example 3.2.2. *Matrices of size 2×2 is a vector space. One possible basis is*

$$\bar{\mathbf{e}}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{\mathbf{e}}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \bar{\mathbf{e}}_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \bar{\mathbf{e}}_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrix

$$\bar{\mathbf{u}} = \begin{pmatrix} 1 & 7 \\ 3 & 2 \end{pmatrix}$$

has coordinates $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 7 \\ 2 \end{pmatrix}$, *since*

$$\bar{\mathbf{u}} = \underbrace{u_1}_{1} \underbrace{\bar{\mathbf{e}}_1}_{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} + \underbrace{u_2}_{3} \underbrace{\bar{\mathbf{e}}_2}_{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}} + \underbrace{u_3}_{7} \underbrace{\bar{\mathbf{e}}_3}_{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} + \underbrace{u_4}_{2} \underbrace{\bar{\mathbf{e}}_4}_{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} = \begin{pmatrix} 1 & 7 \\ 3 & 2 \end{pmatrix}$$

The dimension of the vector space is 4.

Image matrix

$$f = \begin{bmatrix} f(1,1) & f(1,2) & \dots & f(1,N) \\ f(2,1) & f(2,2) & \dots & f(2,N) \\ \vdots & \vdots & \ddots & \vdots \\ f(M,1) & f(M,2) & \dots & f(M,N) \end{bmatrix}$$

$$f(j,\cdot) = [f(j,1)\ f(j,2)\ \dots\ f(j,N)]\ ,$$

$$f(\cdot,k) = \begin{bmatrix} f(1,k) \\ f(2,k) \\ \vdots \\ f(M,k) \end{bmatrix}\ .$$

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Column stacking

$$\widetilde{f} = \begin{bmatrix} f(\cdot, 1) \\ f(\cdot, 2) \\ \vdots \\ f(\cdot, N) \, . \end{bmatrix}$$

$$\widetilde{f+g} = \widetilde{f} + \widetilde{g}, \quad \widetilde{\lambda f} = \lambda \widetilde{f}$$

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Set of images is a vector space

- Images are a vector space (with scalar product)
  - Addition
  - Multiplication by scalar
- Two ways to think of 'images' as vectors (both are the same)
- 1. Column stacking
  - Use column stacking to convert to 'old school' vector R^n
  - Use linear algebra as usual
  - Convert back to matrix form when needed
- 2. Image basis
  - Choose a basis (any basis).
  - Through the use of coordinates, obtain vector representation
  - Use linear algebra as usual
  - Convert back when needed

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Canonical basis

$$\chi(i,j) = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ & \vdots & 1 & \vdots & \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix},$$

with the 1 at position (i,j).
Using this canonical basis we can write

$$f = \sum_{i,j} f(i,j)\chi(i,j) \quad .$$

Idea for image transform:
Choose another basis that is more suitable in some sense.
Image matrices can thus be seen as vectors in a linear space.

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Scalar product of images

Definition

The scalar product of two matrices (images) is defined as

Frobenius inner product (element-wise product)

$$f \cdot g = \sum_{i=1}^{M} \sum_{j=1}^{N} \bar{f}(i,j) g(i,j) \quad .$$

in Matlab:
sum(sum(f.*g))

$x, y \in \mathbb{R}(\mathbb{C})$ are **orthogonal** if $x \cdot y = 0$. This is often written

$$x \perp y \quad \Leftrightarrow \quad x \cdot y = 0 \quad .$$

in Matlab:
sum(sum(f.*f))
OR norm(f, 'fro')

The length or **the norm** of x is defined as

Frobenius inner product (element-wise product)

$$||f|| = \sqrt{f \cdot f} = \sqrt{\sum_{i=1}^{M} \sum_{j=1}^{N} \bar{f}(i,j) f(i,j)}.$$

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Scalar product and norm

Example 3.2.1 (Scalar product and norm)**.** *Let*

$$f = \begin{pmatrix} 1 & 0 \\ -2 & 2 \end{pmatrix}$$

and

$$g = \begin{pmatrix} 4 & 2 \\ -1 & -3 \end{pmatrix}.$$

What is the scalar product $f \cdot g$? What is the norm $||f||$?

1.4 + 0.2 + (-2)(-1) + 2(-3) = 4 + 0 + 2 - 6 = 0

sqrt [1.1 + 0.0 + (-2)(-2) + 2.2 = 1 + 0 + 4 + 4] = sqrt(9) = 3

$$f \cdot g = \sum_{i=1}^{M} \sum_{j=1}^{N} \bar{f}(i,j) g(i,j). \quad ||f|| = \sqrt{f \cdot f} = \sqrt{\sum_{i=1}^{M} \sum_{j=1}^{N} \bar{f}(i,j) f(i,j)}.$$

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Orthonormal basis

Definition

$\{e_1, \ldots, e_n\}$ is an **orthonormal (ON-) basis** in $\mathbb{R}^n$ ($\mathbb{C}^n$) if

$$e_i \cdot e_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

■

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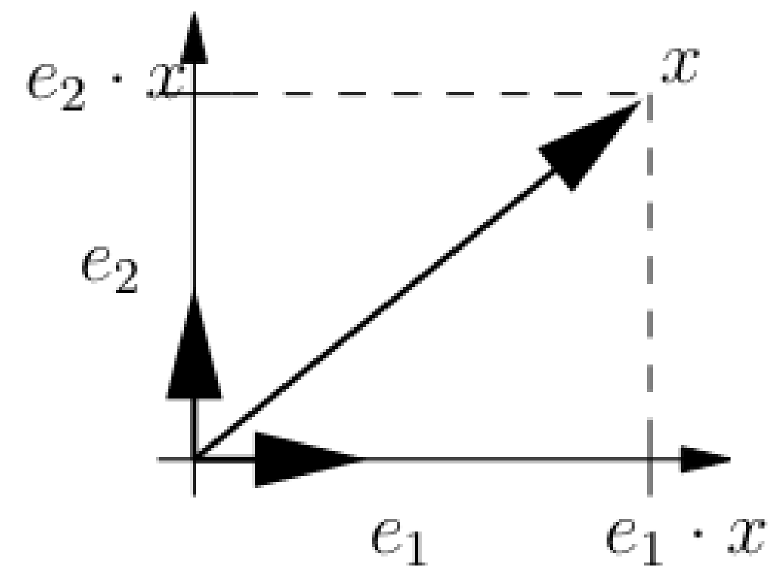
Orthonormal basis



Theorem

Assume that $\{e_1, \ldots, e_n\}$ *is orthonormal (ON) basis and*

$$x = \sum_{i=1}^{n} \xi_i e_i \quad .$$

Then

$$\xi_i = e_i \cdot x = e_i^* x, \quad ||x||^2 = \sum_{i=1}^{n} |\xi_i|^2$$

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Orthogonal projection

Definition

Let $\{a_1, \ldots, a_k\} \in \mathbb{R}^n$, $k \leq n$, span a linear subspace, π, in $\mathbb{R}^n$, i.e.:

$$\pi = \{w | w = \sum_{i=1}^{k} x_i a_i,\ x_i \in \mathbb{R}\} \ .$$

The **orthogonal projection** of $u \in \mathbb{R}^n$ on π is a linear mapping P, such that $u_\pi = Pu$ and defined by

$$\min_{w \in \pi} ||u - w|| = ||u - u_\pi|| \ .$$

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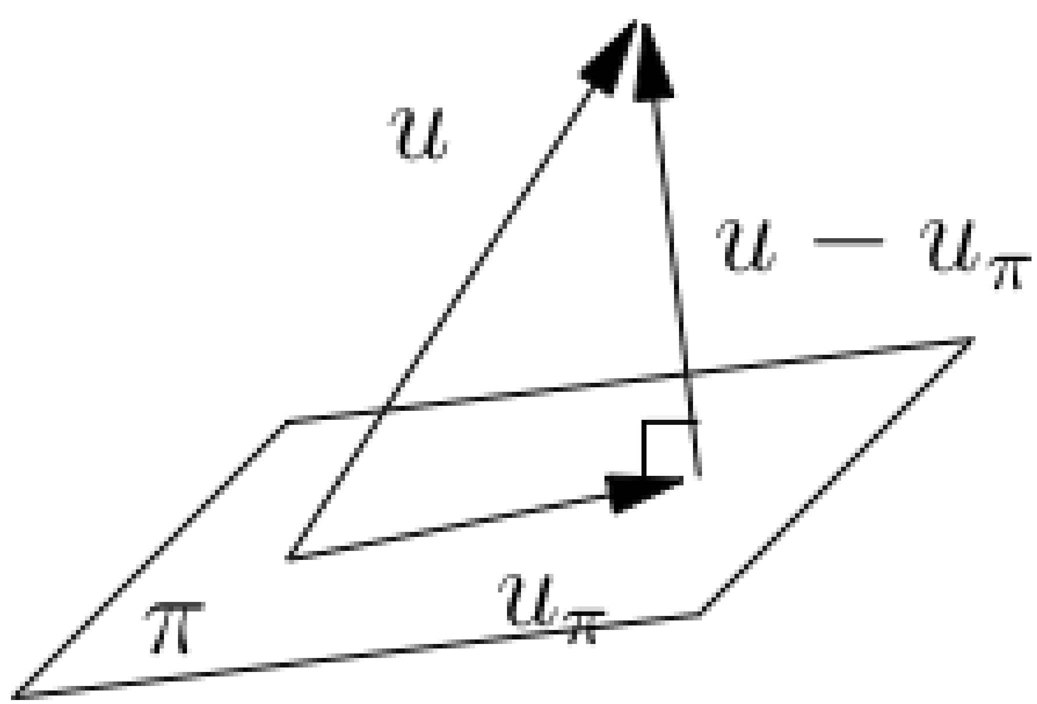
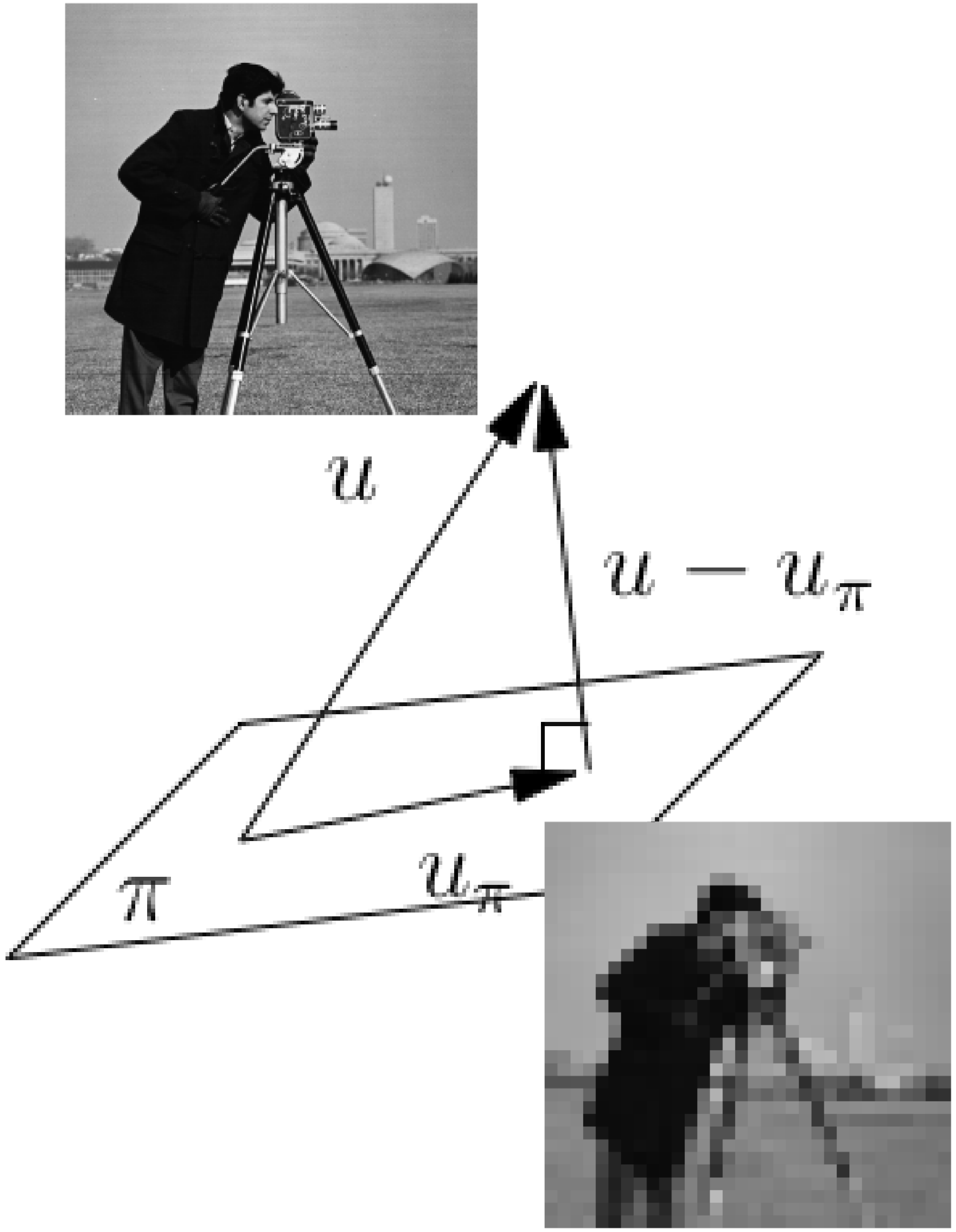
Orthogonal projection

The orthogonal projection is characterized by

1. $u_\pi \in \pi$
2. $u - u_\pi \perp w$ for every $w \in \pi$



Orthogonal projection



Let $a \in \pi$ and $b \in \pi$ be two solutions to the minimisation problem. Set

$$f(t) = \|u - ta - (1-t)b\|^2 = \ldots$$

$$= \|u - b\|^2 + t^2\|a - b\|^2 - 2t(a - b) \cdot (u - b), \quad t \in \mathbb{R}\,.$$

This is a second degree polynomial with minimum in $t = 0$ and $t = 1 \Rightarrow f(t)$ is a constant function and thus $\Rightarrow a = b$.

Let $f(t) = \|u - u_\pi + ta\|^2$, where $a \in \pi$. It follows that $f'(0) = 2(u - u_\pi) \cdot a = 0$, i.e. $(u - u_\pi) \perp a$.
Conversely: Assume $w \in \pi$. The property that $(u - u_\pi) \perp a$, for every $a \in \pi$ gives that

$$\|u - w\|^2 = \|u - u_\pi + u_\pi - w\|^2 =$$

$$\|u - u_\pi\|^2 + \|u_\pi - w\|^2 \geq \|u - u_\pi\|^2,$$

i.e. u_π solves the minimization problem.

Let $A = [a_1 \dots a_k]$ be a $n \times k$ matrix and

$$\pi = \{w | w = Ax,\ x_i \in \mathbb{R}^n\}$$

Lemma

If $\{a_1, \dots, a_k\}$ are linearly independent $\mathbb{R}^n$ then A^*A is invertible.

Proof: Do it on your own. (Use SVD if you are familiar with it.)

■

Theorem

if the columns of A are linearly independent, then the projection of u on π is given by

$$u_\pi = x_1 a_1 + \ldots + x_k a_k, \quad x = (A^* A)^{-1} A^* u \ .$$

Proof: Use the characterization of the projection (above).

$$a_i^*(u - u_\pi) = 0 \quad \Rightarrow$$

$$A^*(u - Ax) = 0 \quad \Rightarrow$$

$$A^* u = A^* A x \quad \Rightarrow \quad x = (A^* A)^{-1} A^* u$$

Definition

$A^+ = (A^*A)^{-1}A^*$ is called the **pseudo-inverse** of A. ■

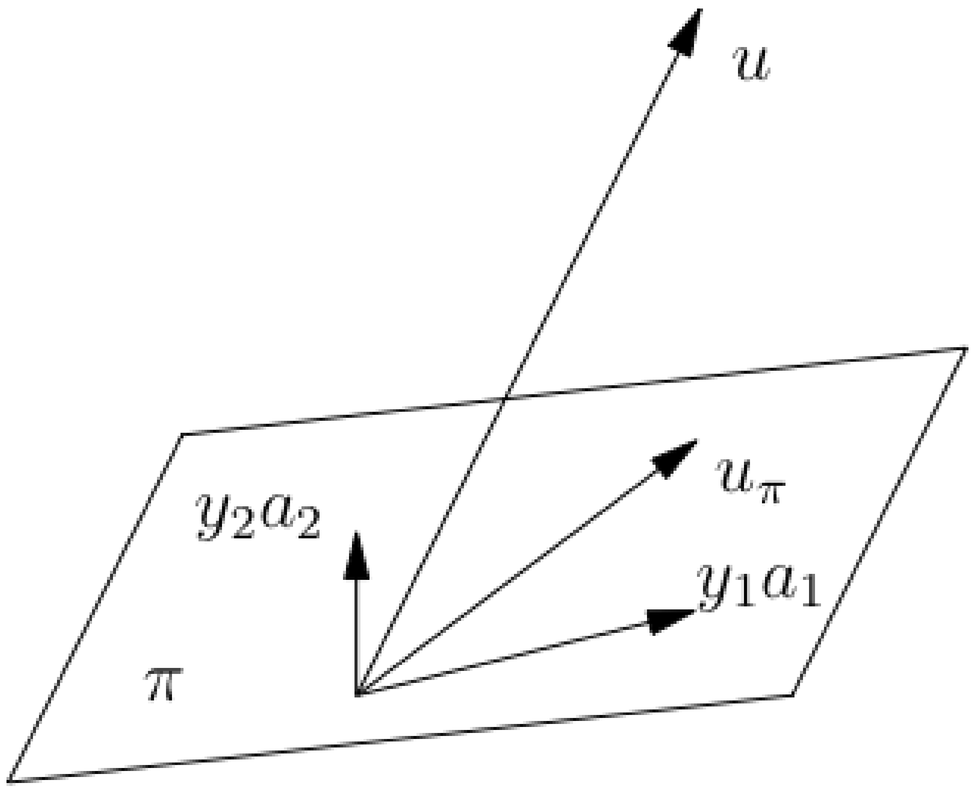
Observe that if A is quadratic and invertible then $A^+ = A^{-1}$.

Theorem

If $\{a_1, \ldots, a_k\}$ are orthonormal, then the projection of u on π is given by

$$u_\pi = y_1 a_1 + \ldots + y_k a_k, \quad y_i = a_i^* u \ .$$

Proof: This follows from $A^*A = I$. ■



Orthogonal projection

Example

What is the orthogonal projection of f

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 7 & 7 \end{pmatrix}$$

onto the space spanned by (e_1, e_2, e_3)

$$e_1 = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, e_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}, e_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}$$

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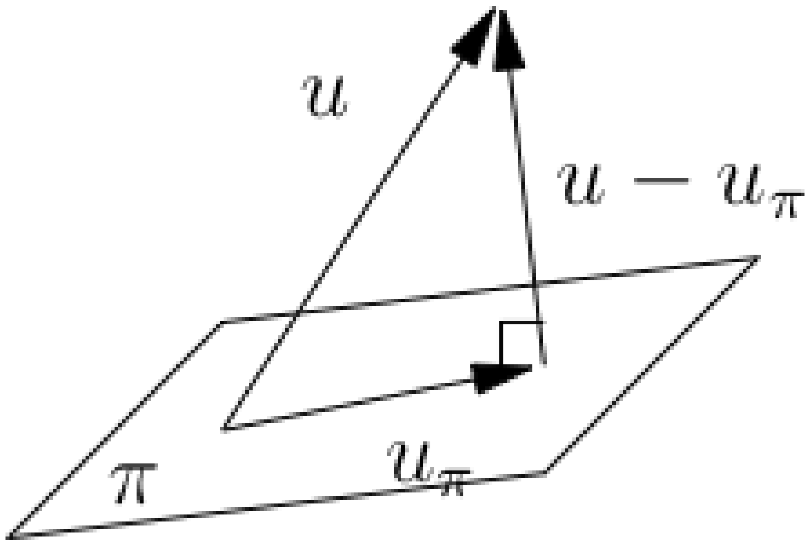

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 7 & 7 \end{pmatrix}$$

Orthogonal projection

$$e_1 = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, e_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}, e_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}$$

Since (e_1, e_2, e_3) is orthonormal the coordinates are
$x_1 = f \cdot e_1 = 14, x_2 = f \cdot e_2 = -15/\sqrt{6}, x_3 = f \cdot e_3 = -4/\sqrt{6}$.
The orthogonal projection is then
$\hat{f} = 14e_1 - 15/\sqrt{6}e_2 - 4/\sqrt{6}e_3$

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 7 & 7 \end{pmatrix}, \hat{f} = \begin{pmatrix} 1.5 & 2\frac{1}{6} & 2\frac{5}{6} \\ 4 & 4\frac{2}{3} & 5\frac{1}{3} \\ 6.5 & 7\frac{1}{6} & 7\frac{5}{6} \end{pmatrix},$$

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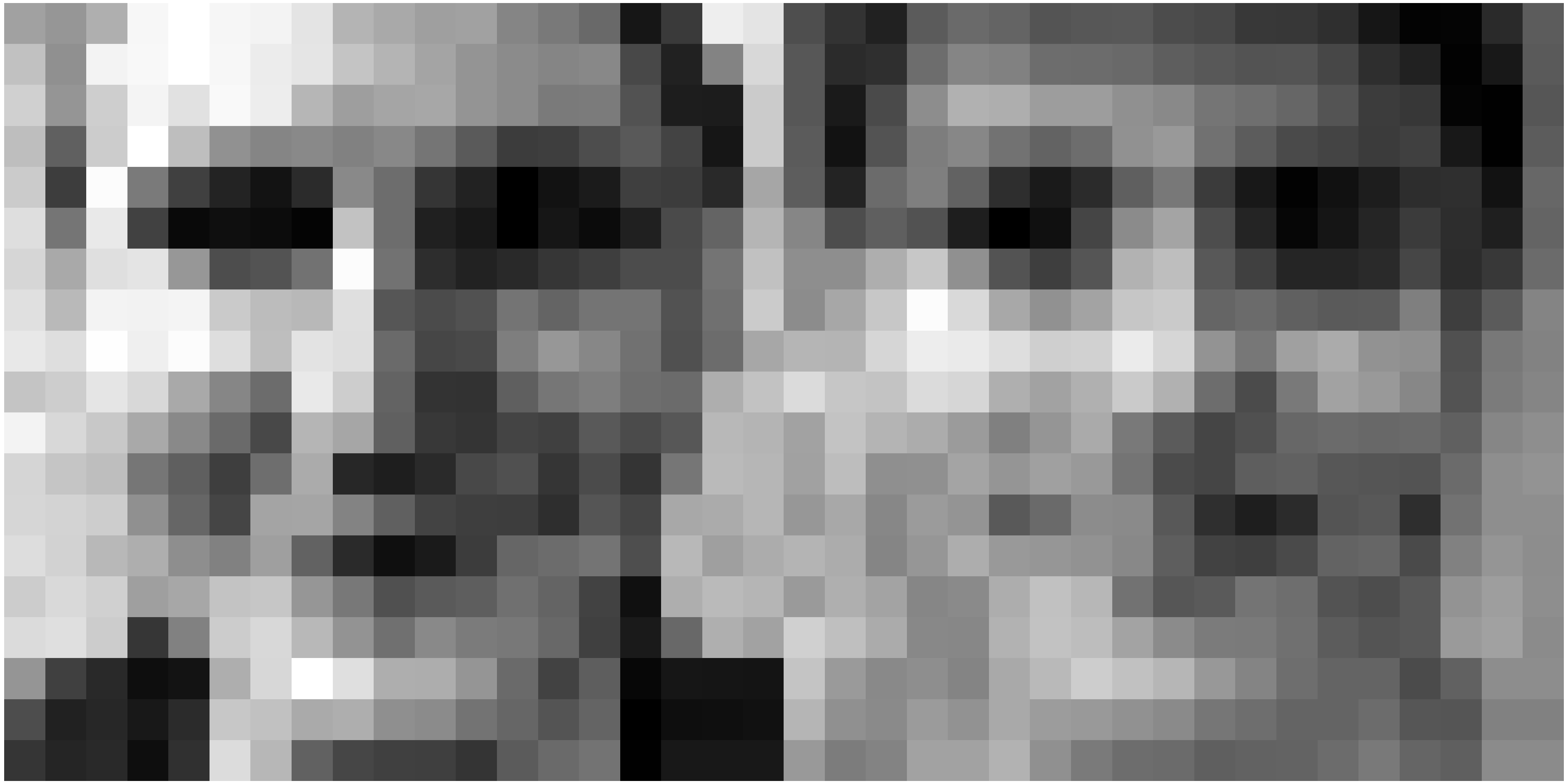
X

What is the orthogonal projection of f

onto the space spanned by (e_1, e_2, e_3)

Since (e_1, e_2, e_3) is orthonormal, the coordinates are
$x_1 = f \cdot e_1 = -2457, x_2 = f \cdot e_2 = 303, x_3 = f \cdot e_3 = -603.$
The orthogonal projection is then $\hat{f} = -2457e_1 + 303e_2 - 603e_3$



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Projection onto affine subspace

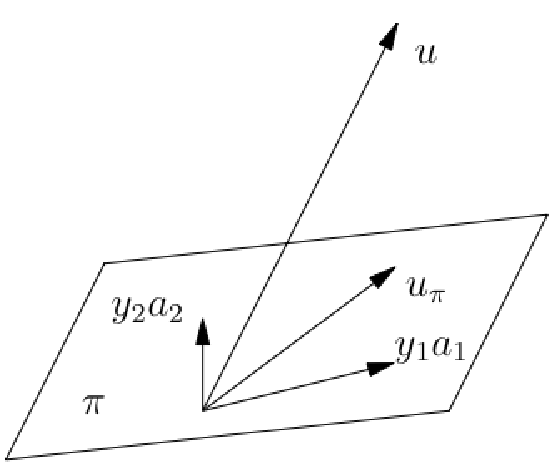
- Previously projection onto linear subspace

$$\pi = \{w \mid w = \sum_1^n x_i a_i = Ax \quad \text{where} \quad x_i \in \mathbb{C}\ (\text{or}\mathbb{R})\}$$

- A linear subspace always contains the zero vector
- How about planes or 'subspaces' that are shifted away from the origin. Such sets are called affine spaces.

$$\pi = \{w \mid w = m + \sum_1^n x_i a_i = Ax + m \quad \text{where} \quad x_i \in \mathbb{C}\ (\text{or}\mathbb{R})\}.$$

- An affine subspace is typically not a linear space



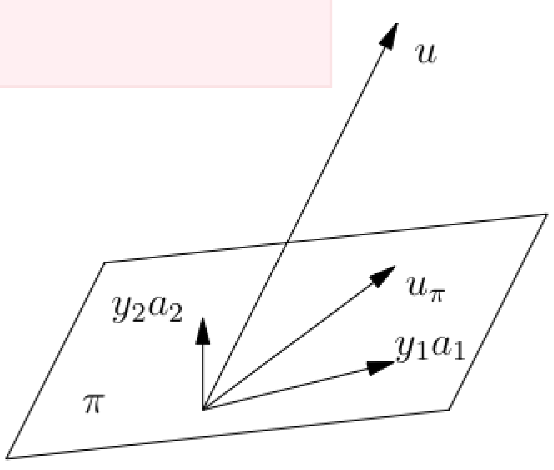
Projection onto affine subspace

- An affine subspace, defined by $m, a_1, \ldots, a_k$.

$$\pi = \{w \mid w = m + \sum_1^n x_i a_i = Ax + m \quad \text{where} \quad x_i \in \mathbb{C}\ (\text{or}\mathbb{R})\}.$$

- Projection of u onto the affine subspace

  - Subtract m, i.e. form $v = u - m$.
  - Project v onto the space spanned by $a_1, \ldots, a_k$, i.e. $v_\pi = A^+ v$.
  - Add m, i.e. form $u_\pi = v_\pi + m$.



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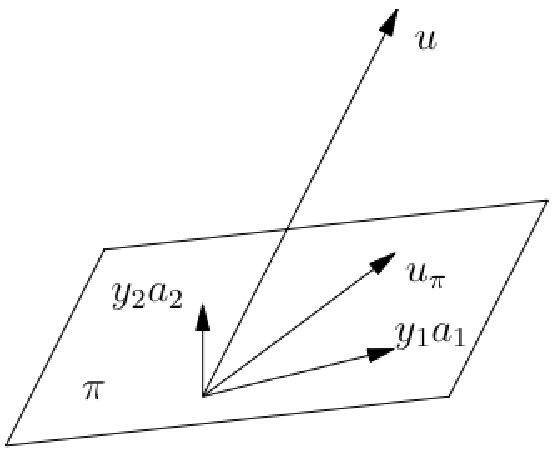
PCA - Principal Component Analysis

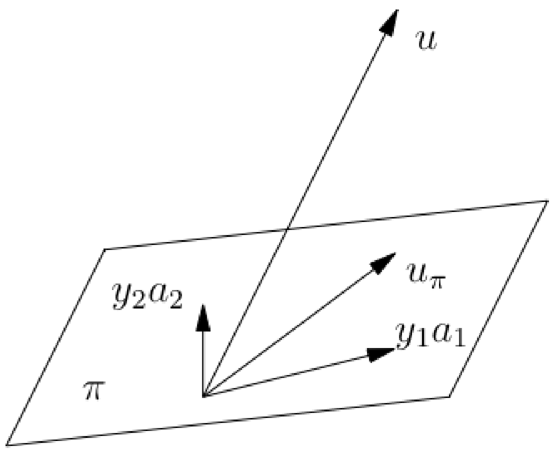


- Orthogonal projection – project an image u on
  - subspace spanned by $a_1, \ldots, a_k$.
  - or affine subspace defined by $m, a_1, \ldots, a_k$.

- How do we find a good subspace?
- Given lots of vectors $x_1, \ldots, x_N$. Find a suitable affine subspace so that the orthogonal projections y_i of x_i are as close to x_i as possible

$$e(\pi) = \sum_{i=1}^{N} ||y_i(\pi) - x_i||^2.$$

PCA - Principal Component Analysis

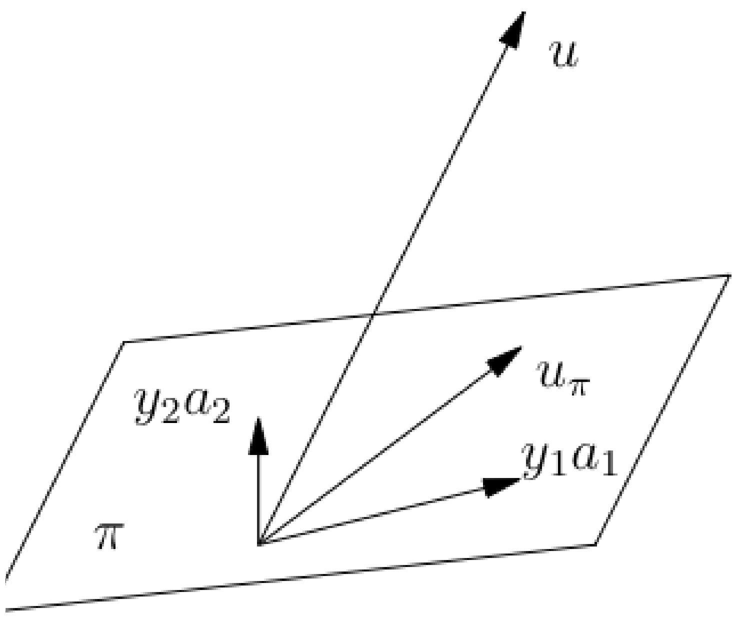


1. Calculate the mean $m = \frac{1}{N}\sum_{i=1}^{N} x_i$.
2. Subtract the mean from all examples $z_i = x_i - m$.
3. Place all of the resulting vectors as columns of a matrix, $M = \begin{pmatrix} z_1 & \dots & z_N \end{pmatrix}$.
4. Factorize M using the singular value decomposition $M = USV^T$.
5. Use the first k columns of U as the basis of the subspace, i.e. $a_i = u_i$, with $U = \begin{pmatrix} u_1 & \dots & u_m \end{pmatrix}$.

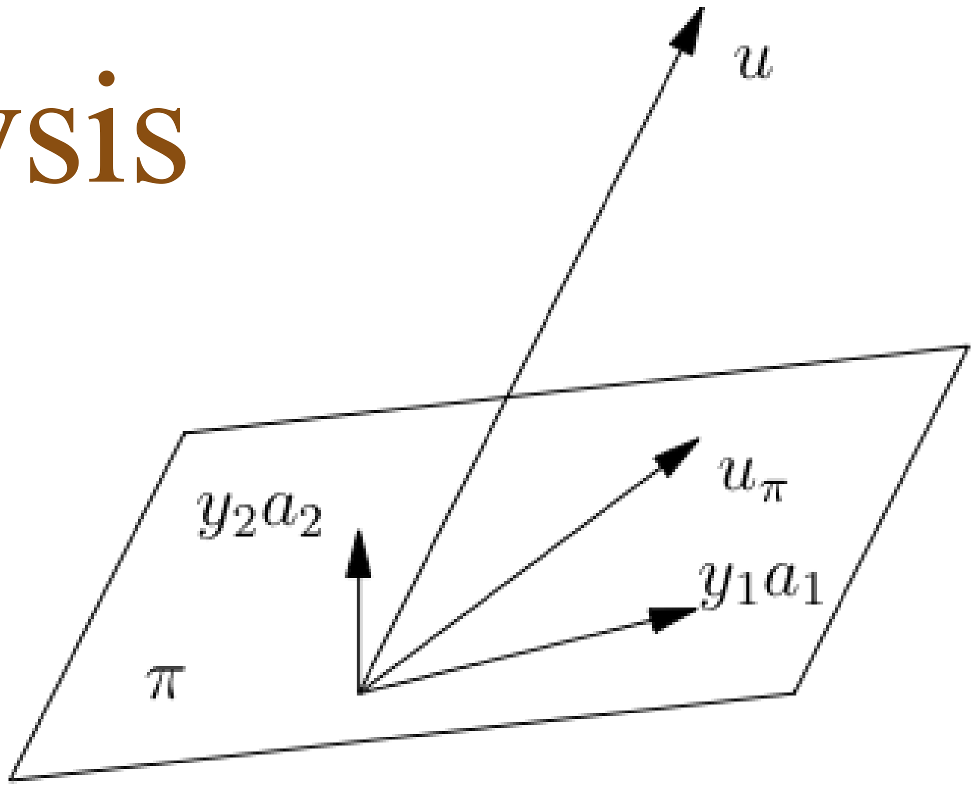
$$\pi = \{w \mid w = m + \sum_1^n x_i a_i = Ax + m \quad \text{where} \quad x_i \in \mathbb{C}\ (\text{or}\mathbb{R})\}.$$

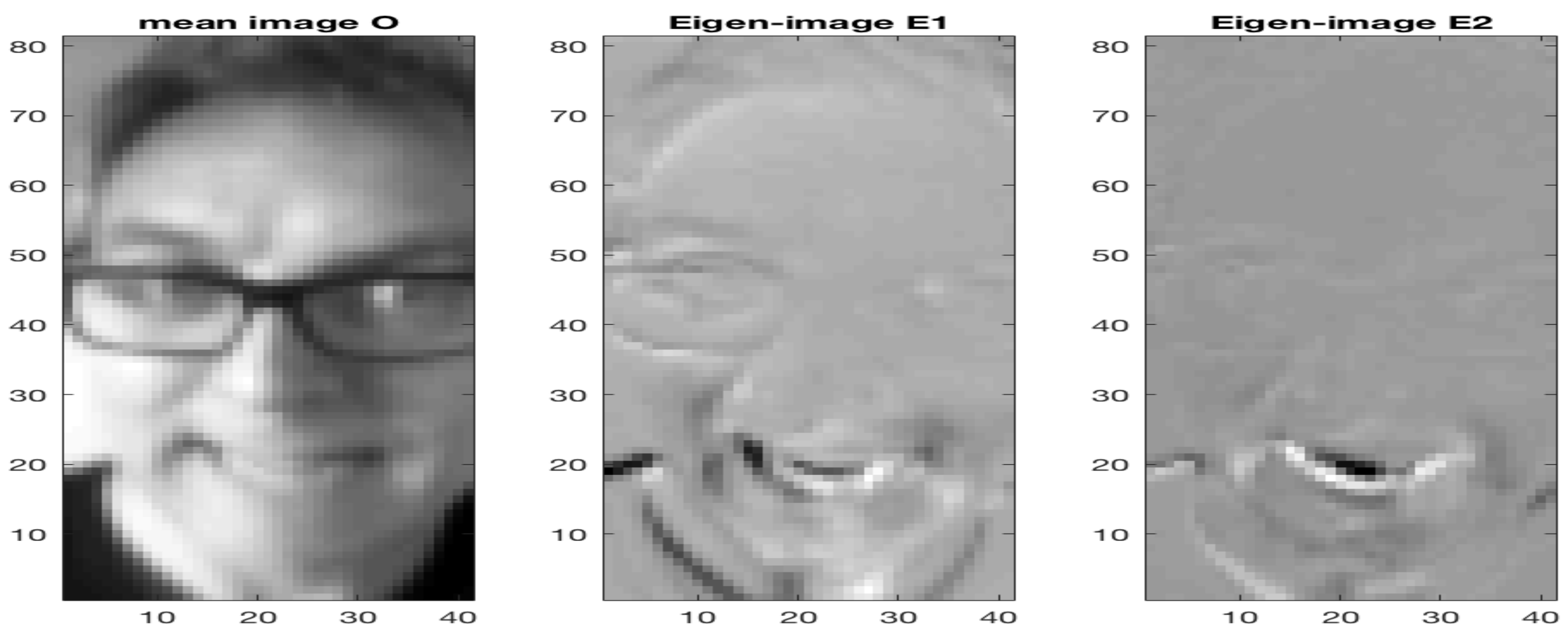
$$e(\pi) = \sum_{i=1}^{N} ||y_i(\pi) - x_i||^2.$$

PCA –"Training"
Given examples, find subspace

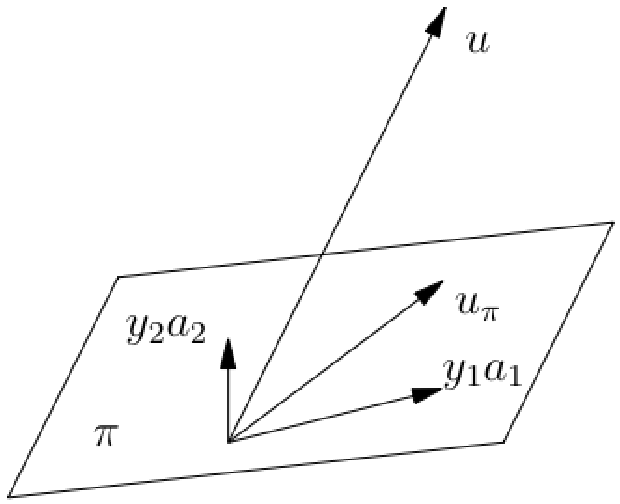


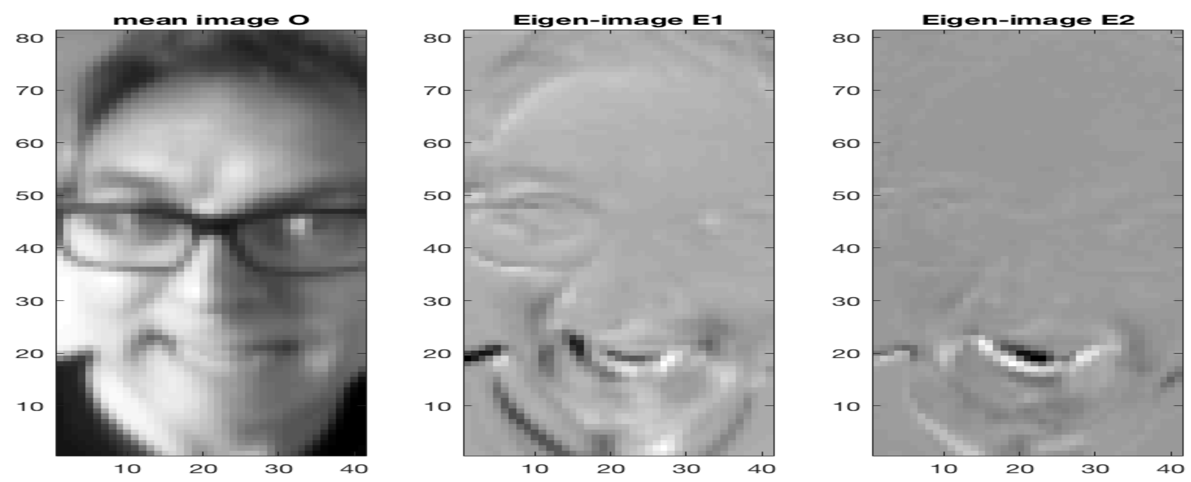
PCA - Principal Component Analysis



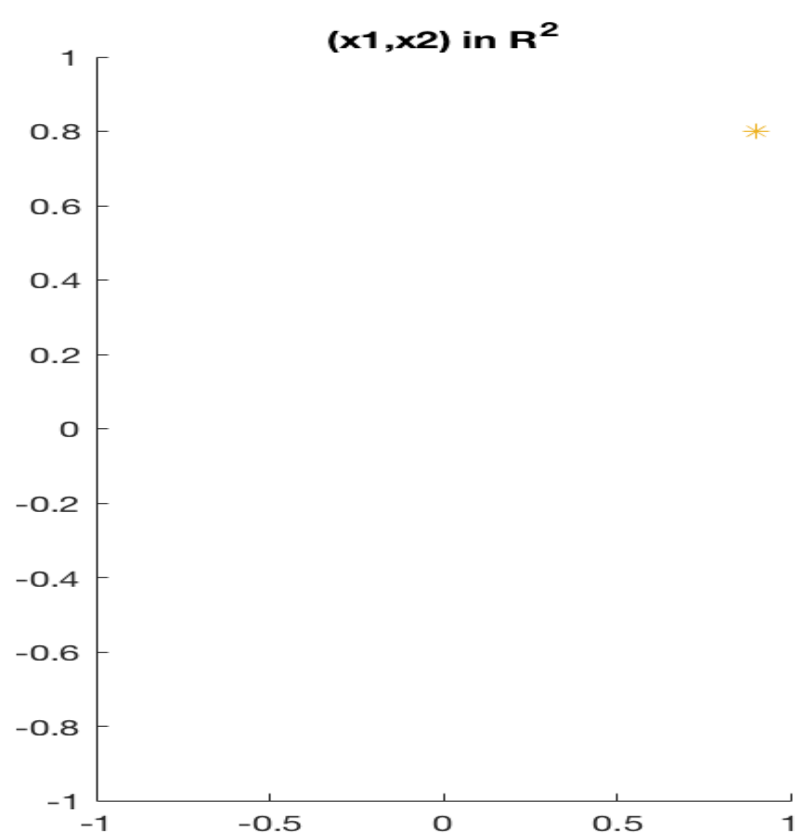


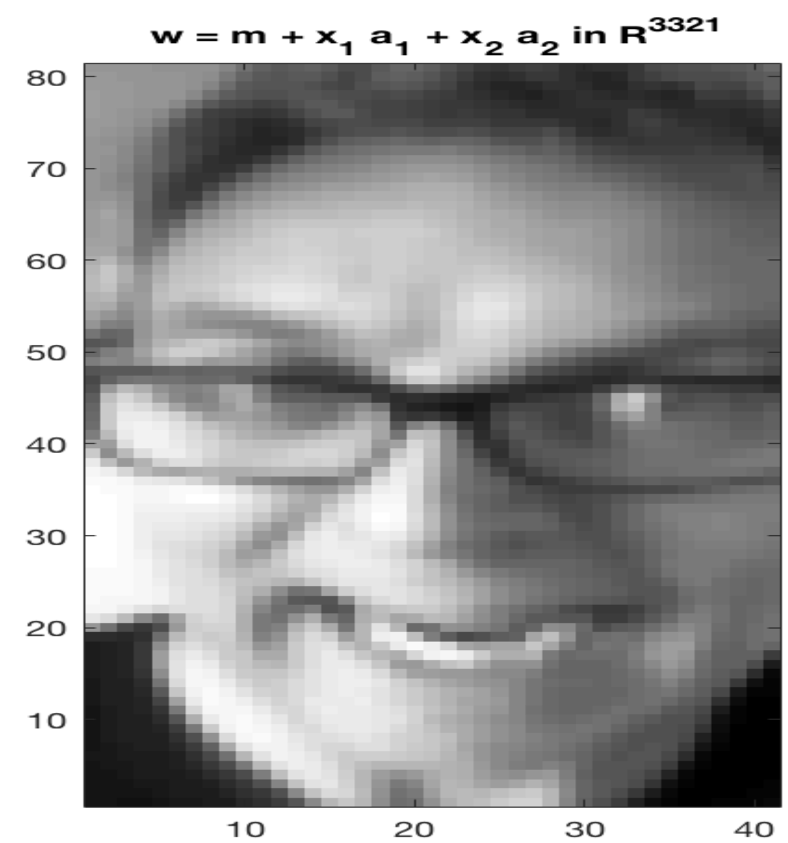
PCA - Principal Component Analysis



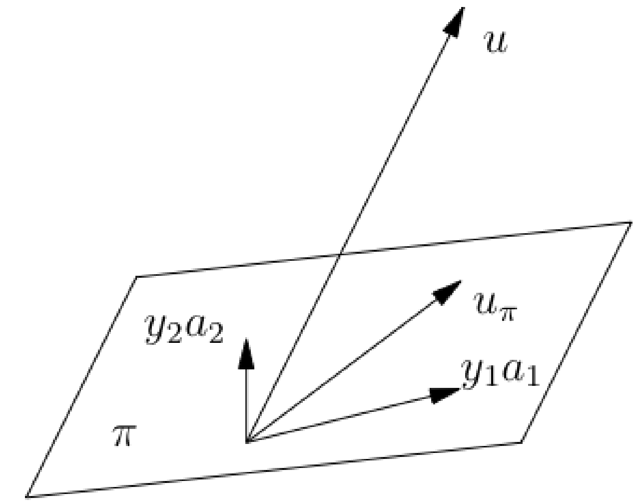


$$w = m + \sum_{1}^{n} x_i a_i$$

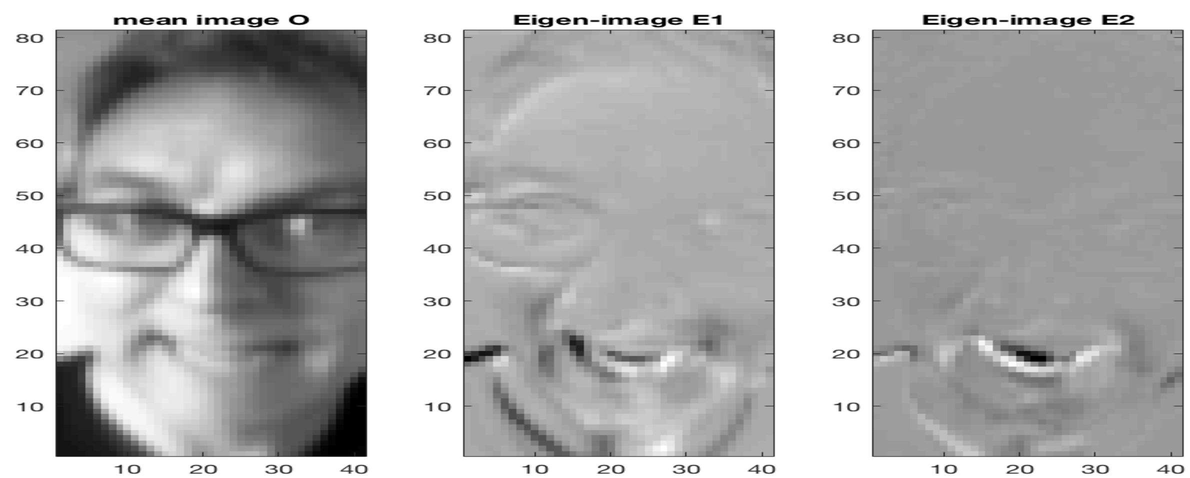


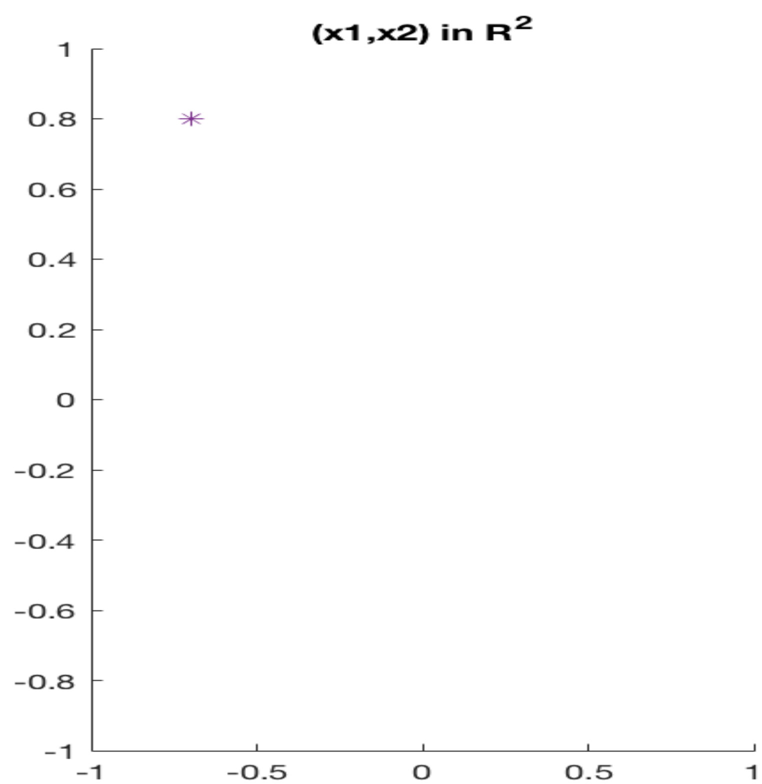


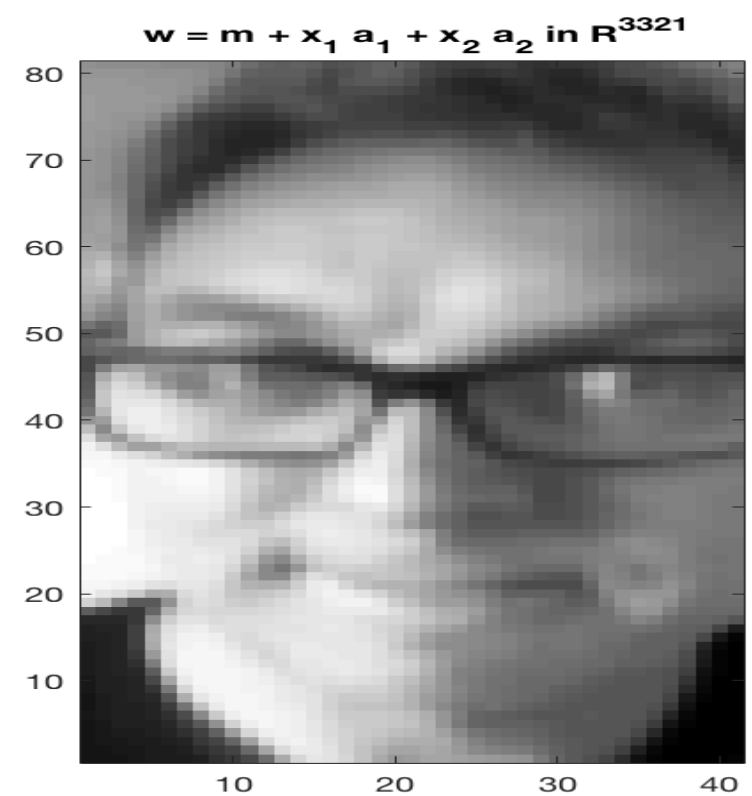
PCA - Principal Component Analysis



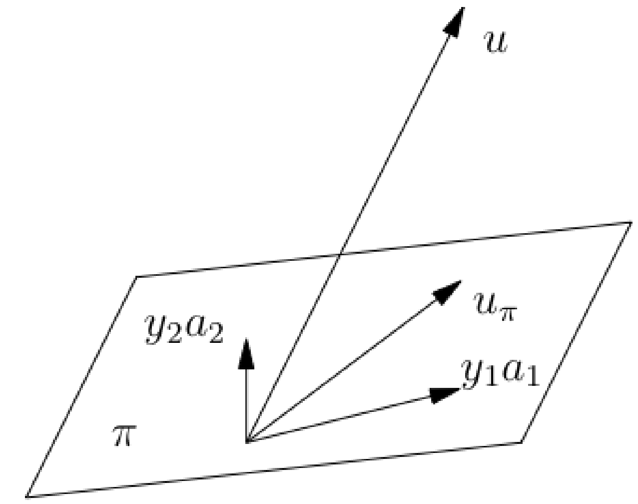
$$w = m + \sum_{1}^{n} x_i a_i$$

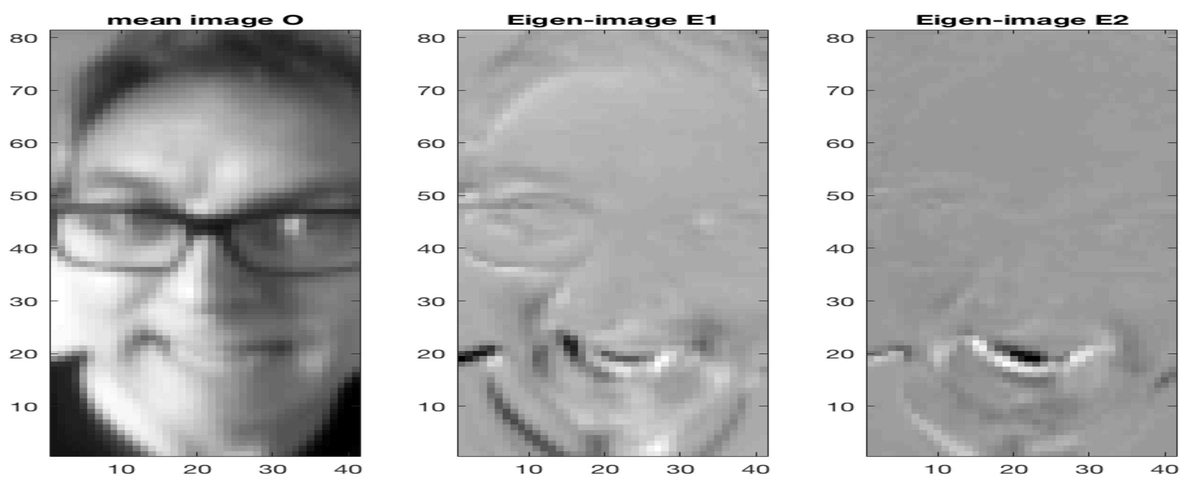




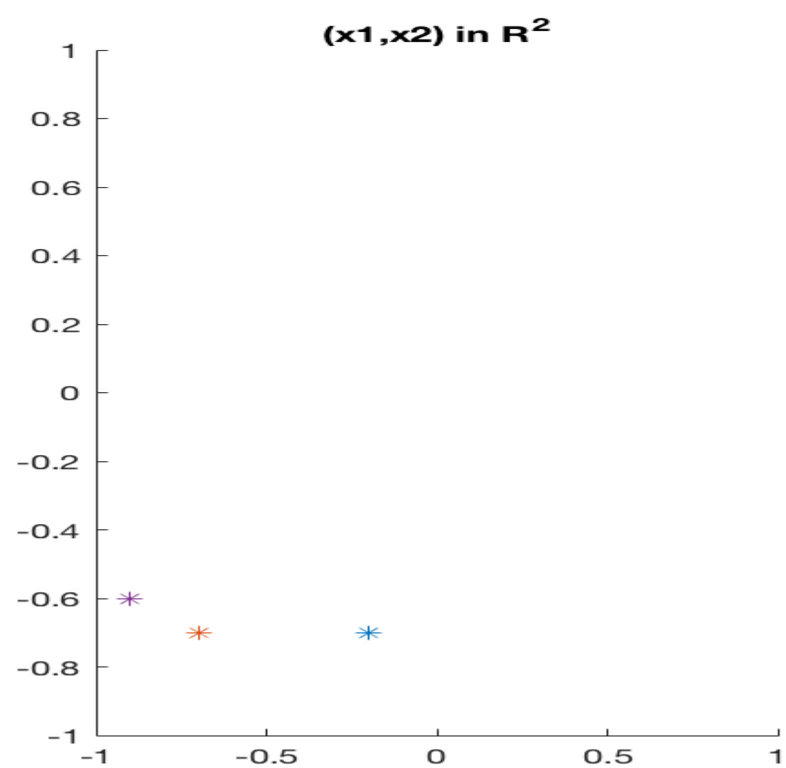


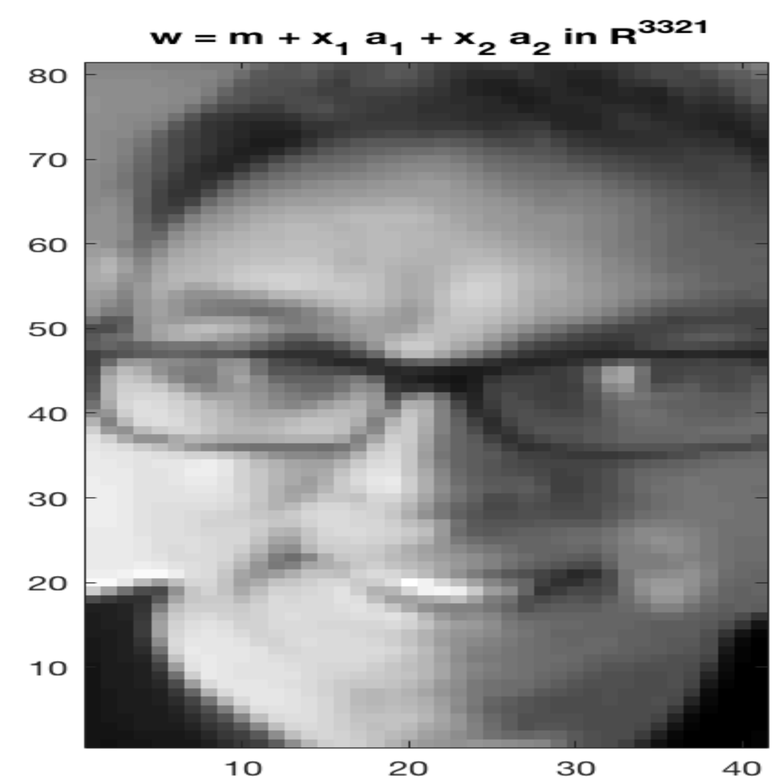
PCA - Principal Component Analysis



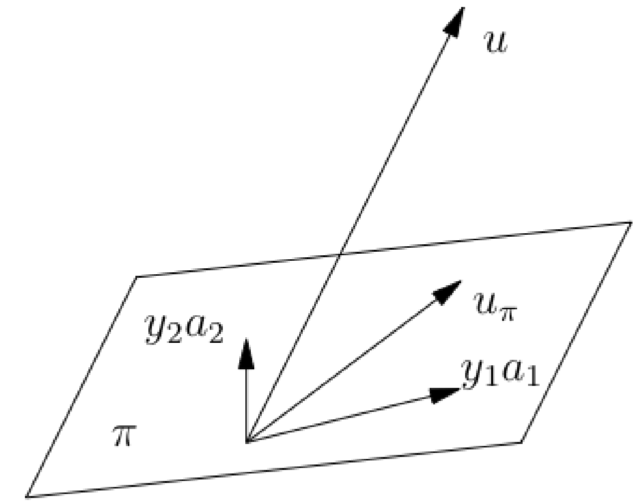


$$w = m + \sum_{1}^{n} x_i a_i$$

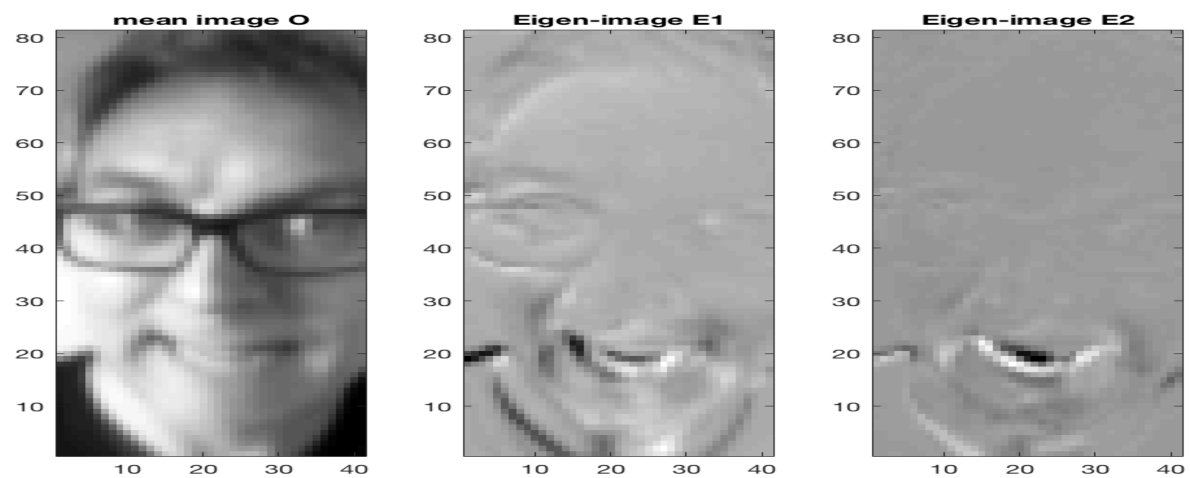


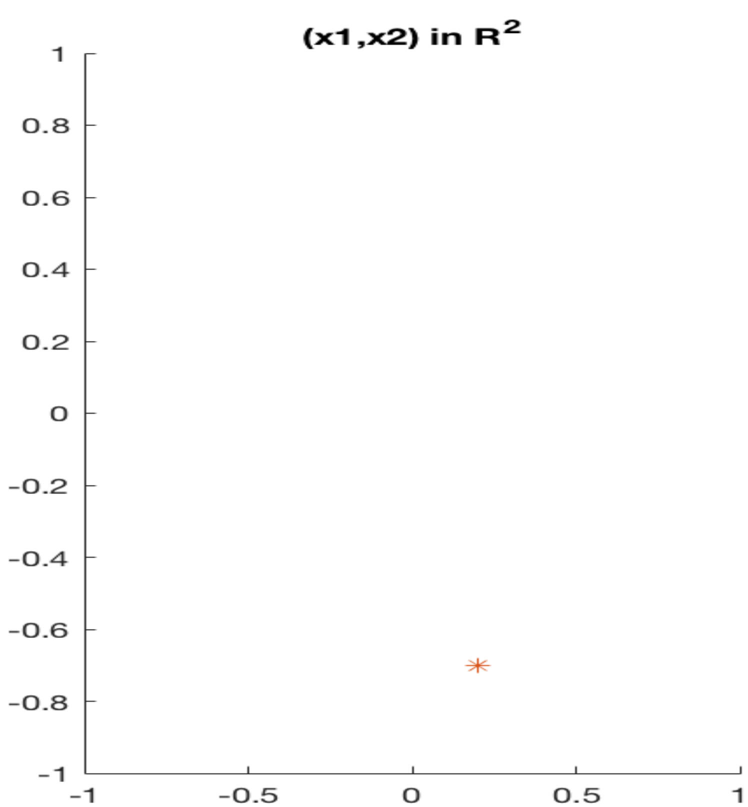


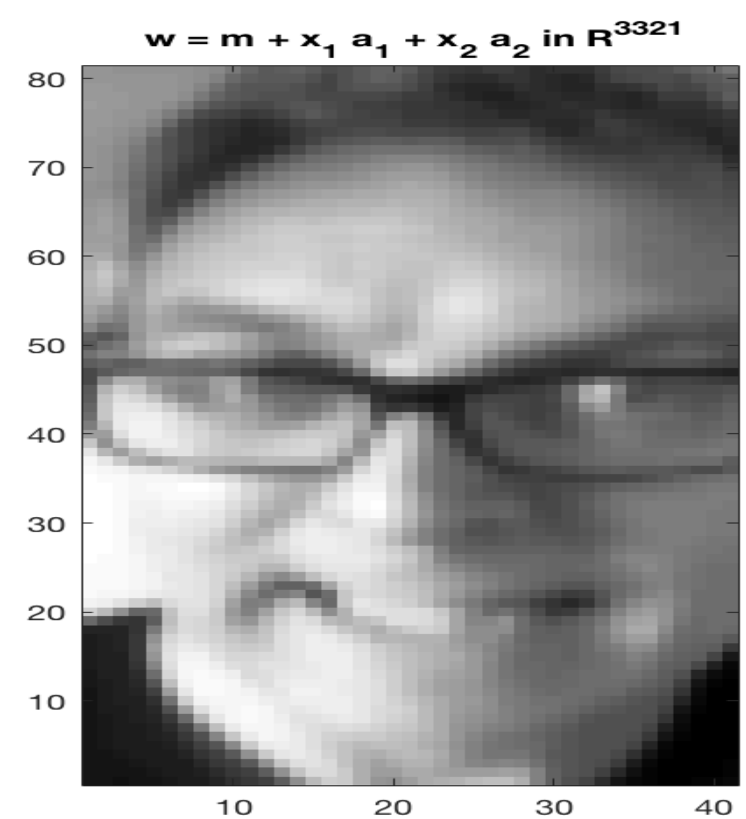
PCA - Principal Component Analysis



$$w = m + \sum_{1}^{n} x_i a_i$$







Overview –
Linear Algebra and FFT

1. Linear Algebra
    1. Vector space – ’A matrix is a vector’ What does this mean?
    2. Basis, coordinates
    3. Scalar product
    4. Projection onto a subspace
    5. Projection onto an affine ‘subspace’
    6. (Principal Component Analysis – Recipe)
    7. Change of basis
2. **Fourier Transform**

Fourier Transform

$$F(u,v) = \sum_{x=1}^{M} \sum_{y=1}^{N} f(x,y)\, e^{-i2\pi((u-1)(x-1)/M+(v-1)(y-1)/N)}$$

- Can be viewed as a change of basis
- Image f –> Fourier Transform F   (and back)
- Has strong connections with convolutions
- (next lecture)
- Useful for image compression
- Useful for image understanding
- Basically a great tool

Fourier Transform

- Definition, is a change of basis, what does is mean
  - Detour (for increased understanding
  - Ordinary Fourier Transform (from previous courses)
  - Examples
  - Properties
- Discrete Fourier Transform – 1D

Image basis example (Walsh)

$$f = \begin{bmatrix} 9 & -1 \\ 5 & 7 \end{bmatrix}$$

$$\Phi_{11} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} /2 \qquad \Phi_{12} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} /2$$

$$\Phi_{21} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} /2 \quad \Phi_{22} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} /2$$

$$x_{ij} = f \cdot \Phi_{ij} = \sum_{\lambda,\mu} f(\lambda,\mu)\Phi_{ij}(\lambda,\mu)$$

$$f = x_{11}\Phi_{11} + x_{21}\Phi_{21} + x_{12}\Phi_{12} + x_{22}\Phi_{22}$$

$$x = \begin{bmatrix} 10 & 4 \\ -2 & 6 \end{bmatrix}$$

- Image f –> Fourier Transform x   (and back)

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Fourier transform as change of image basis

$$x_{ij} = f \cdot \Phi_{ij} = \sum_{\lambda,\mu} f(\lambda,\mu)\Phi_{ij}(\lambda,\mu)$$

$$f = x_{11}\Phi_{11} + x_{21}\Phi_{21} + x_{12}\Phi_{12} + x_{22}\Phi_{22}$$

$$F(u,v) = \sum_{x=1}^{M} \sum_{y=1}^{N} f(x,y)\, e^{-i2\pi((u-1)(x-1)/M+(v-1)(y-1)/N)}$$

$$f(x,y) = \frac{1}{MN} \sum_{u=1}^{M} \sum_{v=1}^{N} F(u,v)\, e^{i2\pi((u-1)(x-1)/M+(v-1)(y-1)/N)}$$

Compare with ordinary Fourier Transform

Definition

Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. The Fourier transformen of f is defined as

$$(\mathcal{F}f)(u) = F(u) = \int_{-\infty}^{+\infty} e^{-i2\pi xu} f(x)dx \quad .$$

■

Theorem

Under the right assumptions on f, the following inversion formula

$$f(x) = \int_{-\infty}^{+\infty} e^{i2\pi ux} F(u)du$$

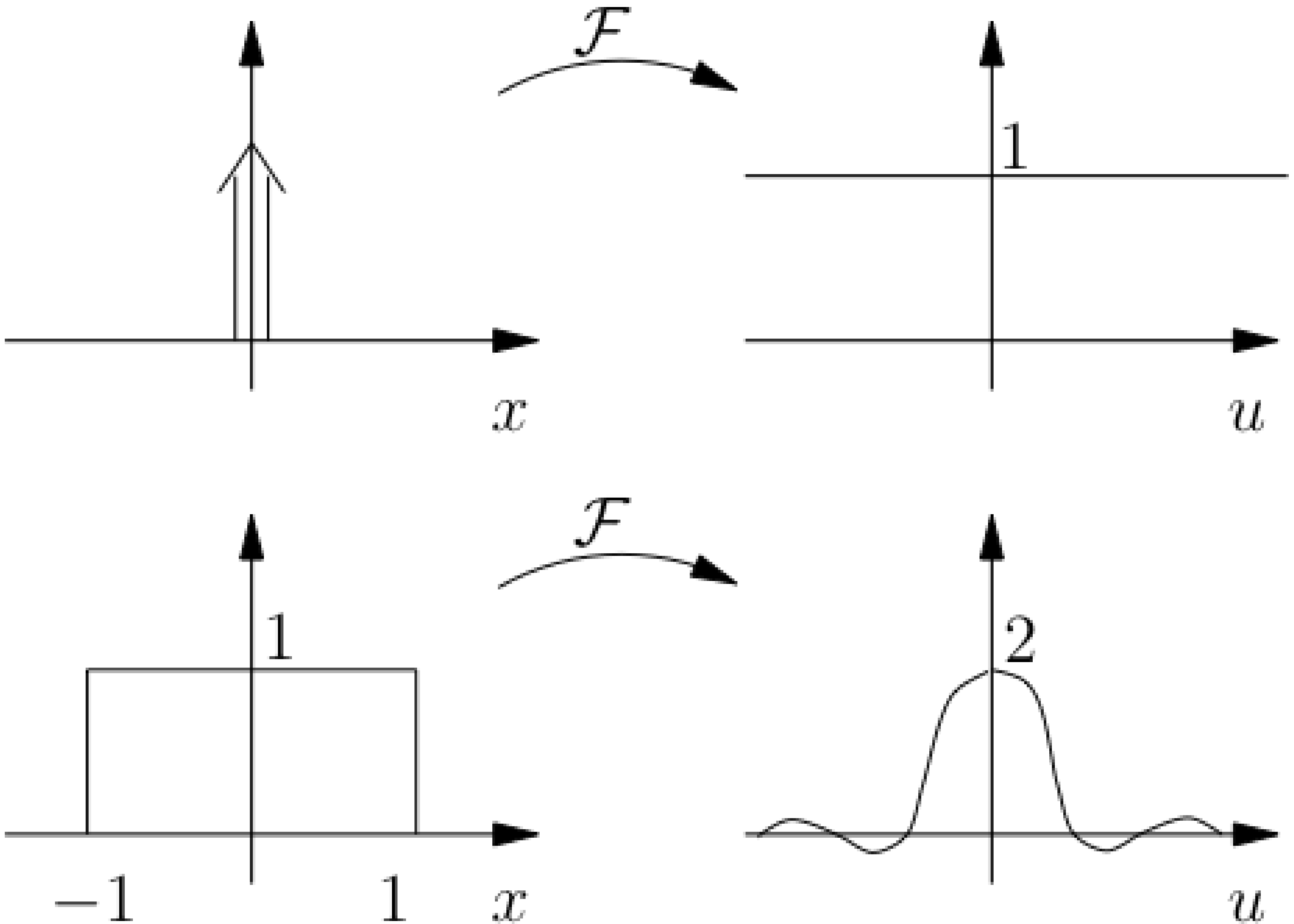
holds.

Examples

$$\delta(x) \mapsto 1(u)$$

$$\mathrm{rect}(x) \mapsto 2\frac{\sin(2\pi u)}{2\pi u} = 2\,\mathrm{sinc}(2\pi u)$$

Examples



Examples

$$
\begin{aligned}
c_1 f_1(x) + c_2 f_2(x) &\mapsto c_1 F_1(u) + c_2 F_2(u) && \text{(linearity)} \\
f(\lambda x) &\mapsto \frac{1}{|\lambda|} F(\frac{u}{\lambda}) && \text{(scaling)} \\
f(x - a) &\mapsto e^{-i2\pi ua} F(u) && \text{(translation)} \\
e^{-i2\pi xa} f(x) &\mapsto F(u + a) && \text{(modulation)} \\
\overline{f(x)} &\mapsto \overline{F(-u)} && \text{(conjugation)} \\
\frac{df}{dx} &\mapsto 2\pi i u F(u) && \text{(differentiation I)} \\
-2\pi i x f(x) &\mapsto \frac{dF}{du} && \text{(differentiation II)}
\end{aligned}
$$

Example: $\delta(x - 1) \mapsto e^{-i2\pi u}$

Discrete Fourier Transform - 1D

$$f = \begin{bmatrix} f(1) \\ \vdots \\ f(N) \end{bmatrix}$$

$$F(u) = \sum_{x=1}^{N} f(x)\, exp[-i2\pi(u-1)(x-1)/N]$$

$$F(u) = \sum_{x=1}^{N} f(x)\omega_N^{(x-1)(u-1)}$$

$$\omega_N = \exp(-i2\pi/N)$$

Discrete Fourier Transform - 1D

Definition

The Fourier Matrix $\mathcal{F}_N$ is given by

$$\mathcal{F}_N = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_N & \omega_N^2 & \dots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \dots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \dots & \omega_N^{(N-1)(N-1)} \end{pmatrix} .$$

$$f \longrightarrow F = \mathcal{F}_N f.$$

Discrete Fourier Transform - 1D

Theorem 3.3.1. *For the Fourier matrix the following holds,*

$$\mathcal{F}\overline{\mathcal{F}} = NI \ .$$

From this we obtain $\mathcal{F}^{-1} = \frac{1}{N}\overline{\mathcal{F}}$ *The inverse Fourier transform is thus*

$$f = \overline{\mathcal{F}}F \Longleftrightarrow f(x) = \frac{1}{N}\sum_{u=1}^{N} F(u)\omega_N^{(x-1)(u-1)}, \quad x = 1, \ldots, N \ .$$

Discrete Fourier Transform - 1D

- Important: DFT assumes that signals are periodic!
- Think of the signal as wrapped periodically

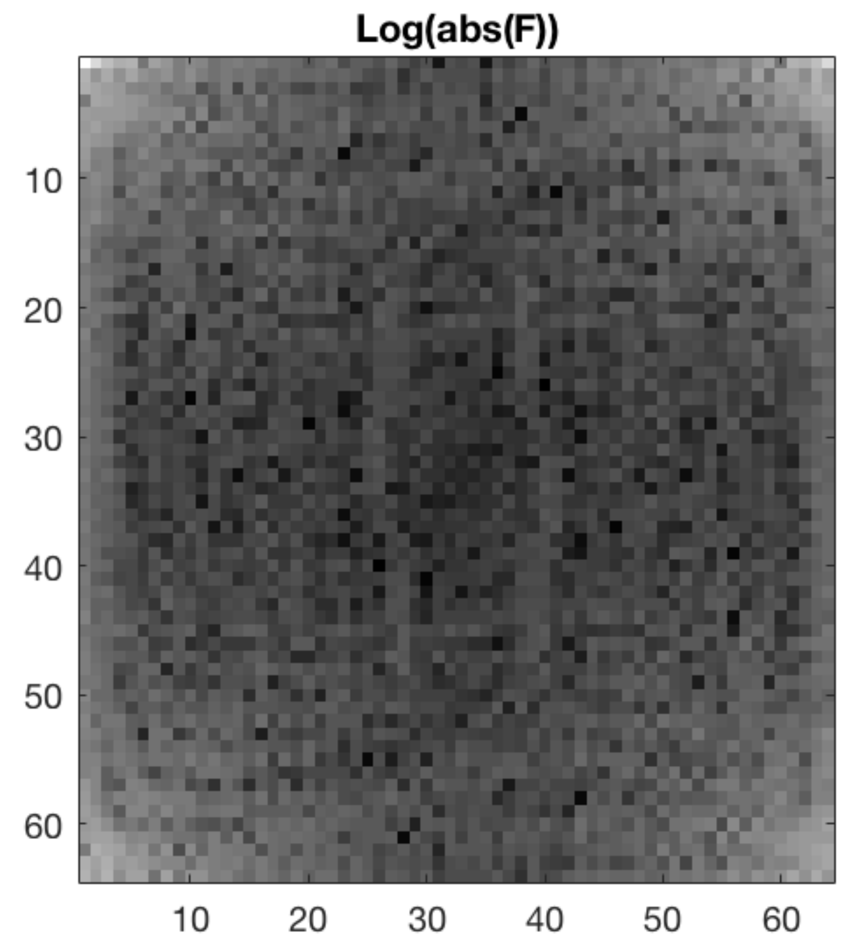
- Fourier transform is complex.
- Plot absolute value and phase

- Low frequencies in the edges/corners.
- Ordinary images typically have large values for low frequencies.

Discrete Fourier Transform - 2D

$$F(u,v) = \sum_{x=1}^{M} \sum_{y=1}^{N} f(x,y)\, e^{-i2\pi((u-1)(x-1)/M+(v-1)(y-1)/N)}$$

$$f(x,y) = \frac{1}{MN} \sum_{u=1}^{M} \sum_{v=1}^{N} F(u,v)\, e^{i2\pi((u-1)(x-1)/M+(v-1)(y-1)/N)}$$



Discrete Fourier Transform - 2D

Let the matrix F represent the Fourier transform of the image $f(x, y)$:

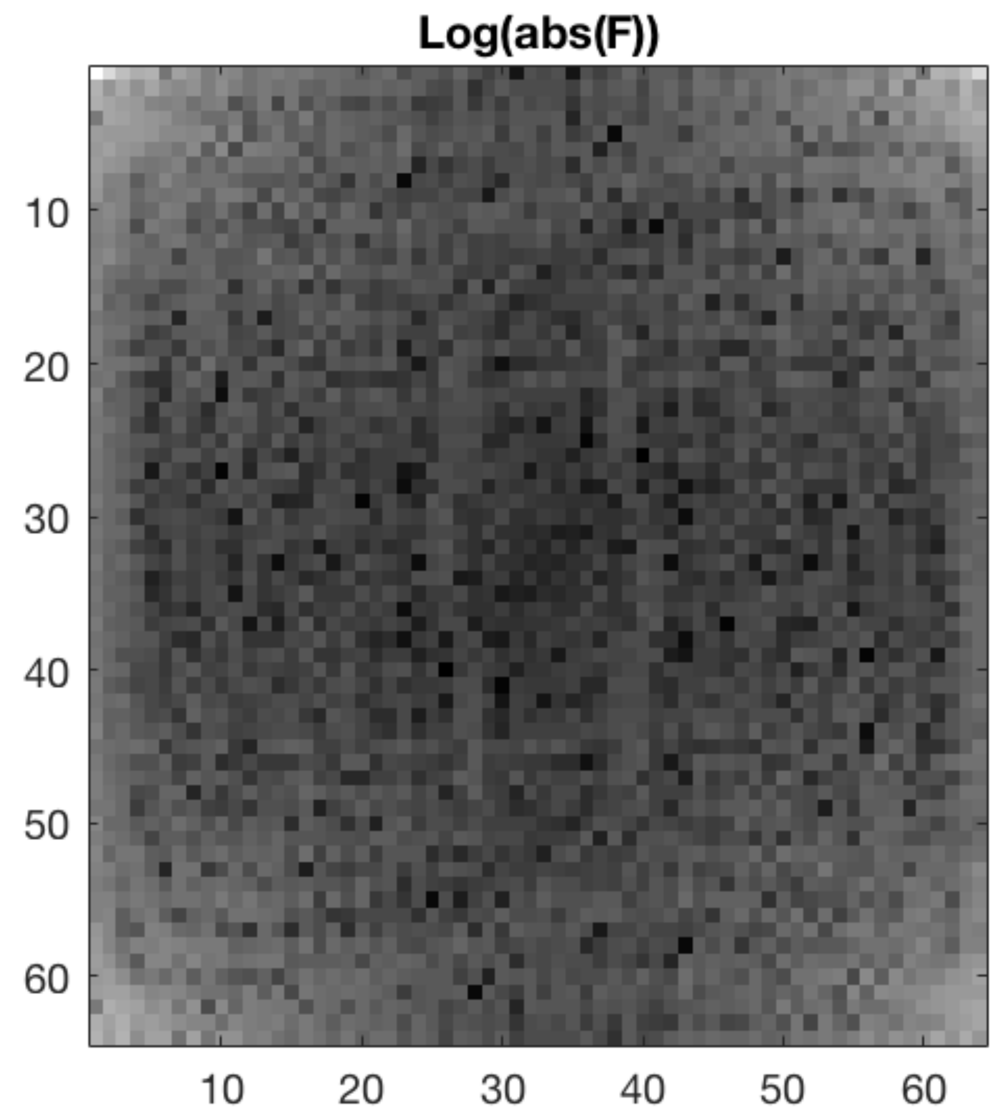
$$F = \mathcal{F}_M f \mathcal{F}_N$$

or

$$F = \mathcal{F}_M (\mathcal{F}_N f^T)^T \quad .$$

i.e. the DFT in two dimensions can be calculated by repeated use of the one-dimensional DFT, first for the rows, then for the columns.

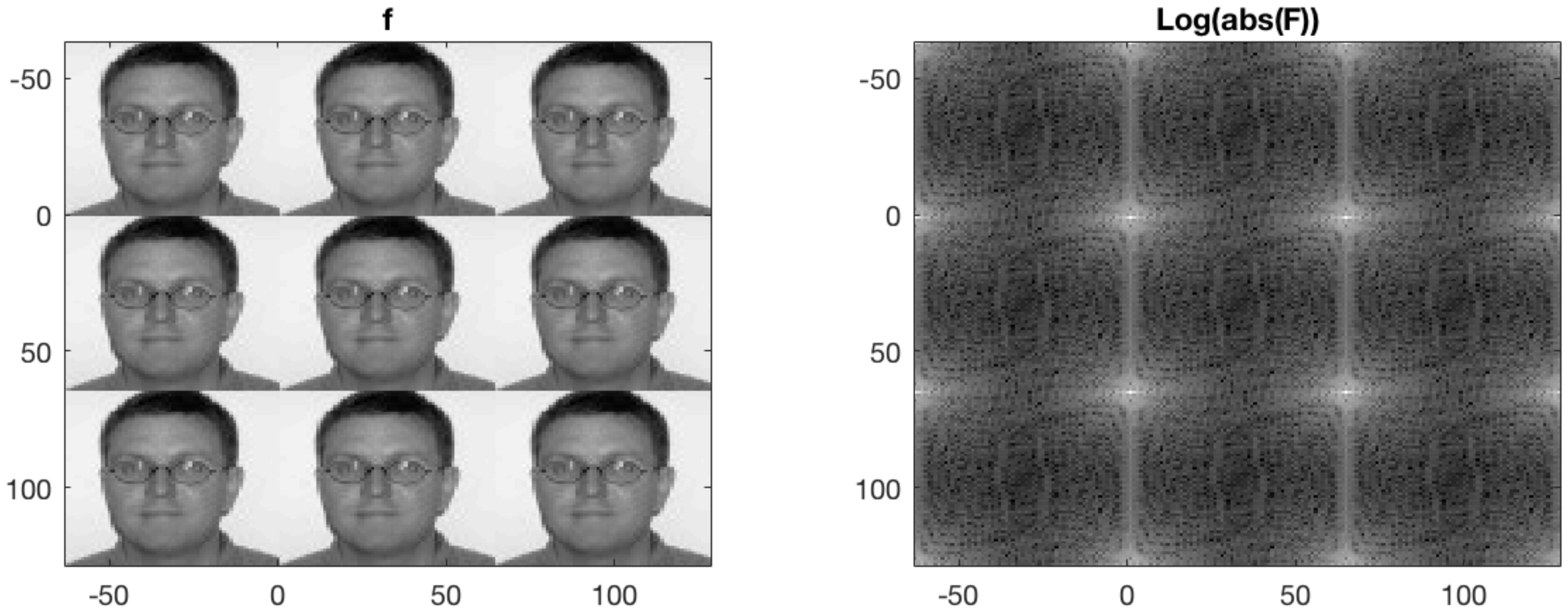
Discrete Fourier Transform - 2D Example



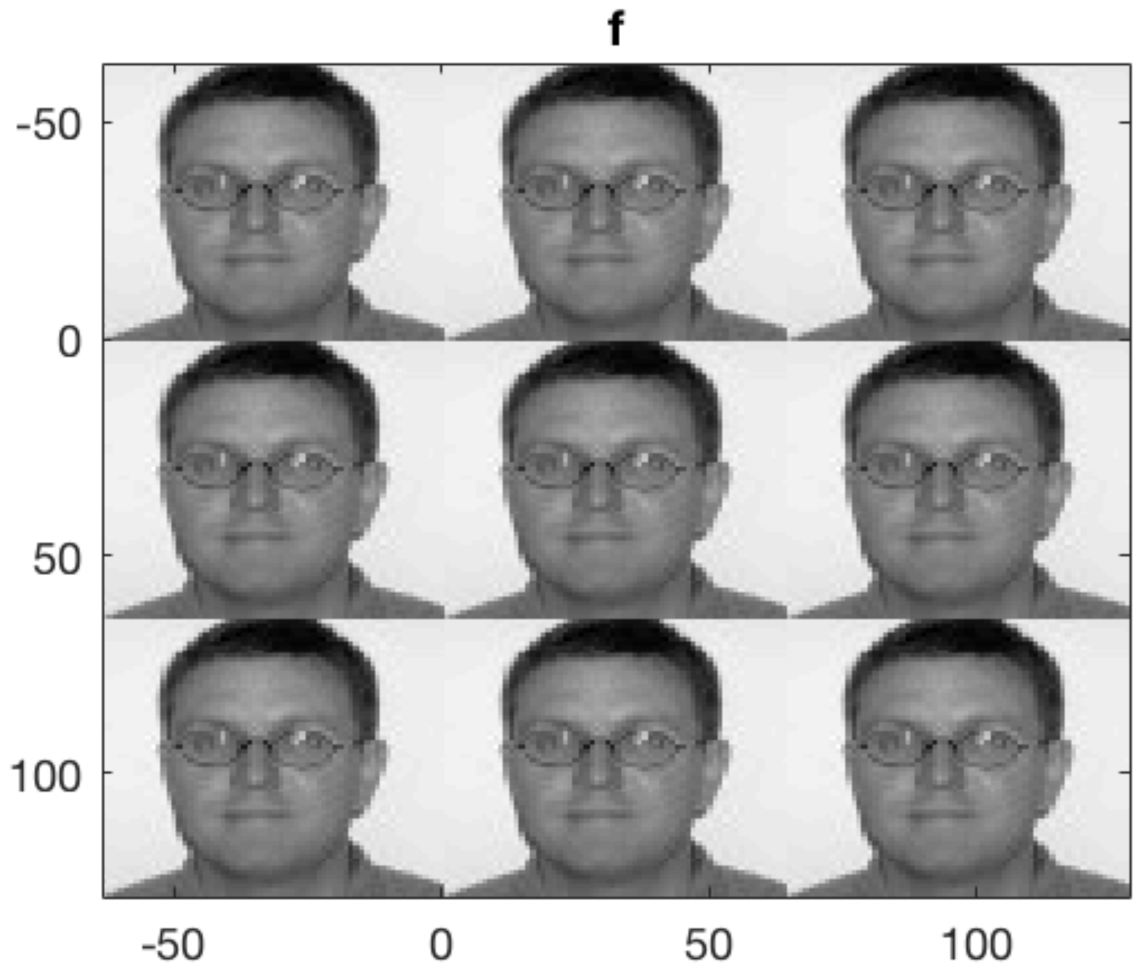
Low frequencies in the edges/corners

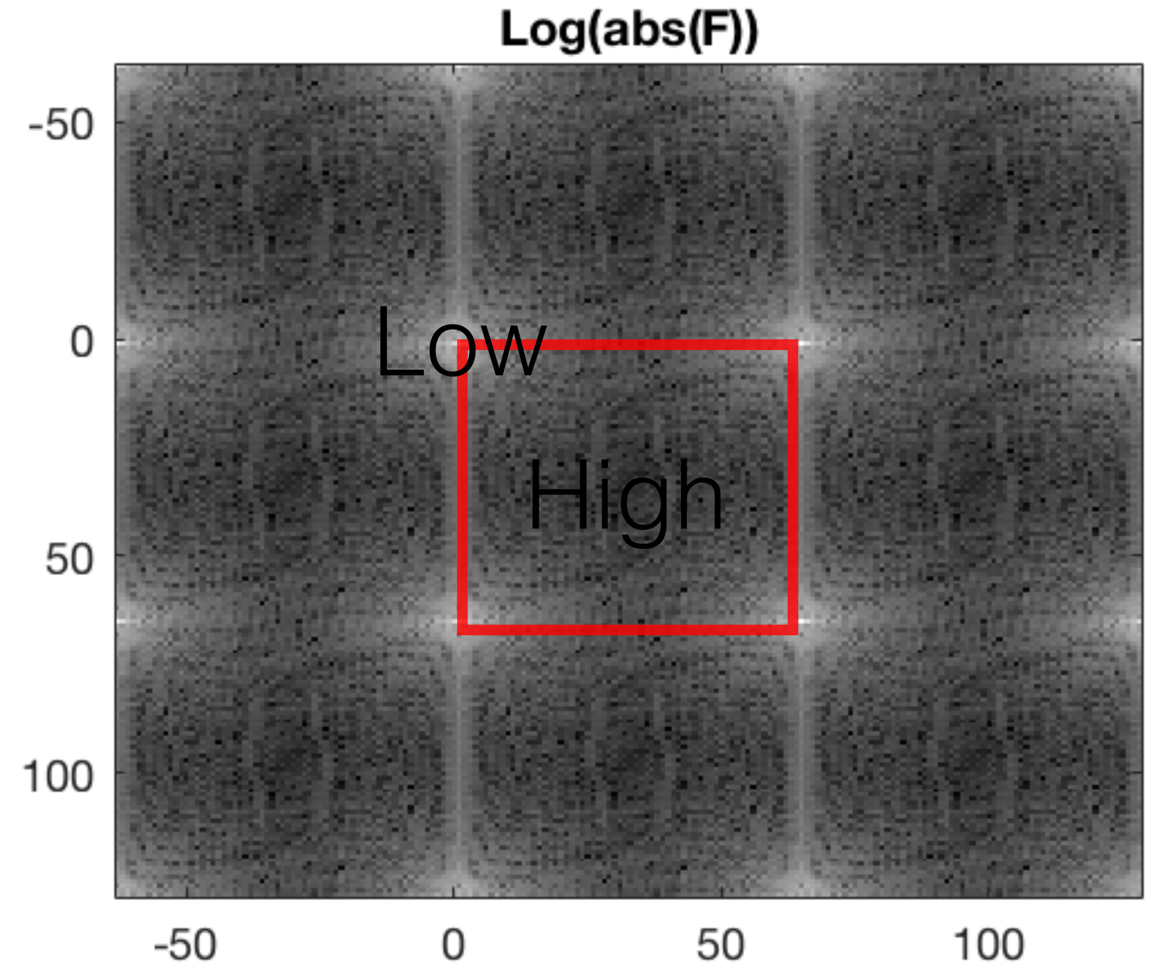
Fourier transform is complex. Plot absolute value and phase

Discrete Fourier Transform - 2D Example – Periodic expansion



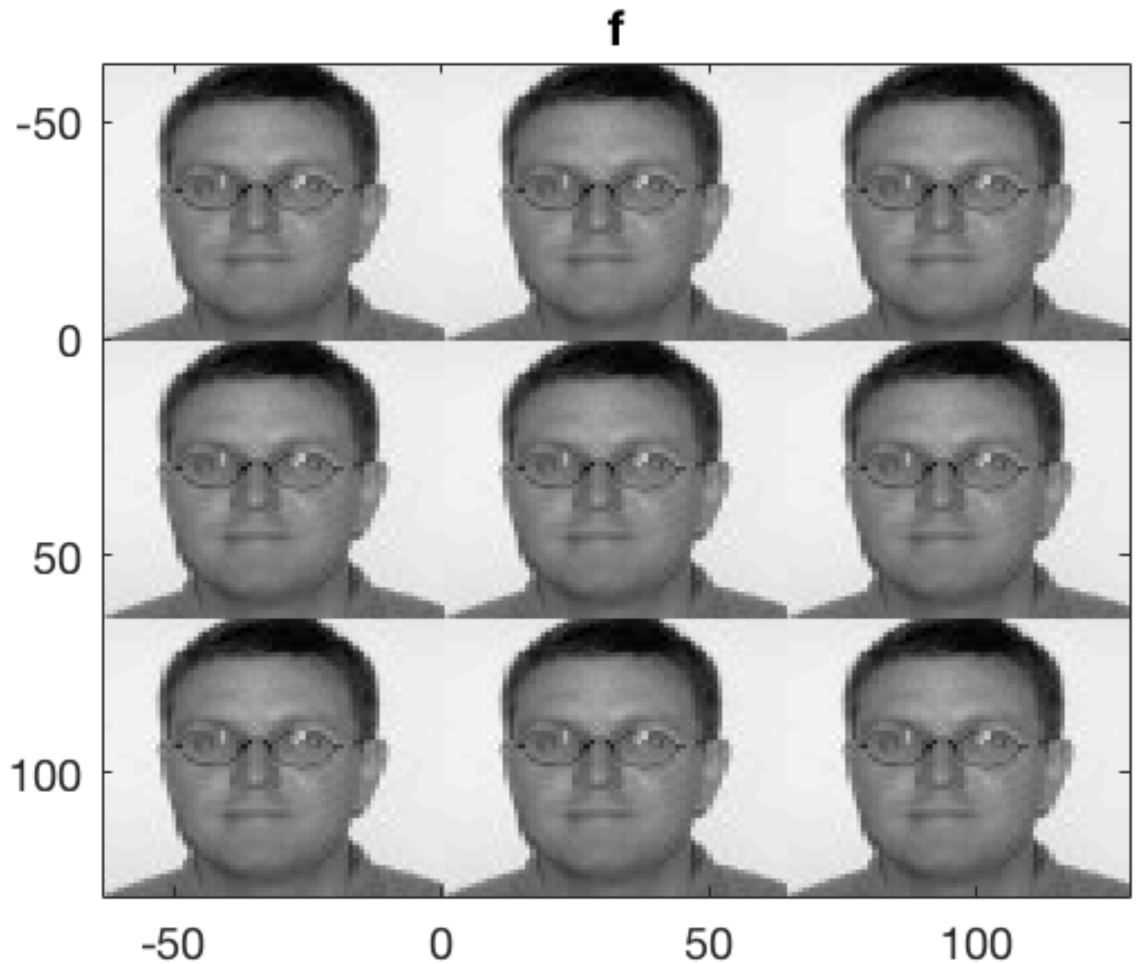
Discrete Fourier Transform - 2D Example – Periodic expansion

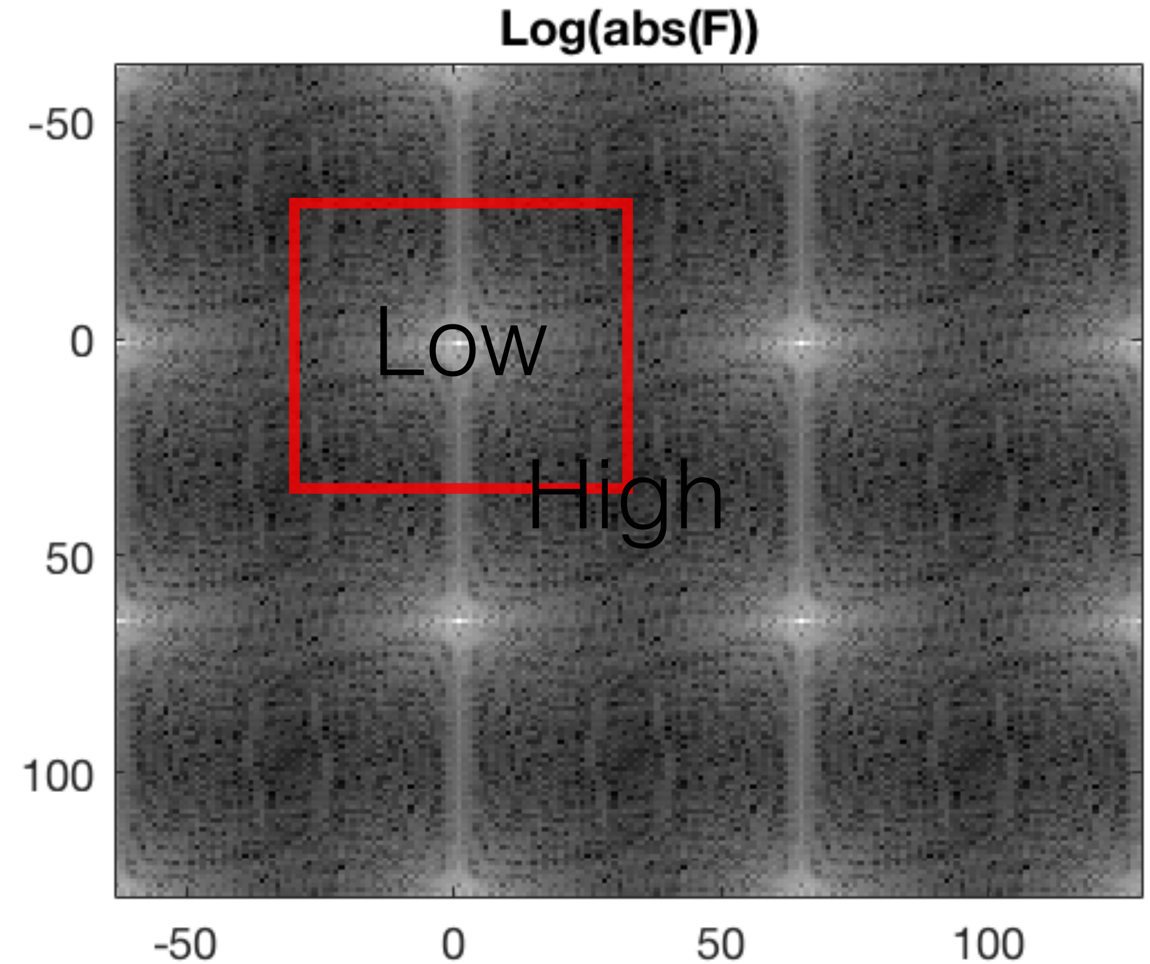




(1-M x 1-N)

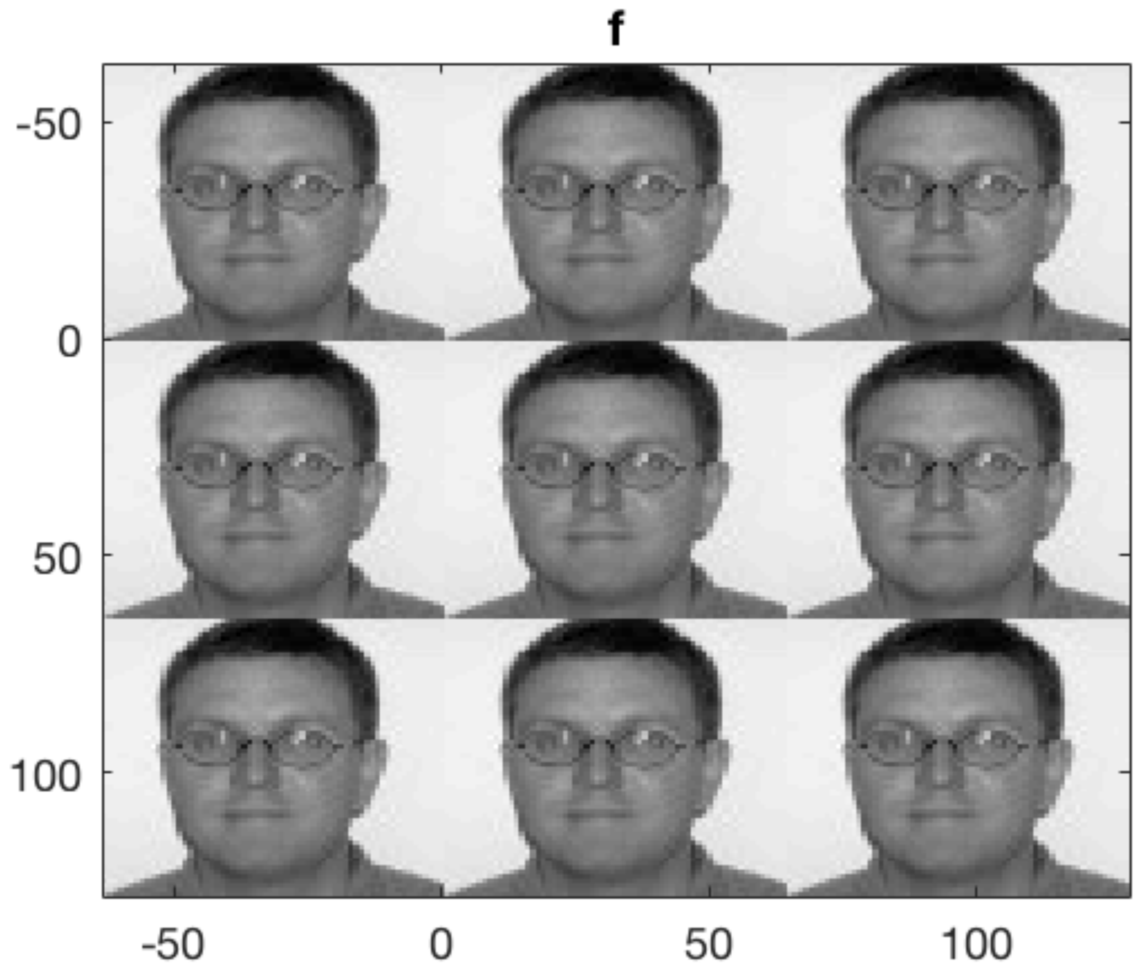
Discrete Fourier Transform - 2D Example – Periodic expansion

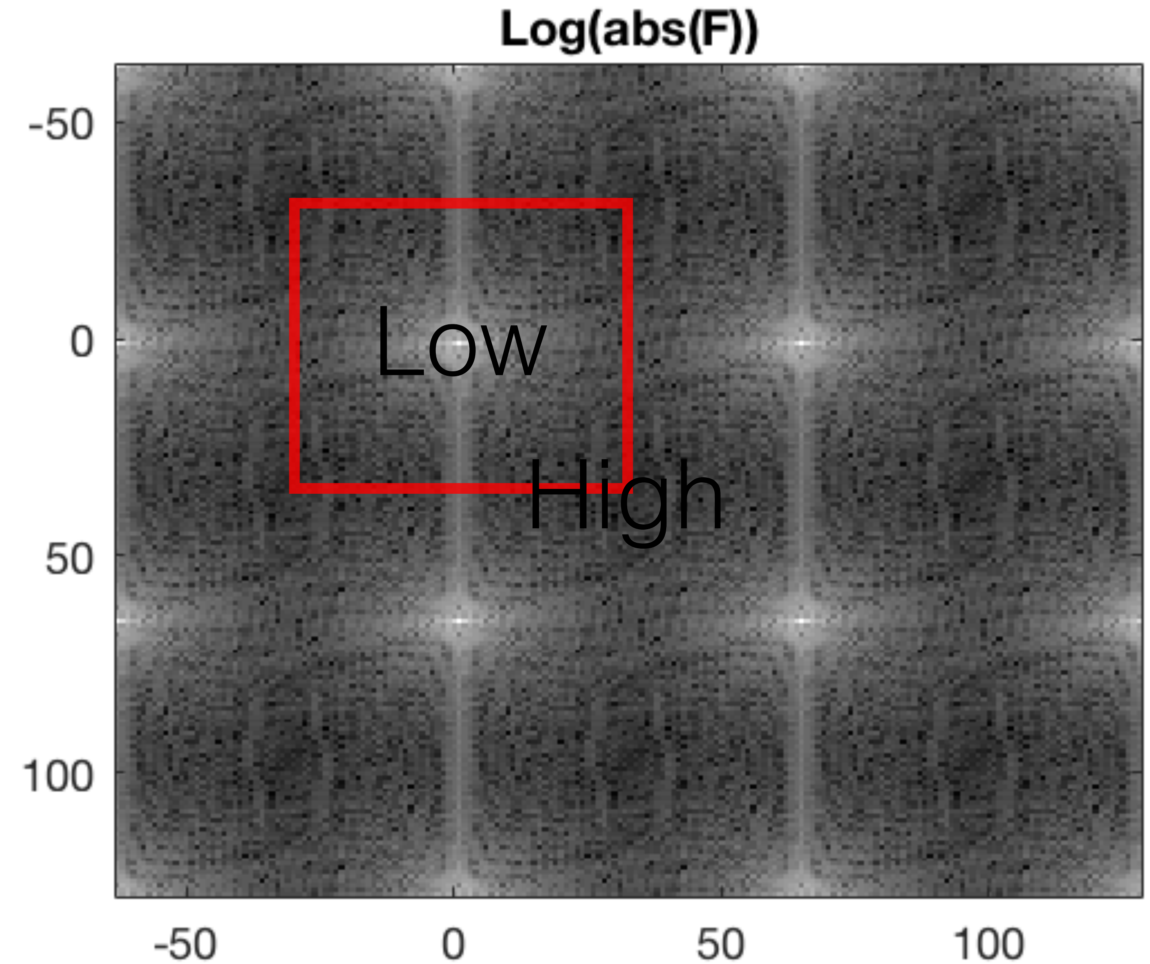




(-M/2 – M/2 x –N/2 – N/2)

Discrete Fourier Transform - 2D Example – Periodic expansion





fftshift

(-M/2 – M/2 x –N/2 – N/2)

- Usually, the gray-levels of the Fourier Transform images are scaled using $c \log(1 + |F(u, v)|)$.
- The middle of the Fourier image (after fftshift) corresponds to low frequencies.
- Outside the middle high components in F corresponds to higher frequencies and the direction corresponds to "edges"in the images with opposite orientation.

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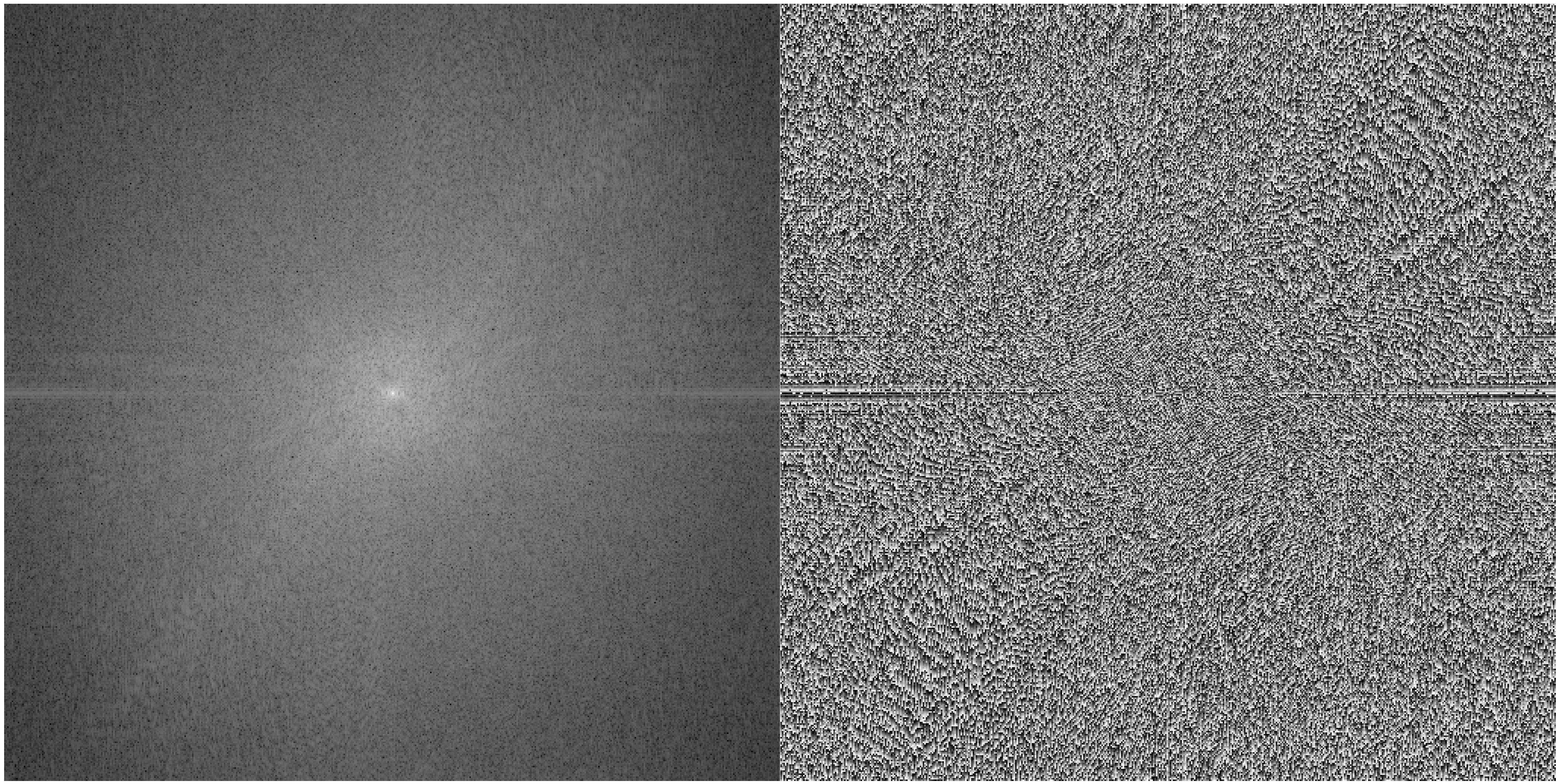
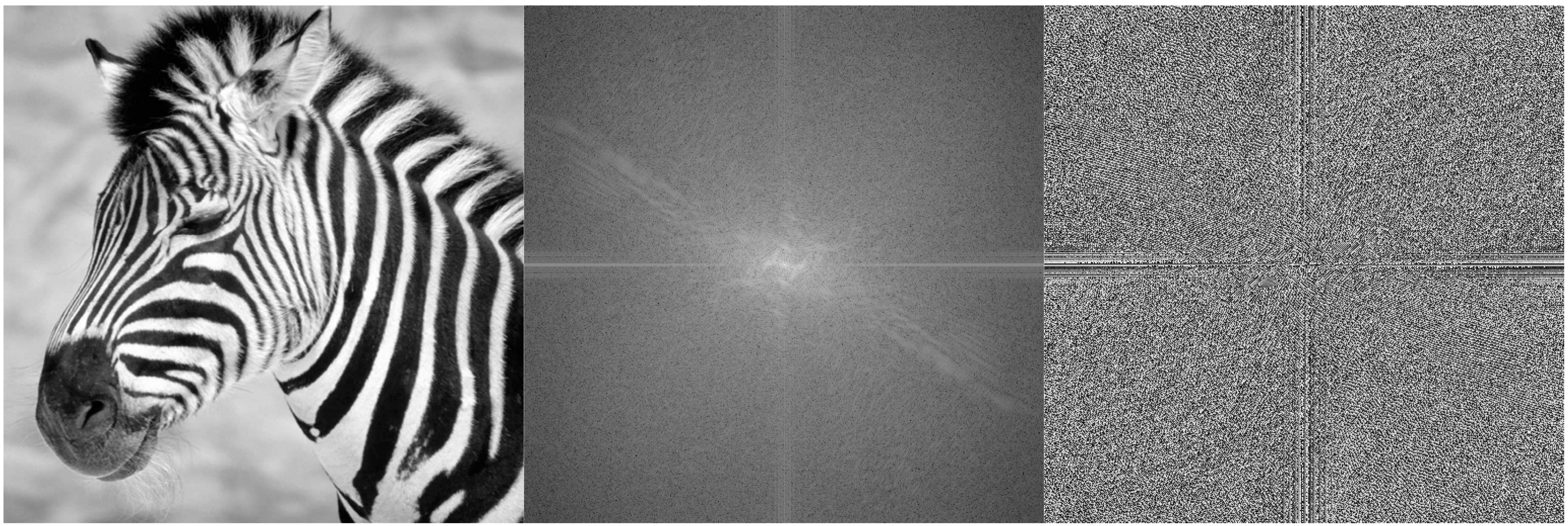
What does the original image look like if this is the Fourier transform?



Left: Magnitude, Right: Phase



Fourier transform

- Image

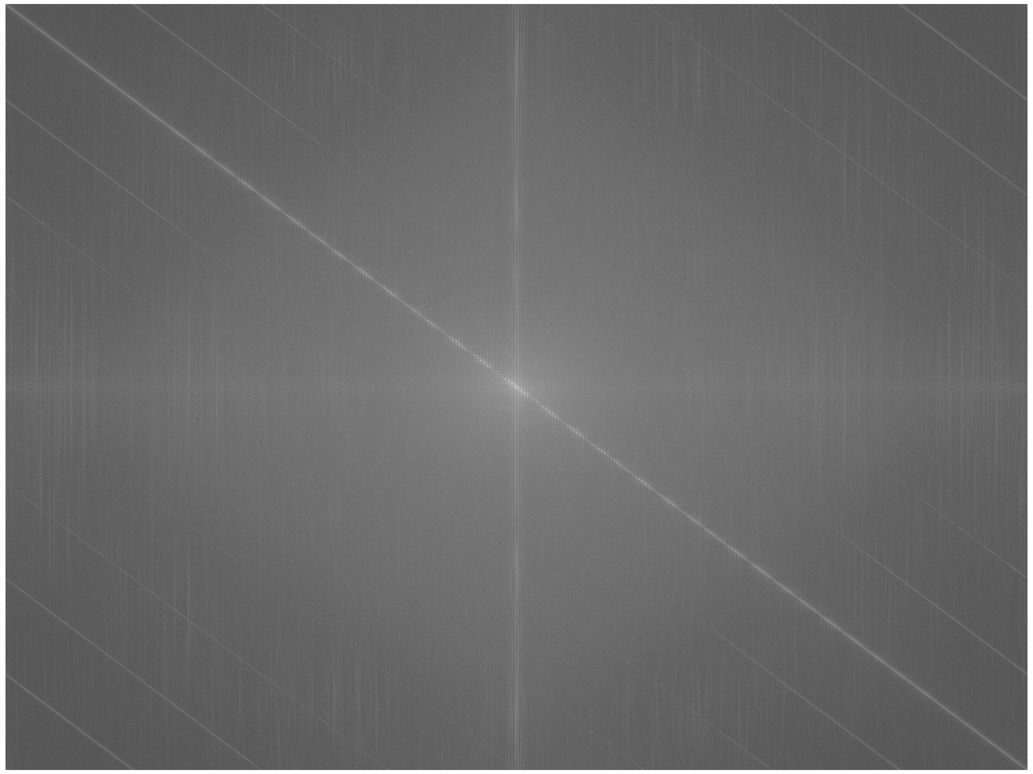
Fourier transform

•Image

•abs(fft2(I))

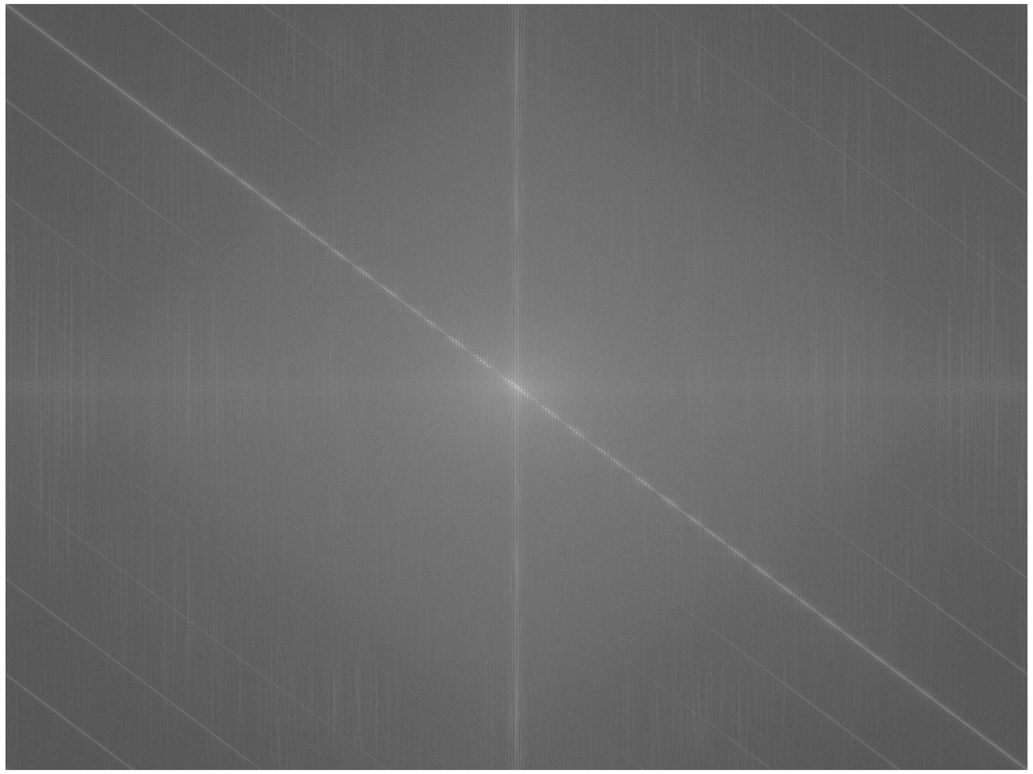
Fourier transform

- Image



- log(abs(fft2(I)))

Edge effects

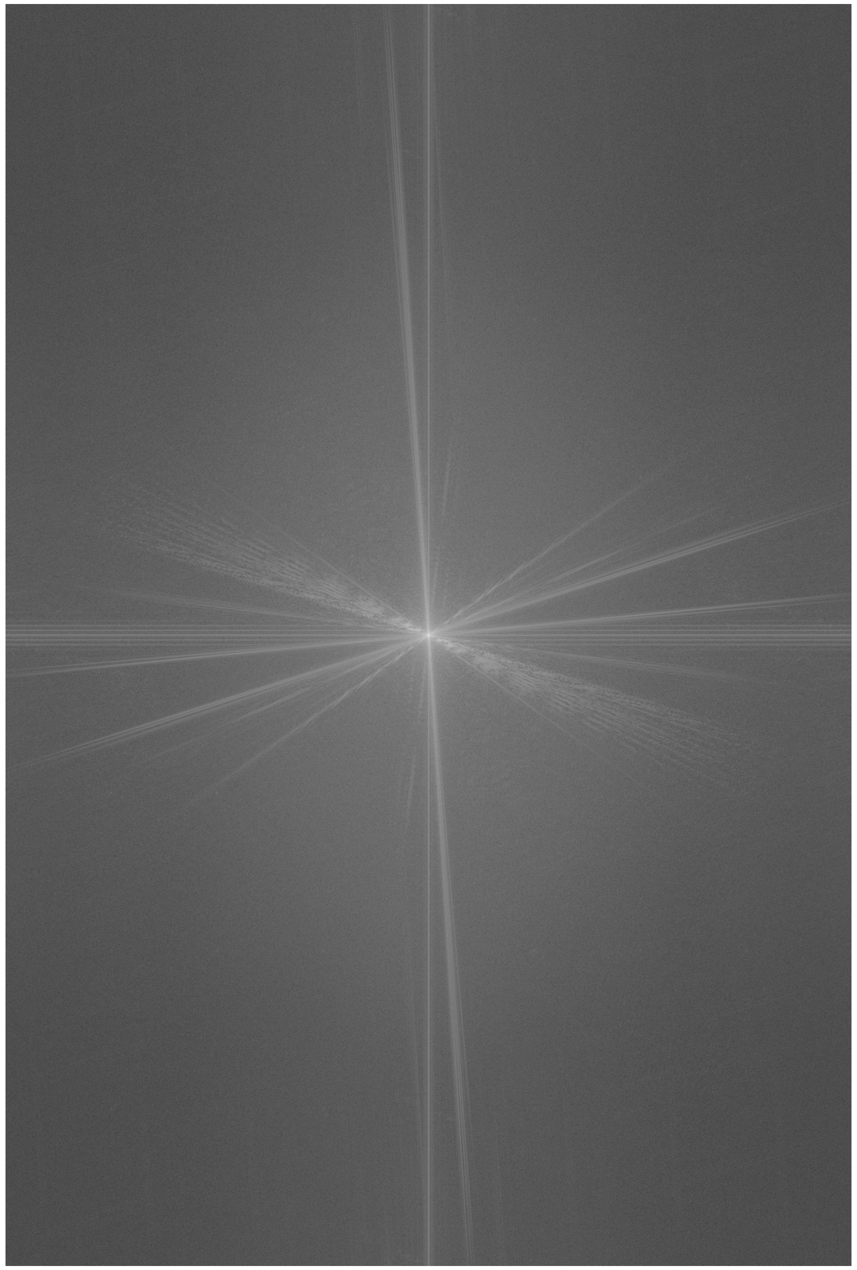


- Image
- log(abs(fft2(I)))

Fourier transform

- Image

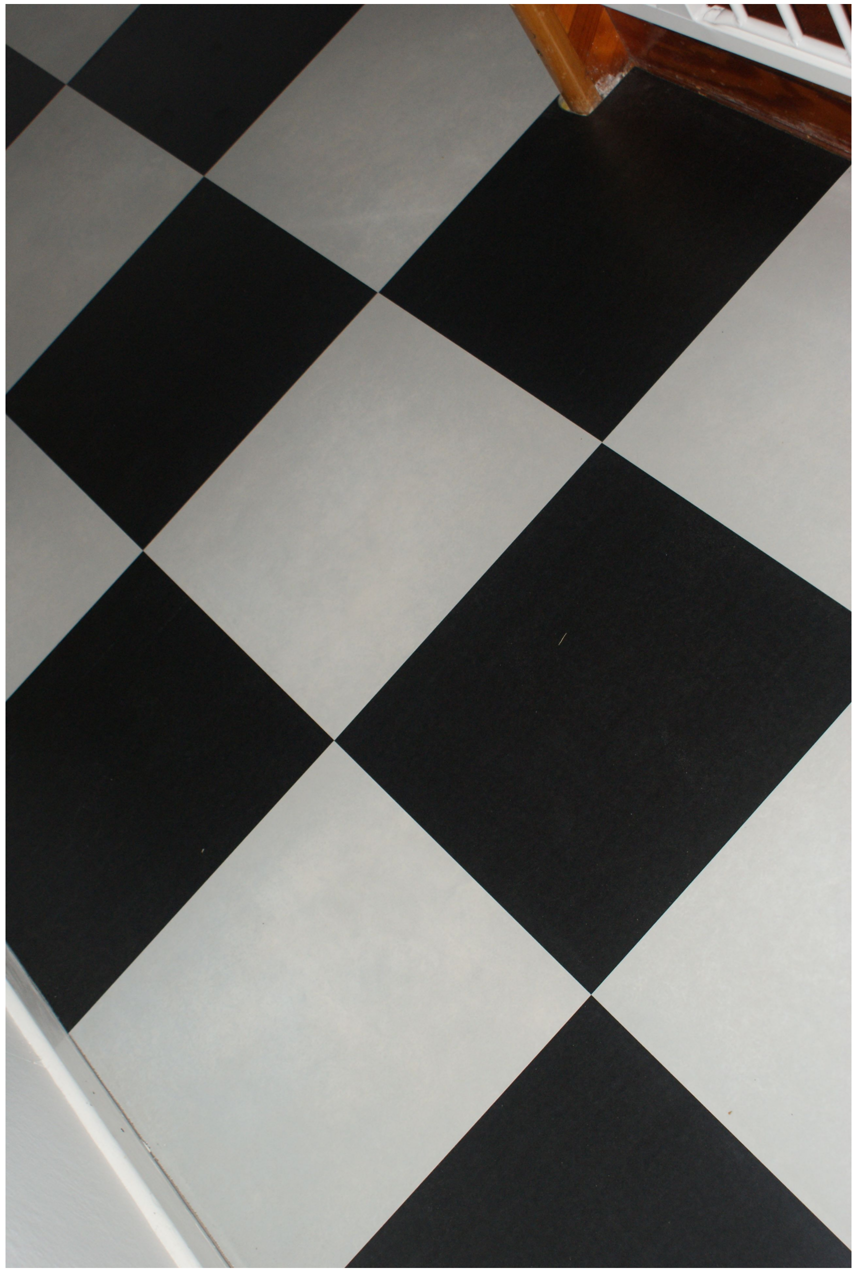
Fourier transform



•Image

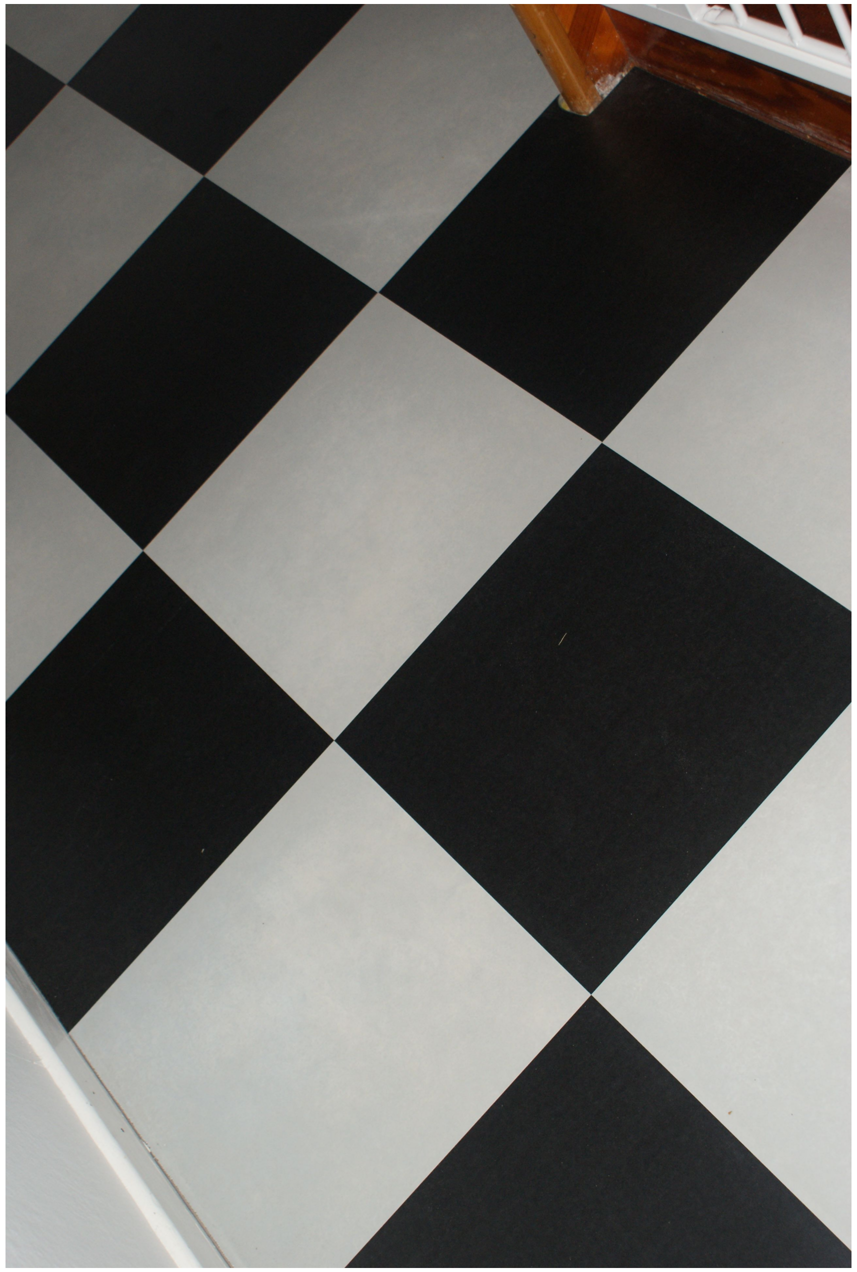
•Fourier transform

Fourier transform

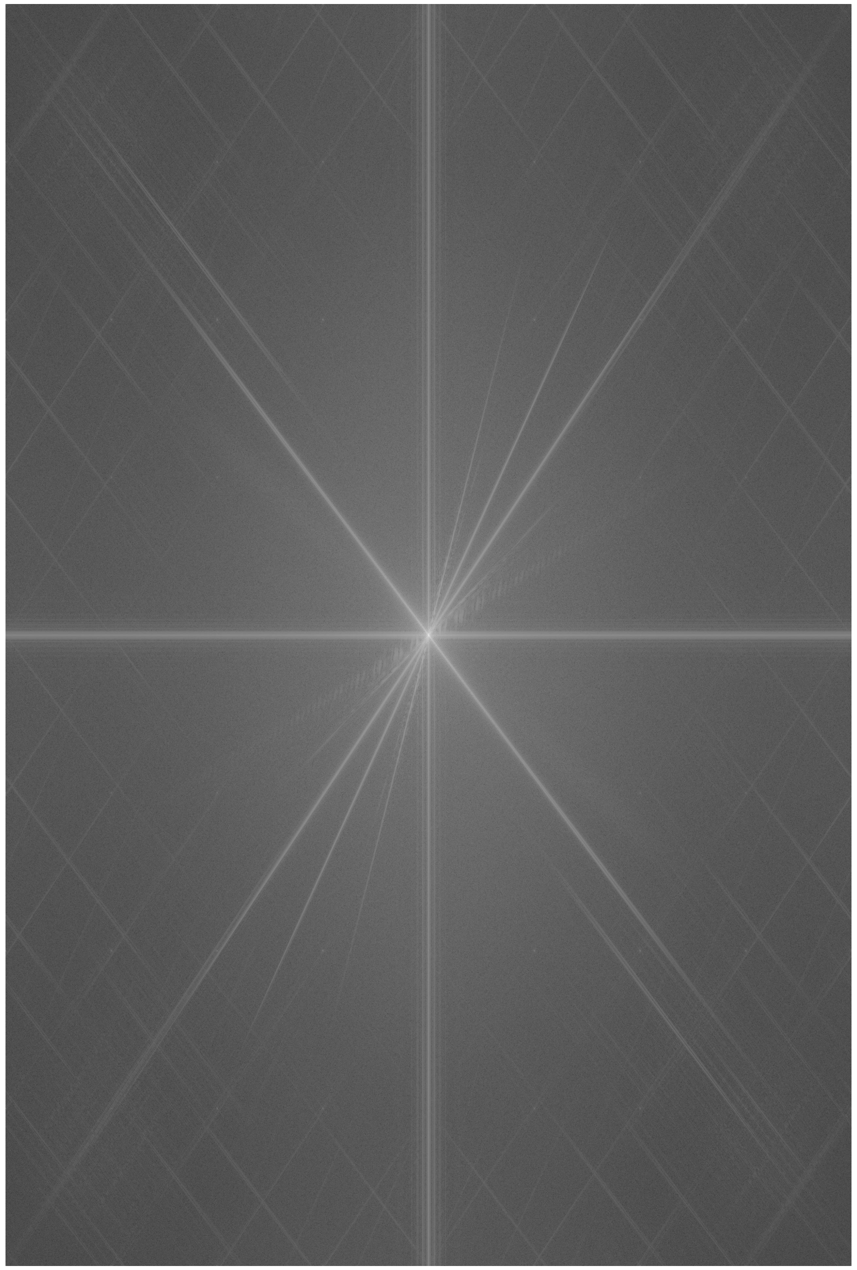


- Image

Fourier transform



•Image



•Fourier transform

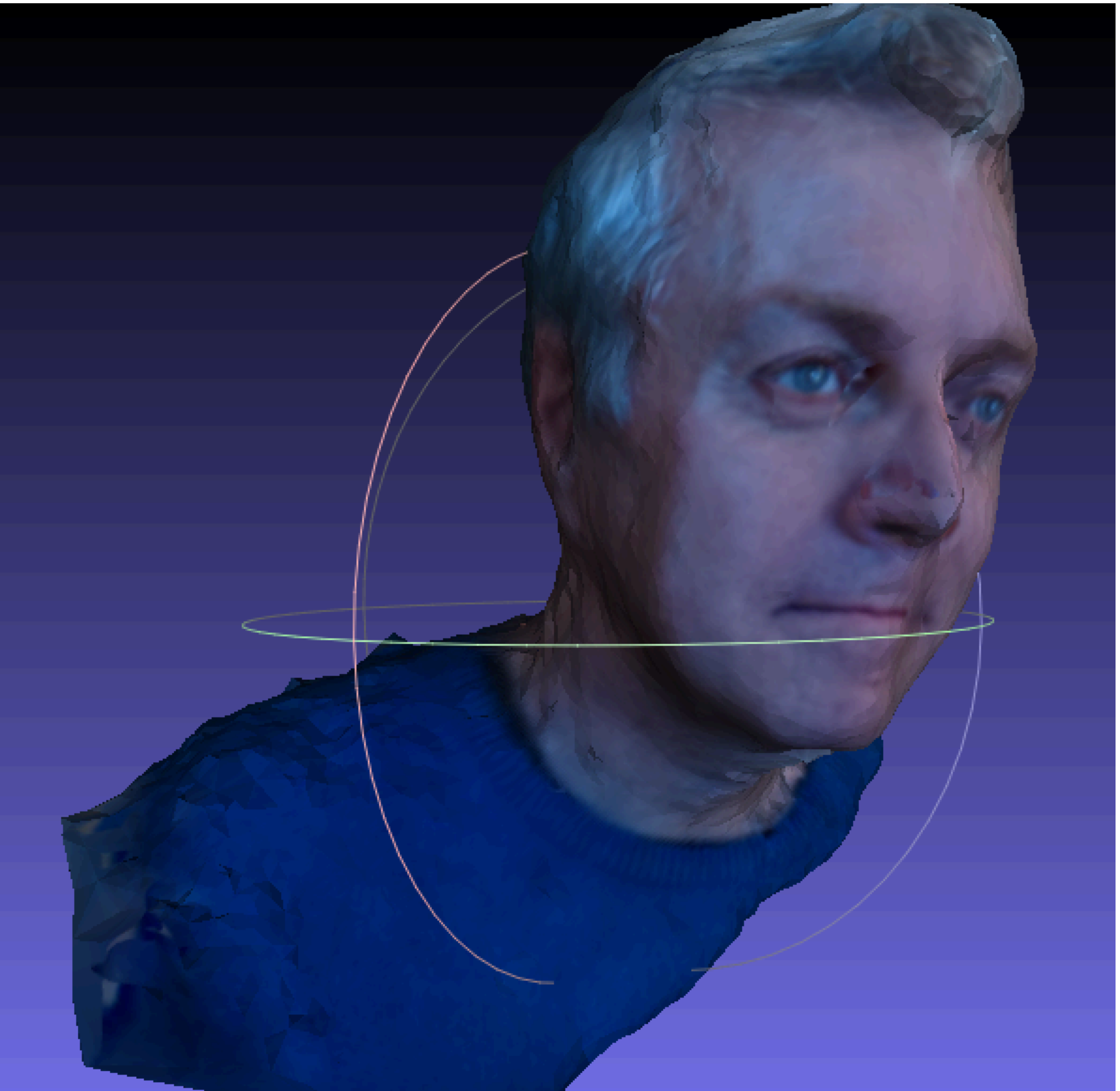
Review

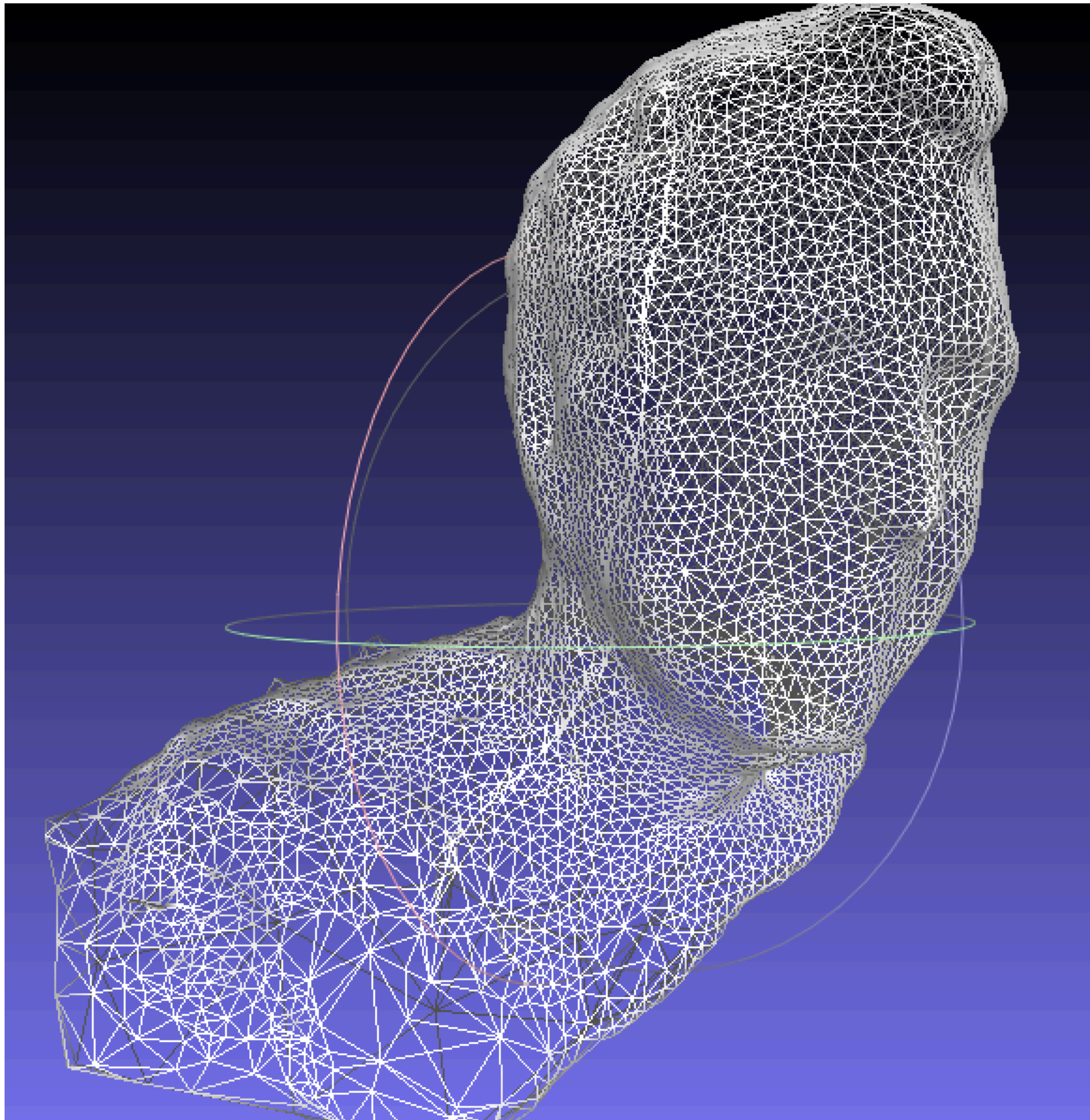
- Linear algebra
  - The space of images is a linear vector space
  - Images are ’vectors’ – in the sense that they are elements of a linear vectors space
  - Can be confusing. Can a matrix be a vector???
- Useful tools
  - Change of basis
  - Projection onto a subspace, onto affine subspace
  - PCA
- Fourier Transform

- Read lecture notes
- Experiment with matlab demo scripts
- Continue with assignment 1

Master’s Thesis Suggestion of the day

- Recognise and label parts of 3D models.
- Eye, mouth, hair, nose, …





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