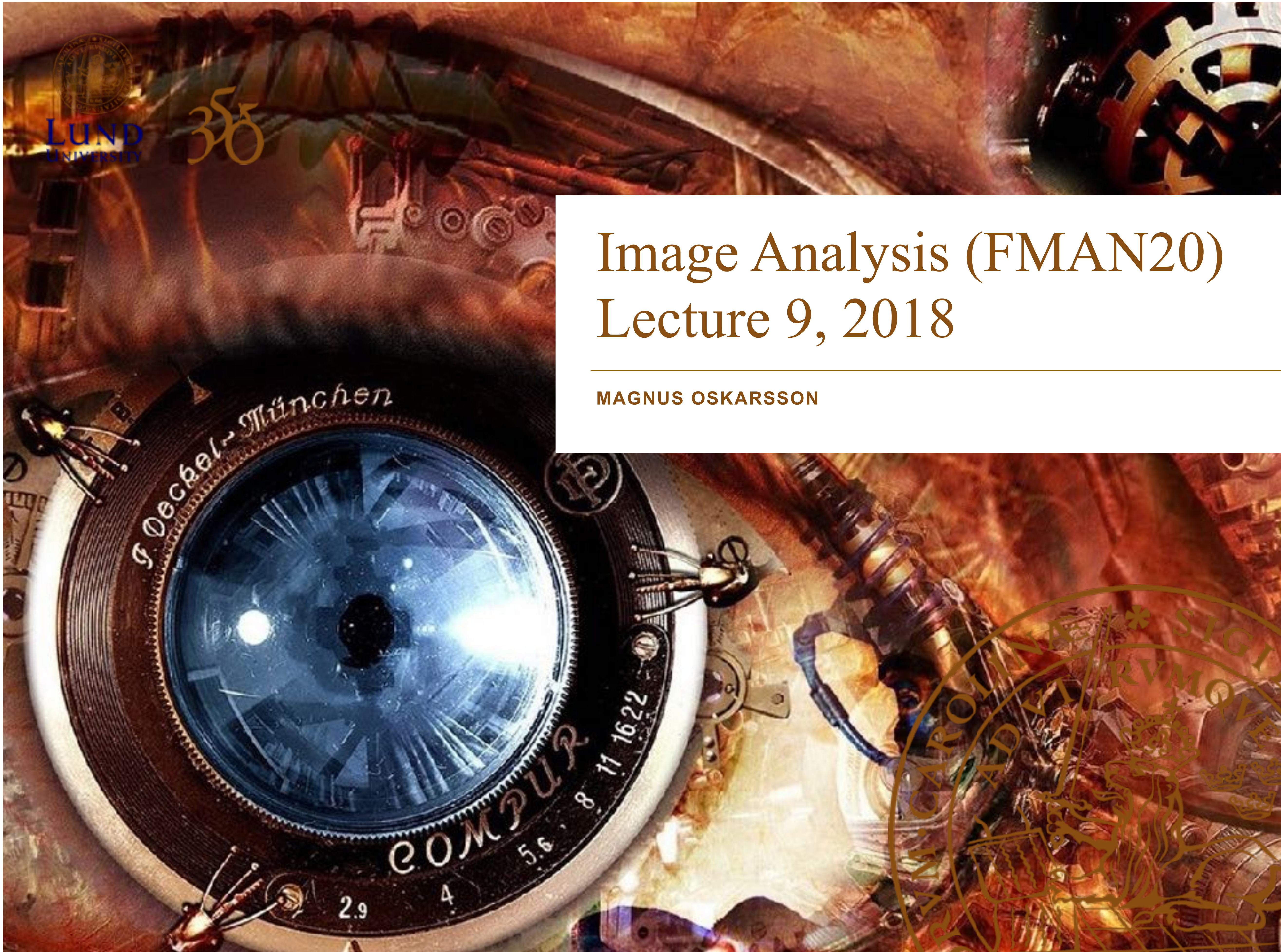

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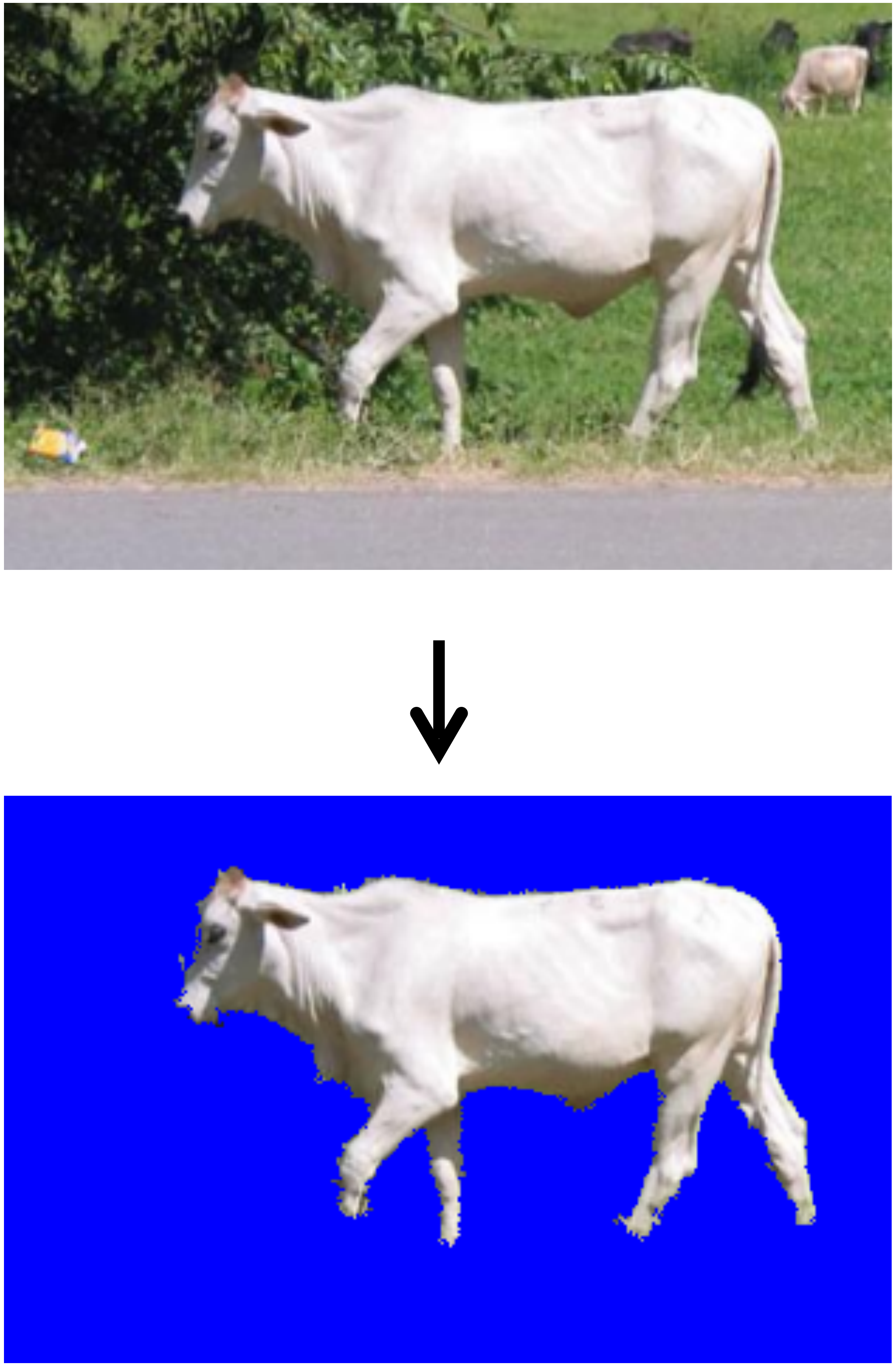
Image Analysis (FMAN20)
Lecture 9, 2018

MAGNUS OSKARSSON



How?

Segmentation using Graphs

- Graphs
- A simple graph based clustering method
- The Mumford-Shah functional
- Graph cuts

Graph theory

A graph $G = (V, E)$ consists of vertices(nodes) V and edges E.
Every edge connects two vertices.
In a directed graph, every edge has an orientation.
In a weighted graph, every edge has a weight (a number).
A graph is connected if one can 'walk' between all pairs of vertices through one or several edges.
Every graph can be split into a disjoint set of connected components.

Graph theory

Weighted graphs can be represented as a matrix. A weighted edge between vertex i and vertex j with v is represented by matrix element (i, j).
For un-directed graphs, half the weight is put at position (i, j) and half in (j, i).
Connected components - blocks in block diagonal matrices.

Graph theoretic clustering

- Represent tokens using a weighted graph.
  - affinity matrix
- Cut up this graph to get subgraphs with strong interior links

Graph theoretic clustering

When solving clustering problems with graph theoretical methods one need a closeness measure $v_{i,j}$, for every pair of nodes (i, j). A large number means that they are close. A small number means that they are different.
The affinity measure depends on which problem one has.
Usual ingredients are

- Distance - e.g. $aff(x, y) = e^{-(x-y)^T(x-y)/(2\sigma_d^2)}$
- Intensity - e.g. $aff(x, y) = e^{-(I(x)-I(y))^T(I(x)-I(y))/(2\sigma_I^2)}$
- Color - e.g. $aff(x, y) = e^{-dist(c(x),c(y))^2/(2\sigma_c^2)}$
- Texture - e.g. $aff(x, y) = e^{-(f(x)-f(y))^T(f(x)-f(y))/(2\sigma_f^2)}$

Graph theoretic clustering

Assume that w_n is a vector of ones for those elements that belong to a particular cluster and zeros otherwise. Then the sum of all weights for edges within a cluster is

$$w_n^T A w_n.$$

By maximizing $w_n^T A w_n$ with the constraint $w_n^T w_n = 1$ one might argue that we maximize clustering.
Maxima with this problem corresponds to stationary points of the Lagrange function.

Graph theoretic clustering

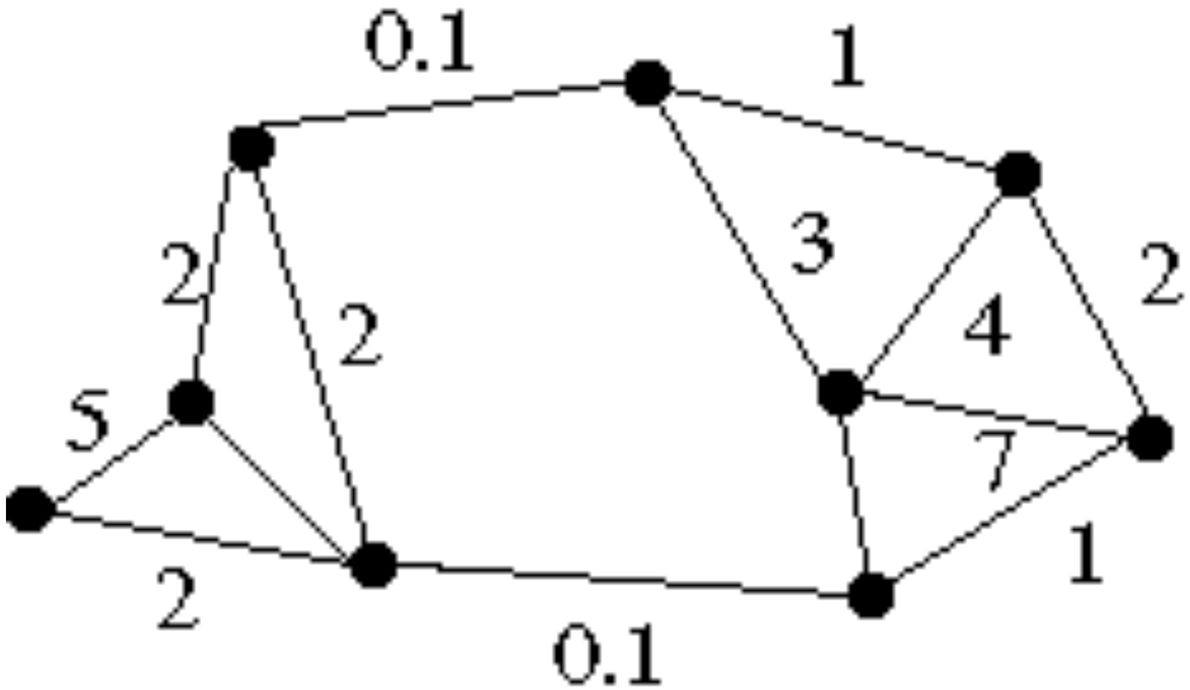
Maximize $w_n^T A w_n$ with constraint $w_n^T w_n = 1$.
Study the Lagrange function

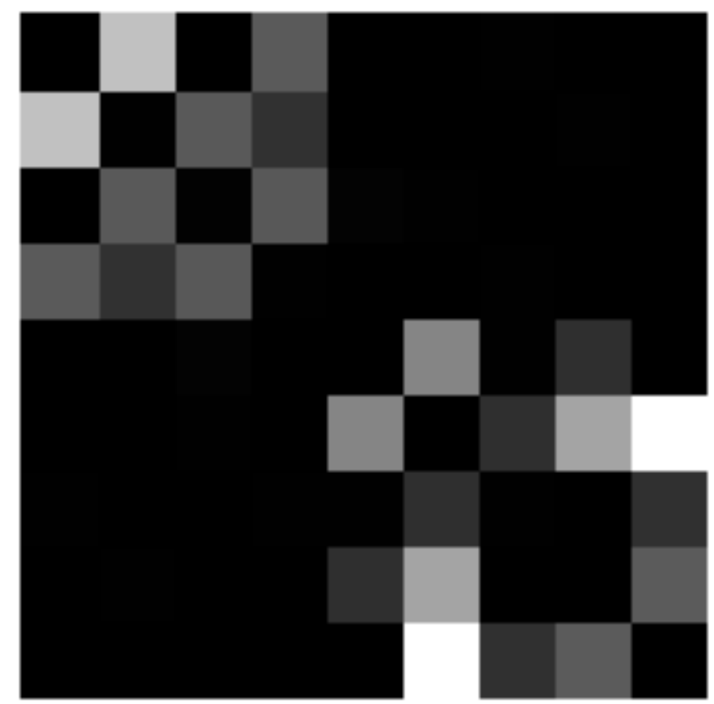
$$L(w_n, \lambda) = w_n^T A w_n + \lambda(w_n^T w_n - 1)$$

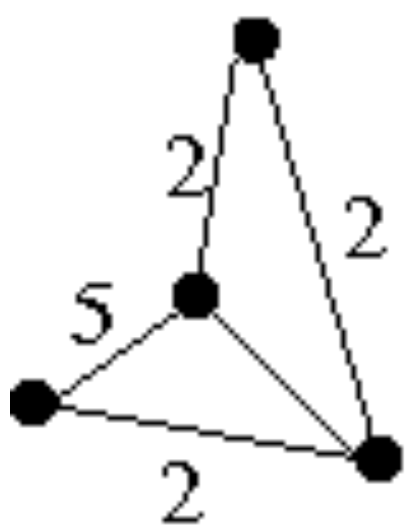
Differentiate and divide with two gives

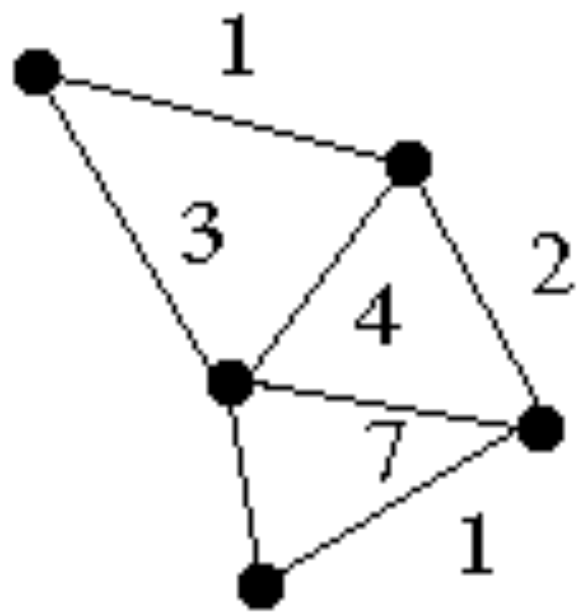
$$A w_n = -\lambda(w_n)$$

This is an eigenvalue problem.

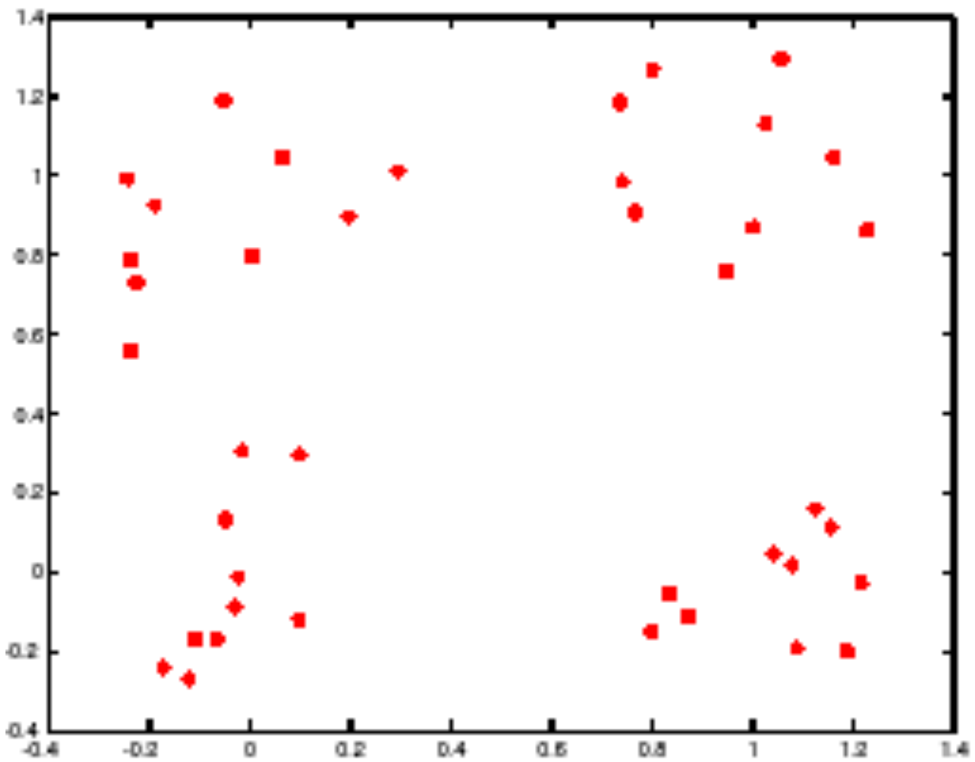

0.1
1
2
2
3
2
4
5
7
2
1
0.1



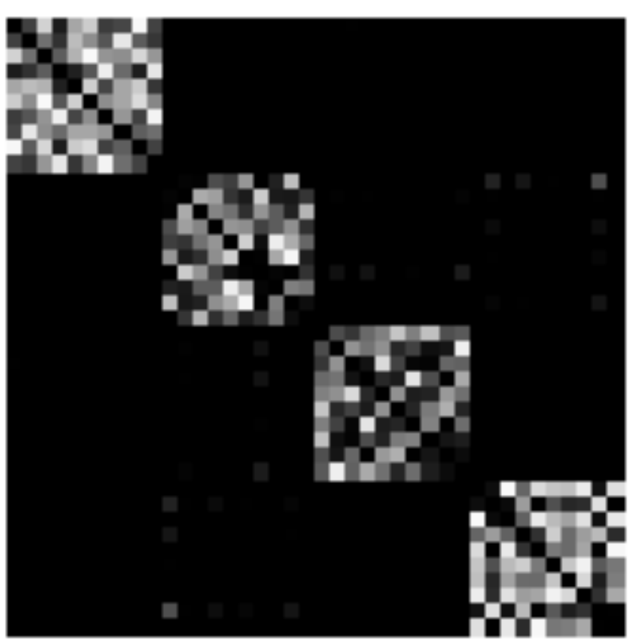

2
2
5
2


1
3
2
4
7
1

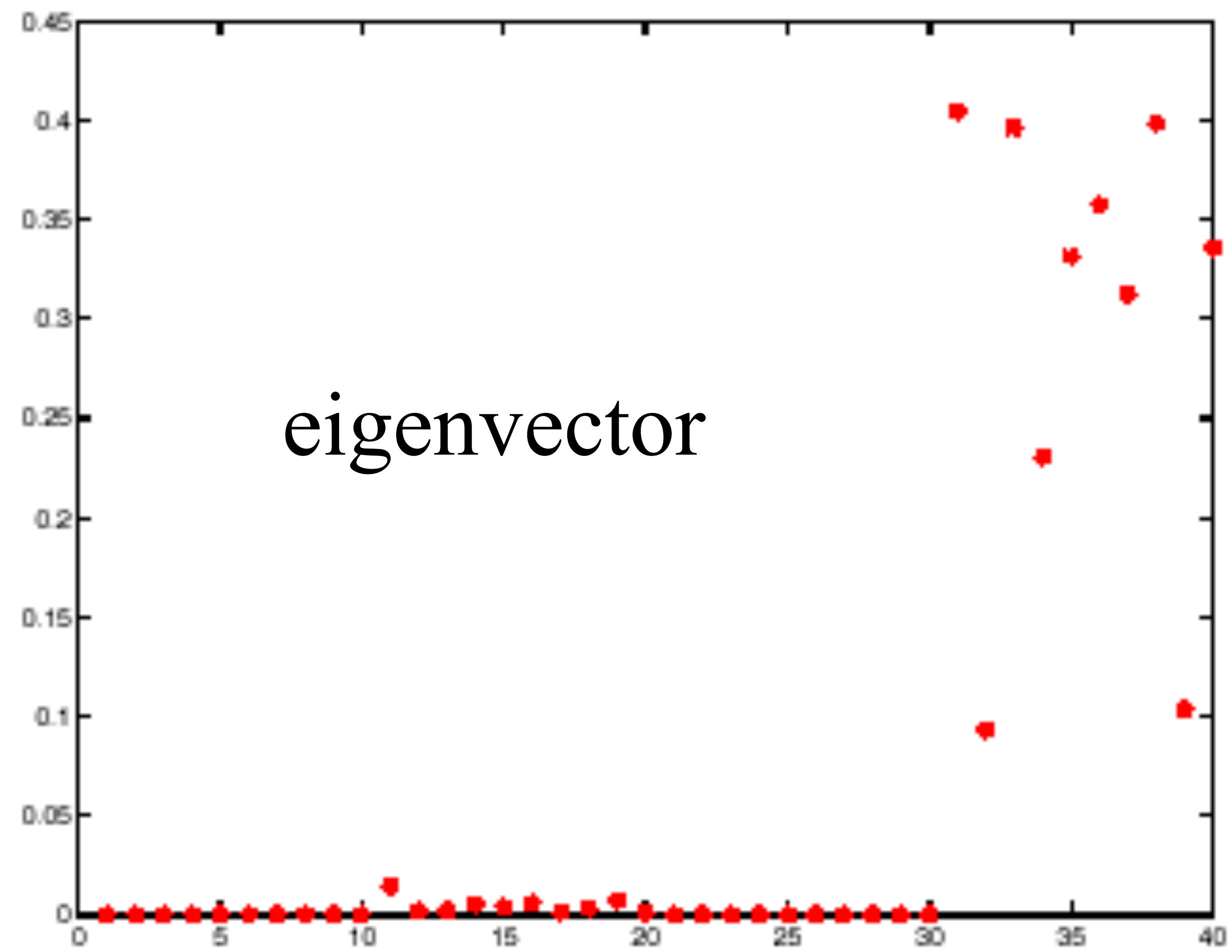
Example eigenvector



points



matrix



eigenvector

More than two segments

- Two options
  - Recursively split each side to get a tree, continuing till the eigenvalues are too small
  - Use the other eigenvectors

3. Segmentation problem The Mumford-Shah functional

- Consists in computing a decomposition of the domain of the image ***f(x,y)***

$$R = \bigcup_{i=1}^{n} R_i$$

1. ***f*** varies smootly and/or slowly within R_i
2. ***f*** varies discontinuously and/or rapidly across most of the boundary Γ between regions R_i

Segmentation problem

- Segmentation problem may be restated as
  - finding optimal approximations of a general function ***f*** by piece-wise smooth functions ***g***, whose restrictions $\boldsymbol{g_i}$ to the regions $\boldsymbol{R_i}$ are differentiable
- Many other applications:
  - Speech recognition
  - Sonar, radar or laser range data
  - MR images and CT scans
  - etc…

Optimal Segmentation

- Mumford and Shah studied 3 functionals which measure the degree of match between an image ***f(x,y)*** and a segmentation.
- First, they defined a general functional ***E*** (the famous Mumford-Shah functional):
  - R_i will be disjoint connected open subsets of the planar domain ***R***, each one with a piece-wise smooth boundary
  - Γ will be the union of the boundaries.

$$R = \bigcup_{i=1}^{n} R_i \bigcup \Gamma$$

Mumford-Shah functional

- Let ***g*** differentiable on $\cup R_i$ and allowed to be discontinuous across Γ.

$$E(g,\Gamma) = \mu^2 \int_R (g-f)^2 dxdy + \int_{R-\Gamma} ||\nabla g||^2 dxdy + \nu|\Gamma|$$

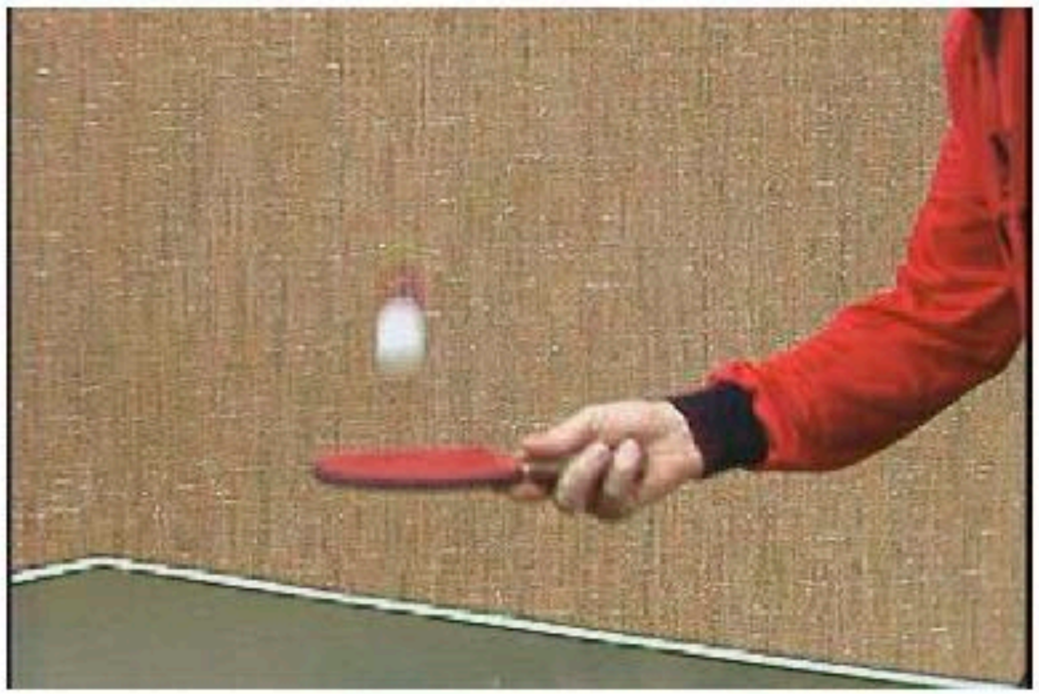
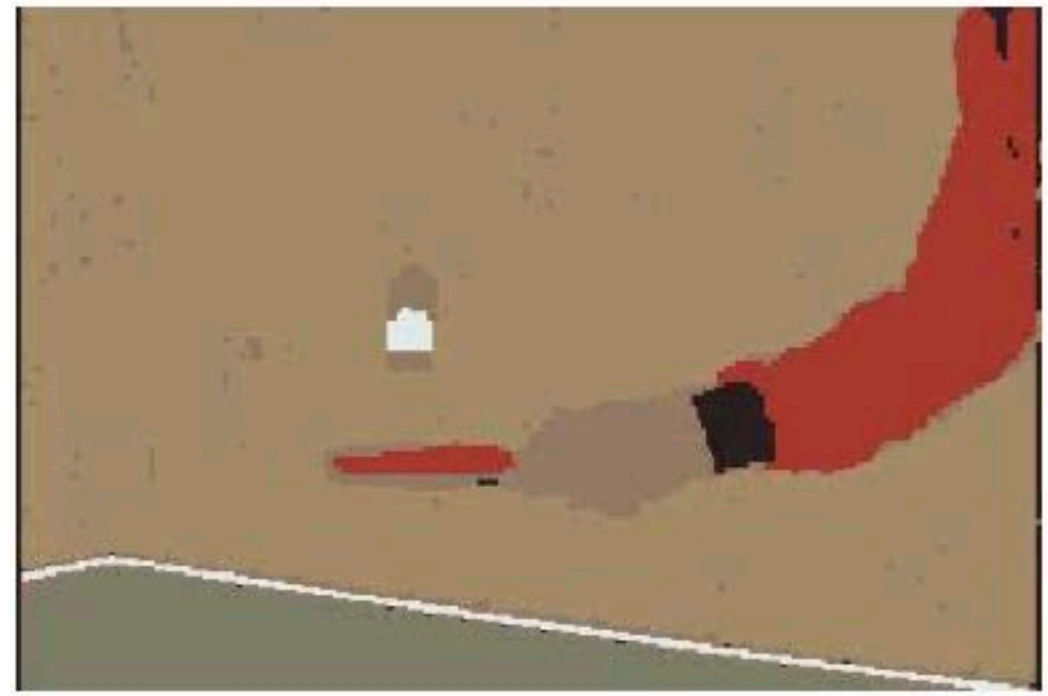
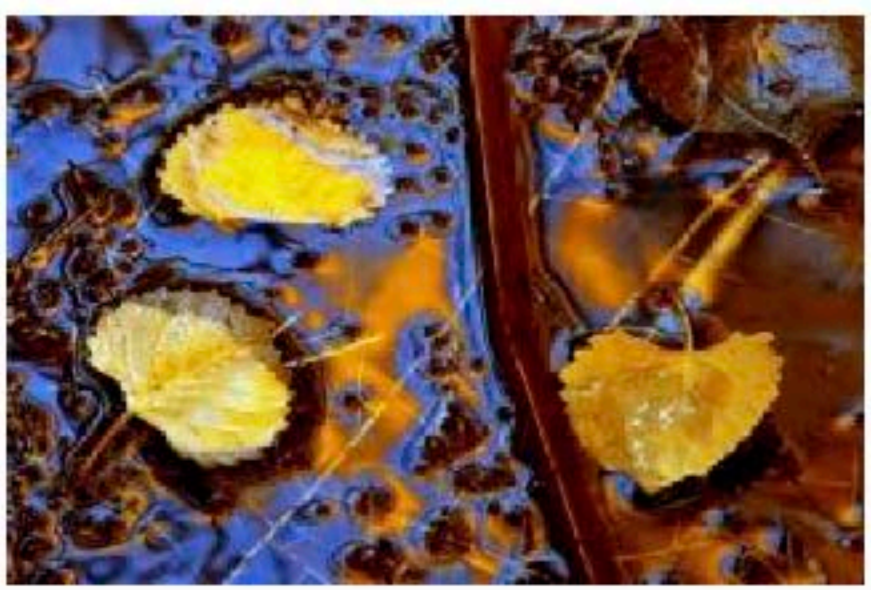
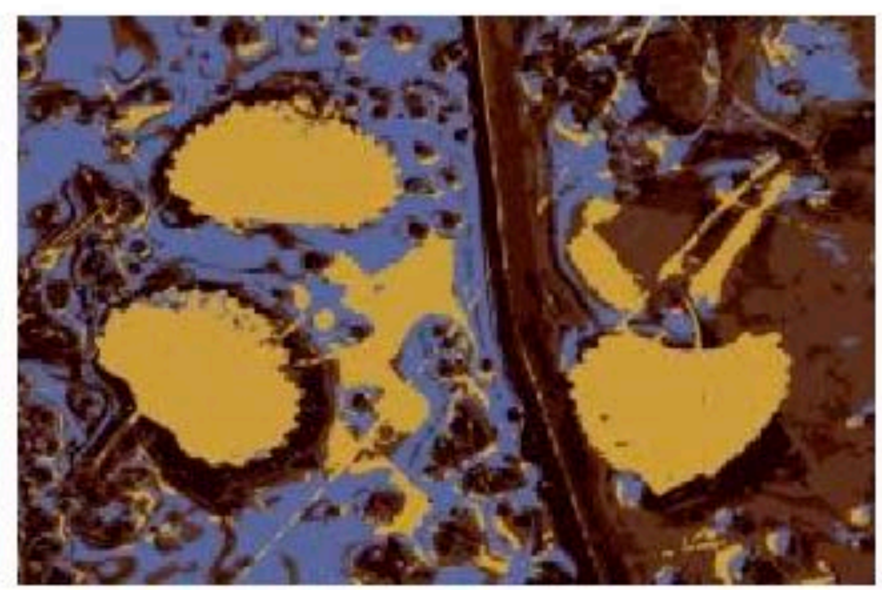
- The smaller ***E***, the better ***(g, Γ)*** segments ***f***
  1. ***g*** approximates ***f***
  2. ***g*** (hence ***f)*** does not vary much on R_is
  3. The boundary *Γ* be as short as possible

- Dropping any term would cause ***inf E=0.***

Cartoon Image

- ***(g, Γ)*** is simply a cartoon of the original image ***f.***
  - ✓ Basically ***g*** is a new image with edges drawn sharply.
  - ✓ The objects are drawn smootly without texture.
  - ✓ ***(g, Γ)*** is essentially an idealization of ***f*** by the sort of image created by an artist.
  - ✓ Such cartoons are perceived correctly as representing the same scene as ***f->g*** is a simplification of the scene containing most of its essential features.

Cartoon Image example

Piecewise constant approximation

- A special case of ***E*** where ***g=a$_i$*** is constant on each open set ***R$_i$***.

$$E_0(a,\Gamma)=\sum_{i=1}^{n}\int_{R_i}(a_i-f)^2dxdy+\nu|\Gamma|$$

- It is minimized in ***a$_i$*** by setting ***a$_i$*** to the mean of ***f*** in ***R$_i$***.

$$a_i=\int_{R_i}f dxdy/\text{area}(R_i)$$

Piecewise constant approximation

- A special case of ***E*** where $g=a_i$ is constant on each open set R_i.

$$E_0(a,\Gamma) = \sum_{i=1}^{n} \int_{R_i} (a_i - f)^2 dxdy + \nu|\Gamma|$$

- It can be proven that minimizing E_0 is well posed:
  - ✓ For any continuous ***f***, there exists a Γ made up of finite number of singular points joined by a finite number of arcs on which E_0 attains a minimum.

Two phase Mumford-Shah functional

$$E_0(a_1, a_2, \Gamma) = \int_{R_1} (a_1 - f)^2 dxdy + \int_{R_2} (a_2 - f)^2 dxdy + \nu|\Gamma|$$

- Energy based on two segments R_1 and R_2
- Assume a_1 and a_2 known
- Regularization based on boundary length

4. Segmentation - Graph Cuts

- Idea:

1. See the segmentation problem as a classification problem
2. Finding the highest a posteriori classification (segmentation) is an optimization problem
3. Construct a graph so that the min-cut problem is equivalent to the optimization problem in (2).
4. Compute a minimum cut that gives the optimal solution.

Note: Min-cut of a graph can be efficiently computed (polynomial time) via max flow algorithms.

A priori probabilities of segmentations

Idea:

We are segmenting some pixels as foreground (1) and some as background (0).

It might be more probable with foreground pixels or the inverse, e.g. P_i(g_i=1)=p1

Assume a priori probabilities that are

P(g) ~ _i p_i(g_i) + \sum_(i,j) \in E p_ij(g_i,g_j)

Note: Min-cut of a graph can be efficiently computed (polynomial time) via max flow algorithms.

Statistical interpretation

Notation:

f – observed image

g – sought, unknown image

$P(g|f)$ - posterior distribution

Using the *Maximum A Posteriori* (*MAP*) principle, we should maximize the posterior distribution.

Bayes rule: $$P(g|f) = \frac{P(f|g)P(g)}{P(f)}$$

Negative logs give:

$$-\log(P(g|f)) = -\log(P(f|g)) - \log(P(g)) + \text{const}$$

Energy:

$$E(f,g) = E_{data}(f,g) + E_{prior}(g)$$

Statistical two-phase Mumford-Shah

Energy: $E(f,g) = E_{data}(f,g) + E_{prior}(g)$

Recall:

$$E_0(a_1, a_2, \Gamma) = \int_{R_1} (a_1 - f)^2 dxdy + \int_{R_2} (a_2 - f)^2 dxdy + \nu|\Gamma|$$

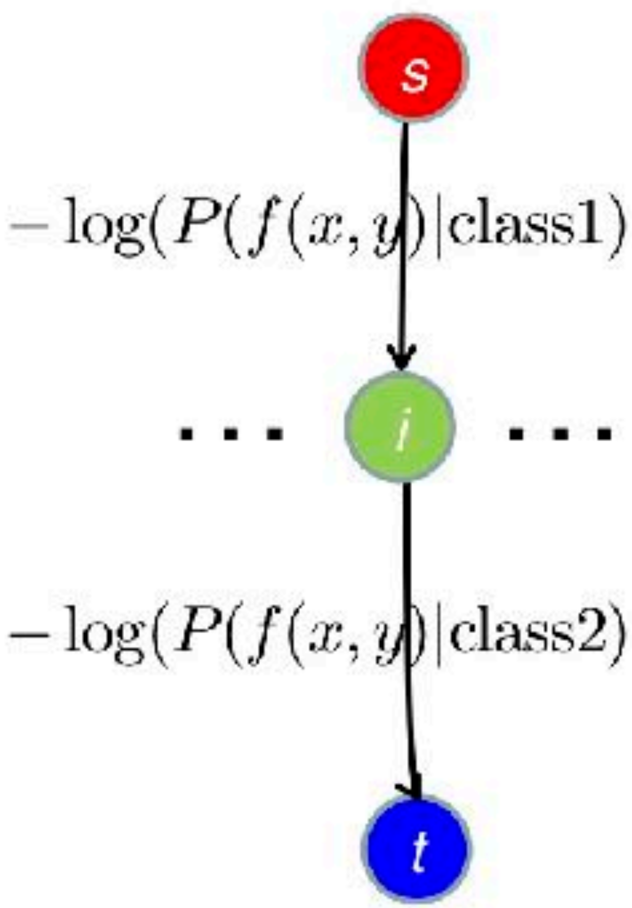
First two data terms: ”reconstructed *g* should be close to data (image) *f*”.
Third term: ”prior knowledge says that shorter curves *g* are preferred”.

More general formulation:

$$E_0(\Gamma) = \int_{R_1} -\log(P(f(x,y)|\text{class1})dxdy + \int_{R_2} -\log(P(f(x,y)|\text{class2})dxdy + \nu|\Gamma|$$

Edge weights for statistical model

Set edge weights such that a cut corresponds to a solution of the optimization problem

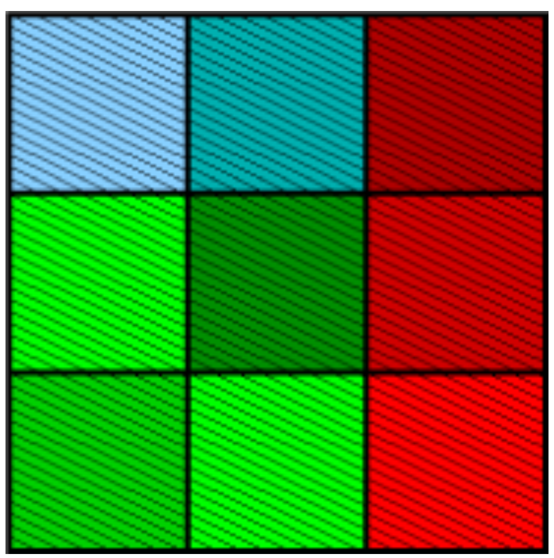


Consider pixel *i*. A cut must contain either:
1. the edge (*s*,*i*), or
2. the edge (*i*,*t*)

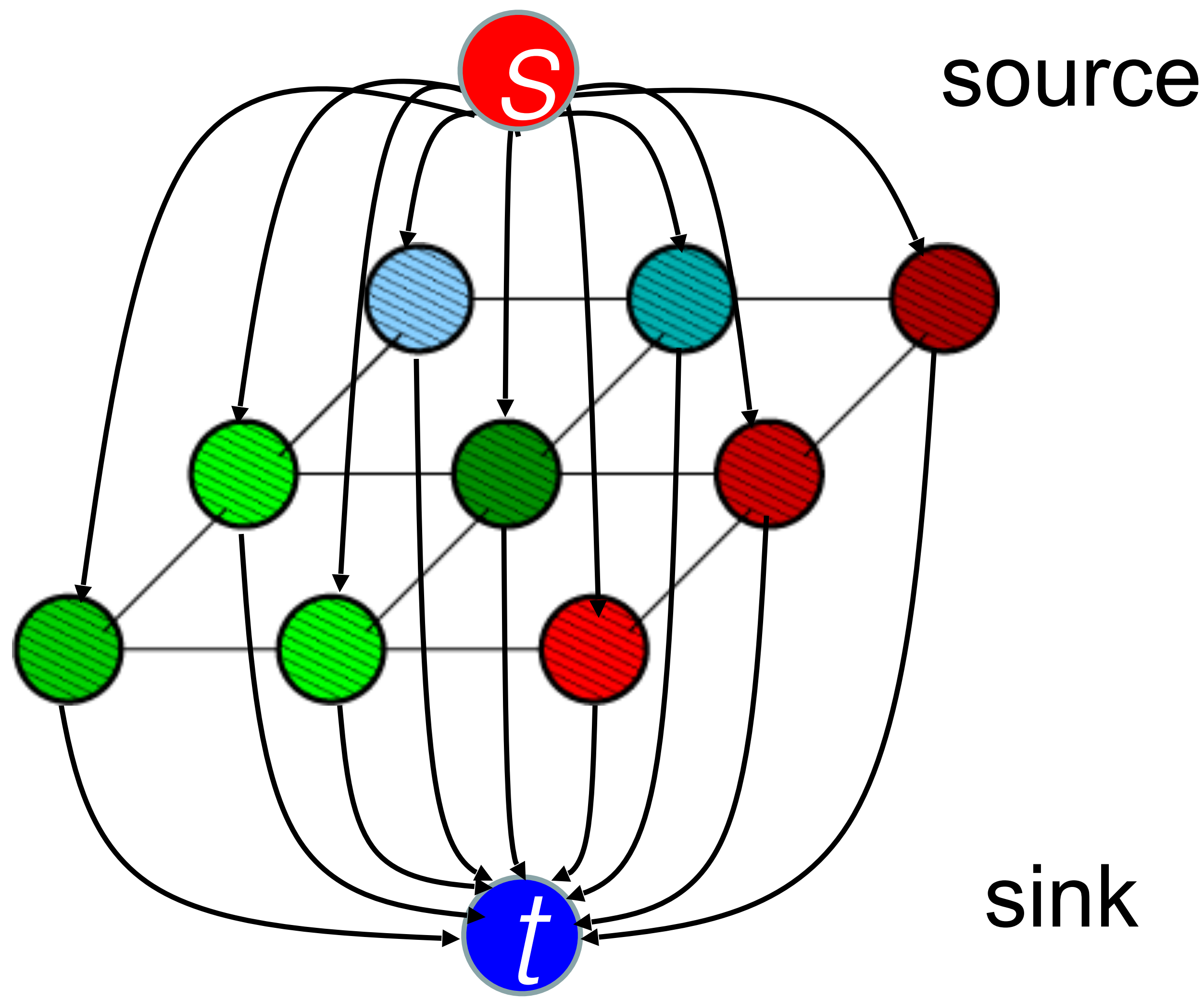
Set edge weights accordingly:
1. $-\log(P(f(x,y)|\text{class1})$ for edge (*s*,*i*),
2. $-\log(P(f(x,y)|\text{class2})$ for edge (*i*,*t*)

$$E_0(\Gamma) = \int_{R_1} -\log(P(f(x,y)|\text{class1})dxdy + \int_{R_2} -\log(P(f(x,y)|\text{class2})dxdy + \nu|\Gamma|$$

Graph representation of images

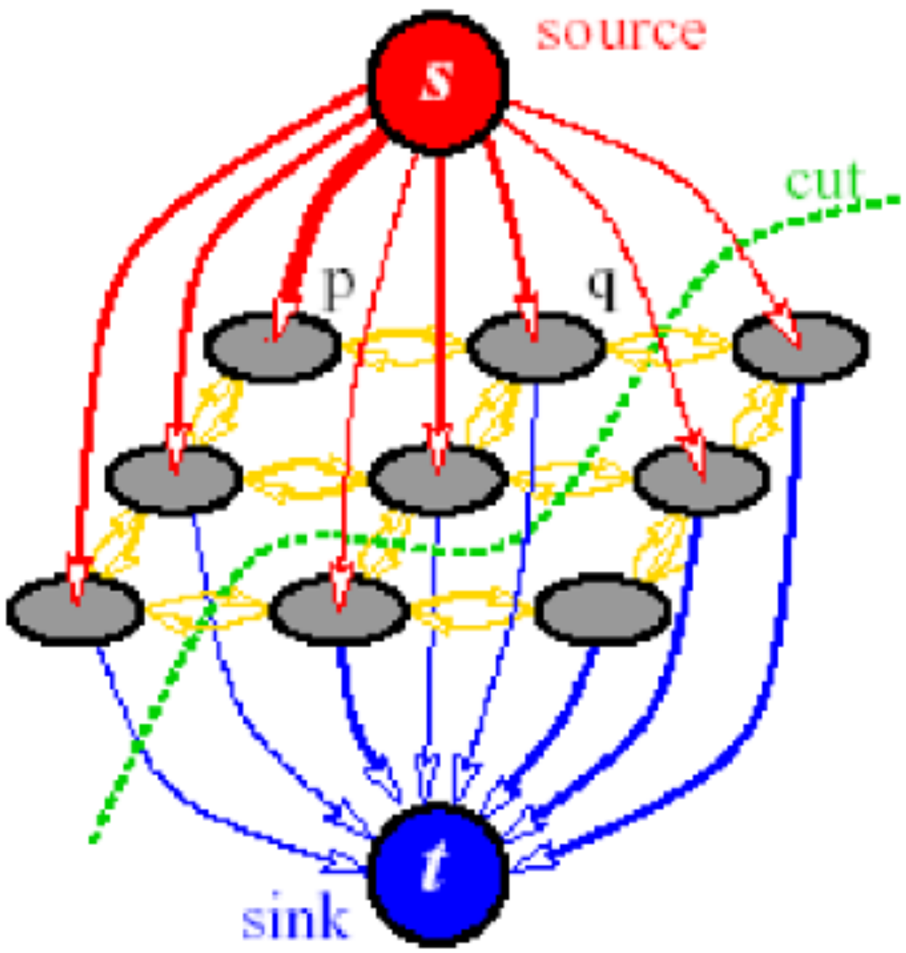


3x3 image



Directed, weighted graph, one vertice for every pixel + source and sink nodes

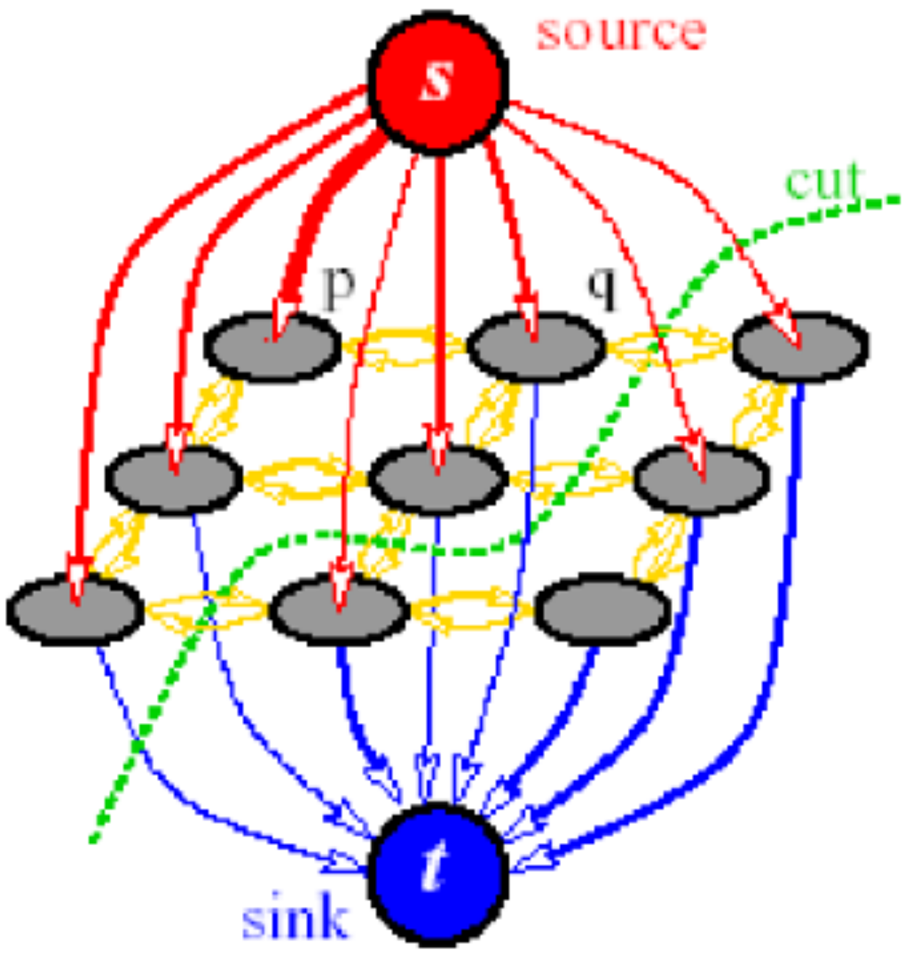
Graph Cuts



Definition: A *cut* (or *s-t cut*) in a graph $G=(V,E)$ is a subset of edges E_c such that there is no path from *s* to *t* when E_c is removed.

Definition: The *cost* of a cut is the sum of all edge weights for the edges in the cut.

Minimum Cuts

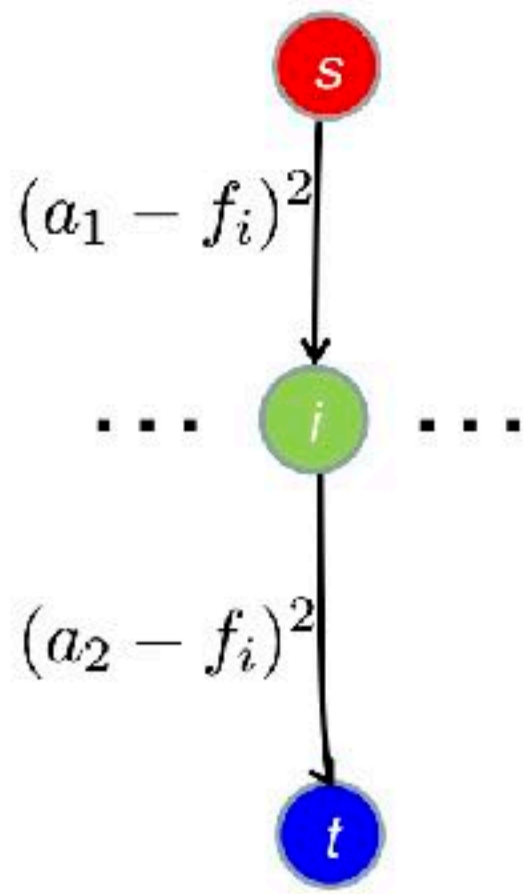


Definition: A *minimum cut* is a cut with minimum cost.

Note: A cut separates all nodes in two sets: (i) nodes that are connected to the source nodes, and
(ii) those that are not.

Edge weights – data term

Set edge weights such that a cut corresponds to a solution of the optimization problem



Consider pixel *i*. A cut must contain either:
1. the edge (*s*,*i*), or
2. the edge (*i*,*t*)

Set edge weights accordingly:
1. $(a_1 - f_i)^2$ for edge (*s*,*i*),
2. $(a_2 - f_i)^2$ for edge (*i*,*t*)

data terms

$$E_0(a_1, a_2, \Gamma) = \int_{R_1} (a_1 - f)^2 dxdy + \int_{R_2} (a_2 - f)^2 dxdy + \nu|\Gamma|$$

In practice

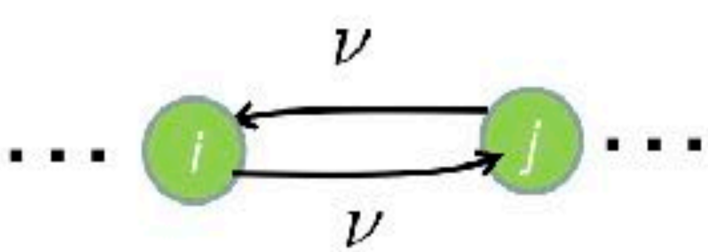
- Determine priors (estimate from example ground truth segmentations)
  - Unary part
    - $a1 = -\log(P(g\_i=1))$
    - $a0 = -\log(P(g\_i=0))$
  - Binary part
    - $b = -\log(P(g\_i \neq g\_j)/P(g\_i=g\_j))$
- Determine dataterm (estimate from example data with ground truth segmentation)
  - $c0 = -\log(P(f\_i \mid g\_i =1))$
  - $c1 = -\log(P(f\_i \mid g\_i =0))$

Put weights in graph

- Determine priors (estimate from example ground truth segmentations)
  - Unary part
    - Log(P(g_i=1)/P(g_i=0))
  - Binary part
    - Log(P(g_i ≠ g_j)/P(g_i=g_j))
- Determine dataterm (estimate from example data with ground truth segmentation)
  - Log(P(f_i | g_i =1))
  - Log(P(f_i | g_i =1))

Edge weights – regularization term

Set edge weights such that a cut corresponds to a solution of the optimization problem



Consider two neighbouring pixels *i* and *j*. If they are in different classes and hence a boundary is passing between them, then a cut must contain either:

1. the edge (*i*,*j*), or
2. the edge (*j*,*i*)

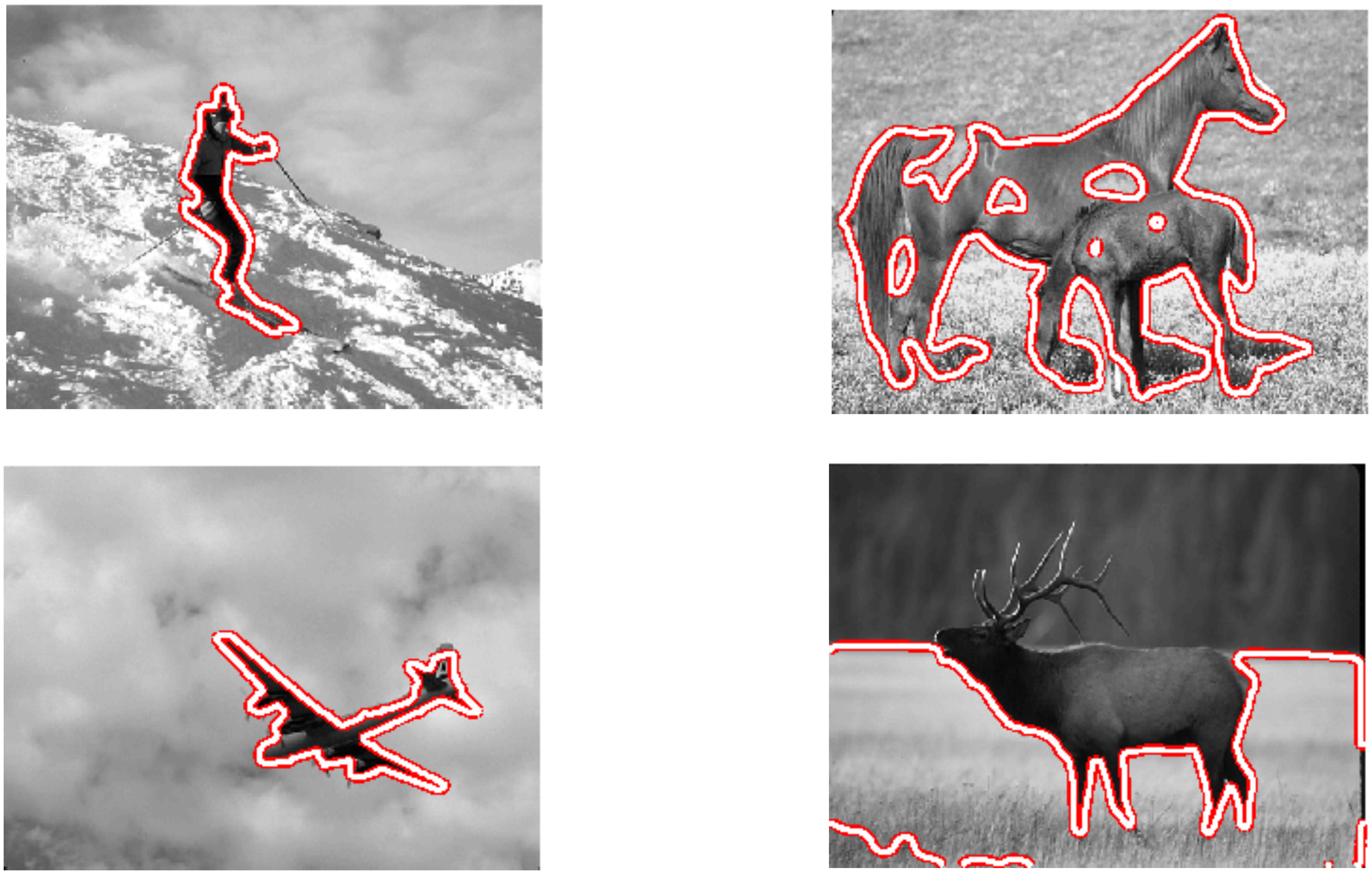
Set edge weights accordingly:

1. ν for edge (*i*,*j*),
2. ν for edge (*j*,*i*)

reg. term

$$E_0(a_1, a_2, \Gamma) = \int_{R_1} (a_1 - f)^2 dxdy + \int_{R_2} (a_2 - f)^2 dxdy + \nu|\Gamma|$$

Results of Two-Class Segmentation



P. Strandmark, F. Kahl, Optimizing Parametric Total Variation Models, *International Conference on Computer Vision*, Sep., Kyoto, Japan 2009.

Example of graph-cut application:

Multi-view volumetric reconstruction

Calibrated images of Lambertian scene

3D model of scene

CVPR'05 slides from Vogiatzis, Torr, Cippola

Review

- Graphs
- Simple graph based segmentation -> eigenvalue problem
- Mumford-Shah functional
- Graph cuts

- Read lecture notes
- Finish assignment 2

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