Image Analysis - Lecture 2

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Lecture 2

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Linear algebra: Linear spaces

The following linear spaces are well-known:

- $\mathbb{R}^n$: all $n \times 1$-matrices, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad x_i \in \mathbb{R}$
- $\mathbb{C}^n$: all $n \times 1$-matrices, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad x_i \in \mathbb{C}$

Basis

Definition

$e_1, \ldots e_n \in \mathbb{R}^n$ is a **basis** in $\mathbb{R}^n$ if

- they are linearly independent
- they span $\mathbb{R}^n$.

■

Example (3D space)

$e_1, e_2, e_3 \in \mathbb{R}^3$ *is a* **basis** *in* $\mathbb{R}^3$ *if they are not located in the same plane.* ■

Canonical Basis

Example (canonical basis)

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \dots \quad e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

is called the **canonical basis** *in* $\mathbb{R}^n$*.*

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 e_1 \ldots + x_n e_n \quad .$$

■

Coordinates

Let $e_1, e_2, \ldots, e_n$ be a basis. Then for every x there is a unique set of scalars ξ_i such that

$$x = \sum_{i=1}^{n} \xi_i e_i \quad .$$

These scalars are called the **coordinates** for x in the basis $e_1, e_2, \ldots, e_n$.

Image matrix

An $M \times N$ image, f, is described by the matrix

$$f = \begin{pmatrix} f(0,0) & \dots & f(0,N-1) \\ \vdots & \ddots & \vdots \\ f(M-1,0) & \dots & f(M-1,N-1) \end{pmatrix}, \quad f(i,j) \in \mathbb{C}$$

$f(i,\cdot)$=i:th row, $f(\cdot,j)$=j:th column.

Row-stacking

Row-stacking:

$$\tilde{f} = \begin{pmatrix} f^T(0,\cdot) \\ f^T(1,\cdot) \\ \vdots \\ f^T(M-1,\cdot) \end{pmatrix} \in \mathbb{R}^{MN}(\mathbb{C}^{MN})$$

Properties:

$$\widetilde{f+g} = \tilde{f} + \tilde{g}$$

$$\widetilde{\lambda f} = \lambda \tilde{f}, \quad \lambda \in \mathbb{R}$$

A linear space!

Canonical basis

$$\chi(i,j) = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ & \vdots & 1 & \vdots & \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix},$$

with the 1 at position (i,j).
Using this canonical basis we can write

$$f = \sum_{i,j} f(i,j)\chi(i,j) \quad .$$

Idea for image transform:
Choose another basis that is more suitable in some sense.
Image matrices can thus be seen as vectors in a linear space.

Scalar product

Definition

Let A be a (complex) matrix. Introduce

$$A^* = (\bar{A})^T \quad .$$

■

Definition

Let x and y be two vectors in $\mathbb{R}^n$ ($\mathbb{C}^n$). **The scalar product** of x and y is defined as

$$x \cdot y = \sum \bar{x}_i y_i = x^* y \quad .$$

■

Orthogonality

Definition

The scalar product of two matrices (images) is defined as

$$f \cdot g = \sum_{i=0}^{M-1} \sum_{j=0}^{N-1} \bar{f}(i,j) g(i,j) \quad .$$

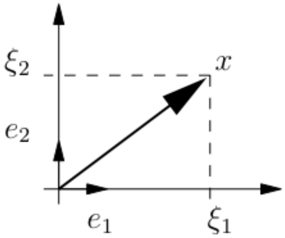
$x, y \in \mathbb{R}(\mathbb{C})$ are **orthogonal** if $x \cdot y = 0$. This is often written

$$x \perp y \quad \Leftrightarrow \quad x \cdot y = 0 \quad .$$

The length or **the norm** of x is defined as

$$||x|| = \sqrt{\sum |x_i|^2} = (x^* x)^{1/2} \quad .$$

Coordinates:



Orthogonal basis (ON-basis)

Definition

$\{e_1, \ldots, e_n\}$ is an **orthonormal (ON-) basis** in $\mathbb{R}^n$ ($\mathbb{C}^n$) if

$$e_i \cdot e_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

■

Coordinates in ON-basis

Theorem

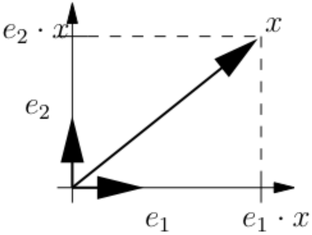
Assume that $\{e_1, \ldots, e_n\}$ *is orthonormal (ON) basis and*

$$x = \sum_{i=1}^{n} \xi_i e_i \quad .$$

Then

$$\xi_i = e_i \cdot x = e_i^* x, \quad ||x||^2 = \sum_{i=1}^{n} |\xi_i|^2$$

Illustration



Orthogonal projection

Definition

Let $\{a_1, \ldots, a_k\} \in \mathbb{R}^n$, $k \leq n$, span a linear subspace, π, in $\mathbb{R}^n$, i.e.:

$$\pi = \{w | w = \sum_{i=1}^{k} x_i a_i,\ x_i \in \mathbb{R}\} \ .$$

The **orthogonal projection** of $u \in \mathbb{R}^n$ on π is a linear mapping P, such that $u_\pi = Pu$ and defined by

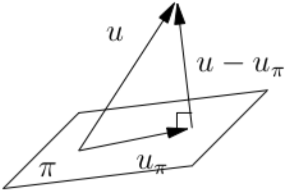
$$\min_{w \in \pi} ||u - w|| = ||u - u_\pi|| \ .$$

■

Orthogonal projection (ctd.)

The orthogonal projection is characterized by

1. $u_\pi \in \pi$
2. $u - u_\pi \perp w$ for every $w \in \pi$



Example: Matrix-basis

What is the orthogonal projection of f

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 7 & 7 \end{pmatrix}$$

onto the space spanned by (e_1, e_2, e_3)

$$e_1 = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, e_2 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix}, e_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix}$$

Example: Matrix-basis (ctd.)

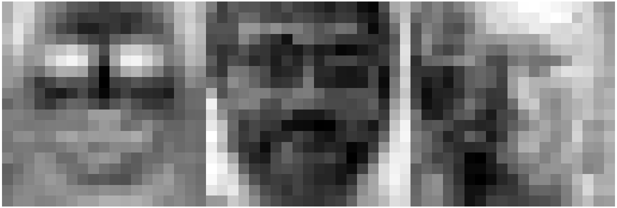
Since (e_1, e_2, e_3) is orthonormal the coordinates are $x_1 = f \cdot e_1 = 14, x_2 = f \cdot e_2 = -15/\sqrt{6}, x_3 = f \cdot e_3 = -4/\sqrt{6}$. The orthogonal projection is then $\hat{f} = 14e_1 - 15/\sqrt{6}e_2 - 4/\sqrt{6}e_3$

$$f = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 7 & 7 \end{pmatrix}, \hat{f} = \begin{pmatrix} 1.5 & 2\frac{1}{6} & 2\frac{5}{6} \\ 4 & 4\frac{2}{3} & 5\frac{1}{3} \\ 6.5 & 7\frac{1}{6} & 7\frac{5}{6} \end{pmatrix},$$

Example: 'Face'-basis

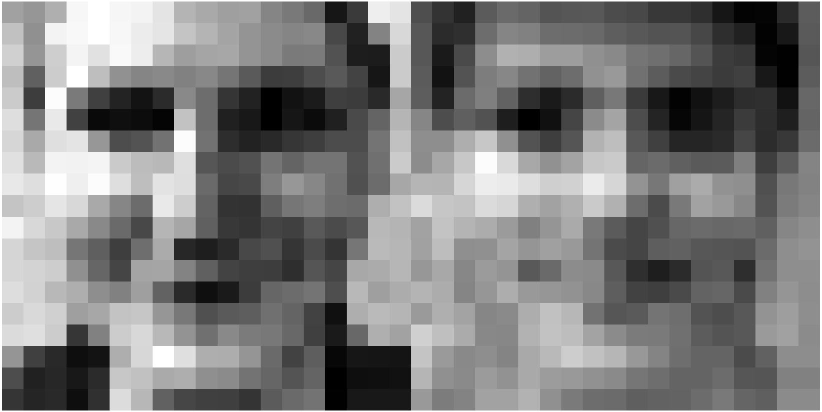
What is the orthogonal projection of f

onto the space spanned by (e_1, e_2, e_3)



Example: 'Face'-basis (ctd.)

Since (e_1, e_2, e_3) is orthonormal, the coordinates are
$x_1 = f \cdot e_1 = -2457, x_2 = f \cdot e_2 = 303, x_3 = f \cdot e_3 = -603.$
The orthogonal projection is then $\hat{f} = -2457 e_1 + 303 e_2 - 603 e_3$



Uniqueness of the projection

Let $a \in \pi$ and $b \in \pi$ be two solutions to the minimisation problem. Set

$$f(t) = \|u - ta - (1-t)b\|^2 = \dots$$

$$= \|u - b\|^2 + t^2\|a - b\|^2 - 2t(a-b)\cdot(u-b), \quad t \in \mathbb{R}\,.$$

This is a second degree polynomial with minimum in $t = 0$ and $t = 1 \Rightarrow f(t)$ is a constant function and thus $\Rightarrow a = b$.

Characterization of the projection

Let $f(t) = \|u - u_\pi + ta\|^2$, where $a \in \pi$. It follows that $f'(0) = 2(u - u_\pi) \cdot a = 0$, i.e. $(u - u_\pi) \perp a$.
Conversely: Assume $w \in \pi$. The property that $(u - u_\pi) \perp a$, for every $a \in \pi$ gives that

$$\|u - w\|^2 = \|u - u_\pi + u_\pi - w\|^2 =$$

$$\|u - u_\pi\|^2 + \|u_\pi - w\|^2 \geq \|u - u_\pi\|^2,$$

i.e. u_π solves the minimization problem.

An important result

Let $A = [a_1 \ldots a_k]$ be a $n \times k$ matrix and

$$\pi = \{w | w = Ax, \; x_i \in \mathbb{R}^n\}$$

Lemma

If $\{a_1, \ldots, a_k\}$ are linearly independent $\mathbb{R}^n$ then $A^* A$ is invertible.

Proof: Do it on your own. (Use SVD if you are familiar with it.)

■

Projection onto the subspace spanned by A

Theorem

if the columns of A are linearly independent, then the projection of u on π is given by

$$u_\pi = x_1 a_1 + \ldots + x_k a_k, \quad x = (A^* A)^{-1} A^* u \ .$$

Proof: Use the characterization of the projection (above).

$$a_i^*(u - u_\pi) = 0 \quad \Rightarrow$$

$$A^*(u - Ax) = 0 \quad \Rightarrow$$

$$A^* u = A^* A x \quad \Rightarrow \quad x = (A^* A)^{-1} A^* u$$

■

The pseudo-inverse

Definition

$A^+ = (A^*A)^{-1}A^*$ is called the **pseudo-inverse** of A. ■

Observe that if A is quadratic and invertible then $A^+ = A^{-1}$.

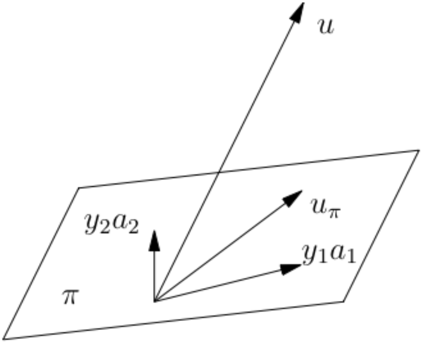
Theorem

If $\{a_1, \ldots, a_k\}$ are orthonormal, then the projection of u on π is given by

$$u_\pi = y_1 a_1 + \ldots + y_k a_k, \quad y_i = a_i^* u \ .$$

Proof: This follows from $A^*A = I$. ■

Illustration



Fourier transform

Definition

Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. The Fourier transformen of f is defined as

$$(\mathcal{F}f)(u) = F(u) = \int_{-\infty}^{+\infty} e^{-i2\pi xu} f(x)dx \quad .$$

■

Theorem

Under the right assumptions on f, the following inversion formula

$$f(x) = \int_{-\infty}^{+\infty} e^{i2\pi ux} F(u)du$$

holds.

Examples

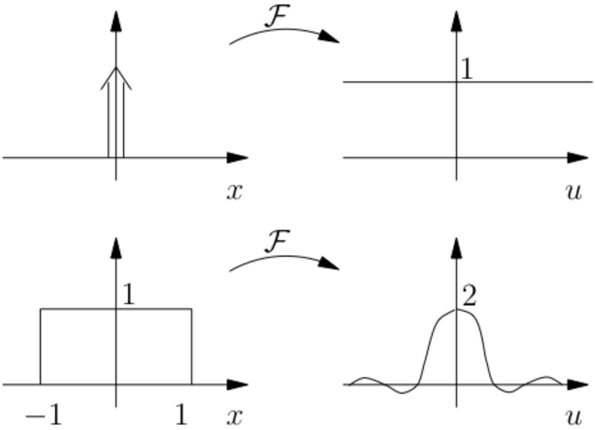
Example

$$\delta(x) \mapsto 1(u)$$

$$\mathrm{rect}(x) \mapsto 2\frac{\sin(2\pi u)}{2\pi u} = 2\,\mathrm{sinc}(2\pi u)$$

■

Illustrations



Properties

$$c_1 f_1(x) + c_2 f_2(x) \mapsto c_1 F_1(u) + c_2 F_2(u) \quad \text{(linearity)}$$

$$f(\lambda x) \mapsto \frac{1}{|\lambda|} F(\frac{u}{\lambda}) \quad \text{(scaling)}$$

$$f(x-a) \mapsto e^{-i2\pi ua} F(u) \quad \text{(translation)}$$

$$e^{-i2\pi xa} f(x) \mapsto F(u+a) \quad \text{(modulation)}$$

$$\overline{f(x)} \mapsto \overline{F(-u)} \quad \text{(conjugation)}$$

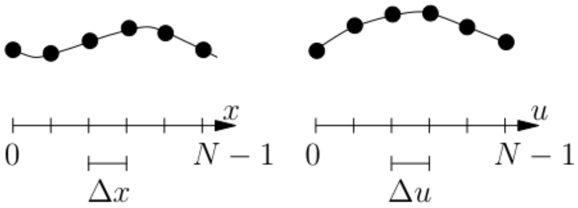
$$\frac{df}{dx} \mapsto 2\pi i u F(u) \quad \text{(differentiation I)}$$

$$-2\pi i x f(x) \mapsto \frac{dF}{du} \quad \text{(differentiation II)}$$

Example: $\delta(x-1) \mapsto e^{-i2\pi u}$

The discrete Fourier transform (DFT)

Sample $f(x)$ and $F(u)$.



The discrete Fourier transform (DFT) (ctd.)

This works particularly well if $\Delta_x \Delta_u = \frac{1}{N}$:

$$\frac{1}{\Delta_x} F(n\Delta_u) \sim \sum_{k=0}^{N-1} e^{-i2\pi kn/N} f(k\Delta_x),$$

$$f(k\Delta_x) \sim \frac{1}{N} \sum_{n=0}^{N-1} e^{i2\pi kn/N} \frac{1}{\Delta_x} F(n\Delta_u).$$

F and f are extended to periodic functions with periods $N\Delta_u$ and $N\Delta_x$ respectively.

Definition of the discrete Fourier transform

Let the vector

$$(f(0), f(1), \ldots, f(N-1)) \ .$$

represent the discretized version of $f(x)$.

Definition

The **discrete Fourier Transformen (DFT)** of f is

$$F(u) = \sum_{k=0}^{N-1} f(k)\omega_N^{ku}, \ u = 0, \ldots, N-1 \ ,$$

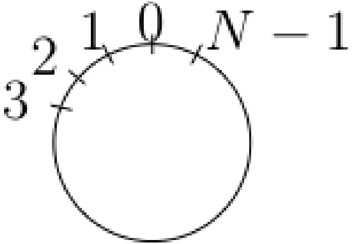
where $\omega_N = e^{-i2\pi/N}$. ■

Represent the sequence $F(u)$ with the vector

$$(F(0), F(1), \ldots, F(N-1)) \ .$$

Important assumption

All sequences are assumed to be period with period N.



Properties of DFT

There are similar formulas for the discrete Fourier Transform (as compared to that of the continuous Fourier Transform), e.g.

$$(f(-k_0), f(1-k_0), \ldots, f(N-1-k_0)) \mapsto$$

$$\mapsto \sum_{k=0}^{N-1} f(k-k_0)\omega^{ku} = [l = k - k_0]$$

$$= \sum_{l=-k_0}^{N-1-k_0} f(l)\omega^{(l+k_0)u} =$$

$$= \omega^{k_0 u} \sum_{l=-k_0}^{N-1-k_0} f(l)\omega^{lu} = [f \text{ periodic}] = \omega^{k_0 u} \sum_{l=0}^{N-1} f(l)\omega^{lu} =$$

$$= \omega^{k_0 u} F(u) = e^{-i2\pi k_0/N} F(u)$$

DFT in matrix form

Let

$$f = \begin{pmatrix} f(0) \\ \vdots \\ f(N-1) \end{pmatrix}, \quad F = \begin{pmatrix} F(0) \\ \vdots \\ F(N-1) \end{pmatrix} .$$

Definition

The Fourier Matrix $\mathcal{F}_N$ is given by

$$\mathcal{F}_N = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_N & \omega_N^2 & \dots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \dots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \dots & \omega_N^{(N-1)(N-1)} \end{pmatrix} .$$

■

Computation of DFT

Theorem

$$F = \mathcal{F}_N f$$

Proof: Use the definition of DFT and of the Fourier Matrix ∎

Consequence: DFT can be computed using matrix multiplication.

Properties of the Fourier matrix

Lemma

$$\overline{\mathcal{F}}_N \mathcal{F}_N = NI \Longleftrightarrow \mathcal{F}_N^{-1} = \frac{1}{N}\overline{\mathcal{F}}_N$$

Proof: Multiply $\overline{\mathcal{F}}_N$ with $\mathcal{F}_N$ and use

$$\omega_N \overline{\omega_N} = 1, \quad \sum_{j=0}^{N-1} (\omega_N^p)^j = \frac{1-\omega_N^{Np}}{1-\omega_N} = 0 \ .$$

■

The inversion formula

This lemma gives us the following inversion formula

Theorem

$$f = \frac{1}{N}\overline{\mathcal{F}}F \iff f(k) = \frac{1}{N}\sum_{u=0}^{N-1} F(u)\omega_N^{-ku}, \quad k = 0, \ldots, N-1$$

Proof:

$$F = \mathcal{F}f \Rightarrow \frac{1}{N}\overline{\mathcal{F}}F = \frac{1}{N}\overline{\mathcal{F}}\mathcal{F}f = If = f \quad .$$

■

Example

Example

$N = 2$, $\omega = -1$:

$$\mathcal{F}_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} .$$

$$f = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow F = \begin{pmatrix} 4 \\ -2 \end{pmatrix} .$$

■

Example

Example

$N = 4$, $\omega = -i$:

$$\mathcal{F}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} .$$

■

Factorization of the Fourier matrix

$N = 2^2 = 4$, $\omega = \omega_4$, $\tau = \omega^2$

$$\mathcal{F}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 \\ 1 & \omega^2 & \omega^4 & \omega^6 \\ 1 & \omega^3 & \omega^6 & \omega^9 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^2 & \omega & \omega^3 \\ 1 & \omega^4 & \omega^2 & \omega^6 \\ 1 & \omega^6 & \omega^3 & \omega^9 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & \tau \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ \omega & \omega\tau \end{pmatrix} \\ \begin{pmatrix} 1 & 1 \\ 1 & \tau \end{pmatrix} & \begin{pmatrix} -1 & -1 \\ -\omega & -\omega\tau \end{pmatrix} \end{pmatrix} P_4$$

Factorization of the Fourier matrix (ctd.)

P_4 denotes a 4×4 permutation matrix (a matrix with zeros and ones, where each row and each column only contains one one).

$$\mathcal{F}_4 = \begin{pmatrix} \mathcal{F}_2 & \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \mathcal{F}_2 \\ \mathcal{F}_2 & -\begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix} \mathcal{F}_2 \end{pmatrix} P_4 = \begin{pmatrix} I & D_2 \\ I & -D_2 \end{pmatrix} \begin{pmatrix} \mathcal{F}_2 & 0 \\ 0 & \mathcal{F}_2 \end{pmatrix} P_4$$

where

$$D_2 = \operatorname{diag}(1, \omega) \ .$$

The Fast Fourier Transform (FFT)

Theorem

The Fourier Matrix can be factorized as

$$\mathcal{F}_{2N} = \begin{pmatrix} I & D_N \\ I & -D_N \end{pmatrix} \begin{pmatrix} \mathcal{F}_N & 0 \\ 0 & \mathcal{F}_N \end{pmatrix} P_{2N} \ ,$$

where

$$D_N = \mathrm{diag}(1, \omega_{2N}, \omega_{2N}^2, \ldots, \omega_{2N}^{N-1}) \ .$$

and P_{2N} *is a permutation matrix of order* $2N \times 2N$ *that maps*

$$(x(0), x(1), \ldots, x(2N-1)) \longrightarrow$$
$$(x(0), x(2), \ldots, x(2N-2), x(1), x(3), \ldots, x(2N-1)) \ .$$

The Fast Fourier Transform (FFT) (ctd.)

Corollary

Calculation of $\mathcal{F}_N f$ thus involves two calculations of $\mathcal{F}_{N/2} f$, which involves 4 calculations of $\mathcal{F}_{N/4} f$, etc.

This algorithm is called the **Fast Fourier Transform.**

Calculational complexity

Let μ_n be the number of multiplications needed for calculating DFT of order 2^n. Factorization gives

$$\mu_n = 2\mu_{n-1} + 2^{n-1} \ .$$

A solution to this recursion formula is

$$\mu_n = \frac{n2^n}{2} = \frac{N \log_2 N}{2} \quad \text{om} \quad N = 2^n \ .$$

Example

$N = 1024 = 2^{10}$

- *FFT gives* $\mu \sim 10^4$ *multiplications.*
- *DFT gives* $N^2 \sim 10^6$ *multiplications.*

■

Two-dimensional Fourier Transform

Definition

Let $f(x, y)$ be a function from $\mathbb{R}^2$ to $\mathbb{R}$. The Fourier transform of f is defined as

$$\mathcal{F}f(u, v) = F(u, v) = \int_{-\infty}^{+\infty} e^{-i2\pi(ux+vy)} f(x, y)dxdy \quad .$$

∎

This can be written (using $\mathbf{u} = (u, v)$, $\mathbf{x} = (x, y)$):

$$F(\mathbf{u}) = \int_{-\infty}^{+\infty} e^{-i2\pi\mathbf{u}\cdot\mathbf{x}} f(\mathbf{x})d\mathbf{x} \quad .$$

The inversion formula in 2D

Theorem

Under certain conditions on f, the following inversion formula

$$f(x,y) = \int_{-\infty}^{+\infty} e^{i2\pi(ux+vy)} F(u,v) dudv$$

holds.

Properties of the 2D Fourier transformation

Properties: (in addition to those for the 1-D Fourier Transform)

$$
\begin{aligned}
f_1(x)f_2(y) &\mapsto F_1(u)F_2(v) \quad \text{(separability)} \\
f(Q\mathbf{x}) &\mapsto F(Q\mathbf{u}) \qquad\quad \text{(rotation)}
\end{aligned}
$$

where Q denotes an orthogonal matrix.

Example

$$
\begin{aligned}
\operatorname{rect}(x)\operatorname{rect}(y) &\mapsto 4\operatorname{sinc}(2\pi u)\operatorname{sinc}(2\pi v) \\
\delta(x)1(y) &\mapsto 1(u)\delta(v) \\
\delta(x)\operatorname{rect}(y) &\mapsto 1(u)2\operatorname{sinc}(2\pi v) \\
f(x-1)+f(x+1) &\mapsto (e^{-i2\pi u}+e^{i2\pi u})F(u) = \\
&= 2\cos(2\pi u)F(u)
\end{aligned}
$$

A useful fact

If f real (usual case for images):

- even $f \mapsto$ real F
- odd $f \mapsto$ imaginary F
- $F(u) = \overline{F(-u)}$

Observe: $F(u, v)$ is in general complex valued. It is common to illustrate the transform with $|F(u, v)|$.

DFT and FFT in two dimensions

The discrete Fourier Transform (DFT) of f is defined as

$$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) e^{-i2\pi(\frac{ux}{M}+\frac{vy}{N})} =$$

$$= \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\omega_M^{ux}\omega_N^{vy} =$$

$$= \begin{pmatrix} 1 & \omega_M^{u} & \omega_M^{2u} & \ldots & \omega_M^{(M-1)u} \end{pmatrix} f \begin{pmatrix} 1 \\ \omega_N^{v} \\ \omega_N^{2v} \\ \vdots \\ \omega_N^{(N-1)v} \end{pmatrix},$$

$$x = 0,\ldots,M-1, \quad y = 0,\ldots N-1 \ .$$

DFT in Matrix form

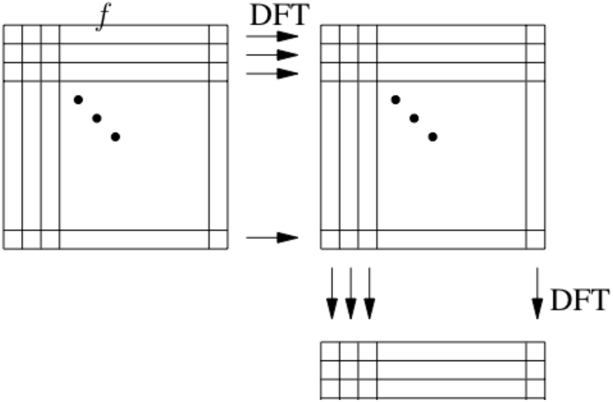
Let the matrix F represent the Fourier transform of the image $f(x, y)$:

$$F = \mathcal{F}_M f \mathcal{F}_N$$

or

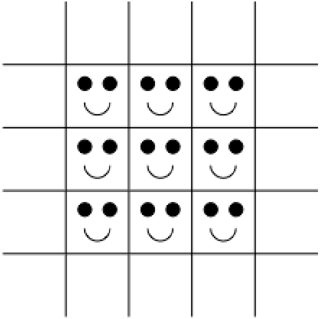
$$F = \mathcal{F}_M (\mathcal{F}_N f^T)^T \quad .$$

i.e. the DFT in two dimensions can be calculated by repeated use of the one-dimensional DFT, first for the rows, then for the columns.


f
DFT
DFT

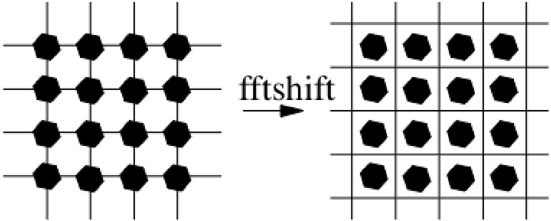
FFT on images

Let the $M \times N$-matrix f represent an image $f(x, y)$.
Extend the image periodically



FFT on images (ctd.)

FFT gives a double periodic function

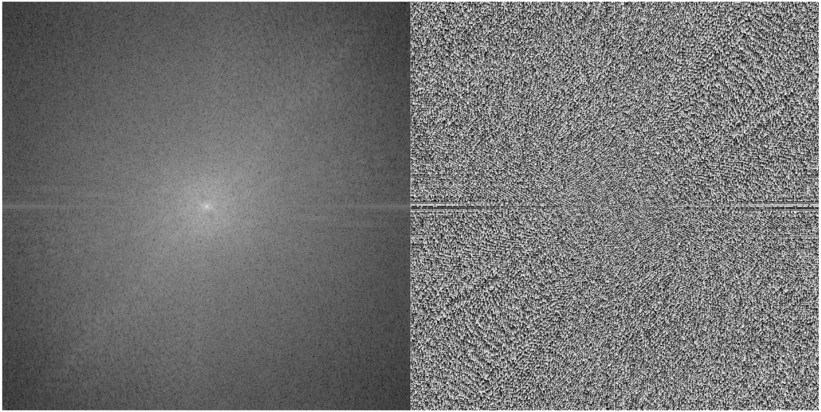


Interpretation of the Fourier Transform

- Usually, the gray-levels of the Fourier Transform images are scaled using $c\log(1 + |F(u,v)|)$.
- The middle of the Fourier image (after fftshift) corresponds to low frequencies.
- Outside the middle high components in F corresponds to higher frequencies and the direction corresponds to "edges"in the images with opposite orientation.

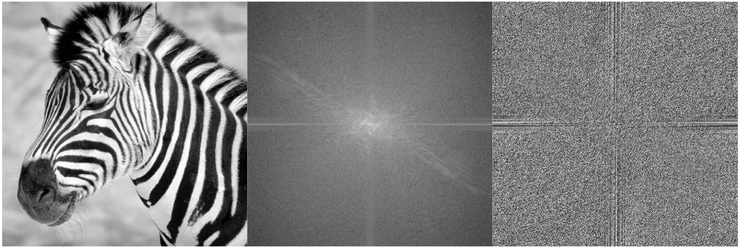
Example

What does the original image look like if this is the Fourier transform?



Left: Magnitude, Right: Phase

Example



Answer

Masters thesis suggestion of the day: Crossword reader/solver

Construct a system for automatic crossword scanning and solution. Idea: Take an image, find squares with text and without, interpret the text. Is it possible to solve crosswords automatically?

Recommended reading

- Forsyth & Ponce: **1. Cameras**. Lecture 1.
- Szeliski: **1. Introduction** and **3.1 Point operators**. Lecture 1.
- Forsyth & Ponce: **7. Linear filters**. Lectures 2-3.
- Szeliski: **3. Image processing**, sections 3.2-3.4. Lectures 2-3.

Review - Lecture 2

- Linear Algebra
- Subspaces
- Projections, Pseudo-inverse
- Image matrix
- Fourier Transform in 1 and 2 dimensions
- Discrete Fourier Transform in 1 and 2 dimensions
- Fast Fourier Transform (FFT)