Ch 5. Linear programming

5.1. Polyhedral set.

Def Polyhedral set $\overset{\text{def}}{=}$ intersection of finitely many closed half-spaces.

From the last lecture:

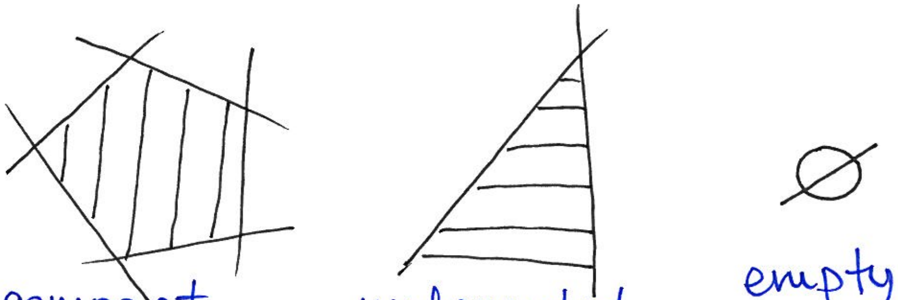
Polyhedral set $\iff$ $\boxed{Ax \le b}$

canonical form (D)

Ex Hyperplane $p^T x = \alpha$ is polyhedral.

$$p^T x = \alpha \iff \begin{cases} p^T x \le \alpha \\ p^T x \ge \alpha \end{cases} \iff A = \begin{bmatrix} p^T \\ -p^T \end{bmatrix}, \ b = \begin{bmatrix} \alpha \\ -\alpha \end{bmatrix}.$$

Ex A polyhedral set can be:



Another way to write $Ax \le b$:

- introduce slack variables $s = b - Ax \ge 0$ to get equality $Ax + s = b$, $s \ge 0$.
- introduce positivity by $x = y - z$, $y, z \ge 0$ (e.g. $y_k = |x_k|$, $z_k = |x_k| - x_k$) to get

$$Ax + s = Ay - Az + s = b, \quad s, y, z \ge 0.$$

Define $A_{new} = [A \ \ -A \ \ I]$ and

$$x_{new} = \begin{bmatrix} y \\ z \\ s \end{bmatrix} \implies \begin{cases} A_{new} \cdot x_{new} = b, \\ x_{new} \ge 0. \end{cases}$$

Thus polyhedral set $\iff$

$$\iff \boxed{\begin{matrix} Ax = b \\ x \ge 0 \end{matrix}}$$

canonical form (P)

Remark: reducing to CF normally increases the dimension.

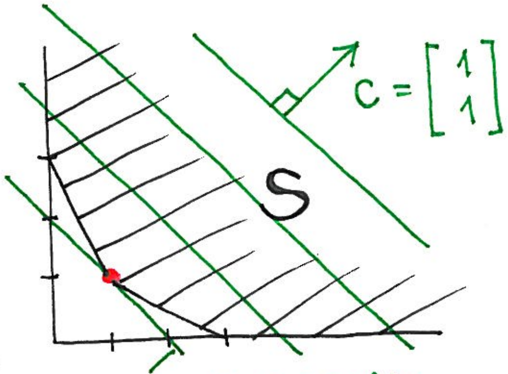
Linear Programming (LP)

$c \in \mathbb{R}^n$, $f(x) = c^T x$, $f : \mathbb{R}^n \to \mathbb{R}$

$$\min_{x \in S} c^T x \quad \text{where } S \text{ - polyhedral} \qquad \text{(LP)}$$

Ex a) $\min(x_1 + x_2)$

$$S : \begin{cases} 2x_1 + x_2 \geqslant 3 \\ x_1 + 2x_2 \geqslant 3 \\ x_1 \geqslant 0,\ x_2 \geqslant 0 \end{cases}$$



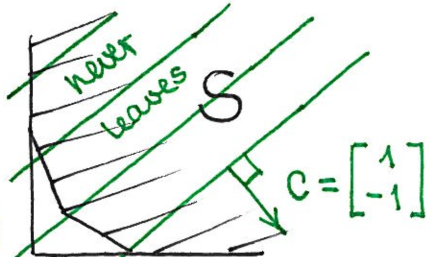
the last possible

$x_0 = (1,1)$; $\min = 2$
(unique solution)

b) $\min(x_1 - x_2)$

the same S

"min" $= -\infty$ (no solution)

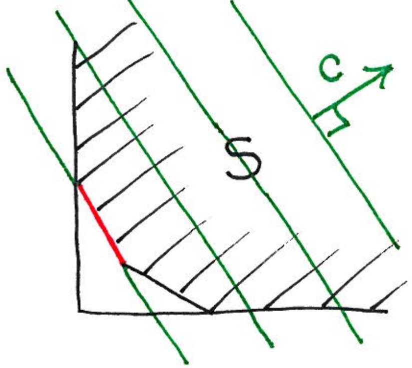


c) $\min(2x_1 + x_2)$

the same S

$x_0 = (t, 3-2t)$, $t \in [0,1]$

$\min = 3$ (many solutions)



d) $\min(x_1 + x_2)$

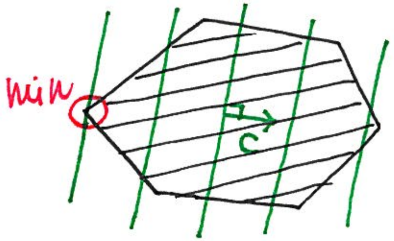
$$S : \begin{cases} -2x_1 - x_2 \geqslant 3 \\ x_1 + 2x_2 \geqslant 3 \\ x_1 \geqslant 0,\ x_2 \geqslant 0 \end{cases} \qquad S = \emptyset$$

(no solution)

Remark: bad things that can happen: $\min = -\infty$ (if S unbounded) and $S = \emptyset$.

- $\min = -\infty \Rightarrow$ too loose constraints.
- $S = \emptyset \Rightarrow$ too tight constraints.

Important feature: if it exists, min is always attained at a corner (= extreme point) Th. 4, p. 162



To solve LP one could check all corners (finitely many), but they are too many.

A possible improvement: consider only those corners that have smaller $f(x)$.

Idea of the algorithm:

1. find a corner point x_0
2. go to another corner x : $f(x) \searrow$

It is called Simplex method (not part of the course)

Another approach is to use barrier function method and duality.

It is called Interior point method.

5.3. Dual problem and duality

A special case first: consider

Primal: $\min\limits_{x \in S} c^T x \qquad S: \begin{cases} Ax \geq b \\ x \geq 0 \end{cases}$

$\updownarrow$

Dual: $\max\limits_{y \in T} b^T y \qquad T: \begin{cases} A^T y \leq c \\ y \geq 0 \end{cases}$

Remark: it is a mutual relation, i.e. dual of the dual = primal.

To see it we rewrite the dual as

$$\max_{y \in T} b^T y = -\min_{y \in T}(-b^T y) \qquad T: \begin{cases} -A^T y \geq -c \\ y \geq 0 \end{cases}$$

looks like primal

$$\text{Dual: } -\max_{x \in S}(-c^T x) \qquad S: \begin{cases} -Ax \leq -b \\ x \geq 0 \end{cases}$$

$$\qquad \| \qquad\qquad\qquad \|$$

$$\min_{x \in S} c^T x \qquad S: \begin{cases} Ax \geq b \\ x \geq 0 \end{cases}$$

Th (weak duality)

$$c^T x \geq b^T y, \quad \forall x \in S, \ \forall y \in T$$

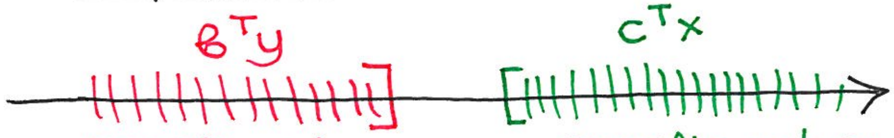
Proof: $c^T x - b^T y = c^T x - y^T b =$

$= c^T x - y^T A x + y^T A x - y^T b =$

$$= \underbrace{(c^T - y^T A)}_{\geq 0} \underbrace{x}_{\geq 0} + \underbrace{y^T}_{\geq 0} \underbrace{(Ax - b)}_{\geq 0} \geq 0. \qquad (*)$$

$\blacksquare$

Interpretation:



Corollary: $S \neq \emptyset, T \neq \emptyset \Rightarrow \min\limits_{x \in S} c^T x \geq \max\limits_{y \in T} b^T y$

Th (Complementary Slackness Principle)

Let $\bar{x} \in S$ and $\bar{y} \in T$. Then

$$c^T \bar{x} = b^T \bar{y} \iff \begin{cases} \bar{x}_k (c - A^T \bar{y})_k = 0, & \forall k \\ \bar{y}_j (A\bar{x} - b)_j = 0, & \forall j \end{cases}$$

Proof: write (*) in the coordinate form

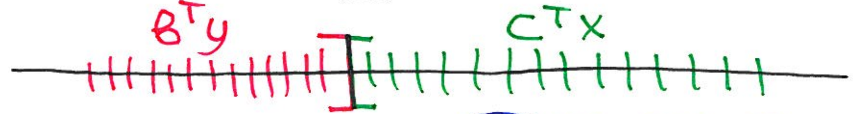
$$c^T \bar{x} - b^T \bar{y} = (c - A^T \bar{y})^T \bar{x} + \bar{y}^T (A\bar{x} - b) =$$

$$= \sum_{k=1}^{n} \underset{\geq 0}{(c - A^T \bar{y})_k}\, \underset{\geq 0}{\bar{x}_k} + \sum_{j=1}^{n} \underset{\geq 0}{\bar{y}_j}\, \underset{\geq 0}{(A\bar{x} - b)_j}.$$

Clearly, $c^T \bar{x} - b^T \bar{y} = 0 \iff$

$\iff$ all terms are zero. ∎

Interpretation: if $\bar{x} \in S$ and $\bar{y} \in T$ are such that $c^T \bar{x} = b^T \bar{y}$ then both are the optimal solutions.



and in this case $\min\limits_{x \in S} c^T x = \max\limits_{y \in T} b^T y$

The general case now: we construct the dual problem by demanding all terms in (*) are ≥ 0, i.e.

$$c^T x - b^T y = \sum_{k=1}^{n} \underbrace{x_k (c - A^T y)_k}_{\geq 0} + \sum_{j=1}^{n} \underbrace{y_j (Ax - b)_j}_{\geq 0}$$

The rule:
$(Ax)_j = b_j \longrightarrow y_j$ free
$(Ax)_j \geq b_j \longrightarrow y_j \geq 0$
$x_k \geq 0 \longrightarrow (A^T y)_k \leq c_k$
x_k free $\longrightarrow (A^T y)_k = c_k$

Ex $\min(2x_1+x_2) \longrightarrow \max(y_1+4y_2)$

$$\begin{cases} x_1+x_2 \geq 1 \\ 3x_1+x_2=4 \\ x_1 \geq 0 \\ x_2 \text{ free} \end{cases} \qquad \begin{cases} y_1+3y_2 \leq 2 \\ y_1+y_2=1 \\ y_1 \geq 0 \\ y_2 \text{ free} \end{cases}$$

How to verify that $\bar{x}=\left(\frac{3}{2},-\frac{1}{2}\right)$ is optimal?

1) Check that $\bar{x}\in S$:

$\bar{x}_1=\frac{3}{2}\geq 0$ - ok!, $A\bar{x}-b=\begin{bmatrix}0\\0\end{bmatrix}\begin{matrix}\geq 0\\ =0\end{matrix}$ - ok!

2) Find a candidate for the dual optimal solution by CSP:

$A\bar{x}-b=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow$ no information.

$\bar{x}_1 \neq 0 \quad \Rightarrow y_1+3y_2=2 \quad \overset{\text{solve}}{\Rightarrow} \bar{y}=\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}$

$\bar{x}_2 \neq 0 \quad \Rightarrow y_1+y_2=1$ - (already known)

3) Test the candidate for $\bar{y}\in T$:

$\bar{y}_1 \geq 0$ - ok!, $A^T\bar{y}-c=\begin{bmatrix}0\\0\end{bmatrix}\begin{matrix}\leq 0\\ =0\end{matrix}$ - ok!

1,2,3) $\Rightarrow \bar{x}$ is the primal optimal (and besides $\bar{y}$ is the dual optimal)

Moreover, $\min\limits_{x\in S} c^Tx = \max\limits_{y\in T} b^Ty$

Is $\min\limits_{x\in S} c^Tx > \max\limits_{y\in T} b^Ty$ possible?

Th (Lemma 2, p. 186)

Let $S\neq\emptyset$. Then

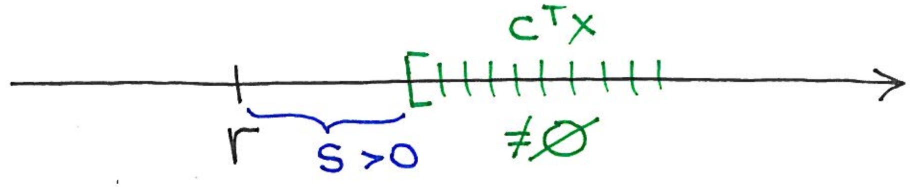
$\forall x\in S: c^Tx > r \iff \exists y\in T: b^Ty > r.$

Proof: $\boxed{\Leftarrow}$ easy by weak duality.

$\forall x\in S: c^Tx \geq b^Ty > r.$

$\boxed{\Rightarrow}$ By Farkas th. (technical).

Let $S=\{Ax=b, x\geq 0\} \Rightarrow T=\{A^Ty\leq c\}$



$\nexists$ solution to
$$\begin{cases} Ax = b \\ x \geq 0 \\ c^T x = r \end{cases}$$
and $\exists$ solution to
$$\begin{cases} Ax = b \\ x \geq 0 \\ c^T x = r + s, \; s \geq 0 \end{cases}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\nexists \begin{cases} \begin{bmatrix} A \\ -c^T \end{bmatrix} x = \begin{bmatrix} b \\ -r \end{bmatrix} \\ x \geq 0 \end{cases} \qquad \exists \begin{cases} \begin{bmatrix} A & 0 \\ -c^T & 1 \end{bmatrix} \begin{bmatrix} x \\ s \end{bmatrix} = \begin{bmatrix} b \\ -r \end{bmatrix} \\ \begin{bmatrix} x \\ s \end{bmatrix} \geq 0 \end{cases}$$

$$\Updownarrow \longleftarrow \text{Farkas} \longrightarrow \Updownarrow$$

$$\exists \begin{cases} \begin{bmatrix} A^T & -c \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \leq 0 \\ \begin{bmatrix} b^T & -r \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} > 0 \end{cases} \qquad \nexists \begin{cases} \begin{bmatrix} A^T & -c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} \leq 0 \\ \begin{bmatrix} b^T & -r \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} > 0 \end{cases}$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\exists \begin{cases} A^T v \leq c \cdot w \\ b^T v > r \cdot w \end{cases} \qquad \nexists \begin{cases} A^T v \leq c \cdot w \\ b^T v > r \cdot w \\ w \leq 0 \end{cases} \Rightarrow$$

$\Rightarrow \exists$ solution to
$$\begin{cases} A^T v \leq c \cdot w \\ b^T v > r \cdot w \end{cases} \text{ with } \underline{\underline{w > 0}} \Rightarrow$$

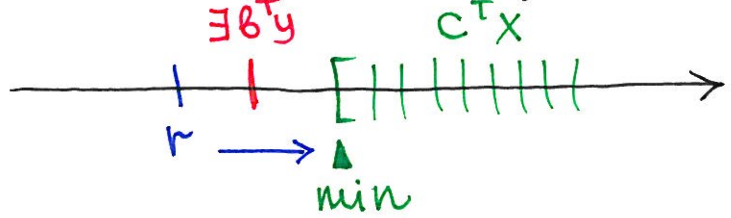
$$\Rightarrow \text{define } y = \frac{v}{w} \Rightarrow \begin{cases} A^T y \leq c \\ b^T y > r \end{cases}$$

■

Corollary (strong duality): Th. 7, p. 184

$$S \neq \emptyset, T \neq \emptyset \Rightarrow \boxed{\min_{x \in S} c^T x = \max_{y \in T} b^T y}$$

Proof: $T \neq \emptyset \Rightarrow$ by the weak duality $\min_{x \in S} c^T x > -\infty$. For any $r < \min$ $\exists y \in T$:



Taking $r \to \min$ gives $\min_{x \in S} c^T x = \max_{y \in T} b^T y$. ■

D \ P	$S \neq \emptyset$, $\min > -\infty$	$S \neq \emptyset$, $\min = -\infty$	$S = \emptyset$
$T \neq \emptyset$, $\max < +\infty$	$\min = \max$		
$T \neq \emptyset$, $\max = +\infty$			
$T = \emptyset$			

– not possible

Read 5.4, Ex. 18, 20.