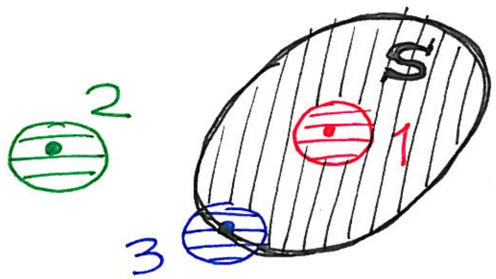
# Ch 4. Convex sets and separation.

## 4.1. Some topological terminology.

- Briefly, read yourself.
- Point locations:

1. interior
2. outer
3. boundary



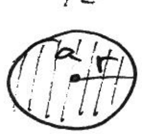

- $int(S)$ = all interior points of $S$,
  $\partial S$ = all boundary of $S$,
  $cl(S) = int(S) \cup \partial S$ = closure of $S$.

Def. $S$ def closed if $S = cl(S)$.

Ex. $S \Rightarrow cl(S)$, $int(S)$

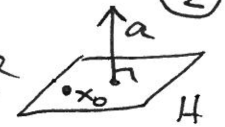
Ex $B_{a,r} = \{x \in \mathbb{R}^n \mid \|x-a\| < r\}$ - open ball

$cl\, B_{a,r} = \{x \in \mathbb{R}^n \mid \|x-a\| \le r\}$ - closed ball





Ex $a^T(x - x_0) = 0$ – hyperplane



$\Updownarrow$

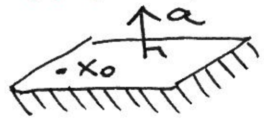
$a^T x = b\, (= a^T x_0) \iff a_1 x_1 + \dots + a_n x_n = b.$

- Hyperplane is closed $[H = cl(H)]$
- Empty interior $[\, int(H) = \emptyset \,]$.

Ex. $a^T(x - x_0) \le 0$ – closed half-space.

$\Updownarrow$

$a^T x \le b \iff a_1 x_1 + \dots + a_n x_n \le b$.



- Boundary $\partial H$ = hyperplane.
- $a^T(x - x_0) < 0$ – open half-space.

Ex. $a_1, \dots, a_m \in \mathbb{R}^n$.

$$\begin{cases} a_1^T x \le b_1, \\ \vdots \\ a_m^T x \le b_m \end{cases} \iff \begin{cases} a_{11} x_1 + \dots + a_{1n} x_n \le b_1, \\ \vdots \\ a_{m1} x_1 + \dots + a_{mn} x_n \le b_m \end{cases} \iff$$

$\iff A x \le b$ polyhedral set

= finite intersection of closed subspaces.

**Def** $S \subset \mathbb{R}^n$. $S \overset{def}{=}$ compact set if $S$ is closed and bounded.

**Th** (Weierstraß)

If $S \subset \mathbb{R}^n$ is compact set and $f: S \to \mathbb{R}$ is continuous function then $\exists\, x_{min} \in S: f(x_{min}) = \min_{x \in S} f(x)$.

**Remark**: very important in Ch. 7.

## 4.2. Convexity

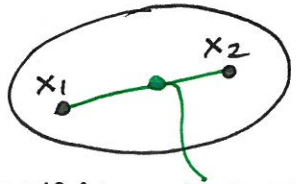
[ **Quantors**: $\exists$ = exists, $\forall$ = for all, $\exists!$ = exists a unique ]

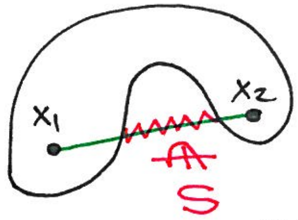
**Def**: $S \subset \mathbb{R}^n$. $S \overset{def}{=}$ convex if

$$\forall x_1, x_2 \in S: \ \lambda x_1 + (1-\lambda) x_2 \in S, \ \forall \lambda \in [0,1]$$



**Interpretation**:




convex $\lambda x_1 + (1-\lambda) x_2$ not convex

**Ex** $H = \{a^T x = b\}$ is convex.

**Proof**: By the definition

$$\forall x_1, x_2 \in H \Rightarrow \begin{cases} a^T x_1 = b \\ a^T x_2 = b \end{cases} \Rightarrow$$

$$\Rightarrow a^T(\lambda x_1 + (1-\lambda) x_2) =$$

$$= \lambda a^T x_1 + (1-\lambda) a^T x_2 =$$

$$= \lambda b + (1-\lambda) b = b \Rightarrow$$

$$\Rightarrow \lambda x_1 + (1-\lambda) x_2 \in H, \ \forall \lambda \in \mathbb{R} \Rightarrow$$

$\Rightarrow$ in particular, $\forall \lambda \in [0,1] \Rightarrow$

$\Rightarrow H$ convex. ∎

**Ex**. Polyhedral set $Ax \leq b$ is convex (similar proof).

Ex $S = \{ H = {}_n\Gamma^n \mid H - \text{pos. semidef} \}$

is convex.

Proof: Take $H_1, H_2 \in S \Rightarrow$

$\Rightarrow x^T H_1 x \geq 0, \; x^T H_2 x \geq 0, \; \forall x \in \mathbb{R}^n \Rightarrow$

$\Rightarrow x^T(\lambda H_1 + (1-\lambda) H_2) x = \underbrace{\lambda}_{\geq 0} \underbrace{x^T H_1 x}_{\geq 0} +$

$+ \underbrace{(1-\lambda)}_{\geq 0} \underbrace{x^T H_2 x}_{\geq 0} \geq 0, \; \forall x \in \mathbb{R}^n. \Rightarrow$

$\Rightarrow \lambda H_1 + (1-\lambda) H_2 \in S, \; \forall \lambda \in [0,1].$ ■

Ex $S = \{ p(x) = \sum_{k=0}^{n} c_k x^k \mid c_k \geq 0, \forall k \}$

is convex.

Proof: Take $p_1, p_2 \in S \Rightarrow$

$\Rightarrow p_1(x) = \sum_{k=0}^{n} \underbrace{c_{1k}}_{\geq 0} x^k, \; p_2(x) = \sum_{k=0}^{n} \underbrace{c_{2k}}_{\geq 0} x^k \Rightarrow$

$\Rightarrow [\lambda p_1 + (1-\lambda) p_2](x) = \lambda p_1(x) + (1-\lambda) p_2(x) =$

$= \sum_{k=0}^{n} \underbrace{(\lambda \overset{\geq 0}{c_{1k}} + (1-\lambda) \overset{\geq 0}{c_{2k}})}_{\geq 0} x^k \in S, \; \forall \lambda \in [0,1].$ ■



Lemma: (Lemma 2, p. 121)

If $S_1, S_2 \subset \mathbb{R}^n$ are convex then

1) $S_1 \cap S_2$ convex,

2) $S_1 + S_2$ convex,

$[S_1 + S_2 = \{ x_1 + x_2 \mid x_1 \in S_1, x_2 \in S_2 \}]$

Proof: $S_1 \cap S_2$ - yourself (Ex 4.8)

$S_1 + S_2$: take $x, y \in S_1 + S_2 \Rightarrow$

$\Rightarrow x = \underset{\in S_1}{x_1} + \underset{\in S_2}{x_2}, \; y = \underset{\in S_1}{y_1} + \underset{\in S_2}{y_2} \Rightarrow$

$\Rightarrow \lambda x + (1-\lambda) y = \lambda(x_1 + x_2) + (1-\lambda)(y_1 + y_2) =$

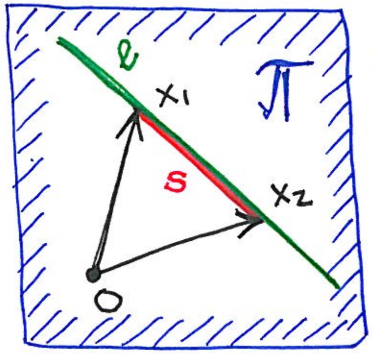
$= \underbrace{(\lambda x_1 + (1-\lambda) y_1)}_{\in S_1} + \underbrace{(\lambda x_2 + (1-\lambda) y_2)}_{\in S_2} \in$

$\in S_1 + S_2, \; \forall \lambda \in [0,1].$ ■

<u>Terminology</u>: $\lambda_1, \dots, \lambda_m \in \mathbb{R}$, $x_1, \dots, x_m \in \mathbb{R}^n$

- $\sum_{k=1}^{m} \lambda_k x_k \overset{\text{def}}{=}$ <u>linear combination</u>,
- $\sum_{k=1}^{m} \lambda_k x_k \overset{\text{def}}{=}$ <u>affine combination</u>, where $\sum_{k=1}^{m} \lambda_k = 1$
- $\sum_{k=1}^{m} \lambda_k x_k \overset{\text{def}}{=}$ <u>convex combination</u> where $\sum_{k=1}^{m} \lambda_k = 1$, all $\lambda_k \geq 0$.

<u>Ex</u> for two vectors $x_1, x_2 \in \mathbb{R}^2$




All linear comb. of $x_1, x_2 \Rightarrow$ the plane $\pi$

All affine comb. of $x_1, x_2 \Rightarrow$ the line $\ell$

All convex comb. of $x_1, x_2 \Rightarrow$ the segment $s$.

$$\lambda_1 x_1 + \lambda_2 x_2 = \underline{\underline{\lambda_1 x_1 + (1-\lambda_1) x_1}}$$

$$\lambda_1 + \lambda_2 = 1 \Leftrightarrow \lambda_2 = 1 - \lambda_1; \quad \lambda_1, \lambda_2 \geq 0 \Leftrightarrow \underline{\underline{0 \leq \lambda_1 \leq 1}}$$



<u>Remark</u>: $S$ convex $\Leftrightarrow$

$\Leftrightarrow$ all convex combinations of points from $S$ belong to $S$.

<u>Proof</u>: by induction on # points

<u>Ex</u> for <u>three</u> points $x_1, x_2, x_3 \in S$

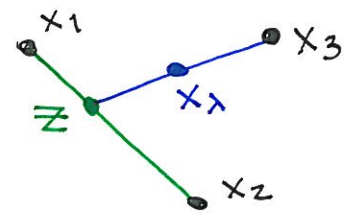
$$x_\lambda = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 \qquad (1)$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 \qquad (2)$$

$$\lambda_1, \lambda_2, \lambda_3 \geq 0 \qquad (3)$$

<u>Case $\lambda_1 + \lambda_2 = 0$</u> $\Leftrightarrow \lambda_1 = \lambda_2 = 0$, $\lambda_3 = 1$

$\Rightarrow x_\lambda = x_3 \in S$




<u>Case $\lambda_1 + \lambda_2 > 0$</u>:

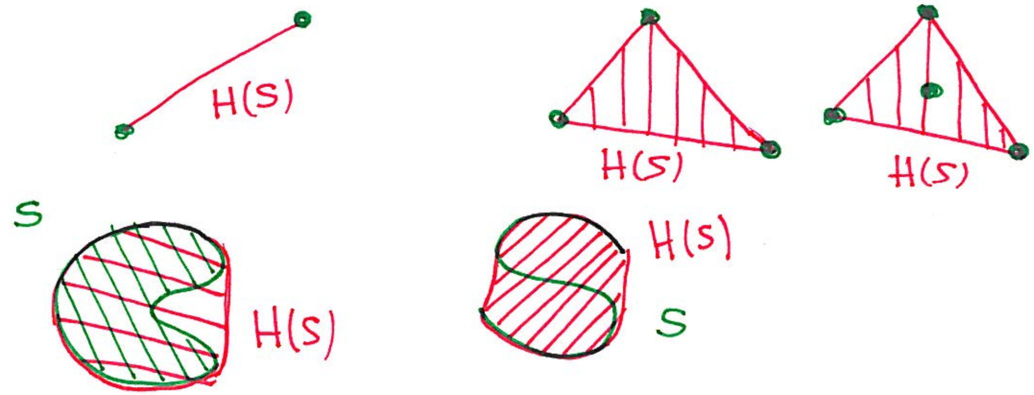
$$x_\lambda = (\lambda_1 + \lambda_2) \underbrace{\left[ \frac{\lambda_1}{\lambda_1 + \lambda_2} x_1 + \frac{\lambda_2}{\lambda_1 + \lambda_2} x_2 \right]}_{\text{conv. comb. } x_1, x_2 \, = \, z \, \in \, S} + \lambda_3 x_3$$

$\Rightarrow x_\lambda =$ conv. comb. of $z, x_3 \Rightarrow x_\lambda \in S$.

**Def** $S \subset \mathbb{R}^n$

$H(S) = \{$all conv. comb. of points of $S\}$ – $\underset{\text{def}}{}$ convex hull (convex hõlje).

**Ex** $S = \{2 \text{ points}\}$ $\{3 \text{ points}\}$ $\{4 \text{ points}\}$




**Remark**: $S$ convex $\Leftrightarrow$ $S = H(S)$

- Lemma 4, p. 123:

$$H(S) = \bigcap_{\substack{T \supset S, \\ T \text{ convex}}} T$$

"the smallest convex set that contains $S$"

- $H(S) = \{$all convex combinations of at most $n+1$ points in $S\}$ (Caratheodory th., p. 124)

## 4.3. Separation of convex sets

**Th** (Theorem 5, p. 129)

$S \subset \mathbb{R}^n$, $S = cl(S) = H(S) \neq \emptyset$, $y \notin S \Rightarrow$

$\Rightarrow \exists!\, x_0 \in S: \|y - x_0\| = \min_{x \in S} \|y - x\|.$

exists unique

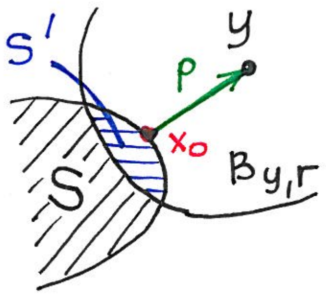
Moreover, for $p = y - x_0$ it holds

$$p^T(x - x_0) \leq 0, \; \forall x \in S. \quad *)$$

**Remark**: $\min_{x \in S} \|y - x\| = dist(y, S)$ – the distance from $y$ to the set $S$.

$*)$ means that the set $S$ is located in the half-space $\{p^T(x - x_0) \leq 0\}$. It will play the crucial role in convex optimization.

<u>Proof:</u> for $r = \text{dist}(y, S) + 1$ define $S' = S \cap B_{y,r} \neq \emptyset$



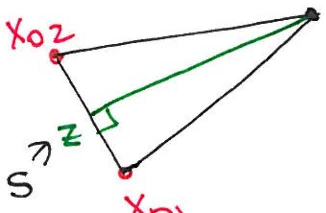

$S'$ compact $\Rightarrow$ by Weierstraß $\exists\, x_0 \in S'$ that solves

$$\min_{x \in S'} \|y - x\|$$

However, $\min\limits_{S} = \min\limits_{S'}$ since all points outside $B_{y,r}$ satisfy $\|y - x\| > r \Rightarrow$ $\Rightarrow$ not interesting for $\min\limits_{x \in S} \|y - x\|$.

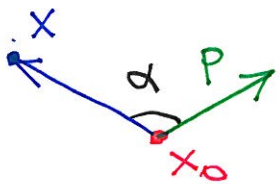
Thus $x_0 \in S' \subset S$ solves $\min\limits_{x \in S} \|y - x\|$ too.

<u>Uniqueness:</u> if $x_{01}, x_{02}$ are two solutions $\Rightarrow$




$\Rightarrow$ from the isosceles $\Delta$ $\Rightarrow$ $\Rightarrow \|z - y\| < \min$ (impossible) ↯

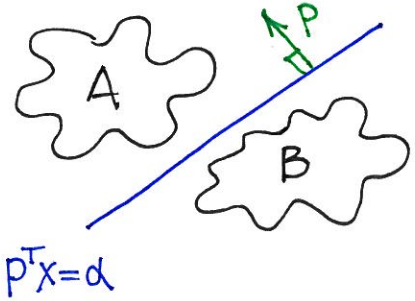
$$p^T(x - x_0) \leq 0 \iff \alpha \geq \frac{\pi}{2}.$$




($\alpha < \frac{\pi}{2} \Rightarrow \|p\|$ not shortest ↯)



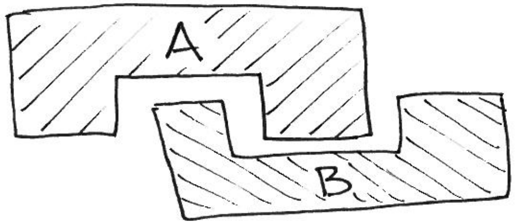
<u>Def</u> $A, B \subset \mathbb{R}^n$. The hyperplane $H = \{p^T x = \alpha\}$ separates $A$ and $B$ if




$p^T x > \alpha,\ \forall x \in cl(A)$,

$p^T x < \alpha,\ \forall x \in cl(B)$.

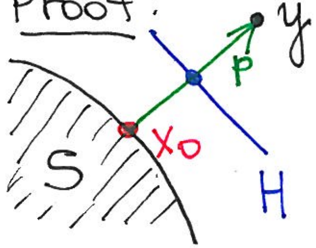
<u>Ex</u> not always possible, e.g.




**Th** (Theorem 6, p. 132)

$S \subset \mathbb{R}^n$. $S = cl(S) = H(S) \neq \emptyset$, $y \notin S \Rightarrow$ $\Rightarrow \exists\, H = \{p^T x = \alpha\}$ that separates the point $y$ and the set $S$.

<u>Proof:</u>




$\exists!\, x_0 \in S : \|y - x_0\| = \min\limits_{x \in S} \|y - x\|$

$p := y - x_0,\ \alpha := \frac{1}{2} p^T (x_0 + y)$.

**Corollary:** • $S = cl(S) = H(S) \neq \emptyset$.

Then $S = \bigcap H^-$.

$H^-$ half space

$H^- \supset S$

**Proof:** "$\subset$" trivial.

"$\supset$" Let $y \notin S \Rightarrow \exists H$ that separates $y$ and $S \Rightarrow \exists H^- \supset S : H^- \not\ni y \Rightarrow$
$\Rightarrow y \notin \bigcap H^- \Rightarrow S \supset \bigcap H^-$. ∎

• $S_1, S_2$ – closed convex $\neq \emptyset$,

$S_1$ is compact $\Rightarrow \exists$ separating $H$.

**Proof:** apply th. to $S_1 - S_2$ and $0$ ∎

• Support hyperplanes exist at each point of $\partial S$ for convex closed $S \neq \emptyset$.

*) facts on this page are not needed for our course, but are important in convex optimization.



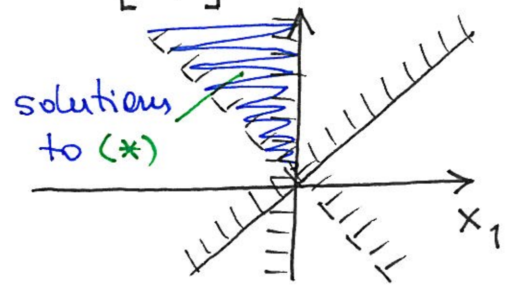
## 4.4. Farkas' theorem.

Consider two systems of inequalities:

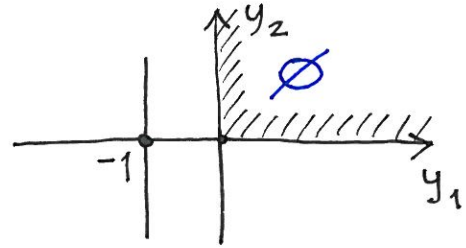
$$(*) \begin{cases} Ax \leq 0 \\ c^T x > 0 \end{cases} \quad \text{and} \quad (**) \begin{cases} A^T y = c \\ y \geq 0 \end{cases}$$

Ex. $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$, $c = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

$$(*) \begin{cases} x_1 - x_2 \leq 0 \\ x_1 \leq 0 \\ 2x_1 + x_2 > 0 \end{cases}$$




$$(**) \begin{cases} y_1 + y_2 = 2 \\ -y_1 = 1 \\ y_1 \geq 0, y_2 \geq 0 \end{cases}$$




**Th** (Farkas)

(*) has a solution $\Leftrightarrow$ (**) has none.

**Proof:** $\boxed{\Rightarrow}$ Assume $\exists y$ : solution to (**). $\Rightarrow$

$$\Rightarrow A^T y = c \Rightarrow c^T x = \underset{\geq 0}{y^T} \underbrace{Ax}_{\leq 0} \leq 0 \;\; ⚡ \;\; (c^T x > 0 \;\; (*))$$

$\boxed{\Leftarrow}$ Define $S = \{ z \in \mathbb{R}^n \mid \exists\, y \geqslant 0 : z = A^T y \}$.

(positive lin. comb. of rows of A)

- $0 \in S \Rightarrow S \neq \emptyset$
- $S$ closed (only $\geqslant, =$), convex (easy by def.).
- (**) has no solution $\Rightarrow c \notin S$.

By separation th.: $\exists\, p \in \mathbb{R}^n : p^T z < \alpha, \ \forall z \in S$

$p^T c > \alpha$.

Let's prove that $p$ solves (*)

$0 \in S \Rightarrow p^T 0 < \alpha \Rightarrow \alpha > 0 \Rightarrow c^T p = p^T c > \alpha > 0$.

$p^T z = p^T A^T y = y^T A p < \alpha, \ \forall y \geqslant 0$ - impossible

unless $Ap \leqslant 0$. Indeed, if $(Ap)_k > 0 \Rightarrow$

$\Rightarrow$ choose $y = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ t \\ \vdots \\ 0 \end{bmatrix} \end{array} \begin{array}{c} \\ \\ \\ k \\ \\ \\ \end{array} \Rightarrow y^T A p = t \cdot (Ap)_k \xrightarrow[t \to \infty]{} \infty$. $\blacksquare$

Read yourself: examples in 4.4,

4.5 Cones and dual cones, in particular,

no solution to (*) $\Leftrightarrow c \in$ dual cone to $Ax \leqslant 0$.