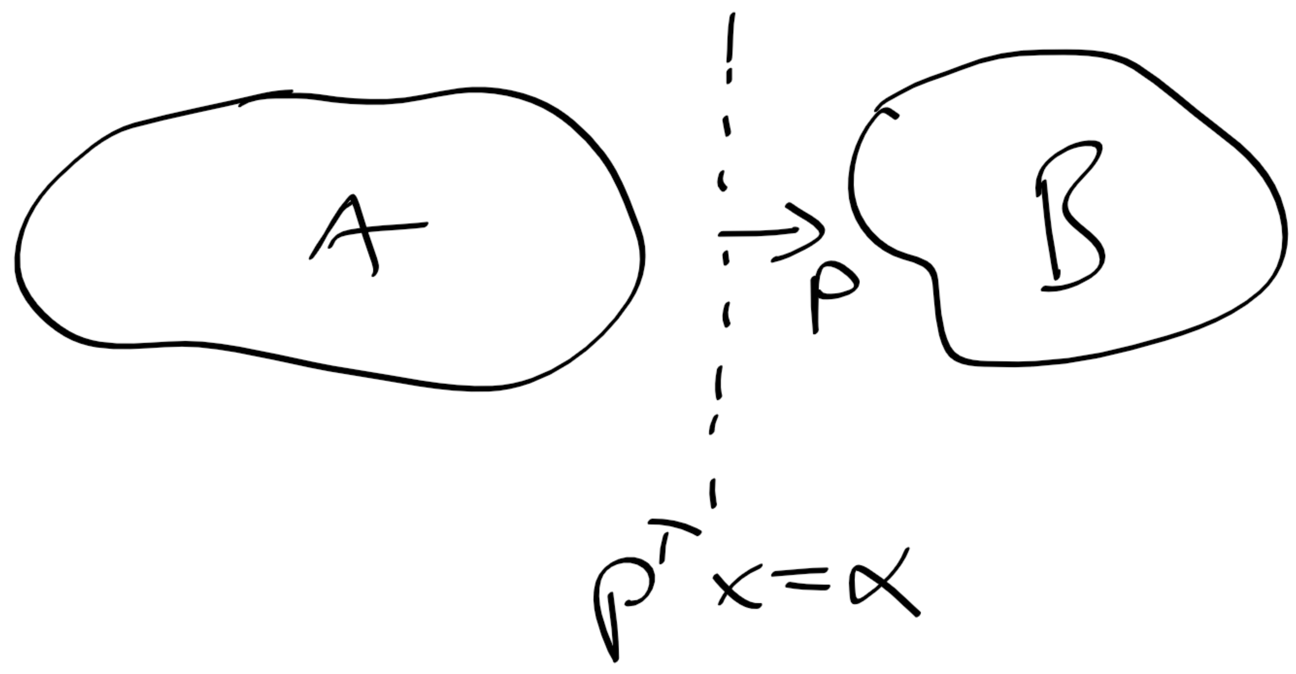
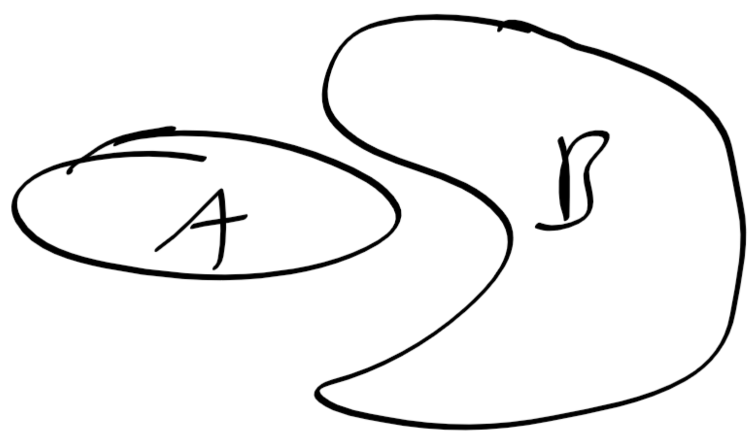
4.3 b Separating planes

Def. The hyperplane $p^T x = \alpha$ separates the sets A and $B \subseteq \mathbb{R}^n$ iff

$$p^T x < \alpha \quad \text{when } x \in cl(A)$$
$$p^T x > \alpha \quad \text{when } x \in cl(B)$$



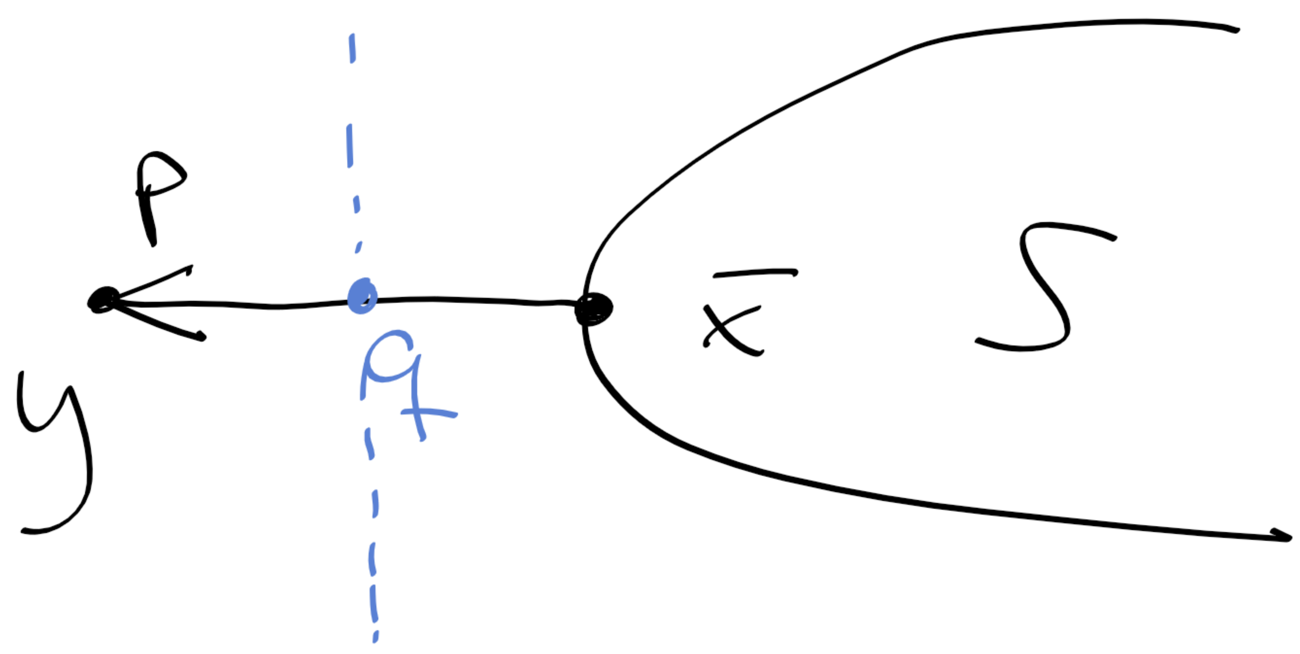


no sep. plane exists

Thm 6: $\emptyset \neq S \subseteq \mathbb{R}^n$ closed and convex. If $y \notin S$, then y and S have a separating hyperplane.

Proof: By Thm 5 $\exists\ \bar{x} \in S$ with least distance to y and $p^T(x - \bar{x}) \leq 0 \ \forall x \in S$ where $p = y - \bar{x}$.

Let $q = \frac{1}{2}(y + \bar{x})$. Then



$p^T(x - q) = 0$ is a separating hyperplane, because

- $p^T(y - q) = p^T(y - \frac{1}{2}y - \frac{1}{2}\bar{x}) = \frac{1}{2}p^T(y - \bar{x})$
$$= \frac{1}{2}p^T p = \frac{1}{2}\|p\|^2 > 0$$

- $x \in S \Rightarrow p^T(x - q) = p^T(x - \bar{x} + \bar{x} - q) = \underbrace{p^T(x - \bar{x})}_{\leq 0} + p^T(\bar{x} - q)$
$$\leq p^T(\bar{x} - q) = p^T(\bar{x} - \tfrac{1}{2}y - \tfrac{1}{2}\bar{x}) = \tfrac{1}{2}p^T(\underbrace{\bar{x} - y}_{-p}) = -\tfrac{1}{2}\|p\|^2 < 0$$

#

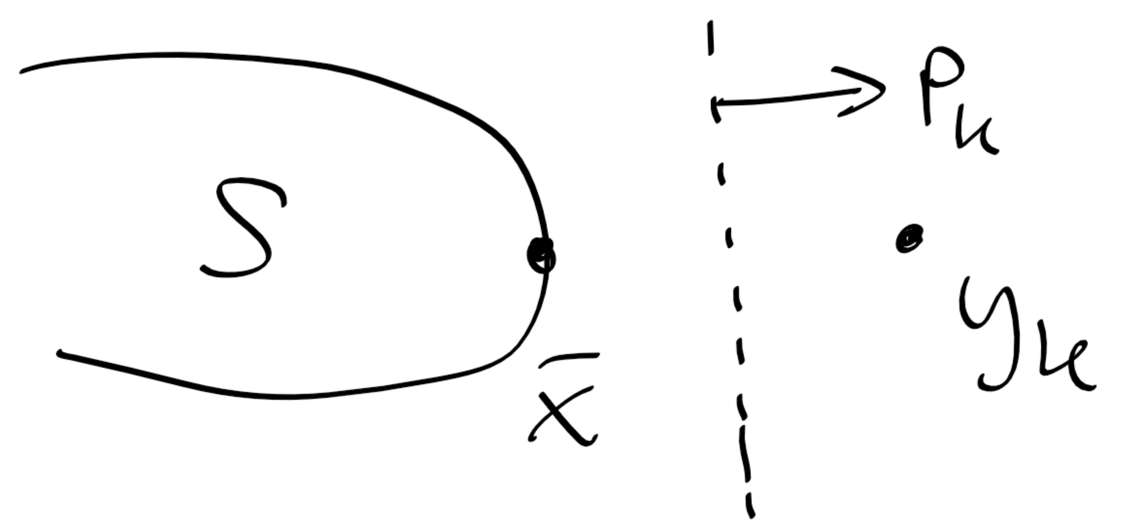
Thm 8: $\emptyset \neq S \subseteq \mathbb{R}^n$ convex and $\bar{x} \in \partial S \Longrightarrow$

$\exists$ support plane of S at $\bar{x}$.

Proof: $\bar{x} \in \partial S \Rightarrow \exists\, y_k \in B_{1/k}(\bar{x}) \cap \complement\, cl(S)$, $k \in \mathbb{N}$

$y_k \to \bar{x}$ as $k \to \infty$. For every k $\exists$ a separating

plane $p_k^T x = \alpha_k$ with $\|p_k\|=1$ eg. $\alpha_k = \frac{1}{2} p_k^T (y_k + \bar{x})$



(*) $p_k^T x < \alpha_k \quad \forall x \in S$

The sequence $\{p_k\}_1^\infty \subseteq \{x \in \mathbb{R}^n : \|x\| = 1\}$ compact, so Bolzano-Weierstraß' Thm gives a subsequence converges to p. Taking limit: $\alpha_k \to \frac{1}{2} p^T(\bar{x} + \bar{x}) = p^T \bar{x} =: \alpha$ and

(*) $\to$ $p^T x \leq \alpha \quad \forall x \in S$ and

$p^T x = \alpha$ is a support plane. #

4.4 Farkas' theorem

Consider two systems of inequalities where c $m\times 1$, a_i $n\times 1$ and

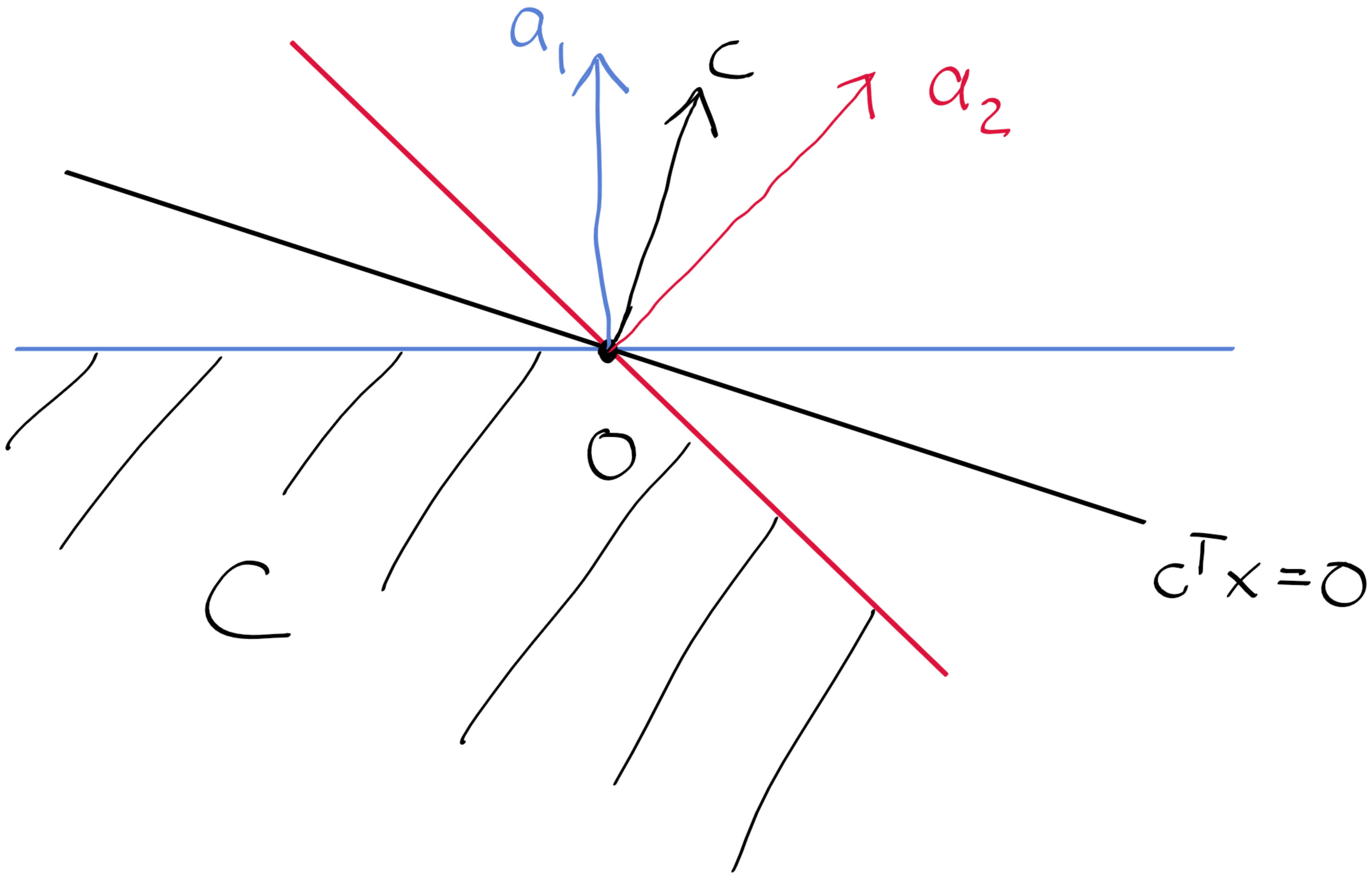
$$A = \begin{pmatrix} a_1^T \\ \vdots \\ a_m^T \end{pmatrix} \quad m\times n$$

$$(*) \quad \begin{cases} Ax \le 0 \\ c^T x > 0 \end{cases} \iff \begin{cases} a_1^T x \le 0 \\ \vdots \\ a_m^T x \le 0 \\ c^T x > 0 \end{cases}$$

$$(**) \quad \begin{cases} A^T y = c \\ y \geq 0 \end{cases} \iff \begin{cases} \sum_{i=1}^{m} y_i a_i = c \\ \text{all } y_i \geq 0 \end{cases} \overset{\text{def.}}{\iff}$$

c is a positive linear combin. of $a_1, \dots, a_m$

$C = \{x : Ax \le 0\}$ intersection of closed half-spaces containing the origin. Geometrically:



no solution of $(*)$

$\iff$ $\begin{cases} Ax \le 0 \\ c^T x \le 0 \end{cases}$ has a sol.

and $c^T x = 0$ is a support plane to C

$\iff$ c is a pos. linear combination of a_1, a_2

$\iff$ $(**)$ has a solution

<u>Farkas' Theorem</u>: Exactly one of $(*)$ and $(**)$ has a solution.

Proof: Assume that both have solutions. Then

$$y \geq 0 \quad \text{and} \quad Ax \leq 0 \quad \Rightarrow \quad \begin{cases} y_1 a_1^T x \leq 0 \\ \vdots \\ y_m a_m^T x \leq 0 \end{cases} \quad \overset{\text{add}}{\Rightarrow}$$

$$0 \geq \sum y_i a_i^T x = y^T A x = (A^T y)^T x = c^T x,$$ which contradicts $c^T x > 0$. Thus

$$(*) \text{ sol.} \Longrightarrow (**) \text{ has no sol.}$$

$$\left(\text{or } (**) \text{ sol} \Rightarrow (*) \text{ no sol}\right)$$

Assume $(**)$ has no solution. Define

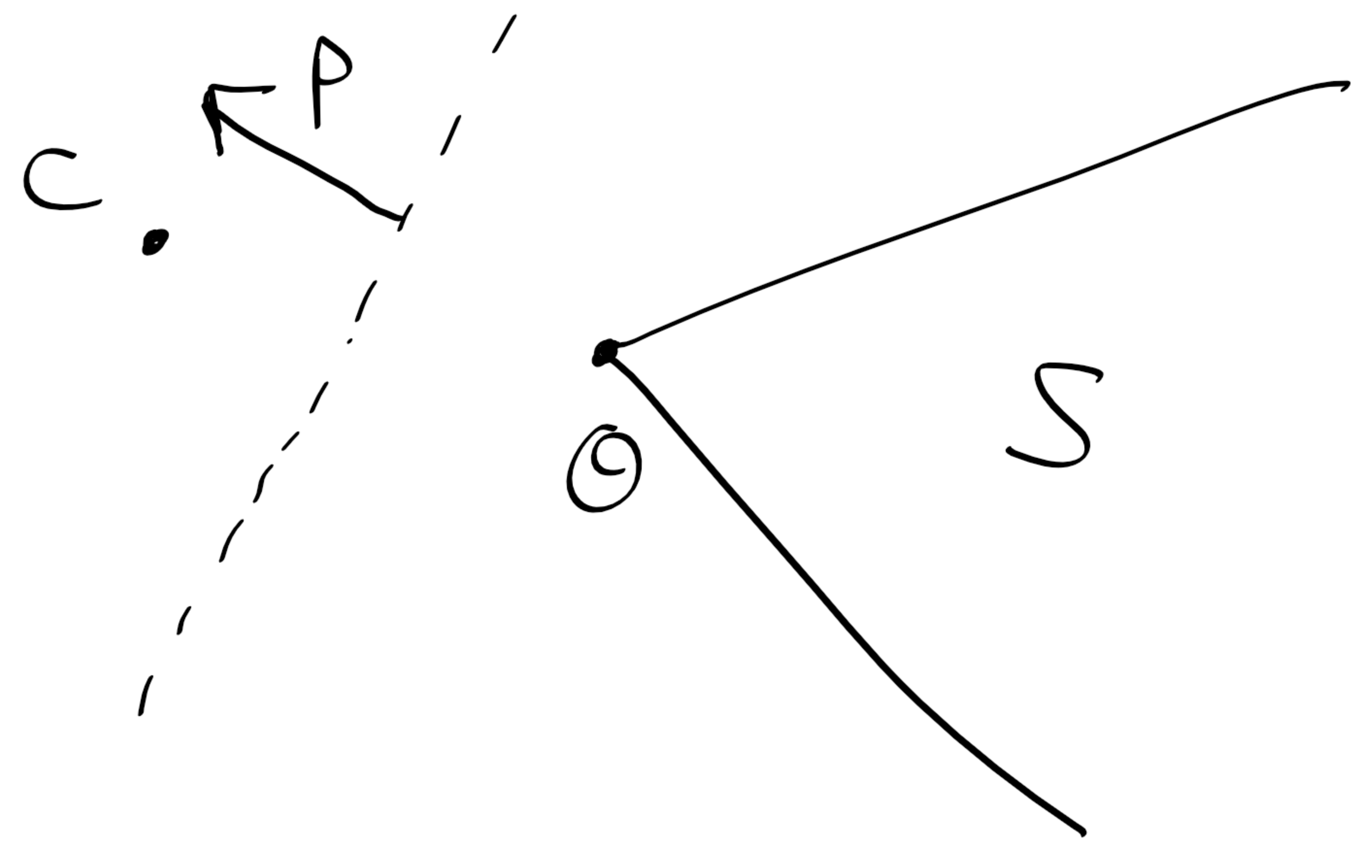
$$S = \left\{ x \in \mathbb{R}^n : x = A^T y = \sum y_i a_i, \text{ all } y_i \geq 0 \right\}$$

Convex set (exercise) and $c \notin S$.

There exists a plane $p^T x = \alpha$ separating $\{c\}$ and S such that

(i) $p^T c > \alpha$

(ii) $p^T x < \alpha \quad \forall x \in S$



Since $0 \in S$, (ii) gives $p^T 0 < \alpha \Leftrightarrow 0 < \alpha$

Hence, $p^T c > 0$ (one ineq. of $(*)$)

Now (ii) $\Leftrightarrow p^T (A^T y) < \alpha \quad \forall y \geq 0$

$$\Leftrightarrow (Ap)^T y < \alpha \quad \forall y \geq 0$$

Since $\alpha > 0$: $\quad Ap \leq 0$ (second ineq. of $(*)$)

(If $(Ap)_i > 0$, then set the other $y_j = 0$ $j \neq i$ and let $y_i \to \infty \Rightarrow (Ap)^T y \to \infty$)

Thus p solves $(*)$. #

Ex. Which half-spaces in $\mathbb{R}^3$ contain the points that satisfy

$$\begin{cases} x_1 + x_2 + x_3 \le 0 \\ x_1 + 2x_2 + 2x_3 \le 0 \end{cases} \iff Ax \le 0 \quad ?$$

Sol: We look for $c \neq 0$ so that $Ax \le 0 \Rightarrow c^T x \le 0$

Equivalently $Ax \le 0$ and $c^T x > 0$ has no sol.

Farkas' gives that the following has a solution:

$$A^T y = c \,, \quad y \ge 0 \iff$$

$$\begin{cases} y_1 + y_2 = c_1 \\ y_1 + 2y_2 = c_2 \\ y_1 + 2y_2 = c_3 \\ \quad y \ge 0 \end{cases} \iff \begin{cases} y_1 = 2c_1 - c_2 \ge 0 \\ y_2 = c_2 - c_1 \ge 0 \\ 0 = c_3 - c_2 \end{cases}$$

We choose $c_3 = 1$; then $c_2 = 1$ and $\frac{1}{2} \le c_1 \le 1$

Answer: The half-spaces are $c_1 x_1 + x_2 + x_3 \le 0$ with $\frac{1}{2} \le c_1 \le 1$