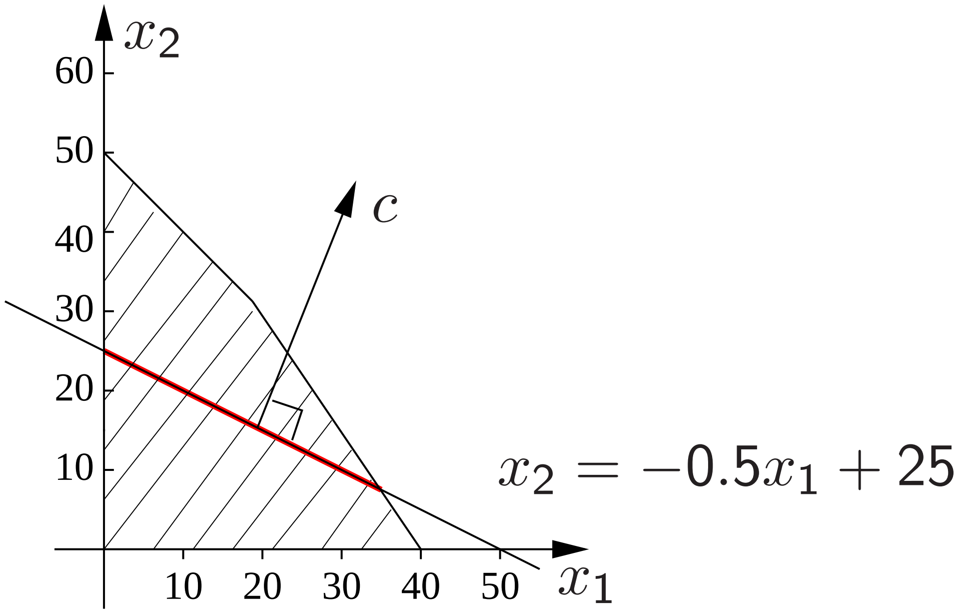
Lecture 2: The Simplex method

1. Repetition of the geometrical simplex method.
2. Linear programming problems on standard form.
3. The Simplex algorithm.
4. How to find an initial basic solution.

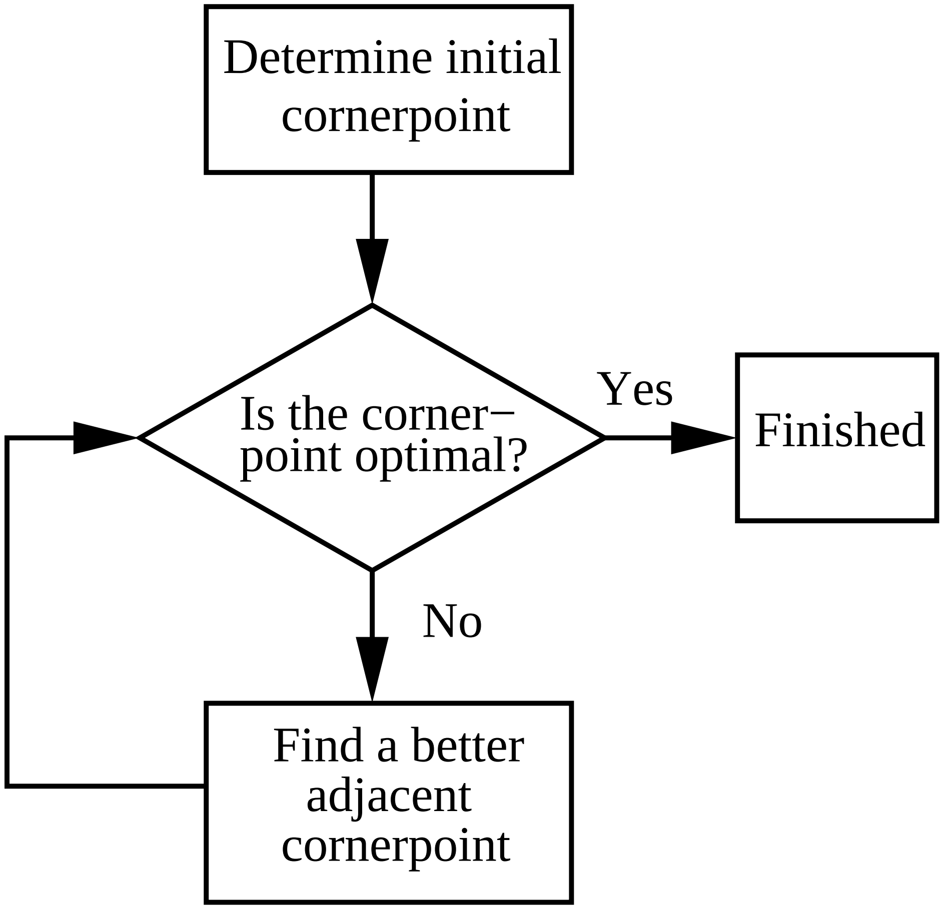
Repetition of the geometrical Simplex method

The product planning problem

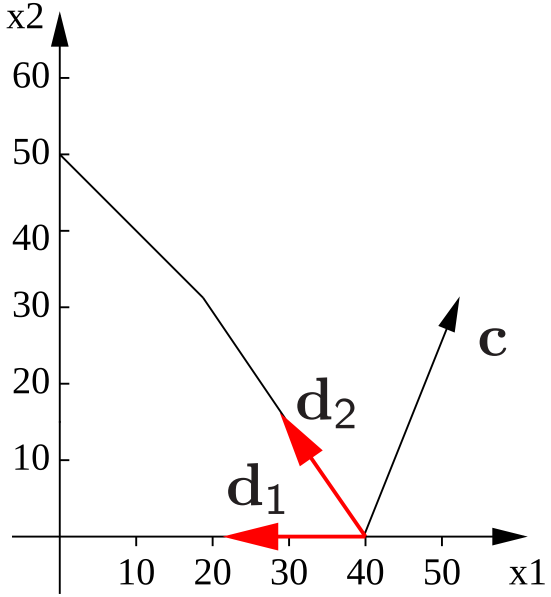
$$\begin{aligned} \text{maximize} \quad & 200x_1 + 400x_2 \\ \text{s.t.} \quad & \tfrac{1}{40}x_1 + \tfrac{1}{60}x_2 \leq 1 \\ & \tfrac{1}{50}x_1 + \tfrac{1}{50}x_2 \leq 1 \\ & x_k \geq 0, \;\; k = 1, 2 \end{aligned}$$



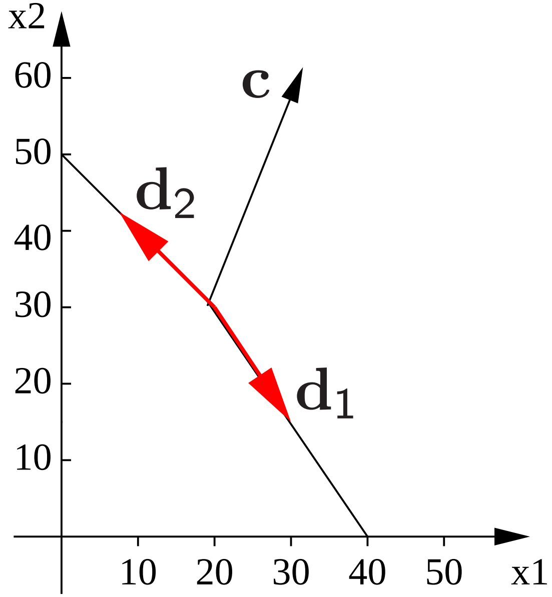
The idéa of the Simplex method is to iteratively search along edges to other corner points for a better objective function value.



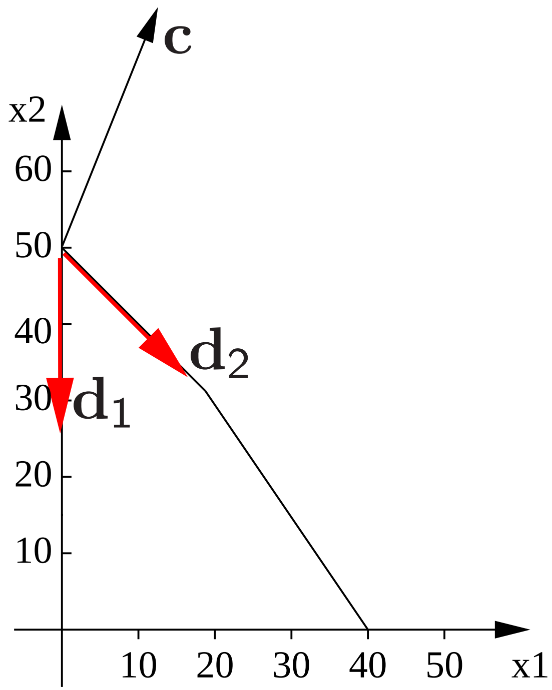
Searching with the geometrical simplex method



$\mathbf{c}^\mathsf{T}\mathbf{d}_1 < 0$
$\mathbf{c}^\mathsf{T}\mathbf{d}_2 > 0$
choose direction d_2



$\mathbf{c}^\mathsf{T}\mathbf{d}_1 < 0$
$\mathbf{c}^\mathsf{T}\mathbf{d}_2 > 0$
choose direction d_2



$\mathbf{c}^\mathsf{T}\mathbf{d}_1 < 0$
$\mathbf{c}^\mathsf{T}\mathbf{d}_2 < 0$
optimal CP

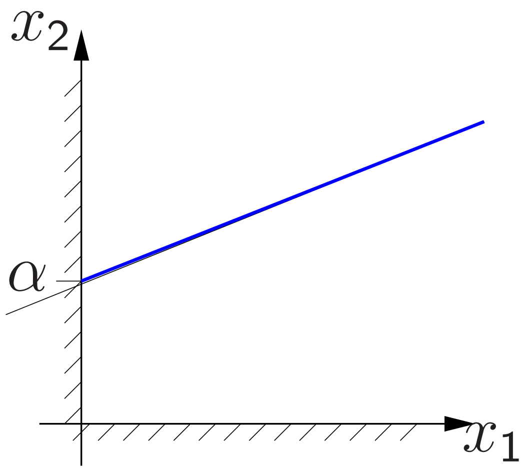
The Product planning example

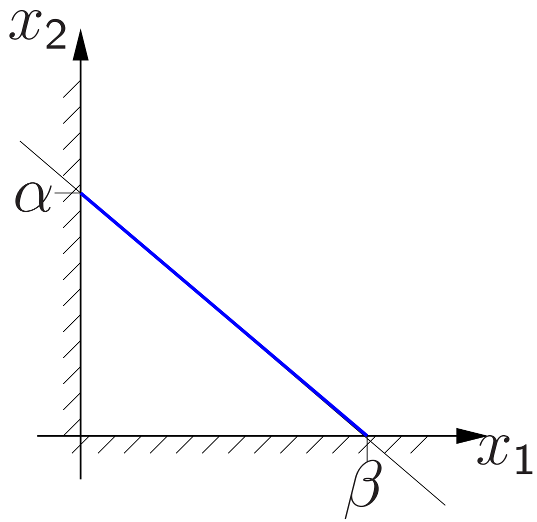
The Standard form for the product planning example.

$$
\begin{aligned}
\text{maximize} \quad & 200x_1 + 400x_2 & = -\text{minimize} \quad & -200x_1 - 400x_2 \\
\text{s.t.} \quad & \tfrac{1}{40}x_1 + \tfrac{1}{60}x_2 \leq 1 & \text{s.t.} \quad & \tfrac{1}{40}x_1 + \tfrac{1}{60}x_2 + x_3 = 1 \\
& \tfrac{1}{50}x_1 + \tfrac{1}{50}x_2 \leq 1 & & \tfrac{1}{50}x_1 + \tfrac{1}{50}x_2 + x_4 = 1 \\
& x_k \geq 0, \;\; k = 1, 2 & & x_k \geq 0, \;\; k = 1, 2, 3, 4.
\end{aligned}
$$

How are corner points represented in the standard form ?

Geometric interpretation for LP on standard form: 2D



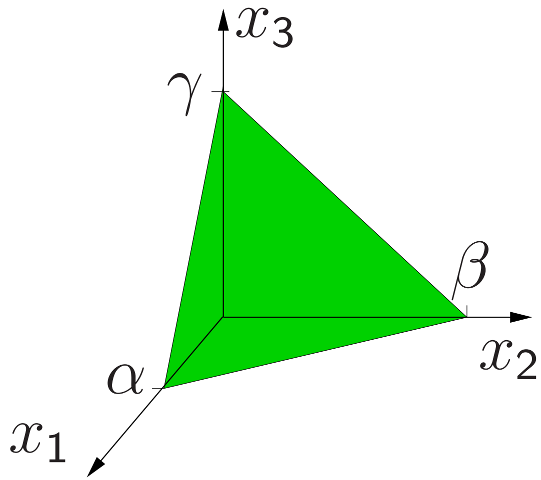


Constraint: $a_{11}x_1 + a_{12}x_2 = b_1$.

$\mathcal{F}$ is a semi-infinite line segment (left), or finite line segment (right).

The optimal $\hat{\mathbf{x}} = (0, a)$, $(\beta, 0)$ or the problem has no finite solution.

Geometric interpretation for LP on standard form: 3D



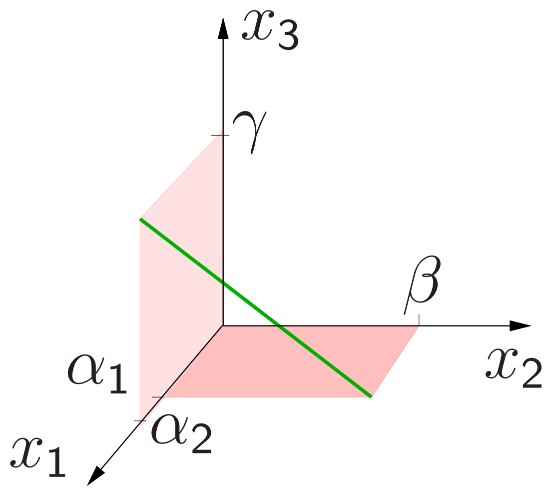
Constraint: $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$.

$\mathcal{F}$ is the intersection of a plane and the first quadrant. (green)

There are three corner points

$$\mathbf{x}^{(1)} = (\alpha, 0, 0), \quad \mathbf{x}^{(2)} = (0, \beta, 0), \quad \mathbf{x}^{(3)} = (0, 0, \gamma),$$

Geometric interpretation for LP on standard form: 3D



Constraints: $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1.$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2.$$

$\mathcal{F}$ is the intersection of a line and the first quadrant. (green)

There are two corner points

$$\mathbf{x}^{(1)} = (\alpha_1, \beta, 0), \quad \mathbf{x}^{(2)} = (\alpha_2, 0, \gamma),$$

Note: $\#$ nonzero elements in $\mathbf{x}^{(k)} = \#$ constraints, in these examples.

The Product planning example

The constraint in the standard form is $\mathbf{A}\mathbf{x} = \mathbf{b}$ where

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{40} & \frac{1}{60} & 1 & 0 \\ \frac{1}{50} & \frac{1}{50} & 0 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

It can be written as

$$\sum_{k=0}^{4} \mathbf{a}_k x_k = \mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \mathbf{a}_3 x_3 + \mathbf{a}_4 x_4 = b$$

If we let, *e.g.*, $x_2 = x_3 = 0$ then $\mathbf{a}_1 x_1 + \mathbf{a}_4 x_4 = \mathbf{A}_\beta \mathbf{x}_\beta = \mathbf{b}$, where

$$\mathbf{A}_\beta = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{40} & 0 \\ \frac{1}{50} & 1 \end{bmatrix}, \qquad \mathbf{x}_\beta = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix}$$

Then we determine

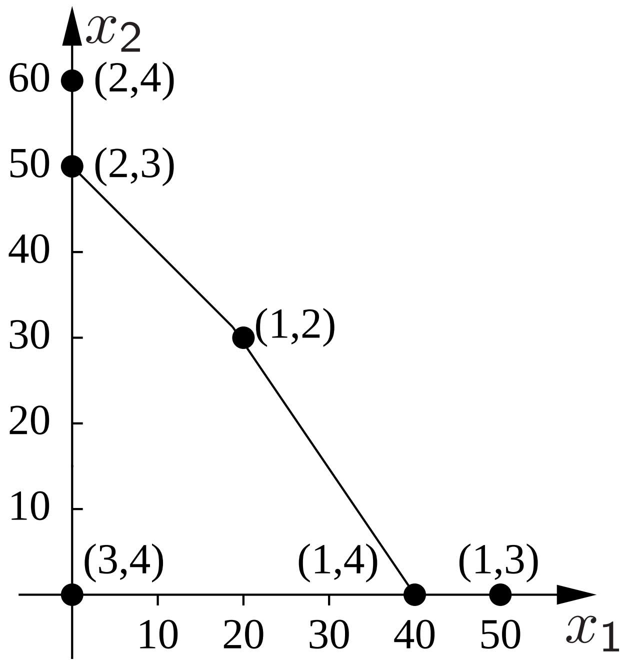
$$\mathbf{x}_\beta = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \mathbf{A}_\beta^{-1}\mathbf{b} = \begin{bmatrix} 40 \\ \frac{1}{5} \end{bmatrix}$$

These values of x_1 and x_4, together with $x_2 = x_3 = 0$, gives a feasible solution, *i.e.*, $\mathbf{Ax} = \mathbf{b}$, where $(x_1, x_2) = (40, 0)$.

<u>This is a corner-point</u>.

The other solutions corresponding to combinations of two columns in $\mathbf{A}$ are depicted in the table on the next slide.

Geometrical illustration of the basic solutions



β	$\mathbf{A}_\beta$	$\mathbf{x}_\beta$	(x_1, x_2)
$(3,4)$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
$(2,4)$	$\begin{bmatrix} \frac{1}{60} & 0 \\ \frac{1}{50} & 1 \end{bmatrix}$	$\begin{bmatrix} 60 \\ -\frac{1}{5} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 60 \end{bmatrix}$
$(1,4)$	$\begin{bmatrix} \frac{1}{40} & 0 \\ \frac{1}{50} & 1 \end{bmatrix}$	$\begin{bmatrix} 40 \\ \frac{1}{5} \end{bmatrix}$	$\begin{bmatrix} 40 \\ 0 \end{bmatrix}$
$(2,3)$	$\begin{bmatrix} \frac{1}{60} & 1 \\ \frac{1}{50} & 0 \end{bmatrix}$	$\begin{bmatrix} 50 \\ \frac{1}{6} \end{bmatrix}$	$\begin{bmatrix} 0 \\ 50 \end{bmatrix}$
$(1,3)$	$\begin{bmatrix} \frac{1}{40} & 1 \\ \frac{1}{50} & 0 \end{bmatrix}$	$\begin{bmatrix} 50 \\ -\frac{1}{4} \end{bmatrix}$	$\begin{bmatrix} 50 \\ 0 \end{bmatrix}$
$(1,2)$	$\begin{bmatrix} \frac{1}{40} & \frac{1}{60} \\ \frac{1}{50} & \frac{1}{60} \end{bmatrix}$	$\begin{bmatrix} 20 \\ 30 \end{bmatrix}$	$\begin{bmatrix} 20 \\ 30 \end{bmatrix}$

In general, it holds that cornerpoints corresponds to so called basic solutions.

The LP-problem in standard form

$$\begin{aligned} \text{minimize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j = b_i, \; i = 1, \ldots, m \\ & x_j \geq 0, \; j = 1, \ldots, n \end{aligned} \qquad \begin{aligned} = \text{minimize} \quad & \mathbf{c}^\mathsf{T} \mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_n \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}, \mathbf{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

The Simplex algorithm

The constraint $\mathbf{Ax} = \mathbf{b}$ can be written: $\sum_{k=1}^{n} \mathbf{a}_k x_k = \mathbf{b}$.

For given vectors of indices of basic variables and non-basic variables

$$\beta = (\beta_1, \ldots, \beta_m) \qquad\qquad \nu = (\nu_1, \ldots, \nu_l), \quad l = n - m$$

we define

$$\mathbf{A}_\beta = \begin{bmatrix} \mathbf{a}_{\beta_1} & \ldots & \mathbf{a}_{\beta_m} \end{bmatrix}, \qquad\qquad \mathbf{A}_\nu = \begin{bmatrix} \mathbf{a}_{\nu_1} & \ldots & \mathbf{a}_{\nu_l} \end{bmatrix},$$

$$\mathbf{c}_\beta = \begin{bmatrix} c_{\beta_1} \\ \vdots \\ c_{\beta_m} \end{bmatrix}, \quad \mathbf{x}_\beta = \begin{bmatrix} x_{\beta_1} \\ \vdots \\ x_{\beta_m} \end{bmatrix} \qquad\qquad \mathbf{c}_\nu = \begin{bmatrix} c_{\nu_1} \\ \vdots \\ c_{\nu_l} \end{bmatrix}, \quad \mathbf{x}_\nu = \begin{bmatrix} x_{\nu_1} \\ \vdots \\ x_{\nu_l} \end{bmatrix}$$

Then $\mathbf{Ax} = \mathbf{A}_\beta \mathbf{x}_\beta + \mathbf{A}_\nu \mathbf{x}_\nu = \mathbf{b}$ and $\mathbf{c}^T \mathbf{x} = \mathbf{c}_\beta^T \mathbf{x}_\beta + \mathbf{c}_\nu^T \mathbf{x}_\nu$.

Definition 4.4: Basic solutions

Suppose β is a basic index tuple.

1. A *basic solution* corresponding to β is a solution to $\mathbf{Ax} = \mathbf{b}$ such that $\mathbf{A}_\beta \mathbf{x}_\beta = \mathbf{b}$ and $\mathbf{x}_\nu = 0$.

2. A *basic feasible solution* corresponding to β is a basic solution $\mathbf{x}$ such that $\mathbf{x}_\beta \geq 0$.

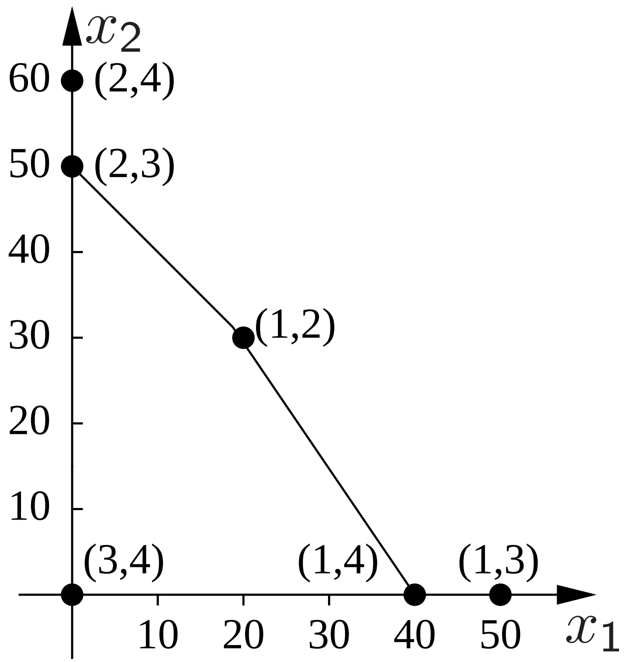
3. A BFS is called non-degenerated if $\mathbf{x}_\beta > 0$

4. A BFS is called degenerated if $\mathbf{x}_{\beta_k} = 0$ for some index β_k.

The basic solution corresponding to β is given by

$$\mathbf{A}_\beta \mathbf{x}_\beta = \mathbf{b} \ \Rightarrow \mathbf{x}_\beta = \mathbf{A}_\beta^{-1} \mathbf{b}$$

The Product planning example



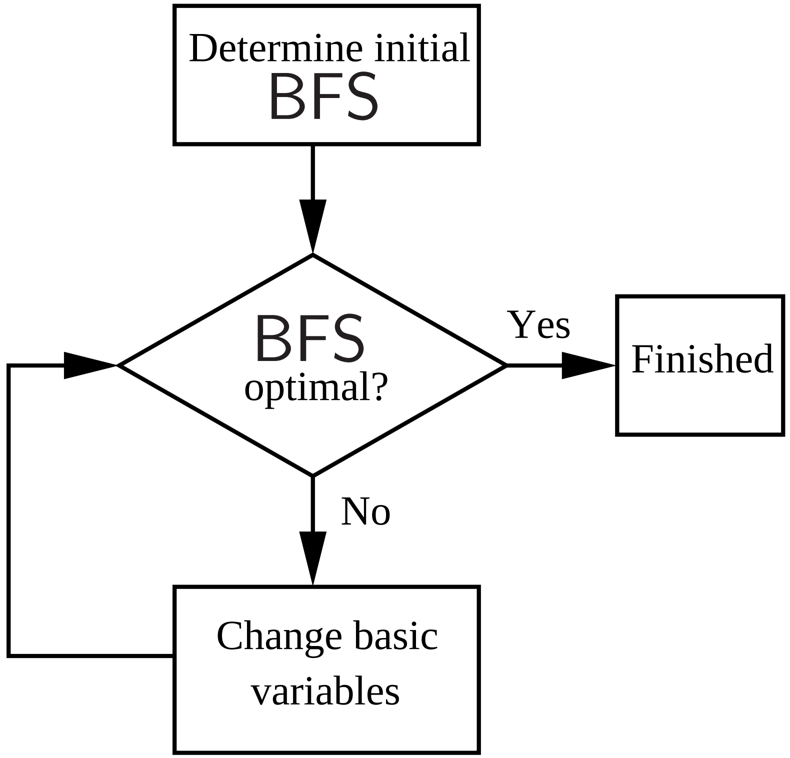
β	$\mathbf{A}_\beta$	$\mathbf{x}_\beta$	ν	$\mathbf{x}_\nu$
$(3,4)$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$	$(1,2)$	$\mathbf{0}$
$(2,4)$	$\begin{bmatrix} \frac{1}{60} & 0 \\ \frac{1}{50} & 1 \end{bmatrix}$	$\begin{bmatrix} 60 \\ -\frac{1}{5} \end{bmatrix}$	$(1,3)$	$\mathbf{0}$
$(1,4)$	$\begin{bmatrix} \frac{1}{40} & 0 \\ \frac{1}{50} & 1 \end{bmatrix}$	$\begin{bmatrix} 40 \\ \frac{1}{5} \end{bmatrix}$	$(2,3)$	$\mathbf{0}$
$(2,3)$	$\begin{bmatrix} \frac{1}{60} & 1 \\ \frac{1}{50} & 0 \end{bmatrix}$	$\begin{bmatrix} 50 \\ \frac{1}{6} \end{bmatrix}$	$(1,4)$	$\mathbf{0}$
$(1,3)$	$\begin{bmatrix} \frac{1}{40} & 1 \\ \frac{1}{50} & 0 \end{bmatrix}$	$\begin{bmatrix} 50 \\ -\frac{1}{4} \end{bmatrix}$	$(2,4)$	$\mathbf{0}$
$(1,2)$	$\begin{bmatrix} \frac{1}{40} & \frac{1}{60} \\ \frac{1}{50} & \frac{1}{60} \end{bmatrix}$	$\begin{bmatrix} 20 \\ 30 \end{bmatrix}$	$(3,4)$	$\mathbf{0}$

Which basic solutions are feasible ?

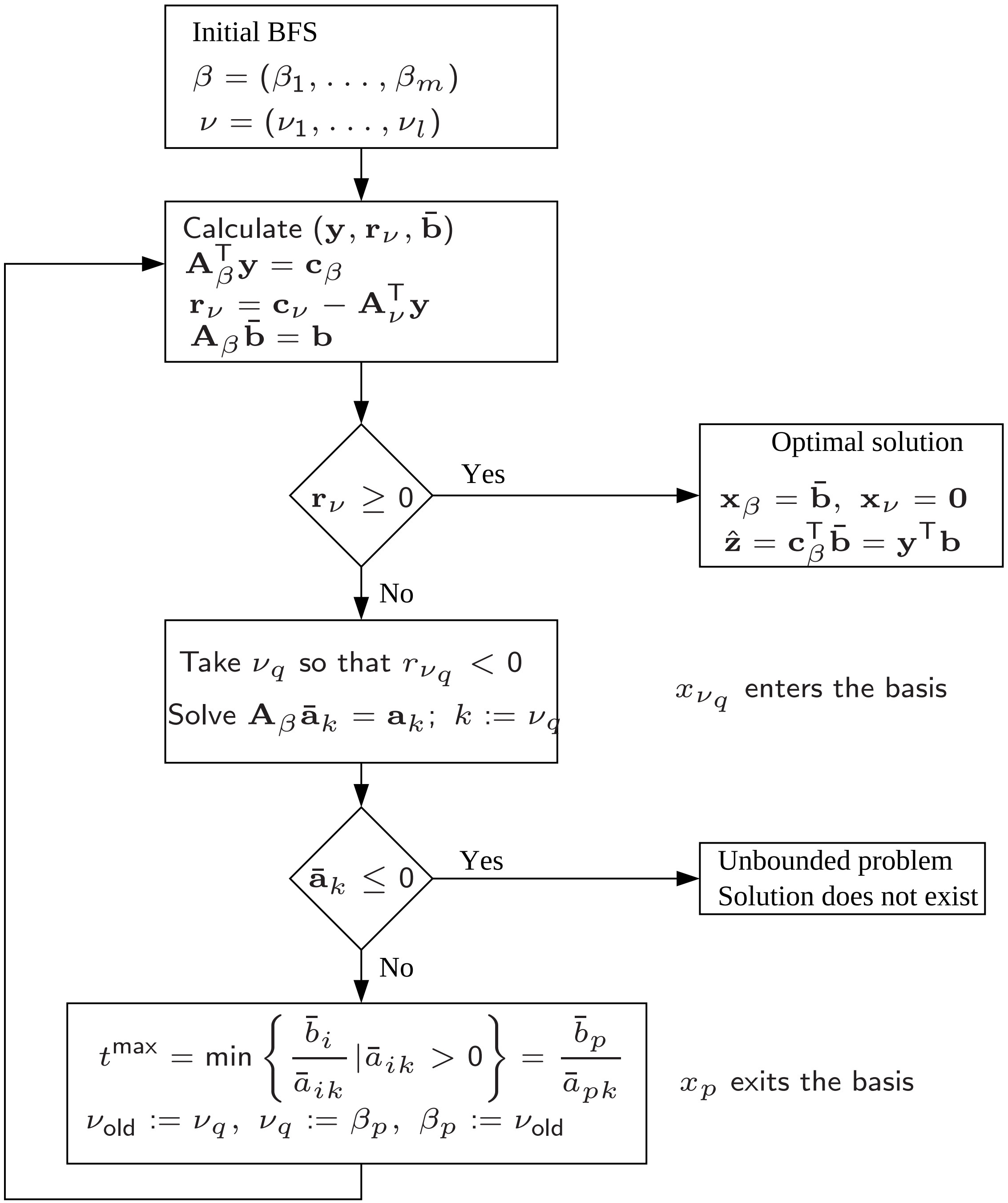
Compare with the geometrical interpretation.

Theorem 4.8 *If a linear program in standard form has a finite optimal solution, then it has an optimal basic feasible solution.*

The Theorem motivates the following Simplex algorithm



The different blocks can be implemented using linear algebra.


Initial BFS
$\beta = (\beta_1, \ldots, \beta_m)$
$\nu = (\nu_1, \ldots, \nu_l)$
Calculate $(\mathbf{y}, \mathbf{r}_\nu, \bar{\mathbf{b}})$
$\mathbf{A}_\beta^\mathsf{T} \mathbf{y} = \mathbf{c}_\beta$
$\mathbf{r}_\nu = \mathbf{c}_\nu - \mathbf{A}_\nu^\mathsf{T} \mathbf{y}$
$\mathbf{A}_\beta \bar{\mathbf{b}} = \mathbf{b}$
$\mathbf{r}_\nu \geq 0$
Yes
Optimal solution
$\mathbf{x}_\beta = \bar{\mathbf{b}}, \ \mathbf{x}_\nu = \mathbf{0}$
$\hat{\mathbf{z}} = \mathbf{c}_\beta^\mathsf{T} \bar{\mathbf{b}} = \mathbf{y}^\mathsf{T} \mathbf{b}$
No
Take ν_q so that $r_{\nu_q} < 0$
Solve $\mathbf{A}_\beta \bar{\mathbf{a}}_k = \mathbf{a}_k; \ k := \nu_q$
x_{ν_q} enters the basis
$\bar{\mathbf{a}}_k \leq 0$
Yes
Unbounded problem
Solution does not exist
No
$t^{\max} = \min \left\{ \frac{\bar{b}_i}{\bar{a}_{ik}} \middle| \bar{a}_{ik} > 0 \right\} = \frac{\bar{b}_p}{\bar{a}_{pk}}$
$\nu_{\text{old}} := \nu_q, \ \nu_q := \beta_p, \ \beta_p := \nu_{\text{old}}$
x_p exits the basis

Simplex applied on the product planning example

The standard form for the product planning example.

$$\begin{bmatrix} \text{minimize} & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0 \end{bmatrix} \quad \text{where} \quad \begin{matrix} \mathbf{c}^T = \begin{bmatrix} -200 & -400 & 0 & 0 \end{bmatrix}, \\ \mathbf{A} = \begin{bmatrix} \frac{1}{40} & \frac{1}{60} & 1 & 0 \\ \frac{1}{50} & \frac{1}{50} & 0 & 1 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{matrix}$$

We want to solve this using simplex.

When we introduce the slack variables it can be shown that if we take these as starting basic variables they form a BFS. (Given that $\mathbf{b} \geq 0$)

Simplex

Let $\beta = \{3, 4\}$ and $\nu = \{1, 2\}$,

$$\mathbf{A}_\beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{A}_\nu = \begin{bmatrix} 1/40 & 1/60 \\ 1/50 & 1/50 \end{bmatrix}$$

$$\mathbf{c}_\beta^T = \begin{bmatrix} 0 & 0 \end{bmatrix}, \qquad \mathbf{c}_\nu^T = \begin{bmatrix} -200 & -400 \end{bmatrix}$$

The basic solution $\mathbf{x}_\beta = \bar{\mathbf{b}} = \mathbf{A}_\beta^{-1}\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is feasible.

The simplex multiplicators are given by the equation $\mathbf{A}_\beta^T y = \mathbf{c}_\beta$ and reduced costs are given by $\mathbf{r}_\nu^T = \mathbf{c}_\nu^T - y^T \mathbf{A}_\nu$, i.e.

$$y = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ and } \mathbf{r}_\nu^T = \begin{bmatrix} -200 & -400 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1/40 & 1/60 \\ 1/50 & 1/50 \end{bmatrix} = \begin{bmatrix} -200 & -400 \end{bmatrix}.$$

Simplex

Since the reduced costs are negative, the current BFS can not be optimal. $\mathbf{r}_{\nu_2}$ is the smallest, so let $\mathbf{x}_{\nu_2} = \mathbf{x}_2$ be a new basic variable.

Which variable should exit the base ?

$\bar{\mathbf{b}}$ is determined from $\mathbf{A}_\beta \bar{\mathbf{b}} = \mathbf{b}$, i.e. $\bar{\mathbf{b}} = \mathbf{A}_\beta^{-1}\mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\bar{\mathbf{a}}_2$ is determined from $\mathbf{A}_\beta \bar{\mathbf{a}}_2 = \mathbf{a}_2$, i.e. $\bar{\mathbf{a}}_2 = \mathbf{A}_\beta^{-1}\mathbf{a}_2 = \begin{bmatrix} 1/60 \\ 1/50 \end{bmatrix}$

Now x_2 should be made as large as possible while $\mathbf{x}_\beta$ is non-negative:

$$\mathbf{x}_\beta = \bar{\mathbf{b}} - \bar{\mathbf{a}}_2 x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/60 \\ 1/50 \end{bmatrix} x_2 \geq 0$$

The largest possible value of x_2 is 50, whereby $x_{\beta_2} = x_4 = 0$.

Therefore, x_4 should be replaced by x_2 in the next basic solution.

Simplex: Iteration 2

Let $\beta = \{3, 2\}$ and $\nu = \{1, 4\}$,

$$\mathbf{A}_\beta = \begin{bmatrix} 1 & 1/60 \\ 0 & 1/50 \end{bmatrix}, \qquad \mathbf{A}_\nu = \begin{bmatrix} 1/40 & 0 \\ 1/50 & 1 \end{bmatrix}$$

$$\mathbf{c}_\beta^T = \begin{bmatrix} 0 & -400 \end{bmatrix}, \qquad \mathbf{c}_\nu^T = \begin{bmatrix} -200 & 0 \end{bmatrix}$$

The basic solution $\mathbf{x}_\beta = \bar{\mathbf{b}} = \mathbf{A}_\beta^{-1}\mathbf{b} = \begin{bmatrix} 1 & -5/6 \\ 0 & 50 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/6 \\ 50 \end{bmatrix}$ is then feasible.

The simplex multiplicators are given by the equation $\mathbf{A}_\beta^T y = \mathbf{c}_\beta$

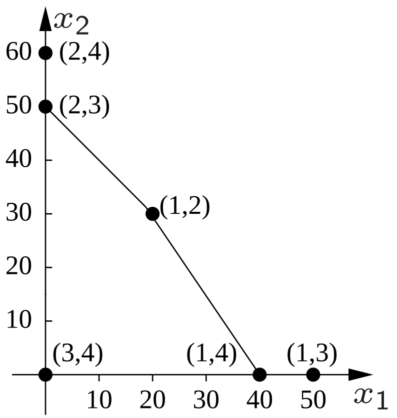
$$y = \mathbf{A}_\beta^{-T}\mathbf{c}_\beta = \begin{bmatrix} 1 & -5/6 \\ 0 & 50 \end{bmatrix}^T \begin{bmatrix} 0 \\ -400 \end{bmatrix} = \begin{bmatrix} 0 \\ -20000 \end{bmatrix}.$$

Simplex: Iteration 2

Reduced costs are given by $\mathbf{r}_\nu^T = \mathbf{c}_\nu^T - y^T \mathbf{A}_\nu$, i.e.

$$\mathbf{r}_\nu^T = \begin{bmatrix} -200 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -20000 \end{bmatrix}^T \begin{bmatrix} 1/40 & 0 \\ 1/50 & 1 \end{bmatrix} = \begin{bmatrix} 200 \; 20000 \end{bmatrix}.$$

Since the reduced costs are positive the current BFS is optimal.



We started at the origin with basic variables (3,4) and then switched to the basic variables (3,2), i.e. the point (0,50), which we argued was optimal previously using geometric methods.

Initial BFS

Often it is non-trivial to find an initial BFS. Then it is possible to solve the problem with a two-phase method. Here we assume that $\mathbf{b} \geq \mathbf{0}$.

Phase 1: Solve the LP-problem

$$\begin{aligned} \text{minimize} \quad & \mathbf{e}^\mathsf{T}\mathbf{v} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} + \mathbf{I}\mathbf{v} = \mathbf{b} \\ & \mathbf{x} \geq 0, \ \mathbf{v} \geq 0 \end{aligned}$$

where $\mathbf{I}$ is the unit matrix and $\mathbf{e} = \begin{bmatrix} 1 & \dots & 1 \end{bmatrix}^\mathsf{T}$. Let the initial BFS have a basis that correspond to $\mathbf{v}$. If the optimal solution $(\hat{\mathbf{v}}, \hat{\mathbf{x}})$ is such that $\hat{\mathbf{v}} = \mathbf{0}$, then $\hat{\mathbf{x}}$ is a BFS to the original problem.

Phase 2: Solve the original problem using the BFS $\hat{\mathbf{x}}$ found in phase 1.

Reading instructions

- Chapters 4 and 5 in the book.