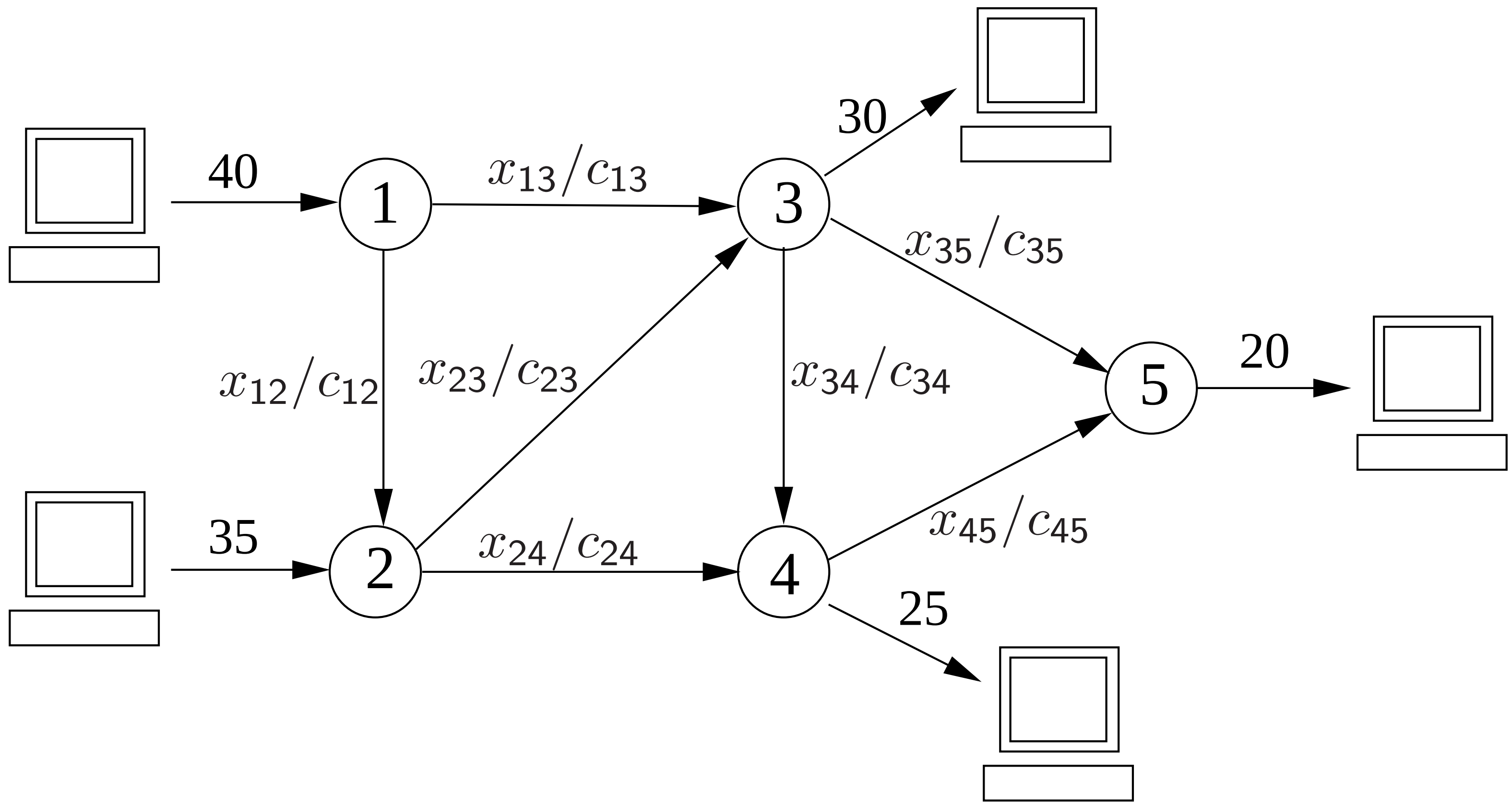
Lecture: Optimization of network flows

1. Minimum cost flow problems in networks
2. Modelling and graph theory.
3. The Simplex method.

The minimum cost flow problem in networks



Data is sent from servers in nodes 1 and 2 to terminals in nodes 3, 4, 5.

The cost for traffic in the link between node i and j is c_{ij} SEK/Kbyte.

We want to minimize the total cost for the data traffic

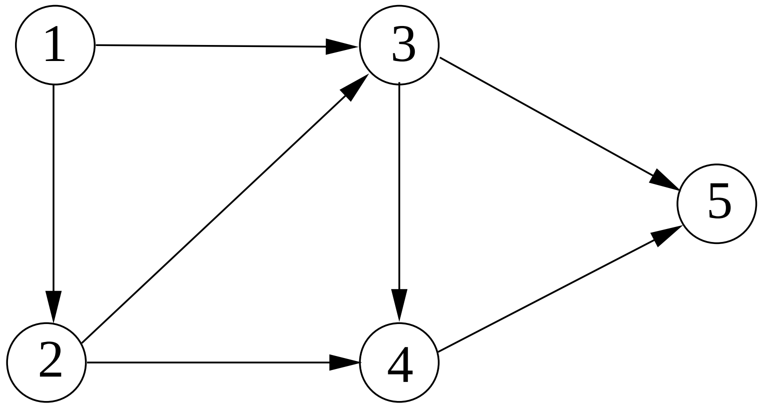
The optimization problem formulation

$$\begin{aligned}
\text{minimize} \quad & \sum_{\text{all arcs}} c_{ij}x_{ij} \\
\text{s.t.} \quad & x_{12} + x_{13} = 40 \\
& -x_{12} + x_{23} + x_{24} = 35 \\
& -x_{13} - x_{23} + x_{34} + x_{35} = -30 \\
& -x_{24} - x_{34} + x_{45} = -25 \\
& -x_{35} - x_{45} = -20 \\
& x_{ij} \geq 0, \ \text{all flows in the links}
\end{aligned}$$

We will address the following questions

- Modelling of the network flow problem
  - Graphs, trees, cycles, spanning trees.
- How can the particular structure of the graph be used in the Simplex method.

Modelling and graph theory



A directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{B})$ consists of a set of nodes $\mathcal{N} = \{1, \ldots, m\}$ and a set of arcs $\mathcal{B}$

- The arc from node i to node j is denoted (i, j).
- We differentiate from the arc (i, j) and the arc (j, i).

In the graph above

$$\mathcal{N} = \{1, \ldots, 5\}$$
$$\mathcal{B} = \{(1,2), (1,3), (2,3), (2,4), (3,4), (3,5), (4,5)\}$$

If we arrange the arcs in $\mathcal{B}$ in some order, *e.g.*

$$\mathcal{B} = \{\underset{\rho_1}{(1,2)}, \underset{\rho_2}{(1,3)}, \underset{\rho_3}{(2,3)}, \underset{\rho_4}{(2,4)}, \underset{\rho_5}{(3,4)}, \underset{\rho_6}{(3,5)}, \underset{\rho_7}{(4,5)}\} \tag{1}$$

then the incidence matrix $\tilde{\mathbf{A}} \in \mathbf{R}^{m \times n}$ of the graph is defined as

$$a_{ij} = \begin{cases} 1, & \text{arc } \rho_j \text{ starting in node } i \\ -1, & \text{arc } \rho_j \text{ ending in node } i \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

It is easy to see that $\mathbf{e}^\mathsf{T}\tilde{\mathbf{A}} = \mathbf{0}$, where $\mathbf{e}^\mathsf{T} = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}$. The rows are thus linearly dependent and it is possible to eliminate the last row to obtain the reduced incidence matrix $\mathbf{A}$.

In our example we have

$$\tilde{\mathbf{A}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 \end{bmatrix}$$

Path and cycle

- A *path* from node s to node t in a directed graph is a connected sequence of arcs $\rho_1, \ldots, \rho_N$ where

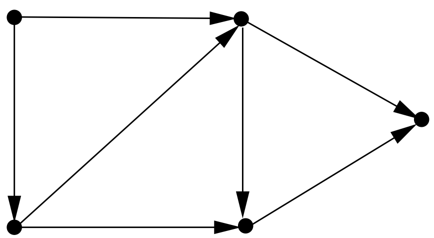
$$\rho_k = (i_{k-1}, i_k), \text{ or } \rho_k = (i_k, i_{k-1}), \text{ and } i_0 = s, \ i_N = t$$

- A *directed path* from node s to node t in a directed graph is a connected sequence of arcs $\rho_1, \ldots, \rho_N$ where

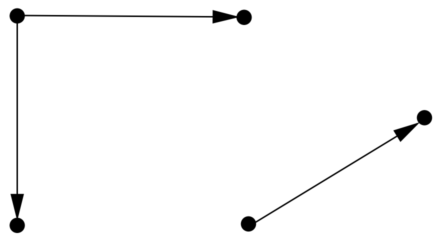
$$\rho_k = (i_{k-1}, i_k), \text{ and } i_0 = s, \ i_N = t$$

- A (directed) *cycle* is a (directed) path from a node to itself.

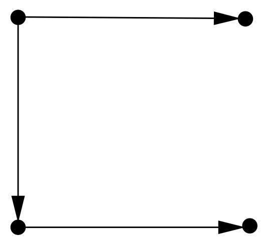
- A graph is called *connected* if there exists a path between each pair of nodes.
- A graph is called *acyclic* if it contains no cycles.
- A *tree* is formed by a connected subset of the graph that contains no cycles (*i.e.*, it is acyclic).
- In a graph with n nodes a tree with $n-1$ arcs form a *spanning tree*.



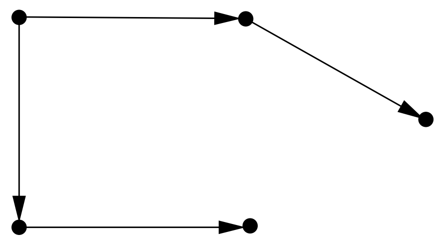
Connected graph
NOT acyclic



Acyclic graph
NOT connected

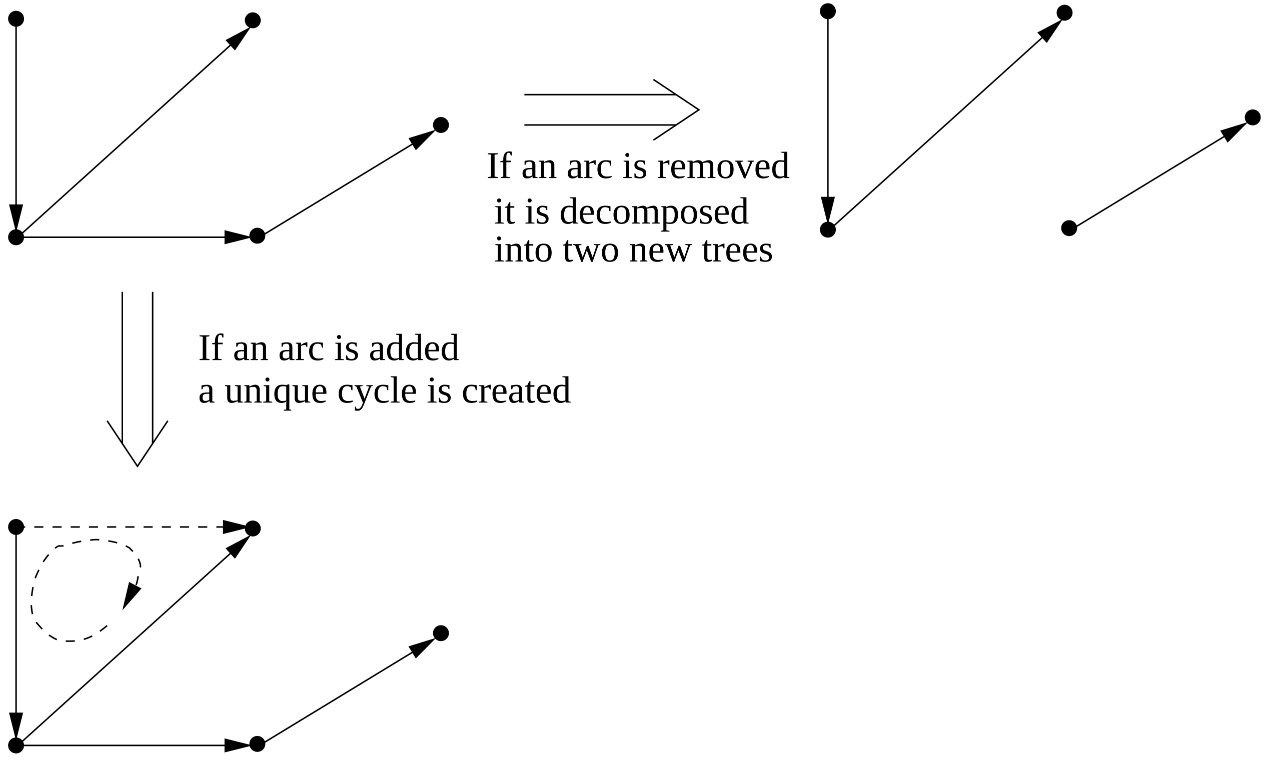


Tree

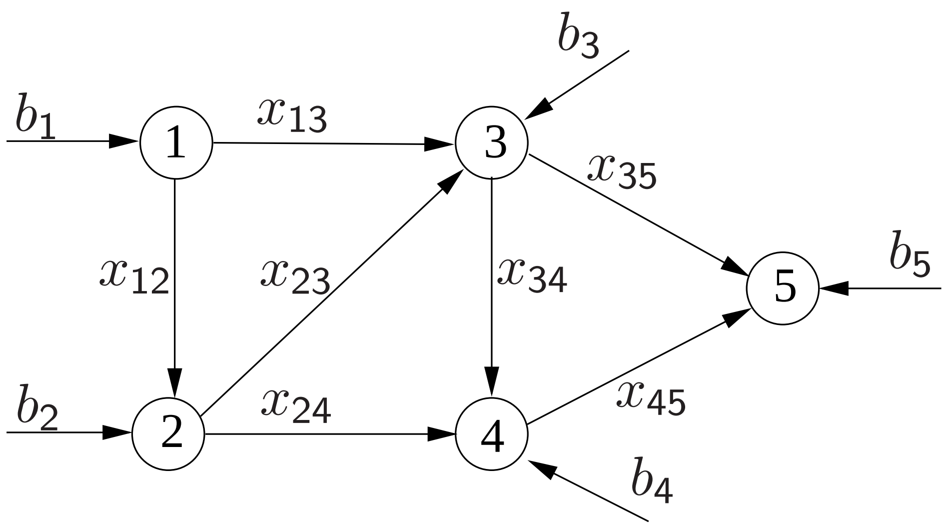


Spanning Tree

- If an arc is added to a spanning tree, then a unique cycle is created.
- If an arc is removed from a spanning tree, then the tree is decomposed into two new trees.



Balance of flow



- x_{ij} denotes the flow (data traffic, oil, e.t.c.) in arc (i, j)
  - If the flow x_{ij} goes in the direction $i \rightarrow j$, then $x_{ij} \geq 0$. Otherwise $x_{ij} \leq 0$.
- b_i denotes external flow in/out to node i. Flow in if $b_i \geq 0$, and flow out if $b_i \leq 0$.

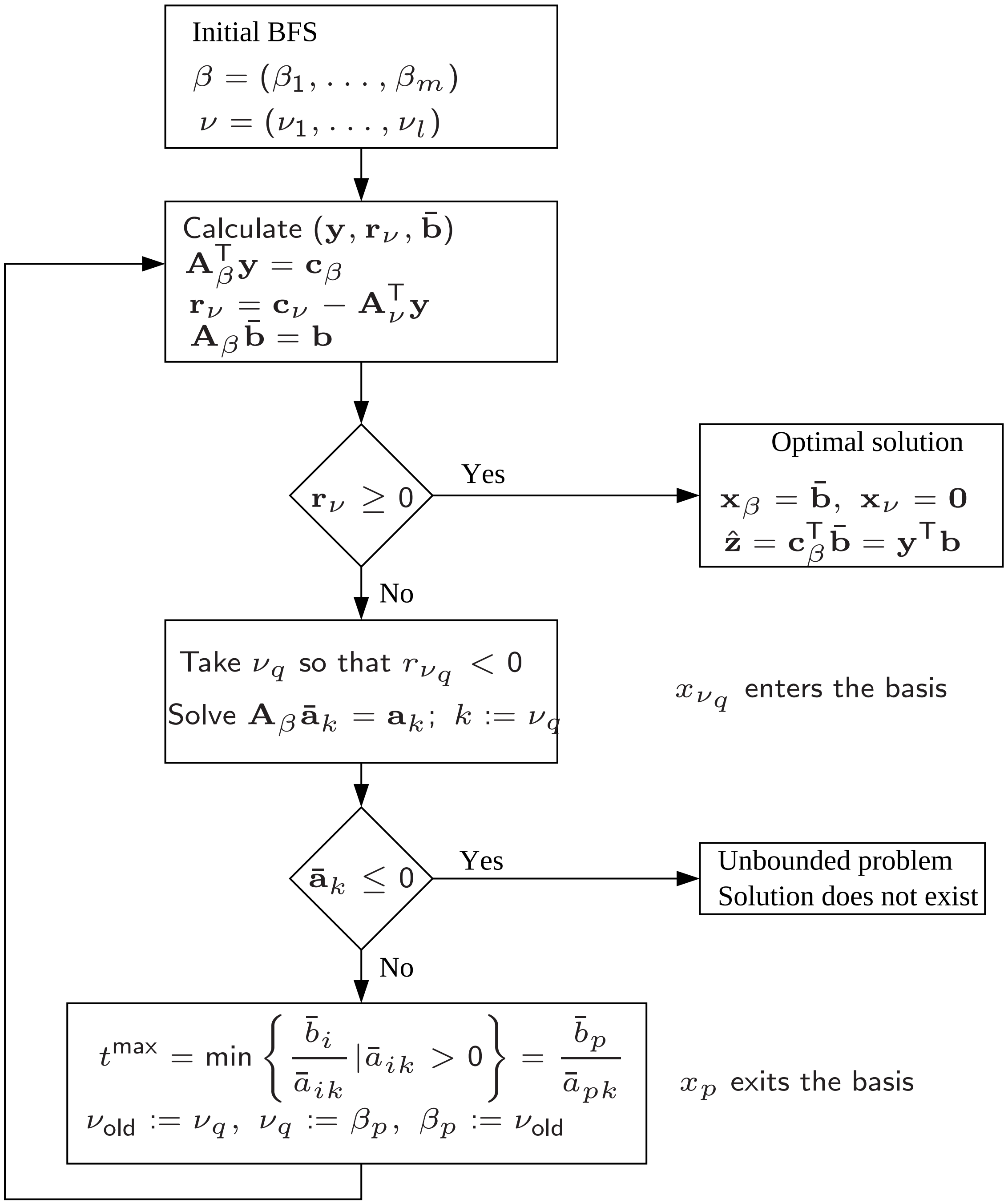
Flow balance in node i (flow out = flow in):

$$\sum_{j=1}^{n} a_{ij}x_{ij} = b_i,\ \ i = 1, \dots, n \quad \Leftrightarrow \quad \tilde{\mathbf{A}}\mathbf{x} = \tilde{\mathbf{b}}$$

In the above balance equation, $\tilde{\mathbf{A}}$ is the incidence matrix of the graph defined in equation (2) and $\mathbf{x}$ is a column vector with the flows x_{ij} sorted in the same order as the arcs in equation (1).

Finally, $\tilde{\mathbf{b}} = \begin{bmatrix} b_1 & \dots & b_n \end{bmatrix}^{\mathsf{T}}$.

The balance equation has solutions iff $\mathbf{e}^{\mathsf{T}}\mathbf{b} = \sum_{i=1}^{n} b_i = 0$. With this assumption we can reduce the flow balance equation by eliminating the last row (*i.e.*, the n:th balance equation is redundant). We arrive at the equation system $\mathbf{A}\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} b_1 & \dots & b_{n-1} \end{bmatrix}^{\mathsf{T}}$.


Initial BFS
$\beta = (\beta_1, \ldots, \beta_m)$
$\nu = (\nu_1, \ldots, \nu_l)$
Calculate $(\mathbf{y}, \mathbf{r}_\nu, \bar{\mathbf{b}})$
$\mathbf{A}_\beta^\mathsf{T} \mathbf{y} = \mathbf{c}_\beta$
$\mathbf{r}_\nu = \mathbf{c}_\nu - \mathbf{A}_\nu^\mathsf{T} \mathbf{y}$
$\mathbf{A}_\beta \bar{\mathbf{b}} = \mathbf{b}$
$\mathbf{r}_\nu \geq 0$
Yes
Optimal solution
$\mathbf{x}_\beta = \bar{\mathbf{b}}, \ \mathbf{x}_\nu = \mathbf{0}$
$\hat{\mathbf{z}} = \mathbf{c}_\beta^\mathsf{T} \bar{\mathbf{b}} = \mathbf{y}^\mathsf{T} \mathbf{b}$
No
Take ν_q so that $r_{\nu_q} < 0$
Solve $\mathbf{A}_\beta \bar{\mathbf{a}}_k = \mathbf{a}_k; \ k := \nu_q$
x_{ν_q} enters the basis
$\bar{\mathbf{a}}_k \leq 0$
Yes
Unbounded problem
Solution does not exist
No
$t^{\max} = \min \left\{ \frac{\bar{b}_i}{\bar{a}_{ik}} \,|\, \bar{a}_{ik} > 0 \right\} = \frac{\bar{b}_p}{\bar{a}_{pk}}$
$\nu_{\text{old}} := \nu_q, \ \nu_q := \beta_p, \ \beta_p := \nu_{\text{old}}$
x_p exits the basis

For network flow problems the simplex algorithm is simplified in the following way:

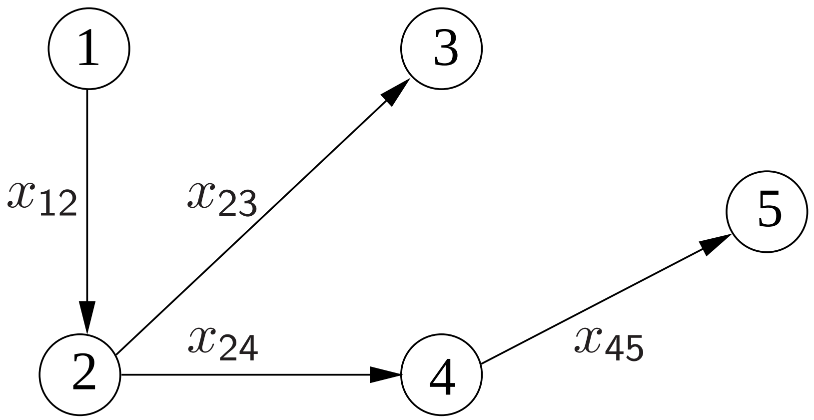
1. Arcs corresponding to a basic index vector (β) corresponds to a spanning tree
2. The values of the basic variables (determined by $\mathbf{A}_\beta \mathbf{x}_\beta = \mathbf{b}$) follows simply by balancing the flows in the spanning tree.
3. The Simplex multiplicators given by $\mathbf{A}_\beta^\mathsf{T} \mathbf{y} = \mathbf{c}_\beta$ are easily determined by the spanning tree.
4. The quotient test is performed by studying the flow in a cycle.
5. The pivot step, (change of basis) is done by exchanging an arc in the spanning tree.

We illustrate this on the example studied earlier.

Basic matrices corresponds to spanning trees

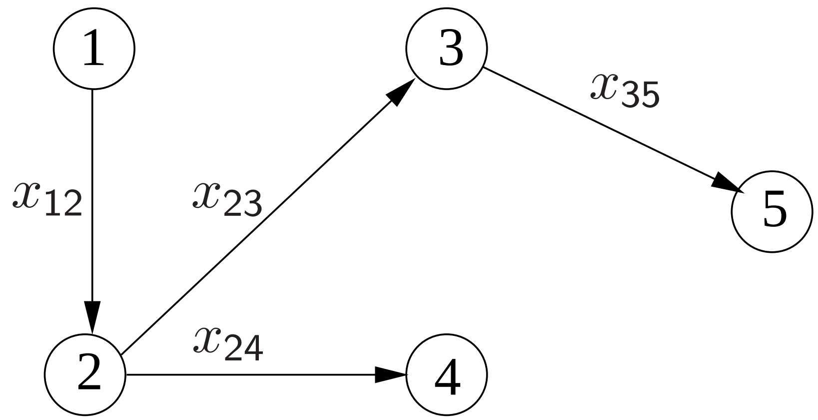
Theorem *$m-1$ columns from the $(m-1)\times n$ matrix $\mathbf{A}$ in a minimum cost flow problem are linearly independent iff the corresponding $m-1$ arcs form a spanning tree.*

The figure shows two basic matrices and the corresponding spanning trees:



$\beta = \{(1,2),(2,3),(2,4),(4,5)\}$

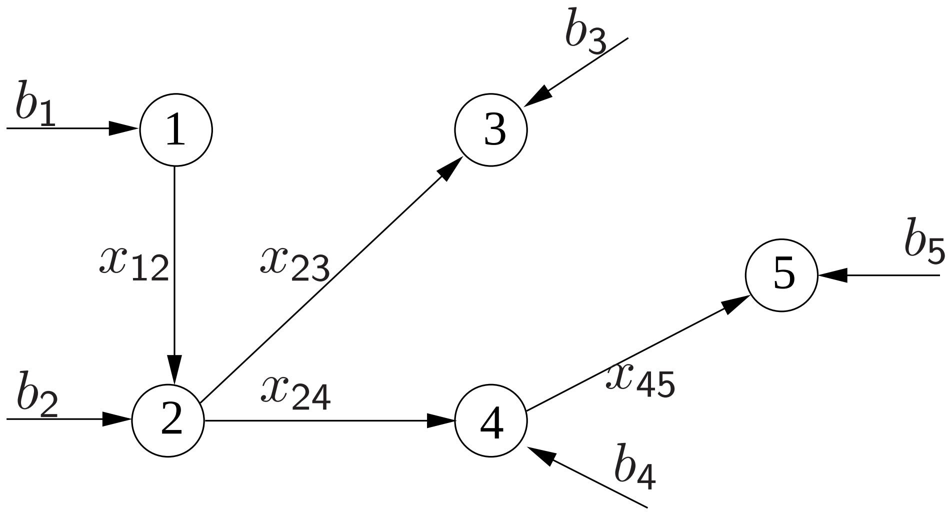
$$\mathbf{A}_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$



$\beta = \{(1,2),(2,3),(2,4),(3,5)\}$

$$\mathbf{A}_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Determining the basic solution $\mathrm{Bx} = \mathrm{b}$



The basic variables can be determined from the flow balance equations in the spanning tree (recall $(b_1, b_2, b_3, b_4) = (40, 35, -30, -25)$, $b_5 = 20$.)

$$\begin{aligned} x_{12} &= b_1 = 40 \\ x_{23} &= -b_3 = 30 \\ x_{24} &= b_2 + x_{12} - x_{23} = 45 \\ x_{45} &= x_{24} + b_4 = 20 \end{aligned} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Note that $x_{45} + x_{35} = 20$, *i.e.*, flow balance in node 5, follows automatically

The basic flow can also be determined through the balance equations "downstreams":

$$
\begin{aligned}
x_{45} &= b_5 = 20\\
x_{24} &= x_{45} - b_4 = 45\\
x_{23} &= -b_3 = 30\\
x_{12} &= x_{23} + x_{24} - b_2 = b_1 = 40
\end{aligned}
$$

Do you see why ?

Note that $\mathbf{x}_\beta = \bar{\mathbf{b}} = \begin{bmatrix} \bar{b}_{12} \\ \bar{b}_{23} \\ \bar{b}_{24} \\ \bar{b}_{45} \end{bmatrix} = \begin{bmatrix} 40 \\ 30 \\ 45 \\ 20 \end{bmatrix} \geq 0$ so we have a feasible basic solution.

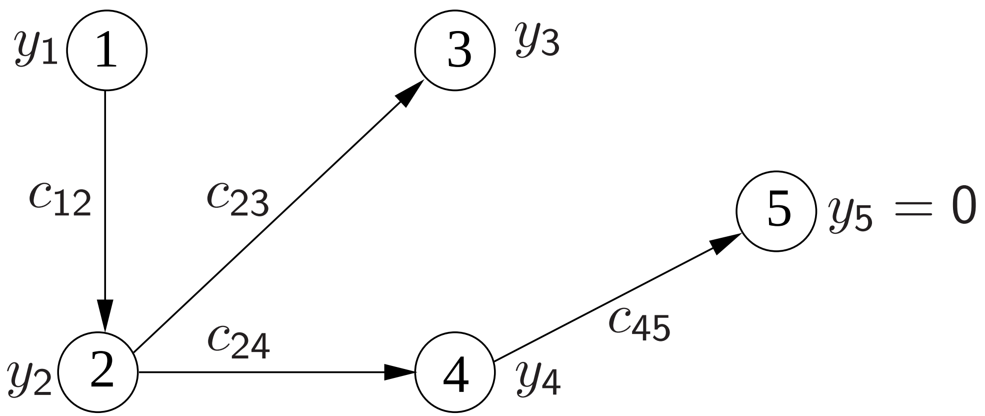
Determining the simplex multiplicators $\mathbf{y}^\top \mathbf{A}_\beta = \mathbf{c}_\beta^\top$

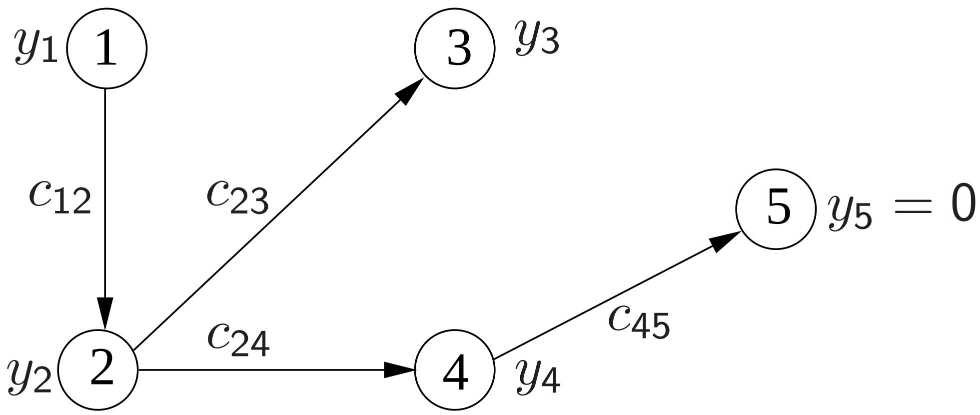
The Simplex multiplicators are determined by the equation $\mathbf{y}^\top \mathbf{A}_\beta = \mathbf{c}_\beta^\top$.

To node $1, \ldots, m-1$ corresponds the simplex multiplicator y_k, $k = 1, \ldots, m-1$. If we let $y_m = 0$ then

$$y_i - y_j = c_{ij}, \text{ for all } (i, j) \in \beta \text{ (all basic arcs)}$$

Hence, the simplex multiplicators can be determined from the network





With $\mathbf{c}^\mathsf{T} = (c_{12}, c_{13}, c_{23}, c_{24}, c_{34}, c_{35}, c_{45}) = (2, 5, 2, 2, 1, 1, 2)$ we obtain (in this order since $y_5 = 0$)

$$
\begin{aligned}
y_4 - y_5 = c_{45} \;&\Rightarrow\; y_4 = c_{45} = 2 \\
y_2 - y_4 = c_{24} \;&\Rightarrow\; y_2 = y_4 + c_{24} = 2 + 2 = 4 \\
y_2 - y_3 = c_{23} \;&\Rightarrow\; y_3 = y_2 - c_{23} = 4 - 2 = 2 \\
y_1 - y_2 = c_{12} \;&\Rightarrow\; y_1 = y_2 + c_{12} = 4 + 2 = 6
\end{aligned}
$$

Determining the reduced costs $\mathbf{r}_\nu = \mathbf{c}_\nu^\top - \mathbf{y}^\top \mathbf{A}_\nu$

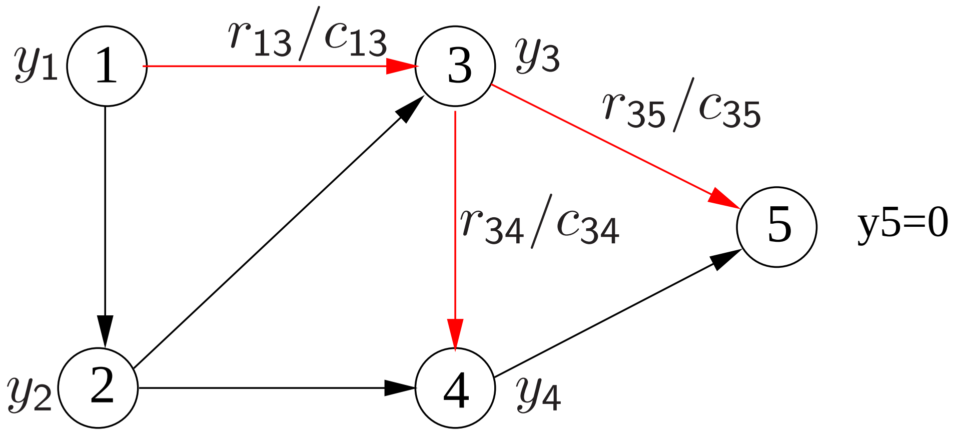
The formula for the reduced costs $\mathbf{r}_\nu = \mathbf{c}_\nu^\top - \mathbf{y}^\top \mathbf{A}_\nu$ results in the equations ($y_m = 0$)

$$r_{ij} = c_{ij} - y_i + y_j \text{ for all } (i,j) \in \nu \text{ (\textit{i.e.}, for all non-basic arcs)}$$

In our example, we have

$$\nu = \{(1,3),(3,4),(4,5)\}, \qquad \mathbf{A}_\nu = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

We can perform the calculations with help of the network



$$r_{13} = c_{13} - y_1 + y_3 = 5 - 6 + 2 = 1$$

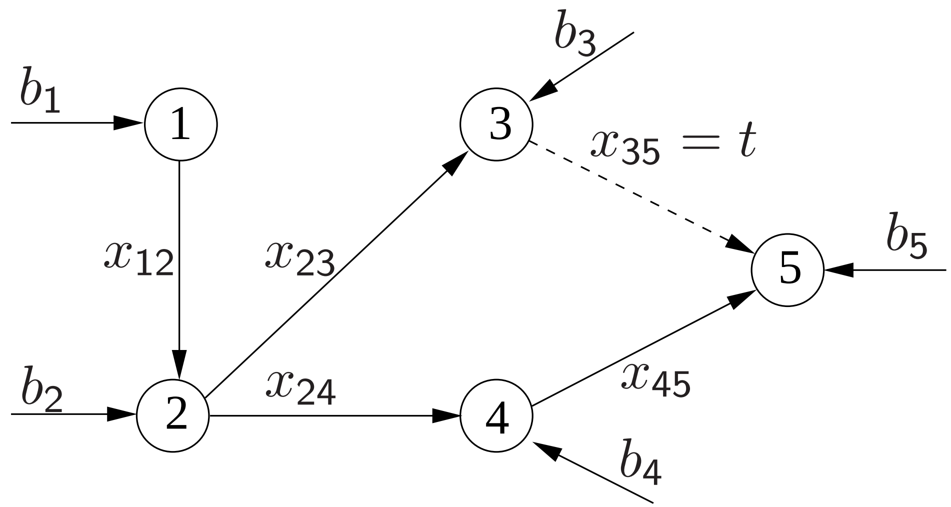
$$r_{34} = c_{34} - y_3 + y_4 = 1 - 2 + 2 = 1$$

$$r_{35} = c_{35} - y_3 + y_5 = 1 - 2 + 0 = -1$$

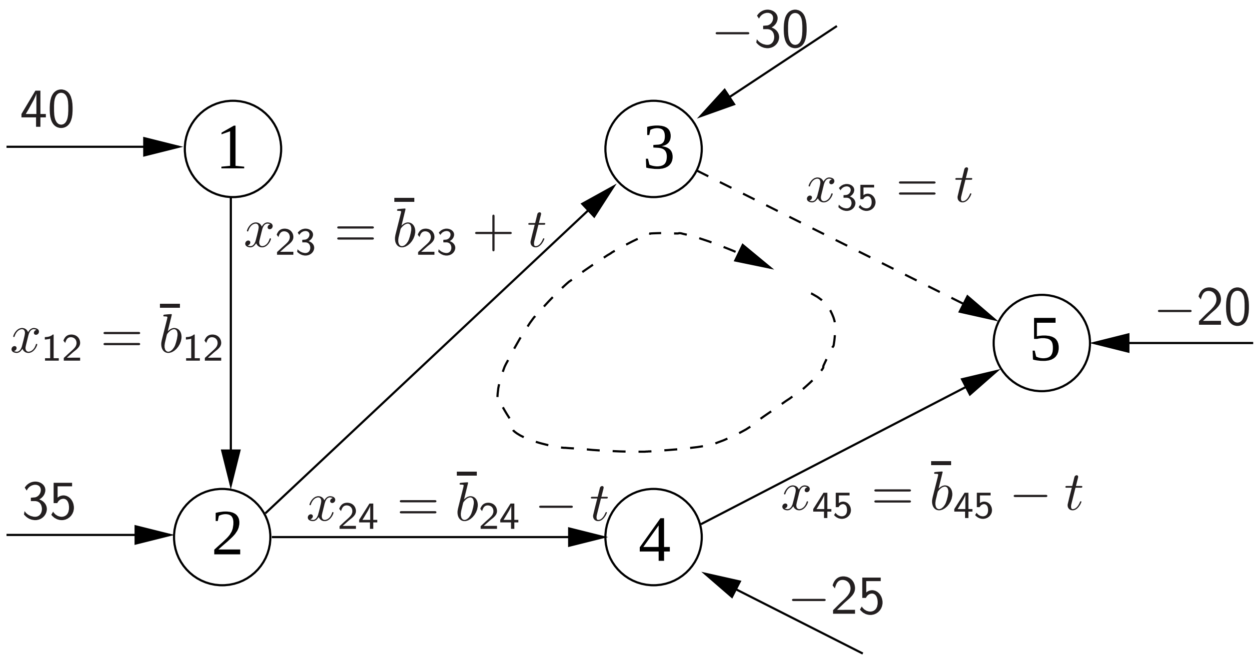
Note that $r_{35} < 0$ and the current basic feasible solution is not optimal.

The Quotient test

Since $r_{35} < 0$ we let $x_{35} = t$, *i.e.*, the arc $(3, 5)$ enters the basis.
The question is now which arc should be removed in order to obtain a new spanning tree corresponding to a new basic matrix.



The test in the usual simplex method $t^{\max} = \min \left\{ \frac{\bar{b}_i}{\bar{a}_{ik}} | \bar{a}_{ik} > 0 \right\} = \frac{\bar{b}_p}{\bar{a}_{pk}}$ is simplified by just looking at the flow balance in the cycle that has been formed.



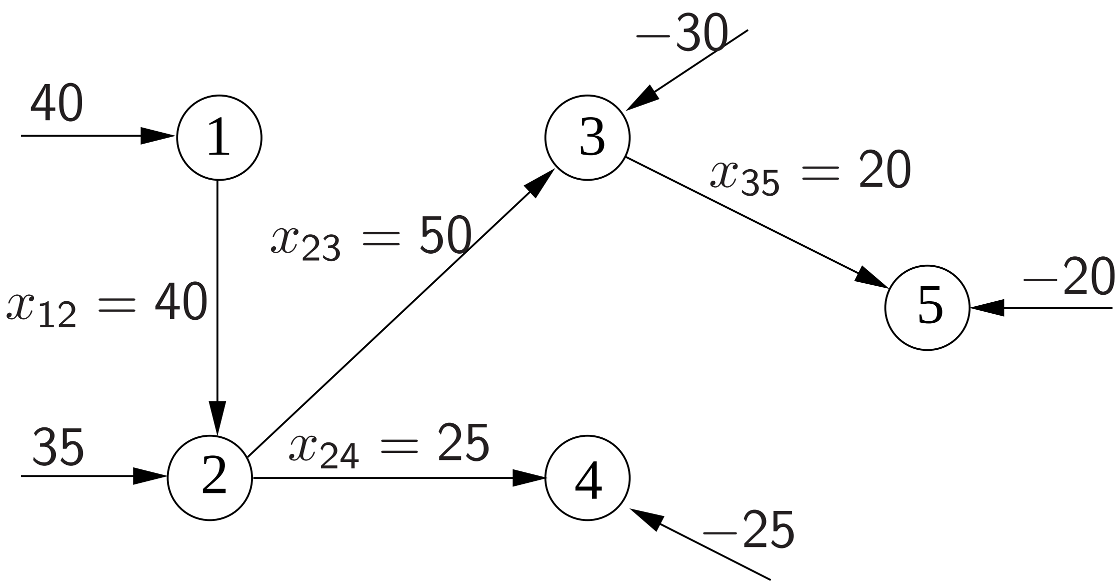
$$x_{23} = \bar{b}_{23} + t = 30 + t$$

$$x_{24} = \bar{b}_{24} - t = 45 - t$$

$$x_{45} = \bar{b}_{45} - t = 20 - t$$

Note that t can be increased until $t^{\text{max}} = 20$ when $x_{45} = 0$.
Then x_{45} exits the basis.

The Pivot step (change of basis)



We have $\beta = \{(1,2),(2,3),(2,4),(3,5)\}$ and

$$\mathbf{x}_\beta = \bar{\mathbf{b}} = \begin{bmatrix} 40 \\ 50 \\ 25 \\ 20 \end{bmatrix} \geq 0$$

which is a new basic feasible solution.

Next iteration

If we repeat the above steps, we get

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \qquad \mathbf{r}_\nu = \begin{bmatrix} r_{13} \\ r_{34} \\ r_{45} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \geq 0$$

Since all the reduced costs are positive, the current basic feasible solution is optimal.

Resumé of the network flow simplex method

For network optimization, most steps in the simplex method are simplified:

(i) A basic index vector β corresponds to the arcs in a spanning tree.

(ii) The basic solution corresponding to β can be determined from the flow balance equations in a spanning tree. The basic solution is feasible if the flow in all basic arcs are non-negative.

(iii) The Simplex multiplicators are determined by $(y_m = 0)$

$$y_i - y_j = c_{ij}, \text{ for all } (i,j) \in \beta \text{ (all basic arcs)}$$

(iv) The reduced costs are determined from

$$r_{ij} = c_{ij} - y_i + y_j \text{ for all } (i,j) \in \nu \text{ (\textit{i.e.} for all non-basic arcs)}$$

(v) If all $r_{ij} \geq 0$ an optimal basic solution has been found.

(vi) If some reduced cost r_{ij} is negative (use the most negative), add the arc (i, j) to the spanning tree corresponding to the basis β. A cycle is formed and we apply a flow $x_{ij} = t \geq 0$. By studying the balance of flow in the cycle, it is easy to see how much t can be increased until one flow in the cycle is zero.

If t can be increased arbitrarily, the optimization problem has no bounded solution.

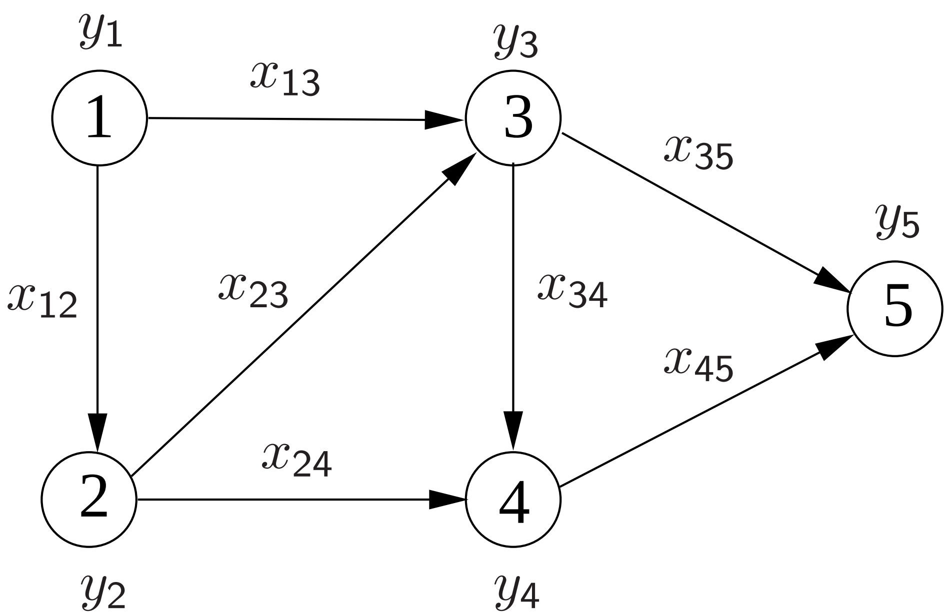
(vii) Update the spanning tree by adding the arc (i, j) to β and removing the arc with zero flow. A new basis is obtained (*i.e.*, a new spanning tree) and a new basic feasible solution.

$(viii)$ Repeat from (iii).

The four fundamental subspaces for the incidence matrix

This part is for illustration only

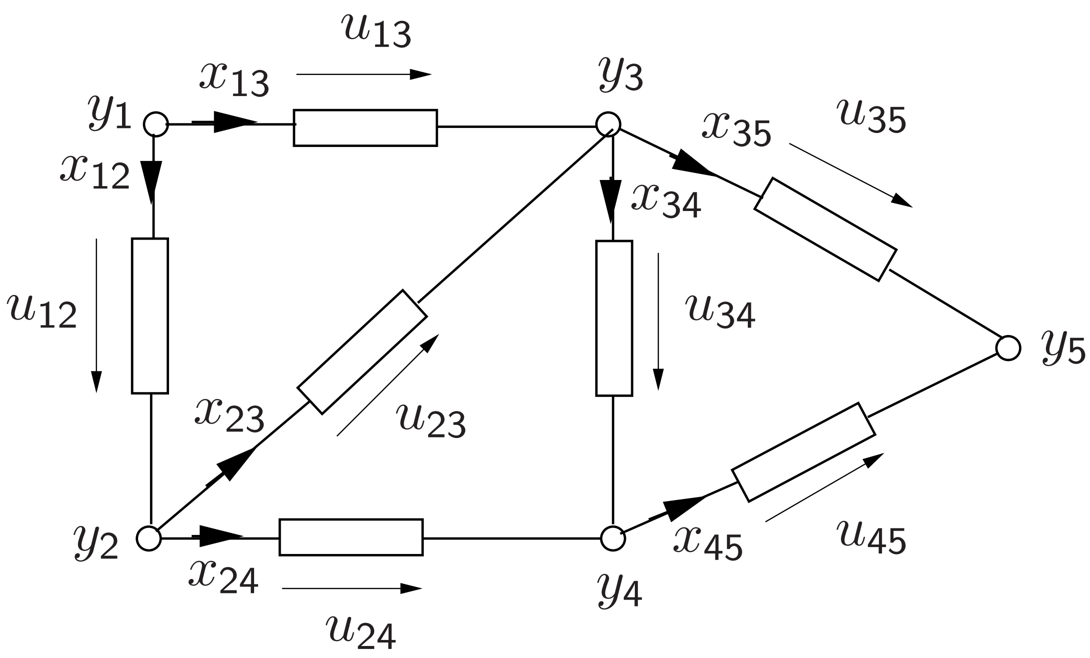
We demonstrate how the four fundamental subspaces for the incidence matrix $\tilde{\mathbf{A}}$ can be interpreted (with an electric analogue) with the help of the associated graph $\mathcal{G} = (\mathcal{N}, \mathcal{B})$. Returning to our example:



To every node j we associate a potential y_j, $j = 1, \ldots, m$.

To every arc (i, j) we associate a flow (current) x_{ij}.

We note that the network corresponds to the electric circuit below:



- y_j, $j = 1, \ldots, 5$ denotes the potential in node j.
- x_{ij} denotes the current in arc (i, j).
- u_{ij} denotes the voltage over the resistor in arc (i, j).

Left nullspace

Interpretation: The equation $\tilde{\mathbf{y}}^\mathsf{T}\tilde{\mathbf{A}} = \mathbf{u}$ corresponds to the physical laws over the arcs (resistors)

$$y_i - y_j = u_{ij} \qquad \text{voltage} = \text{difference in potential}$$

There is no unique solution since we can add an arbitrary constant to all node potentials without changing the voltages over the arcs.

The equation $\mathbf{y}^\mathsf{T}\mathbf{A} = \mathbf{u}$ correspond to the potential y_5 is fixed at zero. (earth)

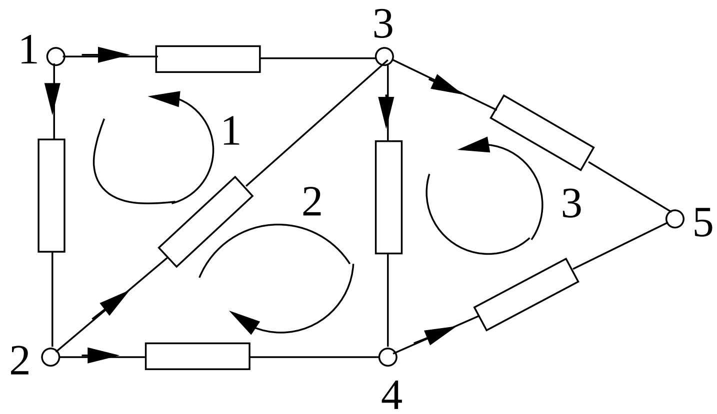
$$\mathcal{N}(\tilde{\mathbf{A}}^\mathsf{T}) = \text{span}\left\{ \mathbf{e} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

If $\tilde{y} \in \mathcal{N}(\tilde{\mathbf{A}}^\mathsf{T})$, then $u = 0$.
i.e., if all potentials are equal, then all voltages are zero.

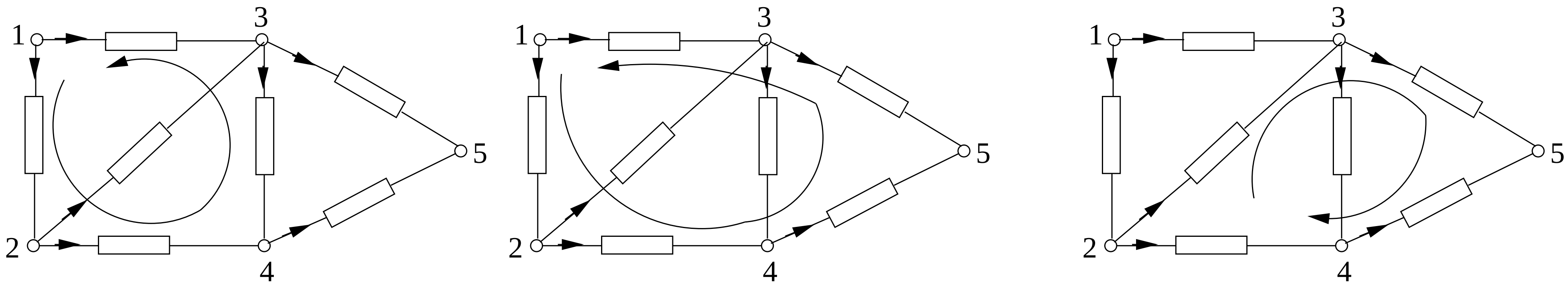
The (right) Null space

$$\mathcal{N}(\tilde{\mathbf{A}}) = \left\{\mathbf{x} \in \mathbf{R}^7 : \tilde{\mathbf{A}}\mathbf{x} = \mathbf{0}\right\} = \text{span}\left\{\begin{bmatrix}1\\-1\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\-1\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\\-1\\1\end{bmatrix}\right\} = \mathcal{N}(\mathbf{A})$$

Interpretation: The Nullspace corresponds to three cycles in the network



Other cycles in the graph are given by linear combinations of the three "basis cycles" in the previous figure.



(basis)cycle 1 − cycle 2 Cycle 1 − cycle 2 + cycle 3 Cycle 2 − cycle 3

The flow in/out of the circuit is not affected by the current $\tilde{\mathbf{x}}$ in a cycle since $\tilde{\mathbf{A}}\tilde{\mathbf{x}} = 0$.

The Rowspace

$$
\begin{aligned}
\mathcal{R}(\tilde{\mathbf{A}}^\mathsf{T}) &= \left\{\mathbf{u} = \tilde{\mathbf{A}}^\mathsf{T}\tilde{\mathbf{y}}\right\} \qquad (= \{\mathbf{u} = \mathbf{A}^\mathsf{T}\mathbf{y}\}) \\
&= \mathcal{N}(\tilde{\mathbf{A}})^\perp \qquad (= \mathcal{N}(\mathbf{A})^\perp) \\
&= \left\{\mathbf{u} \in \mathbf{R}^7 : u_{13} = u_{12} + u_{23};\ u_{24} = u_{23} + u_{34};\ u_{35} = u_{24} + u_{45}\right\}
\end{aligned}
$$

Interpretation: The sum of the voltages over a cycle is equal to zero. This is also called Kirchoff's voltage law.

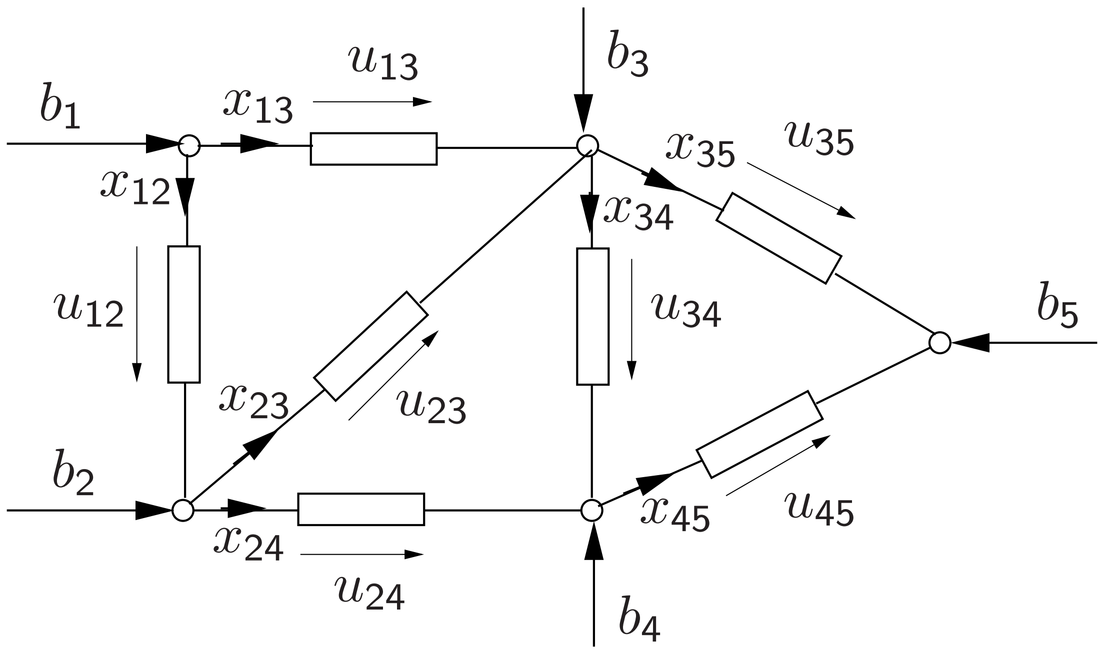
Note that there are more possible cycles, *e.g.*, we have the constraint $u_{12} + u_{24} = u_{13} + u_{34}$, but these constraints follows from linear combinations of the three others (see previous slide).

The Rangespace

$$\mathcal{R}(\tilde{\mathbf{A}}) = \mathcal{N}(\tilde{\mathbf{A}}^\mathsf{T})^\perp = \{\mathbf{b} \in \mathbf{R}^m : \mathbf{e}^\mathsf{T}\mathbf{b} = 0\}$$
$$= \{\mathbf{b} \in \mathbf{R}^m : \sum_{k=1}^{m} b_k = 0\}$$

If b_j is interpreted as an external current that is fed in to node j, then the result says that the sum of these currents must be zero.

Kirchoff's laws



- Kirchoff's Current Law: The sum of the currents in to a node is zero. For the circuit in the figure, this can be written $\tilde{\mathbf{A}}\mathbf{x} = \mathbf{b}$. A necessary assumption is of course that $\sum_{j=1}^{m} b_j = 0$.
- Kirchoff's Voltage Law: The sum of all voltages over a cycle is equal to zero.

Reading instructions

Chapter 7.2 in the book.