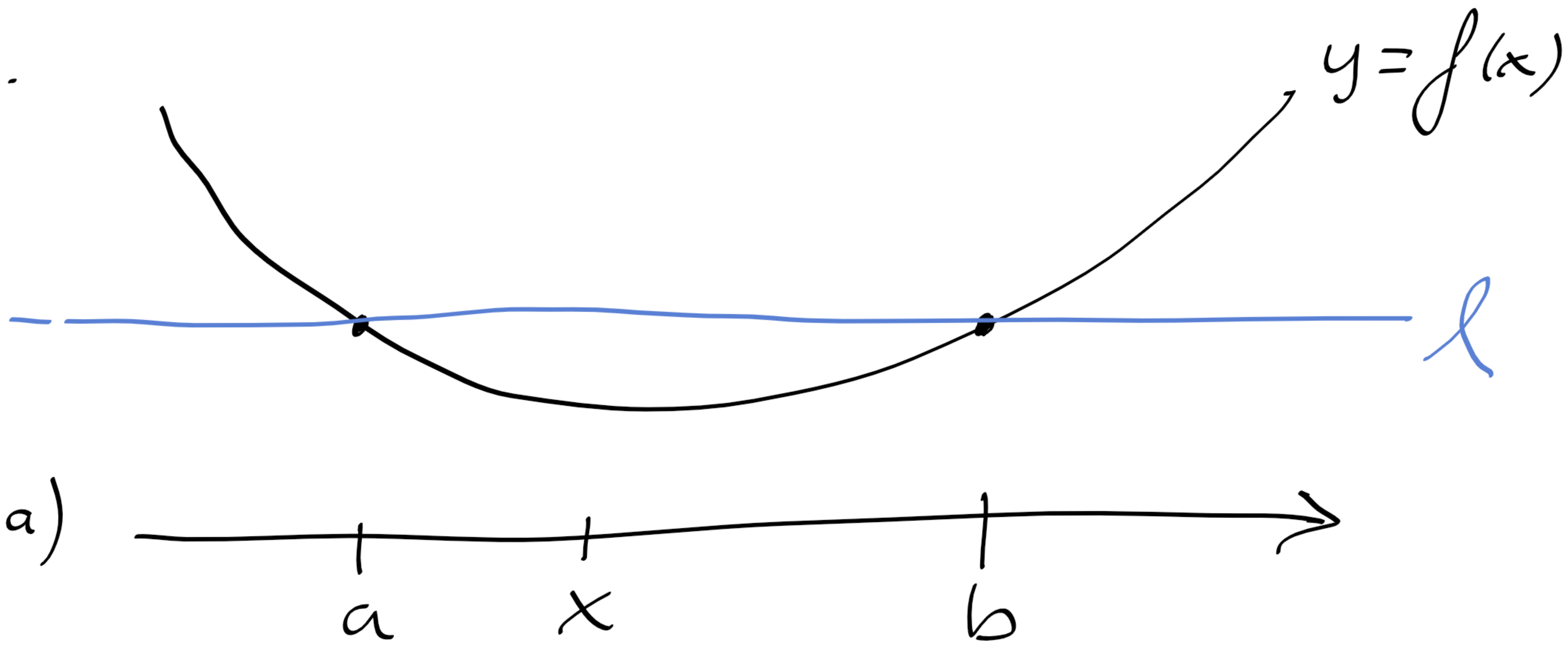
6 Convex functions

6.1 a One variable

Let I be an interval and $f : I \to \mathbb{R}$ and $a < b$, $a, b \in I$.



Def. the slope

$$S(a,b) = \frac{f(b) - f(a)}{b-a} = S(b,a)$$

The line ℓ: $\quad y - f(a) = S(a,b)(x-a)$

Any $x \in (a,b)$ can be written

$$x = a + \lambda(b-a) = (1-\lambda)a + \lambda b \;, \quad \lambda \in (0,1)$$

which on the line ℓ gives

$$y_\ell = f(a) + \frac{f(b)-f(a)}{b-a}\lambda(b-a)$$

$$= f(a) + \lambda\big(f(b) - f(a)\big) = (1-\lambda)f(a) + \lambda f(b)$$

Def. f is convex iff

$$\begin{cases} a, b \in I \\ 0 < \lambda < 1 \end{cases} \Longrightarrow f((1-\lambda)a + \lambda b) \le (1-\lambda)f(a) + \lambda f(b)$$

f is strictly convex iff $\quad <$

f is (strictly) concave iff $-f$ is (strictly) convex.

Prop. (Exerc 6.1) f is convex iff

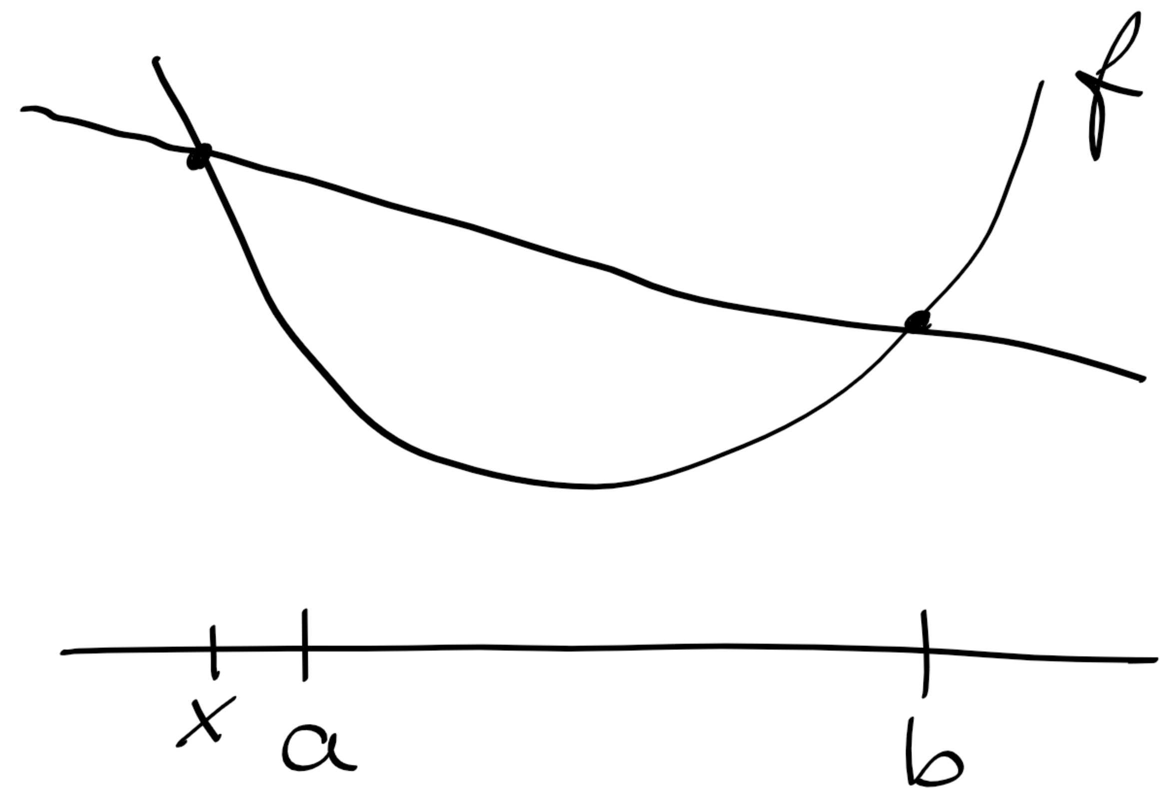
$$\begin{cases} a_1, \dots, a_n \in I \\ \lambda_1, \dots, \lambda_n \ge 0 \\ \sum \lambda_i = 1 \end{cases} \Longrightarrow f\left(\sum \lambda_i a_i\right) \le \sum \lambda_i f(a_i)$$

Lemma 1: f convex $\Rightarrow$

$$f((1-\lambda)a + \lambda b) \geq (1-\lambda) f(a) + \lambda f(b)$$

when $\quad \lambda < 0 \iff x < a$

or $\quad \lambda > 1 \iff x > b$



Proof: Assume $\lambda < 0$. $\quad x = (1-\lambda)a + \lambda b = a + \lambda(b-a)$

$$\Rightarrow \quad a = \underbrace{\frac{1}{1-\lambda}}_{>0} x - \underbrace{\frac{\lambda}{1-\lambda}}_{>0} b \quad \text{and the sum is } 1$$

f is convex gives $\quad f(a) \leq \frac{1}{1-\lambda} f(x) - \frac{\lambda}{1-\lambda} f(b) \iff$

$$f(x) \geq (1-\lambda) f(a) + \lambda f(b)$$

Analogously for $\lambda > 1$ #

Cor. 1: f convex, $a \in I$ local minimizer

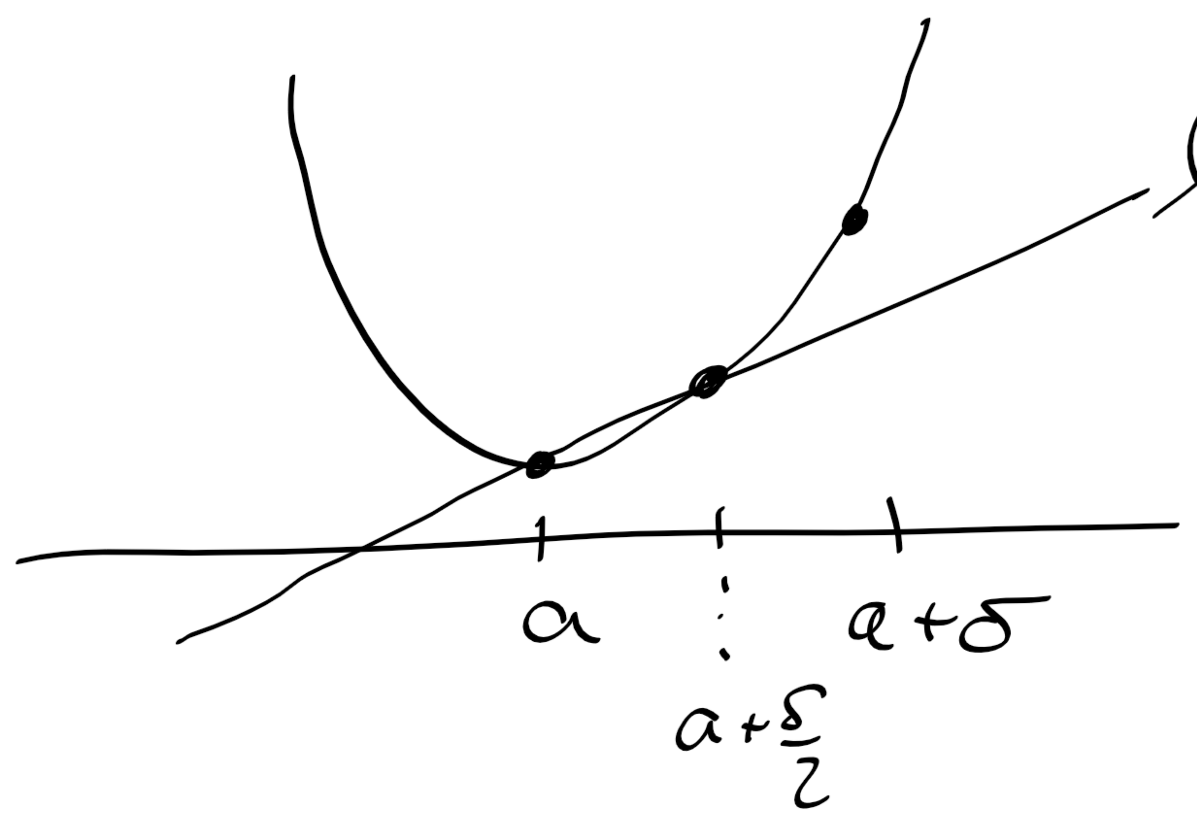
$\Rightarrow$ a is a global minimizer

Proof: If $a \in \mathrm{int}(I)$, then $f(x) \geq f(a)$ for

$a < x < a + \delta$ for some $\delta > 0$

$\Rightarrow f(a + \frac{\delta}{2}) \geq f(a) \Rightarrow$

$$S(a+\tfrac{\delta}{2}, a) = \frac{f(a+\frac{\delta}{2}) - f(a)}{\delta/2} \geq 0 \quad \text{slope of } l$$



Lemma 1 gives

$$f(x) \geq f(a + \tfrac{\delta}{2}) \geq f(a) \quad \forall\, x > a + \frac{\delta}{2}$$

Analogously for $x \leq a$ and if $a \in \partial I$. #

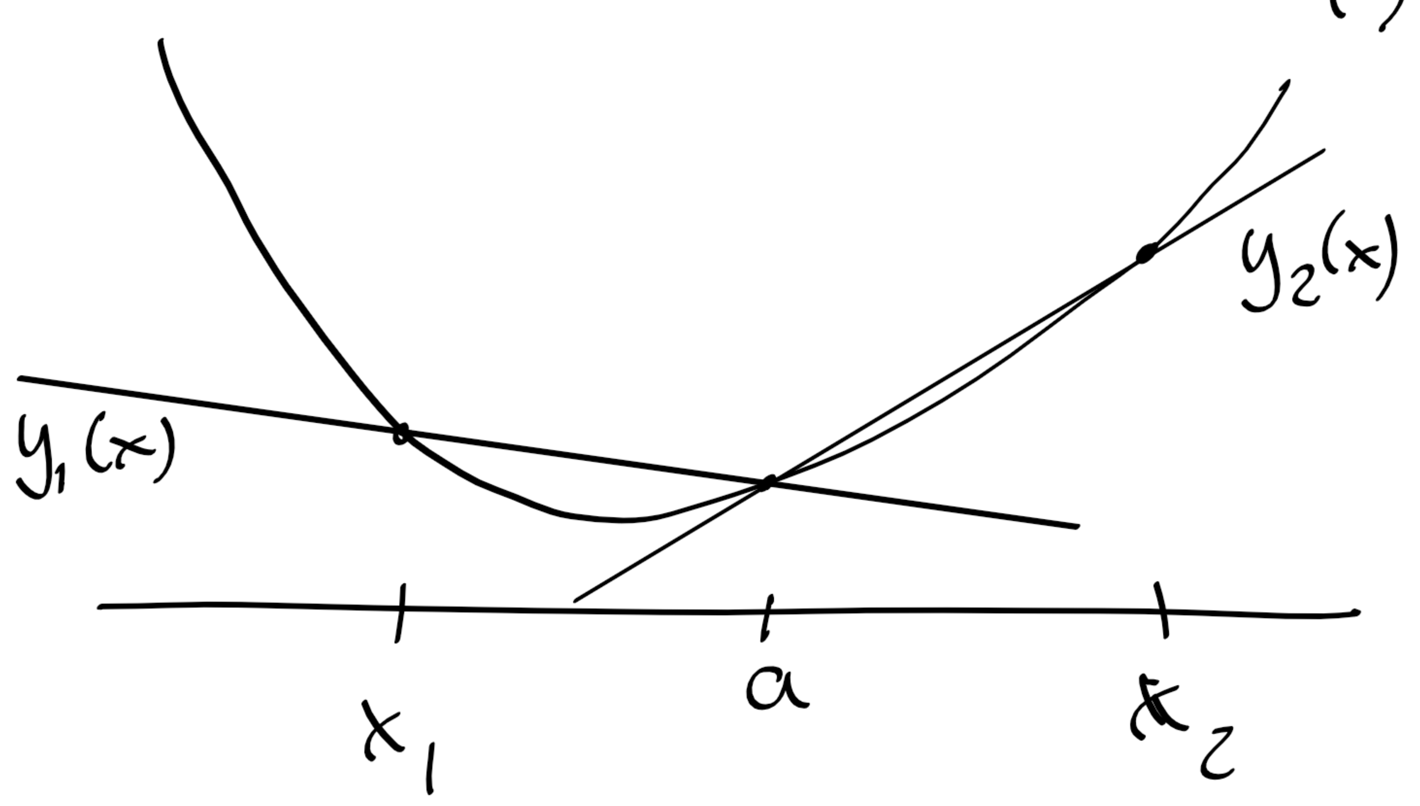
Thm 1. f convex on $I \Rightarrow f$ is continuous on $\mathrm{int}(I)$

Proof: Given $a \in \mathrm{int}(I)$. $\exists\, x_1 < a < x_2$ with $x_1, x_2 \in I$ and straight lines according to the figure.

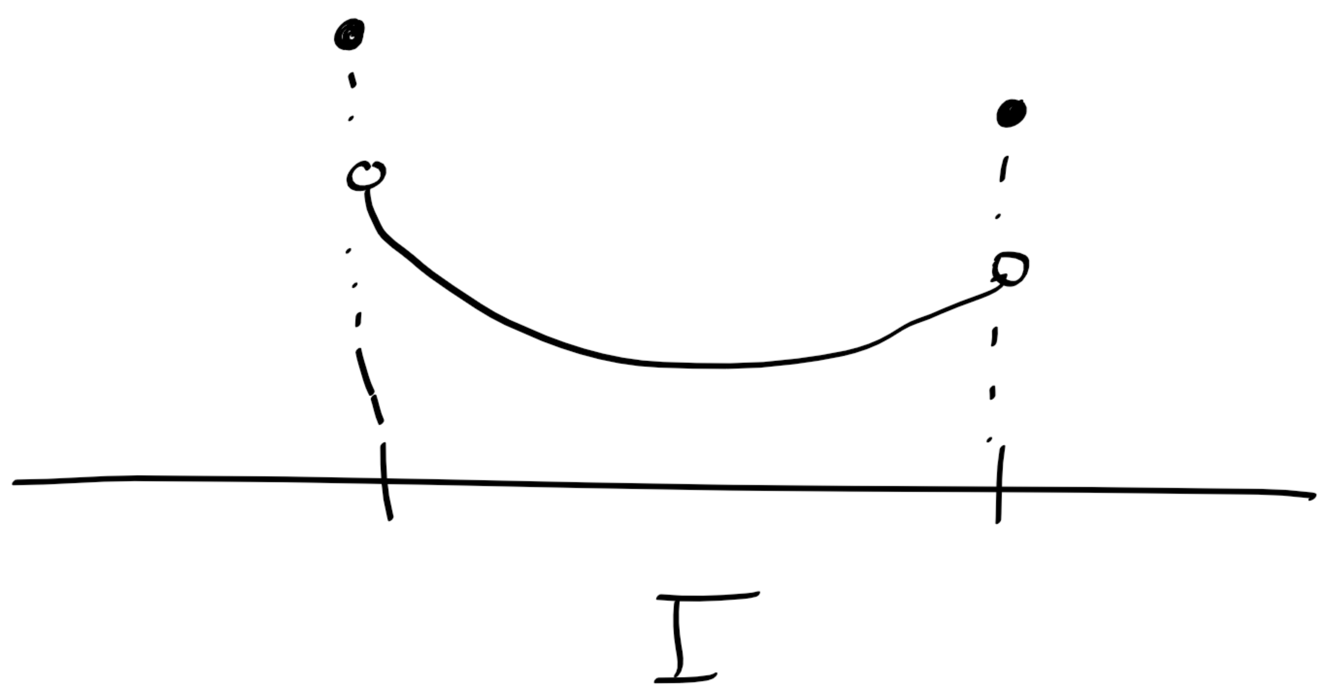
For $a < x < x_2$: $\quad y_1(x) \leq f(x) \leq y_2(x)$

$x_2 \to a^+ \Rightarrow \quad f(a) \leq \lim_{x \to a^+} f(x) \leq f(a)$

Analog. for $x < a$. #

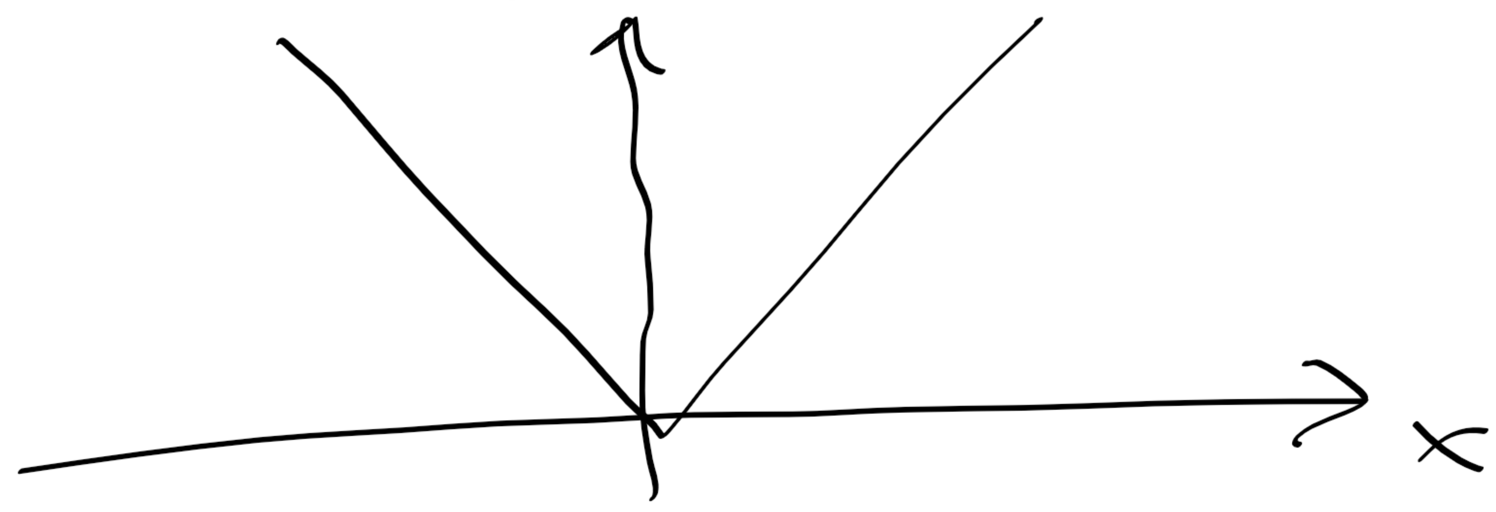


Note: f need not be continuous on ∂I:



What about differentiability?

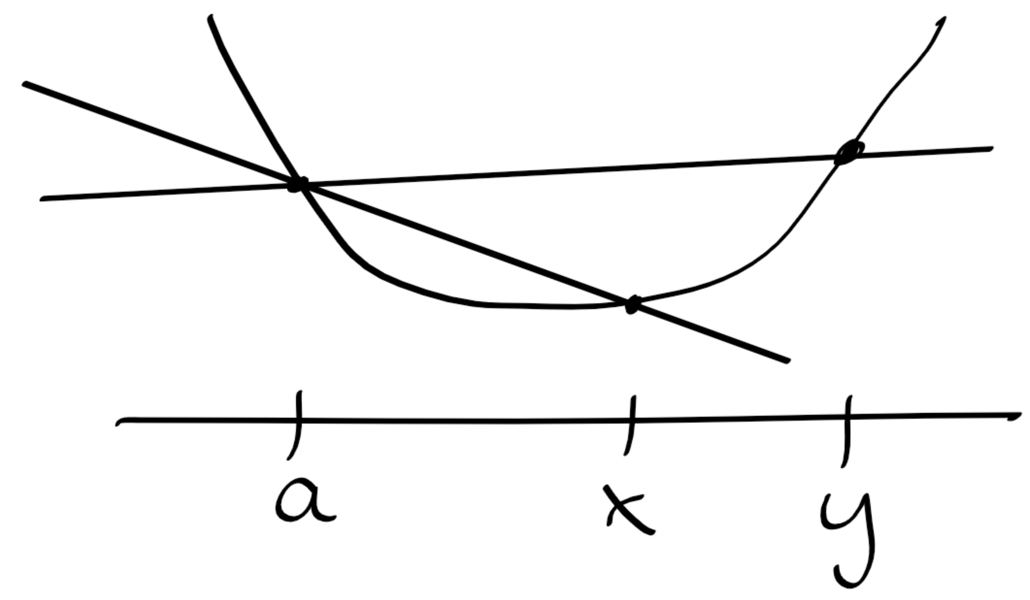
Ex. $f(x) = |x|$



Def. Let $x < y \in I$ (interval) and $f : I \to \mathbb{R}$.

- f is increasing iff $f(x) \leq f(y)$ (non decreasing)
- f is strictly increasing iff $f(x) < f(y)$ (increasing)

6.1b Convex functions of one variable (continued)



Lemma S:

f convex on $I \iff S(a,\cdot)$ is increasing on $I \setminus \{a\}$

$$S(a,b) := \frac{f(b)-f(a)}{b-a}$$

Proof: For $a < x < y$:

$x = (1-\lambda)a + \lambda y = a + \lambda(y-a)$ for some $\lambda \in (0,1)$

f convex $\iff f(x) \leq (1-\lambda)f(a) + \lambda f(y) \iff$

$$S(a,x) - S(a,y) = \frac{f(x)-f(a)}{x-a} - \frac{f(y)-f(a)}{y-a} = \frac{f(x)-f(a)}{\lambda(y-a)} - \lambda\frac{f(y)-f(a)}{y-a}$$

$$= \frac{1}{\lambda(y-a)}\left(f(x) - f(a) - \lambda f(y) + \lambda f(a)\right)$$

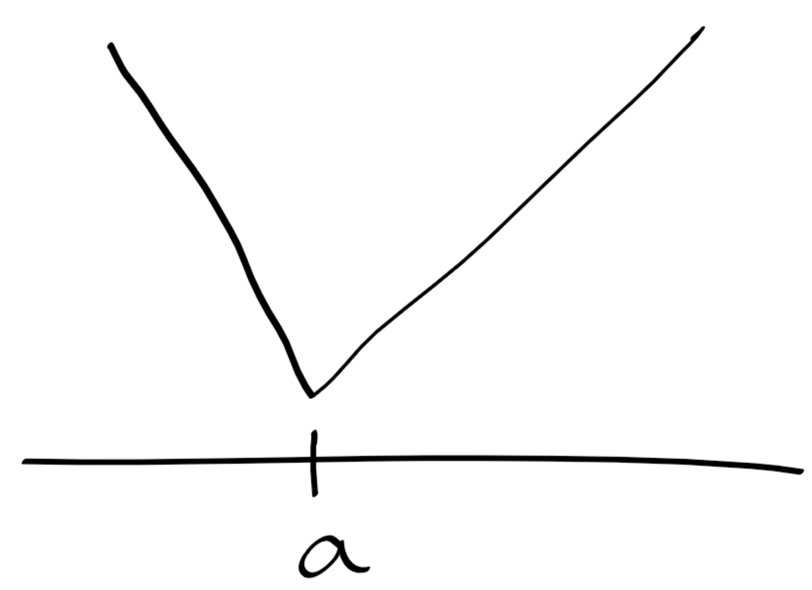
$$= \frac{1}{\lambda(y-a)}\left(f(x) - \left((1-\lambda)f(a) + \lambda f(b)\right)\right) \leq 0$$

Similarly for $x < a$ #

Thm 2. $f : I \to \mathbb{R}$ convex and $a \in \operatorname{int}(I) \implies$ the following limits exist:

$$\lim_{x \to a^+} S(a,x) =: f'_+(a)$$

$$\lim_{x \to a^-} S(a,x) =: f'_-(a)$$



and $f'_-(a) \leq f'_+(a)$

Proof: Let $x < a < y$. Lemma S gives $S(a,x) \leq S(a,y)$

$S(a,x)$ continuous and increasing (bounded above) $\implies \exists \lim_{x \to a^-} S(a,x) \leq S(a,y)$

$\lim_{x \to a^-} S(a,x) \leq \lim_{y \to a^+} S(a,y)$ #

When $f \in C^1$ we extend the slope function

$$S(a,a) = f'(a)$$

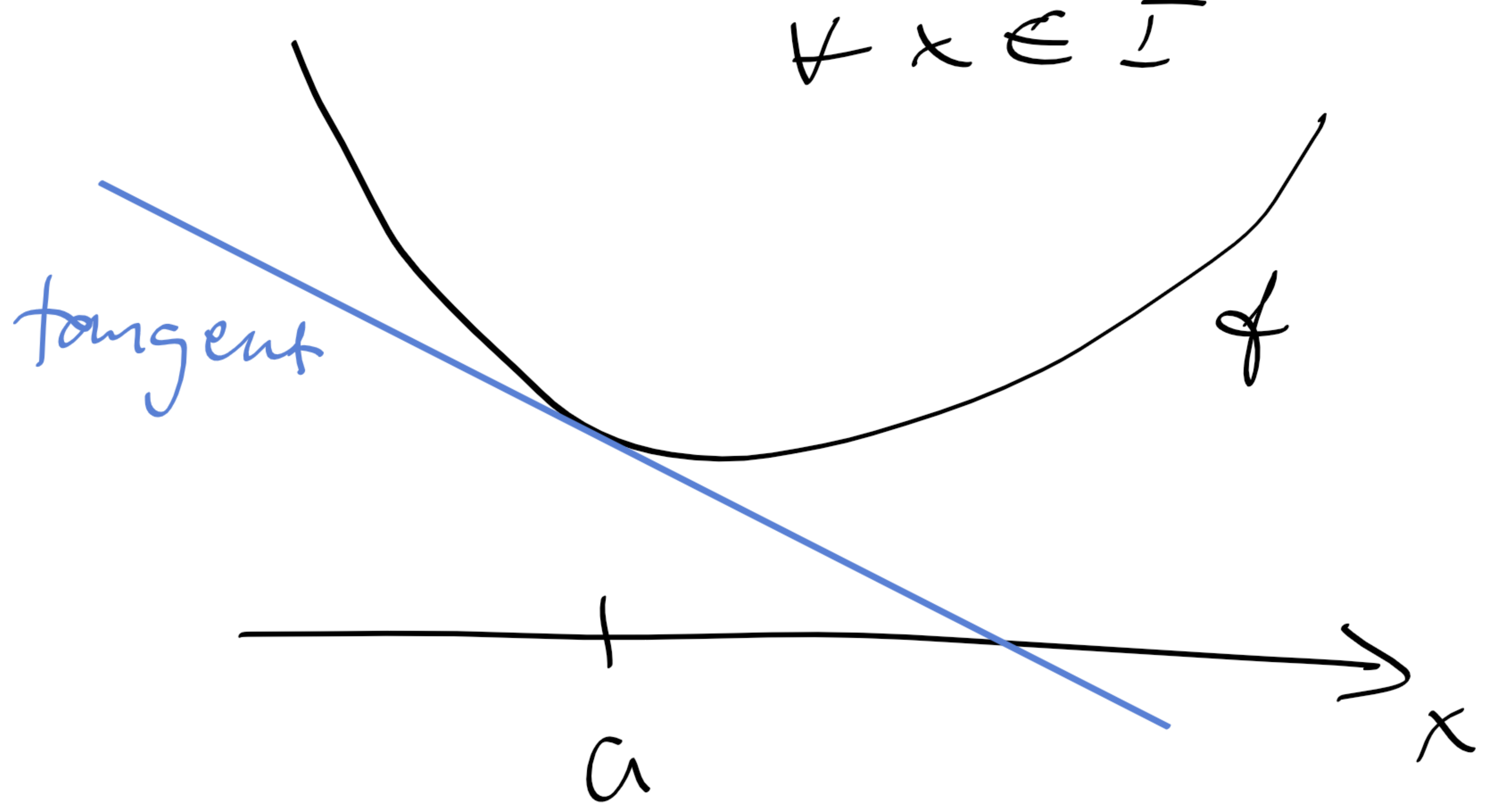
and extend Lemma 5 to:

$$f \text{ convex} \iff S(a,\cdot) \text{ is increasing}$$

Thm 3 (extended): Assume $f \in C^1(I)$. Then

$$f \text{ convex on } I \iff f(x) \geq f(a) + f'(a)(x-a)$$

$\forall\, a \in \operatorname{int}(I)$
$\forall\, x \in I$



Proof: $\boxed{\Rightarrow}$ $x=a$ is trivial.

Assume $x<a$. Then Lemma 5 gives

$$S(a,x) \leq S(a,a) \iff$$

$$\frac{f(x)-f(a)}{x-a} \leq f'(a) \iff$$

$$f(x) - f(a) \geq f'(a)(x-a)$$

Similarly for $x>a$.

$\boxed{\Leftarrow}$ Let $x<y$ and $0<\lambda<1$ and set $z=(1-\lambda)x+\lambda y$

We have
$$f(x) \geq f(z) + f'(z)(x-z) \qquad (a)$$
$$f(y) \geq f(z) + f'(z)(y-z) \qquad (b)$$

Form $(1-\lambda)(a) + \lambda(b)$:

$$(1-\lambda)f(x) + \lambda f(y) \geq f(z) + f'(z)K$$

where
$$K = (1-\lambda)(x-z) + \lambda(y-z)$$
$$= (1-\lambda)x - (1-\lambda)z + \lambda y - \lambda z$$
$$= (1-\lambda)x + \lambda y - z = 0$$

#

Cor. 2. $f \in C^1$ and convex

a stationary point $\Rightarrow$ a global minimizer

Proof: Let $f'(a)=0$ in Thm 3: $f(x) \geq f(a)$ $\forall x \in I$ #

Thm 4. If $f \in C^1$, then

$$f \text{ convex } \Leftrightarrow f' \text{ is increasing}$$

Proof: $\boxed{\Rightarrow}$ $x < y \Rightarrow$ (Lemma 5)

$$f'(x) = S(x,x) \le S(x,y) \le S(y,x) \le f'(y)$$

$\boxed{\Leftarrow}$ Let $a < x < b$ Mean-value theorem gives

$\exists\, \xi, \eta$: $a < \xi < x < \eta < b$:

$$S(a,x) \underset{\substack{\uparrow \\ \text{mean-v. thm}}}{=} f'(\xi) \underset{\substack{\uparrow \\ f' \text{ incr.}}}{\le} f'(\eta) \underset{\substack{\uparrow \\ \text{mean-v thm}}}{=} S(x,b)$$

$\Rightarrow$ $S(x,a) \le S(x,b)$ thus $S(x,\cdot)$ increasing $\overset{\text{Lemma 5}}{\Longleftrightarrow}$ f convex. #

Thm 5: If $f \in C^2$ and convex, then

$$f \text{ convex } \Leftrightarrow f'' \geq 0$$

Proof: Thm 4: f convex $\overset{\text{Thm 4}}{\Longleftrightarrow}$ $g = f'$ increasing $\overset{\text{old}}{\Longleftrightarrow}$ $g' = f'' \geq 0$

$\boxed{\overset{\text{old}}{\Rightarrow}}$ $x < a \Rightarrow g(x) \le g(a) \Rightarrow \dfrac{g(x) - g(a)}{x - a} \geq 0 \Rightarrow g'_-(a) \geq 0$

anal. for $x > a$ etc.

$\boxed{\overset{\text{old}}{\Leftarrow}}$ Mean-value thm: $g(x) - g(a) = \underbrace{g'(\xi)}_{\geq 0} \underbrace{(x - a)}_{\le 0} \le 0$ #

assume $x < a$