Ex Linear Programming.

$$\min c^T x \mid Ax = b,\ x \geq 0.$$

$$f(x) = c^T x,\quad g(x) = -x,\quad h(x) = b - Ax$$

$$L(x,u,v) = c^T x - u^T x + v^T(b - Ax) =$$

$$= (c^T - u^T - v^T A)x + v^T b.$$

$$\theta(u,v) = \inf_{x \in X} L(x,u,v).\ \text{Here } X = \mathbb{R}^n.$$

$$\theta(u,v) = \begin{cases} v^T b & \text{if } c^T - u^T - v^T A = 0, \\ -\infty & \text{otherwise.} \end{cases}$$

Thus $\displaystyle\sup_{(u,v)\in U} \theta(u,v) = \sup_{u = c - A^T v \geq 0} b^T v =$

$$= \sup b^T v \mid A^T v \leq c.$$

the dual LP problem.

Remark: sup = supremum = same as maximum, but always $\exists$ and can be $+\infty$.

- The primal problem:

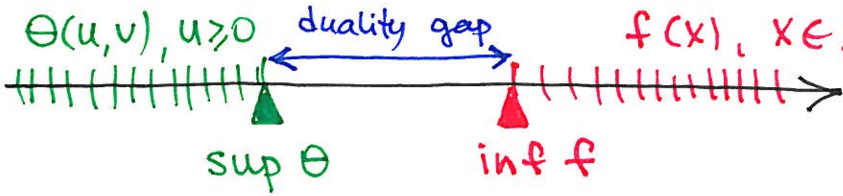
$$\inf f(x) \mid x \in X,\ g(x) \leq 0,\ h(x) = 0.$$

- The dual problem:

$$\sup \theta(u,v) \mid u \geq 0.$$

By the weak duality we get

$$\sup_{u \geq 0} \theta(u,v) \leq \inf_{x \in S} f(x)$$



Remark: in general

- duality gap may be $\neq 0$ (bad).
- $\theta(u,v) = \inf_{x \in X} L(x,u,v)$ is concave (good)
- dual to dual $\neq$ primal.

Ex $\min(x^2+y^2) \mid x^2+y^2 \leq 1 \leftarrow \left\{\begin{array}{l}\text{not active}\\ \text{at min!}\end{array}\right.$

$$\theta(u) = \inf_{\mathbb{R}^2}\left(x^2+y^2+u(x^2+y^2-1)\right) =$$

$$= \inf_{\mathbb{R}^2}\Big((1+u)(x^2+y^2)\Big) - u = -u.$$

new primal

Dual: $\max_{u \geq 0}(-u) = -\min_{u \geq 0} u = -\min_{x \geq 0} x$

New $\tilde{\theta}(u) = \inf_{x \in \mathbb{R}}(x - ux) =$

$$= \inf_{x \in \mathbb{R}}(1-u)x = \begin{cases} 0 & \text{if } u = 1, \\ -\infty & \text{otherwise.} \end{cases}$$

Dual to dual: $\sup_{u=1} 0 \neq$ primal.

Th (Th. 3, p. 298) main result!

$(\bar{x}, \bar{u}, \bar{v})$ – saddle point

$$\Updownarrow$$

$\bar{x}$ is primal feasible and $\theta(\bar{u}, \bar{v}) = f(\bar{x})$.

Proof: $\boxed{\Downarrow}$ By Lemma (Lecture 11, p. 8)

$\bar{x}$ is P-feasible, $\bar{u}^T g(\bar{x}) = 0$, $L(\bar{x}, \bar{u}, \bar{v}) = \min_{x \in X} L(x, \bar{u}, \bar{v})$.

Then $\theta(\bar{u}, \bar{v}) = \inf_{x \in X} L(x, \bar{u}, \bar{v}) = L(\bar{x}, \bar{u}, \bar{v}) =$

$$= f(\bar{x}) + \underbrace{\bar{u}^T g(\bar{x})}_{=0} + \underbrace{\bar{v}^T h(\bar{x})}_{=0} = f(\bar{x}).$$

$\boxed{\Uparrow}$ $\theta(\bar{u}, \bar{v}) = \inf_{x \in X} L(x, \bar{u}, \bar{v}) \leq L(\bar{x}, \bar{u}, \bar{v}) =$

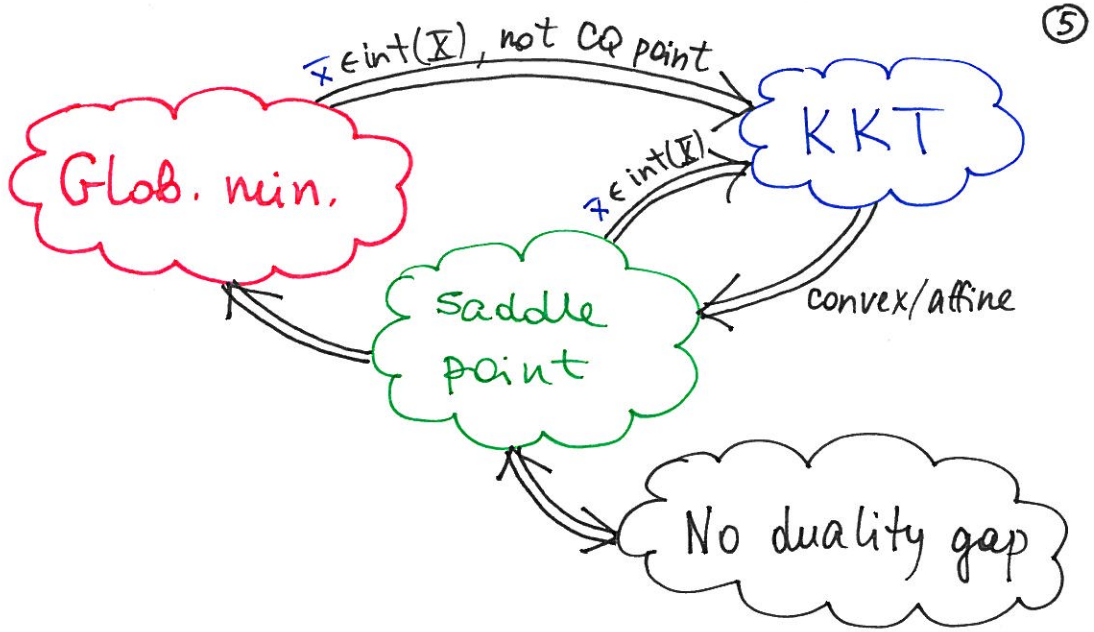
$$= f(\bar{x}) + \underbrace{\bar{u}^T g(\bar{x})}_{\leq 0} + \underbrace{\bar{v}^T h(\bar{x})}_{=0} \leq f(\bar{x}) = \theta(\bar{u}, \bar{v}) \Rightarrow$$

$\Rightarrow$ all inequalities are, in fact, equalities,

i.e. $L(\bar{x}, \bar{u}, \bar{v}) = \min_{x \in X} L(x, \bar{u}, \bar{v})$, $\bar{u}^T g(\bar{x}) = 0 \Rightarrow$

$\Rightarrow (\bar{x}, \bar{u}, \bar{v})$ is the saddle point by the same Lemma above. ∎

Remark: sufficient condition for global min without convexity.



Ex $\min(3x^2 - 4xy + 2y^2) \mid x^2 - y^2 \geq 1$.

Here $X = \mathbb{R}^2$, $g(x,y) = 1 - x^2 + y^2$.

$$L(x,y,u) = 3x^2 - 4xy + 2y^2 + u(1 - x^2 + y^2) =$$

$$= (3-u)x^2 - 4xy + (2+u)y^2 + u =$$

$$= \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} 3-u & -2 \\ -2 & 2+u \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}}_{Q(x,y)} + u.$$

$$\theta(u) = \inf_{x,y} L(x,y,u) = \inf_{x,y} Q(x,y) + u.$$

How to minimize $x^T H x$?

- if $x^T H x \geq 0, \forall x$, i.e. H – pos. semidef., then $\inf_x x^T H x = 0$.
- otherwise $\exists x_0 : x_0^T H x_0 < 0$ and $\inf_x x^T H x = -\infty$ (take $x = t x_0$, $t \to \infty$).

Hence, $\theta(u) = \begin{cases} u & \text{if } Q \text{ pos. semidef.,} \\ -\infty & \text{otherwise.} \end{cases}$

For which $u \geq 0$ is Q pos. semidef. ?

- $\det Q_1 = 3 - u$,
- $\det Q_2 = (3-u)(2+u) - 4 = 2 + u - u^2 = (u+1)(2-u)$.

Necessary: both $\geq 0 \Rightarrow 0 \leq u \leq 2$.

But then $\det Q_1 > 0 \Rightarrow$ sufficient too.

$$\theta(u) = \begin{cases} u & \text{if } 0 \leq u \leq 2, \\ -\infty & \text{otherwise.} \end{cases}$$

<u>The dual problem</u>: $\max\limits_{0\le u\le 2} u$.

Easy to solve: $\bar u = 2$, $\theta(\bar u) = \boxed{2}$.

If we find $\bar x, \bar y : f(\bar x, \bar y) = \boxed{2} \Rightarrow$

$\Rightarrow$ no duality gap $\Rightarrow$ global min.

1) $\bar x, \bar y$ should solve $\inf\limits_{x,y} L(x, y, \bar u)$,

2) $\bar u\, g(\bar x, \bar y) = 0$.

$\bar u = 2 \Rightarrow L(x, y, \bar u) = x^2 - 4xy + 4y^2 + 2 =$

$= (x - 2y)^2 + 2$.

$$\left.\begin{aligned} &1) \Rightarrow \underline{\underline{\bar x = 2\bar y}}. \\ &2) \Rightarrow \bar x^2 - \bar y^2 = 1 \end{aligned}\right\} \Rightarrow 3\bar y^2 = 1 \Leftrightarrow \bar y = \pm\frac{1}{\sqrt{3}}$$

$\Rightarrow \bar x = \pm\frac{2}{\sqrt{3}}$ and $f(\bar x, \bar y) = \frac{12}{3} - \frac{8}{3} + \frac{2}{3} = \boxed{2}$.

By Th. about no duality gap we have that $(\bar x, \bar y)$ is the <u>global minimum</u>.

Geometrical interpretation of duality

($\approx$ Ex. 3, p. 300 + Remark about G)

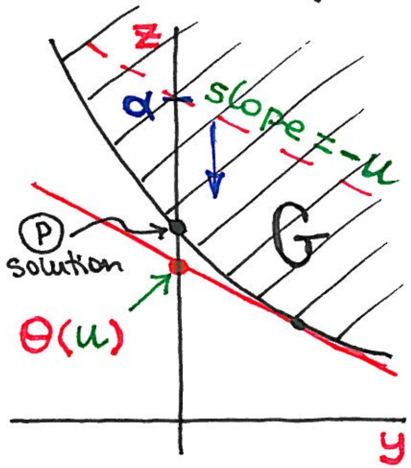
(P) $\min\limits_{x\in S} f(x)$, $S = \{x \in \underline{X} \subset \mathbb{R}^n \mid g(x) \le 0\}$
only <u>one</u> ineq.

Define $G = \{(y, z) \mid \exists x \in \underline{X} : g(x) \le y,\ f(x) \le z\}$

<u>Exercise</u>: $\underline{X}$ convex, f, g convex $\Rightarrow$ G convex.

$\theta(u) = \inf\limits_{x\in \underline{X}} (f(x) + u g(x)) = \min\limits_{(y,z)\in G} (z + uy)$.

["$\le$": $f(x) + u g(x) \le z + uy$; "$\ge$": $(g(x), f(x)) \in G$]



$z + uy = \alpha$: a line with the slope $-u$

$\theta(u) = \min\{\alpha \mid \text{line} \cap G \ne \emptyset\}$

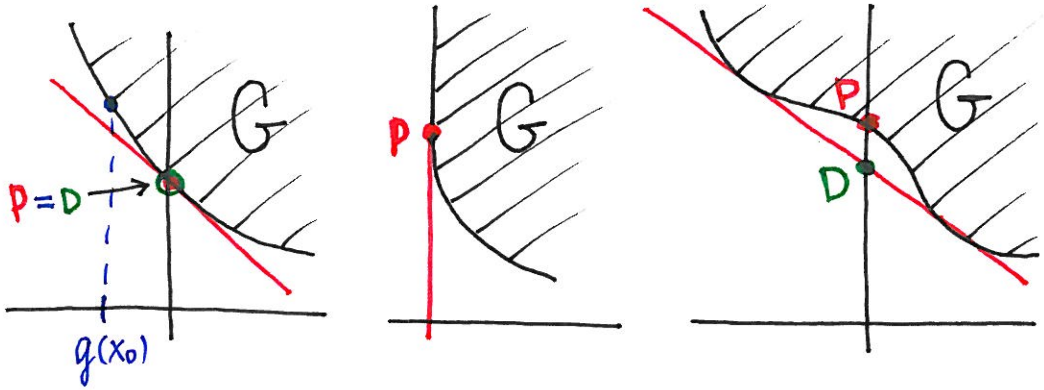
$\alpha = z$ when $y = 0$.

$\theta(u)$ is the z-coordinate of intersection of the tangent line with the z-axis.

Slater condition:

$$\exists\, x_0 \in \underline{X} : \ g(x_0) < 0$$

(P) convex + Slater $\Rightarrow$ no duality gap.



convex + Slater

convex, no Slater

not convex

Remark: Slater $\Rightarrow$ no CQ points.

Ex $\min e^{-x} \ \Big| \ \dfrac{x^2}{y} \le 0, \ y > 0.$

Take $\underline{X} = \{y > 0\}$, $g(x,y) = \dfrac{x^2}{y}$.

$$\nabla g = \begin{bmatrix} \frac{2x}{y} \\ -\frac{x^2}{y^2} \end{bmatrix}, \quad \nabla^2 g = \begin{bmatrix} \frac{2}{y} & -\frac{2x}{y^2} \\ -\frac{2x}{y^2} & \frac{2x^2}{y^3} \end{bmatrix}$$

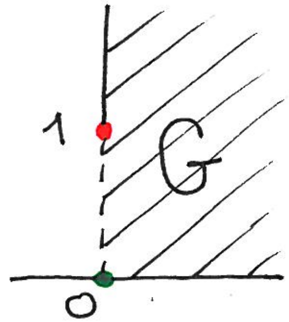
$\frac{2}{y} > 0$, $\frac{4x^2}{y^4} - \frac{4x^2}{y^4} = 0 \Rightarrow$ pos. semidef. $\Rightarrow$ convex

$$\left.\begin{array}{l} L = e^{-x} + u\dfrac{x^2}{y} \geq 0. \\ \text{Take } y = x^3,\ x \to +\infty \Rightarrow L \to 0 \end{array}\right\} \Rightarrow$$

$\Rightarrow \theta(u) = \inf L = 0, \ \forall u \geq 0.$

Dual: $\max\limits_{u \geq 0} \theta(u) = \boxed{0} = D$

Primal: $\dfrac{x^2}{y} \le 0 \overset{y>0}{\Longleftrightarrow} x^2 \le 0 \Leftrightarrow x = 0 \Rightarrow$



$\Rightarrow e^{-x} = e^0 = \boxed{1} = P$

NB: No Slater here.