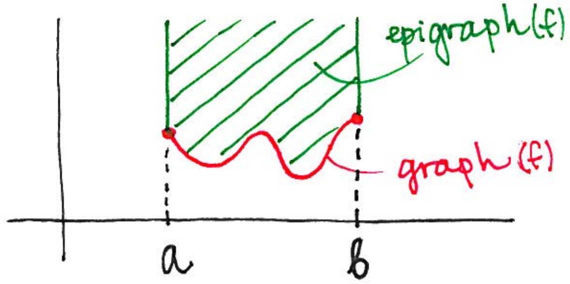
Ch. 6 Convex functions

Let $S \subset \mathbb{R}^n$, $f : S \to \mathbb{R}$

graph$(f) = \{(x,y) \in \mathbb{R}^{n+1} \mid x \in S,\ y = f(x)\}$

epigraph$(f) = \{(x,y) \in \mathbb{R}^{n+1} \mid x \in S,\ y \geq f(x)\}$

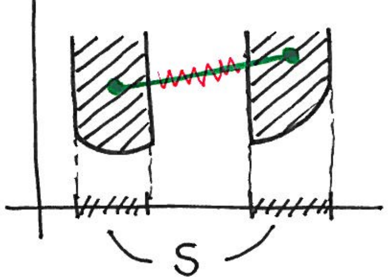
Ex $S = [a, b]$



Def $f \overset{\text{def}}{=}$ convex function if epigraph(f) is a convex set.

Remark: then S must necessarily be convex too.

Ex

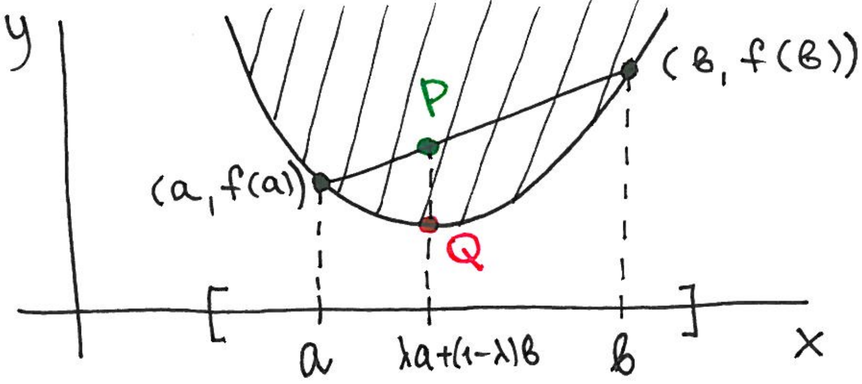


S not convex $\Rightarrow$

$\Rightarrow$ epigraph(f) not convex too.

6.1. Convex functions of one variable.

$S =$ interval I, $f : I \to \mathbb{R}$



$Q = (\lambda a + (1-\lambda) b,\ f(\lambda a + (1-\lambda) b))$,

$P = \lambda \cdot (a, f(a)) + (1-\lambda) \cdot (b, f(b)) =$

$= (\lambda a + (1-\lambda) b,\ \lambda f(a) + (1-\lambda) f(b))$.

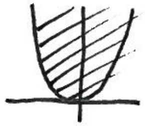
Def (alt.) $f \overset{\text{def}}{=}$ convex on I if

$$\forall a, b \in I : f(\lambda a + (1-\lambda) b) \leq \lambda f(a) + (1-\lambda) f(b),\quad \forall \lambda \in [0,1]$$

Ex. $f(x) = kx + m$ is convex (prove yourself).

Ex $f(x) = x^2$ is convex

Proof: easy to see by epigraph



By the algebraic definition (harder):

$$f(\lambda a + (1-\lambda) b) = (\lambda a + (1-\lambda) b)^2 =$$

$$= \lambda^2 a^2 + 2\lambda(1-\lambda) ab + (1-\lambda)^2 b^2 \leq {}^{*)}$$

$$\leq \lambda^2 a^2 + \lambda(1-\lambda)(a^2 + b^2) + (1-\lambda)^2 b^2 =$$

$$= \lambda a^2 + (1-\lambda) b^2 = \lambda f(a) + (1-\lambda) f(b).$$

$*)$ $(a-b)^2 = a^2 - 2ab + b^2 \geq 0.$

Remark:

- strictly convex if $<$ in the definition.
- $f \overset{\text{def}}{=}$ concave if $(-f)$ convex.
- f - convex function $\Rightarrow$

$\Rightarrow \{x \in S \mid f(x) \leq \text{const}\}$ convex set.

NB Not $\Leftrightarrow$!

Ex $f(x) = -x^2$ is not convex, but

$\{x \in \mathbb{R} \mid f(x) \leq 0\} = \mathbb{R}$ - convex.

Lemma: f, g convex on I $\Rightarrow$

$\Rightarrow h(x) = f(x) + g(x)$ is convex.

Proof: By definition, let $a, b \in I$, $0 \leq \lambda \leq 1$

and estimate $h(\lambda a + (1-\lambda) b) =$

$$= f(\lambda a + (1-\lambda) b) + g(\lambda a + (1-\lambda) b) \leq$$

$$\leq \lambda f(a) + (1-\lambda) f(b) + \lambda g(a) + (1-\lambda) g(b) =$$

$$= \lambda \big(\underbrace{f(a) + g(a)}_{h(a)}\big) + (1-\lambda)\big(\underbrace{f(b) + g(b)}_{h(b)}\big). \quad \blacksquare$$

Remark: we will use the function

$$h(x) = f(x) - f(a) - f'(a)(x-a) =$$

$$= f(x) + \big(\underbrace{-f(a) - f'(a)(x-a)}_{= kx+m \text{ - convex}}\big).$$

Then f convex $\Rightarrow$ h convex.

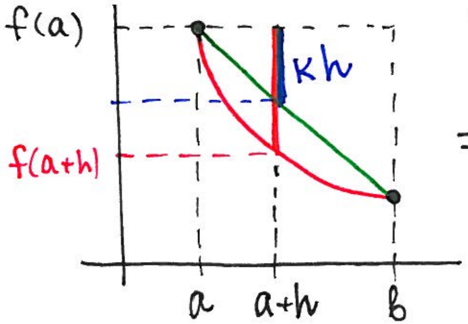
Some properties of convex functions

Corollary 1, p. 199

① If f convex on I then

$$\boxed{\text{local min} \Rightarrow \text{global min}}$$

Proof: let a be a local min.



If a not glob. min $^{*)}$ $\Rightarrow$
$\Rightarrow \exists\, b \in I : f(b) < f(a)$.
Assume $b > a$ (for $b < a$ – similar proof).

Then the slope $= -k < 0$ and

$$f(a+h) \leq f(a) - kh < f(a)$$

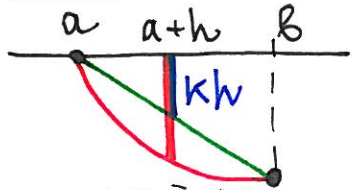
$\forall\, h \in \,]0, b-a[$. ⚡

since loc. min. $\Rightarrow f(a+h) \geq f(a)$ for small h. Hence $*)$ is wrong. ■

①' $f \in C^1(I)$, convex and $f(a) = f'(a) = 0 \Rightarrow$

$$\Rightarrow f(x) \geq 0 \text{ on } I.$$

Proof: let $\exists\, b \in I : f(b) < 0$. $^{*)}$



Assume again $b > a \Rightarrow$

$\Rightarrow$ slope $= -k < 0 \Rightarrow$

$$\Rightarrow \left| \frac{f(a+h) - f(a)}{h} \right| \geq \frac{kh}{h} = k, \quad \forall h \in \,]0, b-a[$$

$\xrightarrow{h \to 0^+} f'(a) \Rightarrow f'(a) \neq 0$ ⚡ $\Rightarrow$

$\Rightarrow$ $*)$ is wrong and $f(x) \geq 0$ on I. ■

Corollary 2, p. 202

② $f \in C^1(I)$, convex. Then

$$\boxed{f'(a) = 0 \Rightarrow a \text{ – global min.}}$$

Proof: apply ①' to $h(x) = f(x) - f(a)$.

h convex, $h(a) = h'(a) = 0 \Rightarrow$

$\Rightarrow h(x) = f(x) - f(a) \geq 0 \Rightarrow f(x) \geq f(a)$. ■
$\forall x \in I$.

≈ Th. 3, p. 202 (only $\Rightarrow$)

3 $f \in C^1(I)$. Then

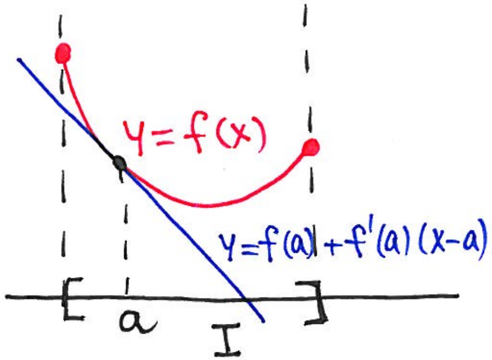
$$f \text{ convex} \iff f(x) \geq f(a) + f'(a)(x-a) \quad \forall\, x, a \in I$$

Interpretation: *)

$$f(x) \geq f(a) + f'(a)(x-a)$$

$$\Updownarrow$$

the graph is above its tangent line.



Proof: $\boxed{\Rightarrow}$ Let $a \in I$ and define

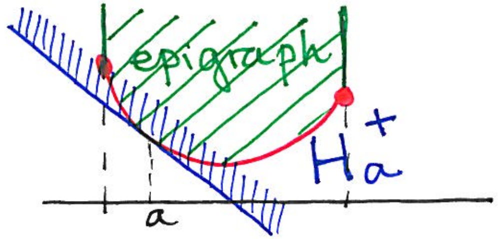
$$h(x) = f(x) - f(a) - f'(a)(x-a).$$

By Remark, h convex, $h(a) = h'(a) = 0 \Rightarrow$

$\Rightarrow$ by 1' $h(x) \geq 0$ on I.

$\boxed{\Leftarrow}$ *) means that

$$\text{epigraph}(f) \subset H_a^+$$



We prove that, in fact

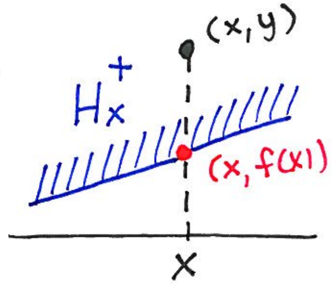
$$\text{epigraph}(f) = \underbrace{\bigcap_{a \in I} H_a^+ \cap \{(x,y) \mid x \in I\}}_{\text{convex}} \Rightarrow f \text{ convex}.$$

"$\subset$" trivial by *), $\forall\, a \in I$.

"$\supset$" Let $(x,y) \in \bigcap_{a \in I} H_a^+$, $x \in I$.

$\Rightarrow (x,y) \subset H_x^+ \Rightarrow y \geq f(x)$,

i.e. $(x,y) \in \text{epigraph}(f)$. ∎



! 4 $f \in C^2(I)$. Then

$$f \text{ convex} \iff f'' \geq 0 \text{ on } I$$

Very important test!

Proof: Let $a \in I$ and define

$$h(x) = f(x) - f(a) - f'(a)(x-a).$$

$\boxed{\Rightarrow}$ h convex, $h'(a) = 0 \Rightarrow$

$\Rightarrow$ by (2) a - glob. min. $\Rightarrow h''(a) \geq 0 \Rightarrow$

$\Rightarrow f''(a) = h''(a) \geq 0.$

$\boxed{\Leftarrow}$ $h''(x) = f''(x) \geq 0 \Rightarrow h' \nearrow$ on $I \Rightarrow$

		a	
h'	$-$	0	$+$
h	$\searrow$	0	$\nearrow$

$\Rightarrow h(x) \geq 0 \Rightarrow$

$$\Rightarrow f(x) \geq f(a) + f'(a)(x-a) \overset{(3)}{\Rightarrow} f \text{ convex.} \blacksquare$$

Ex $f(x) = e^x$ - convex on $\mathbb{R}$.

Proof: $f''(x) = e^x \geq 0$.

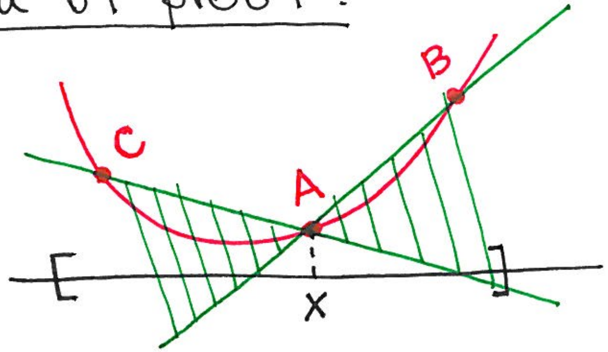
Ex $f(x) = -\ln x$ - convex on $]0, +\infty[$

Proof: $f'(x) = -\frac{1}{x}$, $f''(x) = \frac{1}{x^2} > 0$.

Remark: convex $\not\Rightarrow$ continuous, but

$\boxed{f \text{ convex on } [a,b] \Rightarrow \text{continuous on }]a,b[}$

Idea of proof:



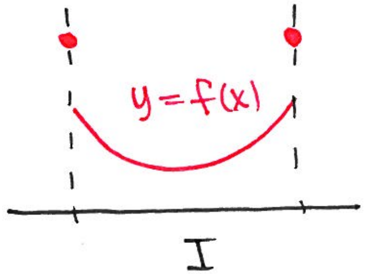
$x \in \text{int}(I) \Rightarrow \exists\, B, C$ on the graph:

$C < A < B$. Convexity implies that

$\boxed{\text{graph between } C, B \subset \text{green cone.}} \Rightarrow$

$\Rightarrow f$ continuous at x.

Ex



The function $f(x)$ is convex, but not continuous at the end points.