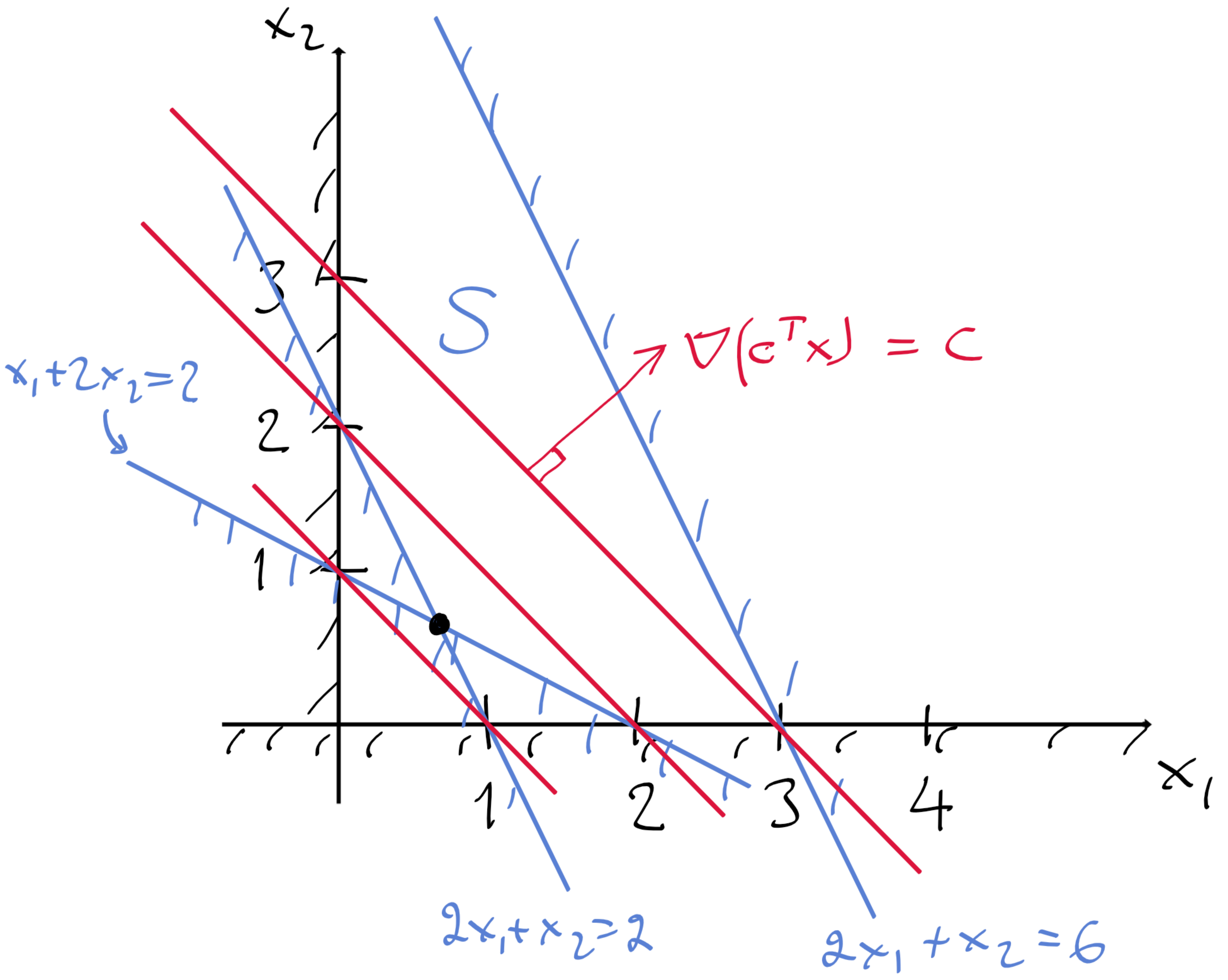
5. Linear programming (LP)

Problem

$$\text{minimize } c^T x$$
$$\text{subject to } x \in S = \{x \in \mathbb{R}^n : Ax \le b\}$$

polyhedral set = intersection of closed half-spaces (convex set)

Ex. $c^T x = x_1 + x_2$, $S = \left\{ x \in \mathbb{R}^2 : \begin{array}{l} x_1 + 2x_2 \ge 2 \\ 2x_1 + x_2 \ge 2 \\ 2x_1 + x_2 \le 6 \\ x_1 \ge 0 \\ x_2 \ge 0 \end{array} \right\}$

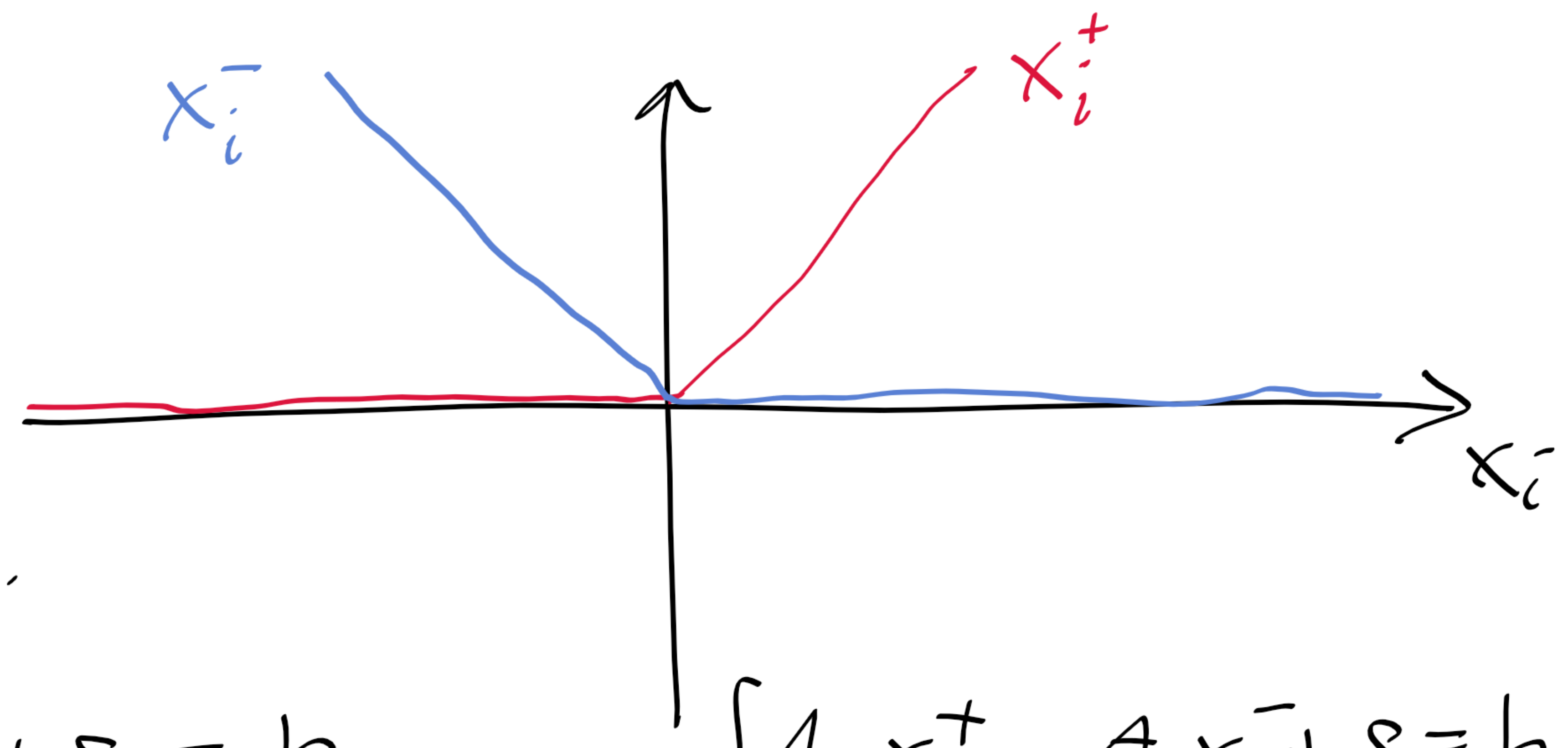


Minimum occurs at an extreme point • (corner). This is always the case for LP problems unless $S = \emptyset$ or $\inf_{x \in S} c^T x = -\infty$ (Thm 4)

Let us rewrite S in the canonical form $\left\{ \begin{array}{l} \tilde{A}\tilde{x} = \tilde{b} \\ \tilde{x} \ge 0 \end{array} \right\}$.

Define slack variables $s := b - Ax \ge 0$ and

$$x_i^+ = \max(0, x_i) \ge 0$$
$$x_i^- = \min(0, -x_i) \ge 0$$



then $x_i = x_i^+ - x_i^-$, $i = 1, \dots, n$.

Now
$$Ax \le b \iff \left\{ \begin{array}{l} Ax + s = b \\ s \ge 0 \end{array} \right. \iff \left\{ \begin{array}{l} Ax^+ - Ax^- + s = b \\ x^+, x^-, s \ge 0 \end{array} \right.$$

$$\iff \left\{ \begin{array}{l} \tilde{A}\tilde{x} = b \\ \tilde{x} \ge 0 \end{array} \right. \text{ with } \tilde{x} = \begin{pmatrix} x^+ \\ x^- \\ s \end{pmatrix} \text{ and } \tilde{A} = \begin{pmatrix} A & -A & I \end{pmatrix}$$

Remark: If you have $|x_i| = x_i^+ + x_i^-$

Canonical form of LP problem:

$$(LP)\qquad \begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

In (LP), there are generally too many extreme point to check. The <u>simplex method</u> on the canonical form (LP) moves from one extreme point to another with lower function values. (Ch. 5.2, not in this course). Problems

- $S = \emptyset$
- S unbounded and $\inf_{x \in S} c^T x = -\infty$
- How to find a starting point.

Duality may help (Ch. 5.3) since (LP) is (CP).

$$\theta(u, v) = \inf_{x \in \mathbb{R}^n} \mathcal{L}(x; u, v) = \inf\left(c^T x - u^T x + v^T(b - Ax)\right)$$

$$= \inf_{x \in \mathbb{R}^n} \left(b^T v + (c - u - A^T v)^T x \right)$$

$$= \begin{cases} b^T v & \text{if } c - A^T v = u \geq 0 \\ -\infty & \text{otherwise} \end{cases}$$

Alt.: Let $X = \{x : x \geq 0\}$ etc. with the same conclusion!

$$(D)\qquad \begin{array}{ll} \text{maximize} & b^T v \\ \text{subject to} & A^T v \leq c \end{array}$$

(Can be "solved" with a barrier method)

If $\bar{v}$ solves (D) and $\bar{x} \in S$ with $b^T \bar{v} = c^T \bar{x}$, then duality theory gives $\bar{x}$ solves (LP).

We compute the dual of (D):

$$\max_{A^T v \leq c} b^T v = - \min_{A^T v \leq c} (-b^T v)$$

$$\theta(\xi) = \inf_{v \in \mathbb{R}^m} \mathcal{L} = \inf \left(-b^T v + \xi^T (A^T v - c) \right)$$

$$= \inf_v \left(-c^T \xi + (A\xi - b)^T v \right) = \begin{cases} -c^T \xi & \text{if } A\xi = b \\ -\infty & \text{otherwise} \end{cases}$$

Thus

$$-\max \theta(\xi) = -\max(-c^T \xi) = \min c^T \xi$$

and the dual of (D) is:

$$\begin{aligned} &\text{minimize} \quad c^T \xi \\ &\text{subject to} \quad A\xi = b \\ &\qquad\qquad\qquad \xi \geq 0 \end{aligned}$$

which is the primal (LP) on canonical form ($\xi = x$).

Note: Complementary slackness of (D):

$$\text{(CS)} \quad \underbrace{x^T}_{\geq 0} \underbrace{(A^T v - b)}_{\leq 0} = 0$$

Ex. (P)

$$\begin{aligned} &\text{minimize} \quad c^T x = \begin{pmatrix} -2 & -2 & 6 & 0 & 0 \end{pmatrix} x \\ &\text{subject to} \quad \begin{pmatrix} -1 & -2 & 2 & -1 & 0 \\ -2 & -1 & 1 & 0 & -1 \end{pmatrix} x = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \Leftrightarrow Ax = b \\ &\qquad\qquad\qquad x \geq 0 \end{aligned}$$

The dual is

(D)

$$\begin{aligned} &\text{maximize} \quad b^T v = -v_1 - v_2 \\ &\text{subject to} \quad \begin{cases} -v_1 - 2v_2 & \leq -2 \\ -2v_1 - v_2 & \leq -2 \\ 2v_1 + v_2 & \leq 6 \\ -v_1 & \leq 0 \\ -v_2 & \leq 0 \end{cases} \Leftrightarrow A^T v \leq c \end{aligned}$$

This is the problem (LP1) in video 1. Solution is $\bar{v}_1 = \bar{v}_2 = \frac{2}{3}$ and $b^T \bar{v} = -\frac{4}{3}$

To find the corresponding $\bar{x}$ of (P) we use

(CS) $\quad \bar{x}^T (A^T \bar{v} - c) = 0$

Since $\bar{v} = \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix}$ we get $\bar{x}_3 = \bar{x}_4 = \bar{x}_5 = 0$.

Then $Ax = b$ gives $\begin{cases} -\bar{x}_1 - 2\bar{x}_2 = -1 \\ -2\bar{x}_1 - \bar{x}_2 = -1 \end{cases} \iff \bar{x}_1 = \bar{x}_2 = \frac{1}{3}$

Hence, $\bar{x} = \begin{pmatrix} 1/3 \\ 1/3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $c^T \bar{x} = -2 \cdot \frac{1}{3} - 2 \cdot \frac{1}{3} + 0 = -\frac{4}{3}$

General duality for LP

(P) minimize $c^T x = (c_1^T \ c_2^T)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ ← vectors

subject to $(A_{11} \ A_{12})\, x \geq b_1$

$(A_{21} \ A_{22})\, x = b_2$

$x_1 \geq 0$

x_2 free

Let $X = \{x : x_1 \geq 0\}$

$$\mathcal{L}(x; u, v) = c_1^T x_1 + c_2^T x_2 + u^T(b_1 - A_{11}x_1 - A_{12}x_2) + v^T(b_2 - A_{21}x_1 - A_{22}x_2)$$

$$= u^T b_1 + v^T b_2 + x_1^T \underbrace{(c_1 - A_{11}^T u - A_{21}^T v)}_{\geq 0} + x_2^T (c_2 - A_{12}^T u - A_{22}^T v)$$

where $u \geq 0$

Let $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and $y = \begin{pmatrix} u \\ v \end{pmatrix}$. Then

$$\mathcal{L}(x; y) = c^T x + y^T (b - Ax) = y^T b + x^T (c - A^T y)$$

$$\theta(y) = \inf_{x_1 \geq 0} \mathcal{L}(x; y) = \begin{cases} b^T y & \text{if } \underline{(A_{11}^T \ A_{21}^T)\, y \leq c_1} \text{ and } (A_{12}^T \ A_{22}^T)\, y = c_2 \\ -\infty & \text{otherwise} \end{cases}$$

(D) maximize $b^T y$

subject to $(A_{11}^T \ A_{21}^T)\, y \leq c_1$

$(A_{12}^T \ A_{22}^T)\, y = c_2$

$u \geq 0$

v free

Thm: The dual of (D) is (P).

Proof: $\max b^T y = -\min(-b^T y)$ and $-(A_{11}^T \ A_{21}^T)\, y \geq -c_1$

$-(A_{12}^T \ A_{22}^T)\, y = -c_2$

$u \geq 0$

v free

The dual of (D) satisfies

$-\max(-c^T x) = \min c^T x$

and $\begin{cases} -(A_{11} \ A_{12})\, x \leq -b_1 \\ -(A_{21} \ A_{22})\, x = -b_2 \end{cases} \Leftrightarrow \begin{cases} (A_{11} \ A_{12})\, x \geq b_1 \\ (A_{21} \ A_{22})\, x = b_2 \end{cases}$ and the L-multiplier is $x_1 \geq 0$

From duality theory:

THEOREM 1. $(\bar{\mathbf{x}}; \bar{\mathbf{u}}, \bar{\mathbf{v}}) \in X \times U$ *is a saddle point* $\Longrightarrow$ $\bar{\mathbf{x}}$ *solves (P) and (KKT) holds.*

THEOREM 3. $(\bar{\mathbf{x}}; \bar{\mathbf{u}}, \bar{\mathbf{v}}) \in X \times U$ *is a saddle point* $\Longleftrightarrow$ $\bar{\mathbf{x}}$ *is feasible and* $\theta(\bar{\mathbf{u}}, \bar{\mathbf{v}}) = f(\bar{\mathbf{x}})$.

THEOREM 4. *If (CQ) holds at a solution $\bar{\mathbf{x}}$ of (CP), then there is no duality gap between (CP) and its dual problem.*

Thm 6 (Ch 5). Equivalent statements for for feasible $\bar{x}$ for (P) and $\bar{y}$ for (D):

(i) $\bar{x}$ solves (P) and $\bar{y}$ solves (D)

(ii) $c^T \bar{x} = b^T \bar{y}$

(iii) (CS) $\begin{cases} \bar{y}^T (b - A\bar{x}) = 0 \\ \bar{x}^T (c - A^T \bar{y}) = 0 \end{cases}$

Proof: (i) => (ii) Thm 4 from duality. (ii) => (i) : Thms 3 and 1 above

(ii) <=> (iii) : $\forall$ feasible x and y :

$$b^T y \le b^T y + x^T (c - A^T y) = L(x; y) = c^T x + y^T (b - Ax) \le c^T x$$

Insert $\bar{x}$ and $\bar{y}$. #