**Lecture: Linear algebra.**

1. Subspaces.
2. Orthogonal complement.
3. The four fundamental subspaces
4. Solutions of linear equation systems
   - The fundamental theorem of linear algebra
5. Determining the fundamental subspaces

# Vector Spaces

A *vector space* $V$ is a set on which *vector addition* $+$ and *scalar multiplication* $\cdot$ are such that

V1 For all $v_1, v_2, v_3 \in V$, $v_1+(v_2+v_3) = (v_1+v_2)+v_3$

V2 Exists a zero vector $0 \in V$, such that $v+0 = 0+v = v$ for all $v \in V$

V3 For each $v \in V$ there exists a unique $-v \in V$ such that
$v+(-v) = (-v)+v = 0$

V4 For all $v_1, v_2 \in V$, $v_1+v_2 = v_2+v_1$

V5 For all $v \in V$, $1 \cdot v = v$

V6 For all $\alpha, \beta \in \mathbb{R}$ and all $v \in V$, $\alpha \cdot (\beta \cdot v) = (\alpha\beta) \cdot v$

V7 For all $\alpha, \beta \in \mathbb{R}$ and all $v \in V$, $(\alpha+\beta) \cdot v = \alpha \cdot v + \beta \cdot v$

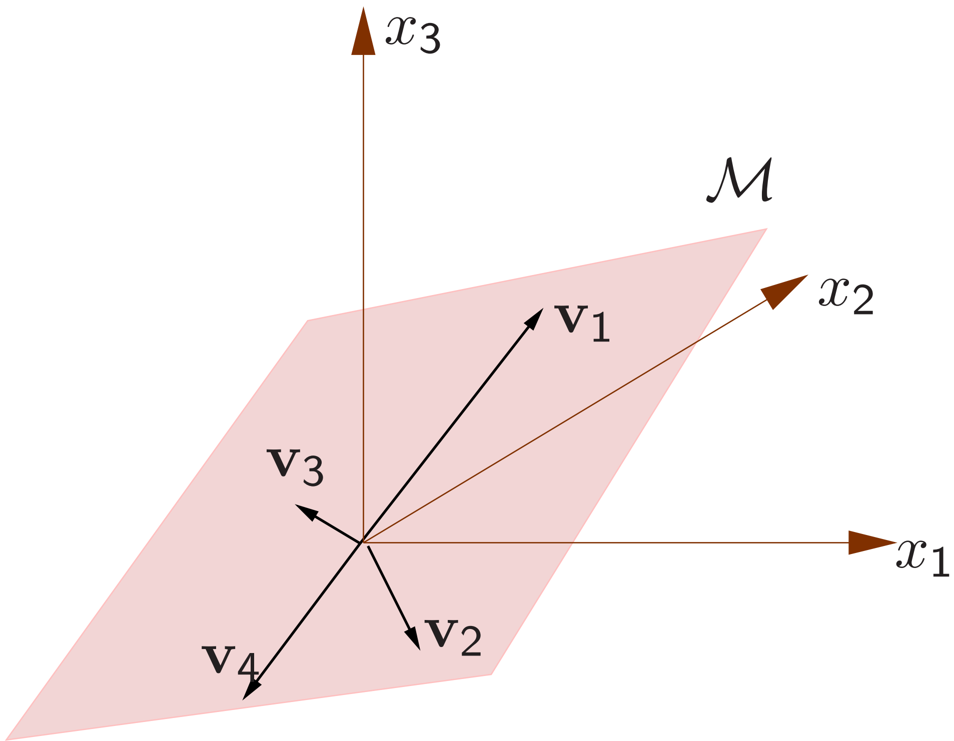
V8 For all $\alpha \in \mathbb{R}$ and all $v_1, v_2 \in V$, $\alpha \cdot (v_1+v_2) = \alpha \cdot v_1 + \alpha \cdot v_2$

# Subspaces

**Definition 1.** *A subset $\mathcal{M} \subset \mathbf{R}^n$ is called a subspace in $\mathbf{R}^n$ if for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{M}$ and $\alpha_1, \alpha_2 \in \mathbf{R}$ it holds that*

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 \in \mathcal{M}$$

Note that it always holds that $0 \in \mathcal{M}$.



## Basis for a subspace

The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent if there exists no scalars $\alpha_1, \cdots, \alpha_k$ (not all zero) such that $\sum_{\ell=1}^{k} \alpha_\ell \mathbf{v}_\ell = 0$.

**Definition 2.** *$\mathcal{M}$ is spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_k$ if for each $\mathbf{v} \in \mathcal{M}$ there are $\alpha_1, \ldots, \alpha_k \in$ **R** such that*

$$\sum_{l=1}^{k} \alpha_l \mathbf{v}_l = \mathbf{v}$$

*If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent, then they form a basis for $\mathcal{M}$ and the dimension of $\mathcal{M}$ is $k$, and it is denoted* $\dim \mathcal{M} = k$.

**Notation:**

$\mathcal{M} = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \{\sum_{l=1}^{k} \alpha_l \mathbf{v}_l : \alpha_l \in \mathbf{R};\ l = 1, \ldots, k\}$.

# Sums of subspaces and orthogonal subspaces

Given two subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$, the subset $\mathcal{M} = \{\mathbf{v}_1 + \mathbf{v}_2 \,:\, \mathbf{v}_1 \in \mathcal{M}_1, \mathbf{v}_2 \in \mathcal{M}_2\}$ is a subspace and we write $\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2$.

Two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal if $\mathbf{v}_1^T\mathbf{v}_2 = \|\mathbf{v}_1\|\|\mathbf{v}_2\| \cos\theta = 0$, *i.e.*, the angle $\theta$ between them is $\pi/2$, and we write $\mathbf{v}_1 \perp \mathbf{v}_2$.

Two subspaces $\mathcal{M}_1$ and $\mathcal{M}_2$ are orthogonal if $\mathbf{v}_1 \perp \mathbf{v}_2$ for all $\mathbf{v}_1 \in \mathcal{M}_1$ and $\mathbf{v}_2 \in \mathcal{M}_2$, and we write $\mathcal{M}_1 \perp \mathcal{M}_2$.

If $\dim \mathcal{M} = \dim \mathcal{M}_1 + \dim \mathcal{M}_2$, then every $\mathbf{v} \in \mathcal{M}$ can be uniquely written on the form $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1 \in \mathcal{M}_1$ and $\mathbf{v}_2 \in \mathcal{M}_2$.

# Direct sums of subspaces

A linear space $\mathcal{M}$ (for example $\mathbf{R}^n$) is a direct sum of the subspaces $\mathcal{M}_1, \mathcal{M}_2$ (i.e. $\mathcal{M}_1, \mathcal{M}_2 \subset \mathcal{M}$) if $\mathcal{M}_1 \perp \mathcal{M}_2$ and $\mathcal{M}_2 + \mathcal{M}_2 = \mathbf{R}^n$. The direct sum is denoted $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$, and every $\mathbf{v} \in \mathcal{M}$ can be written uniquely $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$ for some $\mathbf{v}_1 \in \mathcal{M}_1$ and $\mathbf{v}_2 \in \mathcal{M}_2$.

**Example 1.** *Assume*

$$\begin{aligned}
\mathcal{M}_1 &= \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \\
\mathcal{M}_2 &= \text{span}\{\mathbf{v}_{k+1}, \ldots, \mathbf{v}_l\} \\
\mathcal{M} &= \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_l\}
\end{aligned}$$

*where $\mathbf{v}_1, \ldots, \mathbf{v}_l$ are linearly independent vectors. Then it holds that*

$$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$$

*if $\mathbf{v}_i \perp \mathbf{v}_j$ for all $i = 1, \ldots, k$ and $j = k+1, \ldots, l$.*

# Orthogonal complement

The orthogonal complement to a subspace $\mathcal{M} \subset \mathbf{R}^n$ is defined by

$$\mathcal{M}^\perp = \{\mathbf{w} \in \mathbf{R}^n : \mathbf{w}^T\mathbf{v} = 0,\ \forall\, \mathbf{v} \in \mathcal{M}\}$$

The following holds

- $\mathcal{M}^\perp$ is a subspace
- $\mathbf{R}^n = \mathcal{M} \oplus \mathcal{M}^\perp$.
- If $\dim \mathcal{M} = k$, then $\dim \mathcal{M}^\perp = n - k$.
- $(\mathcal{M}^\perp)^\perp = \mathcal{M}$.

# The four fundamental subspaces

Consider the linear operator $A : \mathbf{R}^n \to \mathbf{R}^m$,

$$\mathbf{A} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{a}}_1 & \hat{\mathbf{a}}_2 & \dots & \hat{\mathbf{a}}_n \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{a}}_1^T \\ \bar{\mathbf{a}}_2^T \\ \vdots \\ \bar{\mathbf{a}}_m^T \end{bmatrix}$$

The Range space: $\mathcal{R}(\mathbf{A}) = \{\mathbf{A}\mathbf{x} | \mathbf{x} \in \mathbf{R}^n\} =$

$$= \left\{ \sum_{j=1}^{n} \hat{\mathbf{a}}_j x_j | x_j \in \mathbf{R}^n,\ j = 1, \dots, n \right\}$$

The Null space: $\mathcal{N}(\mathbf{A}) = \{\mathbf{x} \in \mathbf{R}^n | \mathbf{A}\mathbf{x} = \mathbf{0}\}$

$$= \{\mathbf{x} \in \mathbf{R}^n | \bar{\mathbf{a}}_i^T \mathbf{x} = \mathbf{0},\ i = 1, \dots, m\}$$

The Row space: $\mathcal{R}(\mathbf{A}^\mathsf{T}) = \{\mathbf{A}^\mathsf{T}\mathbf{u} | \mathbf{u} \in \mathbf{R}^m\}$

$$= \left\{ \sum_{i=1}^{m} \bar{\mathbf{a}}_i u_i \middle| u_i \in \mathbf{R}^n,\ i = 1, \ldots, m \right\}$$

The left Null space: $\mathcal{N}(\mathbf{A}^\mathsf{T}) = \{\mathbf{u} \in \mathbf{R}^m | \mathbf{A}^\mathsf{T}\mathbf{u} = \mathbf{0}\}$

$$= \{\mathbf{u} \in \mathbf{R}^m | \hat{\mathbf{a}}_j^\mathsf{T}\mathbf{u} = \mathbf{0},\ j = 1, \ldots, n\}$$

$$= \{\mathbf{u} \in \mathbf{R}^m | \mathbf{u}^\mathsf{T}\mathbf{A} = \mathbf{0}\}$$

- $\mathcal{R}(\mathbf{A})$ and $\mathcal{N}(\mathbf{A}^\mathsf{T})$ are subspaces in $\mathbf{R}^m$
- $\mathcal{R}(\mathbf{A}^\mathsf{T})$ and $\mathcal{N}(\mathbf{A})$ are subspaces in $\mathbf{R}^n$

# The orthogonal complement of the fundamental subspaces

**Theorem 25.1** The following orthogonality relations hold

$(i)$ $\mathcal{R}(\mathbf{A})^{\perp} = \mathcal{N}(\mathbf{A}^{\mathsf{T}})$.

$(ii)$ $\mathcal{R}(\mathbf{A}^{\mathsf{T}})^{\perp} = \mathcal{N}(\mathbf{A})$.

$(iii)$ $\mathcal{N}(\mathbf{A})^{\perp} = \mathcal{R}(\mathbf{A}^{\mathsf{T}})$.

$(iv)$ $\mathcal{N}(\mathbf{A}^{\mathsf{T}})^{\perp} = \mathcal{R}(\mathbf{A})$.

# The Fundamental Theorem of Linear algebra

**Theorem** Let $\mathbf{A} \in \mathbf{R}^{m\times n}$. Then the following direct sums hold

$(a)$ $\mathbf{R}^n = \mathcal{R}(\mathbf{A}^T) \oplus \mathcal{N}(\mathbf{A})$

$(b)$ $\mathbf{R}^m = \mathcal{R}(\mathbf{A}) \oplus \mathcal{N}(\mathbf{A}^\mathsf{T})$

where $\mathcal{R}(\mathbf{A}^\mathsf{T}) \perp \mathcal{N}(\mathbf{A})$ and $\mathcal{R}(\mathbf{A}) \perp \mathcal{N}(\mathbf{A}^\mathsf{T})$.
Furthermore, it holds that $\dim \mathcal{R}(\mathbf{A}) = \dim(\mathbf{A}^\mathsf{T})$.
**Proof:** Since $\mathcal{N}(\mathbf{A}), \mathcal{R}(\mathbf{A}^\mathsf{T}) \subset \mathbf{R}^n$ according to the previous theorem, $\mathcal{R}(\mathbf{A}^\mathsf{T})^\perp = \mathcal{N}(\mathbf{A})$ it holds that $\mathbf{R}^n = \mathcal{R}(\mathbf{A}^T) \oplus \mathcal{N}(\mathbf{A})$.
The proof of $(b)$ follows from the same argument.

# Solution of linear equation systems

Let $\mathbf{A} \in \mathbf{R}^{m \times n}$ and $\mathbf{b} \in \mathbf{R}^n$ and consider the linear equation system

$$\mathbf{Ax} = \mathbf{b} \tag{1}$$

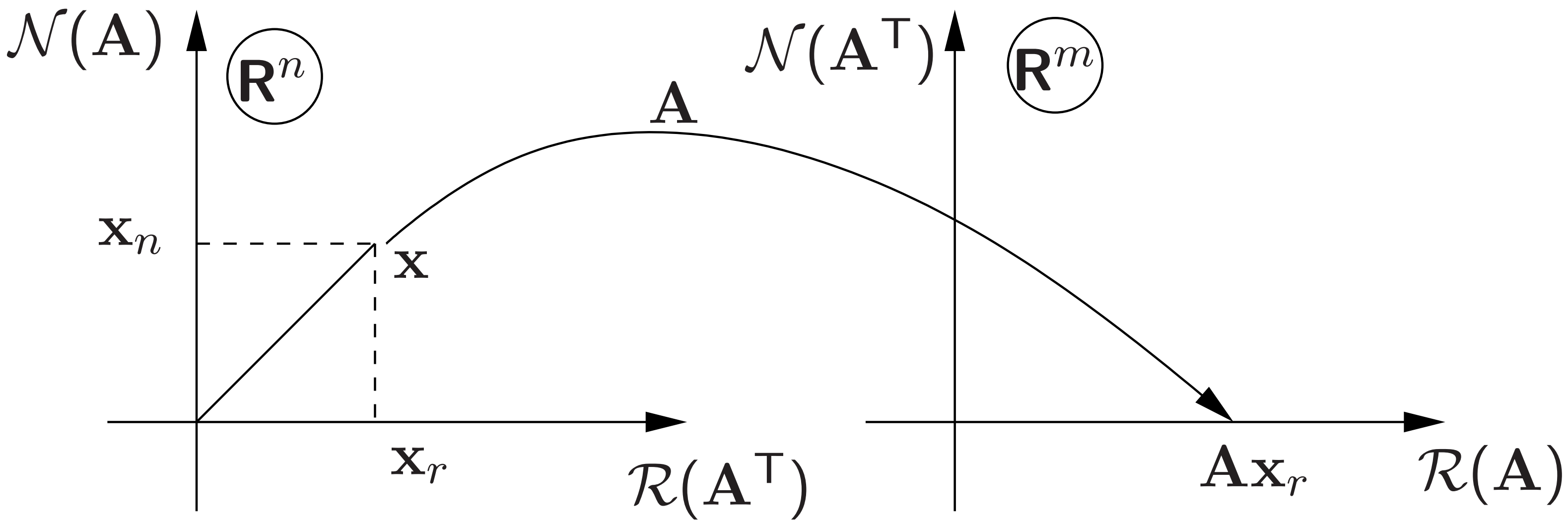
$(A)$ When does there exist a solution ?

$(B)$ Is the solution unique ? Which solution is chosen otherwise ?

$(C)$ How do we construct the solution?

The fundamental theorem of linear algebra answers the questions above.

# Geometric illustration of the fundamental theorem of linear algebra




The picture answers the questions $(A) - (C)$ above.

- The equation system $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a solution iff $\mathbf{b} \in \mathcal{R}(\mathbf{A})$.
- The solution is unique if, and only if, $\mathcal{N}(\mathbf{A}) = 0$. Otherwise, every solution can be decomposed into two components $\mathbf{x} = \mathbf{x}_r + \mathbf{x}_n$ where $\mathbf{x}_r \in \mathcal{R}(\mathbf{A}^\top)$ satisfies $\mathbf{A}\mathbf{x}_r = \mathbf{b}$ and $\mathbf{x}_n \in \mathcal{N}(\mathbf{A})$.
  Often we select the smallest norm solution, *i.e.*, $\mathbf{x}_n = \mathbf{0}$.

The following three slides dicusses the questions $(A) - (C)$ in more detail.

$(A)$ There is a solution to (1) if $\mathbf{b} \in \mathcal{R}(\mathbf{A})$

$(B)$ The solution is unique iff $\mathcal{N}(\mathbf{A}) = \mathbf{0}$. This follows since every $\mathbf{x} \in \mathbf{R}^n$ uniquely can be written $\mathbf{x} = \mathbf{x}_r + \mathbf{x}_n$ where $\mathbf{x}_r \in \mathcal{R}(\mathbf{A}^\mathsf{T})$ and $\mathbf{x}_n \in \mathcal{N}(\mathbf{A})$. We have $\mathbf{A}\mathbf{x} = \mathbf{A}\mathbf{x}_r + \mathbf{A}\mathbf{x}_n = \mathbf{A}\mathbf{x}_r$. The component $\mathbf{x}_n \in \mathcal{N}(\mathbf{A})$ is thus arbitrary and the solution is unique iff $\mathcal{N}(\mathbf{A}) = \mathbf{0}$.

Which solution do we chose if $\mathcal{N}(\mathbf{A}) \neq \mathbf{0}$?
One choice is to take the solution that has the shortest length measured in the Euclidean norm $\|\mathbf{x}\| = \sqrt{\mathbf{x}^\mathsf{T}\mathbf{x}}$. It follows that we should choose $\mathbf{x}_n = \mathbf{0}$.

$(C)$ Assume $\mathbf{b} \in \mathcal{R}(\mathbf{A})$. There are three cases for the construction of solutions

$(i)$ $\mathcal{N}(\mathbf{A}) = \mathbf{0}$ and $m = n$ (quadratic matrix).
Then the solution is determined by $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

$(ii)$ If $\mathcal{N}(\mathbf{A}) \neq 0$, then we choose the minimum norm solution solution given by $\mathbf{x} = \mathbf{A}^\mathsf{T}\mathbf{u}$, where $\mathbf{u} \in \mathbf{R}^m$ solves $\mathbf{A}\mathbf{A}^\mathsf{T}\mathbf{u} = \mathbf{b}$.
If $\mathbf{A}\mathbf{A}^\mathsf{T}$ is invertible then the closed form is $\mathbf{x} = \mathbf{A}^\mathsf{T}(\mathbf{A}\mathbf{A}^\mathsf{T})^{-1}\mathbf{b}$.

$(iii)$ $\mathcal{N}(\mathbf{A}) = \mathbf{0}$ and $m > n$ (overdetermined system).
Then the solution is obtained by $\mathbf{x} = (\mathbf{A}^\mathsf{T}\mathbf{A})^{-1}\mathbf{A}^\mathsf{T}\mathbf{b}$.

# Determining the four fundamental subspaces

Two methods for determining the fundamental subspaces.

- Singular Value Decomposition.
  *The best method from a numerical point of view.*
- Gauss-Jordan's method.
  *We focus on this method*

# Minimal rank factorization

**Theorem 26.1** Let $A \in \mathbb{R}^{m\times n}$ and $A = BC$, where $B \in \mathbb{R}^{m\times r}$ has linearly independent columns and $C \in \mathbb{R}^{r\times n}$ has linearly independent rows. Then $A$ and $A^T$ both have ranges of dimension $r$. Furthermore,

$$\mathcal{N}(A) = \mathcal{N}(C)$$

$$\mathcal{N}(A^T) = \mathcal{N}(B^T)$$

$$\mathcal{R}(A) = \mathcal{R}(B)$$

$$\mathcal{R}(A^T) = \mathcal{R}(C^T)$$

## Summary of Gauss-Jordan's method

Step 1 Perform elementary row operations to transform $\mathbf{A}$ to "staircase form".

$$\mathbf{PA} = \mathbf{T} = \begin{bmatrix} \mathbf{U} \\ \mathbf{O} \end{bmatrix} \tag{2}$$

where $\mathbf{P}$ is a product of elementary row operation matrices, while $\mathbf{U}$ is a staircase matrix on the form

$$\mathbf{U} = \begin{array}{c} \begin{bmatrix} & 1 & * & 0 & * & 0 & * & * & 0 & * & * & * \\ & & & 1 & * & 0 & * & * & 0 & * & * & * \\ & & & & & 1 & * & * & 0 & * & * & * \\ & \mathbf{0} & & & & & & & & & & \\ & & & & & & & & 1 & * & * & * \end{bmatrix} \\ \quad \beta_1 \quad \beta_2 \quad \beta_3 \qquad \beta_r \end{array}$$

Step 1 (cont.) The staircase columns have indices $\beta_1, \ldots, \beta_r$, where $r = \text{rang}(\mathbf{A}) = \dim \mathcal{R}(\mathbf{A})$ (rang of the matrix).

The other columns have the indices $\nu_1, \ldots, \nu_l$, $l = n - r$.

Step 2 Define

$$
\begin{aligned}
\mathbf{A}_\beta &= \begin{bmatrix} \hat{a}_{\beta_1} & \ldots & \hat{a}_{\beta_r} \end{bmatrix} \\
\mathbf{U}_\beta &= \begin{bmatrix} u_{\beta_1} & \ldots & u_{\beta_r} \end{bmatrix} = \mathbf{I}_r \\
\mathbf{U}_\nu &= \begin{bmatrix} u_{\nu_1} & \ldots & u_{\nu_l} \end{bmatrix} \\
\mathbf{S} &= \mathbf{P}^{-1}
\end{aligned}
$$

Note that $\mathbf{U}_\beta$ (the staircase columns in $\mathbf{U}$) is an identity matrix.

Step 2 (cont.) Let $\mathbf{S} = \mathbf{P}^{-1}$, then an equivalent expression for (2) is

$$\mathbf{A} = \mathbf{S}\mathbf{T} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{S}_2 \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{O} \end{bmatrix} = \mathbf{S}_1\mathbf{U}$$

If we consider the columns with indices $\beta_1, \ldots, \beta_r$ we have

$$\mathbf{A}_\beta = \mathbf{S}_1\mathbf{U}_\beta = \mathbf{S}_1,$$

and then $\mathbf{A} = \mathbf{A}_\beta\mathbf{U}$.

<u>Step 3</u> The factorization $\mathbf{A} = \mathbf{A}_\beta \mathbf{U}$ is of the form $\mathbf{A} = \mathbf{BC}$ where

$$\text{m rows}\left\{\underbrace{\begin{bmatrix} & \mathbf{A} & \end{bmatrix}}_{\text{n columns}}\right. = \underbrace{\begin{bmatrix}\mathbf{B}\end{bmatrix}}_{\substack{\text{r linearly}\\ \text{independent columns}}} \left.\begin{bmatrix} & \mathbf{C} & \end{bmatrix}\right\}\text{r linearly independent rows}$$

According to ch. 26

$$\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{A}_\beta \mathbf{U}) = \mathcal{R}(\mathbf{A}_\beta) = \mathsf{span}\{\hat{a}_{\beta_1}, \ldots, \hat{a}_{\beta_r}\}$$

$$\mathcal{N}(\mathbf{A}) = \mathcal{N}(\mathbf{A}_\beta \mathbf{U}) = \mathcal{N}(\mathbf{U}) = \{\mathbf{x} \in \mathbf{R}^n : \mathbf{Ux} = \mathbf{0}\}$$

$$= \{\mathbf{x} \in \mathbf{R}^n : \mathbf{U}_\beta \mathbf{x}_\beta + \mathbf{U}_\nu \mathbf{x}_\nu = \mathbf{0}\}$$

$$= \{\mathbf{x} \in \mathbf{R}^n : \mathbf{x}_\beta = -\mathbf{U}_\nu \mathbf{x}_\nu;\ \mathbf{x}_\nu \in \mathbf{R}^l\}$$

Step 3 (cont.) Furthermore, using the previous factorization

$$\mathcal{R}(\mathbf{A}^{\mathsf{T}}) = \mathcal{R}(\mathbf{U}^{\mathsf{T}}\mathbf{A}_{\beta}^{T}) = \mathcal{R}(\mathbf{U}^{T})$$

Finally,

$$\mathcal{N}(\mathbf{A}^{\mathsf{T}}) = \mathcal{R}(\mathbf{A})^{\perp} = \{\mathbf{y} \in \mathbf{R}^{m} : \mathbf{A}_{\beta}^{\mathsf{T}}\mathbf{y} = \mathbf{0}\}$$

which corresponds to solving an equation system.

# Solving linear equation systems

Existence and uniqueness of a linear equation system

$$\mathbf{Ax} = \mathbf{b}$$

can be determined using

1. the fundamental Theorem of linear algebra
   - Gives theoretical insight and solution formulas.
2. Gauss-Jordan's method
   - Practical method for calculations by hand.

# Gauss-Jordan's method for solving equation systems

Transform the equation system using elementary row operations

$$
\begin{aligned}
& \mathbf{A}\mathbf{x} = \mathbf{b} \\
\Leftrightarrow \quad & \mathbf{P}\mathbf{A}\mathbf{x} = \mathbf{P}\mathbf{b} \\
\Leftrightarrow \quad & \begin{bmatrix} \mathbf{U} \\ \mathbf{0} \end{bmatrix} \mathbf{x} = \begin{bmatrix} \bar{\mathbf{b}}_r \\ \bar{\mathbf{b}}_n \end{bmatrix}
\end{aligned}
$$

where

$$
\mathbf{U} = \begin{bmatrix}
1 & * & 0 & * & 0 & * & * & 0 & * & * & * \\
 & & 1 & * & 0 & * & * & 0 & * & * & * \\
 & & & & 1 & * & * & 0 & * & * & * \\
 & \mathbf{0} & & & & & & \ddots & & & \\
 & & & & & & & 1 & * & * & *
\end{bmatrix}
$$

$$\beta_1 \quad \beta_2 \quad \beta_3 \qquad \beta_r$$

Observe that

1. A solution does not exist, if and only if, $\bar{\mathbf{b}}_n \neq 0$
2. If $\bar{\mathbf{b}}_n = 0$, there exists solutions, and they can all be written on the form $\mathbf{x} = \mathbf{x}_r + \mathbf{x}_n$ where $\mathbf{x}_n \in \mathcal{N}(\mathbf{A})$ and

$$\mathbf{x} = \begin{cases} \mathbf{x}_{\beta_l}, & \text{for coefficients corresponding to staircase columns} \\ 0, & \text{otherwise} \end{cases}$$

where

$$\mathbf{x}_\beta = \begin{bmatrix} x_{\beta_1} \\ \vdots \\ x_{\beta_k} \end{bmatrix} = \bar{\mathbf{b}}_r$$

# Example

Determine the nullspace and rangespace for the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 & 1 & 0 \\ 1 & 2 & 7 & 3 & 1 \\ 2 & 1 & 8 & 1 & 4 \\ -1 & 0 & -3 & 1 & -3 \end{bmatrix}$$

Perform elementary row operations to transform $A$ into staircase form.

# Example

$$A_1 = \begin{bmatrix} 1 & -1 & 1 & 1 & 0 \\ 0 & 3 & 6 & 2 & 1 \\ 0 & 3 & 6 & -1 & 4 \\ 0 & -1 & -2 & 2 & -3 \end{bmatrix} = P_1 A, \quad \text{where } P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ +1 & 0 & 0 & 1 \end{bmatrix}$$

We note that $\beta_1 = 1$.

$$A_2 = \begin{bmatrix} 1 & -1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2/3 & 1/3 \\ 0 & 3 & 6 & -1 & 4 \\ 0 & -1 & -2 & 2 & -3 \end{bmatrix} = P_2 A_1, \quad \text{where } P_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Example

$$A_3 = \begin{bmatrix} 1 & 0 & 3 & 5/3 & 1/3 \\ 0 & 1 & 2 & 2/3 & 1/3 \\ 0 & 0 & 0 & -3 & 3 \\ 0 & 0 & 0 & 8/3 & -8/3 \end{bmatrix} = P_3 A_2, \quad \text{where } P_3 = \begin{bmatrix} 1 & +1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & +1 & 0 & 1 \end{bmatrix}$$

We note that $\beta_2 = 2$.

$$A_4 = \begin{bmatrix} 1 & 0 & 3 & 5/3 & 1/3 \\ 0 & 1 & 2 & 2/3 & 1/3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 8/3 & -8/3 \end{bmatrix} = P_4 A_3, \quad \text{where } P_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

# Example

$$A_5 = \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = P_5 A_4, \quad \text{where } P_4 = \begin{bmatrix} 1 & 0 & -5/3 & 0 \\ 0 & 1 & -2/3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -8/3 & 1 \end{bmatrix}$$

We note that $\beta_3 = 4$, and $\nu_1 = 3$, $\nu_2 = 5$. Rank$(A) = 3$.

Then $P = P_5 P_4 P_3 P_2 P_1$, and $A = P^{-1}T = A_\beta U$ where

$$A_\beta = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 3 \\ 2 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \quad \text{and } U = \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

# Example

The range space is now given by $\mathcal{R}(A) = \mathcal{R}(A_\beta)$ where we know that $A_\beta$ has linearly independent columns.

$$\mathcal{R}(A) = \mathcal{R}(\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 3 \\ 2 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}) = \text{span}(\hat{a}_1, \hat{a}_2, \hat{a}_4)$$

## Example

The nullspace is given by

$$\mathcal{N}(A) = \mathcal{N}(U) = \left\{ I \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} = - \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_3 \\ x_5 \end{bmatrix} \right\}$$

$$\mathcal{N}(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 1 \\ -0 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -2 \\ -1 \\ 0 \\ --1 \\ 1 \end{bmatrix} x_5 \right\} = \text{span} \left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

# Example

We also know that

$$\mathcal{R}(A^T) = \mathcal{R}(U^T) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

and

$$\mathcal{N}(A^T) = \mathcal{N}(A_\beta^T) = \cdots = \text{span}\left\{ \begin{bmatrix} -2 \\ -5 \\ 8 \\ 9 \end{bmatrix} \right\}$$

follows by similar calculations as for $\mathcal{N}(A)$.

# Reading instructions

- Chapter 23-26.