Proving that a global minimum exists

$$(P) \quad \begin{cases} \text{minimize} & f(x) \\ \text{subject to} & x \in S \end{cases}$$

where f is continuous.

1) If S is compact, use Weierstraß' thm.

2) If S is not compact, then add constraints of the form $g_i(x) \leq 0$ so the region becomes compact and use Weierstraß. Prove that f is large on $g_i(x) = 0$, so that the added inequalities are inactive at the minimizer (put them in X).

   Examples: i) $|x_i| \leq R$ with R large.

   ii) $x_1^2 + x_2^2 + \dots + x_n^2 \leq R$ — " —.

   iii) Take a point $\tilde{x} \in S$ and compute the value $f(\tilde{x}) = c$. Add the constraint $f(x) \leq c$ and prove that this gives a compact region. Clearly the minimizer is inside this region.

3) If (P) is a convex problem, then use KKT theory to find one point. This is a global solution.

4) Use duality.