Lecture 9: Dynamics: Euler-Lagrange Equations

- Computing Kinetic and Potential Energies

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- Equations of Motion

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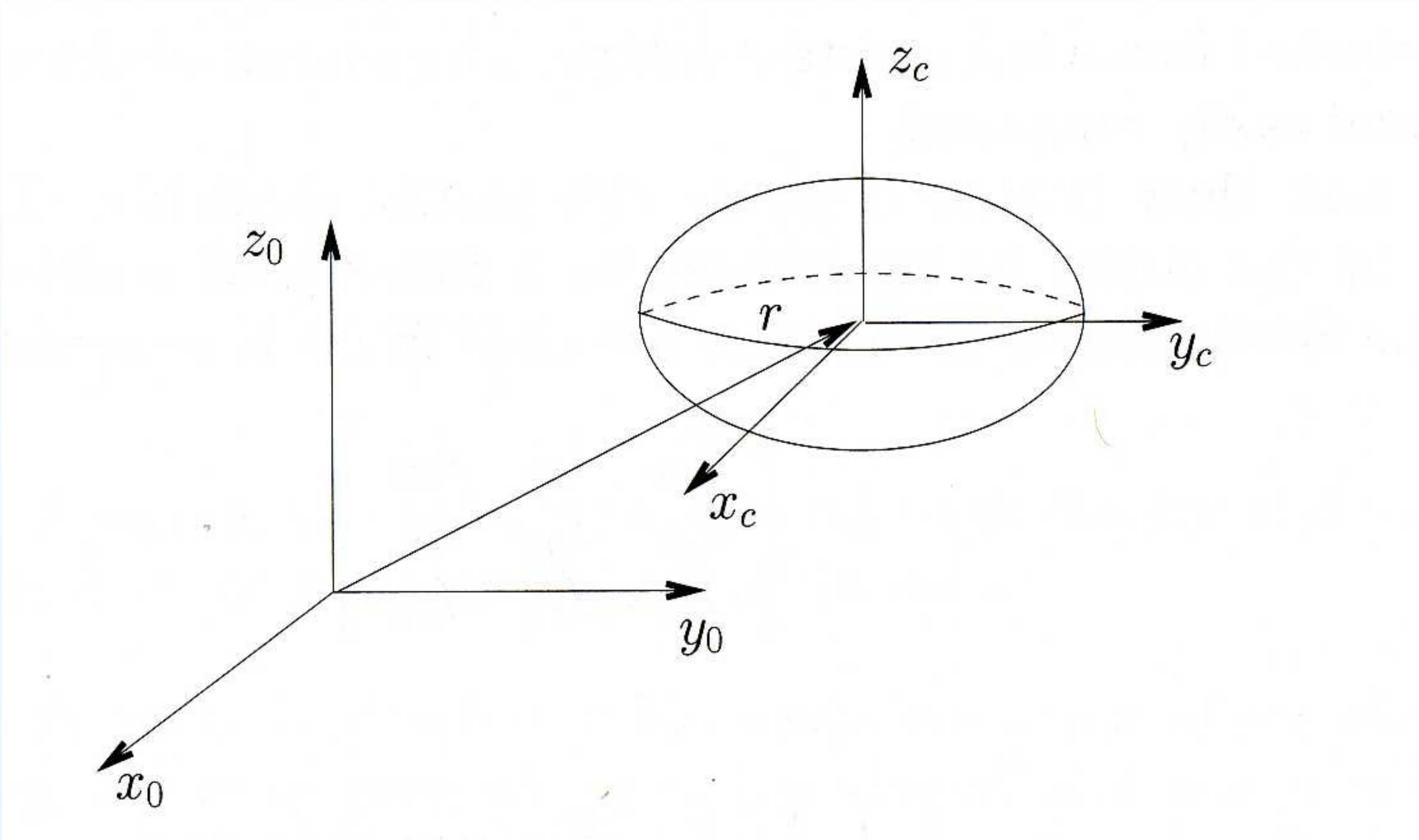
Computing Kinetic Energy



Rigid body has 6 degrees of freedom. Its kinetic energy consists of kinetic energy of rotation and kinetic energy of translation

$$\mathcal{K} = \frac{1}{2} m |v|^2 + \frac{1}{2} \omega^T \mathcal{I} \omega$$

Computing Kinetic Energy

We know how to compute the angular velocity

$$\boldsymbol{S(\omega) = \frac{d}{dt} R(t) R^T(t) \quad \longrightarrow \quad \omega}$$

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Computing Kinetic Energy

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The 3×3 matrix $\mathcal{I}$ is called the tensor of inertia

In the body frame it is constant $I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$ and

computed as

$$\begin{aligned} I_{xx} &= \int\int\int (y^2 + z^2)\rho(x, y, z)dxdydz \\ I_{yy} &= \int\int\int (x^2 + z^2)\rho(x, y, z)dxdydz \\ I_{zz} &= \int\int\int (y^2 + x^2)\rho(x, y, z)dxdydz \end{aligned}$$

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$$\begin{aligned} I_{xy} = I_{yx} &= -\int\int\int xy\rho(x,y,z)dxdydz \\ I_{xz} = I_{zx} &= -\int\int\int xz\rho(x,y,z)dxdydz \\ I_{yz} = I_{zy} &= -\int\int\int yz\rho(x,y,z)dxdydz \end{aligned}$$

Computing Kinetic Energy

We know how to compute the angular velocity

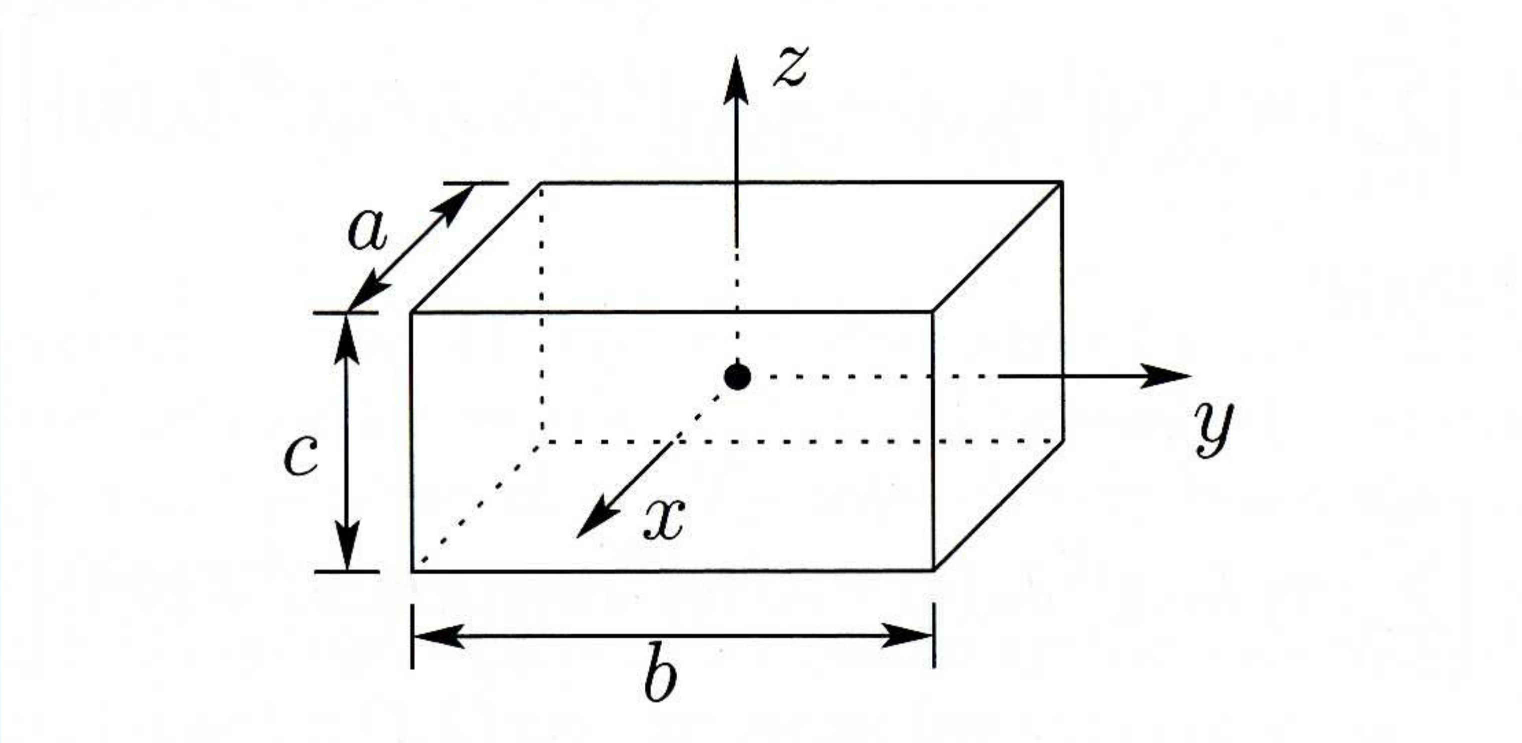
$$S(\omega) = \frac{d}{dt}R(t)R^T(t) \quad \longrightarrow \quad \omega$$

The 3×3 matrix $\mathcal{I}$ is called the tensor of inertia

To compute the tensor of inertia in the inertia frame, we can use the formula

$$\mathcal{I} = R(t)IR^T(t)$$

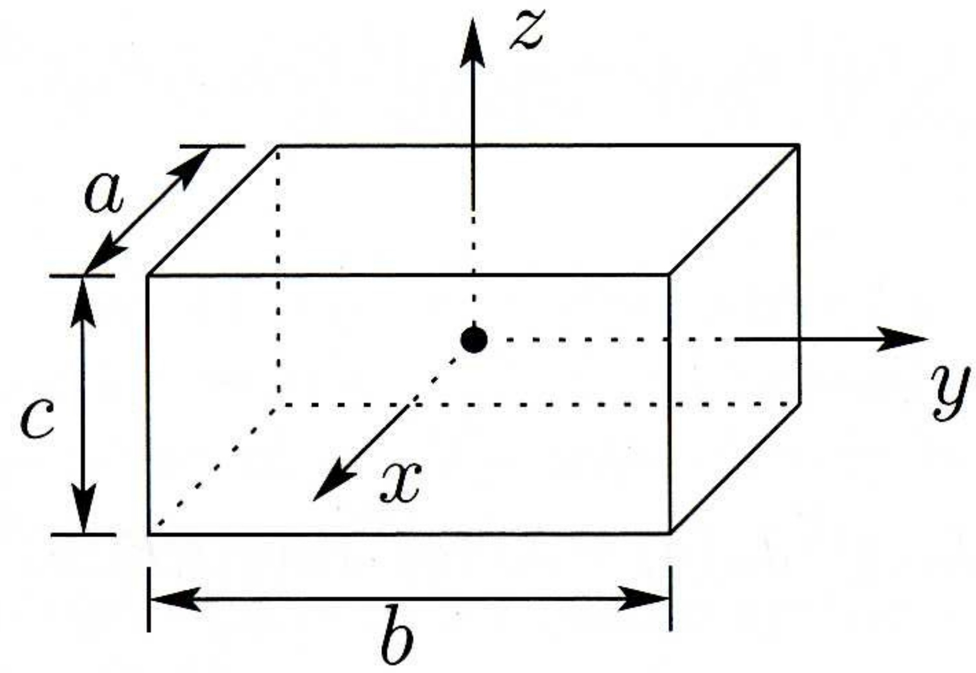
Computing Kinetic Energy



Rectangular brick with uniform mass density. Let us compute

$$\boldsymbol{I_{xx} = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}(y^2+z^2)\rho(x,y,z)dxdydz = ???}$$

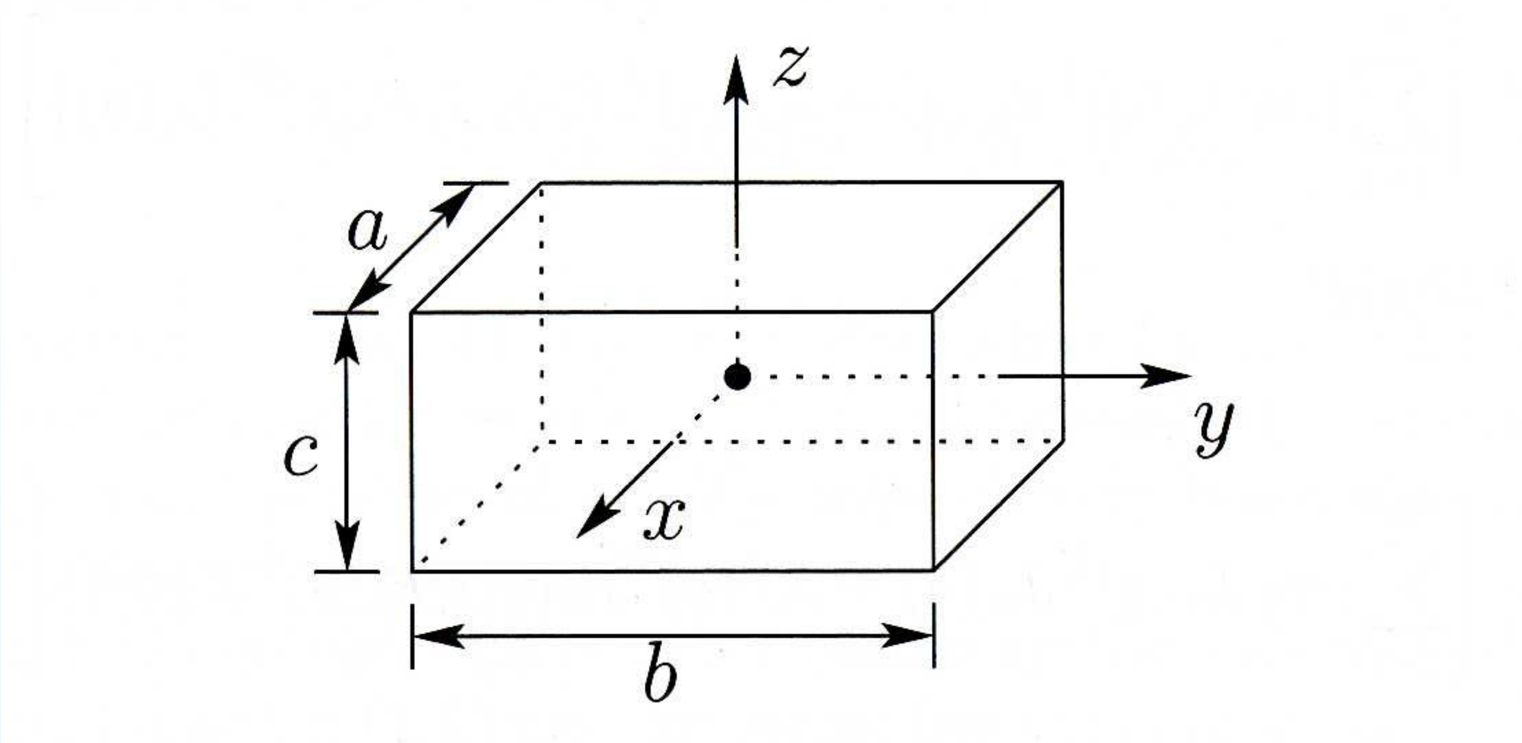
Computing Kinetic Energy



Rectangular brick with uniform mass density. Let us compute

$$
\begin{aligned}
I_{xx} &= \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}(y^2+z^2)\rho(x,y,z)\,dx\,dy\,dz \\
&= \rho\frac{abc}{12}(b^2+c^2) = \frac{m}{12}(b^2+c^2)
\end{aligned}
$$

Computing Kinetic Energy



Rectangular solid brick with uniform mass density. In the same way

$$I_{yy} = \frac{m}{12}(a^2+c^2), \quad I_{zz} = \frac{m}{12}(a^2+b^2), \quad I_{xy} = I_{xz} = I_{yz} = 0$$

Computing Kinetic Energy for n-Link Robot

To use the formula

$$\mathcal{K} = \frac{1}{2} m |v|^2 + \frac{1}{2} \omega^T \mathcal{I} \omega$$

we need to express

- $v = \dot{r}$ as function of generalized coordinates q and velocities $\dot{q}$;
- ω as function of generalized coordinates q and velocities $\dot{q}$;

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The final form of kinetic energy is

$$\mathcal{K} = \frac{1}{2} \dot{q}^T \left[\sum_{i=1}^{k} m_i J_{v_i}(q)^T J_{v_i}(q) + J_{\omega_i}(q)^T R_i(q) I R_i(q)^T J_{\omega_i}(q) \right] \dot{q}$$

Computing Potential Energy for n-Link Robot

Potential energy of i^{th}-link is

$$\mathcal{P}_i = m_i g^T r_{ci}$$

where r_{ci} is the position of its center of mass

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The total potential energy of the robot is then

$$\mathcal{P} = \sum_{i=1}^{k} \mathcal{P}_i = \sum_{i=1}^{k} m_i g^T r_{ci}$$

Lecture 9: Dynamics: Euler-Lagrange Equations

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- Equations of Motion
- Examples
- Properties of Equations of Motion

Equations of Motion

We have seen that

- In general, kinetic energy is

$$\mathcal{K} = \frac{1}{2}\dot{q}^T\left[\sum_{i=1}^{k} m_i J_{v_i}(q)^T J_{v_i}(q) + J_{\omega_i}(q)^T R_i(q) I R_i(q)^T J_{\omega_i}(q)\right]\dot{q}$$

$$= \frac{1}{2}\dot{q}^T D(q)\dot{q} = \sum_{i,j} d_{ij}(q)\dot{q}_i\dot{q}_j$$

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$$= \frac{1}{2}\dot{q}^T D(q)\dot{q} = \sum_{i,j} d_{ij}(q)\dot{q}_i\dot{q}_j$$

- If generalized forces are potential, then $\psi_j = -\frac{\partial \mathcal{P}}{\partial q_j} + \tau_j$
- We can introduce a scalar function $\mathcal{L} = \mathcal{K} - \mathcal{P}$ and write the equation of motion in compact form

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = \tau_j$$

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$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = \tau_j = \frac{d}{dt}\frac{\partial (\mathcal{K} - \mathcal{P})}{\partial \dot{q}_j} - \frac{\partial (\mathcal{K} - \mathcal{P})}{\partial q_j}$$

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Equations of Motion

The equations of motion have a particular structure

$$\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_k} - \frac{\partial(\mathcal{K} - \mathcal{P})}{\partial q_k} = \tau_k, \quad k = 1, \ldots, n$$

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Indeed

$$\frac{\partial \mathcal{K}}{\partial \dot{q}_k} = \frac{\partial}{\partial \dot{q}_k}\left[\frac{1}{2}\dot{q}^T D(q)\dot{q}\right] = \sum_{j=1}^{n} d_{kj}\dot{q}_j \Rightarrow \frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_k} = \frac{d}{dt}\left[\sum_{j=1}^{n} d_{kj}\dot{q}_j\right]$$

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and

$$\frac{d}{dt}\left[\sum_{j=1}^{n} d_{kj}\dot{q}_j\right] = \sum_{j=1}^{n} d_{kj}\ddot{q}_j + \sum_{j=1}^{n} \tfrac{d}{dt}\left[d_{kj}(q)\right]\dot{q}_j$$

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$$\begin{aligned} \frac{d}{dt}\left[\sum_{j=1}^{n} d_{kj}\dot{q}_j\right] &= \sum_{j=1}^{n} d_{kj}\ddot{q}_j + \sum_{j=1}^{n} \frac{d}{dt}\left[d_{kj}(q)\right]\dot{q}_j \\ &= \sum_{j=1}^{n} d_{kj}\ddot{q}_j + \sum_{j=1}^{n}\left(\sum_{i=1}^{n}\frac{\partial d_{kj}(q)}{\partial q_i}\dot{q}_i\right)\dot{q}_j \end{aligned}$$

Equations of Motion

The equations of motion have a particular structure

$$\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_k} - \frac{\partial(\mathcal{K} - \mathcal{P})}{\partial q_k} = \tau_k, \quad k = 1, \ldots, n$$

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and

$$\begin{aligned} \frac{d}{dt}\left[\sum_{j=1}^{n} d_{kj}\dot{q}_j\right] &= \sum_{j=1}^{n} d_{kj}\ddot{q}_j + \sum_{j=1}^{n} \frac{d}{dt}\left[d_{kj}(q)\right]\dot{q}_j \\ &= \sum_{j=1}^{n} d_{kj}\ddot{q}_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j}\right)\dot{q}_i\dot{q}_j \end{aligned}$$

Equations of Motion

The second term of the equations of motion is equal to

$$\begin{aligned}
\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial q_k} &= \frac{\partial}{\partial q_k}\left[\frac{1}{2}\dot{q}D(q)\dot{q}-\mathcal{P}\right] = \frac{1}{2}\dot{q}\left[\frac{\partial}{\partial q_k}D(q)\right]\dot{q}-\frac{\partial}{\partial q_k}\mathcal{P} \\
&= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j-\frac{\partial}{\partial q_j}\mathcal{P}
\end{aligned}$$

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To sum up, the equations of motion are

$$\sum_{j=1}^{n} d_{kj}\ddot{q}_j+\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial d_{kj}}{\partial q_i}+\frac{\partial d_{ki}}{\partial q_j}\right)\dot{q}_i\dot{q}_j-$$

$$-\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j+\frac{\partial}{\partial q_k}\mathcal{P}=\tau_k$$

Equations of Motion

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To sum up, the equations of motion are

$$\boxed{\sum_{j=1}^{n} d_{kj}\ddot{q}_j+\sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i\dot{q}_j+g_k(q)=\tau_k}$$

with

$$c_{ijk}(q)=\frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial d_{kj}}{\partial q_i}+\frac{\partial d_{ki}}{\partial q_j}-\frac{\partial d_{ij}}{\partial q_j}\right), \quad g_k(q)=\frac{\partial}{\partial q_k}\mathcal{P}$$

Equations of Motion

The second term of the equations of motion is equal to

$$
\begin{aligned}
\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial q_k} &= \frac{\partial}{\partial q_k}\left[\frac{1}{2}\dot{q}D(q)\dot{q} - \mathcal{P}\right] = \frac{1}{2}\dot{q}\left[\frac{\partial}{\partial q_k}D(q)\right]\dot{q} - \frac{\partial}{\partial q_k}\mathcal{P} \\
&= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n}\frac{\partial d_{ij}}{\partial q_k}\dot{q}_i\dot{q}_j - \frac{\partial}{\partial q_j}\mathcal{P}
\end{aligned}
$$

To sum up, the equations of motion

$$
\boxed{\sum_{j=1}^{n} d_{kj}\ddot{q}_j + \sum_{j=1}^{n}\sum_{i=1}^{n} c_{ijk}(q)\dot{q}_i\dot{q}_j + g_k(q) = \tau_k}
$$

in vectorial form are

$$
\boxed{D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau}
$$

Lecture 11: Dynamics: Euler-Lagrange Equations

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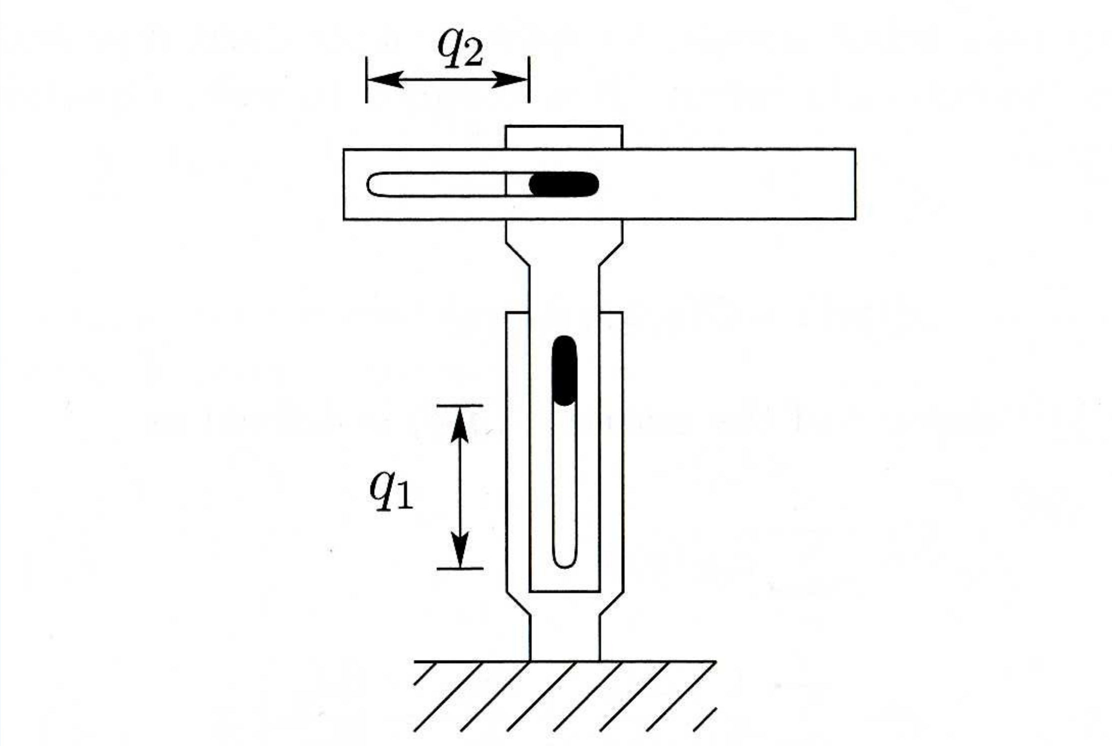
Example: Two-Link Cartesian Manipulator



For this system we need

- to solve forward kinematics problem;
- to compute manipulator Jacobian;
- to compute kinetic and potential energies and the Euler-Lagrange equations

Forward Kinematics and Jacobian

DH parameters for computing homogeneous transformations

$$T(q_i) = \mathrm{Rot}_{z,\theta} \cdot \mathrm{Trans}_{z,d} \cdot \mathrm{Trans}_{x,a} \cdot \mathrm{Rot}_{x,\alpha}$$

are

Forward Kinematics and Jacobian

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are

$$T_1^0 : \quad \theta = 0, \quad d = q_1, \quad a = 0, \quad \alpha = -\frac{\pi}{2}$$

$$T_2^1 : \quad \theta = 0, \quad d = q_2, \quad a = 0, \quad \alpha = 0$$

Forward Kinematics and Jacobian

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The kinetic energy of the system is

$$\mathcal{K} = \tfrac{1}{2}\left[m_1 v_{c1}^2 + \omega_1^T \mathcal{I}_1 \omega_1\right] + \tfrac{1}{2}\left[m_2 v_{c2}^2 + \omega_2^T \mathcal{I}_2 \omega_2\right]$$

and

$$\begin{aligned} v_{c1} &= \left[J_{v1}^{(1)} \; J_{v1}^{(2)}\right] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = J_{v1}^{(1)} \dot{q}_1 + J_{v1}^{(2)} \dot{q}_2 \\ v_{c2} &= \left[J_{v2}^{(1)} \; J_{v2}^{(2)}\right] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = J_{v2}^{(1)} \dot{q}_1 + J_{v2}^{(2)} \dot{q}_2 \end{aligned}$$

Forward Kinematics and Jacobian

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and

$$\omega_1 = \left[J_{\omega_1}^{(1)} \; J_{\omega_1}^{(2)}\right] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = J_{\omega_1}^{(1)} \dot{q}_1 + J_{\omega_1}^{(2)} \dot{q}_2$$

$$\omega_2 = \left[J_{\omega_2}^{(1)} \; J_{\omega_2}^{(2)}\right] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = J_{\omega_2}^{(1)} \dot{q}_1 + J_{\omega_2}^{(2)} \dot{q}_2$$

Forward Kinematics and Jacobian

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To compute the Jacobian we can use the DH-frames, i.e

$$J_v^{(i)} = \begin{cases} z_{i-1}^0, & \text{for prismatic joint} \\ z_{i-1}^0 \times \left[o_c^0 - o_{i-1}^0 \right], & \text{for revolute joint} \end{cases}$$

$$J_\omega^{(i)} = \begin{cases} 0, & \text{for prismatic joint} \\ z_{i-1}^0, & \text{for revolute joint} \end{cases}$$

Forward Kinematics and Jacobian

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$$\Rightarrow J_{v1} = [\vec{z}_0^0, 0] = \left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right], \; J_{v2} = [\vec{z}_0^0, \vec{z}_1^0] = \left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right]$$

Forward Kinematics and Jacobian (Cont'd)

To sum up:

- Angular velocities $\boldsymbol{\omega_1}$ and $\boldsymbol{\omega_2}$ of both links are zeros

Forward Kinematics and Jacobian (Cont'd)

To sum up:

- Angular velocities ω_1 and ω_2 of both links are zeros
- Linear velocities of centers of mass are

$$
\begin{aligned}
v_{c1} &= \left[J_{v1}^{(1)}, J_{v1}^{(2)}\right] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{q}_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \dot{q}_2 \\
v_{c2} &= \left[J_{v2}^{(1)}, J_{v2}^{(2)}\right] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{q}_1 + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \dot{q}_2
\end{aligned}
$$

Forward Kinematics and Jacobian (Cont'd)

To sum up:

- Angular velocities ω_1 and ω_2 of both links are zeros
- Linear velocities of centers of mass are

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\begin{aligned}
v_{c1} &= \left[J_{v1}^{(1)},\ J_{v1}^{(2)}\right] \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{q}_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \dot{q}_2 \\
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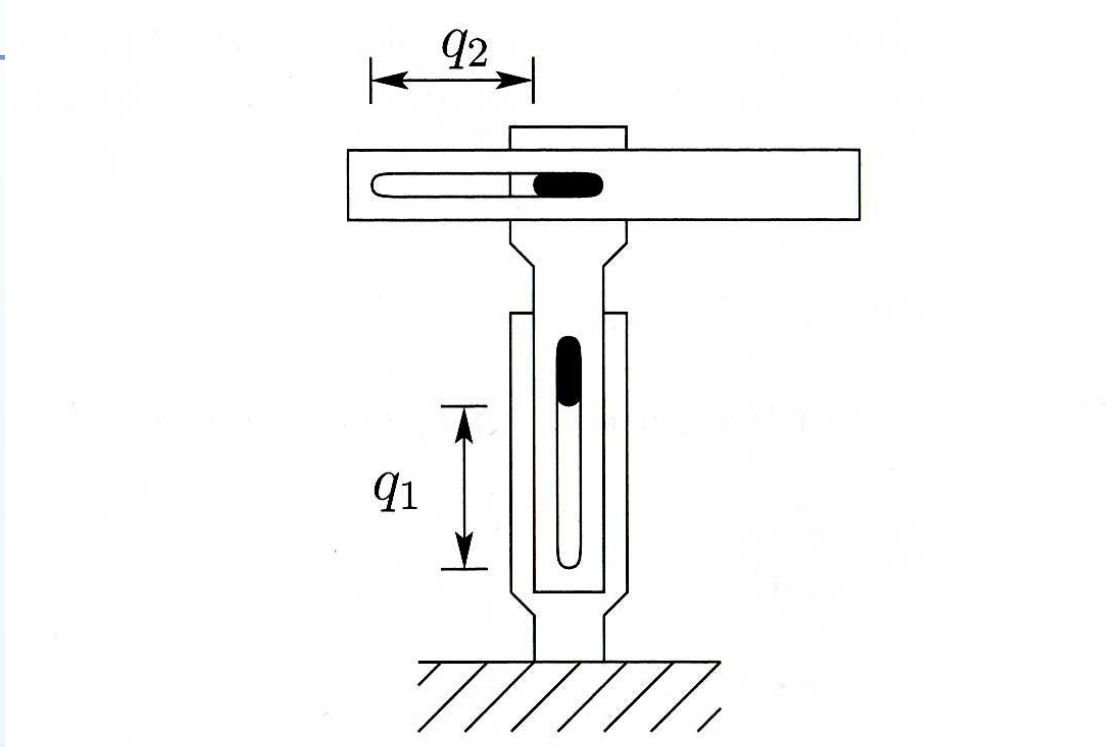
\end{aligned}
$$

- The kinetic energy is

$$
\mathcal{K} = \tfrac{1}{2} m_1 v_{c1}^2 + \tfrac{1}{2} m_2 v_{c2}^2 = \frac{1}{2} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}^T \begin{bmatrix} m_1 + m_2 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}
$$

Potential Energy (PE) for Two-Link Cartesian Manipulator



Observations

- PE is independent of the second link position;
- It depends on the height of center of mass of robot;
- $\mathcal{P} = g \cdot (m_1 + m_2) \cdot q_1 + Const$

Euler-Lagrange Equations for 2-Link Cartesian Manipulator

Given the kinetic $\mathcal{K}$ and potential $\mathcal{P}$ energies, the dynamics are

$$\frac{d}{dt}\left[\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial \dot{q}}\right] - \frac{\partial(\mathcal{K}-\mathcal{P})}{\partial q} = \tau$$

Euler-Lagrange Equations for 2-Link Cartesian Manipulator

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$$\frac{d}{dt}\left[\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial\dot{q}}\right]-\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial q}=\tau$$

With kinetic and potential energies

$$\mathcal{K}=\frac{1}{2}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}^T\begin{bmatrix}m_1+m_2 & 0\\ 0 & m_2\end{bmatrix}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix},\ \mathcal{P}=g\left(m_1+m_2\right)q_1+C$$

Euler-Lagrange Equations for 2-Link Cartesian Manipulator

Given the kinetic $\mathcal{K}$ and potential $\mathcal{P}$ energies, the dynamics are

$$\frac{d}{dt}\left[\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial \dot{q}}\right] - \frac{\partial(\mathcal{K}-\mathcal{P})}{\partial q} = \tau$$

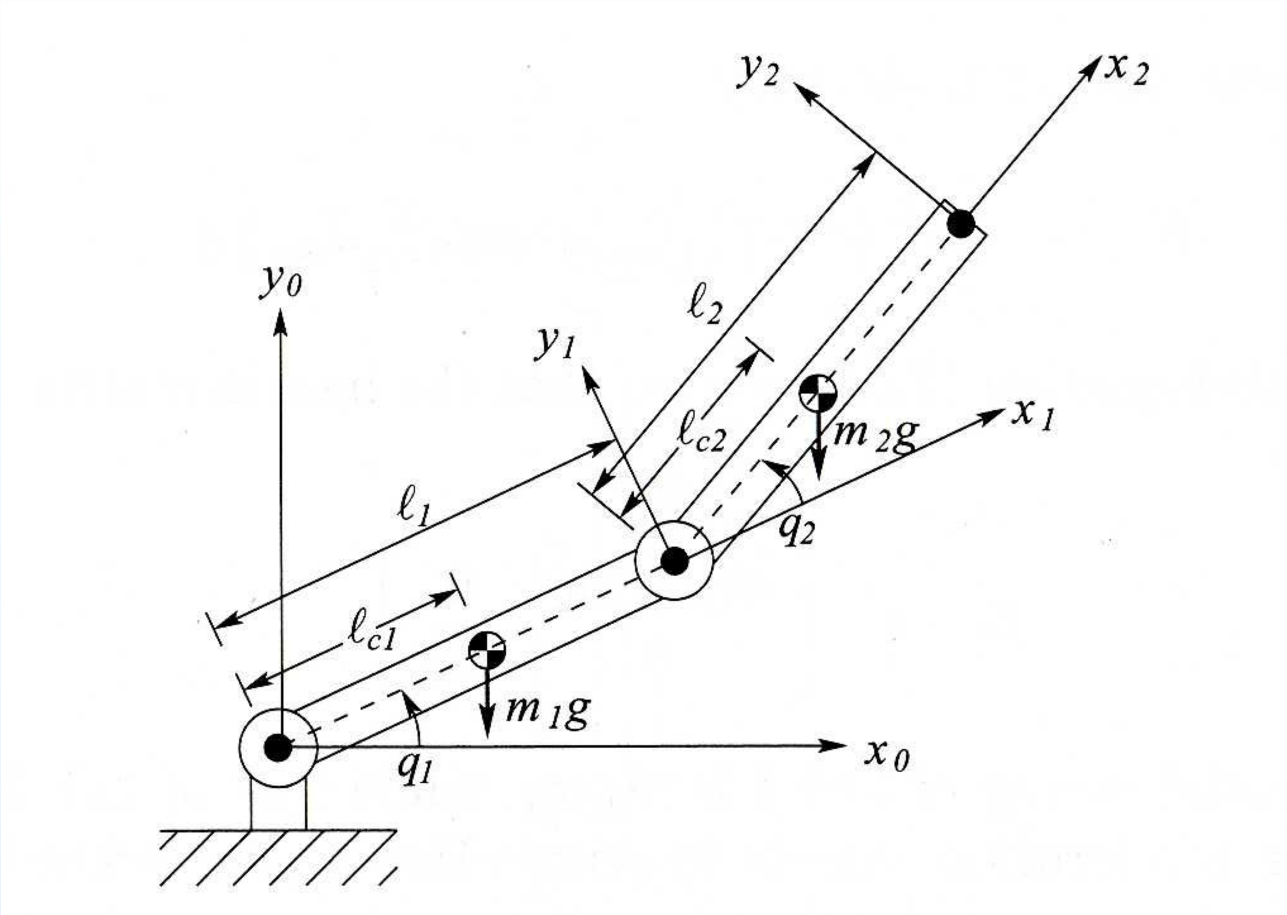
With kinetic and potential energies

$$\mathcal{K} = \tfrac{1}{2}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}^T \begin{bmatrix} m_1+m_2 & 0\\ 0 & m_2\end{bmatrix}\begin{bmatrix}\dot{q}_1\\ \dot{q}_2\end{bmatrix}, \ \mathcal{P} = g\,(m_1+m_2)\,q_1 + C$$

the Euler-Lagrange equations are

$$\begin{aligned}(m_1+m_2)\ddot{q}_1 + g(m_1+m_2) &= \tau_1\\ m_2\ddot{q}_2 &= \tau_2\end{aligned}$$

Example: Planar Elbow Manipulator



For this system we need

- to compute forward kinematics and manipulator Jacobian;
- to compute kinetic and potential energies and the Euler-Lagrange equations

Forward Kinematics and Jacobian

DH parameters for computing homogeneous transformations

$$T(q_i) = \mathrm{Rot}_{z,\theta} \cdot \mathrm{Trans}_{z,d} \cdot \mathrm{Trans}_{x,a} \cdot \mathrm{Rot}_{x,\alpha}$$

are

Forward Kinematics and Jacobian

DH parameters for computing homogeneous transformations

$$T(q_i) = \mathrm{Rot}_{z,\theta} \cdot \mathrm{Trans}_{z,d} \cdot \mathrm{Trans}_{x,a} \cdot \mathrm{Rot}_{x,\alpha}$$

are

$$\begin{aligned} T_1^0 : \quad & \theta = q_1, \quad d = 0, \quad a = l_1, \quad \alpha = 0 \\ T_2^1 : \quad & \theta = q_2, \quad d = 0, \quad a = l_2, \quad \alpha = 0 \end{aligned}$$

Forward Kinematics and Jacobian

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The kinetic energy of the system is

$$\mathcal{K} = \frac{1}{2}\left[m_1 v_{c1}^2 + \omega_1^T \mathcal{I}_1 \omega_1\right] + \frac{1}{2}\left[m_2 v_{c2}^2 + \omega_2^T \mathcal{I}_2 \omega_2\right]$$

and

$$\begin{aligned} v_{c1} &= \left[J_{v1}^{(1)}\ J_{v1}^{(2)}\right]\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = J_{v1}^{(1)}\dot{q}_1 + J_{v1}^{(2)}\dot{q}_2 \\ v_{c2} &= \left[J_{v2}^{(1)}\ J_{v2}^{(2)}\right]\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = J_{v2}^{(1)}\dot{q}_1 + J_{v2}^{(2)}\dot{q}_2 \end{aligned}$$

Forward Kinematics and Jacobian

DH parameters for computing homogeneous transformations

$$T(q_i) = \mathrm{Rot}_{z,\theta} \cdot \mathrm{Trans}_{z,d} \cdot \mathrm{Trans}_{x,a} \cdot \mathrm{Rot}_{x,\alpha}$$

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$$\begin{aligned} T_1^0 : &\quad \theta = q_1, \quad d = 0, \quad a = l_1, \quad \alpha = 0 \\ T_2^1 : &\quad \theta = q_2, \quad d = 0, \quad a = l_2, \quad \alpha = 0 \end{aligned}$$

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$$\mathcal{K} = \frac{1}{2}\left[m_1 v_{c1}^2 + \omega_1^T \mathcal{I}_1 \omega_1\right] + \frac{1}{2}\left[m_2 v_{c2}^2 + \omega_2^T \mathcal{I}_2 \omega_2\right]$$

and

$$\begin{aligned} \omega_1 &= \left[J_{\omega_1}^{(1)}\ J_{\omega_1}^{(2)}\right]\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = J_{\omega_1}^{(1)}\dot{q}_1 + J_{\omega_1}^{(2)}\dot{q}_2 \\ \omega_2 &= \left[J_{\omega_2}^{(1)}\ J_{\omega_2}^{(2)}\right]\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = J_{\omega_2}^{(1)}\dot{q}_1 + J_{\omega_2}^{(2)}\dot{q}_2 \end{aligned}$$

Forward Kinematics and Jacobian

DH parameters for computing homogeneous transformations

$$T(q_i) = \mathrm{Rot}_{z,\theta} \cdot \mathrm{Trans}_{z,d} \cdot \mathrm{Trans}_{x,a} \cdot \mathrm{Rot}_{x,\alpha}$$

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To compute the Jacobian we can use the DH-frames, i.e

$$J_v^{(i)} = \begin{cases} z_{i-1}^0, & \text{for prismatic joint} \\ z_{i-1}^0 \times \left[o_c^0 - o_{i-1}^0 \right], & \text{for revolute joint} \end{cases}$$

$$J_\omega^{(i)} = \begin{cases} 0, & \text{for prismatic joint} \\ z_{i-1}^0, & \text{for revolute joint} \end{cases}$$

Forward Kinematics and Jacobian (Cont'd)

The formula

$$\boldsymbol{J_v^{(i)}} = \begin{cases} z_{i-1}^0, & \text{for prismatic joint} \\ z_{i-1}^0 \times \left[o_c^0 - o_{i-1}^0 \right], & \text{for revolute joint} \end{cases}$$

gives

$$J_{v1}^{(1)} = \vec{z}_0 \times (\vec{o}_{c_1} - \vec{o}_0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} l_{c1} \cos q_1 \\ l_{c1} \sin q_1 \\ 0 \end{bmatrix}$$

Forward Kinematics and Jacobian (Cont'd)

The formula

$$\boldsymbol{J_v^{(i)}} = \begin{cases} \boldsymbol{z_{i-1}^0}, & \text{for prismatic joint} \\ \boldsymbol{z_{i-1}^0} \times \left[\boldsymbol{o_c^0} - \boldsymbol{o_{i-1}^0}\right], & \text{for revolute joint} \end{cases}$$

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Forward Kinematics and Jacobian (Cont'd)

The formula

$$J_v^{(i)} = \begin{cases} z_{i-1}^0, & \text{for prismatic joint} \\ z_{i-1}^0 \times \left[o_c^0 - o_{i-1}^0\right], & \text{for revolute joint} \end{cases}$$

gives

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$$\begin{aligned} J_{v2}^{(1)} &= \vec{z}_0 \times (\vec{o}_{c2} - \vec{o}_0) \\ &= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \left(\begin{bmatrix} l_1\cos q_1 \\ l_1\sin q_1 \\ 0 \end{bmatrix} + \begin{bmatrix} l_{c_2}\cos(q_1+q_2) \\ l_{c_2}\sin(q_1+q_2) \\ 0 \end{bmatrix} \right) \end{aligned}$$

Forward Kinematics and Jacobian (Cont'd)

The formula

$$\boldsymbol{J_v^{(i)}} = \begin{cases} \boldsymbol{z_{i-1}^0}, & \text{for prismatic joint} \\ \boldsymbol{z_{i-1}^0 \times \left[o_c^0 - o_{i-1}^0\right]}, & \text{for revolute joint} \end{cases}$$

gives

$$\boldsymbol{J_{v1}^{(1)}} = \vec{\boldsymbol{z}}_0 \times (\vec{\boldsymbol{o}}_{c_1} - \vec{\boldsymbol{o}}_0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} l_{c_1} \cos q_1 \\ l_{c_1} \sin q_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_{c_1} \sin q_1 \\ l_{c_1} \cos q_1 \\ 0 \end{bmatrix}$$

$$\boldsymbol{J_{v2}^{(1)}} = \vec{\boldsymbol{z}}_0 \times (\vec{\boldsymbol{o}}_{c2} - \vec{\boldsymbol{o}}_0) = \begin{bmatrix} -l_1 \sin q_1 - l_{c_2} \sin(q_1 + q_2) \\ l_1 \cos q_1 + l_{c_2} \cos(q_1 + q_2) \\ 0 \end{bmatrix}$$

Forward Kinematics and Jacobian (Cont'd)

The formula

$$\boldsymbol{J_v^{(i)}} = \begin{cases} z_{i-1}^0, & \text{for prismatic joint} \\ z_{i-1}^0 \times \left[o_c^0 - o_{i-1}^0 \right], & \text{for revolute joint} \end{cases}$$

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$$J_{v1}^{(1)} = \vec{z}_0 \times (\vec{o}_{c_1} - \vec{o}_0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} l_{c_1} \cos q_1 \\ l_{c_1} \sin q_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_{c_1} \sin q_1 \\ l_{c_1} \cos q_1 \\ 0 \end{bmatrix}$$

$$J_{v2}^{(1)} = \vec{z}_0 \times (\vec{o}_{c2} - \vec{o}_0) = \begin{bmatrix} -l_1 \sin q_1 - l_{c_2} \sin(q_1 + q_2) \\ l_1 \cos q_1 + l_{c_2} \cos(q_1 + q_2) \\ 0 \end{bmatrix}$$

$$J_{v2}^{(2)} = \vec{z}_1 \times (\vec{o}_{c2} - \vec{o}_1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} l_{c_2} \cos(q_1 + q_2) \\ l_{c_2} \sin(q_1 + q_2) \\ 0 \end{bmatrix}$$

Forward Kinematics and Jacobian (Cont'd)

The formula

$$\boldsymbol{J_v^{(i)}} = \begin{cases} z_{i-1}^0, & \text{for prismatic joint} \\ z_{i-1}^0 \times \left[o_c^0 - o_{i-1}^0\right], & \text{for revolute joint} \end{cases}$$

gives

$$J_{v1}^{(1)} = \vec{z}_0 \times (\vec{o}_{c_1} - \vec{o}_0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} l_{c_1} \cos q_1 \\ l_{c_1} \sin q_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_{c_1} \sin q_1 \\ l_{c_1} \cos q_1 \\ 0 \end{bmatrix}$$

$$J_{v2}^{(1)} = \vec{z}_0 \times (\vec{o}_{c2} - \vec{o}_0) = \begin{bmatrix} -l_1 \sin q_1 - l_{c_2} \sin(q_1 + q_2) \\ l_1 \cos q_1 + l_{c_2} \cos(q_1 + q_2) \\ 0 \end{bmatrix}$$

$$J_{v2}^{(2)} = \vec{z}_1 \times (\vec{o}_{c2} - \vec{o}_1) = \begin{bmatrix} -l_{c_2} \sin(q_1 + q_2) \\ l_{c_2} \cos(q_1 + q_2) \\ 0 \end{bmatrix}$$

Forward Kinematics and Jacobian (Cont'd)

The formula

$$\boldsymbol{J}_{\boldsymbol{\omega}}^{(i)} = \begin{cases} \mathbf{0}, & \text{for prismatic joint} \\ \boldsymbol{z}_{i-1}^{0}, & \text{for revolute joint} \end{cases}$$

gives

$$\boldsymbol{J}_{\boldsymbol{\omega}_1}^{(1)} = \vec{\boldsymbol{z}}_{\mathbf{0}} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}$$

Forward Kinematics and Jacobian (Cont'd)

The formula

$$J_{\omega}^{(i)} = \begin{cases} 0, & \text{for prismatic joint} \\ z_{i-1}^{0}, & \text{for revolute joint} \end{cases}$$

gives

$$J_{\omega_1}^{(1)} = \vec{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$J_{\omega_2}^{(1)} = \vec{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Forward Kinematics and Jacobian (Cont'd)

The formula

$$J_\omega^{(i)} = \begin{cases} 0, & \text{for prismatic joint} \\ z_{i-1}^0, & \text{for revolute joint} \end{cases}$$

gives

$$\begin{aligned} J_{\omega_1}^{(1)} &= \vec{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ J_{\omega_2}^{(1)} &= \vec{z}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ J_{\omega_2}^{(2)} &= \vec{z}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

Forward Kinematics and Jacobian (Cont'd)

To sum up, the kinetic energy $\mathcal{K}$ is

$$
\begin{aligned}
\mathcal{K} &= \frac{1}{2}\left[m_1 v_{c1}^2 + \omega_1^T \mathcal{I}_1 \omega_1\right] + \frac{1}{2}\left[m_2 v_{c2}^2 + \omega_2^T \mathcal{I}_2 \omega_2\right] \\
&= \frac{1}{2}\left[m_1 \left(J_{v_1}^{(1)} \dot{q}_1\right)^2 + I_1 \left(J_{\omega_1}^{(1)} \dot{q}_1\right)^2\right] + \\
&\qquad + \frac{1}{2}\left[m_2 \left(J_{v_2}^{(1)} \dot{q}_1 + J_{v_2}^{(2)} \dot{q}_2\right)^2 + I_2 \left(J_{\omega_2}^{(1)} \dot{q}_1 + J_{\omega_2}^{(2)} \dot{q}_2\right)^2\right]
\end{aligned}
$$

Forward Kinematics and Jacobian (Cont'd)

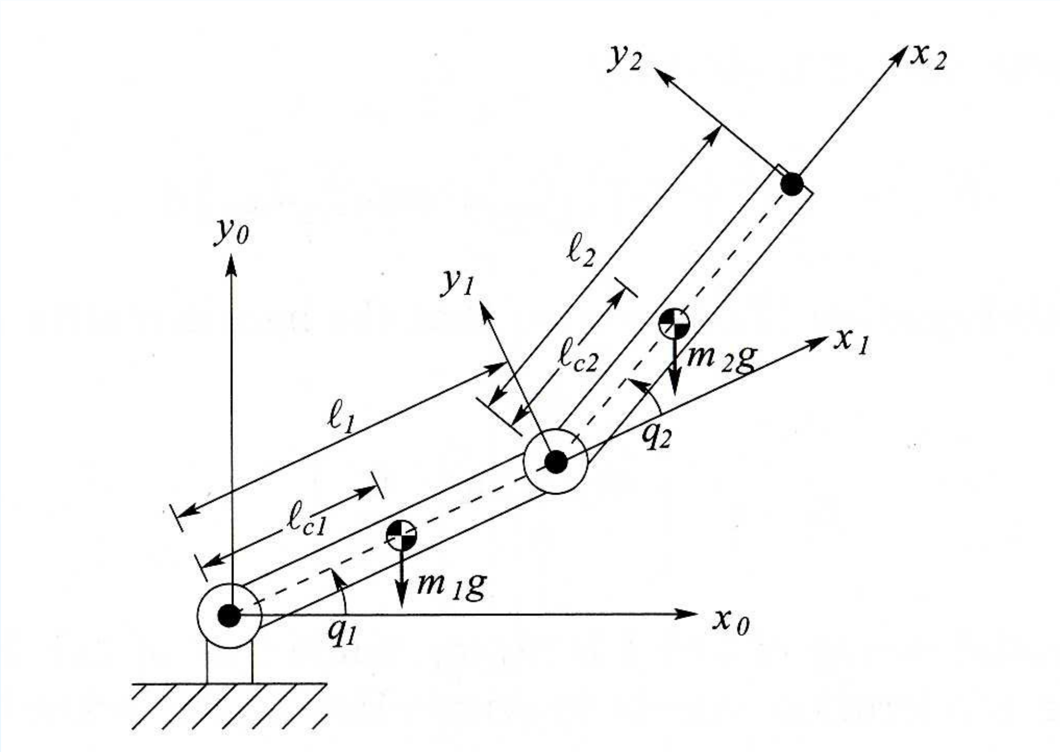
To sum up, the kinetic energy $\mathcal{K}$ is

$$
\begin{aligned}
\mathcal{K} &= \tfrac{1}{2}\left[m_1 v_{c1}^2 + \omega_1^T \mathcal{I}_1 \omega_1\right] + \tfrac{1}{2}\left[m_2 v_{c2}^2 + \omega_2^T \mathcal{I}_2 \omega_2\right] \\
&= \tfrac{1}{2}\left[m_1\left(J_{v_1}^{(1)}\dot{q}_1\right)^2 + I_1\left(J_{\omega_1}^{(1)}\dot{q}_1\right)^2\right] + \\
&\quad + \tfrac{1}{2}\left[m_2\left(J_{v_2}^{(1)}\dot{q}_1 + J_{v_2}^{(2)}\dot{q}_2\right)^2 + I_2\left(J_{\omega_2}^{(1)}\dot{q}_1 + J_{\omega_2}^{(2)}\dot{q}_2\right)^2\right] \\
&= \tfrac{1}{2}\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix}^T \begin{bmatrix} d_{11} & d_{12} \\ d_{12} & d_{22}\end{bmatrix}\begin{bmatrix}\dot{q}_1 \\ \dot{q}_2\end{bmatrix}
\end{aligned}
$$

with

$$
\begin{aligned}
d_{11} &= m_1 l_{c_1}^2 + m_2\left(l_1^2 + l_{c_2}^2 + 2 l_1 l_{c_2}\cos q_2\right) + I_1 + I_2 \\
d_{12} &= m_2\left(l_{c_2}^2 + l_1 l_{c_2}\cos q_2\right) + I_2 \\
d_{22} &= m_2 l_{c_2}^2 + I_2
\end{aligned}
$$

Potential Energy (PE) for Two-Link Elbow Manipulator



- PE of the 1st link is $\mathcal{P}_1 = m_1 g y_{c_1} = m_1 g l_{c_1} \sin q_1$
- PE of the 2nd link is
  $\mathcal{P}_2 = m_1 g y_{c_2} = m_2 g \left(l_1 \sin q_1 + l_{c_2} \sin(q_1 + q_2) \right)$
- Total PE is $\mathcal{P}_1 + \mathcal{P}_2$

Lecture 12: Dynamics: Euler-Lagrange Equations

- Computing Kinetic and Potential Energies
- Equations of Motion
- Examples
- Properties of Equations of Motion

Passivity Relation

Given a mechanical system

$$\boxed{\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = \tau} \Leftrightarrow \boxed{D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau}$$

with

$$\mathcal{L} = \frac{1}{2}\dot{q}^T D(q)\dot{q} - P(q)$$

Passivity Relation

Given a mechanical system

$$\boxed{\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = \tau} \quad \Leftrightarrow \quad \boxed{D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau}$$

with

$$\mathcal{L} = \frac{1}{2}\dot{q}^T D(q)\dot{q} - P(q)$$

Its energy is given by

$$\mathcal{H} = \frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)$$

Passivity Relation

Given a mechanical system

$$\boxed{\frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}}\right] - \frac{\partial \mathcal{L}}{\partial q} = \tau} \Leftrightarrow \boxed{D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau}$$

with

$$\mathcal{L} = \frac{1}{2}\dot{q}^T D(q)\dot{q} - P(q)$$

Its energy is given by

$$\mathcal{H} = \frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)$$

What will happen with $\frac{d}{dt}\mathcal{H}$?

Passivity Relation (Cont'd)

Differentiating $\mathcal{H}$ along a solution of the system, we have

$$\begin{aligned} \frac{d}{dt}\mathcal{H} &= \frac{d}{dt}\left[\frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)\right] \\ &= \frac{1}{2}\ddot{q}^T D(q)\dot{q} + \frac{1}{2}\dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \end{aligned}$$

Passivity Relation (Cont'd)

Differentiating $\mathcal{H}$ along a solution of the system, we have

$$
\begin{aligned}
\frac{d}{dt}\mathcal{H} &= \frac{d}{dt}\left[\frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)\right] \\
&= \frac{1}{2}\ddot{q}^T D(q)\dot{q} + \frac{1}{2}\dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q}
\end{aligned}
$$

Passivity Relation (Cont'd)

Differentiating $\mathcal{H}$ along a solution of the system, we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{H} &= \frac{d}{dt}\left[\frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)\right] \\
&= \frac{1}{2}\ddot{q}^T D(q)\dot{q} + \frac{1}{2}\dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T \left[\tau - C(q,\dot{q})\dot{q} - g(q)\right] + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q}
\end{aligned}$$

Here we use the Euler-Lagrange equations

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

Passivity Relation (Cont'd)

Differentiating $\mathcal{H}$ along a solution of the system, we have

$$
\begin{aligned}
\frac{d}{dt}\mathcal{H} &= \frac{d}{dt}\left[\frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)\right] \\
&= \frac{1}{2}\ddot{q}^T D(q)\dot{q} + \frac{1}{2}\dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T\left[\tau - C(q,\dot{q})\dot{q} - g(q)\right] + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T\tau + \dot{q}^T\left(\frac{1}{2}\frac{d}{dt}\left[D(q)\right] - C(q,\dot{q})\right)\dot{q} + \dot{q}^T\left(\frac{\partial \mathcal{P}}{\partial q} - g(q)\right)
\end{aligned}
$$

Passivity Relation (Cont'd)

Differentiating $\mathcal{H}$ along a solution of the system, we have

$$
\begin{aligned}
\frac{d}{dt}\mathcal{H} &= \frac{d}{dt}\left[\frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)\right] \\
&= \frac{1}{2}\ddot{q}^T D(q)\dot{q} + \frac{1}{2}\dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T\left[\tau - C(q,\dot{q})\dot{q} - g(q)\right] + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T\tau + \dot{q}^T\left(\frac{1}{2}\frac{d}{dt}\left[D(q)\right] - C(q,\dot{q})\right)\dot{q} + \dot{q}^T \underbrace{\left(\frac{\partial \mathcal{P}}{\partial q} - g(q)\right)}_{=0}
\end{aligned}
$$

Passivity Relation (Cont'd)

Differentiating $\mathcal{H}$ along a solution of the system, we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{H} &= \frac{d}{dt}\left[\frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)\right] \\
&= \frac{1}{2}\ddot{q}^T D(q)\dot{q} + \frac{1}{2}\dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T\left[\tau - C(q,\dot{q})\dot{q} - g(q)\right] + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T\tau + \dot{q}^T \underbrace{\left(\frac{1}{2}\frac{d}{dt}\left[D(q)\right] - C(q,\dot{q})\right)}_{=0}\dot{q}
\end{aligned}$$

Passivity Relation (Cont'd)

Differentiating $\mathcal{H}$ along a solution of the system, we have

$$
\begin{aligned}
\frac{d}{dt}\mathcal{H} &= \frac{d}{dt}\left[\frac{1}{2}\dot{q}^T D(q)\dot{q} + P(q)\right] \\
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&= \dot{q}^T D(q)\ddot{q} + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T \left[\tau - C(q,\dot{q})\dot{q} - g(q)\right] + \frac{1}{2}\dot{q}^T \frac{d}{dt}\left[D(q)\right]\dot{q} + \dot{q}^T \frac{\partial \mathcal{P}}{\partial q} \\
&= \dot{q}^T \tau + \dot{q}^T \underbrace{\left(\frac{1}{2}\frac{d}{dt}\left[D(q)\right] - C(q,\dot{q})\right)}_{=0}\dot{q} \\
&= \dot{q}^T \tau
\end{aligned}
$$

Passivity Relation (Cont'd)

The differential relation

$$\frac{d}{dt}\mathcal{H} \quad = \quad \dot{q}^T \boldsymbol{\tau}$$

can be integrated, so that

$$\begin{aligned} \int_0^T \frac{d}{dt}\mathcal{H}(q(t), \dot{q}(t))dt &= \mathcal{H}(q(T), \dot{q}(T)) - \mathcal{H}(q(0), \dot{q}(0)) \\ &= \int_0^T \dot{q}(t)^T \boldsymbol{\tau}(t) dt \end{aligned}$$

Passivity Relation (Cont'd)

The differential relation

$$\frac{d}{dt}\mathcal{H} \quad = \quad \dot{q}^T \boldsymbol{\tau}$$

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$$\Rightarrow \int_0^T \dot{q}(t)\boldsymbol{\tau}(t)dt \geq -\mathcal{H}(q(0), \dot{q}(0))$$

Passivity Relation (Cont'd)

The differential relation

$$\frac{d}{dt}\mathcal{H} \quad = \quad \dot{q}^{T}\tau$$

can be integrated, so that

$$\begin{aligned}\int_0^T \frac{d}{dt}\mathcal{H}(q(t), \dot{q}(t))dt &= \mathcal{H}(q(T), \dot{q}(T)) - \mathcal{H}(q(0), \dot{q}(0)) \\ &= \int_0^T \dot{q}(t)^{T}\tau(t)dt\end{aligned}$$

$$\boxed{\Rightarrow \int_0^T \dot{q}(t)\tau(t)dt \geq -\mathcal{H}(q(0), \dot{q}(0))}$$

These relations are called

- passivity (dissipativity) relation
- passivity (dissipativity) relation in the integral form

Skew Symmetry of $\dot{D}(q) - C(q, \dot{q})$

To check that

$$N = \frac{d}{dt}\left[D(q)\right] - 2C(q, \dot{q}), \quad N^T = -N$$

look at $(k, j)^{th}$-component

$$\frac{d}{dt} d_{kj} - 2c_{kj} \;=\; \sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i}\dot{q}_i - \sum_{i=1}^{n}\left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]\dot{q}_i$$

Skew Symmetry of $\dot{D}(q) - C(q, \dot{q})$

To check that

$$N = \frac{d}{dt}\left[D(q)\right] - 2C(q, \dot{q}), \quad N^T = -N$$

look at $(k, j)^{th}$-component

$$\begin{aligned} \frac{d}{dt} d_{kj} - 2c_{kj} &= \sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i} \dot{q}_i - \sum_{i=1}^{n} \left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right] \dot{q}_i \\ &= \sum_{i=1}^{n} \left\{ \frac{\partial d_{kj}}{\partial q_i} - \left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right] \right\} \dot{q}_i \end{aligned}$$

Skew Symmetry of $\dot{D}(q) - C(q, \dot{q})$

To check that

$$N = \frac{d}{dt}\left[D(q)\right] - 2C(q, \dot{q}), \quad N^T = -N$$

look at $(k, j)^{th}$-component

$$\begin{aligned}
\frac{d}{dt} d_{kj} - 2c_{kj} &= \sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i} \dot{q}_i - \sum_{i=1}^{n} \left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right] \dot{q}_i \\
&= \sum_{i=1}^{n} \left\{ \frac{\partial d_{kj}}{\partial q_i} - \left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right] \right\} \dot{q}_i \\
&= \sum_{i=1}^{n} \left\{ \frac{\partial d_{ij}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j} \right\} \dot{q}_i
\end{aligned}$$

Skew Symmetry of $\dot{D}(q) - C(q,\dot{q})$

To check that

$$N = \frac{d}{dt}\left[D(q)\right] - 2C(q,\dot{q}), \quad N^T = -N$$

look at $(k,j)^{th}$-component

$$\begin{aligned}
\frac{d}{dt}d_{kj} - 2c_{kj} &= \sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i}\dot{q}_i - \sum_{i=1}^{n}\left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]\dot{q}_i \\
&= \sum_{i=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i} - \left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]\right\}\dot{q}_i \\
&= \sum_{i=1}^{n}\left\{\frac{\partial d_{ij}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_i = \sum_{i=1}^{n}\left\{\frac{\partial d_{ji}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_i
\end{aligned}$$

Skew Symmetry of $\dot{D}(q) - C(q, \dot{q})$

To check that

$$N = \frac{d}{dt}\left[D(q)\right] - 2C(q,\dot{q}), \quad N^T = -N$$

look at $(k, j)^{th}$-component

$$\begin{aligned}
\frac{d}{dt}d_{kj} - 2c_{kj} &= \sum_{i=1}^{n} \frac{\partial d_{kj}}{\partial q_i}\dot{q}_i - \sum_{i=1}^{n}\left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]\dot{q}_i \\
&= \sum_{i=1}^{n}\left\{\frac{\partial d_{kj}}{\partial q_i} - \left[\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k}\right]\right\}\dot{q}_i \\
&= \sum_{i=1}^{n}\left\{\frac{\partial d_{ij}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_i = \sum_{i=1}^{n}\left\{\frac{\partial d_{ji}}{\partial q_k} - \frac{\partial d_{ki}}{\partial q_j}\right\}\dot{q}_i
\end{aligned}$$

$$\Rightarrow \qquad n_{kj} = -n_{jk}$$