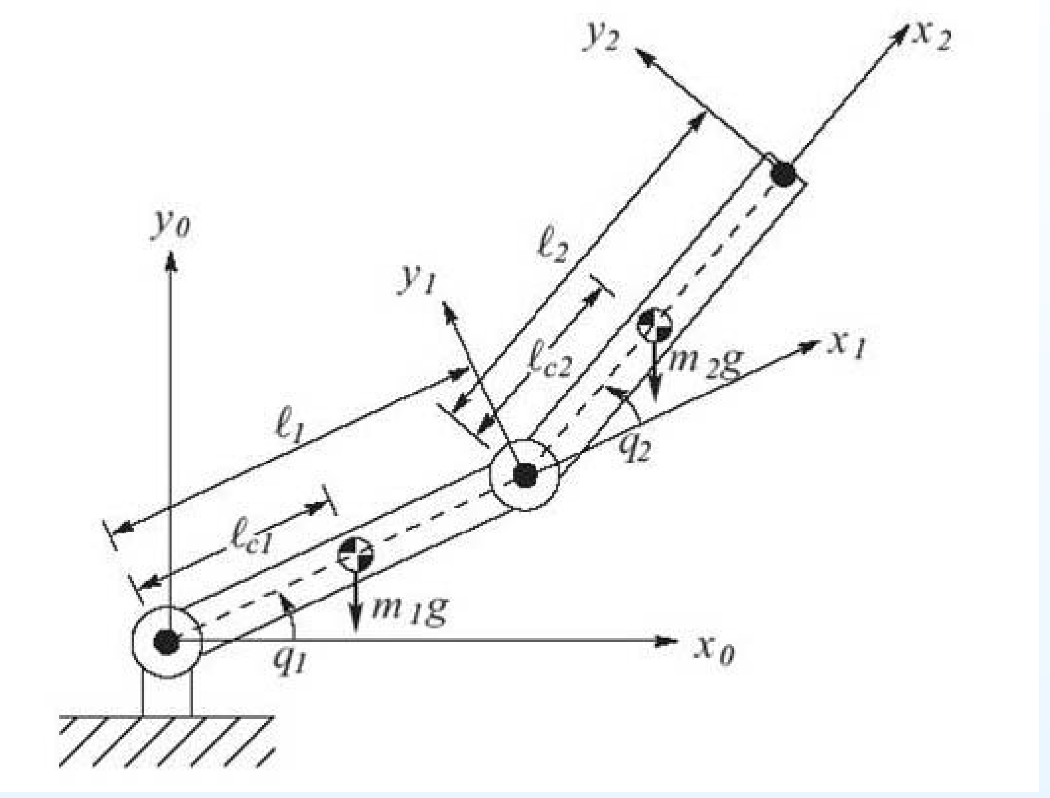
Lecture 10: Dynamics: Euler-Lagrange Equations

- Examples

Lecture 10: Dynamics: Euler-Lagrange Equations

- Examples
- Properties of Equations of Motion

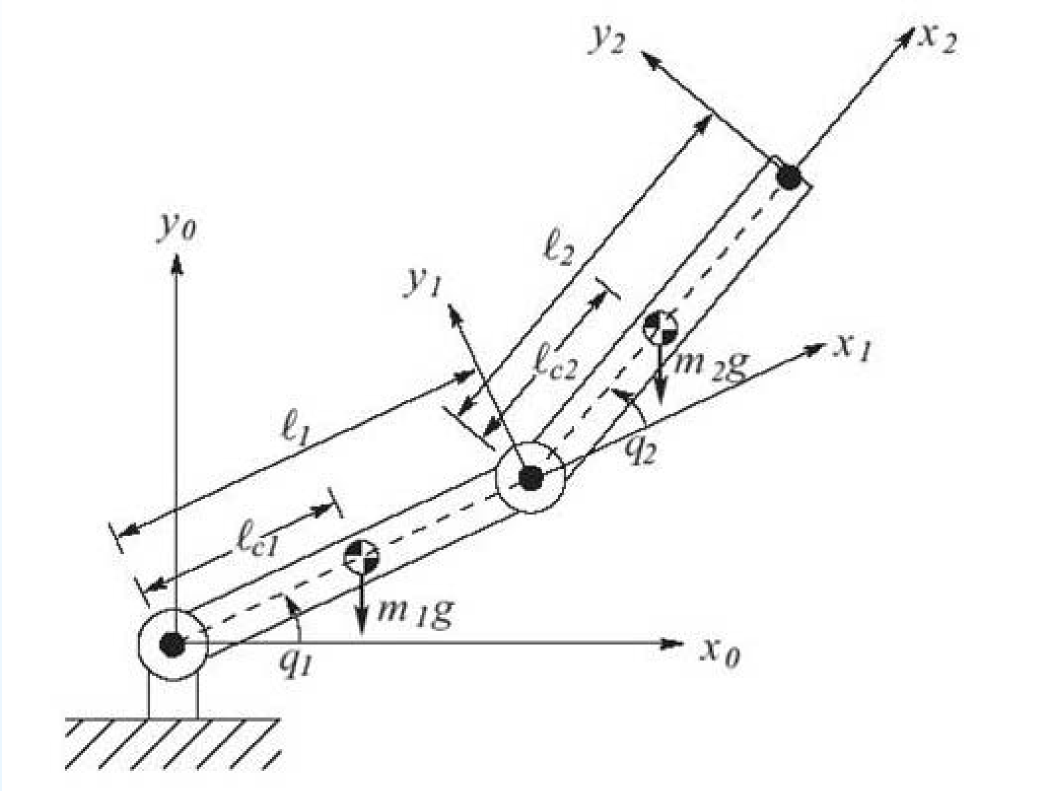
Example: Double Link Pendulum



Steps in deriving equations of motion

- Write the kinetic energy $\mathcal{K}$;
- Write the potential energy $\mathcal{P}$;
- Use them to obtain the Euler-Lagrange equations

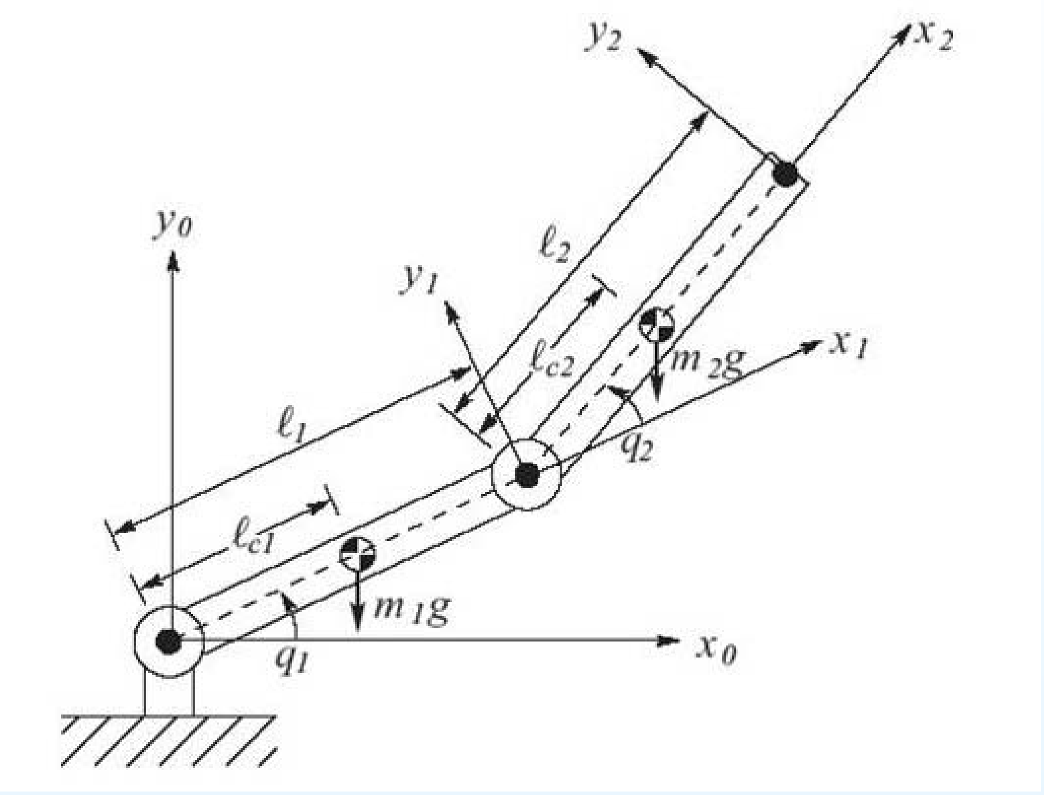
Example: Double Link Pendulum



Steps in deriving equations of motion

- $\mathcal{K} = \frac{1}{2}\left[m_1 v_{c1}^2 + \omega_1^T \mathcal{I}_1 \omega_1\right] + \frac{1}{2}\left[m_2 v_{c2}^2 + \omega_2^T \mathcal{I}_2 \omega_2\right]$;
- Write the potential energy $\mathcal{P}$;
- Use them to obtain the Euler-Lagrange equations

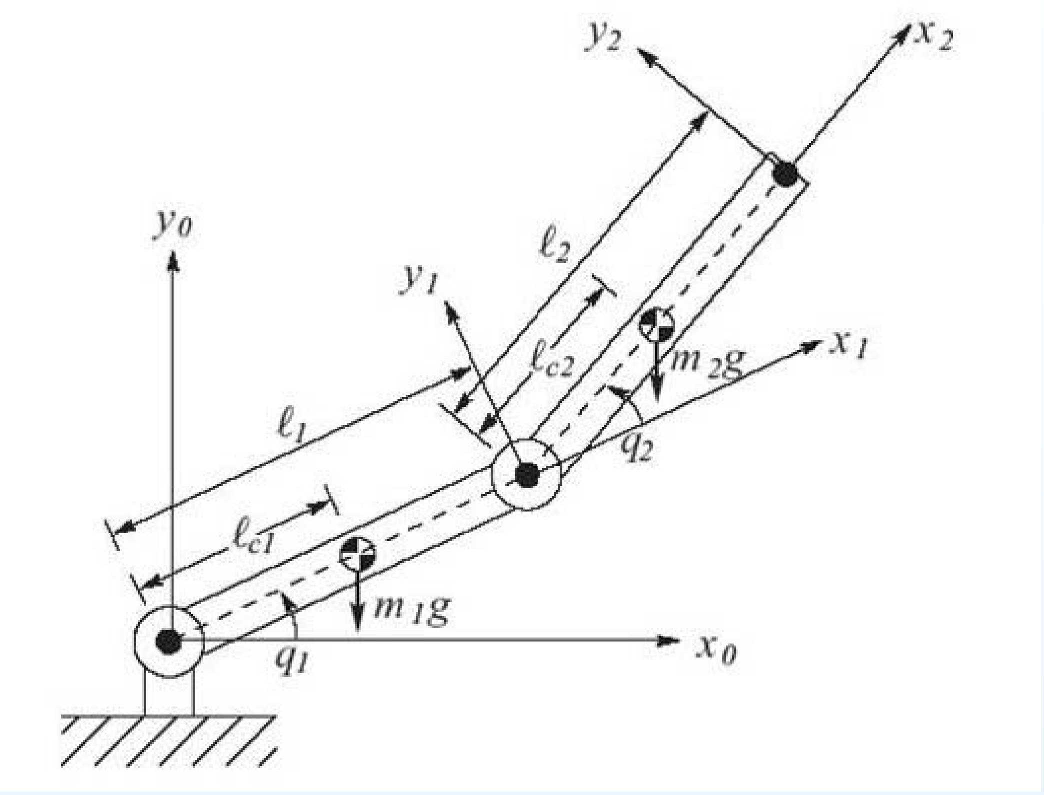
Example: Double Link Pendulum



Steps in deriving equations of motion

- $\mathcal{K} = \frac{1}{2}\left[m_1 v_{c1}^2 + \omega_1^T \mathcal{I}_1 \omega_1\right] + \frac{1}{2}\left[m_2 v_{c2}^2 + \omega_2^T \mathcal{I}_2 \omega_2\right];$
- $\mathcal{P} = m_1 \cdot g \cdot y_{c1} + m_2 \cdot g \cdot y_{c2};$
- Use them to obtain the Euler-Lagrange equations

Example: Double Link Pendulum



Steps in deriving equations of motion

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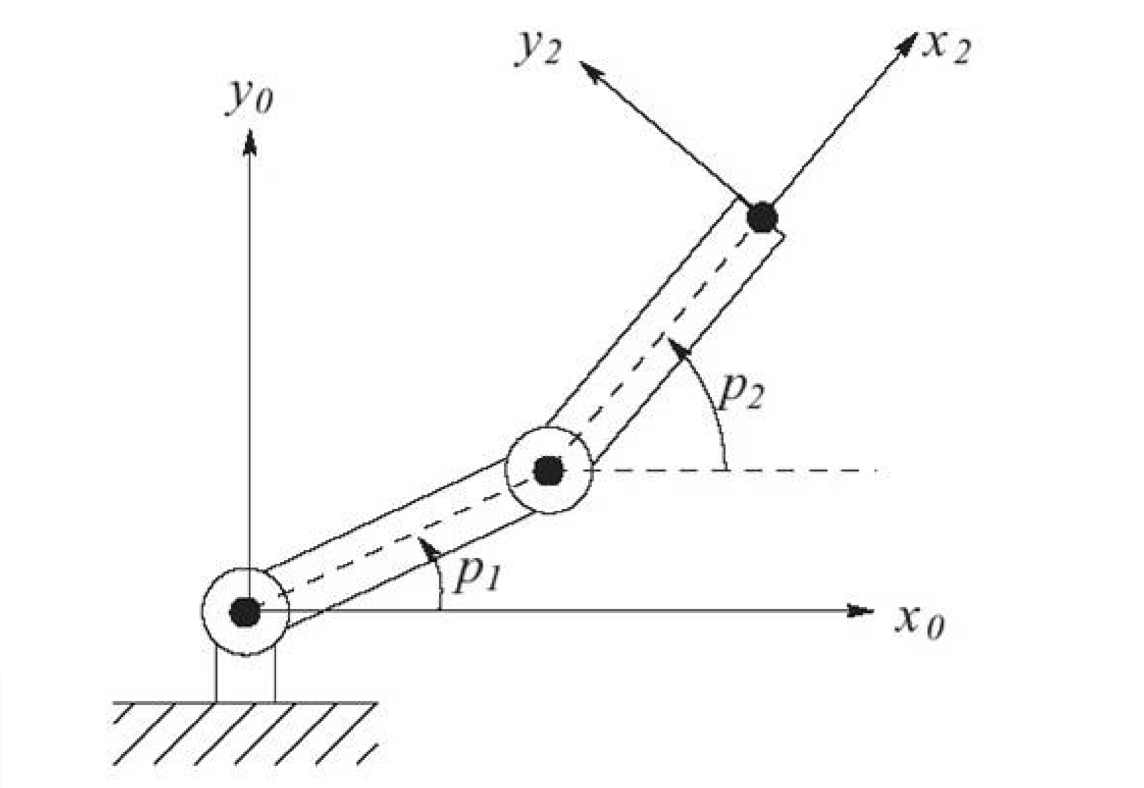
- $\mathcal{P} = m_1 \cdot g \cdot y_{c1} + m_2 \cdot g \cdot y_{c2};$
- $\frac{d}{dt}\left[\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial \dot{q}_1}\right] - \frac{\partial(\mathcal{K}-\mathcal{P})}{\partial q_1} = \tau_1$, $\frac{d}{dt}\left[\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial \dot{q}_2}\right] - \frac{\partial(\mathcal{K}-\mathcal{P})}{\partial q_2} = \tau_2$

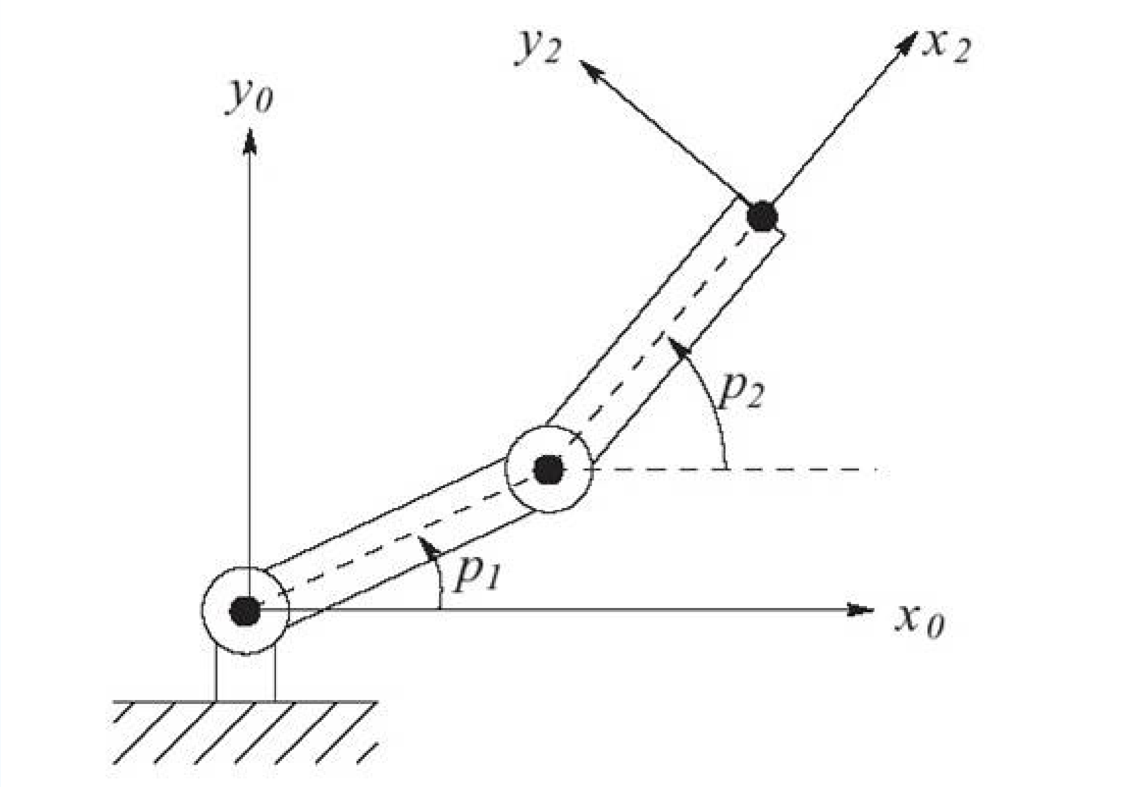
Example: Double Link Pendulum



Steps in deriving equations of motion in different coordinates

- Write the kinetic energy $\mathcal{K}(p_1, p_2, \dot{p}_1, \dot{p}_2)$;
- Write the potential energy $\mathcal{P}(p_1, p_2)$;
- Use them to obtain the Euler-Lagrange equations

Example: Double Link Pendulum

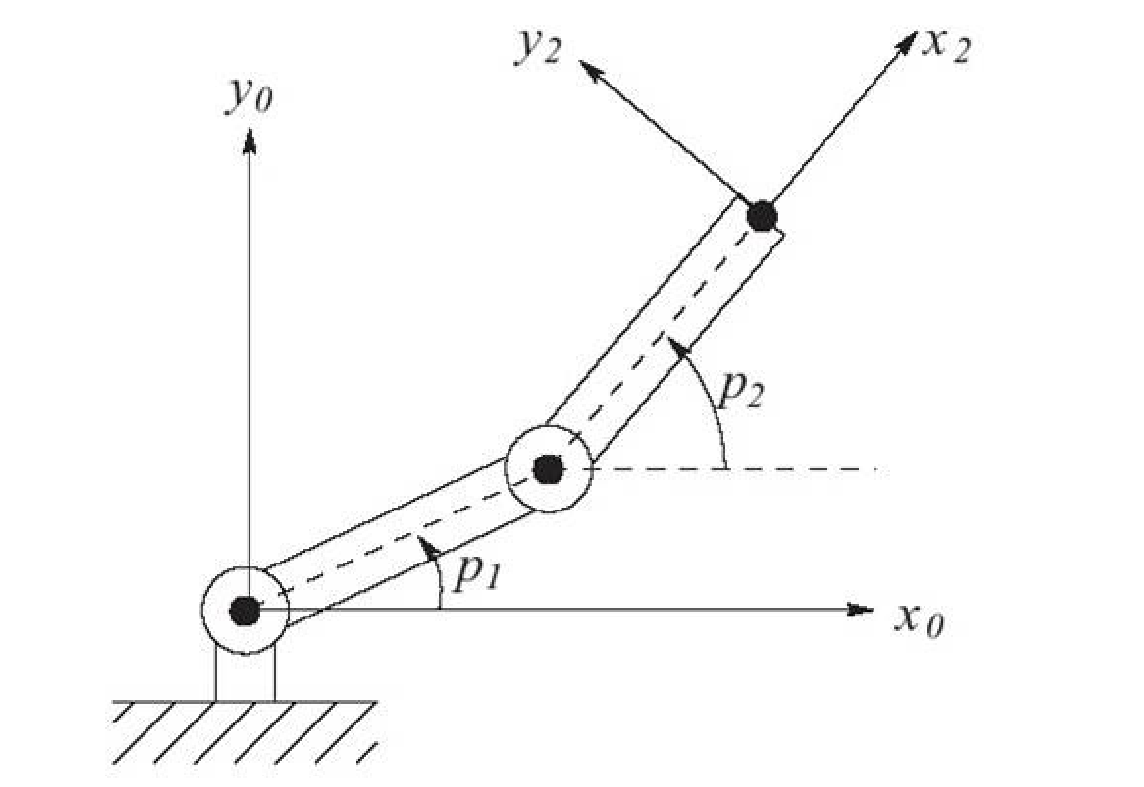


Steps in deriving equations of motion in different coordinates

$$\mathcal{K} \;=\; \frac{1}{2}\left[m_1 v_{c_1}^2 + \omega_1^T \mathcal{I}_1 \omega_1\right] + \frac{1}{2}\left[m_2 v_{c_2}^2 + \omega_2^T \mathcal{I}_2 \omega_2\right]$$

$$x_{c_1} = l_{c_1}\cos(p_1), \qquad y_{c_1} = l_{c_1}\sin(p_1)$$

Example: Double Link Pendulum



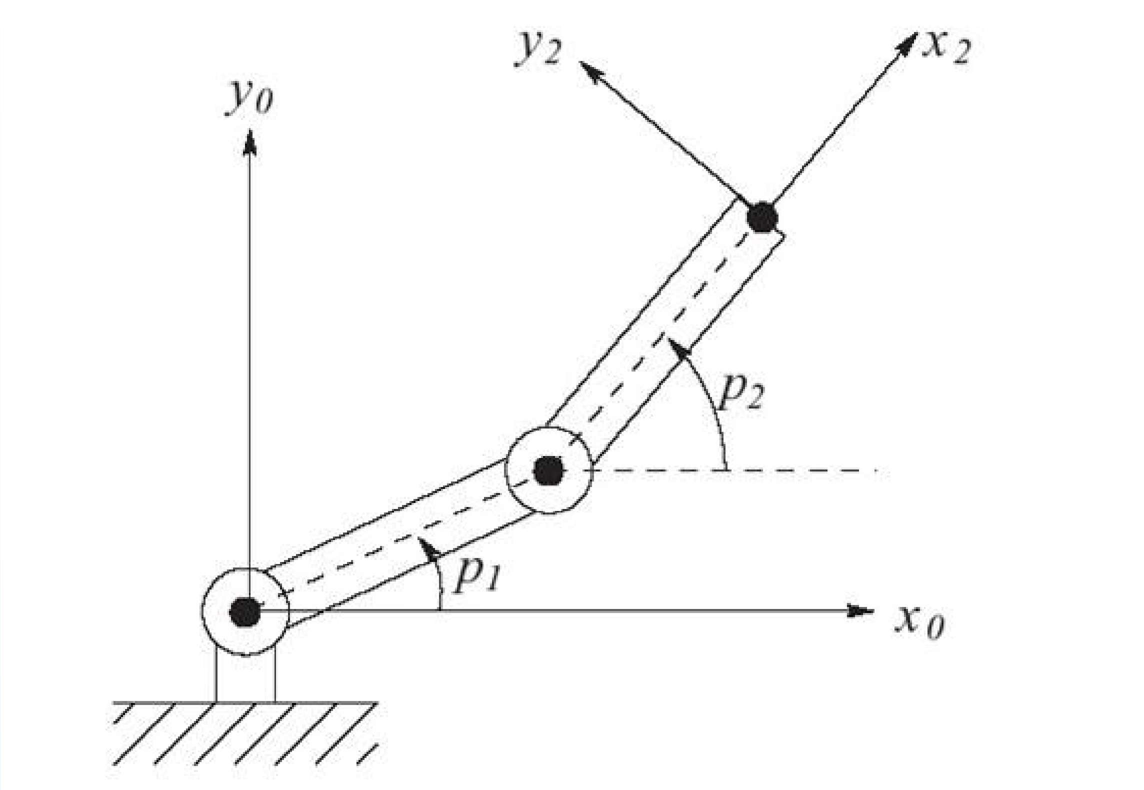
Steps in deriving equations of motion in different coordinates

$$\mathcal{K} = \frac{1}{2}\left[m_1 v_{c_1}^2 + \omega_1^T \mathcal{I}_1 \omega_1\right] + \frac{1}{2}\left[m_2 v_{c_2}^2 + \omega_2^T \mathcal{I}_2 \omega_2\right]$$

$$x_{c_1} = l_{c_1}\cos(p_1), \qquad y_{c_1} = l_{c_1}\sin(p_1)$$

$$\dot{x}_{c_1} = -l_{c_1}\sin(p_1)\cdot\dot{p}_1, \qquad \dot{y}_{c_1} = l_{c_1}\cos(p_1)\cdot\dot{p}_1$$

Example: Double Link Pendulum

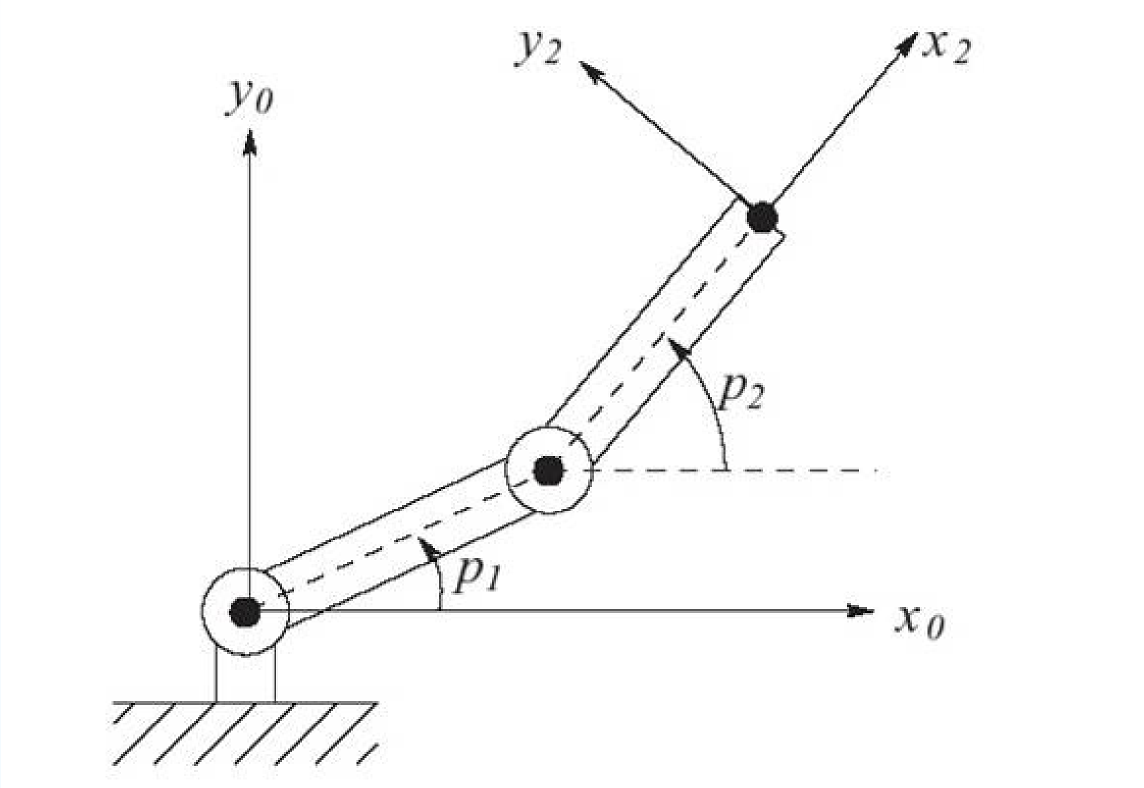


Steps in deriving equations of motion in different coordinates

$$\mathcal{K} = \frac{1}{2}\left[m_1 v_{c_1}^2 + \omega_1^T \mathcal{I}_1 \omega_1\right] + \frac{1}{2}\left[m_2 v_{c_2}^2 + \omega_2^T \mathcal{I}_2 \omega_2\right]$$

$$\boxed{\omega_1 = \dot{p}_1 \cdot \vec{k}}$$

Example: Double Link Pendulum



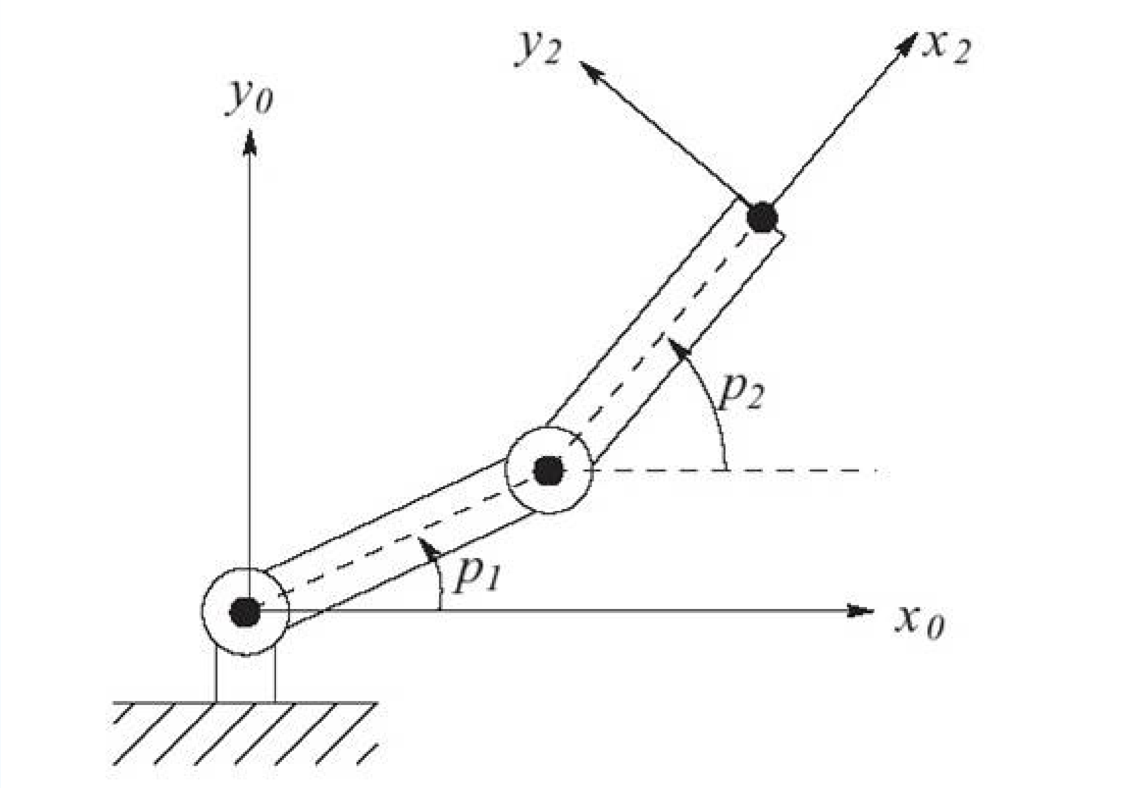
Steps in deriving equations of motion in different coordinates

$$\mathcal{K} = \frac{1}{2}\left[m_1 v_{c_1}^2 + \omega_1^T \mathcal{I}_1 \omega_1\right] + \frac{1}{2}\left[m_2 v_{c_2}^2 + \omega_2^T \mathcal{I}_2 \omega_2\right]$$

$$x_{c_2} = l_1 \cos(p_1) + l_{c_2} \cos(p_2)$$

$$\dot{x}_{c_2} = -l_1 \sin(p_1) \cdot \dot{p}_1 - l_{c_2} \sin(p_2) \cdot \dot{p}_2$$

Example: Double Link Pendulum



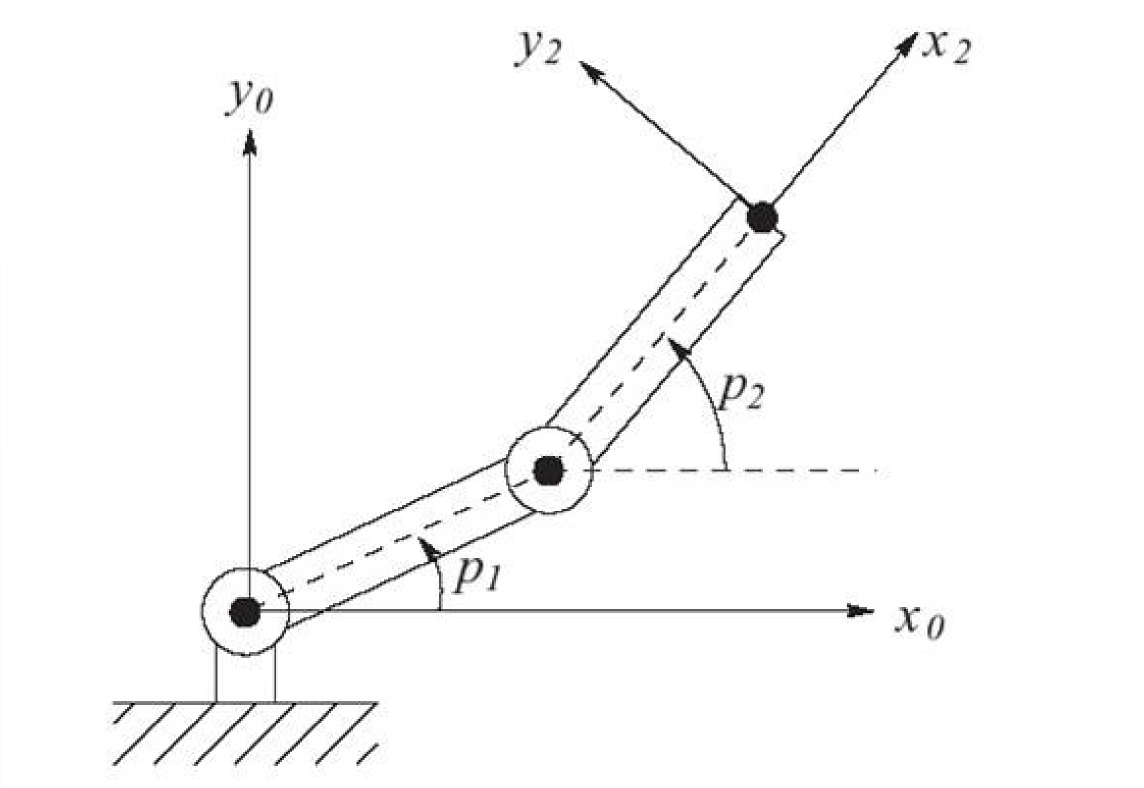
Steps in deriving equations of motion in different coordinates

$$\mathcal{K} = \frac{1}{2}\left[m_1 v_{c_1}^2 + \omega_1^T \mathcal{I}_1 \omega_1\right] + \frac{1}{2}\left[m_2 v_{c_2}^2 + \omega_2^T \mathcal{I}_2 \omega_2\right]$$

$$y_{c_2} = l_1 \sin(p_1) + l_{c_2} \sin(p_2)$$

$$\dot{y}_{c_2} = l_1 \cos(p_1) \cdot \dot{p}_1 + l_{c_2} \cos(p_2) \cdot \dot{p}_2$$

Example: Double Link Pendulum

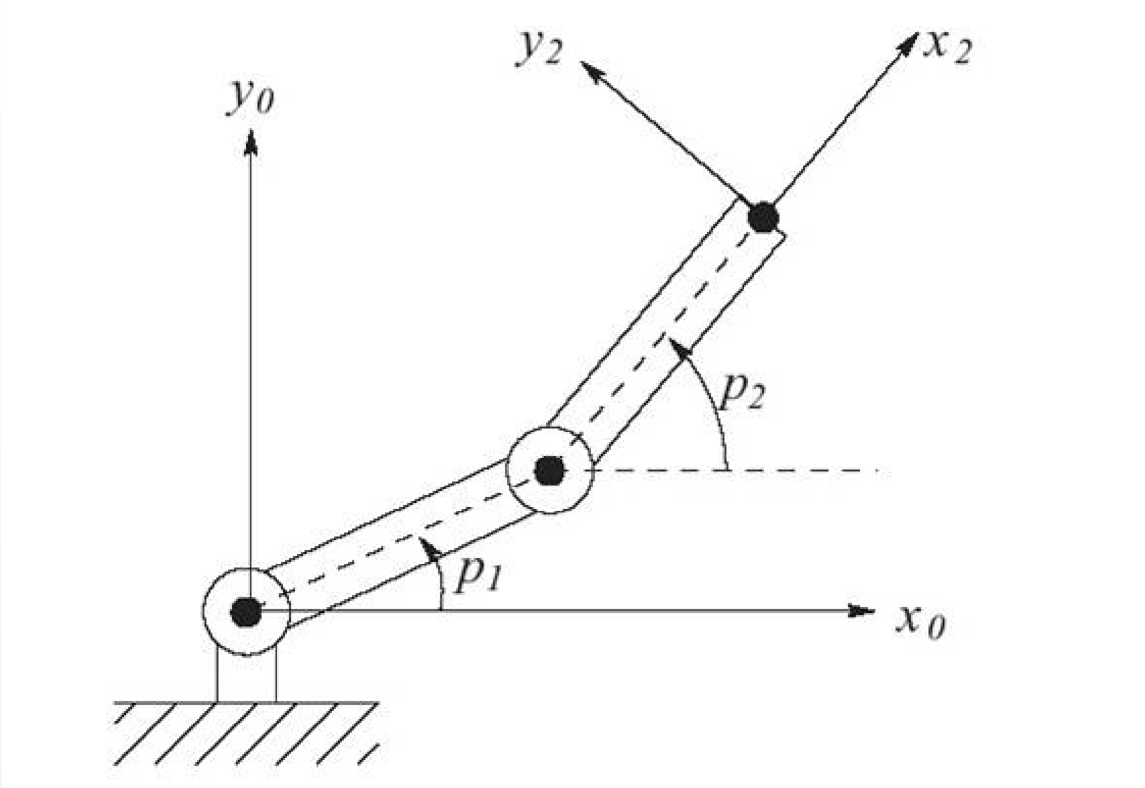


Steps in deriving equations of motion in different coordinates

$$\mathcal{K} \;=\; \frac{1}{2}\left[m_1 v_{c_1}^2 + \omega_1^T \mathcal{I}_1 \omega_1\right] + \frac{1}{2}\left[m_2 v_{c_2}^2 + \omega_2^T \mathcal{I}_2 \omega_2\right]$$

$$\boxed{\omega_2 = \dot{p}_2 \cdot \vec{k}}$$

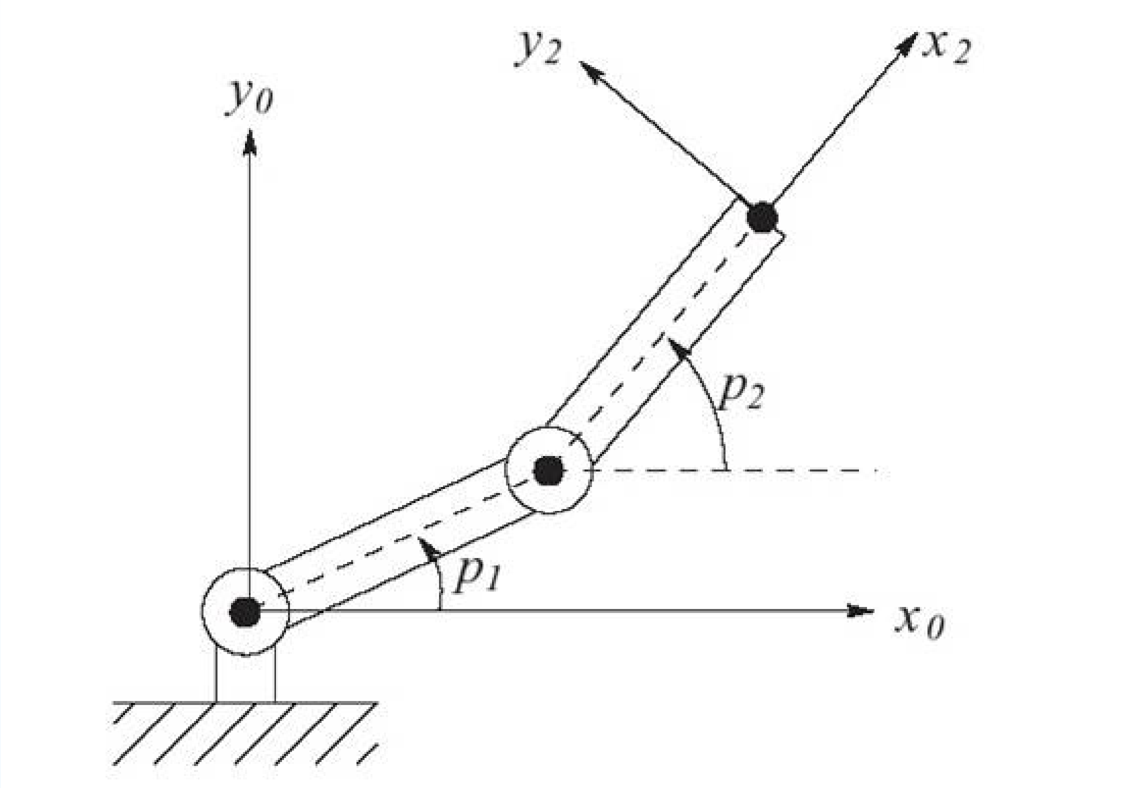
Example: Double Link Pendulum



Steps in deriving equations of motion in different coordinates

$$\begin{aligned}
\mathcal{K} &= \frac{1}{2}\left[m_1 v_{c_1}^2 + \omega_1^T \mathcal{I}_1 \omega_1\right] + \frac{1}{2}\left[m_2 v_{c_2}^2 + \omega_2^T \mathcal{I}_2 \omega_2\right] \\
&= \frac{1}{2}\begin{bmatrix}\dot{p}_1 \\ \dot{p}_2\end{bmatrix}^T D(p_1, p_2)\begin{bmatrix}\dot{p}_1 \\ \dot{p}_2\end{bmatrix}
\end{aligned}$$

Example: Double Link Pendulum

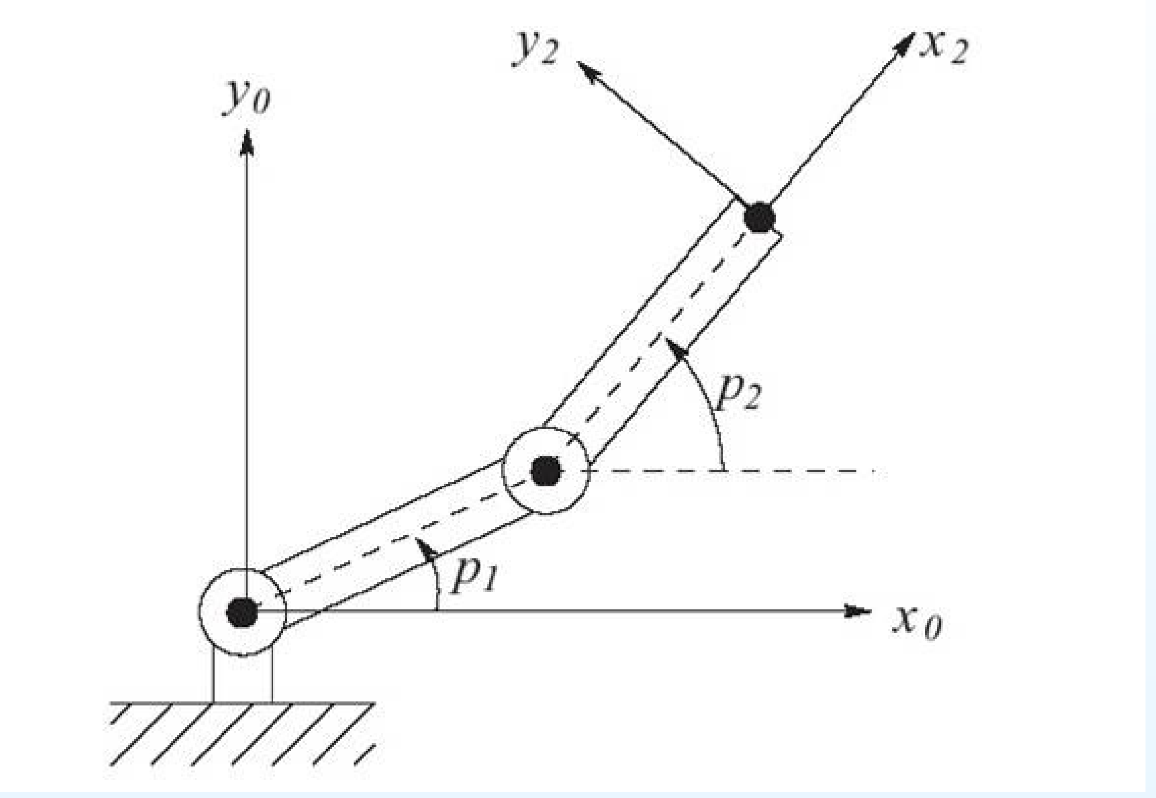


Steps in deriving equations of motion in different coordinates

$$\mathcal{K} = \frac{1}{2}\left[m_1 v_{c_1}^2 + \omega_1^T \mathcal{I}_1 \omega_1\right] + \frac{1}{2}\left[m_2 v_{c_2}^2 + \omega_2^T \mathcal{I}_2 \omega_2\right]$$

$$= \frac{1}{2}\begin{bmatrix}\dot{p}_1\\ \dot{p}_2\end{bmatrix}^T \begin{bmatrix} m_1 l_{c_1}^2 + m_2 l_1^2 + I_1 & m_2 l_1 l_{c_2}\cos(p_2 - p_1) \\ m_2 l_1 l_{c_2}\cos(p_2 - p_1) & m_2 l_{c_2}^2 + I_2 \end{bmatrix}\begin{bmatrix}\dot{p}_1\\ \dot{p}_2\end{bmatrix}$$

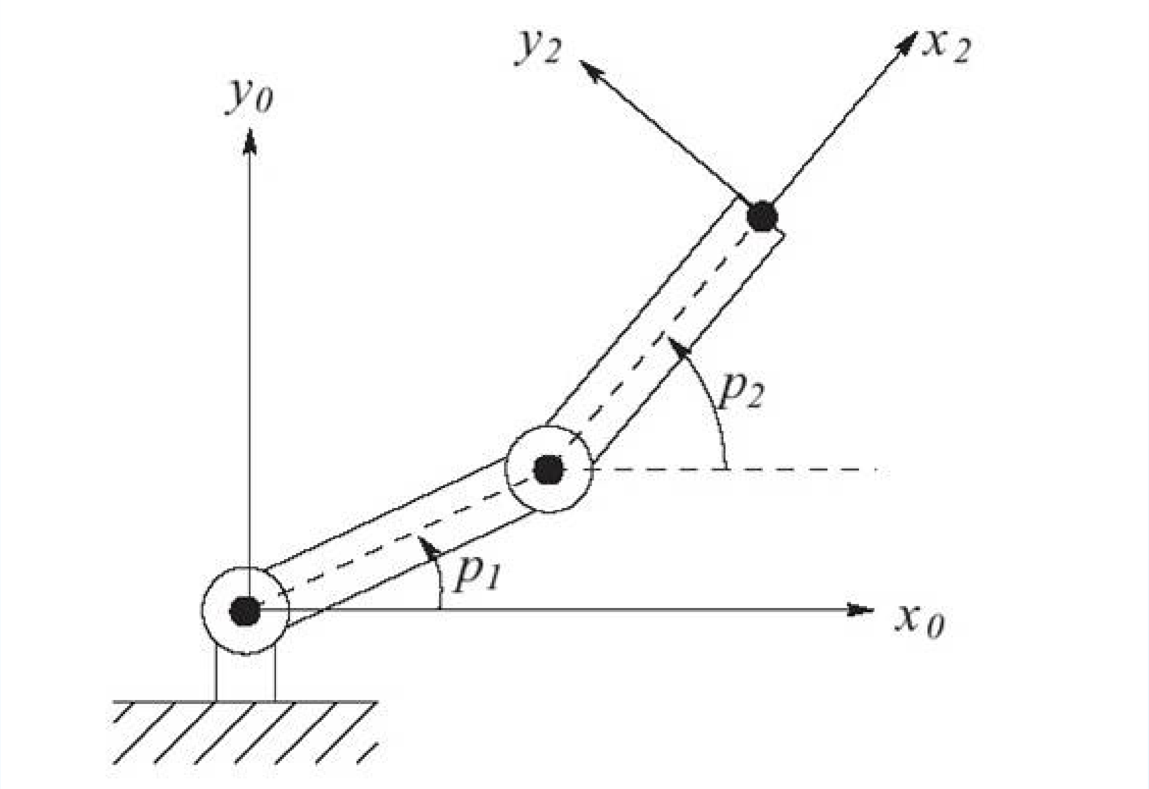
Example: Double Link Pendulum



Steps in deriving equations of motion in different coordinates

$$\mathcal{P} = m_1 \cdot g \cdot y_{c_1} + m_2 \cdot g \cdot y_{c_2}$$

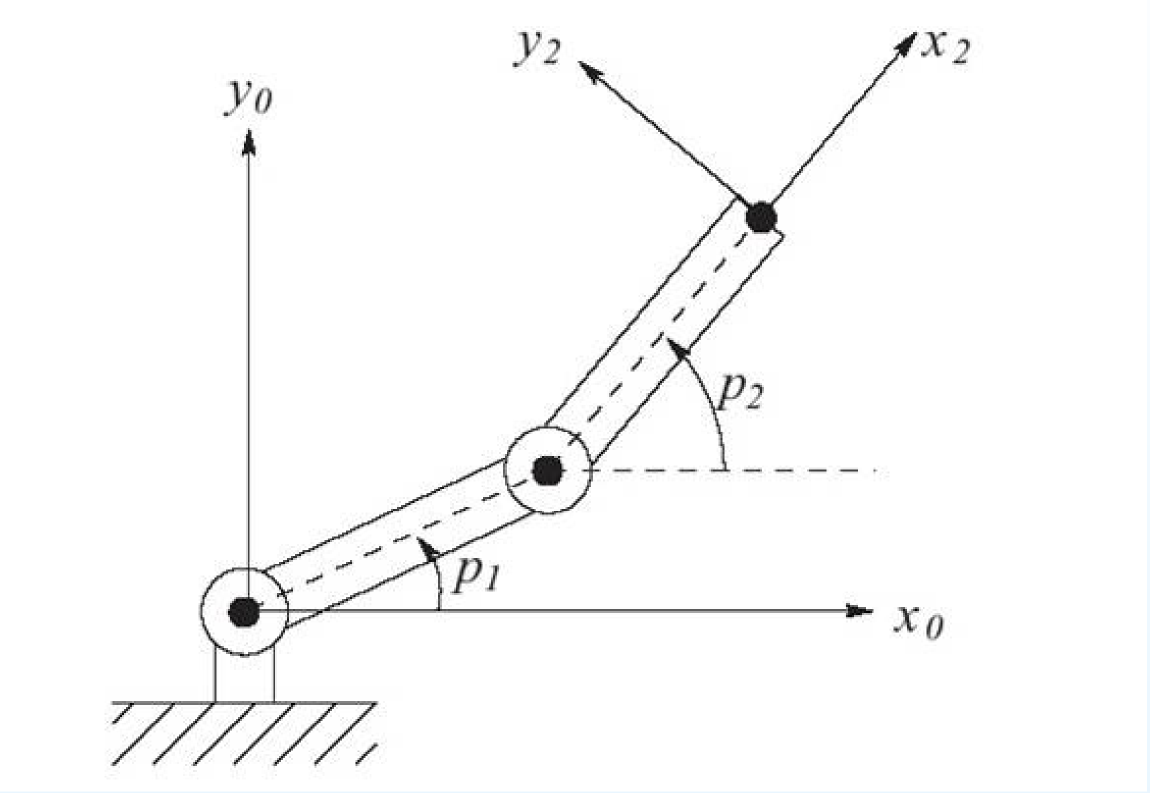
Example: Double Link Pendulum



Steps in deriving equations of motion in different coordinates

$$\begin{aligned}\mathcal{P} &= m_1 \cdot g \cdot y_{c_1} + m_2 \cdot g \cdot y_{c_2} \\ &= m_1 \cdot g \cdot l_{c_1} \cdot \sin(p_1) + m_2 \cdot g \cdot \left[l_1 \cdot \sin(p_1) + l_{c_2} \cdot \sin(p_2)\right]\end{aligned}$$

Example: Double Link Pendulum

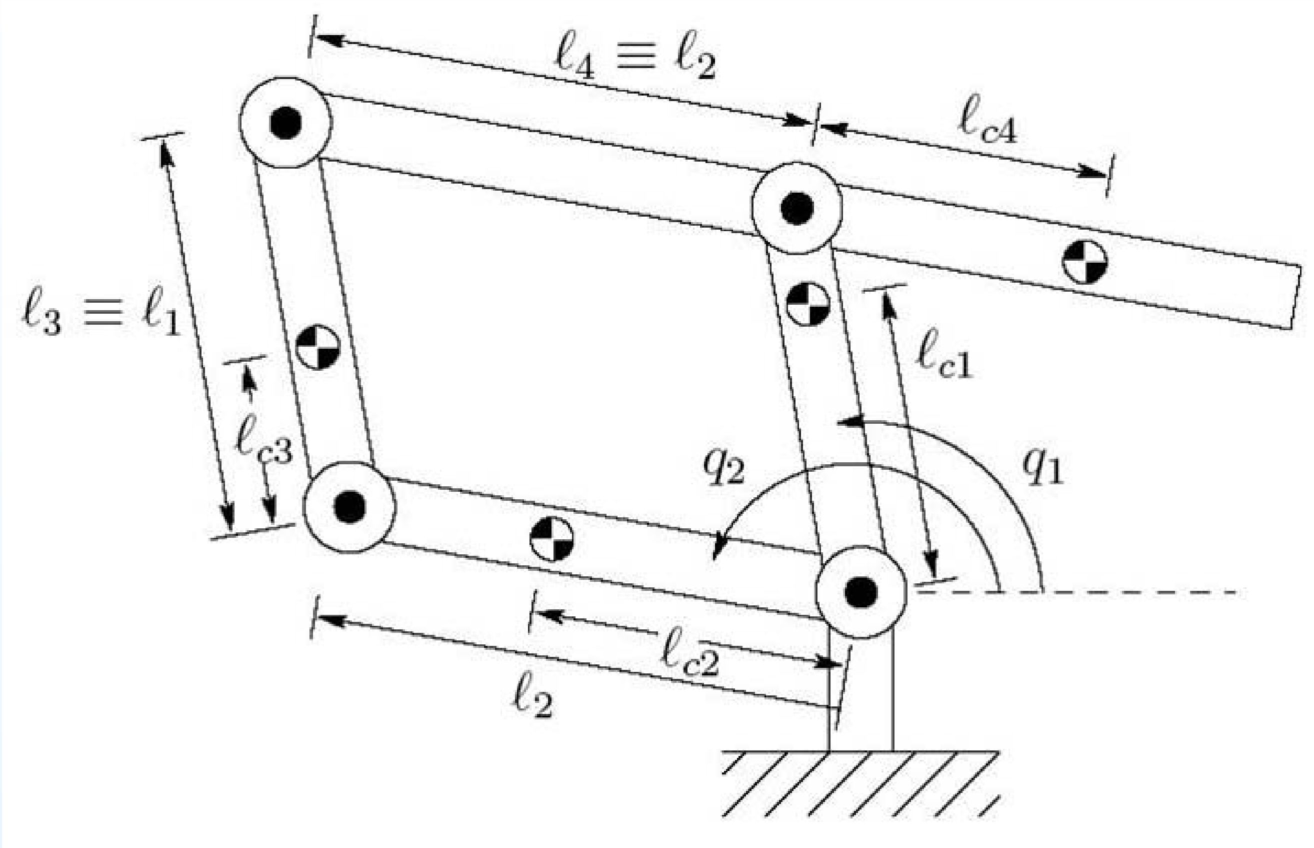


Steps in deriving equations of motion in different coordinates

$$\frac{d}{dt}\left[\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial \dot{p}_1}\right] - \frac{\partial(\mathcal{K}-\mathcal{P})}{\partial p_1} = \tau_1$$

$$\frac{d}{dt}\left[\frac{\partial(\mathcal{K}-\mathcal{P})}{\partial \dot{p}_2}\right] - \frac{\partial(\mathcal{K}-\mathcal{P})}{\partial p_2} = \tau_2$$

Example: Mechanism with a Closed Kinematic Chain



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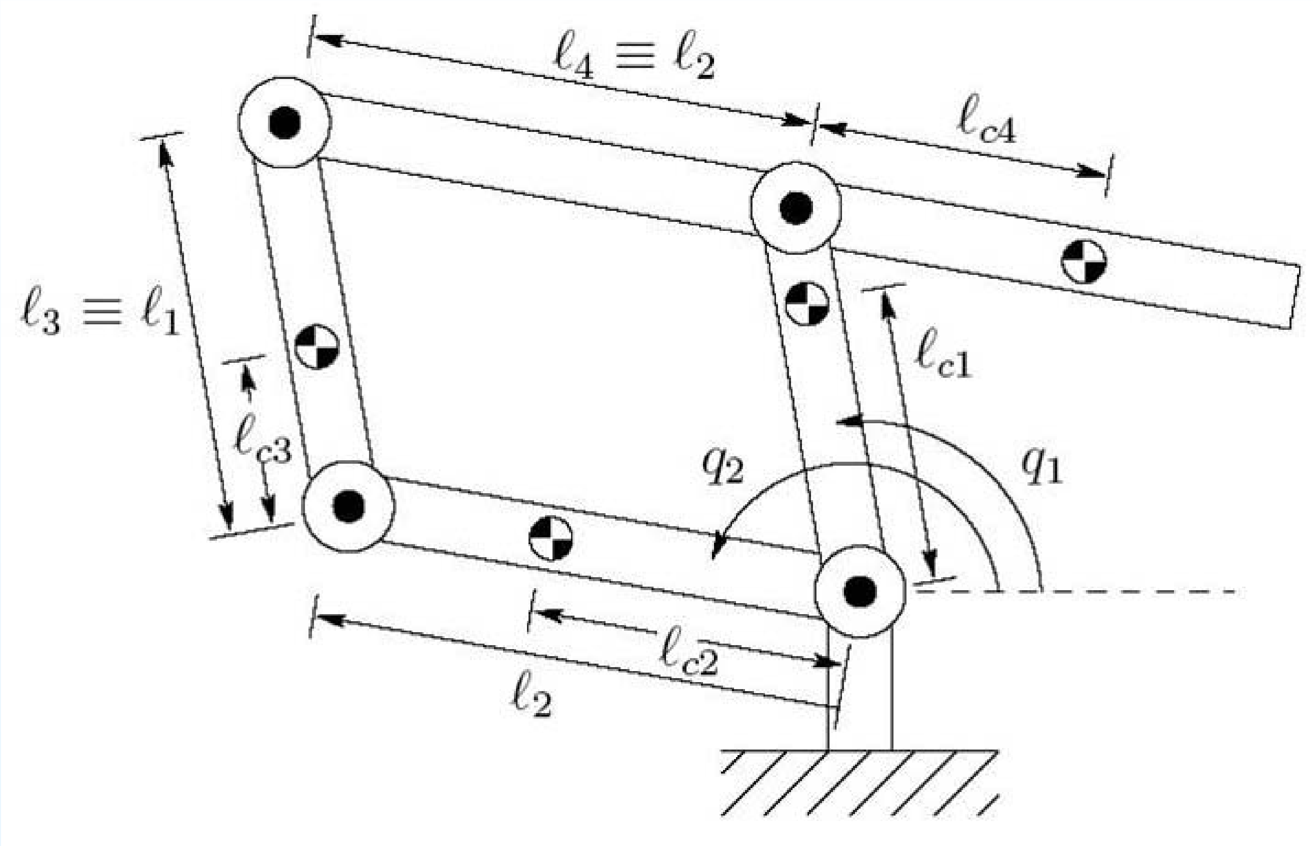
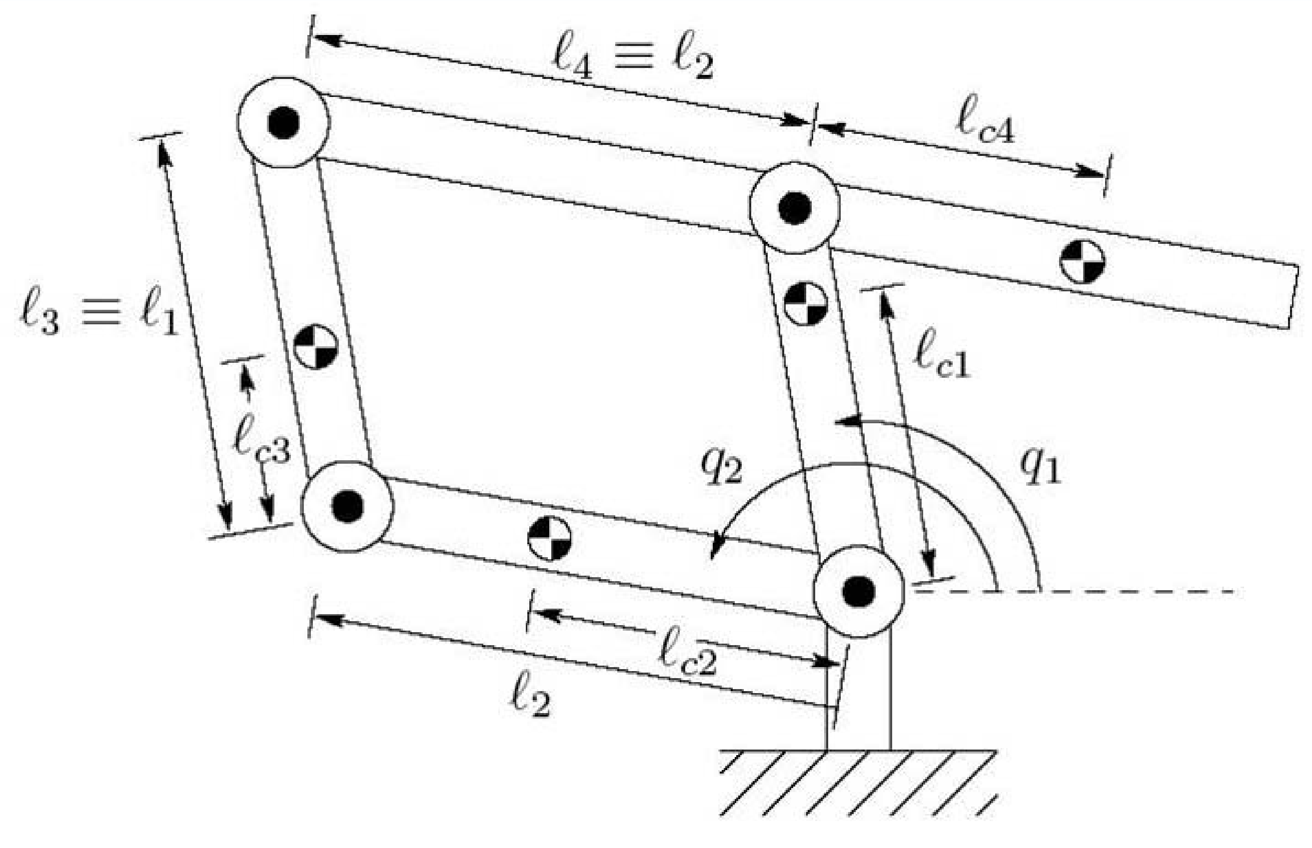


Steps in deriving equations of motion

$$\mathcal{K} = \frac{1}{2}\left[m_1 v_{c_1}^2 + \omega_1^T \mathcal{I}_1 \omega_1\right] + \frac{1}{2}\left[m_2 v_{c_2}^2 + \omega_2^T \mathcal{I}_2 \omega_2\right] + $$

$$+\frac{1}{2}\left[m_3 v_{c_3}^2 + \omega_3^T \mathcal{I}_3 \omega_3\right] + \frac{1}{2}\left[m_4 v_{c_4}^2 + \omega_4^T \mathcal{I}_4 \omega_4\right]$$

Example: Mechanism with a Closed Kinematic Chain

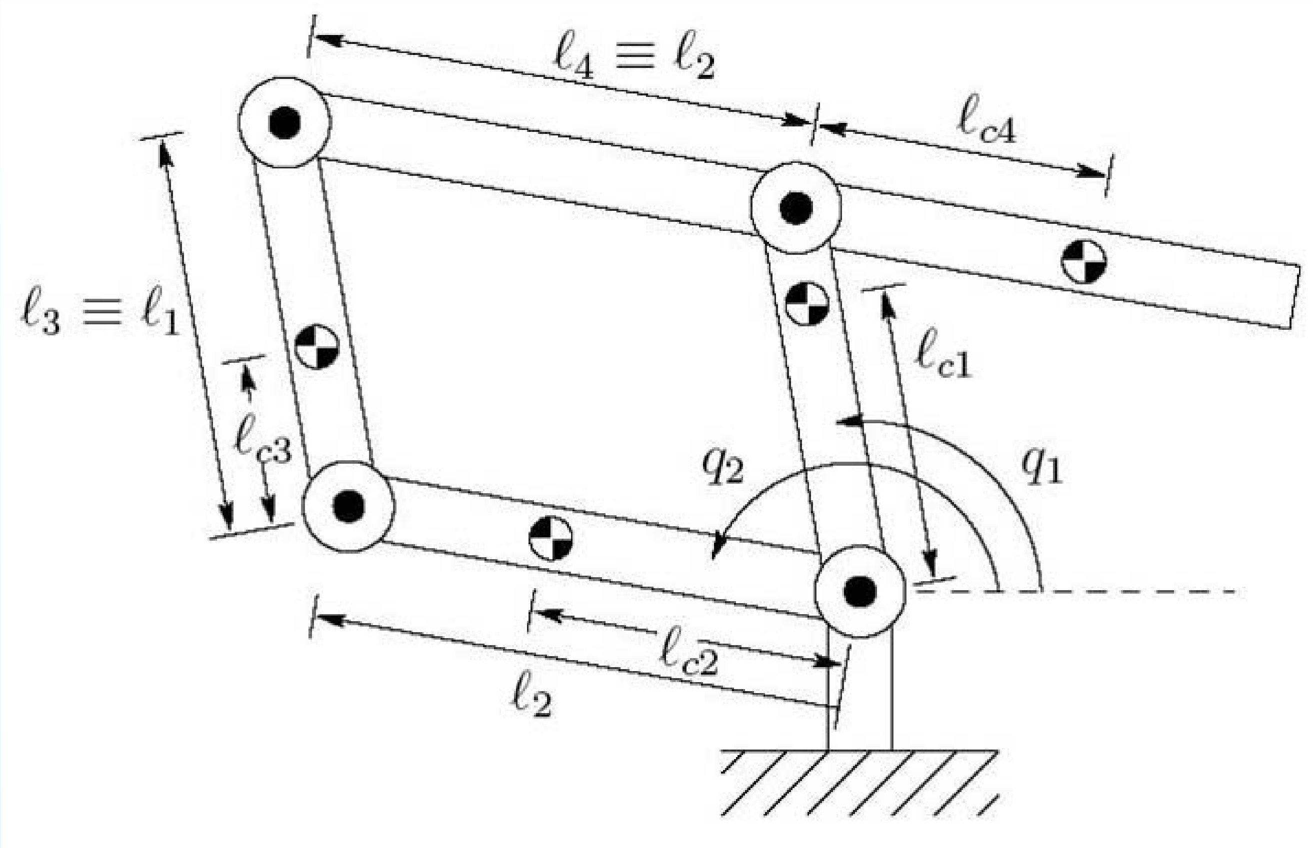


Steps in deriving equations of motion

$$x_{c_1} = l_{c_1} \cos q_1 , \qquad y_{c_1} = l_{c_1} \sin q_1$$

$$x_{c_2} = l_{c_2} \cos q_2 , \qquad y_{c_2} = l_{c_2} \sin q_2$$

Example: Mechanism with a Closed Kinematic Chain

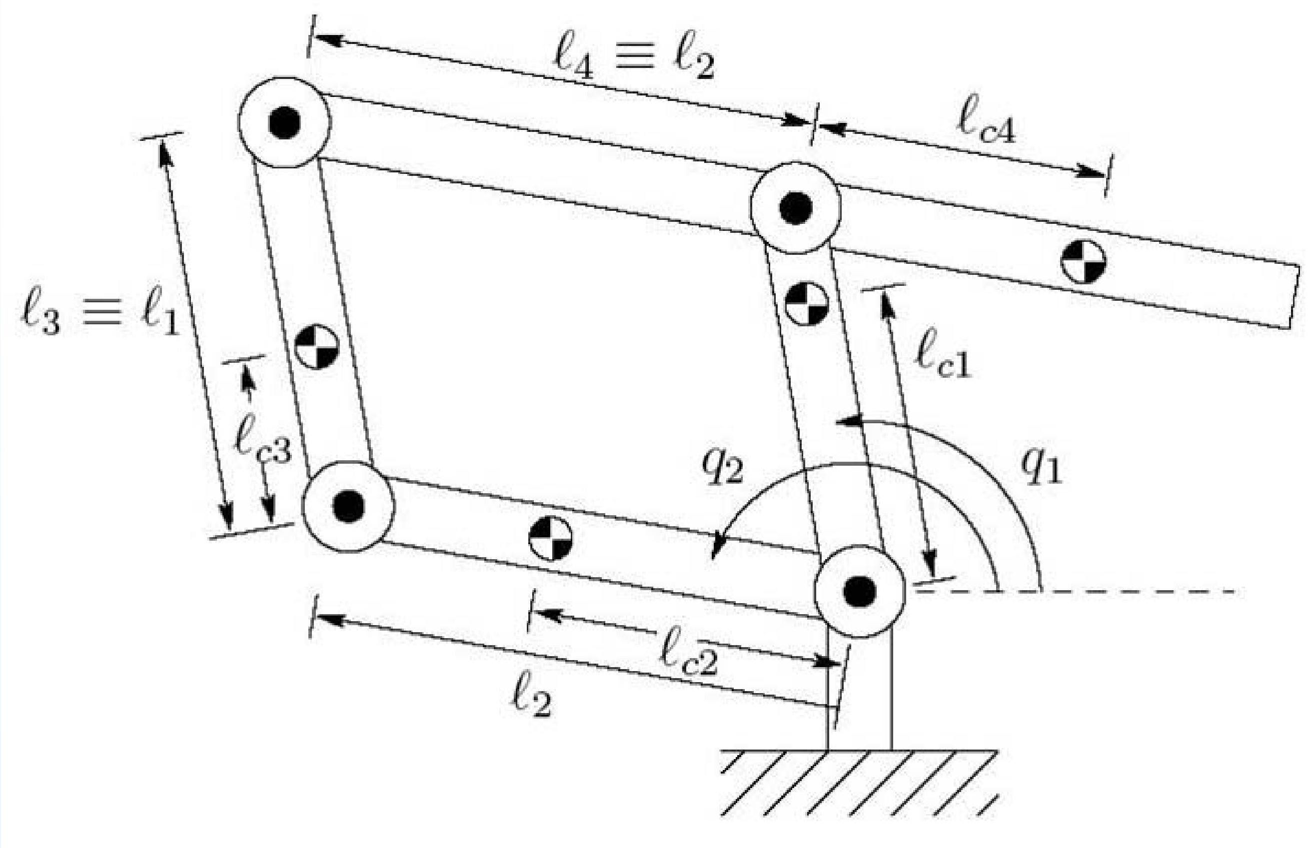


Steps in deriving equations of motion

$$x_{c_3} = l_2 \cos q_2 + l_{c_3} \cos q_1$$

$$y_{c_3} = l_2 \sin q_2 + l_{c_3} \sin q_1$$

Example: Mechanism with a Closed Kinematic Chain

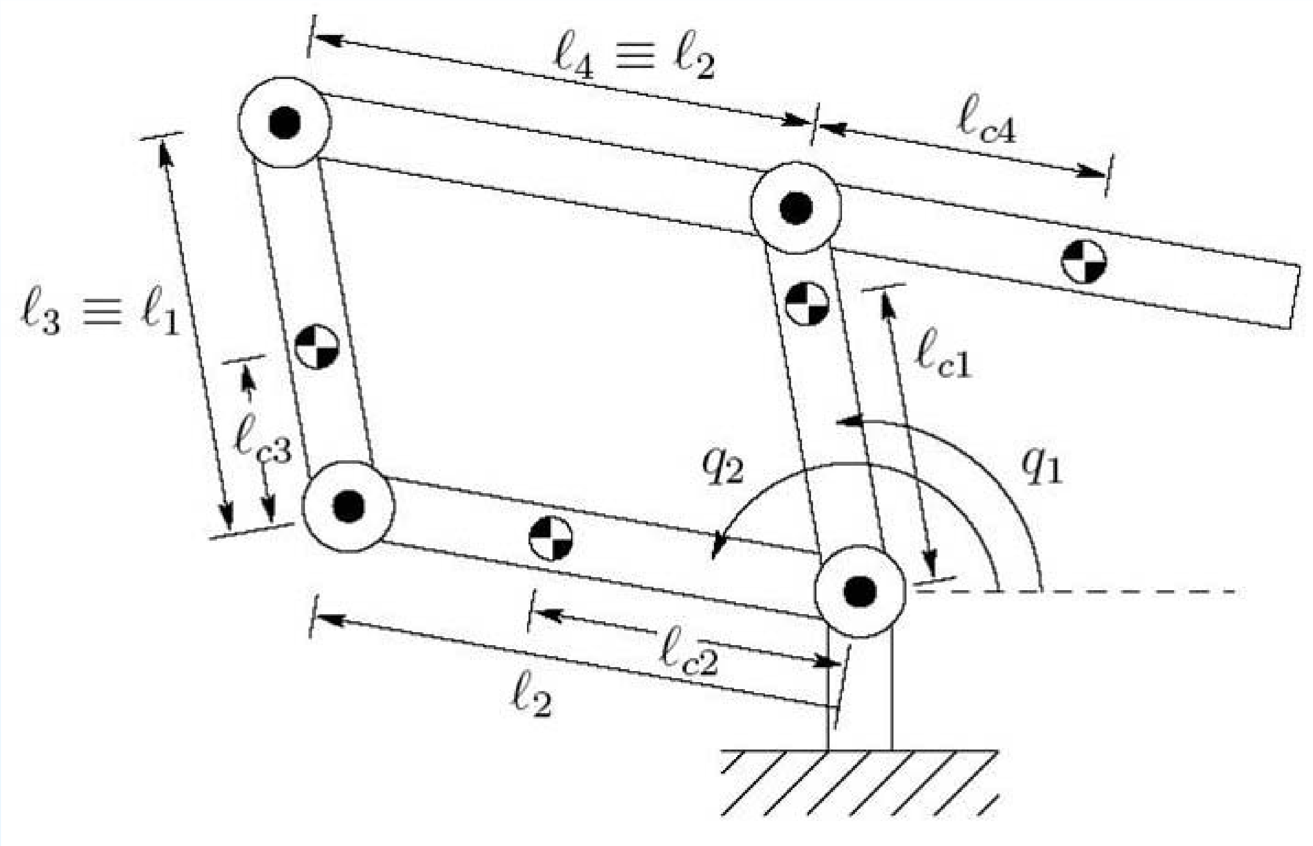


Steps in deriving equations of motion

$$x_{c_4} = l_1 \cos q_1 - l_{c_4} \cos q_2$$

$$y_{c_4} = l_1 \sin q_1 - l_{c_4} \sin q_2$$

Example: Mechanism with a Closed Kinematic Chain

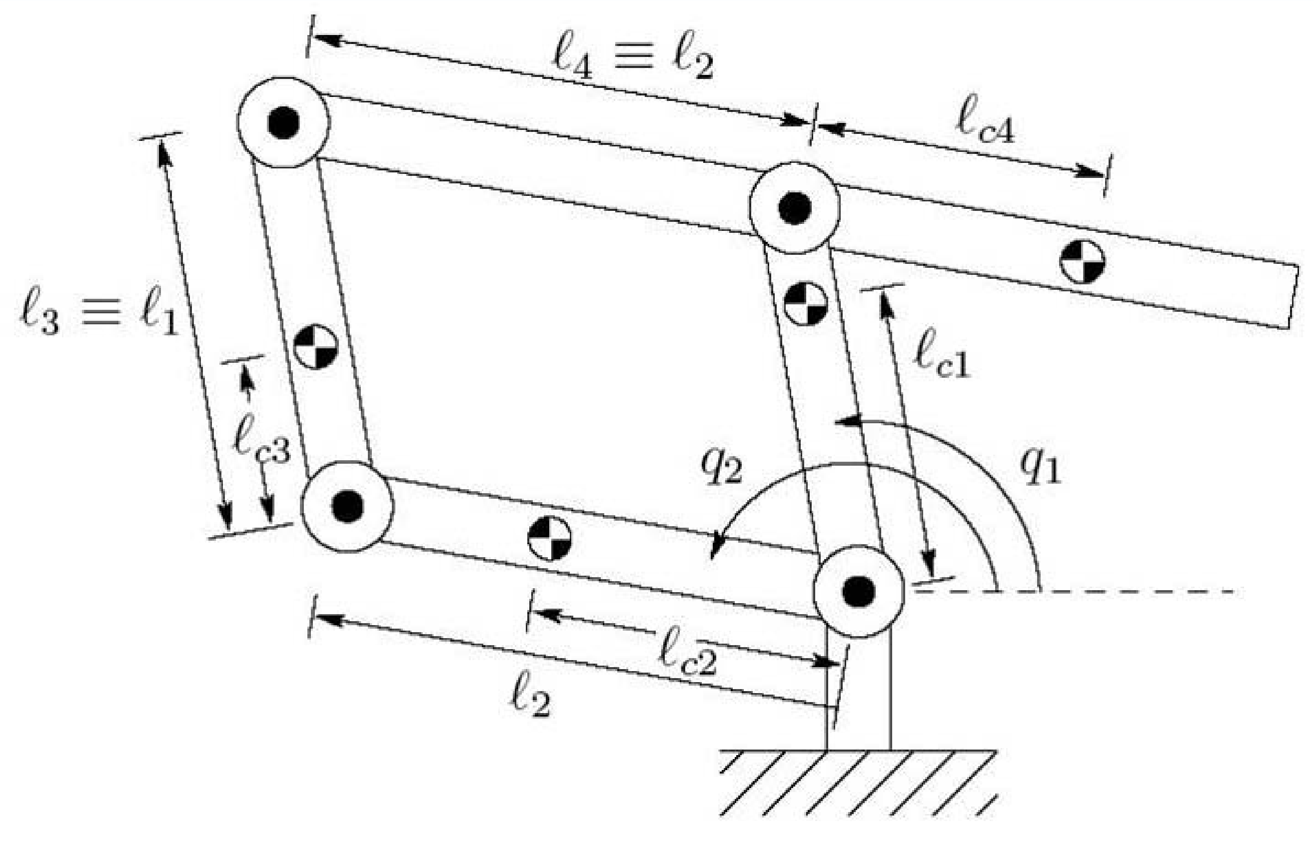


Steps in deriving equations of motion

$$\omega_1 = \omega_3 = \dot{q}_1 \vec{k}$$

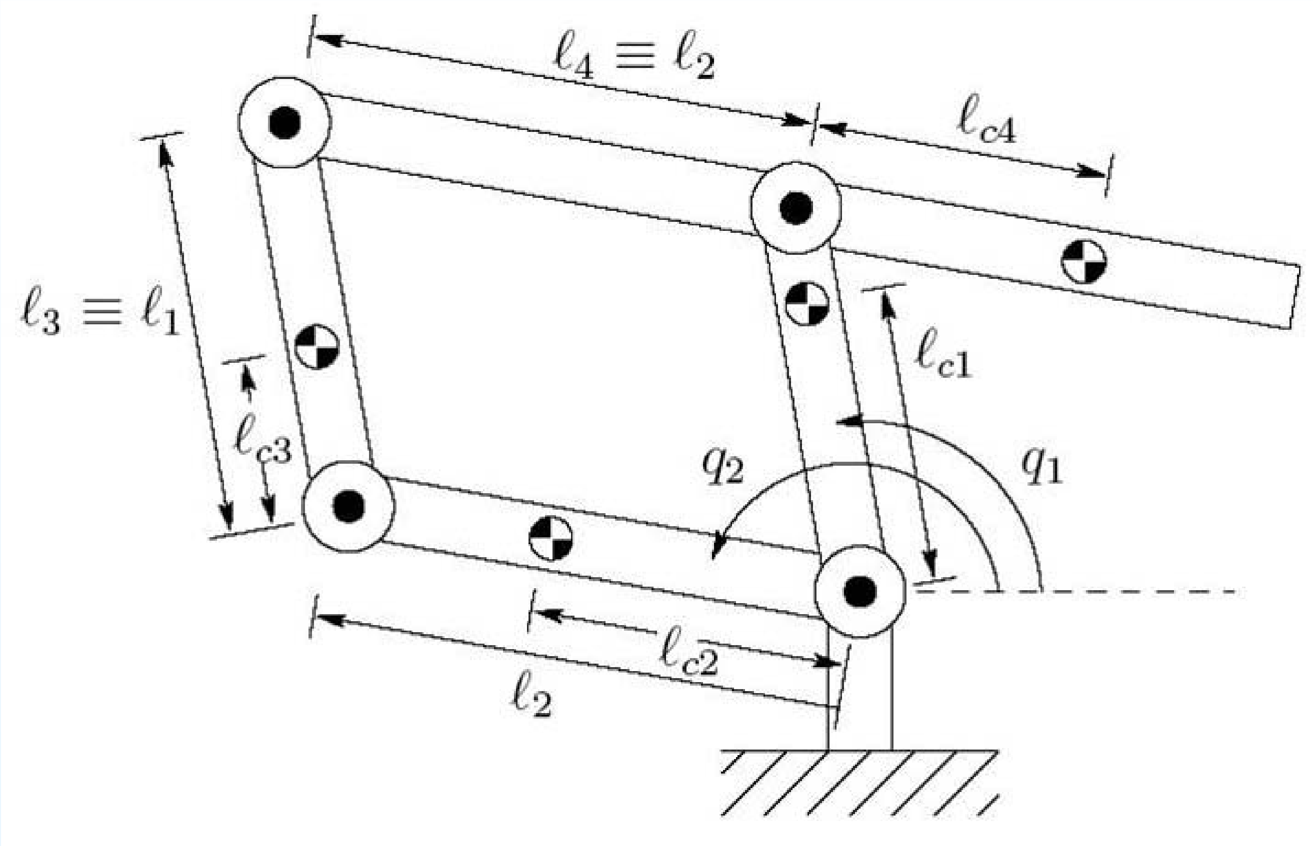
$$\omega_2 = \omega_4 = \dot{q}_2 \vec{k}$$

Example: Mechanism with a Closed Kinematic Chain



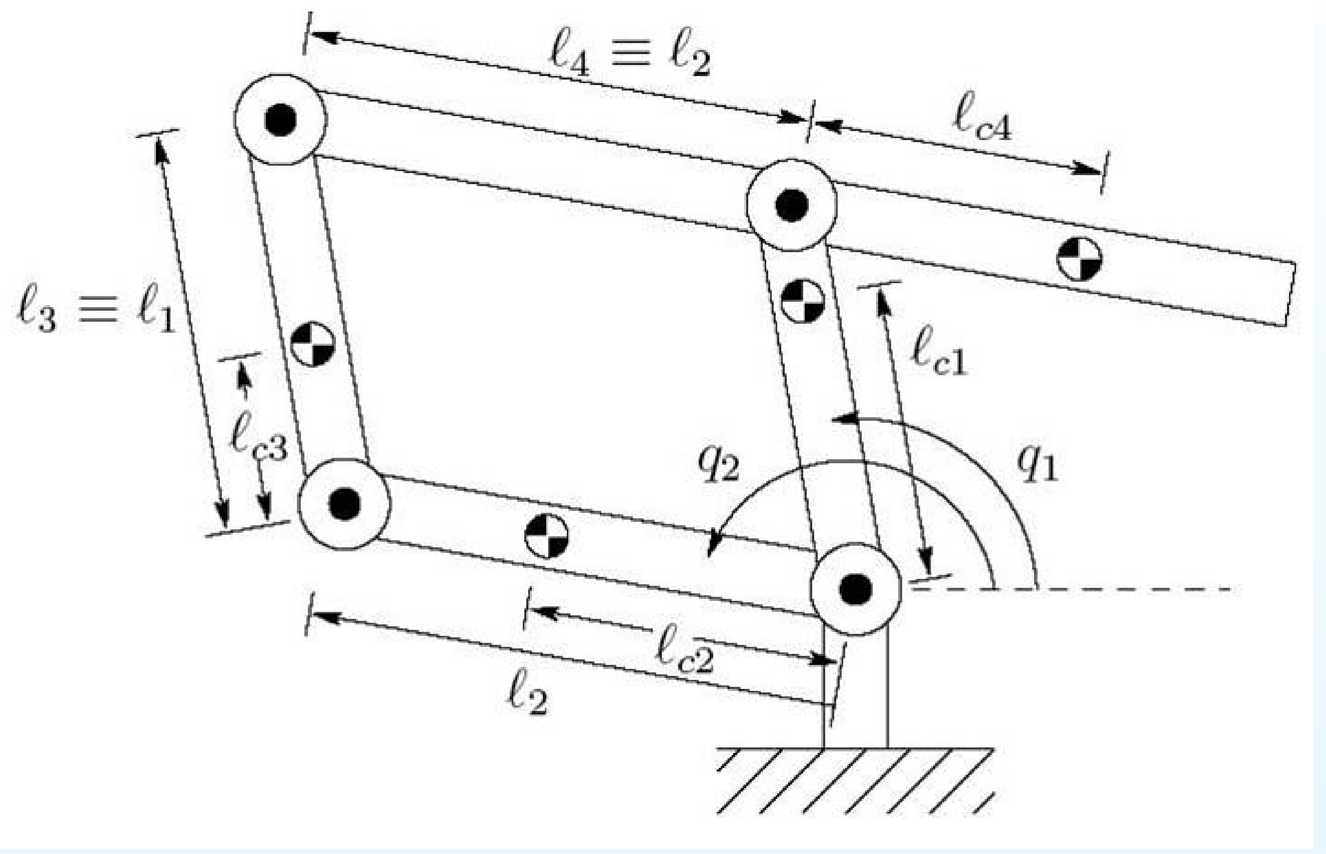
$$
\begin{aligned}
\mathcal{K} &= \frac{1}{2}\left[m_1 v_{c_1}^2 + \omega_1^T \mathcal{I}_1 \omega_1\right] + \frac{1}{2}\left[m_2 v_{c_2}^2 + \omega_2^T \mathcal{I}_2 \omega_2\right] + \\
&\qquad + \frac{1}{2}\left[m_3 v_{c_3}^2 + \omega_3^T \mathcal{I}_3 \omega_3\right] + \frac{1}{2}\left[m_4 v_{c_4}^2 + \omega_4^T \mathcal{I}_4 \omega_4\right] \\
&= \frac{1}{2}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}^T \begin{bmatrix} d_{11}(q_1, q_2) & d_{12}(q_1, q_2) \\ d_{12}(q_1, q_2) & d_{22}(q_1, q_2) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}
\end{aligned}
$$

Example: Mechanism with a Closed Kinematic Chain



$$
\begin{aligned}
d_{11} &= m_1 l_{c_1}^2 + m_3 l_{c_3}^2 + m_4 l_4^2 + I_1 + I_3 \\
d_{12} &= \boxed{[m_3 l_2 l_{c_3} - m_4 l_1 l_{c_4}]} \cdot \cos(q_2 - q_1) \\
d_{22} &= m_2 l_{c_2}^2 + m_3 l_2^2 + m_4 l_{c_4}^2 + I_2 + I_4
\end{aligned}
$$

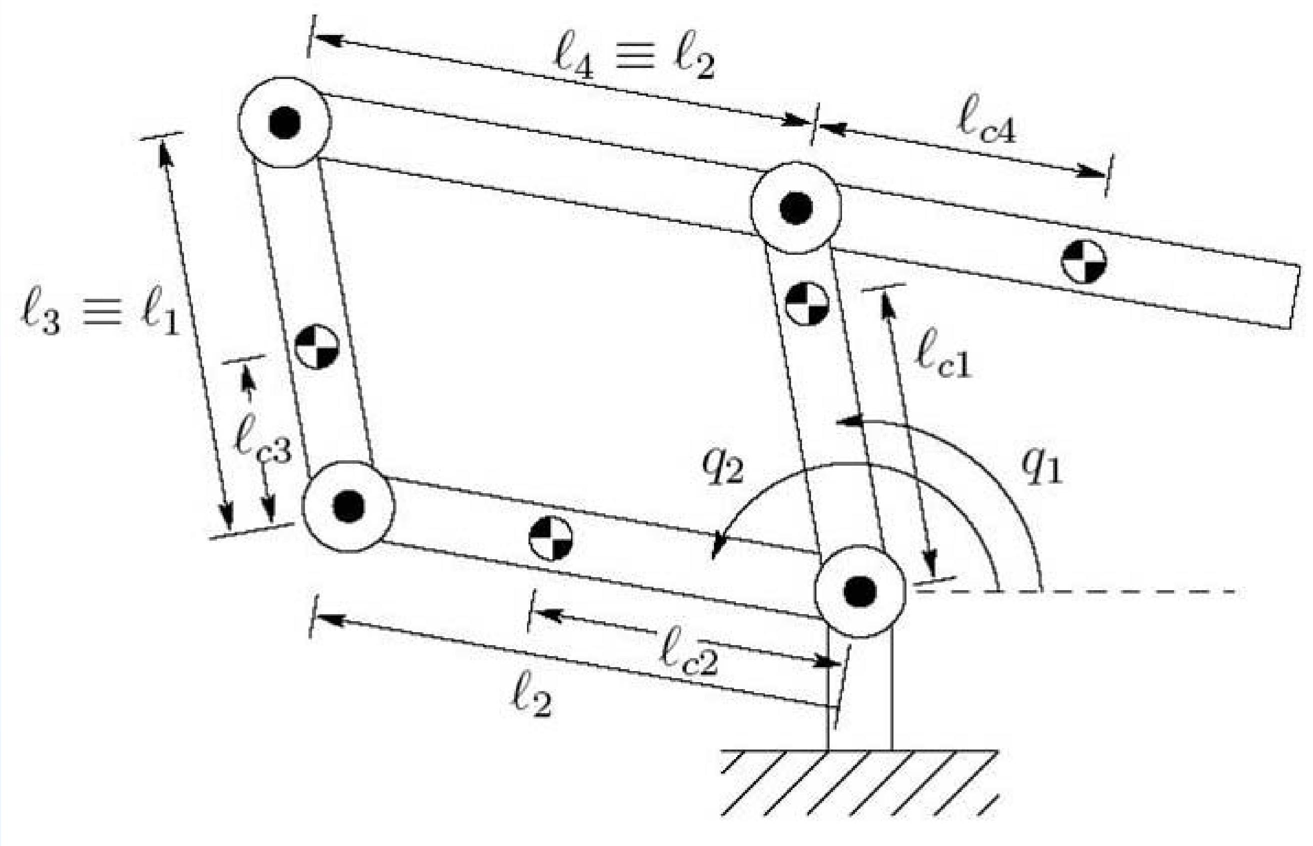
Example: Mechanism with a Closed Kinematic Chain



Steps in deriving equations of motion

$$
\begin{aligned}
\mathcal{P} &= \sum_{i=1}^{4} m_i \cdot g \cdot y_{c_i} \\
&= [m_1 l_{c_1} + m_3 l_{c_3} + m_4 l_1] \cdot g \cdot \sin q_1 + \\
&\qquad + [m_2 l_{c_2} + m_3 l_2 - m_4 l_{c_4}] \cdot g \cdot \sin q_2
\end{aligned}
$$

Example: Mechanism with a Closed Kinematic Chain



If $\boxed{m_3 l_2 l_{c_3} - m_4 l_1 l_{c_4} = 0}$, then the equations of motion

$$d_{11} \cdot \ddot{q}_1 + \frac{\partial}{\partial q_1}\mathcal{P} = \tau_1 \qquad d_{22} \cdot \ddot{q}_2 + \frac{\partial}{\partial q_2}\mathcal{P} = \tau_2$$

are decoupled

Lecture 10: Dynamics: Euler-Lagrange Equations

- Examples
- Properties of Equations of Motion
  - Passivity
  - Bounds on a Matrix of Inertia
  - Linearity in Parameters

Bounds on a Matrix of Inertia

The kinetic energy of any mechanical system

$$\mathcal{K}(q, \dot{q}) = \frac{1}{2}\dot{q}^{T} D(q)\dot{q}$$

is non-negative

Bounds on a Matrix of Inertia

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Hence $D(q)$ is symmetric and non-negative matrix

Bounds on a Matrix of Inertia

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Is it true that $D(q)$ is positive definite?

Bounds on a Matrix of Inertia

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is non-negative

Hence $D(q)$ is symmetric and non-negative matrix

Is it true that $D(q)$ is positive definite?

For the point-mass spherical pendulum

$$\mathcal{K} = \frac{1}{2}ml^2\left[\sin^2(\theta)\dot{\phi}^2 + \dot{\theta}^2\right] = \frac{1}{2}\begin{bmatrix}\dot{\phi} \\ \dot{\theta}\end{bmatrix}^T \begin{bmatrix} ml^2\sin^2\theta & 0 \\ 0 & ml^2\end{bmatrix}\begin{bmatrix}\dot{\phi} \\ \dot{\theta}\end{bmatrix}$$

Lecture 10: Dynamics: Euler-Lagrange Equations

- Examples
- Properties of Equations of Motion
    - Passivity
    - Bounds on a Matrix of Inertia
    - Linearity in Parameters

Linearity of Dynamics in Parameters

Consider the equations of motion of mechanical system

$$\boldsymbol{D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau}$$

and suppose that we have measured

- Control torque $\boldsymbol{\tau = \tau(t)}$, $\boldsymbol{t \in [0, T]}$
- Positions, velocities and accelerations

$$\boldsymbol{q = q(t), \quad \dot{q} = \dot{q}(t), \quad \ddot{q} = \ddot{q}(t), \quad t \in [0, T]}$$

Linearity of Dynamics in Parameters

Consider the equations of motion of mechanical system

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau$$

and suppose that we have measured

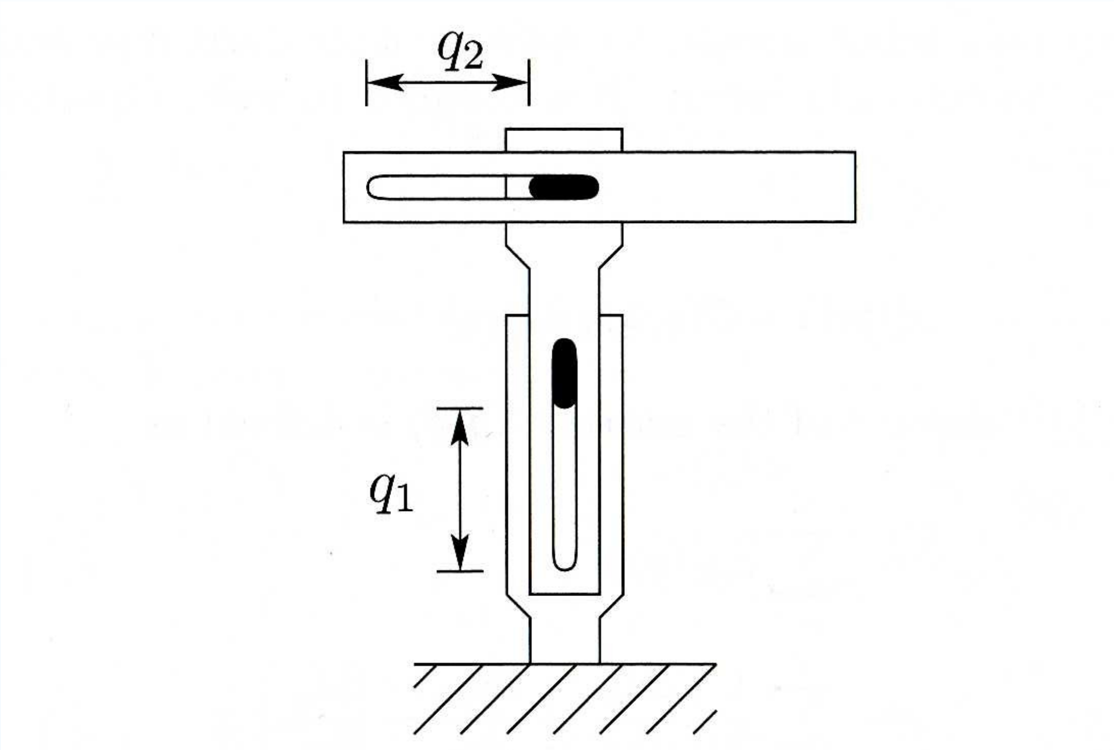
- Control torque $\tau = \tau(t)$, $t \in [0, T]$
- Positions, velocities and accelerations

$$q = q(t), \quad \dot{q} = \dot{q}(t), \quad \ddot{q} = \ddot{q}(t), \quad t \in [0, T]$$

Can we reconstruct parameters

(masses, coefficients of inertia matrices, dimensions) of the system?

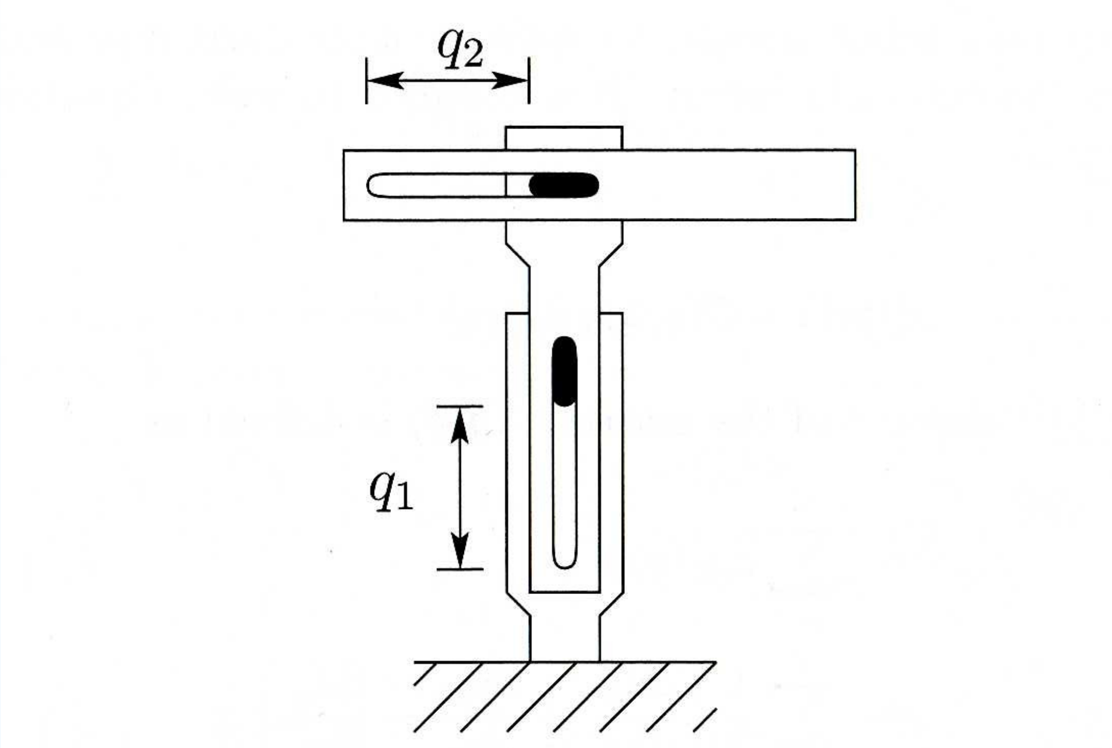
Example: Two Link Cartesian Manipulator



The equations of motion are

$$(m_1 + m_2)\ddot{q}_1 + g(m_1 + m_2) = \tau_1, \qquad m_2\ddot{q}_2 = \tau_2$$

Example: Two Link Cartesian Manipulator

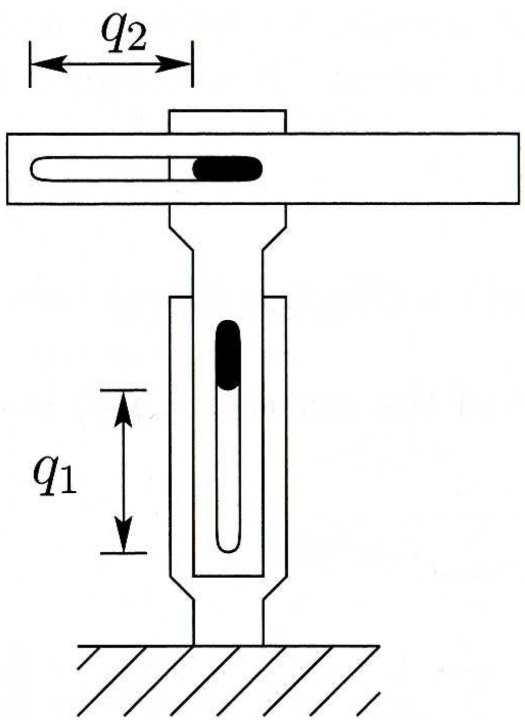


$$(m_1 + m_2)\ddot{q}_1 + g(m_1 + m_2) = \tau_1, \qquad m_2\ddot{q}_2 = \tau_2$$

Let us rewrite them as

$$\begin{bmatrix} \ddot{q}_1(t) + g & 0 \\ 0 & \ddot{q}_2(t) \end{bmatrix} \begin{bmatrix} m_1 + m_2 \\ m_2 \end{bmatrix} = \begin{bmatrix} \tau_1(t) \\ \tau_2(t) \end{bmatrix}$$

Example: Two Link Cartesian Manipulator



The equation for parameters is then

$$\underbrace{\begin{bmatrix} \ddot{q}_1(t) + g & 0 \\ 0 & \ddot{q}_2(t) \end{bmatrix}}_{=REGRESSOR} \underbrace{\begin{bmatrix} m_1 + m_2 \\ m_2 \end{bmatrix}}_{UNKNOWN CONSTANTS} = \begin{bmatrix} \tau_1(t) \\ \tau_2(t) \end{bmatrix}$$