Lecture: Dynamics: Euler-Lagrange Equations

- Examples

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- Holonomic Constraints and Virtual Work

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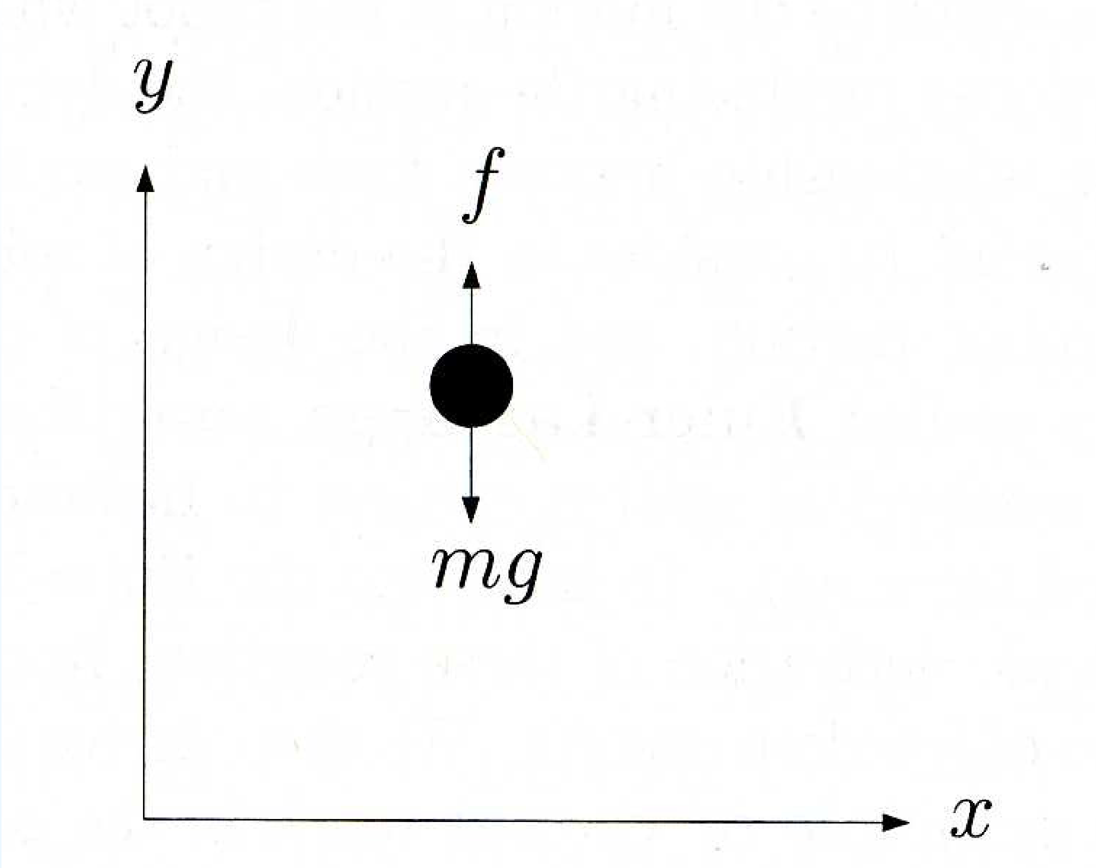
Example



The 2^{nd} Newton law for the particle is

$$\boxed{m\frac{d^2}{dt^2}y = \sum F_i = f - mg}$$

- f is an external force;
- mg is the force acting on the particle due to gravity.

Example

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Terms of the equation of motion can be represented as

$$m\frac{d^2}{dt^2}y \quad = \quad \frac{d}{dt}\left(m\frac{d}{dt}y\right) = \frac{d}{dt}\left(m\frac{\partial}{\partial \dot{y}}\left[\frac{1}{2}\,\dot{y}^2\right]\right) = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{y}}\mathcal{K}\right)$$

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with kinetic/potential energies defined by $\mathcal{K} = \frac{1}{2}m\dot{y}^2$, $\mathcal{P} = mgy$

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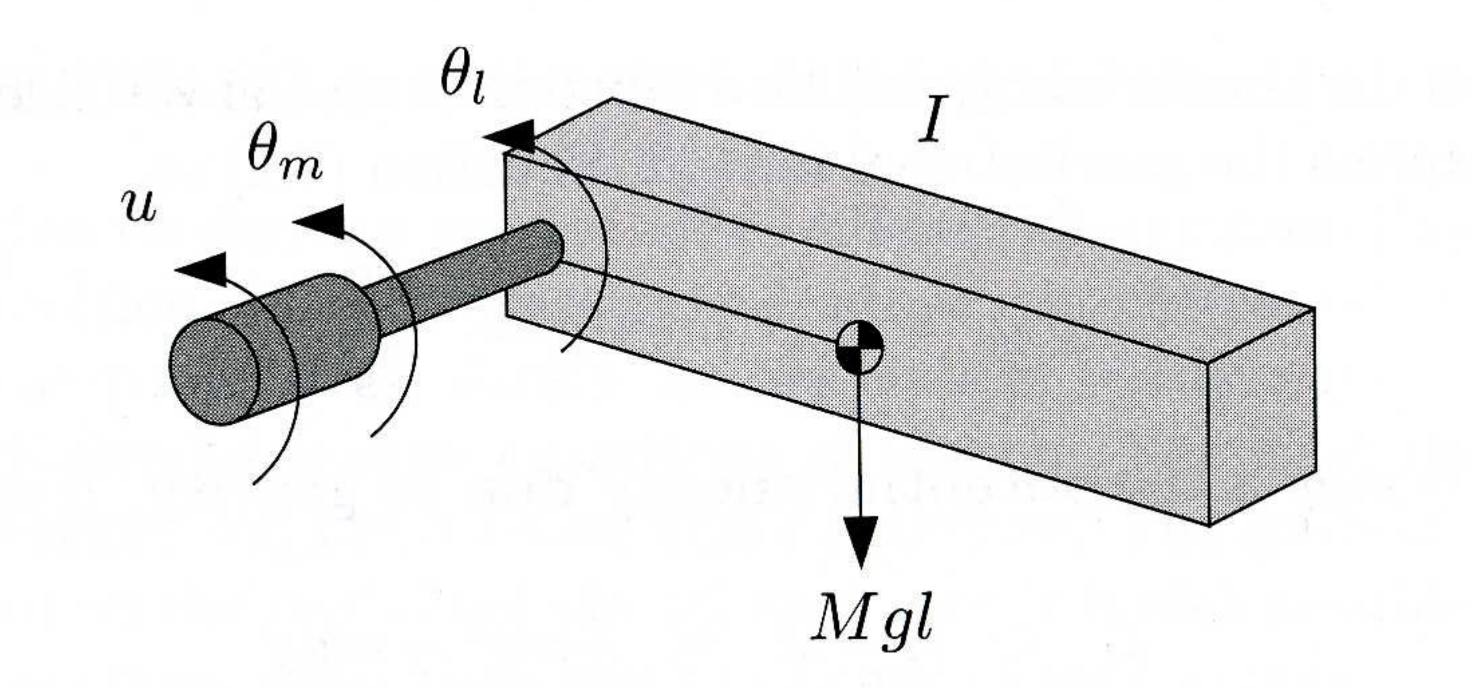
with kinetic/potential energies defined by $\mathcal{K}=\frac{1}{2}m\dot{y}^2$, $\mathcal{P}=mgy$

Then the second Newton law can be rewritten as

$$\boxed{\frac{d}{dt}\left(\frac{\partial}{\partial \dot{y}}\mathcal{L}\right) - \frac{\partial}{\partial y}\mathcal{L} = f} \quad \text{with} \quad \mathcal{L} = \mathcal{K} - \mathcal{P}$$

where the function $\mathcal{L}(y, \dot{y})$ is called the Lagrangian.

Example:



A rigid link (θ_l) coupled through a gear to DC motor ($\theta_m = r\theta_l$):

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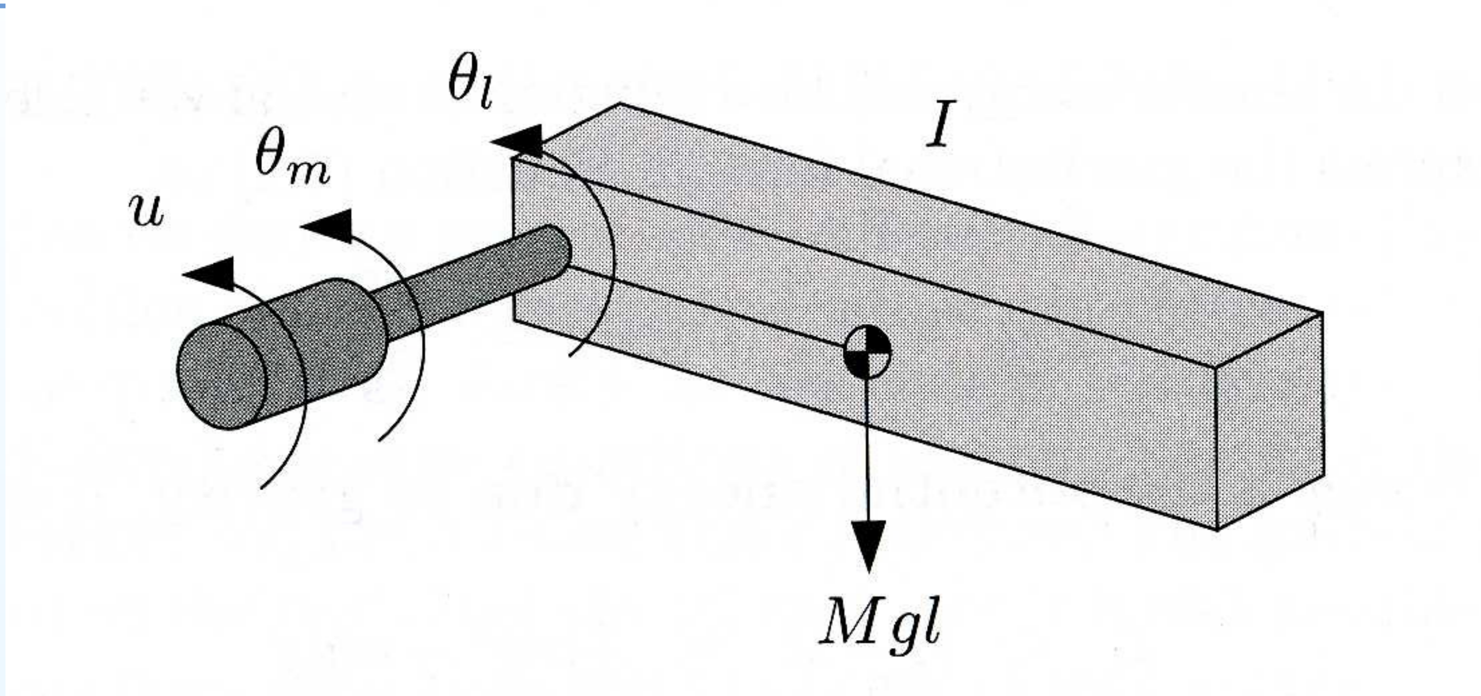

A rigid link (θ_l) coupled through a gear to DC motor ($\theta_m = r\theta_l$):

- Kinetic energy: $\mathcal{K} = \frac{1}{2}J_m\dot{\theta}_m^2 + \frac{1}{2}J_l\dot{\theta}_l^2 = \frac{1}{2}\left(r^2 J_m + J_l\right)\dot{\theta}_l^2$

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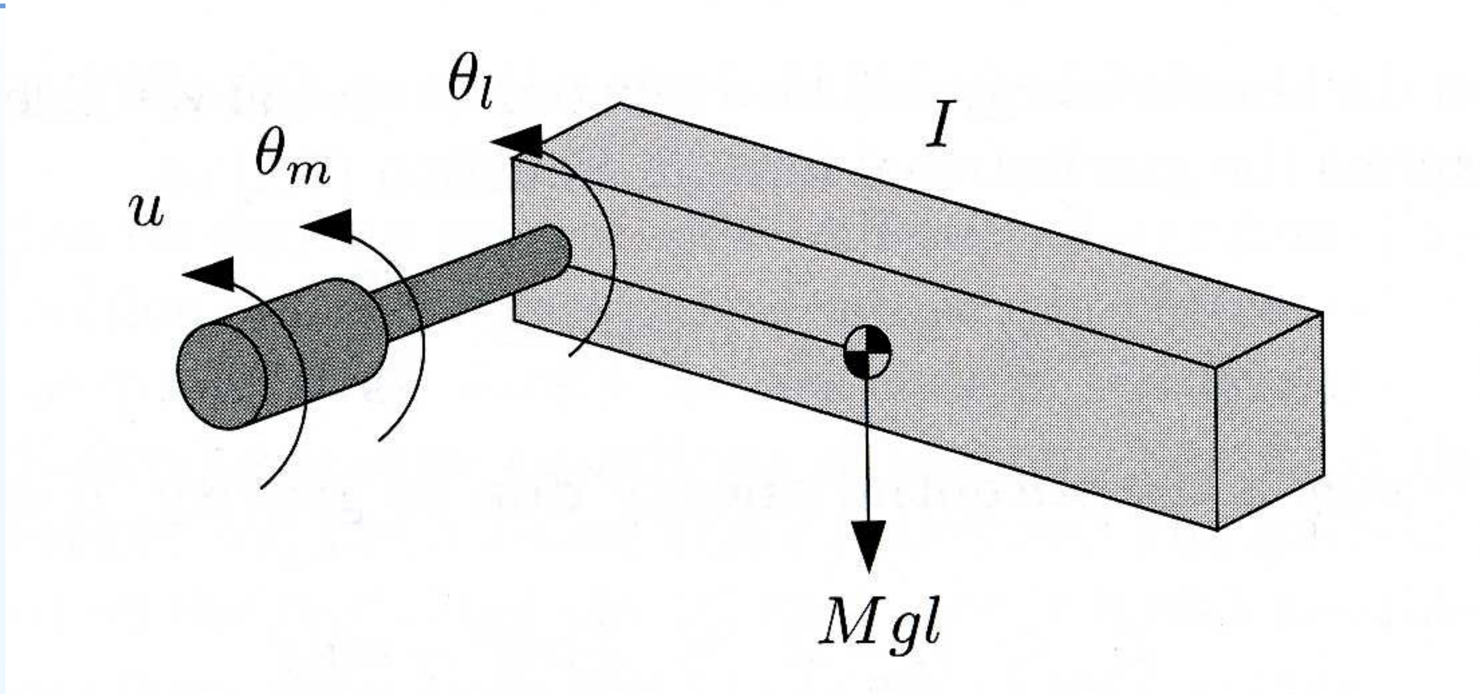

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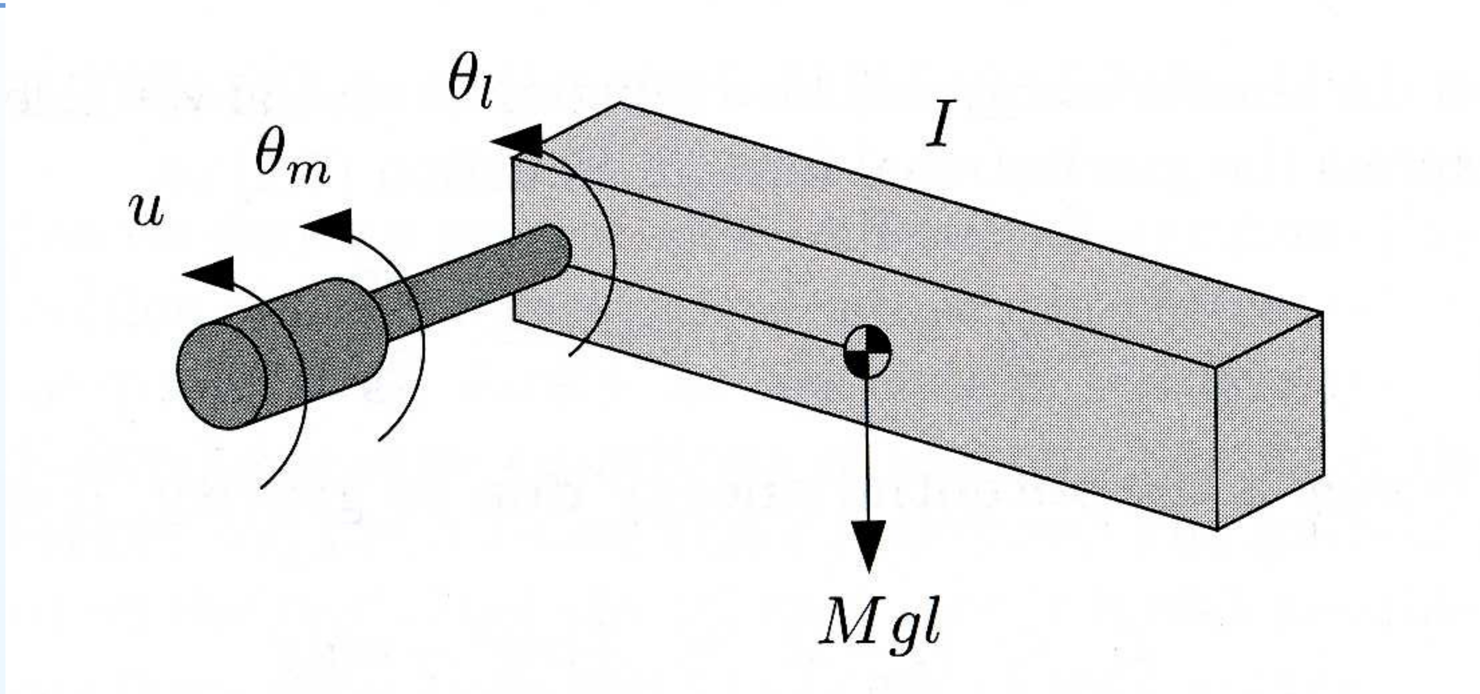

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- Potential energy: $\mathcal{P} = Mgl\left(1 - \cos\theta_l\right)$
- the Lagrangian is $\mathcal{L} = \mathcal{K} - \mathcal{P}$ and the dynamics are

$$\boxed{\frac{d}{dt}\left(\frac{\partial}{\partial \dot{\theta}_l}\mathcal{L}\right) - \frac{\partial}{\partial \theta_l}\mathcal{L} = ru}$$

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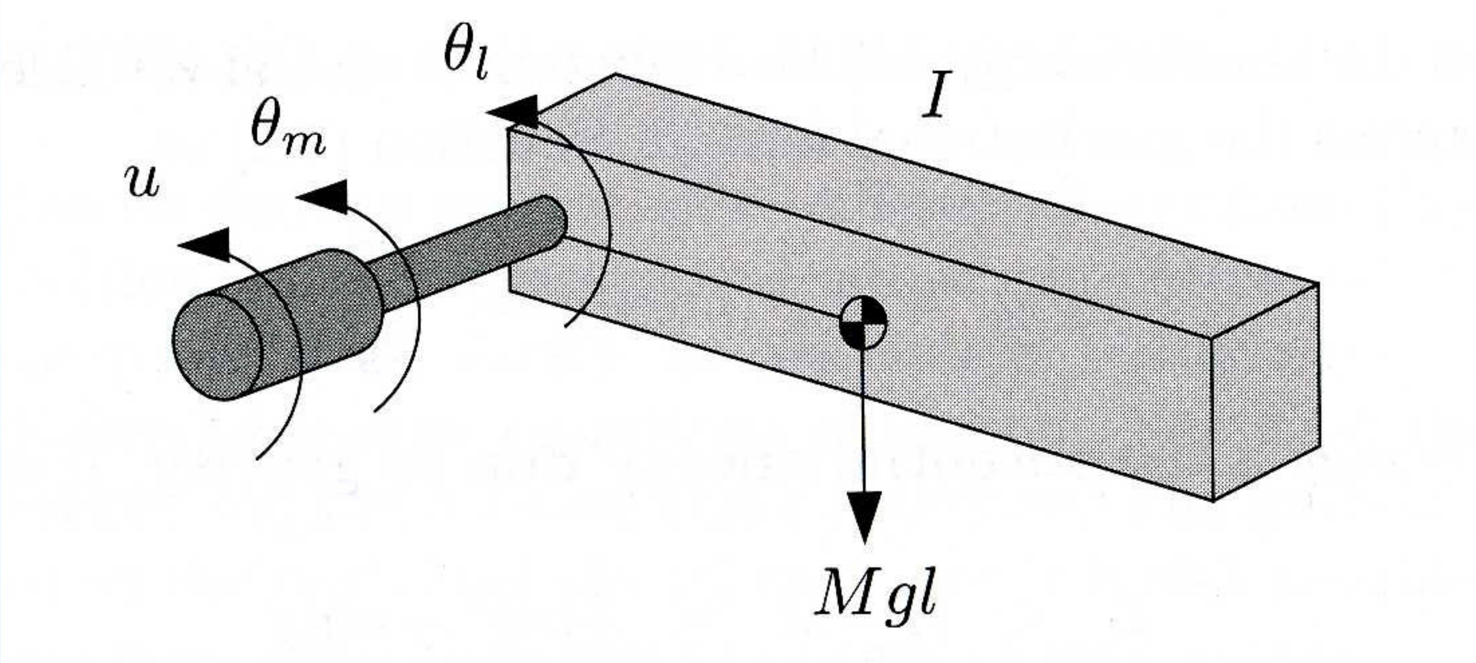

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$$\boxed{\left(r^2 J_m + J_l\right)\ddot{\theta}_l + Mgl\sin\theta_l = ru}$$

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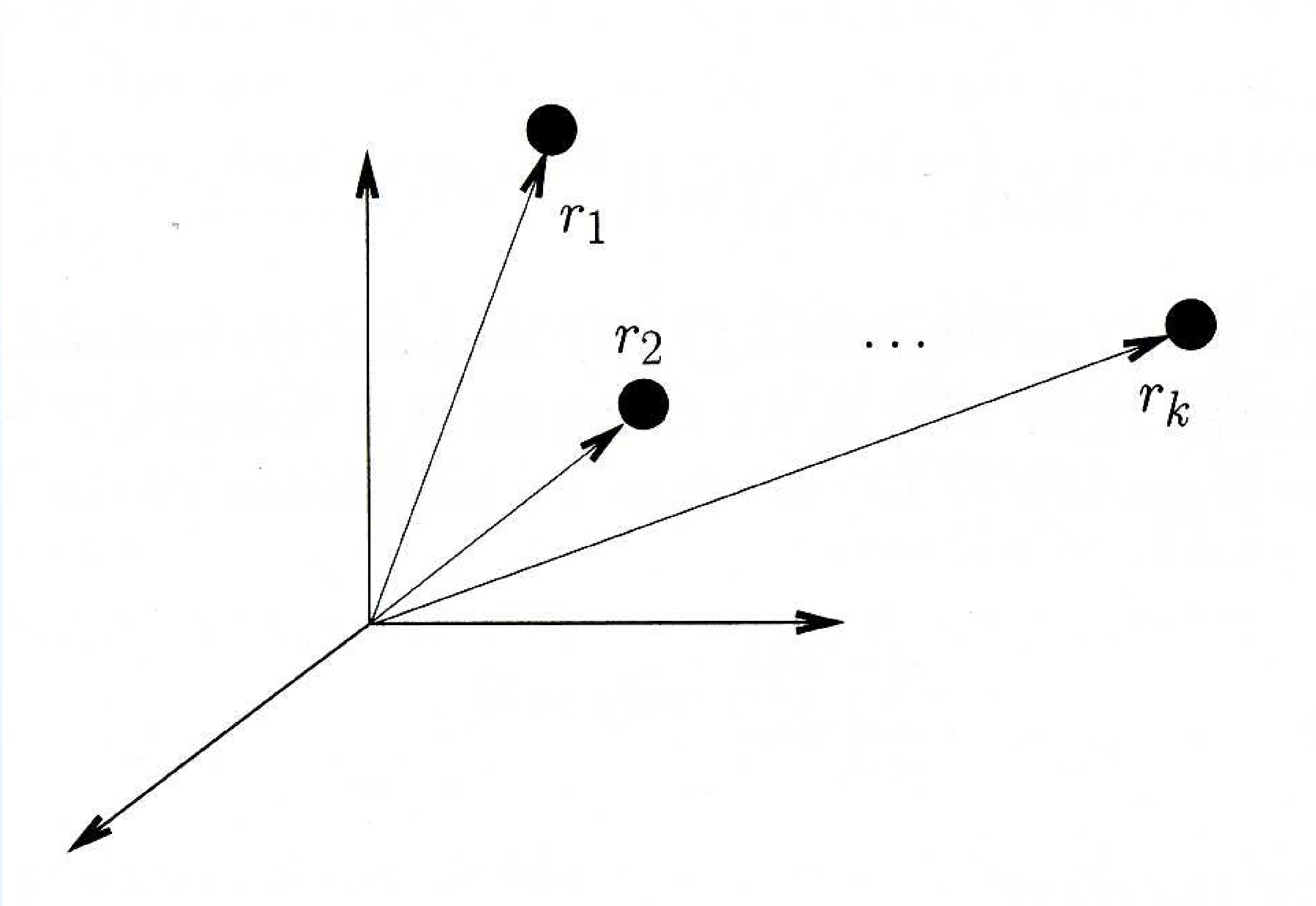
Concept of Holonomic Constraint



Unconstrained system of $\boldsymbol{k}$ particles has $\boldsymbol{3k}$ degrees of freedom.

The number of DoF is less, if the particles are constrained

Concept of Holonomic Constraint

A constraint imposed on k particles (with coordinates $r_1, r_2, \ldots,$ $r_k \in \mathbb{R}^3$) is called holonomic, if it is of the form

$$g_i(r_1, r_2, \ldots, r_k) = 0, \qquad i = 1, 2, \ldots, l$$

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For example, given two particles joined by massless rigid wire of length l, then

$$r_1, r_2 \in \mathbb{R}^3 : \left\| r_1 - r_2 \right\|^2 = \left(r_1 - r_2 \right)^T \left(r_1 - r_2 \right) = l^2$$

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Presence of constraint implies presence a force

(called constraint force), that forces this constraint to hold.

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$$g_i(r_1, r_2, \ldots, r_k) = 0, \qquad i = 1, 2, \ldots, l$$

Differentiating the constraint function $g_i(\cdot)$ with respect to time, we obtain new constraint

$$\tfrac{d}{dt} g_i(r_1, r_2, \ldots, r_k) = \frac{\partial g_i}{\partial r_1} \tfrac{d}{dt} r_1 + \cdots + \frac{\partial g_i}{\partial r_k} \tfrac{d}{dt} r_k = 0$$

or

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The constraint of the form

$$\omega_1(r_1, \dots r_k) dr_1 + \cdots + \omega_k(r_1, \dots r_k) dr_k = 0$$

is called non-holonomic if it cannot be integrated back.

Concept of Generalized Coordinates

If the system is subject to holonomic constraint then

- If system consists of k particles, then it may be possible to express their coordinates as functions of fewer than $3k$ variables

$$r_1 = r_1(q_1, \dots, q_n),\ r_2 = r_2(q_1, \dots, q_n),\ \dots,$$
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- The smallest set of variables is called generalized coordinates
- This smallest number is called a number of degree of freedom
- If the system consists of an infinite number of particles, then it might have finite number of degrees of freedom

Concept of Virtual Displacement

Given a system of k-particles and a holonomic constraint

$$g_i(r_1, r_2, \ldots, r_k) = 0, \qquad i = 1, 2, \ldots, l$$

or the same

$$\frac{\partial g_i}{\partial r_1} dr_1 + \cdots + \frac{\partial g_i}{\partial r_k} dr_k = 0, \quad i = 1, 2, \ldots, l$$

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By definition a set of infinitesimal displacements

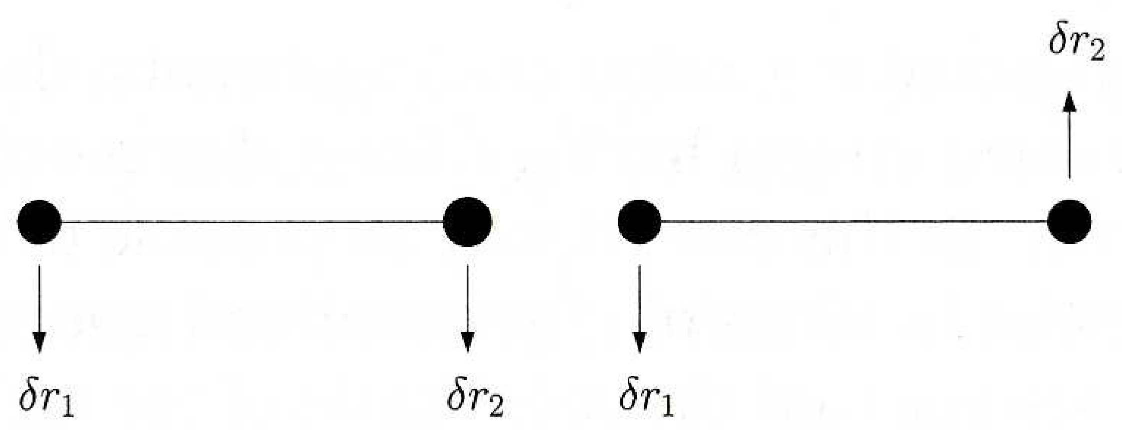
$$\delta r_1, \delta r_2, \ldots, \delta r_k$$

that are consistent with the constraint, i.e.

$$\frac{\partial g_i}{\partial r_1} \delta r_1 + \cdots + \frac{\partial g_i}{\partial r_k} \delta r_k = 0, \quad i = 1, 2, \ldots, l$$

are called virtual displacements

Concept of Virtual Displacement



Virtual displacements of a rigid bar. Such infinitesimal motions do not destroy the constraint

$$\left(\boldsymbol{r_1} - \boldsymbol{r_2}\right)^T \left(\boldsymbol{r_1} - \boldsymbol{r_2}\right) = \boldsymbol{l^2}$$

if $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ are perturbed

$$\boldsymbol{r_1} \longrightarrow \left(\boldsymbol{r_1} + \boldsymbol{\delta r_1}\right) \qquad \boldsymbol{r_2} \longrightarrow \left(\boldsymbol{r_2} + \boldsymbol{\delta r_2}\right)$$

that is

$$\left(\left(\boldsymbol{r_1} + \boldsymbol{\delta r_1}\right) - \left(\boldsymbol{r_2} + \boldsymbol{\delta r_2}\right)\right)^T \left(\left(\boldsymbol{r_1} + \boldsymbol{\delta r_1}\right) - \left(\boldsymbol{r_2} + \boldsymbol{\delta r_2}\right)\right) = \boldsymbol{l^2}$$

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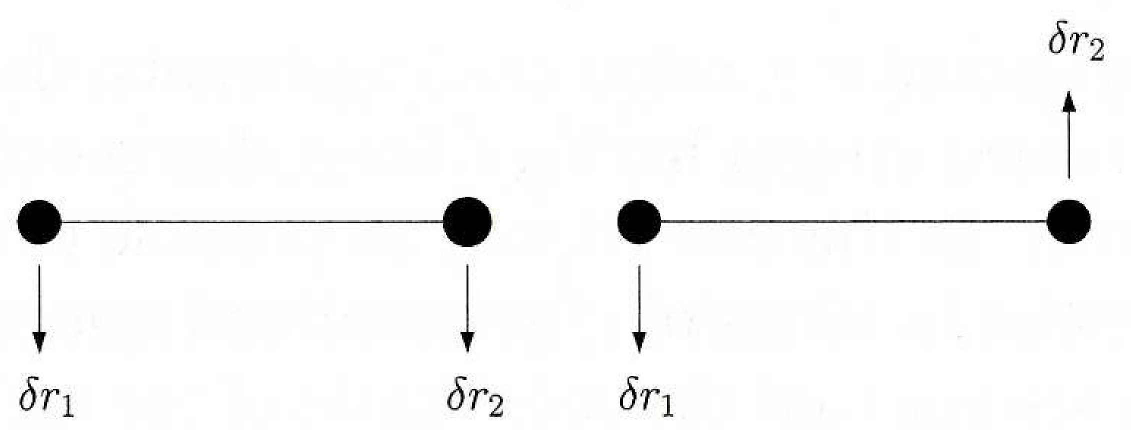

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Consider a system of k-particles, suppose that

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$$\sum_i \left(f_i^c + f_i^e\right) = 0$$

Then the work done by all forces applied to i^{th}-particle along each set of virtual displacement is zero, i.e.

$$\boxed{0} = \sum_i \left(f_i^c + f_i^e\right) \delta r_i = \underbrace{\sum_i f_i^c \delta r_i}_{=0} + \boxed{\sum_i f_i^e \delta r_i}$$

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Then the work done by all forces applied to i^{th}-particle along each set of virtual displacement is zero if we add the inertia forces

$$\mathbf{0} = \sum_i \left(f_i^e - \frac{d}{dt} [m_i \dot{r}_i] \right) \delta r_i$$

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Steps to be done

- Rewrite $\sum_i f_i^e \delta r_i$ as function of generalized coordinates q;
- Rewrite $\sum_i \frac{d}{dt}\left[m_i \dot{r}_i\right] \delta r_i$ as function of generalized coordinates q

D'Alembert Principle

Virtual displacements are computed as

$$\boldsymbol{\delta r_i} = \sum_{j=1}^{n} \frac{\partial \boldsymbol{r_i}}{\partial q_j} \boldsymbol{\delta q_j}, \quad i = 1, \ldots, k$$

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Then

$$\begin{aligned} \sum_{i=1}^{k} f_i^e \delta r_i &= \sum_{i=1}^{k} f_i^e \left(\sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{k} f_i^e \frac{\partial r_i}{\partial q_j} \right) \delta q_j \\ &= \sum_{j=1}^{n} \psi_j \delta q_j \end{aligned}$$

The functions ψ_j are called generalized forces

D'Alembert Principle

The second term can be rewritten as

$$\sum_{i=1}^{k} \frac{d}{dt}\left[m_i \dot{\boldsymbol{r}}_i\right] \boldsymbol{\delta r_i} \quad = \quad \sum_{i=1}^{k} m_i \ddot{\boldsymbol{r}}_i \boldsymbol{\delta r_i} = \sum_{i=1}^{k} m_i \ddot{\boldsymbol{r}}_i \left(\sum_{j=1}^{n} \frac{\partial \boldsymbol{r}_i}{\partial q_j} \boldsymbol{\delta q_j} \right)$$

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$$= \sum_{i=1}^{k} \sum_{j=1}^{n} m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} \delta q_j$$

$$\frac{d}{dt}\left[m_i \dot{r}_i \frac{\partial r_i}{\partial q_j} \right] = m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} + m_i \dot{r}_i \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j} \right]$$

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$$\Rightarrow \quad \sum_{i=1}^{k} m_i \ddot{r}_i \frac{\partial r_i}{\partial q_j} = \sum_{i=1}^{k} \left\{ \frac{d}{dt}\left[m_i \dot{r}_i \frac{\partial r_i}{\partial q_j} \right] - m_i \dot{r}_i \frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j} \right] \right\}$$

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$$= \sum_{j=1}^{n} \left[\sum_{i=1}^{k} \left\{ \frac{d}{dt} \left[m_i \dot{r}_i \frac{\partial r_i}{\partial q_j} \right] - m_i \dot{r}_i \frac{d}{dt} \left[\frac{\partial r_i}{\partial q_j} \right] \right\} \right] \boldsymbol{\delta q_j}$$

$$v_i = \dot{r}_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \dot{q}_j \quad \Rightarrow \quad \frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}$$

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The second term can be rewritten as

$$\sum_{i=1}^{k} \frac{d}{dt}\left[m_i \dot{r}_i\right] \delta r_i = \sum_{i=1}^{k} m_i \ddot{r}_i \delta r_i = \sum_{i=1}^{k} m_i \ddot{r}_i \left(\sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \delta q_j \right)$$

$$= \sum_{j=1}^{n} \left[\sum_{i=1}^{k} \left\{ \frac{d}{dt} \left[m_i \dot{r}_i \frac{\partial r_i}{\partial q_j} \right] - m_i \dot{r}_i \frac{d}{dt} \left[\frac{\partial r_i}{\partial q_j} \right] \right\} \right] \delta q_j$$

$$v_i = \dot{r}_i = \sum_{j=1}^{n} \frac{\partial r_i}{\partial q_j} \dot{q}_j \quad \Rightarrow \quad \frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}$$

$$\frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j}\right] = \sum_{l=1}^{n} \frac{\partial^2 r_i}{\partial q_j \partial q_l} \dot{q}_l = \frac{\partial}{\partial q_j} \left[\sum_{l=1}^{n} \frac{\partial r_i}{\partial q_l} \dot{q}_l \right] = \frac{\partial v}{\partial q_j}$$

D'Alembert Principle

The second term can be rewritten as

$$\begin{aligned}\sum_{i=1}^{k}\tfrac{d}{dt}\left[m_i\dot{r}_i\right]\delta r_i \quad &= \quad \sum_{i=1}^{k} m_i\ddot{r}_i\delta r_i = \sum_{i=1}^{k} m_i\ddot{r}_i\left(\sum_{j=1}^{n}\frac{\partial r_i}{\partial q_j}\delta q_j\right) \\ &= \sum_{j=1}^{n}\left[\sum_{i=1}^{k}\left\{\frac{d}{dt}\left[m_i\dot{r}_i\frac{\partial r_i}{\partial q_j}\right] - m_i\dot{r}_i\frac{d}{dt}\left[\frac{\partial r_i}{\partial q_j}\right]\right\}\right]\delta q_j \\ &= \sum_{j=1}^{n}\left[\sum_{i=1}^{k}\left\{\frac{d}{dt}\left[m_i v_i\frac{\partial v_i}{\partial \dot{q}_j}\right] - m_i v_i\frac{\partial v_i}{\partial q_j}\right\}\right]\delta q_j \\ &= \sum_{j=1}^{n}\left[\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j}\right]\delta q_j\end{aligned}$$

where

$$\mathcal{K} = \sum_{i=1}^{k}\frac{1}{2}m_i\left|v_i\right|^2$$

D'Alembert Principle

To summarize, the equation

$$0 = \sum_i \left(f_i^e - \tfrac{d}{dt}\left[m_i \dot{r}_i\right] \right) \delta r_i$$

with

$$\sum_{i=1}^{k} \tfrac{d}{dt}\left[m_i \dot{r}_i\right] \delta r_i = \sum_{j=1}^{n} \left[\frac{d}{dt} \frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j} \right] \delta q_j, \quad \sum_{i=1}^{k} f_i^e \delta r_i = \sum_{j=1}^{n} \psi_j \delta q_j$$

is

$$\sum_{j=1}^{n} \left\{ \frac{d}{dt} \frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j} - \psi_j \right\} \delta q_j = 0$$

D'Alembert Principle

To summarize, the equation

$$\mathbf{0} = \sum_i \left(f_i^e - \tfrac{d}{dt}\left[m_i \dot{r}_i\right] \right) \delta r_i$$

with

$$\sum_{i=1}^{k} \tfrac{d}{dt}\left[m_i \dot{r}_i\right] \delta r_i = \sum_{j=1}^{n} \left[\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j} \right] \delta q_j, \quad \sum_{i=1}^{k} f_i^e \delta r_i = \sum_{j=1}^{n} \psi_j \delta q_j$$

is

$$\sum_{j=1}^{n} \left\{ \frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j} - \psi_j \right\} \delta q_j = \mathbf{0}$$

If δq_j are independent then we obtain equations

$$\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j} = \psi_j, \quad j = 1, \ldots n$$

D'Alembert Principle

To summarize, the equation

$$0 = \sum_i \left(f_i^e - \tfrac{d}{dt}\left[m_i \dot{r}_i\right] \right) \delta r_i$$

with

$$\sum_{i=1}^{k} \tfrac{d}{dt}\left[m_i \dot{r}_i\right] \delta r_i = \sum_{j=1}^{n} \left[\frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j} \right] \delta q_j, \quad \sum_{i=1}^{k} f_i^e \delta r_i = \sum_{j=1}^{n} \psi_j \delta q_j$$

is

$$\sum_{j=1}^{n} \left\{ \frac{d}{dt}\frac{\partial \mathcal{K}}{\partial \dot{q}_j} - \frac{\partial \mathcal{K}}{\partial q_j} - \psi_j \right\} \delta q_j = 0$$

If ψ_j functions are particular form then the equations are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_j} - \frac{\partial \mathcal{L}}{\partial q_j} = \tau_j, \quad \psi_j = -\frac{\partial \mathcal{P}}{\partial q_j} + \tau_j, \quad \mathcal{L} = \mathcal{K} - \mathcal{P}$$