

# CSE276C - Calculus of Variation

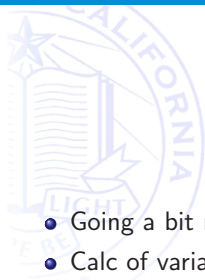
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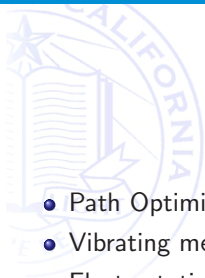
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# Introduction



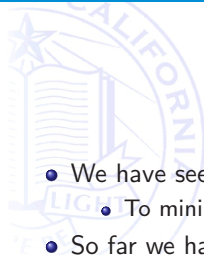
- Going a bit more abstract today.
- Calc of variations is tightly coupled to mechanics
- We will only covers the very basics
- Entire courses at UCSD - MATH201C

# Applications



- Path Optimization
- Vibrating membranes
- Electrostatics
- Machine vision - reconstruction
- Vision - image flow, ...

# Introduction (cont)

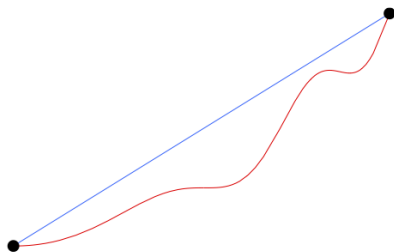
- 
- We have seen the principle
    - To minimize  $P$  is to solve  $P' = 0$
  - So far we have looked at finite dimensional problems
    - $f: \mathcal{R}^n \rightarrow \mathcal{R}$

Looking at  $N$  numbers to minimize  $f$

- In infinite dimensional problems we are considering an continuum
- What about functionals - (functions of functions)?

# Example

- Suppose we connect two points in the plane  $(x_0, y_0)$  and  $(x_1, y_1)$  by a curve of the form  $y = y(x)$ .



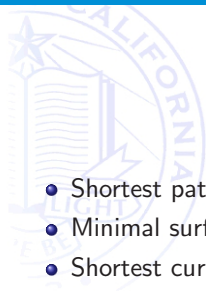
- The length of the curve can be written

$$L(y) = \int_{x_0}^{x_1} \sqrt{1 + (y')^2} dx$$

$L$  is a functional.

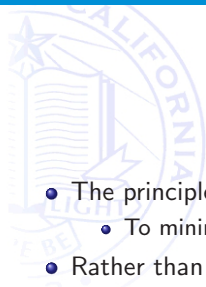
- Find the shortest curve between the two points.

# Similar problems



- Shortest path connecting a non-planar curve, say sphere
- Minimal surface of revolution generated by a connected curve
- Shortest curve with a given area below it
- Closed curve of a given perimeter that encloses the largest area
- Shape of a string hanging from two points under gravity
- Path of light travelling through an inhomogenous curve

# Euler's Equation



- The principle of
  - To minimize  $P$  is to solve  $P' = 0$
- Rather than solving the integral it is an advantage to consider the differential equation.
- The differential equation is called Euler Equation.
- We will derive it shortly

# Consider for a minute

- Suppose  $f : \mathcal{R}^n \rightarrow \mathcal{R}$  what does it mean for  $x^*$  to be a local extremum of  $f$ ?



# Consider for a minute

- Suppose  $f : \mathcal{R}^n \rightarrow \mathcal{R}$  what does it mean for  $x^*$  to be a local extremum of  $f$ ?
  - ① We must have  $f(x) \geq f(x^*)$  for every  $x$  in some neighborhood
  - ② A necessary condition  $\nabla f(x^*) = 0$  i.e., that  $\frac{\partial f}{\partial x_i} = 0$  for all  $i$ .
- For  $P$  the equivalent would be say
  - ①  $P : C^2(\mathcal{R}^n) \rightarrow \mathcal{R}$  and
  - ②  $f \rightarrow P(f)$
- what does it mean for  $f^*$  to be an extremum of  $P$ ?

# Optimal functional?

- What would be conditional for a functional?
  - ① We need  $P(f) \geq P(f^*)$  for every functional close to  $f^*$ 
    - So what is a neighborhood of a function?
  - ② Need a generalized gradient

$$P(f^* + \delta f) \approx P(f^*)$$

- Still very hand wavy

# Simplest problem

- Lets start with a simple problem
- Minimize  $J(y) = \int_{x_0}^{x_1} F(x, y, y') dx$  with  $y, F \in C^2$
- Suppose  $y^*$  minimizes  $J$  it would then be true
  - 1 In a neighborhood of  $y^*$  then  $J(y) \geq J(y^*)$
  - 2  $\delta J = 0$  for a variation  $\delta y$  is

$$\delta J(y^*) = J(y^* + \delta y) - J(y^*)$$

- What are the necessary conditions for this to be valid

# Neighborhood Evaluation

- Lets start by showing optimality in a neighborhood
- Let  $y \in C^2[x_0, x_1]$  such that  $y(x_0) = y(x_1) = 0$
- Let  $\epsilon \in \mathcal{R}$  be a value
- Lets consider a one-parameter family of functions

$$y(x) = y^*(x) + \epsilon y(x)$$

- Where  $y^*$  is the (unknown) optimal function
- Define  $\Phi : \mathcal{R} \rightarrow \mathcal{R}$  by

$$\Phi(\epsilon) = \int_{x_0}^{x_1} F(x, y, y') dx$$

- If  $|\epsilon|$  is small enough then all variants of  $y^* + \epsilon y$  lie in a small neighborhood of  $y^*$ , therefore  $\Phi$  attains a local minimum at  $\epsilon = 0$
- Thus it must be true that  $\Phi'(0) = 0$

# So what is $\Phi'$ ?

- We know that

$$\Phi(\epsilon) = \int_{x_0}^{x_1} F(x, y, y') dx$$

- So it must be true that

$$\Phi'(\epsilon) = \frac{d}{d\epsilon} \int_{x_0}^{x_1} F(x, y, y') dx$$

- Given that we have a  $C^2$  domain we can reverse the order of integration and differentiation, so that

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or

$$\Phi'(\epsilon) = \int_{x_0}^{x_1} \left( \frac{\partial}{\partial y} F(x, y^* + \epsilon y, y^{*'} + \epsilon y') y + \frac{\partial}{\partial y'} F(x, y^* + \epsilon y, y^{*'} + \epsilon y') y' \right) dx$$

- We know that

$$\Phi'(0) = 0 = \int_{x_0}^{x_1} \left( \frac{\partial}{\partial y} F(x, y^*, y^{*'}) y + \frac{\partial}{\partial y'} F(x, y^*, y^{*'}) y' \right) dx$$

# Still more $\Phi'$

- We can write this more compactly

$$\Phi'(0) = \int_{x_0}^{x_1} (F_y y + F_{y'} y') dx$$

- Using integration by parts we get

$$\begin{aligned} \int_{x_0}^{x_1} F_{y'} y' dx &= F_{y'} y \Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} y \frac{d}{dx} F_{y'} dx \\ &= - \int_{x_0}^{x_1} y \frac{d}{dx} F_{y'} dx \end{aligned}$$

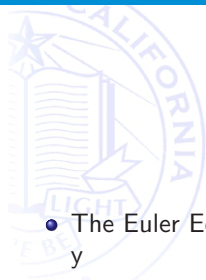
with this we can rewrite

$$\Phi'(0) = \int_{x_0}^{x_1} \left[ F_y - \frac{d}{dx} F_{y'} \right] y dx = 0$$

as this has to apply for any function  $y$  it must be true that

$$F_y - \frac{d}{dx} F_{y'} = 0 \text{ on } [x_0, x_1]$$

- This is called Euler's Equation



- The Euler Equation is essentially a “directional derivative” in the direction of  $y$
- Going back to earlier -  $\delta J$  is finding a function  $y^*$  where  $J$  is stationary.
- We are only considering the basics here.



# Shortest path problem

- Remember the initial question of shortest path?
- Recall:

$$L(y) = \int_{x_0}^{x_1} \sqrt{1 + y'^2} dx$$

with  $y_0 = y(x_0)$  and  $y_1 = y(x_1)$

- So  $F(x, y, y') = \sqrt{1 + y'^2}$

$$F_y = 0 \text{ and } F_{y'} = \frac{y'}{\sqrt{1 + y'^2}}$$

- Euler's Equation reduces to

$$\frac{d}{dx} \frac{y'}{\sqrt{1 + y'^2}} = 0$$

# The shortest path?



- So

$$\frac{y'}{\sqrt{1 + y'^2}} = c$$

- we can rewrite

$$y'^2 = c^2(1 + y'^2)$$

$$y' = \pm \frac{c}{\sqrt{1 - c^2}} = m \text{ just a constant}$$

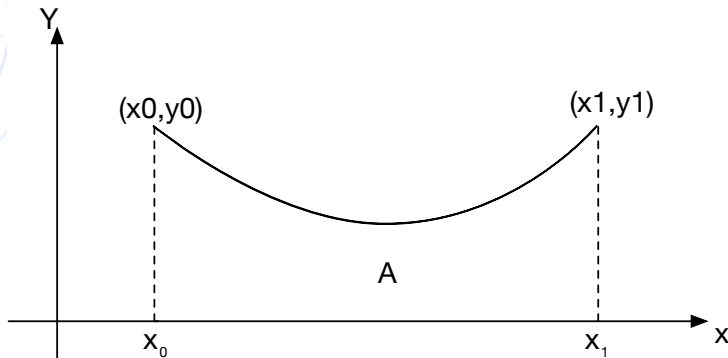
$$y' = m$$

$$y = mx + b$$

surprise it is the equation for a straight line!

# How about constrained optimization?

- Supposed we are supposed to find shortest curve with a fixed area below?



- The area is given to be  $A$  and we have end-points?

# Constrained optimization

- Our objective is then to optimize

$$L(y) = \int_{x_0}^{x_1} \sqrt{1 + y'^2} dx$$

$$A = \int_{x_0}^{x_1} y dx$$

- where the second term is our constraint
- An instance of a general class of problems called isoperimetric problems

# Isoperimetric problems

- The simplified formulation is

Minimize  $J(y) = \int_{x_0}^{x_1} F(x, y, y') dx$

Subject to  $K(y) = c$

where  $K(y) = \int_{x_0}^{x_1} G(x, y, y') dx$

# Constrained Optimization (cont.)

- We can use a combination of variational techniques and Lagrange multipliers to solve such problems
- We can define two functions

$$\begin{aligned}\Phi(\epsilon_1, \epsilon_2) &= \int_{x_0}^{x_1} F(x, y^* + \epsilon_1 y + \epsilon_2 \xi, y^{*'} + \epsilon_1 y' + \epsilon_2 \xi') dx \\ \Psi(\epsilon_1, \epsilon_2) &= \int_{x_0}^{x_1} G(x, y^* + \epsilon_1 y + \epsilon_2 \xi, y^{*'} + \epsilon_1 y' + \epsilon_2 \xi') dx\end{aligned}$$

- Here  $y^*$  is the unknown function and  $y$  and  $\xi$  are two  $C^2$  functions that vanish at the end-points
- So we want to minimize  $\Phi$  subject to the constraint  $\Psi$ . We know there is a local minimum at  $\epsilon_1 = \epsilon_2 = 0$

# Constrained Optimization (Cont.)

- Using a Lagrange approach we can form the function

$$E(\epsilon_1, \epsilon_2, \lambda) = \Phi(\epsilon_1, \epsilon_2) + \lambda(\Psi(\epsilon_1, \epsilon_2) - c)$$

- At the local minimum -  $\nabla E = 0$
- In other words there is a  $\lambda_0$  such that

$$\begin{aligned} \frac{\partial}{\partial \epsilon_1} E(0, 0, \lambda_0) &= 0 & \frac{\partial}{\partial \epsilon_2} E(0, 0, \lambda_0) &= 0 \\ \frac{\partial}{\partial \lambda} E(0, 0, \lambda_0) &= 0 \end{aligned}$$

# Constrained Optimization - let's compute

- Interchanging differentiation and integration we get

$$\frac{\partial}{\partial \epsilon_1} E(0, 0, \lambda_0) = \int_{x_0}^{x_1} (F_y y + F_{y'} y' + \lambda_0 G_y y + \lambda_0 G_{y'} y') dx$$



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- We can do integration by parts and as  $y$  vanishes at end-points we see that

$$\frac{\partial}{\partial \epsilon_1} E(0, 0, \lambda_0) = \int_{x_0}^{x_1} \left( \left[ F_y - \frac{d}{dx} F_{y'} \right] + \lambda_0 \left[ G_y - \frac{d}{dx} G_{y'} \right] \right) y dx$$

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- Similarly:

$$\frac{\partial}{\partial \epsilon_2} E(0, 0, \lambda_0) = \int_{x_0}^{x_1} \left( \left[ F_y - \frac{d}{dx} F_{y'} \right] + \lambda_0 \left[ G_y - \frac{d}{dx} G_{y'} \right] \right) \xi dx$$

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- As before we can conclude

$$\left[ F_y - \frac{d}{dx} F_{y'} \right] + \lambda_0 \left[ G_y - \frac{d}{dx} G_{y'} \right] = 0$$

# Back to our example

- So we can utilize

$$\begin{aligned} F(x, y, y') &= \sqrt{1 + y'^2} & G(x, y, y') &= y \\ F_{y'} &= 0 & G_y &= 1 \\ F_{y'} &= \frac{y'}{\sqrt{1 + y'^2}} & G_{y'} &= 0 \end{aligned}$$

- We want to satisfy the differential equation

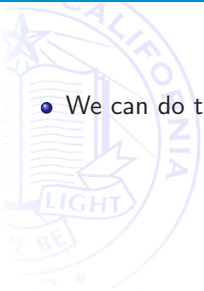
$$-\frac{d}{dx} \frac{y'}{\sqrt{1 + y'^2}} + \lambda_0 = 0$$

- Or

$$\begin{aligned} \frac{y'}{\sqrt{1 + y'^2}} &= \lambda_0 x + c \\ \frac{y'^2}{1 + y'^2} &= (\lambda_0 x + c)^2 \\ y'^2 &= \frac{(\lambda_0 x + c)^2}{1 - (\lambda_0 x + c)^2} \\ y' &= \pm \frac{\lambda_0 x + c}{\sqrt{1 - (\lambda_0 x + c)^2}} \end{aligned}$$

## Example (cont.)

- We can do the integration


$$\begin{aligned}y(x) &= \pm \int \frac{\lambda_0 x + c}{\sqrt{1 - (\lambda_0 x + c)^2}} \\&\quad \text{substitute } u = \lambda_0 x + c \text{ and } du = \lambda_0 dx \\&= \pm \int \frac{u}{\sqrt{1 - u^2}} du = \pm \left[ -\sqrt{1 - u^2} + k \right] \\&= \pm \left[ -\frac{1}{\lambda} \sqrt{1 - (\lambda_0 x + c)^2} - \frac{k}{\lambda_0} \right]\end{aligned}$$

- This can be rewritten to

$$\left( y \pm \frac{k}{\lambda_0} \right)^2 + \left( x + \frac{c}{\lambda_0} \right)^2 = \frac{1}{\lambda_0}$$

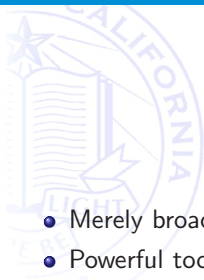
- That is a circle arc!

# Extensions



- For multiple variable you can formulate it similar to the simple case
- Ex: Shortest path in a multiple dimensional space
- Ex: Light ray tracing through non-homogenous media
- You would extend Euler's Equation to have more terms

# Summary



- Merely broached calculus of variation
- Powerful tool for optimization and derivation of analytical models
- Models for airplane wings, elastic membranes
- Important to consider it part of your toolbox