CSE276C - Integration of Functions

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Introduction

- Interested in integration of function to allow estimation of future value
- Lots of potential applications in robotics
  - Position estimation
  - Path optimization
  - Image restoration
- Consider both end-point and boundary value problems, which anchors the problem

Introduction - Setting the stage

- We are trying to solve

$$I = \int_a^b f(x)dx$$

- trying to solve I = y(b) for the equation

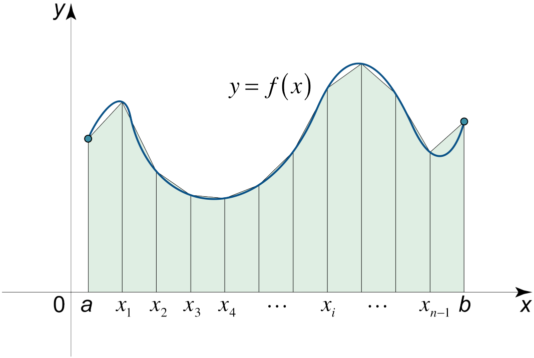
$$\frac{\partial y}{\partial x} = f(x)$$

- with the boundary condition

$$y(a) = 0$$

- Objective to generate a good estimate of y(b) with a reasonable number of evaluations
- Emphasis on 1D problems, but in most cases generalization is straight forward

Setting the stage



Basic use of Simpson's rule

- Consider equally spaces data points

$$x_i = x_0 + ih \;\; i = 0, 1, ..., N$$

- the function is evaluated at x_i

$$f_i = f(x_i)$$

- The Newton-Cotes rules is then

$$\int_{x_0}^{x_1} f(x)dx = \frac{f_1 + f_0}{2} h + O(f'' h^3)$$

- The Simpson rules is

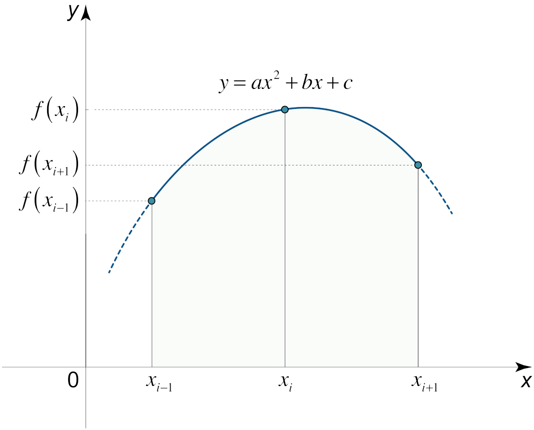
$$\int_{x_0}^{x_2} f(x)dx = \frac{h}{3}(f_0 + 4f_1 + f_2) + O(h^5 f^{(4)})$$

- which is exact to the 3rd degree
- The Simpson $\frac{3}{8}$ rule

$$\int_{x_0}^{x_3} f(x)dx = \frac{h}{8}(3f_0 + 9f_1 + 9f_2 + 3f_3)$$

- There are a series of rules for higher order, check literature

Simpson's Rule



Simpson / Trapezoid Rules

- Clearly the local rules can be chained into a longer evaluation
- $(x_0, x_1), (x_1, x_2), \ldots, (x_{N-1}, x_N)$ to get an extended trapezoid form

$$\int_{x_0}^{x_N} f(x)dx = h(\frac{1}{2}f_0 + f_1 + f_2 + \ldots + f_{N-1} + \frac{1}{2}f_N)$$

- The error estimate is

$$O\left(\frac{(x_N - x_0)f''}{N^2}\right)$$

Trapezoid Rule - Strategy?

- How can you effective use the trapezoid rule?

Trapezoid Rule - Strategy?

- How can you effective use the trapezoid rule?
- Use of a coarse to fine strategy and watch convergence
- This is termed Romberg integration in numerical toolboxes
- In general these methods generate good accuracy for proper functions?

Handling of improper function

- What is an improper function?

Handling of improper function

- What is an improper function?
  1. Integrand goes to a finite value but cannot be evaluated at a point, such as

$$\frac{sinx}{x} \text{ at } x = 0$$

  2. Upper limit is ∞ or lower limit is $-\infty$
  3. Has a singularity at a boundary point, e.g.,

$$x^{-1/2} \text{ at } x = 0$$

  4. Has a singularity within the interval at a known location
  5. Has a singularity within the interval at an unknown location
- If the value is infinite, e.g.,

$$\int_0^\infty x^{-1} dx \text{ or } \int_{-\infty}^\infty cosxdx$$

it is not improper but impossible

The Euler-Maclaurin Summation Formula

- We can write the basic Simpson's rule as

$$
\begin{aligned}
\int_a^b f(x)dx \quad = \quad & \tfrac{h}{2}\left[f(a) + 2\textstyle\sum_{k=1}^{N-1} f(a+kh) + f(b)\right] \\
& - \textstyle\sum_{k=1}^{N/2} \tfrac{h^{2k}B_{2k}}{(2k)!}[f^{(2k-1)}(b) - f^{(2k-1)}(a)] \\
& - \textstyle\sum_{k=0}^{N-1} \tfrac{h^{2k+1}B_{2k}}{(2k)!} f^{(2k)}(a+kh+\theta h)
\end{aligned}
$$

- where 2m first derivatives are continuous over (a,b). h = (a-b)/N and $\theta \in (0,1)$
- So what are the B's?
- They are Bernoulli numbers

$$
\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!}
$$

- example values

$$
\begin{aligned}
B_0 &= 1 \\
B_2 &= \tfrac{1}{6} \\
B_4 &= -\tfrac{1}{30}
\end{aligned}
$$

- Enables you to compute an estimate of the error for a particular integration
- Other integration functions have similar error functions - decreasing with

Extended Mid-point Formulation

- In many cases using the mid-point is a valuable alternative

$$\int_{x_0}^{x_{N-1}} f(x)dx = h(f_{1/2} + f_{3/2} + \ldots + f_{N-3/2}) + O(\frac{1}{N^2})$$

- When combined with the Euler-Maclaurin you get

$$\begin{aligned}\int_{x_0}^{x_{N-1}} f(x)dx &= h(f_{1/2} + f_{3/2} + \ldots + f_{N-3/2}) \\ &+ \tfrac{B_2 h^2}{4}(f'_{N-1} - f'_0) + \ldots + \tfrac{B_{2k} h^{2k}}{(2k)!}(f^{(2k)}_{N-1} - f^{(2k)}_0) + \ldots\end{aligned}$$

- We can do this recursively to estimate convergence

Handling improper integrals

- A trick for improper integrals is to do variable substitution to eliminate a challenge
- Say one of the values is at $-\infty$ or ∞ we can substitute

$$\int_a^b f(x)dx = \int_{1/b}^{1/a} \frac{1}{t^2} f\left(\frac{1}{t}\right) dt$$

Variable substitution

- More generally we can do variable substitution as

$$I = \int_a^b f(x)dx = \int_c^d f(x(t))\frac{dx}{dt}dt$$

- An example is the Schwartz tanh rule

$$x = \frac{1}{2}(b+a) + \frac{1}{2}(b-a)\tanh(t) \; x \in [a,b] \text{ and } t \in [-\infty, \infty]$$

- where

$$\frac{\partial x}{\partial t} = \frac{1}{2}(b-a)sech^2(t) = \frac{2}{b-a}(b-t)(t-a)$$

- *sech*() converges very rapidly for $t \to \infty$ which allows for integration close to singularities

Gauss Integration

- Sometimes uniform sampling is not ideal
- A Gauss model may be an alternative
- The idea is

$$\int_a^b W(x)f(x)dx \approx \sum_{j=0}^{N-1} W_j f(x_j)$$

- For polynomials this can be an exact approximation
- We can approximate f(x) with a Gaussian Mixture and choose weights to match

$$f(x) \approx \sum_{k=0}^{N} W_k N(x|x_k, \sigma_k)$$

- If you have a function with variable dynamics it makes sense to partition the integration into intervals and use Romberg integration on each interval, i.e.

$$
\begin{aligned}
I &= \int_a^b f(x)dx \\
&= \int_a^m f(x)dx + \int_m^b f(x)dx
\end{aligned}
$$

- Rule 1 of data analysis understand your data

Summary

- Simple linear approximations are effective for well-behaved functions
- The order of your approximation can vary according to function complexity
- Using Bernoulli functions we can approximate the estimated error
- Recursive estimation with error monitoring is often effective
- Do a function analysis first to make sure function is proper
- Next time we will discuss integration of ODE with standard methods such as Runga-Kutta, Step-size variation, etc.

Questions