CSE276C - Roots of Polynomials

Henrik I. Christensen

Computer Science and Engineering
University of California, San Diego
http://cri.ucsd.edu

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Outline

Introduction

- Last time we looked at direct search for roots
- Bracketing was the way to limit the search domain
- Brent's method was a simple strategy to do search
- What if we have a polynomial?
  1. Can we find the roots?
  2. Can we simplify the polynomial?
- Lets explore this

Outline

Low order polynomials

- We have closed form solutions to roots of polynomials up to degree 4
- Quadratics

$$ax^2 + bx + c = 0, \; a \neq 0$$

has two roots

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we have real unique, dual or imaginary solutions

Cubics

- The cubic equation
$$x^3 + px^2 + qx + r = 0$$
can be reduced using substitution
$$x = y - \frac{p}{3}$$
to the form
$$y^3 + ay + b = 0$$
where
$$\begin{aligned} a &= \tfrac{1}{3}(3q - p^2) \\ b &= \tfrac{1}{27}(2p^3 - 9pq + 27r) \end{aligned}$$
the condensed form has 3 roots
$$\begin{aligned} y_1 &= A + B \\ y_2 &= -\tfrac{1}{2}(A+B) + \tfrac{i\sqrt{3}}{2}(A-B) \\ y_3 &= -\tfrac{1}{2}(A+B) - \tfrac{i\sqrt{3}}{2}(A-B) \end{aligned}$$
where
$$A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \qquad B = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

Cubic (cont)

- We have three cases:
  1. $\frac{b^2}{4} + \frac{a^3}{27} > 0$: one real root and two conjugate roots
  2. $\frac{b^2}{4} + \frac{a^3}{27} = 0$: three real roots of which at least two are equal
  3. $\frac{b^2}{4} + \frac{a^3}{27} < 0$: three real roots and unequal roots

Quartics

- For the equation

$$x^4 + px^3 + qx^2 + rx + s = 0$$

we can apply a similar trick

$$x = y - \frac{p}{4}$$

to get

$$y^4 + ay^2 + by + c = 0$$

where

$$\begin{aligned} a &= q - \tfrac{3p^2}{8} \\ b &= r + \tfrac{p^3}{8} - \tfrac{pq}{2} \\ c &= s - \tfrac{4p^4}{256} + \tfrac{p^2q}{16} - \tfrac{pr}{4} \end{aligned}$$

Quartics (cont.)

- The reduced equation can be factorized into

$$z^3 - qz^2 + (pr - 4s)z + (4sq - r^2 - p^2 s) = 0$$

if we can estimate z_1 of the above cubic then

$$\begin{aligned} x_1 &= -\tfrac{p}{4} + \tfrac{1}{2}(R + D) \\ x_2 &= -\tfrac{p}{4} + \tfrac{1}{2}(R - D) \\ x_3 &= -\tfrac{p}{4} - \tfrac{1}{2}(R + E) \\ x_4 &= -\tfrac{p}{4} - \tfrac{1}{2}(R - D) \end{aligned}$$

where

$$\begin{aligned} R &= \sqrt{\tfrac{1}{4}p^2 - q + z_1} \\ D &= \sqrt{\tfrac{3}{4}p^2 - R^2 - 2Q + \tfrac{1}{4}(4pq - 8r - p^3)R^{-1}} \\ E &= \sqrt{\tfrac{3}{4}p^2 - R^2 - 2Q - \tfrac{1}{4}(4pq - 8r - p^3)R^{-1}} \end{aligned}$$

Outline

Root Counting

- Consider a polynomial of degree n:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

- if a_i are real the roots are real or complex conjugate pairs.
- p(x) has n roots
- Descartes rules of sign:
  - "The number of positive real zeroes in a polynomial function p(x) is the same or less than by an even numbers as the number of changes in the sign of the coefficients. The number of negative real zeroes of the p(x) is the same as the number of changes in sign of the coefficients of the terms of p(-x) or less than this by an even number"
- Consider

$$p(x) = x^5 + 4x^4 - 3x^2 + x - 6$$

- So it must have 3 or 1 postive root and
- and it must have 2 or 0 negative roots

Sturms theorem

- We can derive a sequence of polynomials
- Let f(x) be a polynomial. Denote the original $f_0(x)$ and the derivative $f'(x) = f_1(x)$. Consider

$$
\begin{array}{rcl}
f_0(x) & = & q_1(x)f_1(x) - f_2(x) \\
f_1(x) & = & q_2(x)f_2(x) - f_3(x) \\
 & \vdots & \\
f_{k-2}(x) & = & q_{k-1}(x)f_{k-1}(x) - f_k(x) \\
f_{k-1}(x) & = & q_k(x)f_k(x)
\end{array}
$$

- The theorem
  - The number of distinct real zeros of a polynomial f(x) with real coefficients in (a, b) is equal to the excess of the number of changes of sign in the sequence $f_0(a), \ldots, f_{k1}(a),\ f_k(a)$ over the number of changes of sign in the sequence $f_0(b), \ldots, f_{k1}(b), f_k(b)$.

Sturm - example

- Consider the polynomial

$$x^5 + 5x^4 - 20x^2 - 10x + 2 = 0$$

The Sturm functions are then

$$\begin{aligned}
f_0(x) &= x^5 + 5x^4 - 20x^2 - 10x + 2 \\
f_1(x) &= x^4 + 4x^3 - 8x - 2 \\
f_2(x) &= x^3 + 3x^2 - 1 \\
f_3(x) &= 3x^2 + 7x + 1 \\
f_4(x) &= 17x + 11 \\
f_5(x) &= 1
\end{aligned}$$

Sturm example (cont)

	$-\infty$	-10	-5	-4	-3	-2	-1	0	1	2	5	10	∞
f_0	$-$	$-$	$-$	$-$	$+$	$-$	$-$	$+$	$-$	$+$	$+$	$+$	$+$
f_1	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$
f_2	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$-$	$+$	$+$	$+$	$+$	$+$
f_3	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$+$
f_4	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$+$
f_5	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
var.	5	5	5	5	4	3	3	2	1	0	0	0	0

- So roots between (-4, -3), (-3, -2), (-1,0), (0,1) and (1,2)

Outline

Deflation

- Once you have a root r you can deflate a polynomial

$$p(x) = (x - r)q(x)$$

- As the degree decreases the complexity of root finding is simplified.
- One can use Horner's scheme

$$p(x) = b_0 + (x - r)(b_n x^{n-1} + \ldots + b_2 x + b_1)$$

as r is a root $b_0 = 0$ so

$$q(x) = b_n x^{n-1} + \ldots + b_2 x + b_1$$

Outline

Newton's Method

- Remember we can do root search/refinement

$$x_{k+1} = x_k - \frac{p(x_k)}{p'(x)}$$

we know that

$$p(x) = p(t) + (x - t)q(x)$$

So p'(t) = q(t) or

$$q(x) = \frac{p(x)}{x - t}$$

- If p(x) has double roots it could be a challenge

Outline

Müllers Method

- Newton's Method is local and sensitive to seed guess
- Müllers method is more global
- Based on a quadratic interpolation
- Assume you have three estimates of the root: x_{k-2}, x_{k-1}, x_k
- Interpolation polynomial

$$p(x) = f(x_k) + f[x_{k-1}, x_k](x - x_k) + f[x_{k-2}, x_{k-1}, x_k](x - x_k)(x - x_{k-1})$$

- Using the equality

$$(x - x_k)(x - x_{k-1}) = (x - x_k)^2 + (x - x_k)(x_k - x_{k-1})$$

we get

$$p(x) = f(x_k) + b(x - x_k) + a(x - x_k)^2$$

which we can solve for p(x) = 0

Outline

Summary

- Frequently using a polynomial refactorization is more stable
- A way to compress data into a semantic form
- For lower order polynomials we have closed for solutions
- We can use Descartes rules, ... to bracket roots
- We can find roots and reduce polynomials
- Newton's method is a simple local rule, but could be noisy
- Mullers method is a way to solve it more generally
- Lots of methods available for special cases