CSE276C - Integration of ODEs

Henrik I. Christensen

Computer Science and Engineering
University of California, San Diego
http://cri.ucsd.edu

October 2020

Outline

Introduction

- For integration of a set of ordinary differential equations you can always reduce it into a set of first order differential equations.
- Example

$$\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} = r(x)$$

- which can be rewritten

$$\begin{aligned} \tfrac{dy}{dx} &= z(x) \\ \tfrac{dz}{dx} &= r(x) - q(x)z(x) \end{aligned}$$

- where z is a new variable

Small example

- Consider a simple motion of a mass when actuated by a mass

$$F(u_1) = m\frac{d^2u_1}{dt^2}$$

- We can rewrite this as

$$\frac{d^2u_1}{dt^2} = \frac{1}{m}F(u_1)$$

- We can introduce $u_2 = \frac{du_1}{dt}$ to generate

$$\begin{array}{rcl}\frac{du_1}{dt} & = & u_2 \\ \frac{du_2}{dt} & = & \frac{1}{m}F(u_1)\end{array}$$

OR

$$\frac{du}{dt} = f(u,t) \text{ with } u = \left(\begin{array}{c} u_1 \\ u_2 \end{array}\right)$$

where

$$f = \left(\begin{array}{c} u_2 \\ \frac{F(u_1)}{m} \end{array}\right)$$

Introduction (cont)

- The generic problem is thus a set of couple 1st order differential equations

$$\frac{dy_i(x)}{dx} = f_i(x_i, y_1, y_2, \ldots, y_n)$$

- There are three major approaches:
  1. Runge-Kutta: Euler type propagation
  2. Richardson extrapolation / Burlirsch-Stoer: extrapolation type estimation with small step sizes
  3. Predictor-Corrector: extrapolation with correction.
- Runge-Kutta most widely adopted for "generic" problems. Great if function evaluation is cheap
- Burlirsch-Stoer generates higher precision
- Predictor-Corrector is historically interesting, but rarely used today

Outline

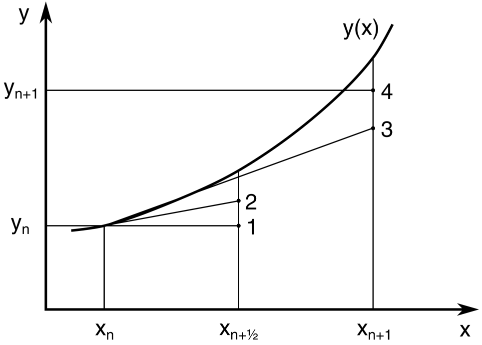
Runge-Kutta

- The forward Euler method is specified as

$$y_{n+1} = y_n + hf(x_n, y_n)$$

with $x_{n+1} = x_n + h$

- A problem is that the integration is asymmetric



Runge-Kutta - Stepped Up

- We can use a mid-point to get a closer estimate, i.e.,

$$\begin{aligned} k_1 &= hf(x_n, y_n) \\ k_2 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1) \\ y_{n+1} &= y_n + k_2 + O(h^3) \end{aligned}$$

- We can easily extend to richer models. A typical example is the fourth order model

$$\begin{aligned}
k_1 &= hf(x_n, y_n) \\
k_2 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1) \\
k_3 &= hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2) \\
k_4 &= hf(x_n + h, y_n + k_3) \\
y_{n+1} &= y_n + \tfrac{1}{6}k_1 + \tfrac{1}{3}k_2 + frac13k_3 + \tfrac{1}{6}k_4 + O(h^5)
\end{aligned}$$

- By far the most frequently used RK method for ODE integration
- Requires four function evaluations for every step

Adaptive Runge-Kutta

- Could we adjust the step-size?
- Estimation of performance adds an overhead
- What would be an obvious solution?

Adaptive Runge-Kutta

- Could we adjust the step-size?
- Estimation of performance adds an overhead
- What would be an obvious solution?
  1. Do a full step
  2. Do a half step
  3. Compare (could be recursive)
  4. Next
- In general no one goes beyond 5th order Runge-Kutta

PI step control of RK

- Could we use PI control to track stepsize?

PI step control of RK

- Could we use PI control to track stepsize?
- How about

$$h_{n+1} = S h_n \text{err}_n^{\alpha} \text{err}_{n-1}^{\beta}$$

  where S is a scale factor. α and β are gain factors
- Typical default values $\alpha = \frac{1}{k} - 0.75\beta$ and $\beta = \frac{0.4}{k}$ and k is an integer that designates order of the integrator

Outline

Richardson Extrapolation / Burlirsch-Stoer

- Aimed at smooth functions
- Generates best precision with minimal effort
- Things to consider
  1. Does not do well on functions w. table lookup or interpolation
  2. Not well suited for functions with singulaties within intg range
  3. Not well suited for "expensive" functions
- The approach is based on three ideas
  1. Final answer is based on selection of (adaptive) stepsize just like Romberg
  2. Use of rational functions for extrapolation (allow larger h)
  3. Integration method reply on use of even functions
- Typically the steps size H is large and h is 100+ steps

Burlirsch-Stoer - The details

- Consider a modified mid-point strategy

$$x_{n+1} = x_n + H$$

but with substeps

$$h = \frac{H}{n}$$

Outline

Outline