

CSE276C - Roots of Polynomials

Henrik I. Christensen

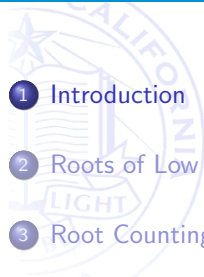


Computer Science and Engineering
University of California, San Diego

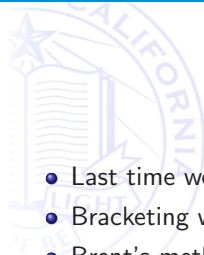
<http://cri.ucsd.edu>

October 2020

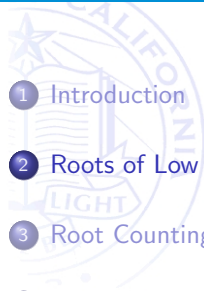
Outline

- 
- 1 Introduction
 - 2 Roots of Low Order Polynomials
 - 3 Root Counting
 - 4 Deflation
 - 5 Newton's Method
 - 6 Müller's Method
 - 7 Summary

Introduction

- 
- Last time we looked at direct search for roots
 - Bracketing was the way to limit the search domain
 - Brent's method was a simple strategy to do search
 - What if we have a polynomial?
 - 1 Can we find the roots?
 - 2 Can we simplify the polynomial?
 - Lets explore this

Outline

- 
- 1 Introduction
 - 2 Roots of Low Order Polynomials
 - 3 Root Counting
 - 4 Deflation
 - 5 Newton's Method
 - 6 Müller's Method
 - 7 Summary

Low order polynomials

- We have closed form solutions to roots of polynomials up to degree 4
- Quadratics

$$ax^2 + bx + c = 0, \quad a \neq 0$$

has two roots

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we have real unique, dual or imaginary solutions

Cubics

- The cubic equation

$$x^3 + px^2 + qx + r = 0$$

can be reduced using substitution

$$x = y - \frac{p}{3}$$

to the form

$$y^3 + ay + b = 0$$

where

$$\begin{aligned} a &= \frac{1}{3}(3q - p^2) \\ b &= \frac{1}{27}(2p^3 - 9pq + 27r) \end{aligned}$$

the condensed form has 3 roots

$$\begin{aligned} y_1 &= A + B \\ y_2 &= -\frac{1}{2}(A + B) + \frac{i\sqrt{3}}{2}(A - B) \\ y_3 &= -\frac{1}{2}(A + B) - \frac{i\sqrt{3}}{2}(A - B) \end{aligned}$$

where

$$A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \quad B = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

Cubic (cont)

• We have three cases:

- ① $\frac{b^2}{4} + \frac{a^3}{27} > 0$: one real root and two conjugate roots
- ② $\frac{b^2}{4} + \frac{a^3}{27} = 0$: three real roots of which at least two are equal
- ③ $\frac{b^2}{4} + \frac{a^3}{27} < 0$: three real roots and unequal roots

Quartics

- For the equation

$$x^4 + px^3 + qx^2 + rx + s = 0$$

we can apply a similar trick

$$x = y - \frac{p}{4}$$

to get

$$y^4 + ay^2 + by + c = 0$$

where

$$\begin{aligned} a &= q - \frac{3p^2}{8} \\ b &= r + \frac{p^3}{8} - \frac{pq}{2} \\ c &= s - \frac{4p^4}{256} + \frac{p^2q}{16} - \frac{pr}{4} \end{aligned}$$

Quartics (cont.)

- The reduced equation can be factorized into

$$z^3 - qz^2 + (pr - 4s)z + (4sq - r^2 - p^2s) = 0$$

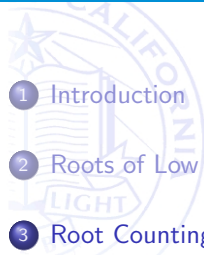
if we can estimate z_1 of the above cubic then

$$\begin{aligned}x_1 &= -\frac{p}{4} + \frac{1}{2}(R + D) \\x_2 &= -\frac{p}{4} + \frac{1}{2}(R - D) \\x_3 &= -\frac{p}{4} - \frac{1}{2}(R + E) \\x_4 &= -\frac{p}{4} - \frac{1}{2}(R - D)\end{aligned}$$

where

$$\begin{aligned}R &= \sqrt{\frac{1}{4}p^2 - q + z_1} \\D &= \sqrt{\frac{3}{4}p^2 - R^2 - 2Q + \frac{1}{4}(4pq - 8r - p^3)R^{-1}} \\E &= \sqrt{\frac{3}{4}p^2 - R^2 - 2Q - \frac{1}{4}(4pq - 8r - p^3)R^{-1}}\end{aligned}$$

Outline

- 
- 1 Introduction
 - 2 Roots of Low Order Polynomials
 - 3 Root Counting
 - 4 Deflation
 - 5 Newton's Method
 - 6 Müller's Method
 - 7 Summary

Root Counting

- Consider a polynomial of degree n :

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

- if a_i are real the roots are real or complex conjugate pairs.
- $p(x)$ has n roots
- Descartes rules of sign:
 - "The number of positive real zeroes in a polynomial function $p(x)$ is the same or less than by an even numbers as the number of changes in the sign of the coefficients. The number of negative real zeroes of the $p(x)$ is the same as the number of changes in sign of the coefficients of the terms of $p(-x)$ or less than this by an even number"

- Consider

$$p(x) = x^5 + 4x^4 - 3x^2 + x - 6$$

- So it must have 3 or 1 positive root and
- and it must have 2 or 0 negative roots

Sturms theorem

- We can derive a sequence of polynomials
- Let $f(x)$ be a polynomial. Denote the original $f_0(x)$ and the derivative $f'(x) = f_1(x)$. Consider

$$\begin{aligned}f_0(x) &= q_1(x)f_1(x) - f_2(x) \\f_1(x) &= q_2(x)f_2(x) - f_3(x) \\&\vdots \\f_{k-2}(x) &= q_{k-1}(x)f_{k-1}(x) - f_k(x) \\f_{k-1}(x) &= q_k(x)f_k(x)\end{aligned}$$

- The theorem
 - The number of distinct real zeros of a polynomial $f(x)$ with real coefficients in (a, b) is equal to the excess of the number of changes of sign in the sequence $f_0(a), \dots, f_{k1}(a), f_k(a)$ over the number of changes of sign in the sequence $f_0(b), \dots, f_{k1}(b), f_k(b)$.

Sturm - example

- Consider the polynomial

$$x^5 + 5x^4 - 20x^2 - 10x + 2 = 0$$

The Sturm functions are then

$$f_0(x) = x^5 + 5x^4 - 20x^2 - 10x + 2$$

$$f_1(x) = x^4 + 4x^3 - 8x - 2$$

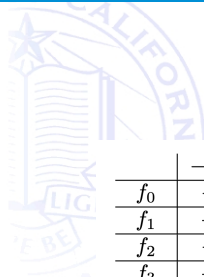
$$f_2(x) = x^3 + 3x^2 - 1$$

$$f_3(x) = 3x^2 + 7x + 1$$

$$f_4(x) = 17x + 11$$

$$f_5(x) = 1$$

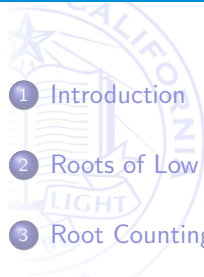
Sturm example (cont)



	$-\infty$	-10	-5	-4	-3	-2	-1	0	1	2	5	10	∞
f_0	-	-	-	-	+	-	-	+	-	+	+	+	+
f_1	+	+	+	+	-	-	-	-	-	+	+	+	+
f_2	-	-	-	-	-	+	+	-	+	+	+	+	+
f_3	+	+	+	+	+	-	-	+	+	+	+	+	+
f_4	-	-	-	-	-	-	-	+	+	+	+	+	+
f_5	+	+	+	+	+	+	+	+	+	+	+	+	+
var.	5	5	5	5	4	3	3	2	1	0	0	0	0

- So roots between $(-4, -3)$, $(-3, -2)$, $(-1, 0)$, $(0, 1)$ and $(1, 2)$

Outline

- 
- 1 Introduction
 - 2 Roots of Low Order Polynomials
 - 3 Root Counting
 - 4 Deflation
 - 5 Newton's Method
 - 6 Müller's Method
 - 7 Summary

Deflation

- Once you have a root r you can deflate a polynomial

$$p(x) = (x - r)q(x)$$

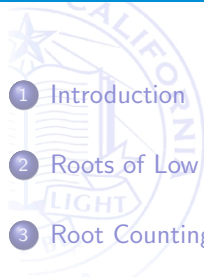
- As the degree decreases the complexity of root finding is simplified.
- One can use Horner's scheme

$$p(x) = b_0 + (x - r)(b_n x^{n-1} + \dots + b_2 x + b_1)$$

as r is a root $b_0 = 0$ so

$$q(x) = b_n x^{n-1} + \dots + b_2 x + b_1$$

Outline

- 
- 1 Introduction
 - 2 Roots of Low Order Polynomials
 - 3 Root Counting
 - 4 Deflation
 - 5 Newton's Method
 - 6 Müller's Method
 - 7 Summary

Newton's Method

- Remember we can do root search/refinement

$$x_{k+1} = x_k - \frac{p(x_k)}{p'(x_k)}$$

we know that

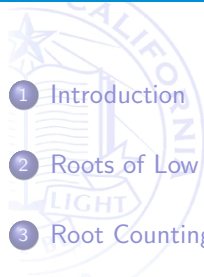
$$p(x) = p(t) + (x - t)q(x)$$

So $p'(t) = q(t)$ or

$$q(x) = \frac{p(x)}{x - t}$$

- If $p(x)$ has double roots it could be a challenge

Outline

- 
- 1 Introduction
 - 2 Roots of Low Order Polynomials
 - 3 Root Counting
 - 4 Deflation
 - 5 Newton's Method
 - 6 Müller's Method
 - 7 Summary

Müllers Method

- Newton's Method is local and sensitive to seed guess
- Müllers method is more global
- Based on a quadratic interpolation
- Assume you have three estimates of the root: x_{k-2}, x_{k-1}, x_k
- Interpolation polynomial

$$p(x) = f(x_k) + f[x_{k-1}, x_k](x - x_k) + f[x_{k-2}, x_{k-1}, x_k](x - x_k)(x - x_{k-1})$$

- Using the equality

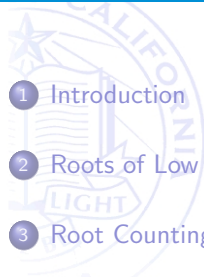
$$(x - x_k)(x - x_{k-1}) = (x - x_k)^2 + (x - x_k)(x_k - x_{k-1})$$

we get

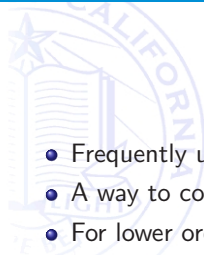
$$p(x) = f(x_k) + b(x - x_k) + a(x - x_k)^2$$

which we can solve for $p(x) = 0$

Outline

- 
- 1 Introduction
 - 2 Roots of Low Order Polynomials
 - 3 Root Counting
 - 4 Deflation
 - 5 Newton's Method
 - 6 Müller's Method
 - 7 Summary

Summary

- 
- Frequently using a polynomial refactorization is more stable
 - A way to compress data into a semantic form
 - For lower order polynomials we have closed for solutions
 - We can use Descartes rules, ... to bracket roots
 - We can find roots and reduce polynomials
 - Newton's method is a simple local rule, but could be noisy
 - Mullers method is a way to solve it more generally
 - Lots of methods available for special cases