

CSE276C - Roots of Polynomials

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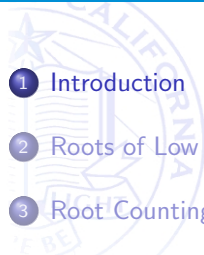


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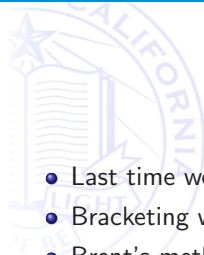
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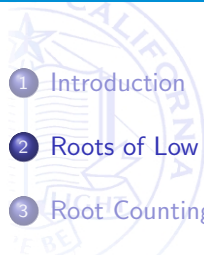
Outline

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- 1 Introduction
 - 2 Roots of Low Order Polynomials
 - 3 Root Counting
 - 4 Bounds on Roots
 - 5 Deflation
 - 6 Newton's Method
 - 7 Müller's Method
 - 8 Summary

Introduction

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- Last time we looked at direct search for roots
 - Bracketing was the way to limit the search domain
 - Brent's method was a simple strategy to do search
 - What if we have a polynomial?
 - 1 Can we find the roots?
 - 2 Can we simplify the polynomial?
 - Lets explore this

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Low order polynomials

- We have closed form solutions to roots of polynomials up to degree 4
- Quadratics

$$ax^2 + bx + c = 0, \quad a \neq 0$$

has two roots

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we have real unique, dual or imaginary solutions

Cubics

- The cubic equation

$$x^3 + px^2 + qx + r = 0$$

can be reduced using substitution

$$x = y - \frac{p}{3}$$

to the form

$$y^3 + ay + b = 0$$

where

$$\begin{aligned} a &= \frac{1}{3}(3q - p^2) \\ b &= \frac{1}{27}(2p^3 - 9pq + 27r) \end{aligned}$$

the condensed form has 3 roots

$$\begin{aligned} y_1 &= A + B \\ y_2 &= -\frac{1}{2}(A + B) + \frac{i\sqrt{3}}{2}(A - B) \\ y_3 &= -\frac{1}{2}(A + B) - \frac{i\sqrt{3}}{2}(A - B) \end{aligned}$$

where

$$A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \quad B = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

Cubic (cont)



• We have three cases:

- ① $\frac{b^2}{4} + \frac{a^3}{27} > 0$: one real root and two conjugate roots
- ② $\frac{b^2}{4} + \frac{a^3}{27} = 0$: three real roots of which at least two are equal
- ③ $\frac{b^2}{4} + \frac{a^3}{27} < 0$: three real roots and unequal roots

Quartics

- For the equation

$$x^4 + px^3 + qx^2 + rx + s = 0$$

we can apply a similar trick

$$x = y - \frac{p}{4}$$

to get

$$y^4 + ay^2 + by + c = 0$$

where

$$\begin{aligned} a &= q - \frac{3p^2}{8} \\ b &= r + \frac{p^3}{8} - \frac{pq}{2} \\ c &= s - \frac{4p^4}{256} + \frac{p^2q}{16} - \frac{pr}{4} \end{aligned}$$

Quartics (cont.)

- The reduced equation can be factorized into

$$z^3 - qz^2 + (pr - 4s)z + (4sq - r^2 - p^2s) = 0$$

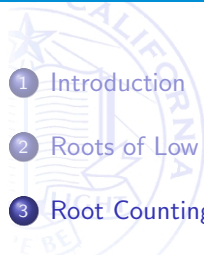
if we can estimate z_1 of the above cubic then

$$\begin{aligned}x_1 &= -\frac{p}{4} + \frac{1}{2}(R + D) \\x_2 &= -\frac{p}{4} + \frac{1}{2}(R - D) \\x_3 &= -\frac{p}{4} - \frac{1}{2}(R + E) \\x_4 &= -\frac{p}{4} - \frac{1}{2}(R - D)\end{aligned}$$

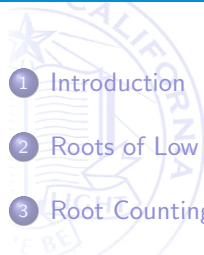
where

$$\begin{aligned}R &= \sqrt{\frac{1}{4}p^2 - q + z_1} \\D &= \sqrt{\frac{3}{4}p^2 - R^2 - 2Q + \frac{1}{4}(4pq - 8r - p^3)R^{-1}} \\E &= \sqrt{\frac{3}{4}p^2 - R^2 - 2Q - \frac{1}{4}(4pq - 8r - p^3)R^{-1}}\end{aligned}$$

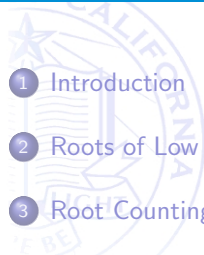
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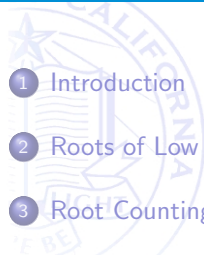
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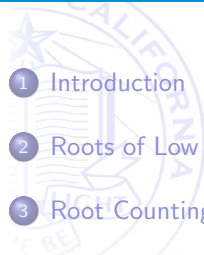
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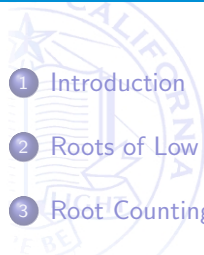
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