CSE276C - Roots of Polynomials

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Outline

Introduction

- Last time we looked at direct search for roots
- Bracketing was the way to limit the search domain
- Brent's method was a simple strategy to do search
- What if we have a polynomial?
  1. Can we find the roots?
  2. Can we simplify the polynomial?
- Lets explore this

Outline

Low order polynomials

- We have closed form solutions to roots of polynomials up to degree 4
- Quadratics

$$ax^2 + bx + c = 0, \ a \neq 0$$

has two roots

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we have real unique, dual or imaginary solutions

Cubics

- The cubic equation

$$x^3 + px^2 + qx + r = 0$$

can be reduced using substitution

$$x = y - \frac{p}{3}$$

to the form

$$y^3 + ay + b = 0$$

where

$$\begin{aligned} a &= \tfrac{1}{3}(3q - p^2) \\ b &= \tfrac{1}{27}(2p^3 - 9pq + 27r) \end{aligned}$$

the condensed form has 3 roots

$$\begin{aligned} y_1 &= A + B \\ y_2 &= -\tfrac{1}{2}(A+B) + \tfrac{i\sqrt{3}}{2}(A-B) \\ y_3 &= -\tfrac{1}{2}(A+B) - \tfrac{i\sqrt{3}}{2}(A-B) \end{aligned}$$

where

$$A = \sqrt[3]{-\frac{b}{2} + \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}} \qquad B = \sqrt[3]{-\frac{b}{2} - \sqrt{\frac{b^2}{4} + \frac{a^3}{27}}}$$

Cubic (cont)

- We have three cases:
  1. $\frac{b^2}{4} + \frac{a^3}{27} > 0$: one real root and two conjugate roots
  2. $\frac{b^2}{4} + \frac{a^3}{27} = 0$: three real roots of which at least two are equal
  3. $\frac{b^2}{4} + \frac{a^3}{27} < 0$: three real roots and unequal roots

Quartics

- For the equation
$$x^4 + px^3 + qx^2 + rx + s = 0$$
we can apply a similar trick
$$x = y - \frac{p}{4}$$
to get
$$y^4 + ay^2 + by + c = 0$$
where
$$\begin{aligned} a &= q - \tfrac{3p^2}{8} \\ b &= r + \tfrac{p^3}{8} - \tfrac{pq}{2} \\ c &= s - \tfrac{4p^4}{256} + \tfrac{p^2 q}{16} - \tfrac{pr}{4} \end{aligned}$$

Quartics (cont.)

- The reduced equation can be factorized into

$$z^3 - qz^2 + (pr - 4s)z + (4sq - r^2 - p^2 s) = 0$$

if we can estimate z_1 of the above cubic then

$$\begin{aligned} x_1 &= -\tfrac{p}{4} + \tfrac{1}{2}(R + D) \\ x_2 &= -\tfrac{p}{4} + \tfrac{1}{2}(R - D) \\ x_3 &= -\tfrac{p}{4} - \tfrac{1}{2}(R + E) \\ x_4 &= -\tfrac{p}{4} - \tfrac{1}{2}(R - D) \end{aligned}$$

where

$$\begin{aligned} R &= \sqrt{\tfrac{1}{4}p^2 - q + z_1} \\ D &= \sqrt{\tfrac{3}{4}p^2 - R^2 - 2Q + \tfrac{1}{4}(4pq - 8r - p^3)R^{-1}} \\ E &= \sqrt{\tfrac{3}{4}p^2 - R^2 - 2Q - \tfrac{1}{4}(4pq - 8r - p^3)R^{-1}} \end{aligned}$$

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