

CSE276C - Optimization

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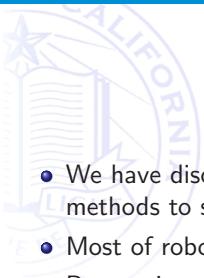
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October 2021

Outline

- 
- 1 Introduction
 - 2 Bracket based methods
 - 3 Downhill Simplex
 - 4 Powell's Method
 - 5 Conjugate Descent/Gradient
 - 6 Stochastic Search
 - 7 Dynamic Programming
 - 8 Summary

Introduction



- We have discussed approximation and root finding. We can leverage these methods to study optimization.
- Most of robotics is about optimization
- Best trajectory between two points
- Best fit of a model to a swarm of data
- Optimal coverage of an area for fire monitoring
- Energy efficient travel from San Diego to Hawaii by water



- Numerical Recipes: Chapter 10
- Numerical Renaissance: Chap 14-16. (Part III)

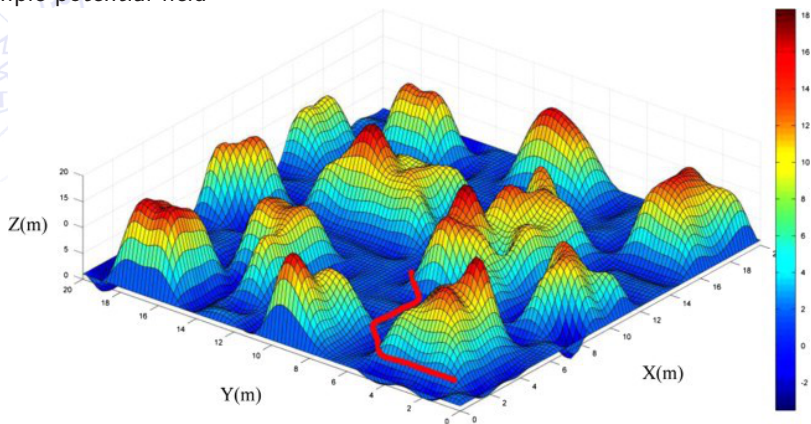
Example 1



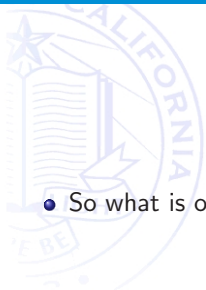
- Optimization of trajectories at high speed

Path Planning

- Example potential field

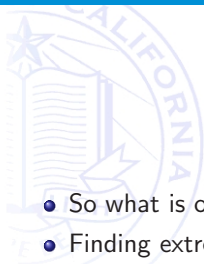


Optimization



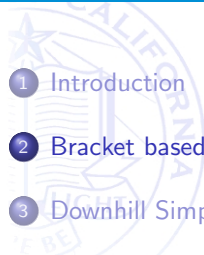
- So what is optimization?

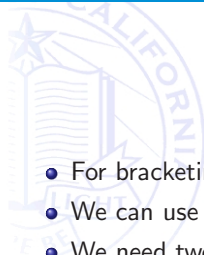
Optimization

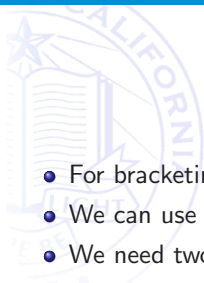


- So what is optimization?
- Finding extrema for a function over a domain
- Minimum or maximum is immaterial as we can use f or $-f$
- In many cases we will have local and global extrema
- Consider both deterministic and stochastic approaches

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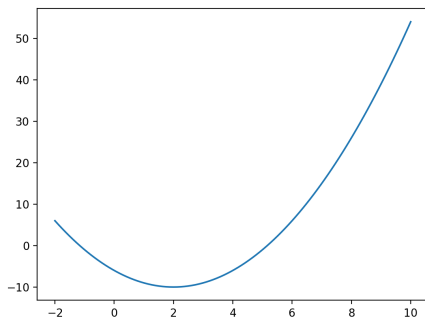
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- For bracketing of roots we use bi-section as a basis.
 - We can use a similar technique to find an extremum
 - We need two points to bracket a root!
 - How many points do we need to bracket an extremum?



- For bracketing of roots we use bi-section as a basis.
- We can use a similar technique to find an extremum
- We need two points to bracket a root!
- How many points do we need to bracket an extremum?
- We need three points to bracket.
- If we have a triplet $a < b < c$. If $f(b)$ is smaller than $f(a)$ and $f(c)$, then we have a minimum within $[a, c]$

Golden Section

- Pick a point between (a,b) or (b,c) and evaluate
- Suppose $x \in (b, c)$ and $f(x) < f(b)$ then our new triple is (b, x, c)
- Consider the function



- How would you choose a new value of x ?

Golden Section (cont.)

- Consider (a, b, c)

$$\frac{b-a}{c-a} = w \qquad \frac{c-b}{c-a} = 1 - w$$

- Lets assume $x \in (b, c)$ and

$$\frac{x-b}{c-a} = z$$

- The next bracket is then $w+z$ or $1-w$

Golden Section (cont.)

- Consider (a, b, c)

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- Lets assume $x \in (b, c)$ and

$$\frac{x-b}{c-a} = z$$

- The next bracket is then $w+z$ or $1-w$
- If we want to make the intervals equal

$$z = 1 - 2w \text{ when } w < \frac{1}{2}$$

- z should be the same distance from b and c and b is from a and c

$$\frac{z}{1-w} = w$$

- we can rewrite to replace z and get the equation

$$w^2 - 3w + 1 = 0 \Rightarrow w = \frac{3 - \sqrt{5}}{2} \approx 0.38197$$

- Widely used to select iteration strategies

Parabolic Interpolation

- We covered Brent's method in root finding and in interpolation
- If we have a triple (a, b, c) and the values $f(a)$, $f(b)$, $f(c)$ we can generate a 2nd order interpolation

$$x = b - \frac{1}{2} \frac{(b-a)^2[f(b) - f(c)] - (b-c)^2[f(b) - f(a)]}{(b-a)[f(b) - f(c)] - (b-c)[f(b) - f(a)]}$$

- When would this fail?

Parabolic Interpolation

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- When would this fail?
- When the triple pair is co-linear!
- The remedy is to use golden section when a co-linear case is seen

1-D search w. derivative information

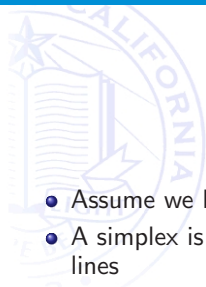


- If we have the triple (a, b, c) and $f(a)$, $f(b)$, $f(c)$
- In addition we have $f'(b)$
- You can use the sign of $f'(b)$ to choose the next bracket

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- 
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Simplex Method



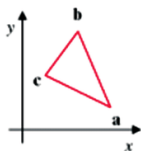
- Assume we have no gradient information or access to formal model.
- A simplex is N dimensions is composed of $N+1$ points. Connected by straight lines
 - A 2D simplex is a triangle
 - A 3D simplex is a tetrahedron.
- We have $N+1$ points x_1, \dots, x_{N+1}

Downhill Simplex Algorithm

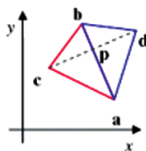
- Initial simple
 - Order the values of the vertices: $f(x_1) \leq f(x_2) \leq \dots \leq f(x_{N+1})$
- Compute x_0 , the centroid of all points except x_{N+1}
- **Reflection** compute $x_r = x_0 + \alpha(x_0 - x_{N+1})$, with $\alpha > 0$ if the reflection is better than $f(x_{N+1})$ replace. Restart
- **Expansion** if $f(x_r) < f(x_1)$ compute $x_e = x_0 + \gamma(x_r - x_0)$ if $f(x_e) < f(x_r)$ replace x_{N+1} else replace x_{N+1} with x_r . Restart
- **Contraction** If $f(x_r) > f(x_N)$ compute $x_c = x_0 + \rho(x_{N+1} - x_0)$ with $\rho < .5$. If $f(x_c) < f(x_{N+1})$ replace and restart
- **Shrink** Replaces all points except x_1 with $x_i = x_1 + \sigma(x_i - x_1)$ and restart
- Terminate when update is below a threshold.

Simplex illustration

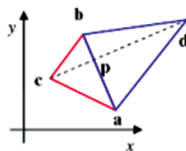
Downhill Simplex Method



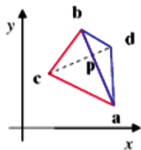
Initial simplex with vertices a, b, c , so that $f(a) < f(b) < f(c)$



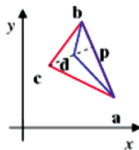
Reflection:
 $d-p = -(c-p)$ with $d-c$ perpendicular to $b-a$.



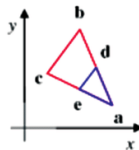
Reflection & expansion:
 $d-p = -2(c-p)$ with $d-c$ perpendicular to $b-a$.



Reflection & contraction:
 $d-p = -1/2(c-p)$ with $d-c$ perpendicular to $b-a$.



Contraction:
 $d-p = 1/2(c-p)$ with $d-c$ perpendicular to $b-a$.



Multiple contraction:
 $(d-a)/(b-a) = (e-a)/(c-a)$

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Powell's Method

- Assume you have an n-dimensional function $f(\vec{x})$ and a starting point P_0 .
- We can use the local gradient to search for an extremum
- We can generate a new estimate

$$P_{new} = P_{old} + \lambda \vec{n}$$

- Locally we can generate a Taylor expansion

$$f(x) = f(P) + \sum_i \frac{\partial f}{\partial x_i} x_i + \frac{1}{2} \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} x_i x_j + \dots$$

or

$$f(x) \approx \vec{c} - b\vec{x} + \frac{1}{2} \vec{x}^T A \vec{x}$$

where

$$\begin{aligned} \vec{c} &= f(P) \\ b &= -\nabla f_P \\ A_{ij} &= \frac{\partial^2 f}{\partial x_i \partial x_j} \text{ Hessian Matrix} \end{aligned}$$

- Also remember

$$\nabla f = A\mathbf{x} - b$$

at an extremum

Powell's Method

- Initialize N unit vectors

$$u_i = e_i \quad i \in 1 \dots N$$

- 1 Start at point P_0
 - 2 For $i=1$ to N
 - 3 Move along P_i from P_{i-1} along u_i
 - 4 New $u_i = u_{i+1}$
 - 5 Set $u_N = P_n - P_0$
 - 6 Move P_n to minimum value
 - 7 Make $P_0 = P_n$
- Might generate linear degenerate solutions

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Conjugate gradient descent

- If we have the gradient from

$$f(x) \approx \vec{c} - b\vec{x} + \frac{1}{2}\vec{x}^T A \vec{x}$$

- We can do a steepest descent

- 1 Start at P_0
- 2 Compute $\nabla f(P_i)$
- 3 move in the direction of gradient to point P_i
- 4 repeat

- We can construct a set of conjugate vectors

$$\begin{aligned} g_{i+1} &= g_i - \lambda A h_i \\ h_{i+1} &= g_{i+1} + \gamma_i h_i \\ \lambda_i &= \frac{g_i g_i}{h_i A h_i} \\ \gamma_i &= \frac{g_{i+1} g_{i+1}}{g_i g_i} \end{aligned}$$

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Stochastic Search

- So far we have used direct functional values for optimization.
- The search has been deterministic
- Sometimes the search space is too large
- What if we use a sampling based approach?
- Some possible examples
 - Traveling salesman
 - Layout of silicon for chips
- Loosely based on Boltzmann distribution

$$P(E) = \exp(-E/kT)$$

- where E is energy/entropy, T is temperature, and k is the Boltzmann constant.

Metropolis Algorithm

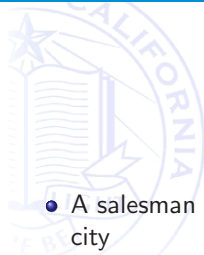
- Transformed into an algorithm by 1953 by Metropolis
- **Algorithm**
- Let $s = s_0$
- For $k = 0$ to k_{max}
 - $T = \text{temperature}(k/k_{max})$
 - Pick random neighbor $s_{new} = \text{neighbor}(T)$
 - If $(P(S, T) \leq \text{random}(0, 1))$
 - $s = s_{new}$
- Return S

Simulated Annealing



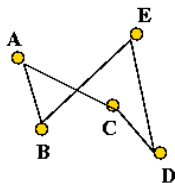
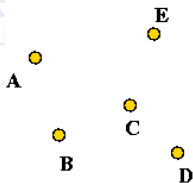
- 1 Description of possible configurations
- 2 A way to generate random perturbation of a configuration
- 3 An objective function whose minimization is the objective
- 4 A control variable that is lowered over times.

Example - traveling salesman

- 
- A salesman has to visit N cities at locations (x_i, y_i) returning to the original city
 - Each city to be visited only once
 - Minimize the travel route
 - Problem in the optimal sense is known to be NP-hard.

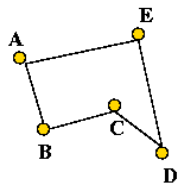
Simple Example - Traveling Salesman

Input:



A non-optimal tour:

A B E D C



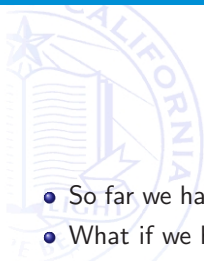
The optimal tour:

A B C D E

Outline

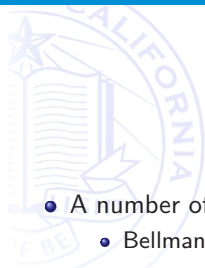
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Dynamic Programming



- So far we have considered functional optimization and stochastic optimization
- What if we have a limited set of action to optimize across?
- Say optimizing a set of actions to traverse a graph?
- A strategy to could be
 - Generate a cost-map across the state space
 - Backtrack to find the optimal set of actions

Dynamic programming

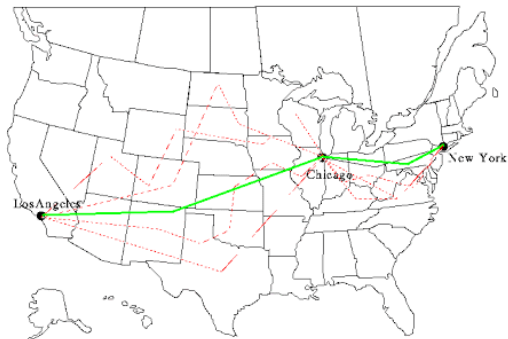


- A number of different names / approaches has been used
 - Bellman, Dijkstra, Viterbi, ...
- Selection a state space for optimization
- Identifying a set of possible actions
- Formulation of an objective function

Example navigation

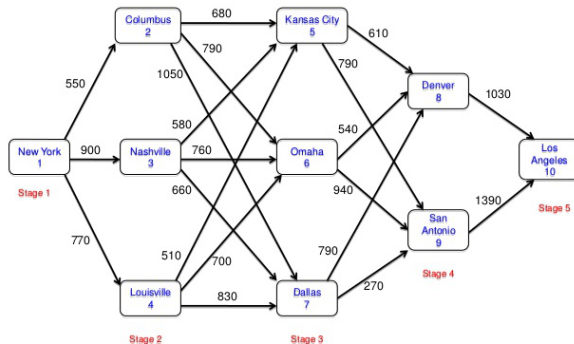


TRIVIAL EXAMPLE OF BELLMAN'S OPTIMALITY PRINCIPLE



Example navigation

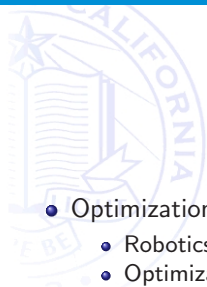
Shortest Path: network figure



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- 
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Summary



- Optimization is a key objective in robotics
 - Robotics in many cases is about formulation of a graph
 - Optimization of an objective function across the graph
- Considered deterministic and stochastic approaches to optimization
- Covered the basics and gave an impression of the fundamentals



Questions