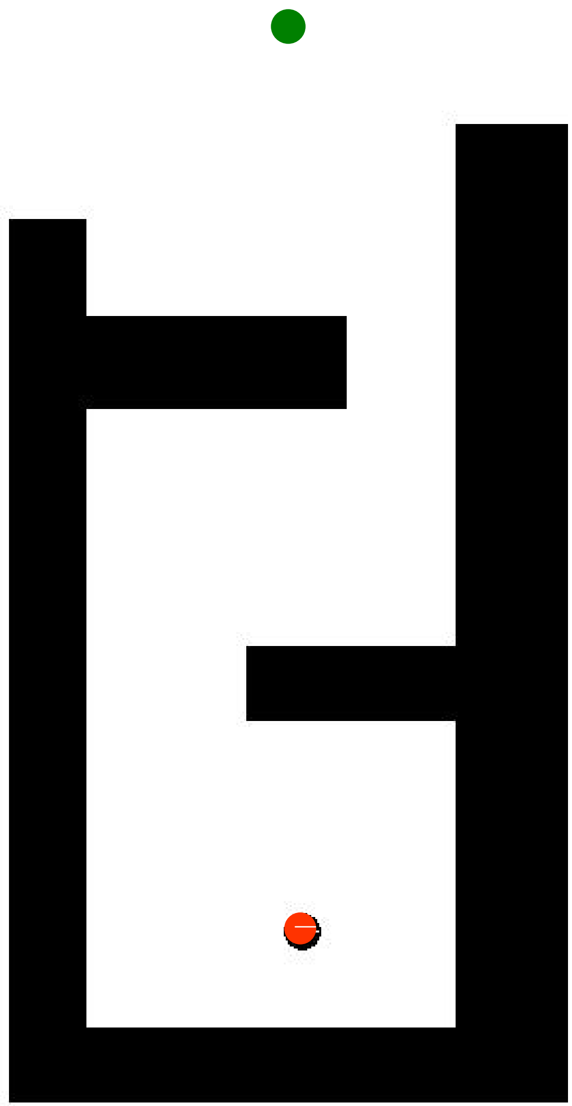
Robotic Motion Planning: Configuration Space

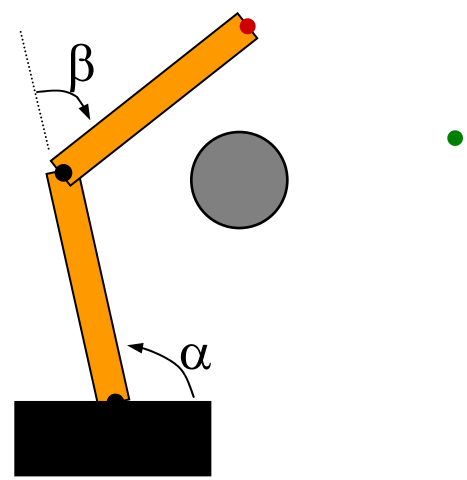
Henrik I Christensen

Adopted from Howie Choset
http://www.cs.cmu.edu/~choset

What if the robot is not a point?

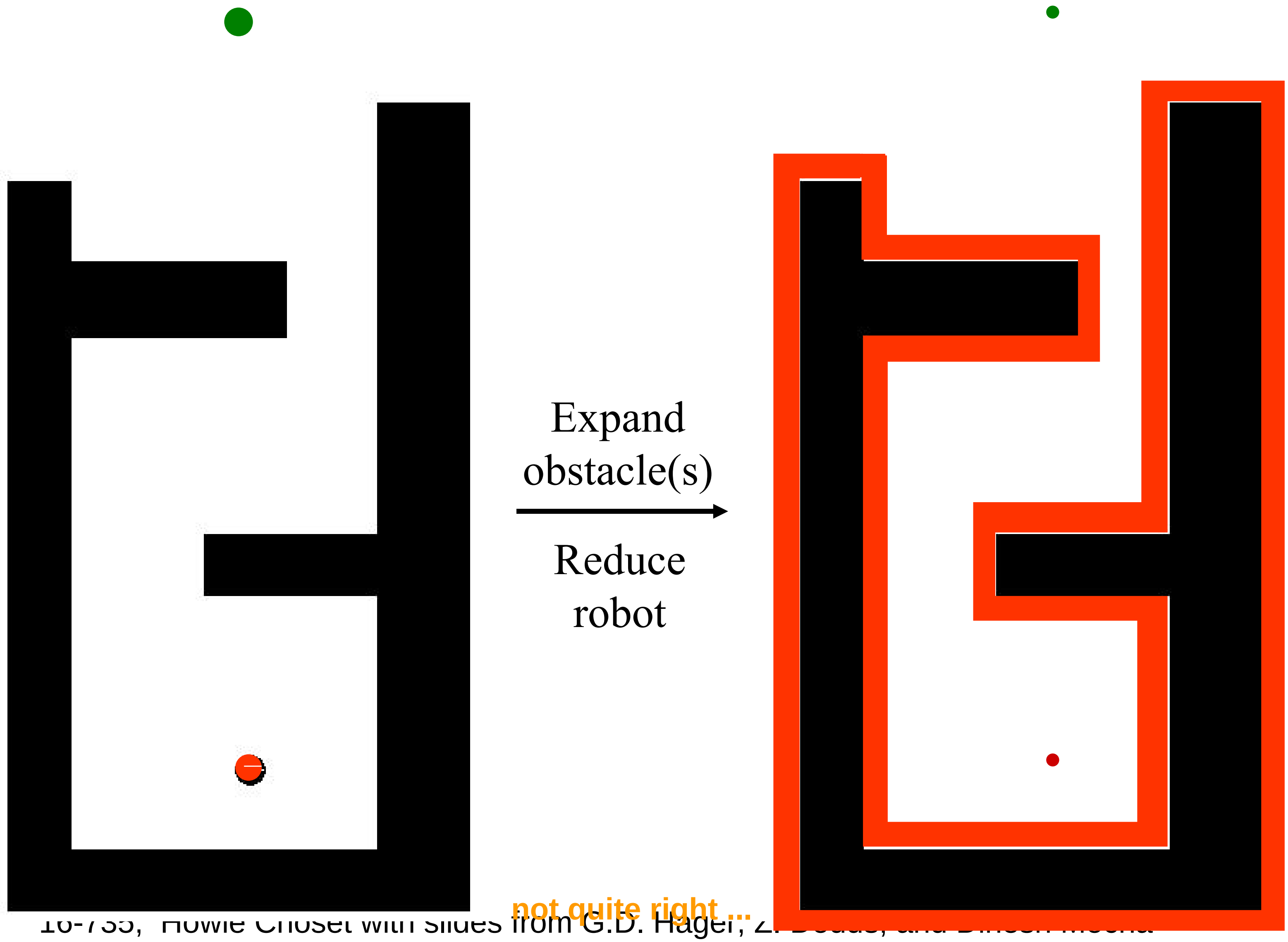


The Scout should probably not be modeled as a point...



Nor should robots with extended linkages that may contact obstacles...

What is the position of the robot?



Configuration Space

- A key concept for motion planning is a **configuration**:
  - *a complete specification of the position of every point in the system*

- A simple example: a robot that translates but does not rotate in the plane:

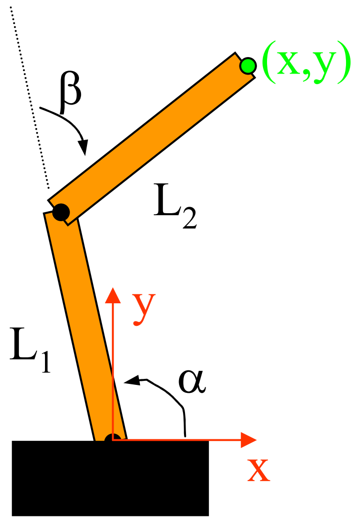
  - what is a sufficient representation of its configuration?

- The space of all configurations is the **configuration space** or **C-space.**

C-space formalism:
Lozano-Perez '79

Robot Manipulators

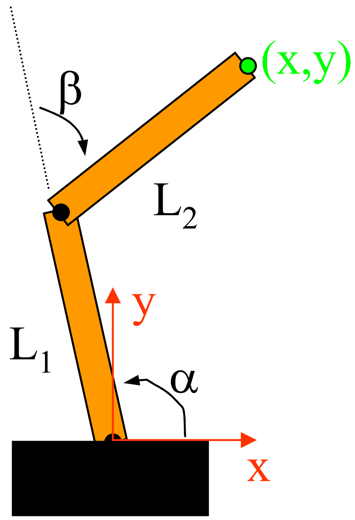
What are this arm's forward kinematics?



(How does its position depend on its joint angles?)

Robot Manipulators

What are this arm's forward kinematics?



Find **(x,y)** in terms of α and β ...

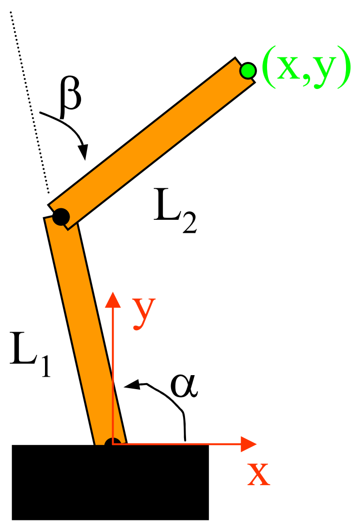
Keeping it "simple"

$c_\alpha = \cos(\alpha)$, $s_\alpha = \sin(\alpha)$

$c_\beta = \cos(\beta)$, $s_\beta = \sin(\beta)$

$c_+ = \cos(\alpha+\beta)$, $s_+ = \sin(\alpha+\beta)$

Manipulator kinematics



Keeping it "simple"

$$c_\alpha = \cos(\alpha)\ ,\ s_\alpha = \sin(\alpha)$$

$$c_\beta = \cos(\beta)\ ,\ s_\beta = \sin(\beta)$$

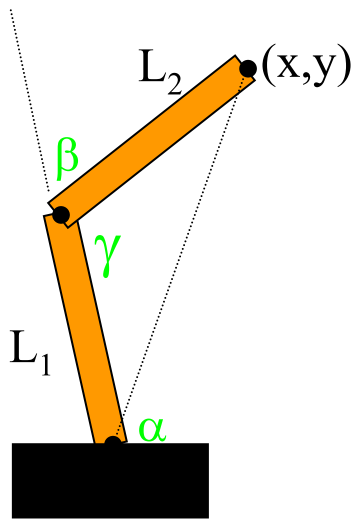
$$c_+ = \cos(\alpha+\beta)\ ,\ s_+ = \sin(\alpha+\beta)$$

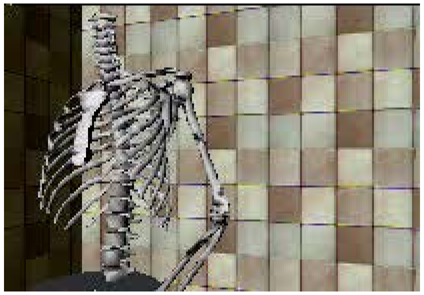
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} L_1 c_\alpha \\ L_1 s_\alpha \end{pmatrix} + \begin{pmatrix} L_2 c_+ \\ L_2 s_+ \end{pmatrix}$$ Position

Inverse Kinematics

Inverse kinematics -- finding joint angles from Cartesian coordinates

via a geometric or algebraic approach...



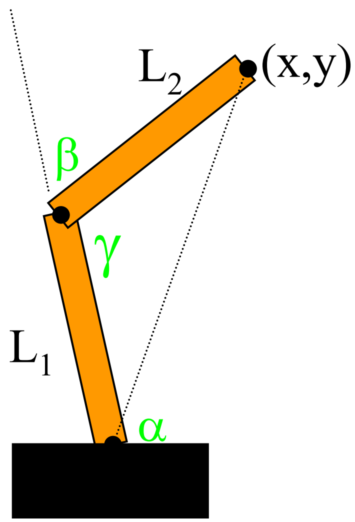


Given (x,y) and L_1 and L_2 , what are the values of α, β, γ ?

Inverse Kinematics

Inverse kinematics -- finding joint angles from Cartesian coordinates

via a geometric or algebraic approach...



$$\gamma = \cos^{-1}\left(\frac{x^2 + y^2 - L_1^{\,2} - L_2^{\,2}}{2L_1L_2}\right)$$

$$\beta = 180 - \gamma$$

$$\alpha = \sin^{-1}\left(\frac{L_2 \sin(\gamma)}{x^2 + y^2}\right) + \tan^{-1}(y/x)$$

atan2(y,x)

(1,0) = 1.3183, -1.06

(-1,0) = 1.3183, 4.45

But it's not usually this ugly...

Puma Inv. Kinematics

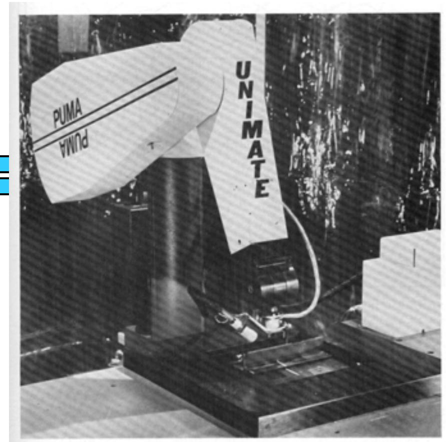


FIGURE 3.17 The Unimation PUMA 560.

```
%
% Solve for theta(1)

r=sqrt(Px^2 + Py^2);
if (n1 == 1),
    theta(1)= atan2(Py,Px) + asin(d3/r);
else
    theta(1)= atan2(Py,Px) + pi - asin(d3/r);
end

%
% Solve for theta(2)

V114= Px*cos(theta(1)) + Py*sin(theta(1));
r=sqrt(V114^2 + Pz^2);
Psi = acos((a2^2-d4^2-a3^2+V114^2+Pz^2)/
           (2.0*a2*r));
theta(2) = atan2(Pz,V114) + n2*Psi;

%
% Solve for theta(3)

num = cos(theta(2))*V114+sin(theta(2))*Pz-a2;
den = cos(theta(2))*Pz - sin(theta(2))*V114;
theta(3) = atan2(a3,d4) - atan2(num, den);
```

```
% Solve for theta(4)

V113 = cos(theta(1))*Ax + sin(theta(1))*Ay;
V323 = cos(theta(1))*Ay - sin(theta(1))*Ax;
V313 = cos(theta(2)+theta(3))*V113 +
       sin(theta(2)+theta(3))*Az;
theta(4) = atan2((n4*V323),(n4*V313));

% Solve for theta(5)

num = -cos(theta(4))*V313 - V323*sin(theta(4));
den = -V113*sin(theta(2)+theta(3)) +
       Az*cos(theta(2)+theta(3));
theta(5) = atan2(num,den);

% Solve for theta(6)

V112 = cos(theta(1))*Ox + sin(theta(1))*Oy;
V132 = sin(theta(1))*Ox - cos(theta(1))*Oy;
V312 = V112*cos(theta(2)+theta(3)) +
       Oz*sin(theta(2)+theta(3));
V332 = -V112*sin(theta(2)+theta(3)) +
        Oz*cos(theta(2)+theta(3));
V412 = V312*cos(theta(4)) - V132*sin(theta(4));
V432 = V312*sin(theta(4)) + V132*cos(theta(4));
num = -V412*cos(theta(5)) - V332*sin(theta(5));
den = - V432;
theta(6) = atan2(num,den);
```

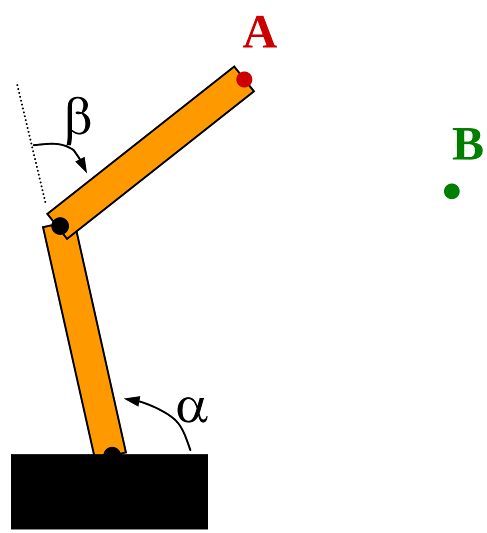
It's usuall much worse!

Some Other Examples of C-Space

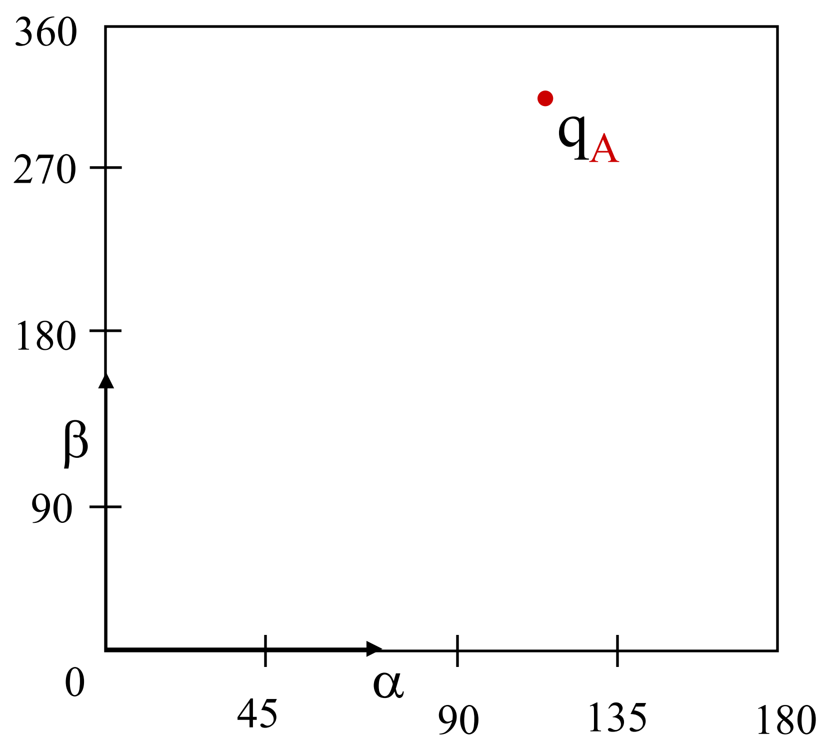
- A rotating bar fixed at a point
  - what is its C-space?
  - what is its workspace
- A rotating bar that translates along the rotation axis
  - what is its C-space?
  - what is its workspace
- A two-link manipulator
  - what is its C-space?
  - what is its workspace?
  - Suppose there are joint limits, does this change the C-space?
  - The workspace?

Configuration Space

Where can we put $\bullet\, q_B$?



An obstacle in the robot's workspace



Torus

(wraps horizontally and vertically)

Obstacles in C-Space

- Let q denote a point in a configuration space Q
- The path planning problem is to find a mapping c:[0,1]$\rightarrow$ Q s.t. no configuration along the path intersects an obstacle
- Recall a workspace obstacle is WO_i
- A *configuration space obstacle* QO_i is the set of configurations q at which the robot intersects WO_i, *that is*
  - $\mathcal{QO}_i = \{q \in \mathcal{Q} \mid R(q) \bigcap \mathcal{WO}_i \neq \emptyset\}.$
- The *free configuration space (*or just *free space*) $\mathcal{Q}_{\text{free}}$ *is*

$$\mathcal{Q}_{\text{free}} = \mathcal{Q} \backslash \left(\bigcup \mathcal{QO}_i \right).$$

The free space is generally an open set

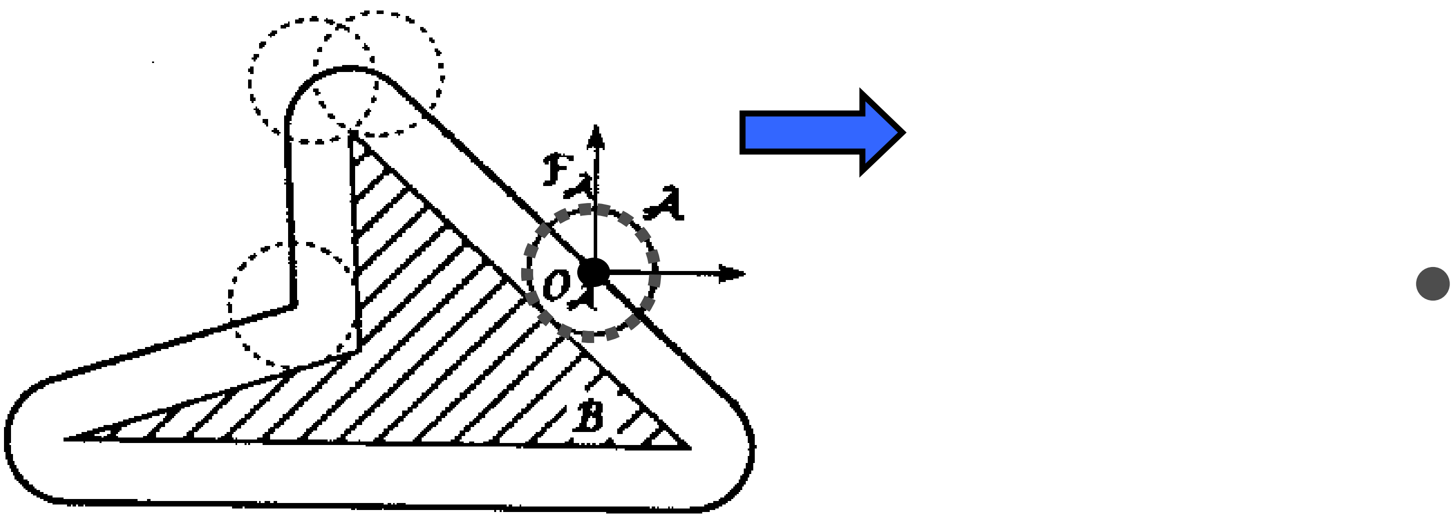
A *free path* is a mapping c:[0,1]$\rightarrow$ Q_{free}

A *semifree path* is a mapping c:[0,1]$\rightarrow$ cl(Q_{free})

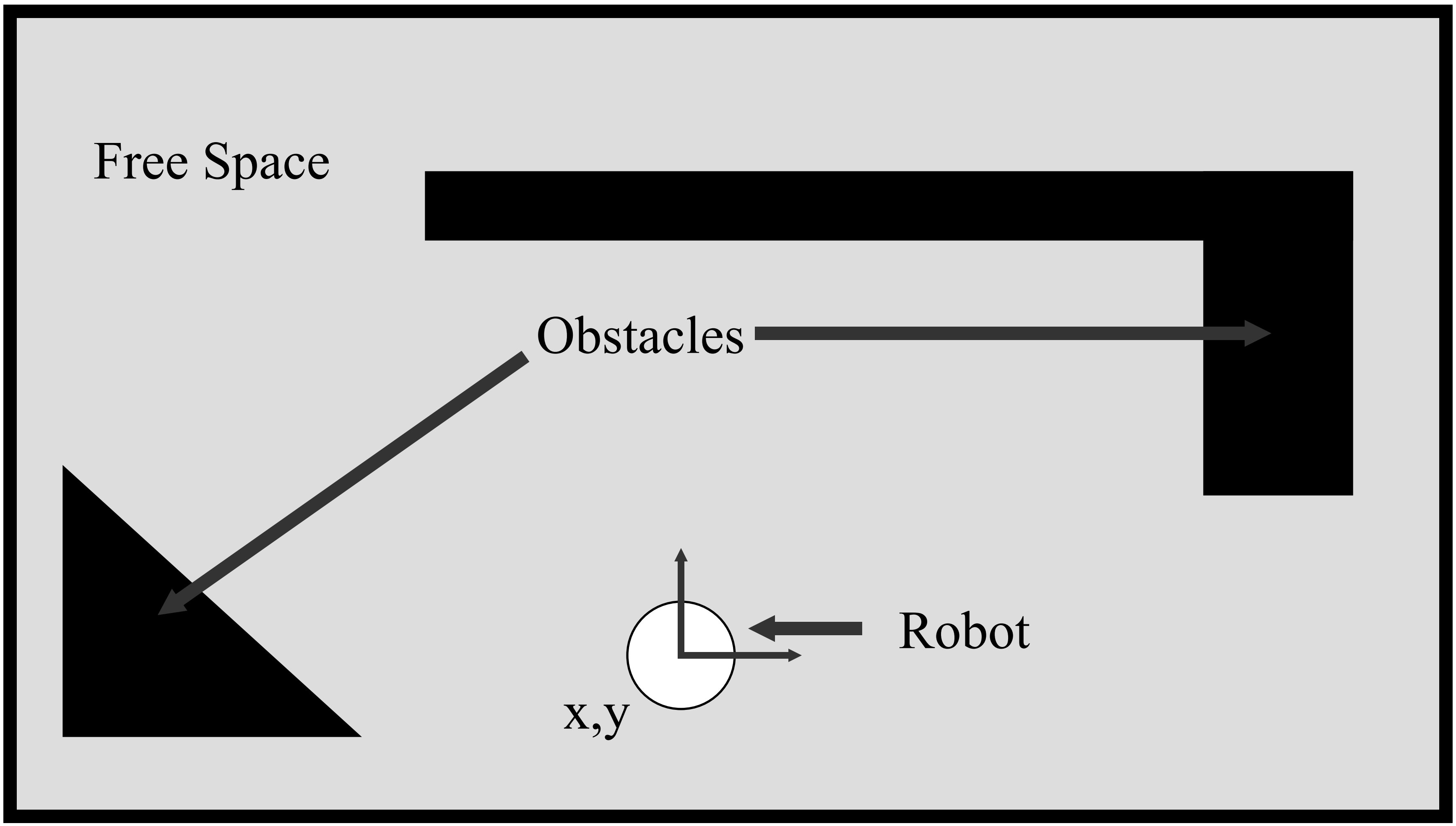
Disc in 2-D workspace

workspace
workspace

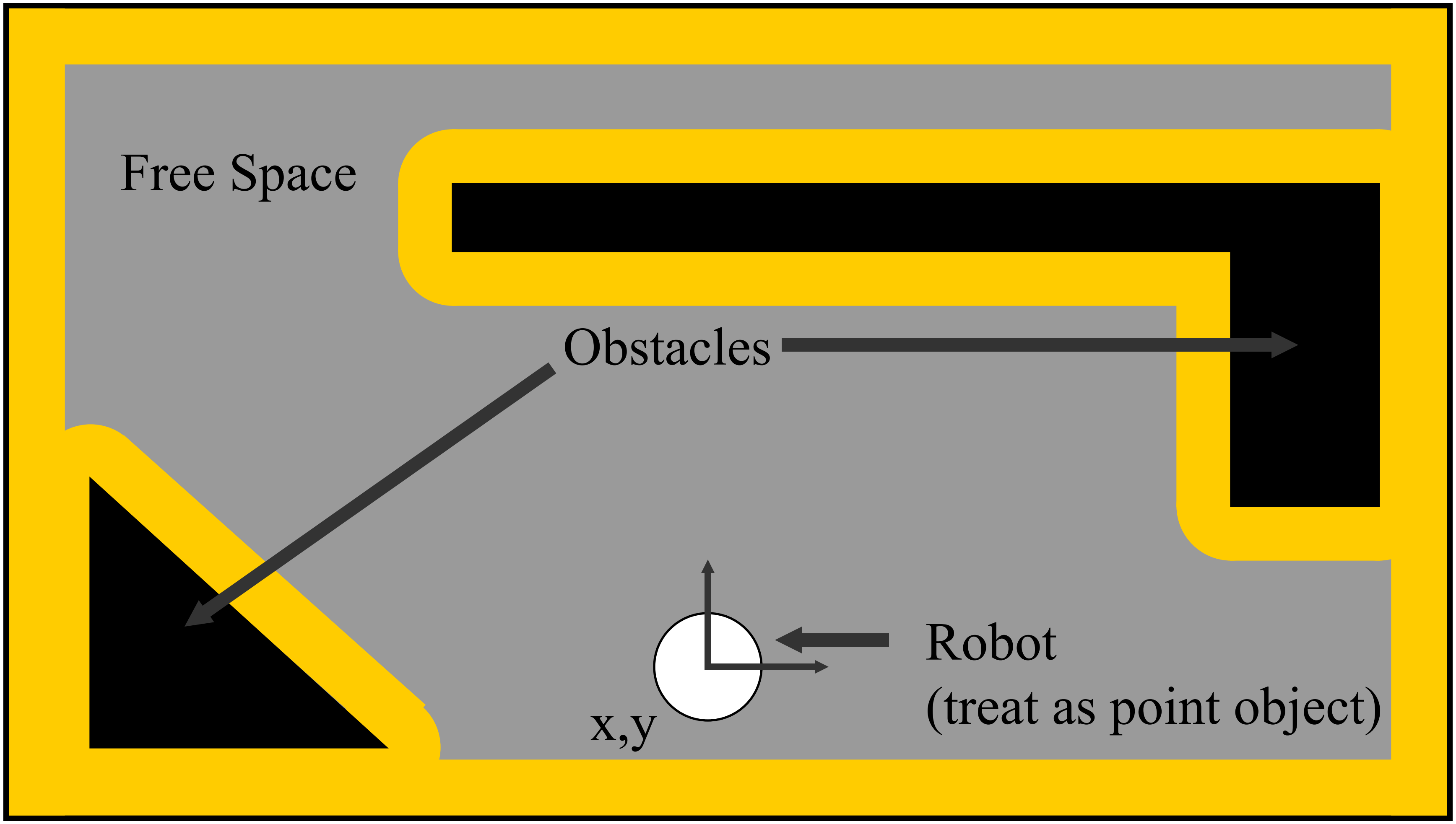
configuration space



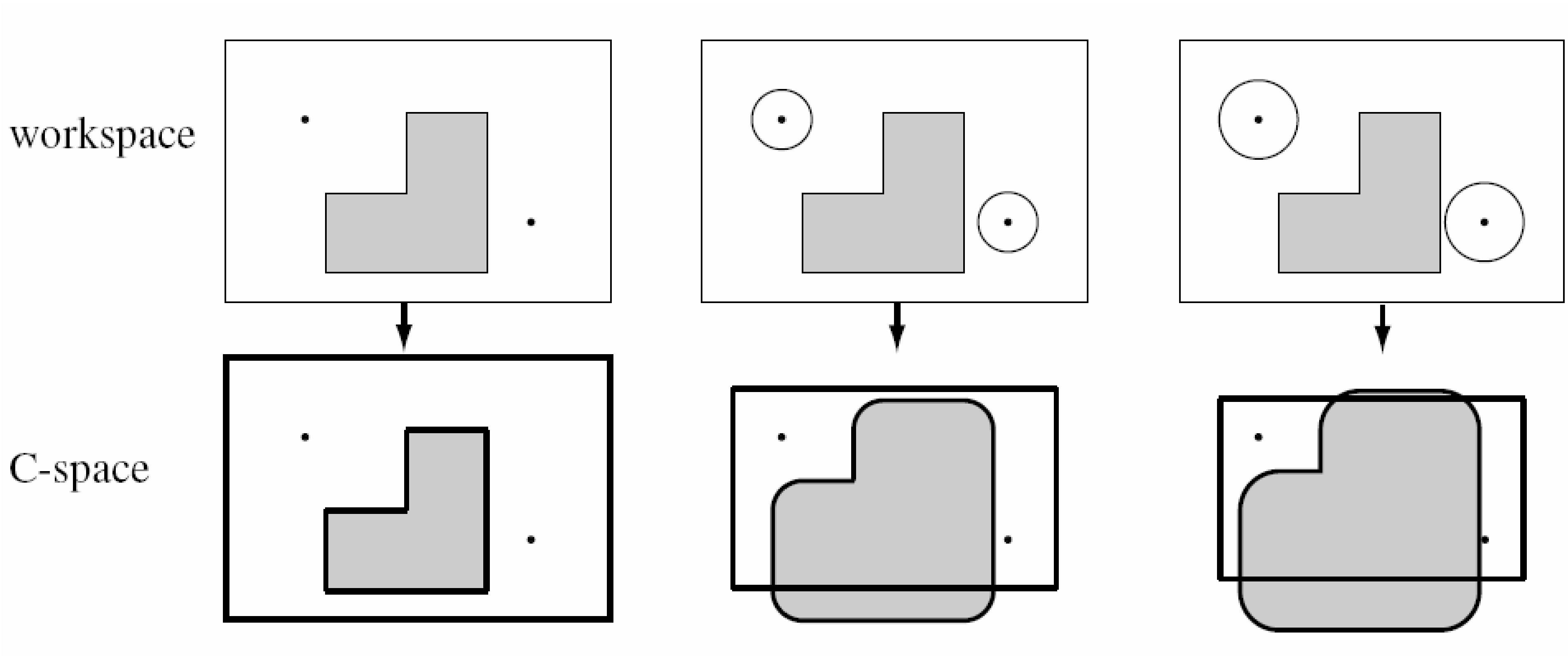
Example of a World (and Robot)



Configuration Space: Accommodate Robot Size



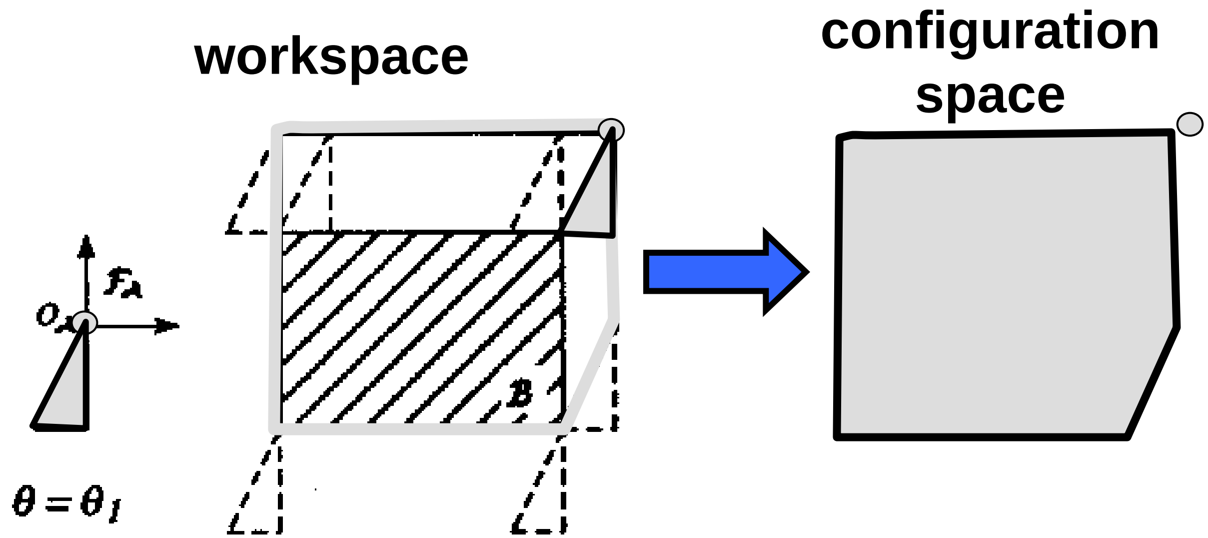
Trace Boundary of Workspace



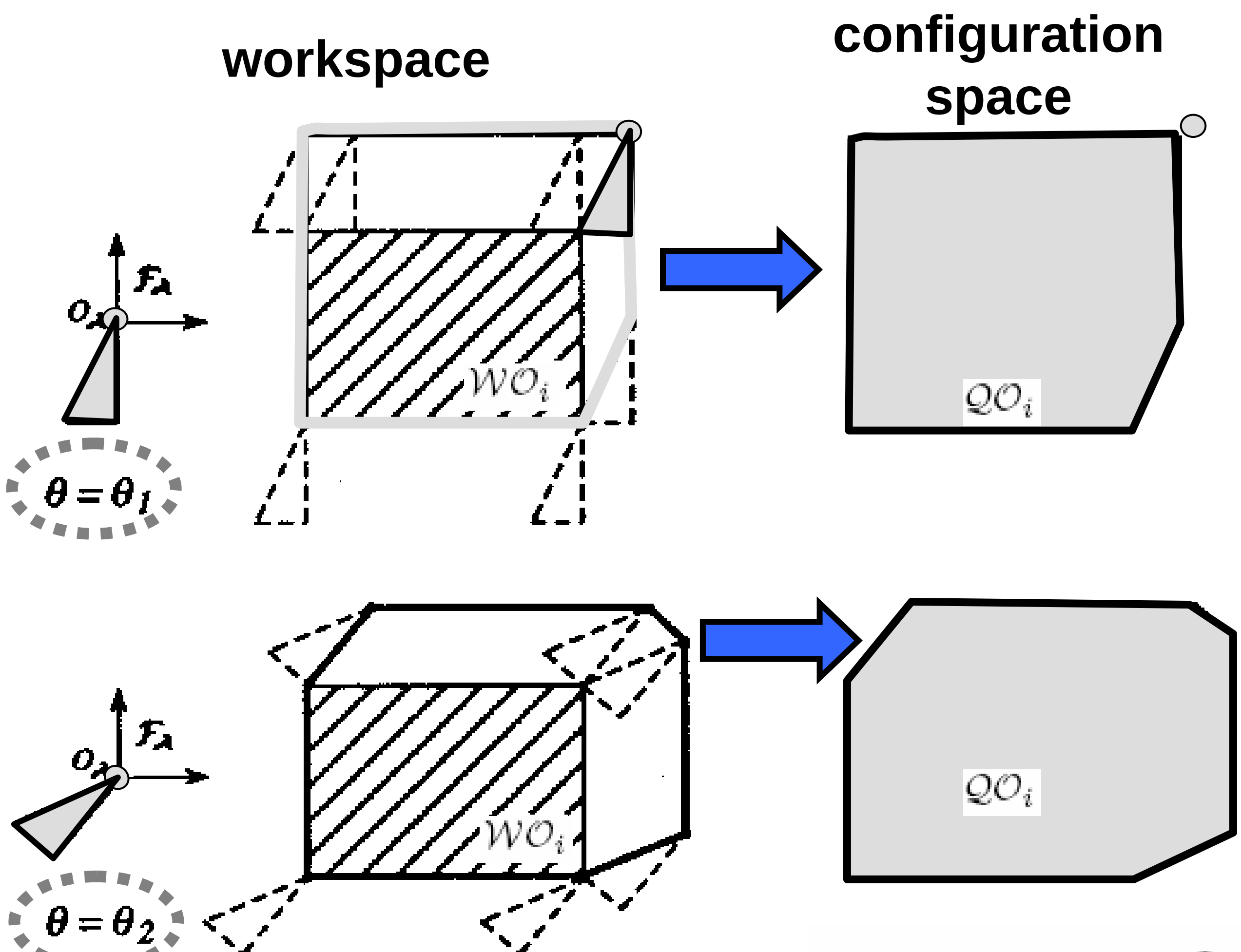
$$\mathcal{QO}_i = \{q \in \mathcal{Q} \mid R(q) \bigcap \mathcal{WO}_i \neq \emptyset\}.$$

Pick a reference point…

Polygonal robot translating in 2-D workspace

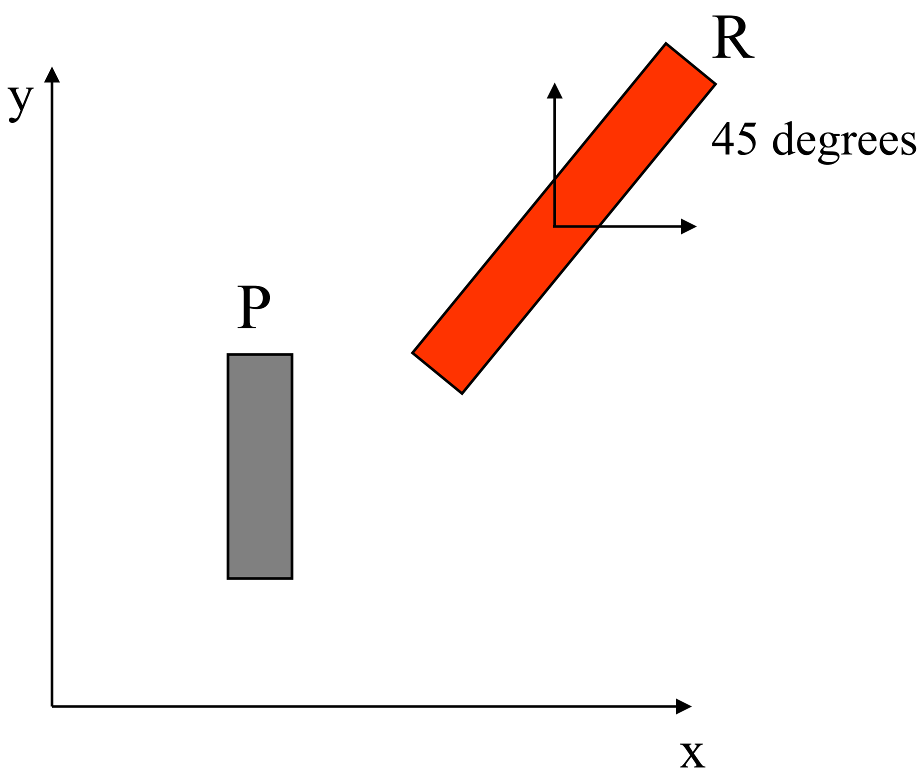


Polygonal robot translating & rotating in 2-D workspace



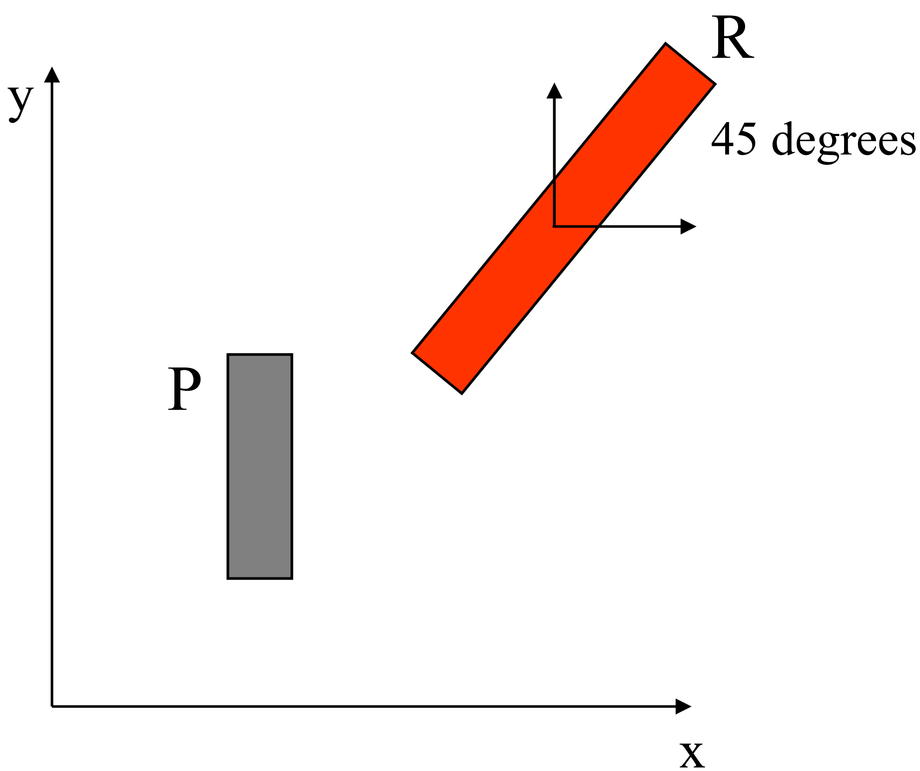
$$\mathcal{QO}_i = \{q \in \mathcal{Q} \mid R(q) \bigcap \mathcal{WO}_i \neq \emptyset\}.$$

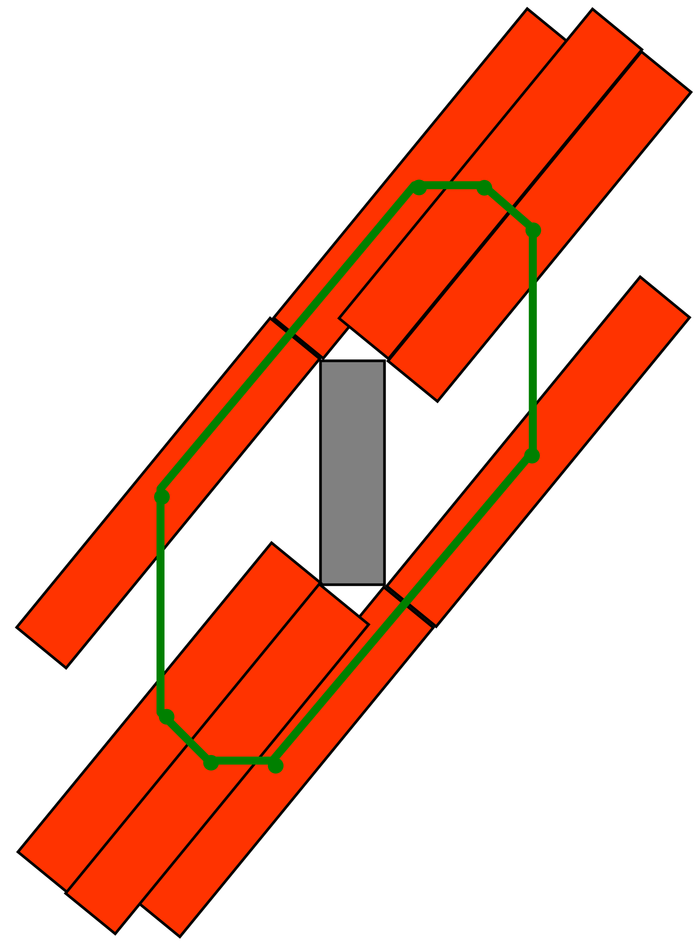
Any reference point



Any reference ~~point~~ configuration

Taking the cross section of configuration space in which the robot is rotated 45 degrees...

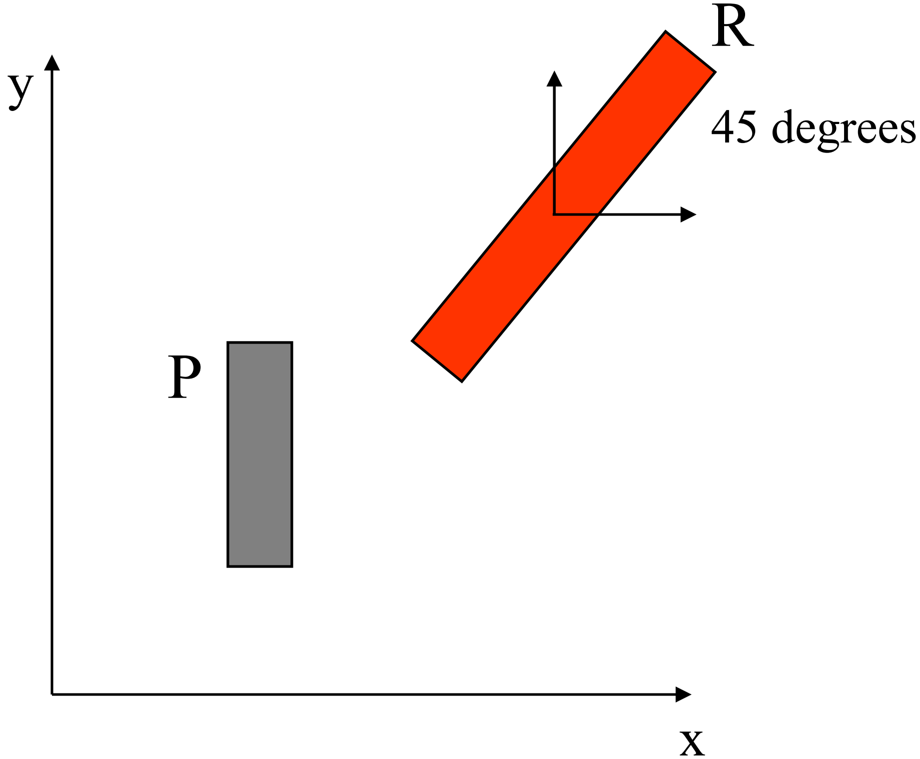


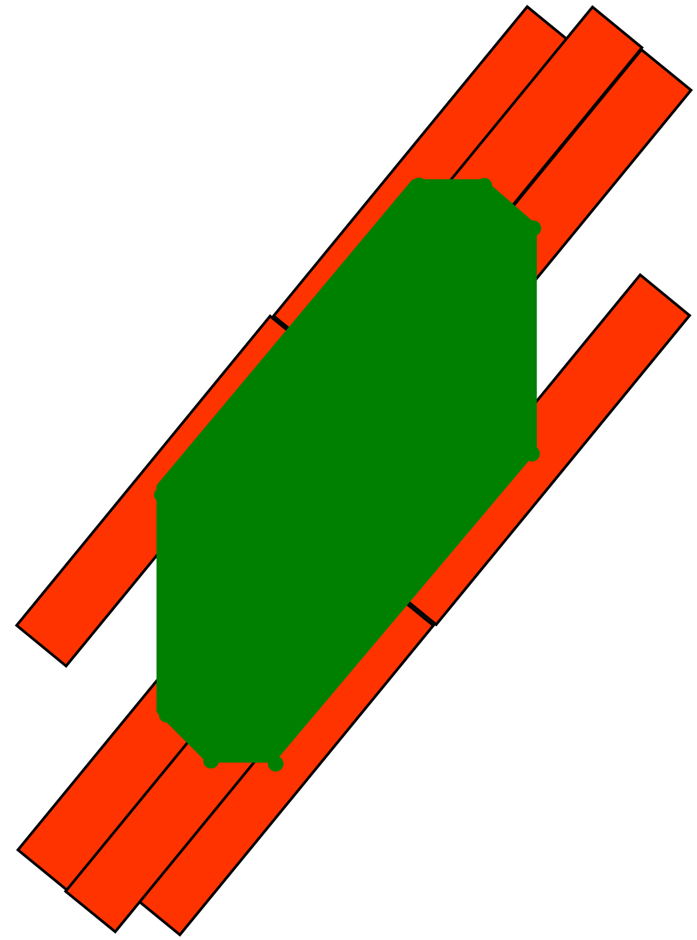


How many sides does P ⊕R have?

Any reference ~~point~~ configuration

Taking the cross section of configuration space in which the robot is rotated 45 degrees...



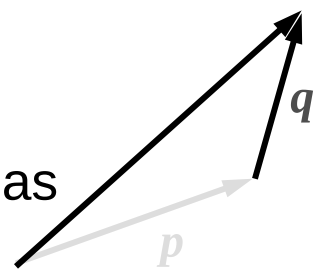


How many sides does P ⊕R have?

Minkowski sum

- The **Minkowski sum** of two sets P and Q, denoted by $P \oplus Q$, is defined as

$$P \oplus Q = \{\, p+q \mid p \in P,\ q \in Q \,\}$$



- Similarly, the **Minkowski difference** is defined as

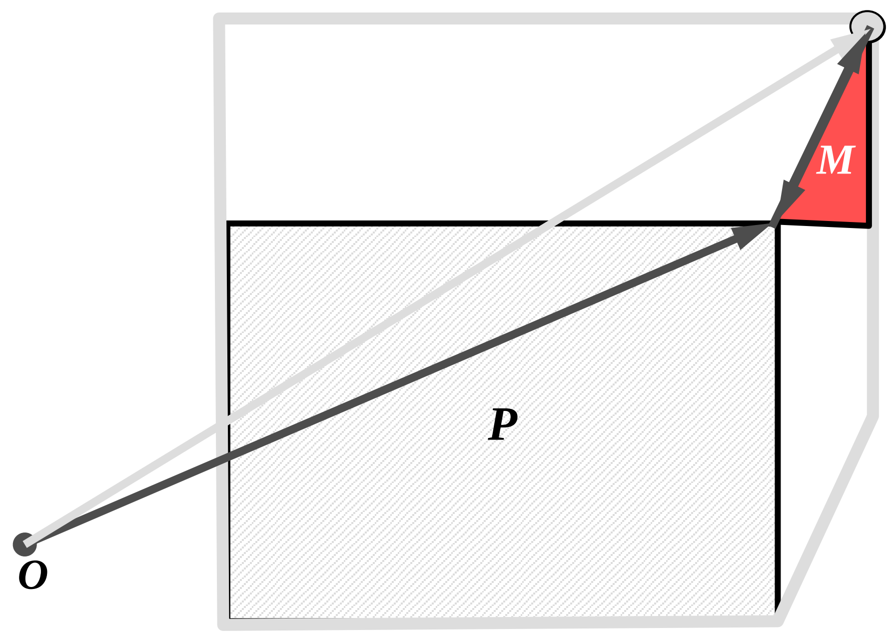
$$P \ominus Q = \{\, p-q \mid p \in P,\ q \in Q \,\}$$

Minkowski sum of convex polygons

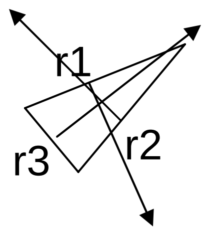
- The Minkowski sum of two convex polygons P and Q of m and n vertices respectively is a convex polygon $P \oplus Q$ of $m + n$ vertices.
  - The vertices of $P \oplus Q$ are the “sums” of vertices of P and Q.

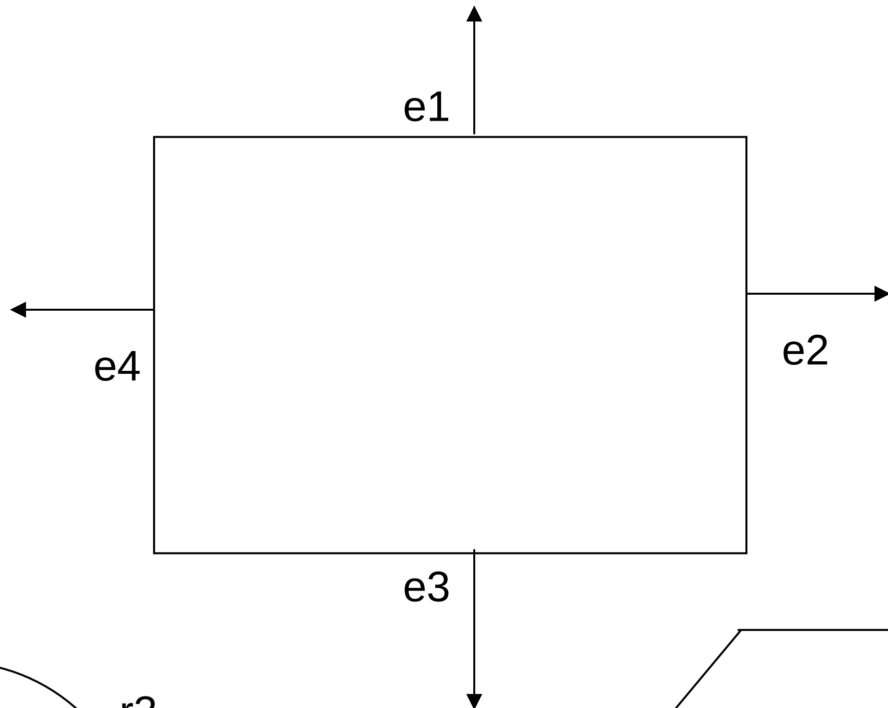
Observation

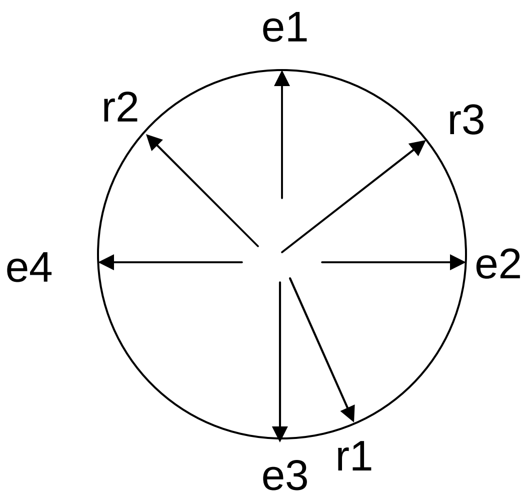
- If P is an obstacle in the workspace and M is a moving object. Then the C-space obstacle corresponding to P is $P \ominus M$.

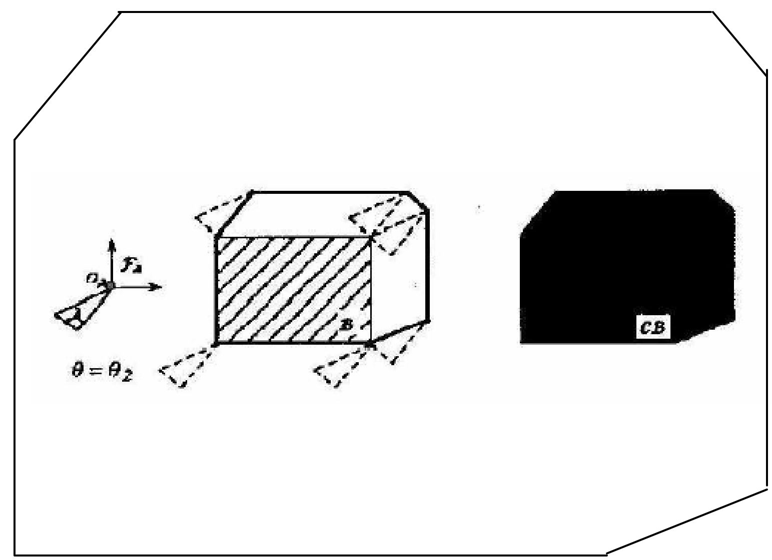


Star Algorithm: Polygonal Obstacles

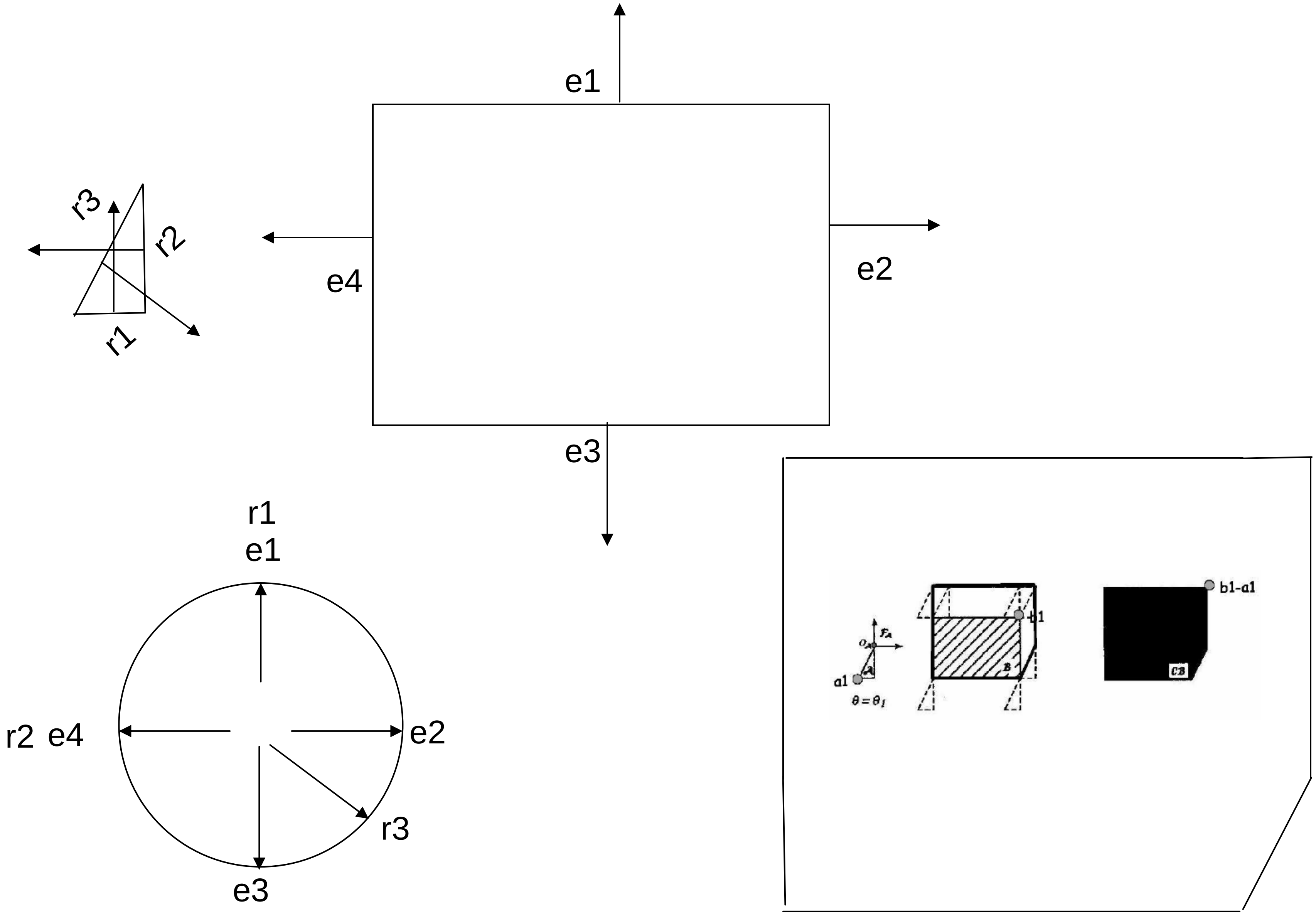






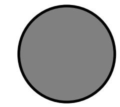


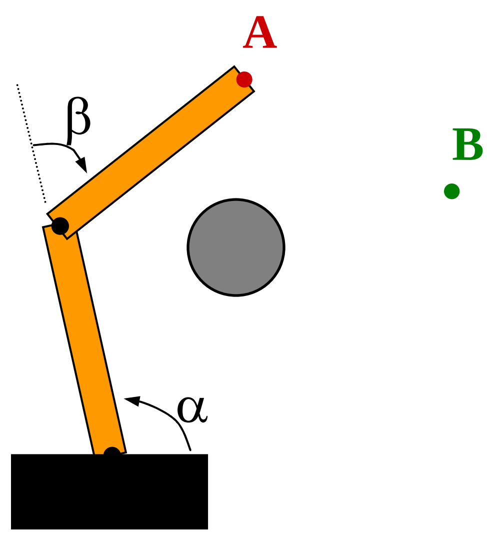
Star Algorithm



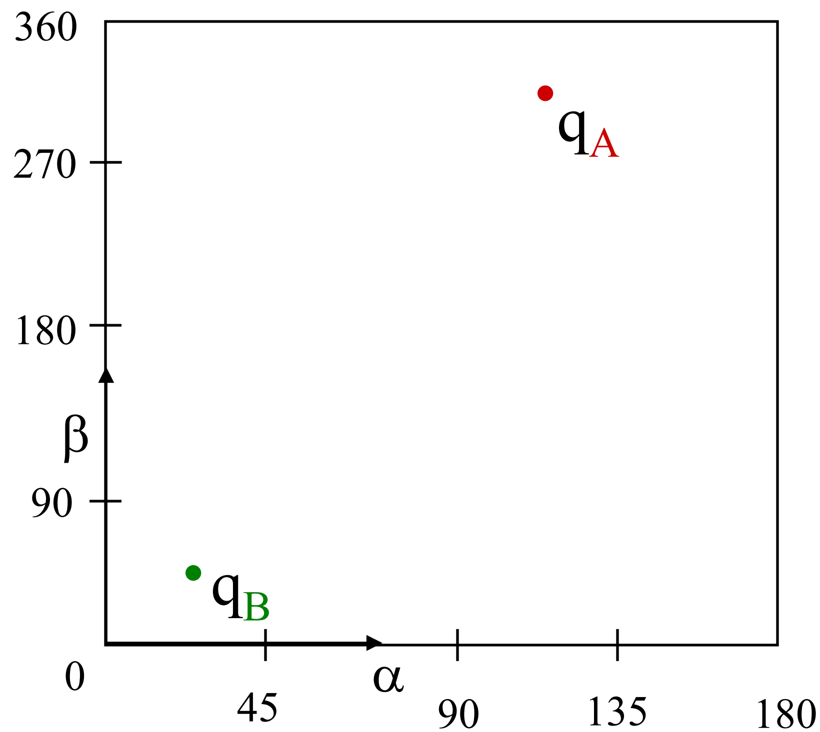
16-735, Howie Choset with slides from G.D. Hager, Z. Dodds, and Dinesh Mocha

Configuration Space "Quiz"

Where do we put  ?



An obstacle in the robot's workspace



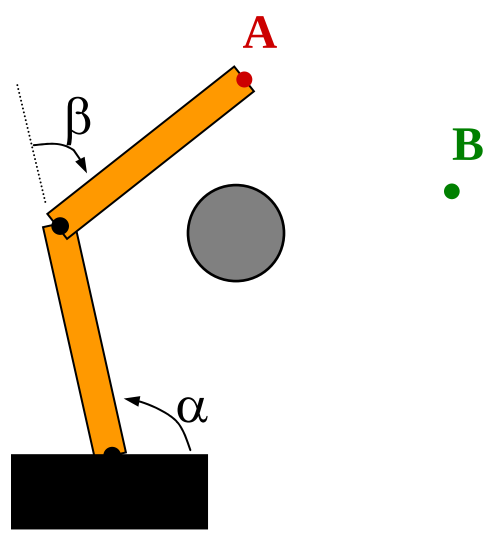
Torus

(wraps horizontally and vertically)

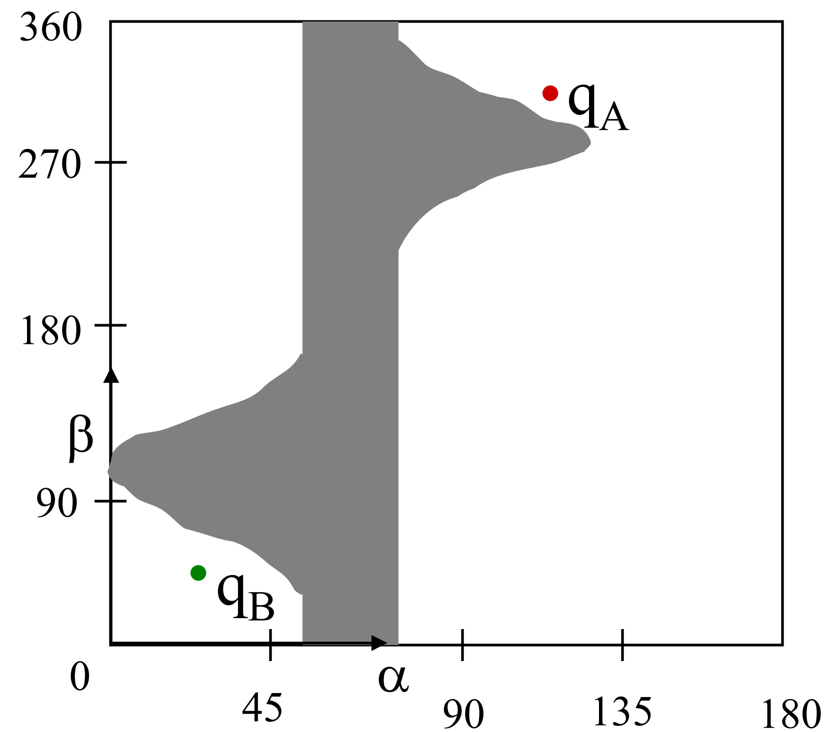
Configuration Space Obstacle

How do we get from **A** to **B** ?

Reference *configuration*

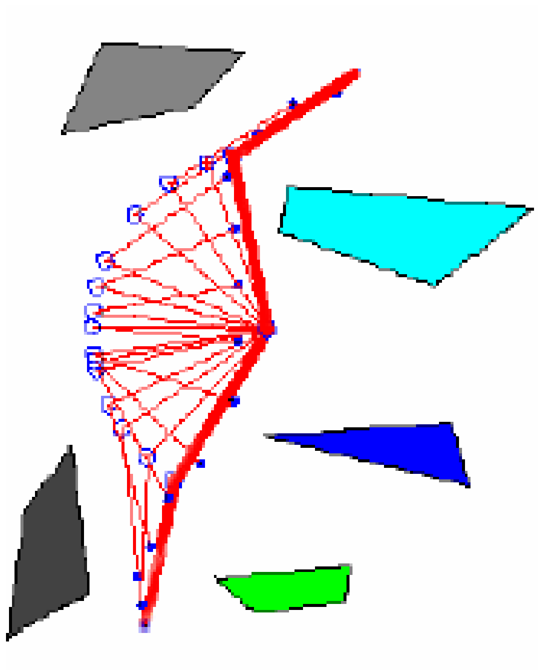


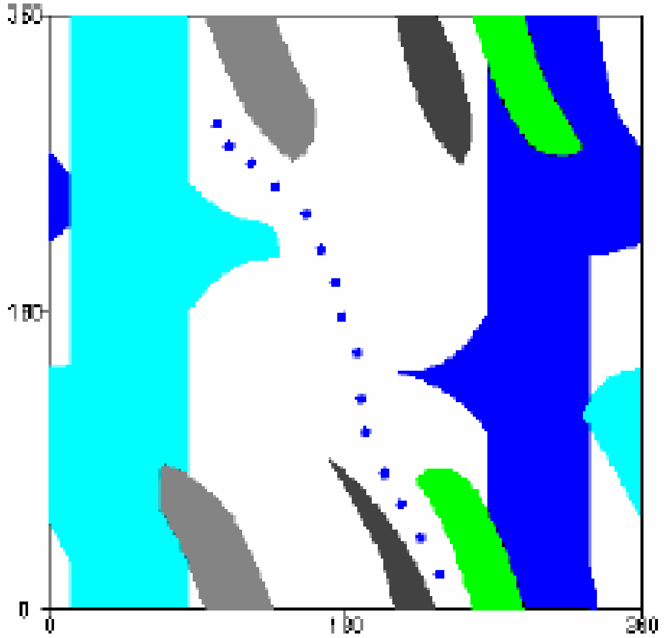
An obstacle in the robot's workspace



The C-space representation of this obstacle…

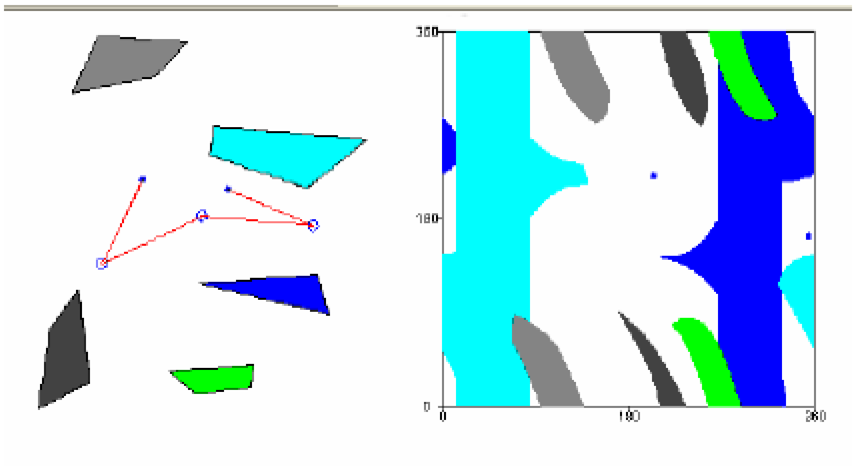
Two Link Path

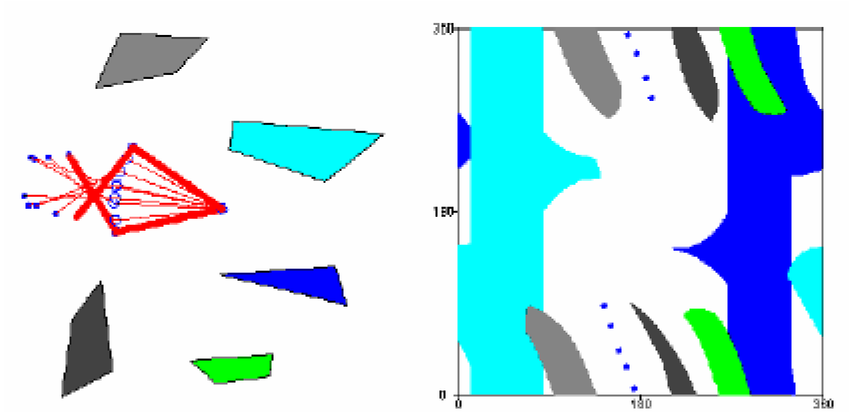




Thanks to Ken Goldberg

Two Link Path





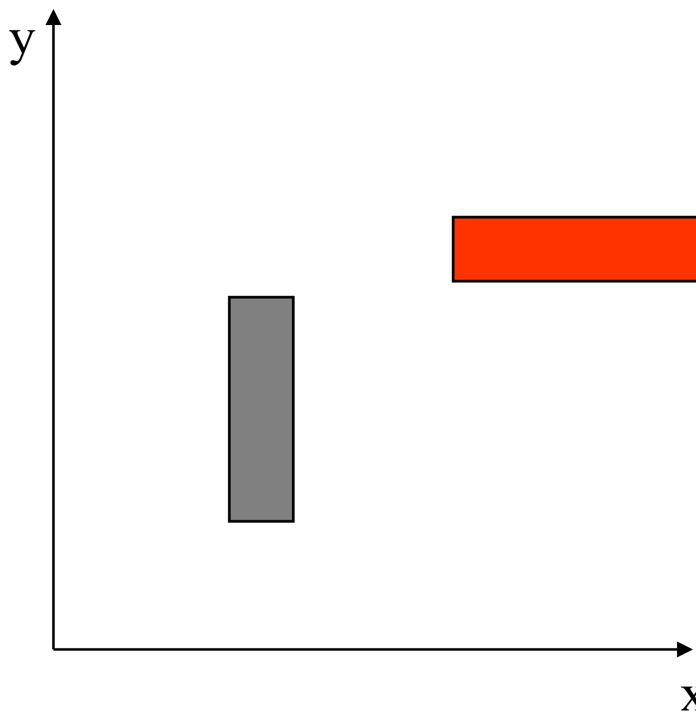
Properties of Obstacles in C-Space

- If the robot and WO_i *are* __________*, then*
  - *Convex then* QO_i is convex
  - *Closed then* QO_i is closed
  - *Compact then* QO_i is compact
  - *Algebraic then* QO_i is algebraic
  - *Connected then* QO_i is connected

Additional dimensions

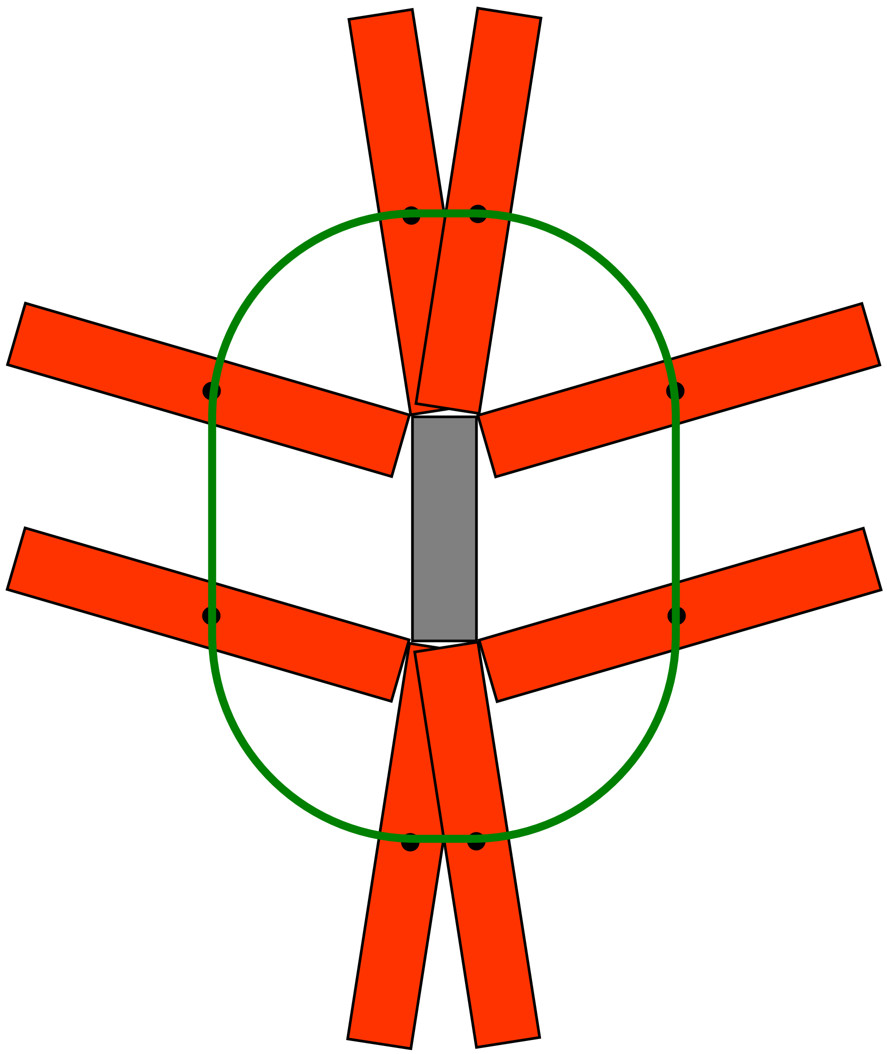
What would the configuration space of a rectangular robot (red) in this world look like? Assume it can translate *and* rotate in the plane.

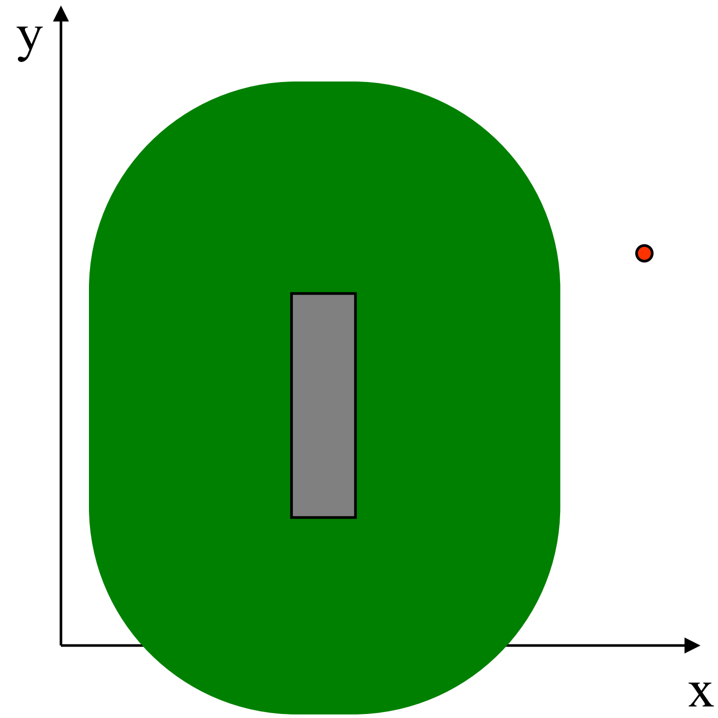
(The blue rectangle is an obstacle.)



a 2d possibility

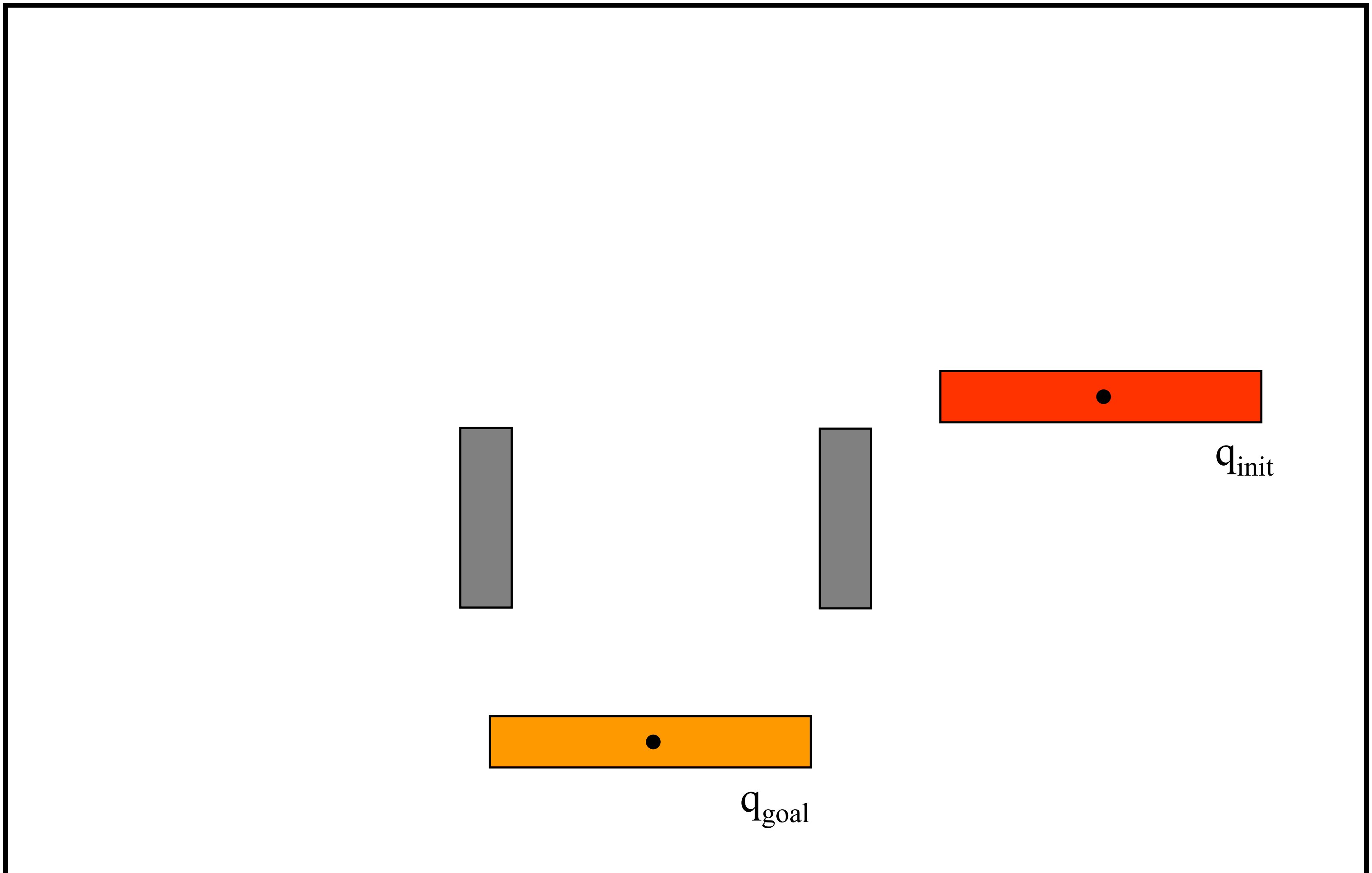
2d projection...





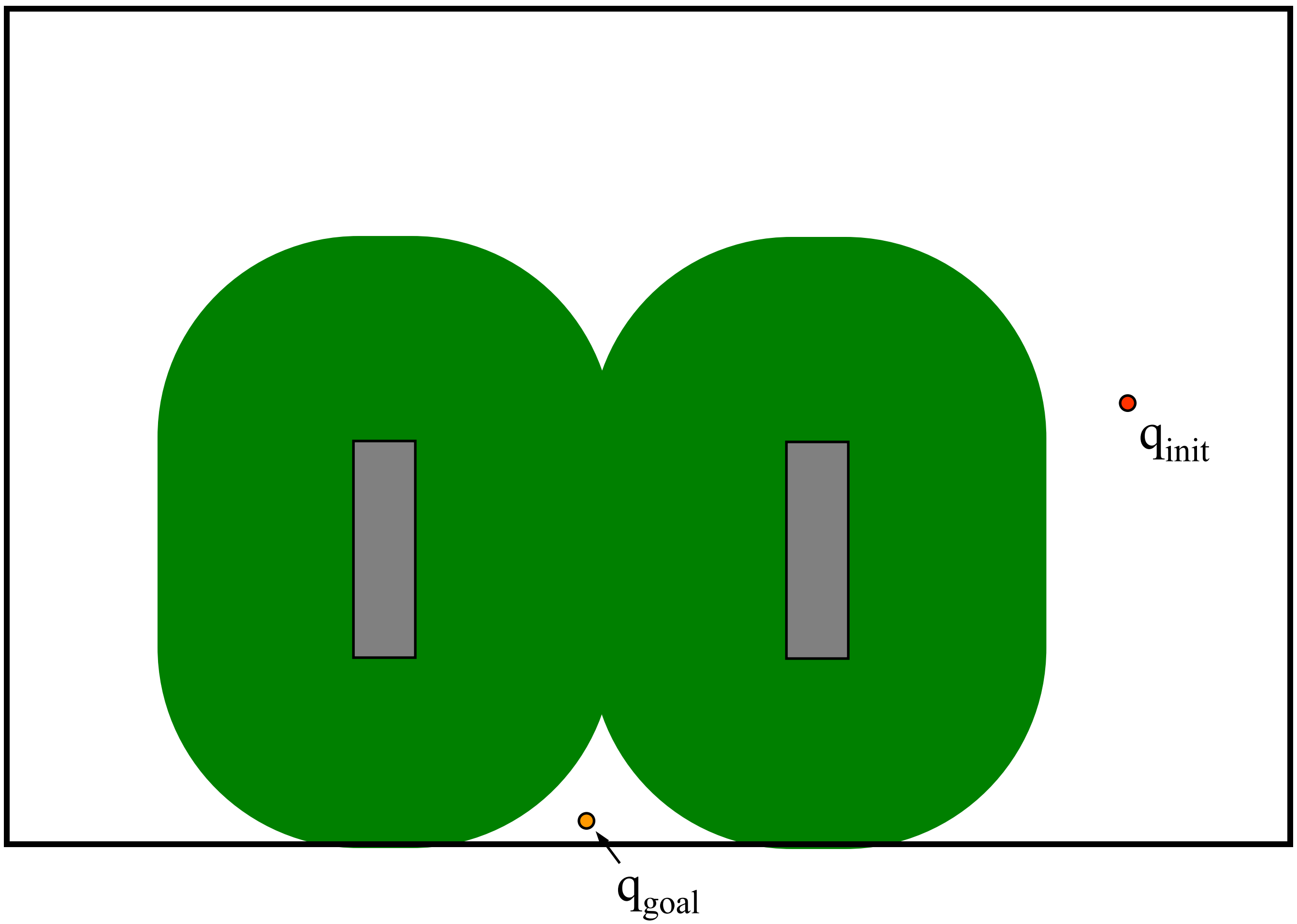
Why not keep it this simple?

A problem?



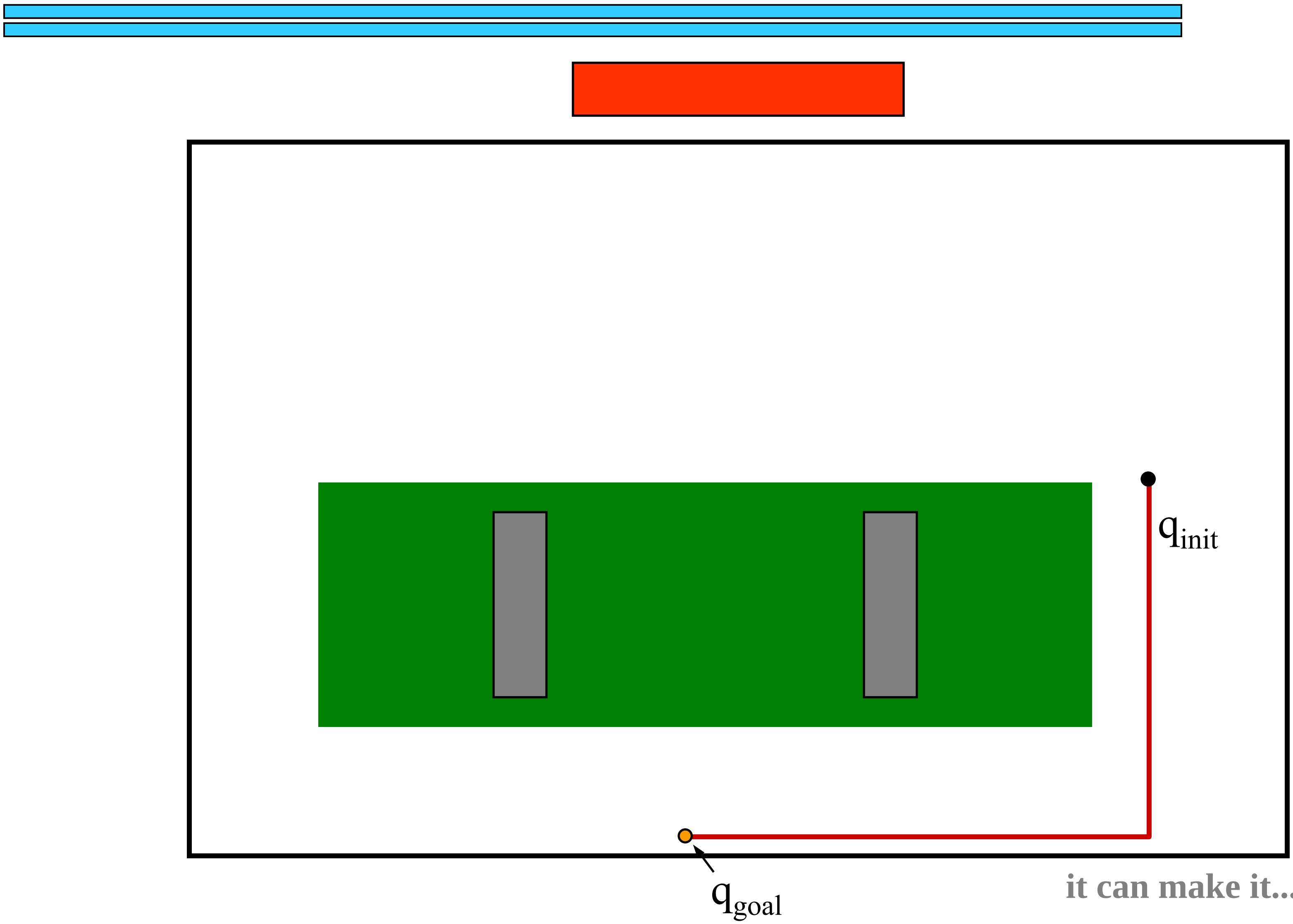
http://www.math.berkeley.edu/~sethian/Applets/java_files_robotic_legal/robotic_legal.html with otherwise straightforward paths

Requires one more d…



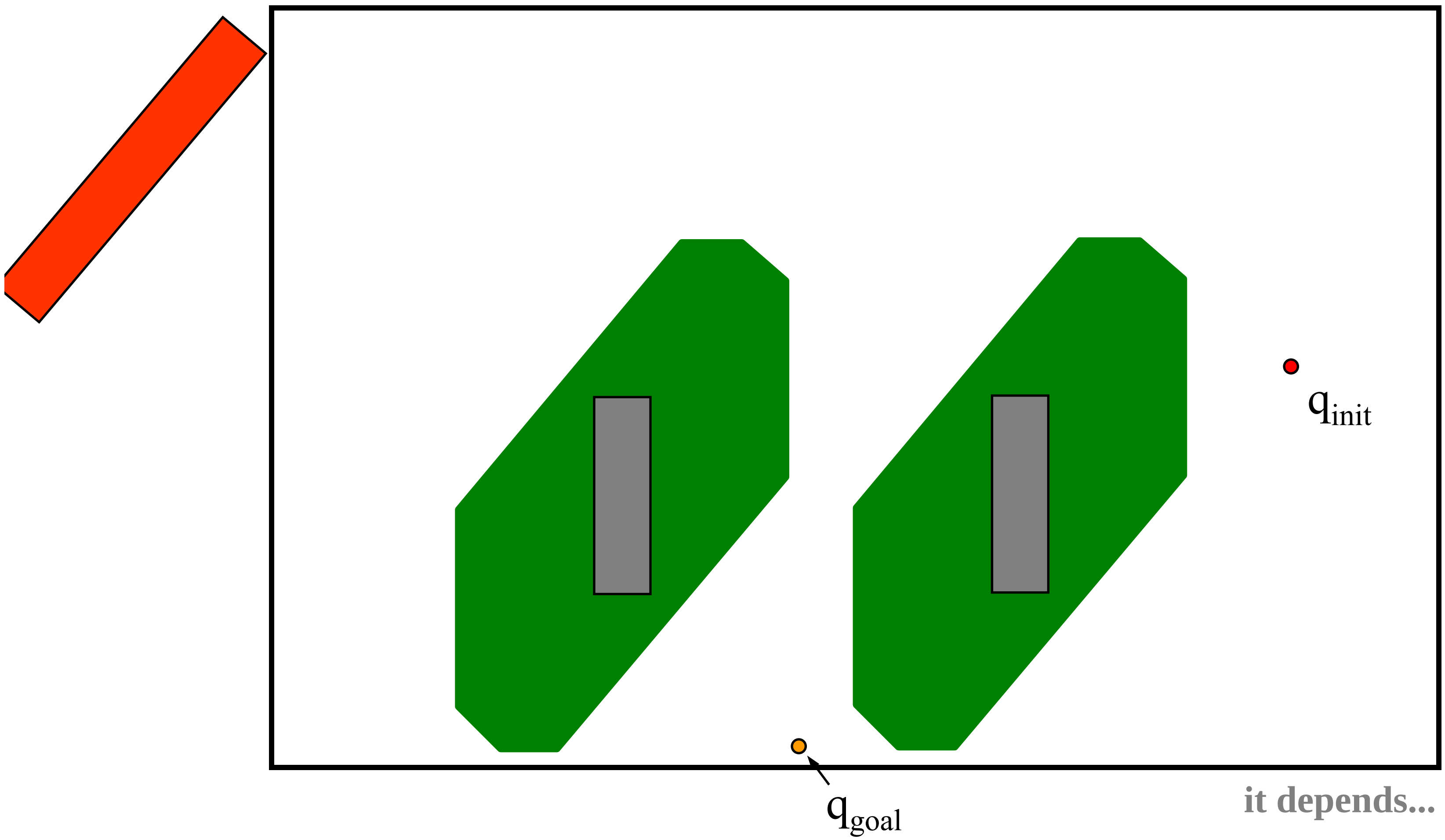
too conservative !
what instead?

When the robot is at one orientation



it can make it...

When the robot is at another orientation

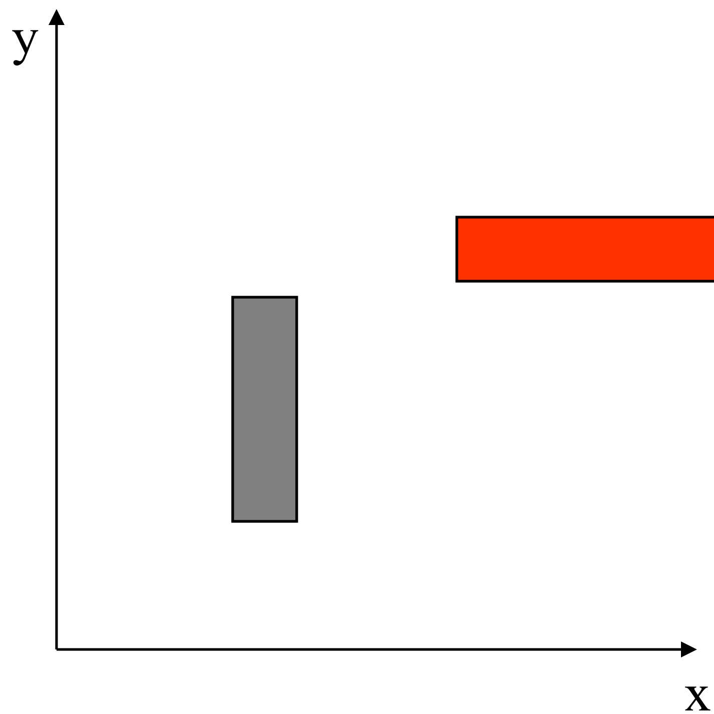


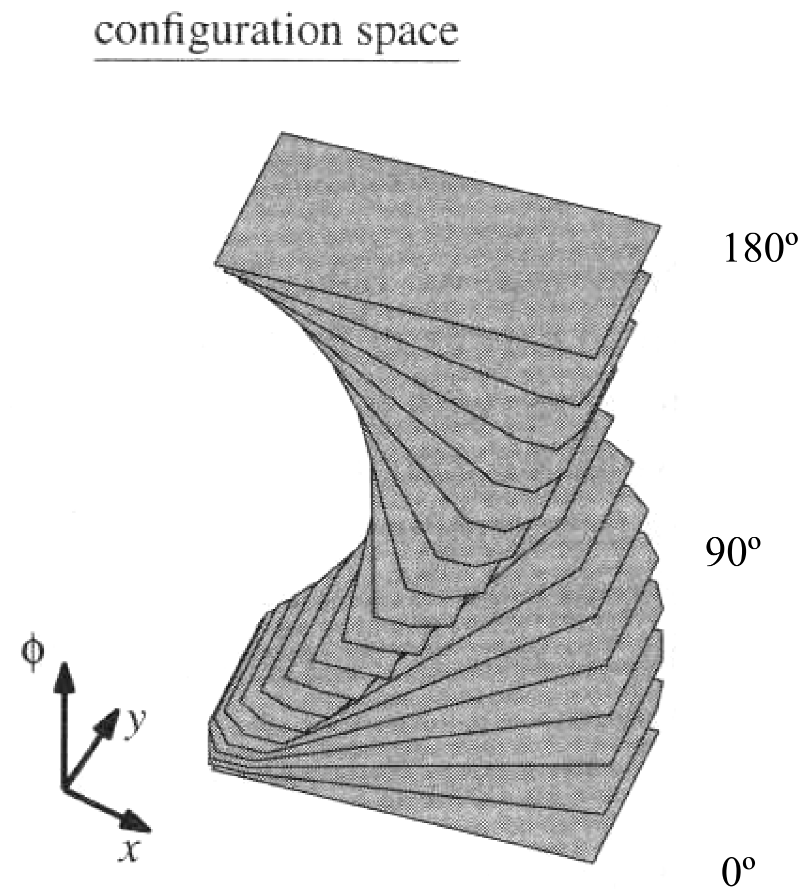
it depends...

Additional dimensions

What would the configuration space of a rectangular robot (red) in this world look like?

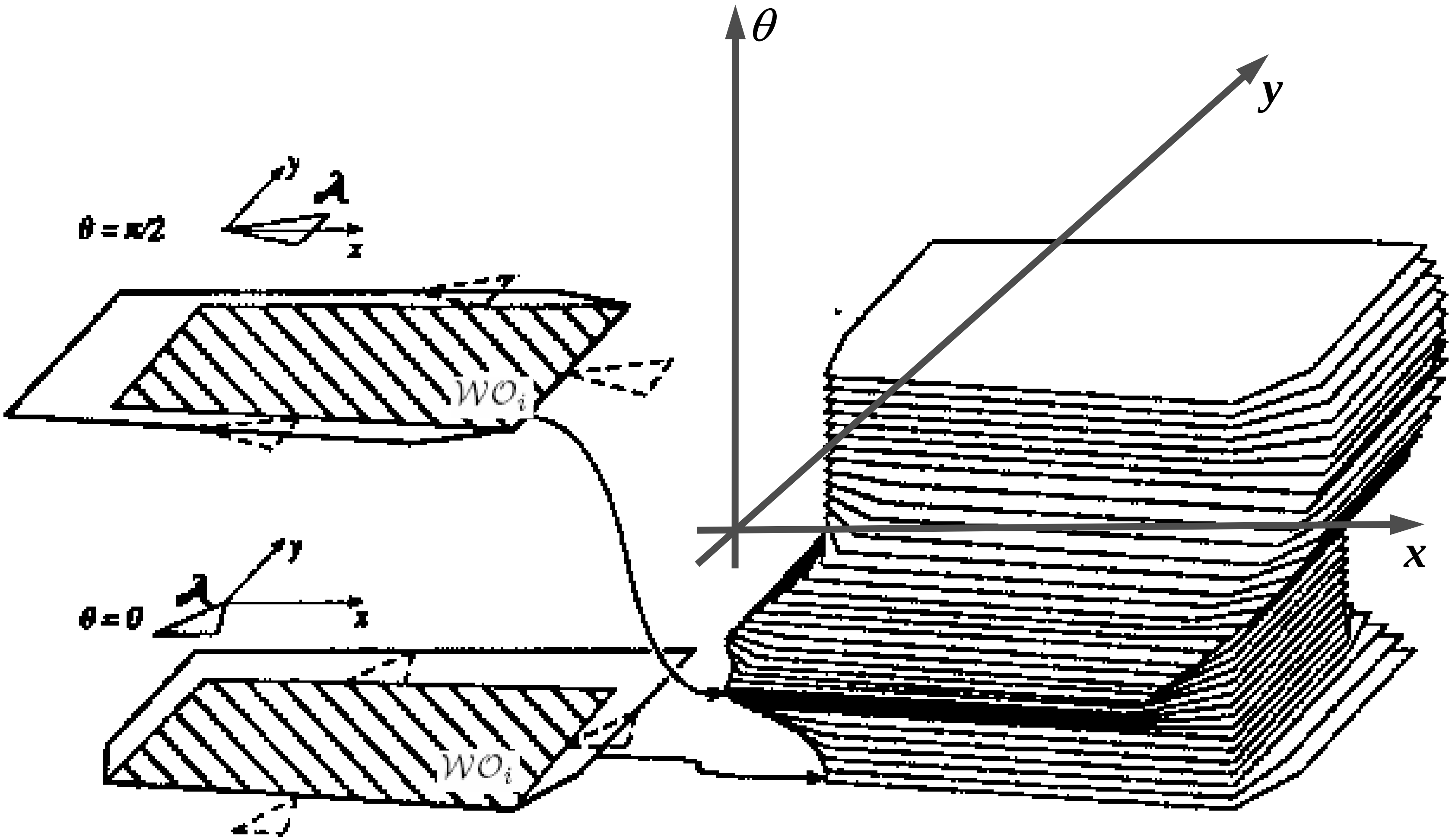
(The obstacle is blue.)



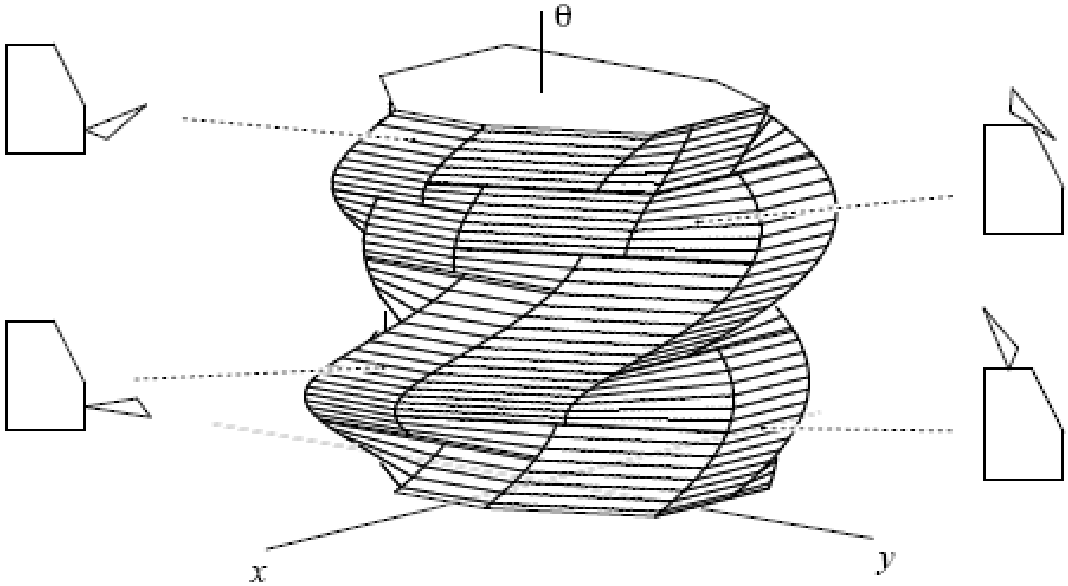


this is twisted...

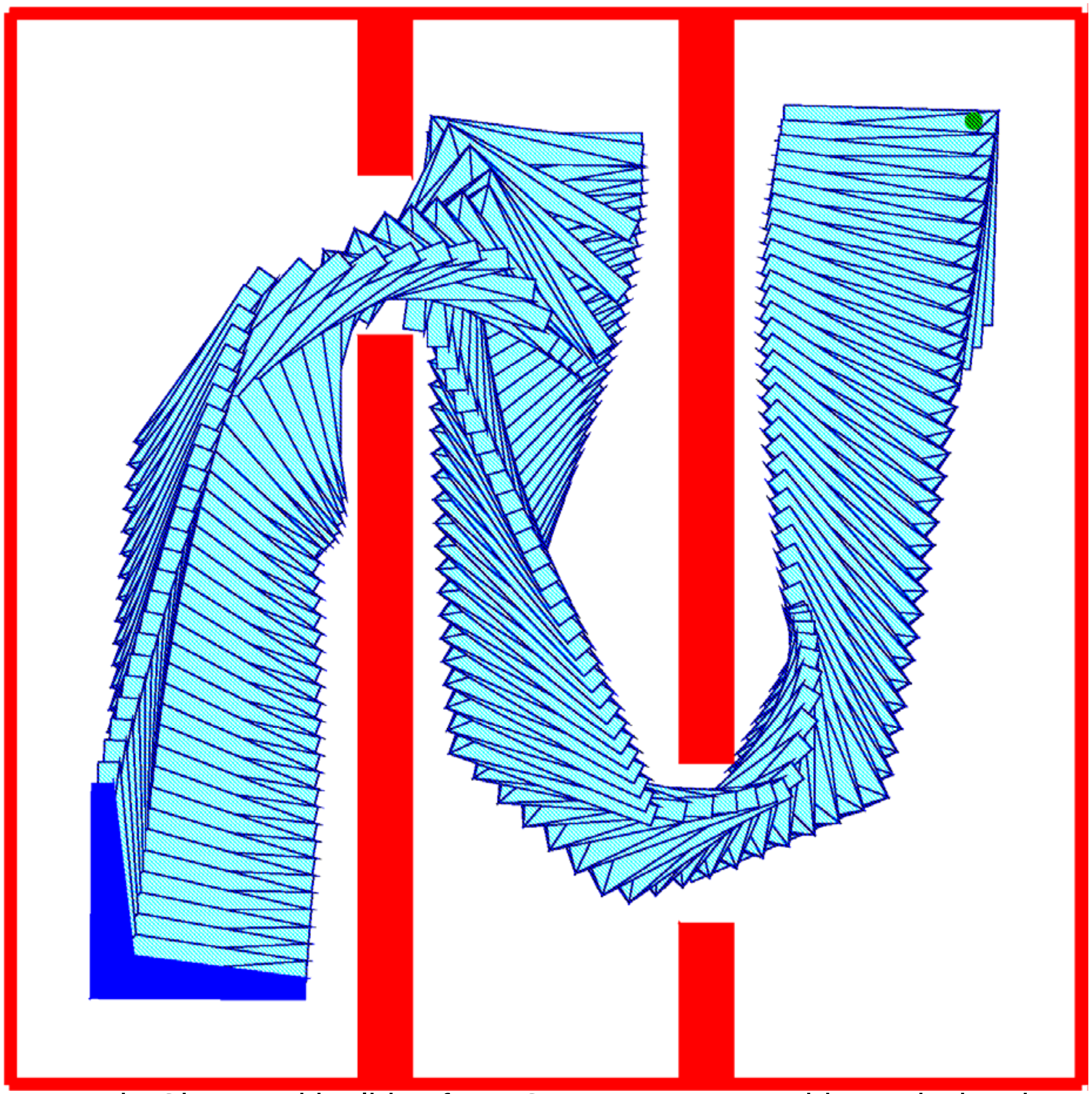
Polygonal robot translating & rotating in 2-D workspace



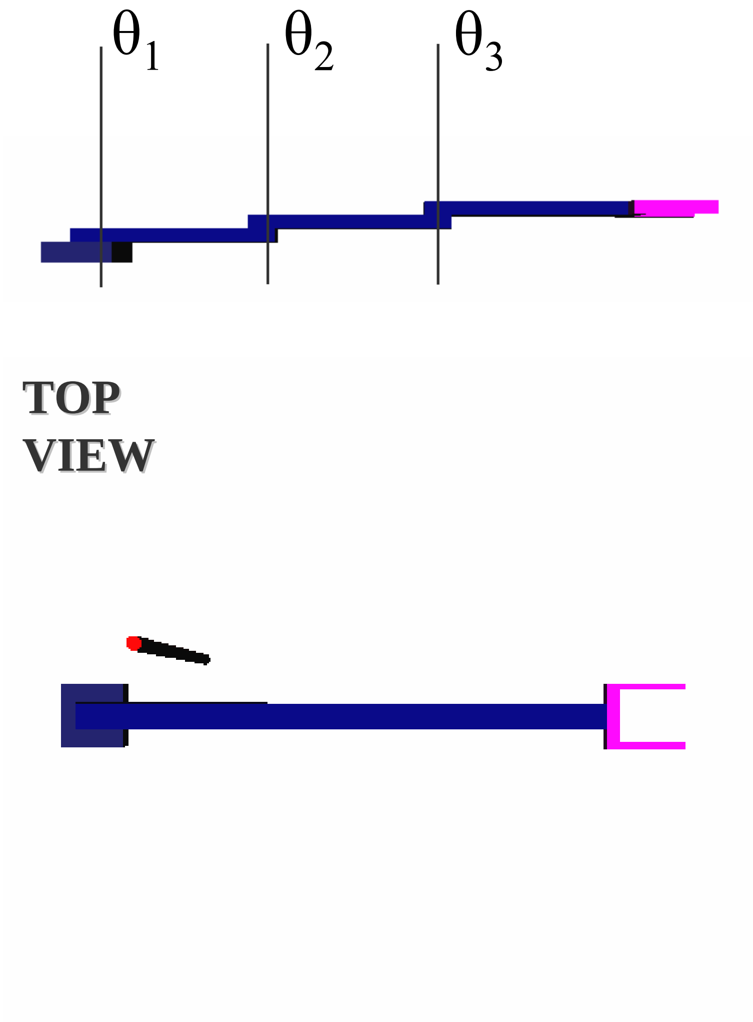
SE(2)



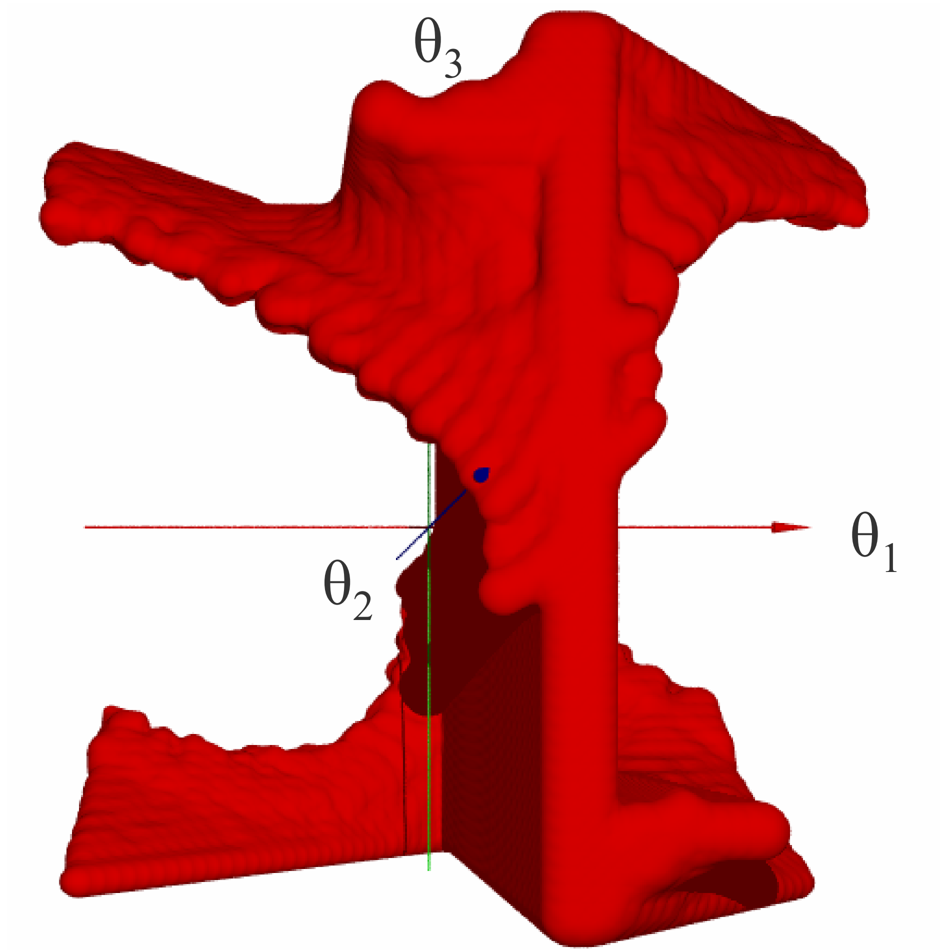
2D Rigid Object



The Configuration Space (C-space)

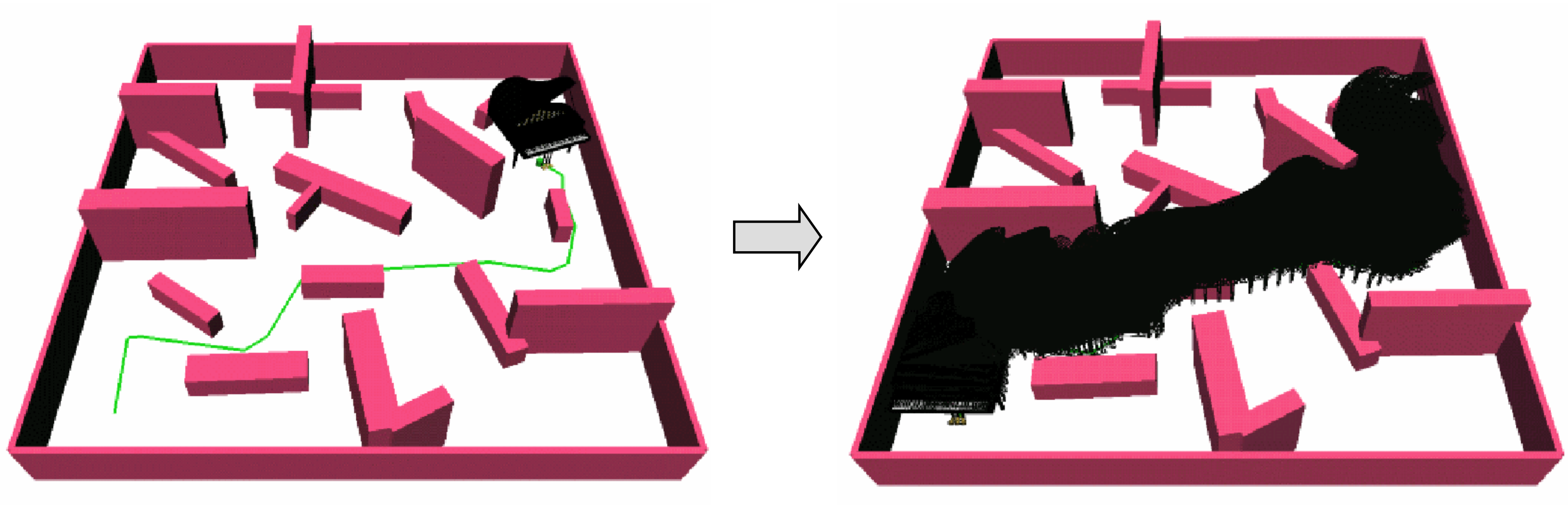


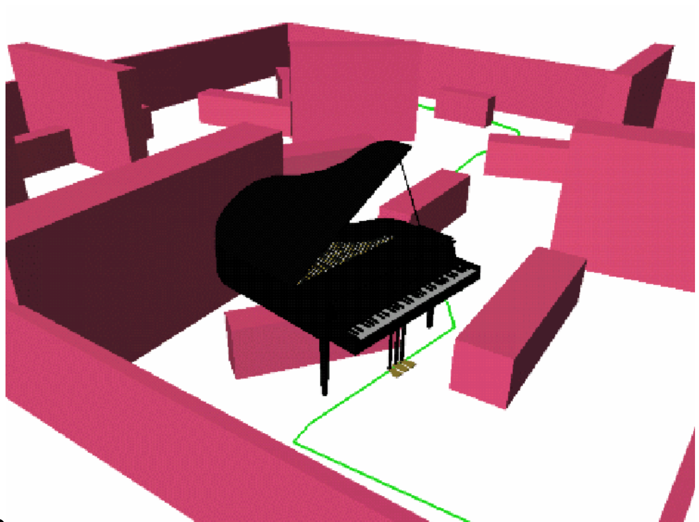
workspace



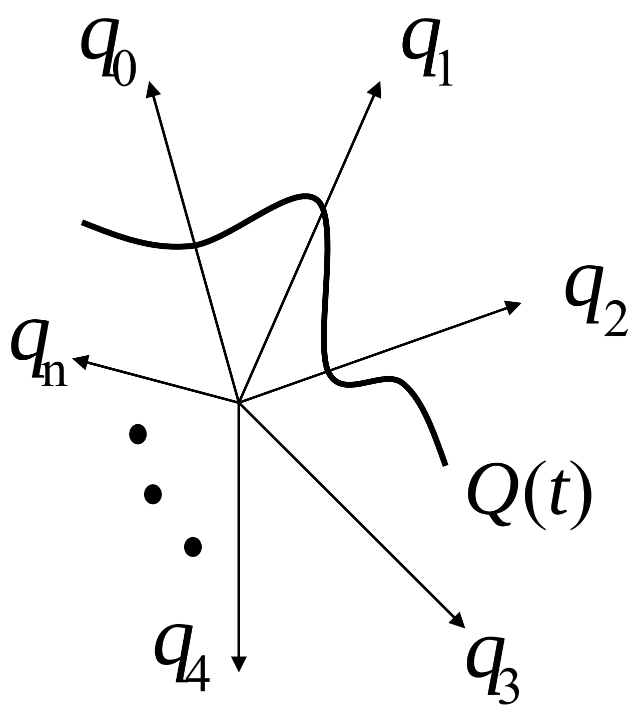
C-space

Moving a Piano

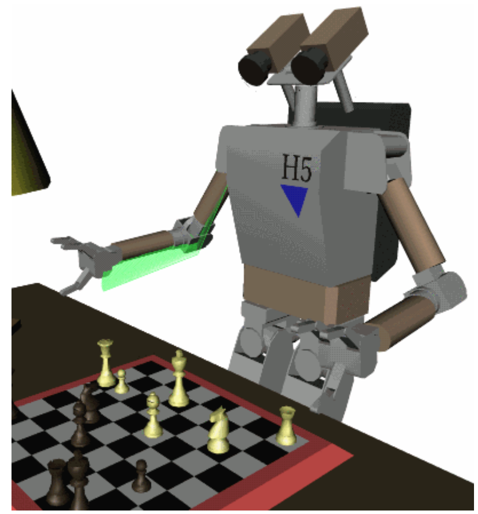




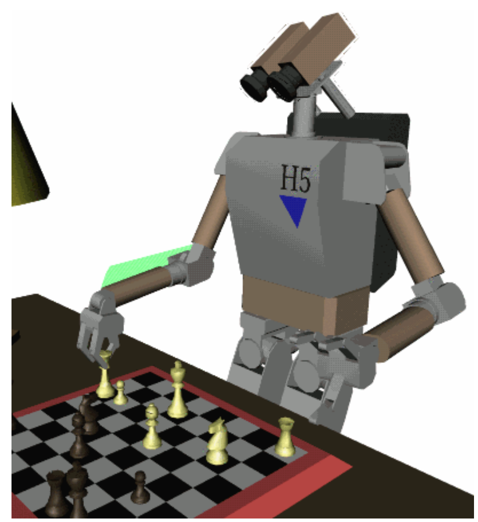
Configuration Space (C-space)



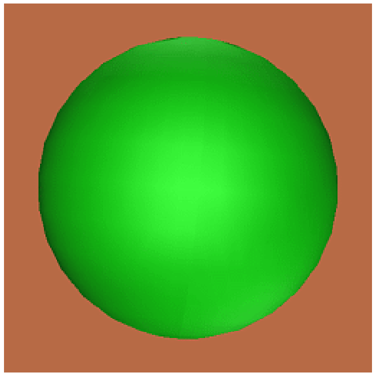
$$Q(t) = \begin{bmatrix} q_0(t) \\ \vdots \\ q_n(t) \end{bmatrix} \quad t \in [0, T]$$



INIT:
$Q(0)$



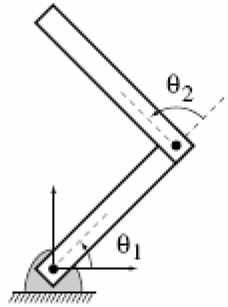
GOAL:
$Q(T)$



Topology?

Sphere?

Torus?



(a)

2R manipulator

Configuration space

Why study the Topology

- Extend results from one space to another: spheres to stars
- Impact the representation
- Know where you are
- Others?

The Topology of Configuration Space

- Topology is the “intrinsic character” of a space
- Two space have a different topology if cutting and pasting is required to make them the same (e.g. a sheet of paper vs. a mobius strip)
  - think of rubber figures --- if we can stretch and reshape “continuously” without tearing, one into the other, they have the same topology
- A basic mathematical mechanism for talking about topology is the homeomorphism.

Homeo- and Diffeomorphisms

- Recall mappings:
  - ϕ: S $\rightarrow$ T
  - If each elements of ϕ goes to a unique T, ϕ is *injective* (or 1-1)
  - If each element of T has a corresponding preimage in S, then ϕ is *surjective* (or onto).
  - If ϕ is surjective and injective, then it is bijective (in which case an inverse, ϕ^{-1} exists).
  - ϕ is *smooth* if derivatives of all orders exist (we say ϕ is C^{∞})
- If ϕ: S $\rightarrow$ T is a bijection, and both ϕ and ϕ^{-1} are continuous, ϕ is a *homeomorphism;* if such a ϕ exists, S and T are *homeomorphic.*
- If homeomorphism where both ϕ and ϕ^{-1} are smooth is a *diffeomorphism.*

Some Examples

- How would you show a square and a rectangle are diffeomorphic?
- How would you show that a circle and an ellipse are diffeomorphic (implies both are topologically S^1)
- Interestingly, a "racetrack" is not diffeomorphic to a circle
  - composed of two straight segments and two circular segments
  - at the junctions, there is a discontinuity; it is therefore not possible to construct a smooth map!
  - How would you show this (hint, do this for a function on $\Re^1$ and think about the chain rule)
  - Is it homeomorphic?

Local Properties

$B_\epsilon(p) = \{p' \in \mathcal{M} \mid d(p, p') < \epsilon\}$ Ball

$p \in \mathcal{M}$ $\quad \mathcal{U} \subseteq \mathcal{M}$ with $p \in \mathcal{U}$ such that for every $p' \in \mathcal{U}$, $\overline{B}_\epsilon(p') \subset \mathcal{U}$. Neighborhood

Manifolds

- A space S *locally diffeomorphic* (homeomorphic) to a space T if each p$\in$ S there is a neighborhood containing it for which a diffeomorphism (homeomorphism) to some neighborhood of T exists.

- S^1 is locally diffeomorphic to $\Re^1$

- The sphere is locally diffeomorphic to the plane (as is the torus)

- A set S is a *k-dimensional manifold* if it is locally **homeomorphic** to $\Re^k$

Charts and Differentiable Manifolds

- A Chart is a pair (U,ϕ) such that U is an open set in a k-dimensional manifold and ϕ is a diffeomorphism from U to some open set in $\Re^k$
  - think of this as a “coordinate system” for U (e.g. lines of latitude and longitude away form the poles).
  - The inverse map is a parameterization of the manifold

- Many manifolds require more than one chart to cover (e.g. the circle requires at least 2)

- An *atlas* is a set of charts that
  - cover a manifold
  - are smooth where they overlap (the book defines the notion of C^∞ related for this; we will take this for granted).

- A set S is a *differentiable manifold of dimension n* if there exists an atlas from S to $\Re^n$
  - For example, this is what allows us (locally) to view the (spherical) earth as flat and talk about translational velocities upon it.

Some Minor Notational Points

- $\Re^1 \times \Re^1 \times ... \times \Re^1 = \Re^n$
- $S^1 \times S^1 \times ... \times S^1 \neq S^n$ (= T^n, the n-dimensional torus)
- S^n is the n-dimensional sphere

- Although S^n is an n-dimensional manifold, it is not a manifold of a single chart --- there is no single, smooth, invertible mapping from S^n to R^n ---
  – they are not ??morphic?

Examples

Type of robot	Representation of Q
Mobile robot translating in the plane	$\mathbb{R}^2$
Mobile robot translating and rotating in the plane	$SE(2)$ or $\mathbb{R}^2 \times S^1$
Rigid body translating in the three-space	$\mathbb{R}^3$
A spacecraft	$SE(3)$ or $\mathbb{R}^3 \times SO(3)$
An n-joint revolute arm	T^n
A planar mobile robot with an attached n-joint arm	$SE(2) \times T^n$

$S^1 \times S^1 \times \ldots \times S^1$ (n times) $= T^n$, the n-dimensional torus

$S^1 \times S^1 \times \ldots \times S^1$ (n times) $\neq S^n$, the n-dimensional sphere in $\mathbb{R}^{n+1}$

$S^1 \times S^1 \times S^1 \neq SO(3)$

$SE(2) \neq \mathbb{R}^3$

$SE(3) \neq \mathbb{R}^6$

What is the Dimension of Configuration Space?

- The dimension is the number of parameter necessary to uniquely specify configuration

- One way to do this is to explicitly generate a parameterization (e.g with our 2-bar linkage)

- Another is to start with too many parameters and add (independent) constraints
  – suppose I start with 4 points in the plane (= 8 parameters), A, B, C, D
  – Rigidity requires d(A,B) = c_1 (1 constraints)
  – Rigidity requires d(A,C) = c_2 and d(B,C) = c_3 (2 constraints)
  – Rigidity requires d(A,D) = c_4 and d(B,D) = c_5 and ??? (?? constraints)

  – HOW MANY D.O.F?

- QUIZ:
  – HOW MANY DOF DO YOU NEED TO MOVE FREELY IN 3-space?

What is the Dimension of Configuration Space?

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- Another is to start with too many parameters and add (independent) constraints
  - suppose I start with 4 points in the plane (= 8 parameters), A, B, C, D
  - Now, require $||A-B|| = c_1$ and $||C-D|| = c_2$ (2 constraints)
  - Now, require B = C (? constraints)
  - Now, fix A = 0 (? constraints)
  - HOW MANY D.O.F?

- QUIZ:
  - HOW MANY DOF DO YOU NEED TO MOVE FREELY IN 3-space?
    - 3+3
  - HOW MANY in 4-space?

More on dimension

$\mathbb{R}^1$ and $SO(2)$ are one-dimensional manifolds;

$\mathbb{R}^2$, S^2 and T^2 are two-dimensional manifolds;

$\mathbb{R}^3$, $SE(2)$ and $SO(3)$ are three-dimensional manifolds;

$\mathbb{R}^6$, T^6 and $SE(3)$ are six-dimensional manifolds.

More Example Configuration Spaces (contrasted with workspace)

- Holonomic robot in plane:
  - workspace $\Re^2$
  - configuration space $\Re^2$

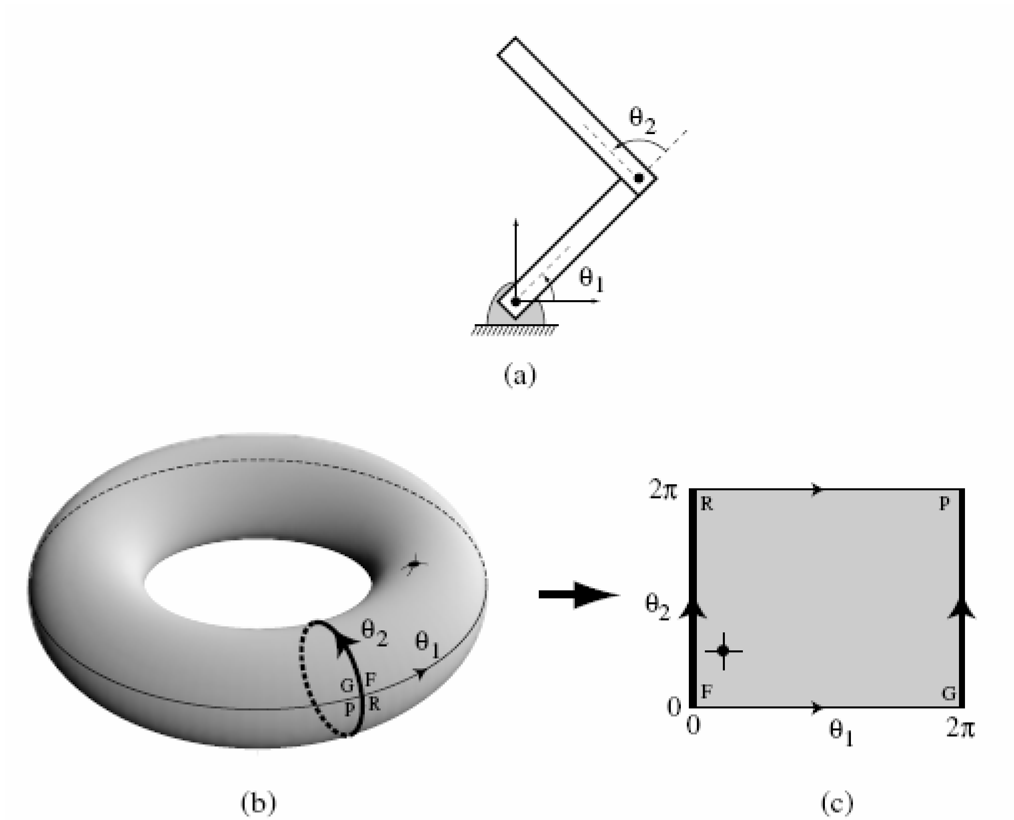
- 3-joint revolute arm in the plane
  - Workspace, a torus of outer radius L1 + L2 + L3
  - configuration space T^3

- 2-joint revolute arm with a prismatic joint in the plane
  - workspace disc of radius L1 + L2 + L3
  - configuration space $T2 \times \Re$

- 3-joint revolute arm mounted on a mobile robot (holonomic)
  - workspace is a “sandwich” of radius L1 + L2 + L3
  - $\Re^2 \times T^3$

- 3-joint revolute arm floating in space
  - workspace is $\Re^3$
  - configuration space is T^3

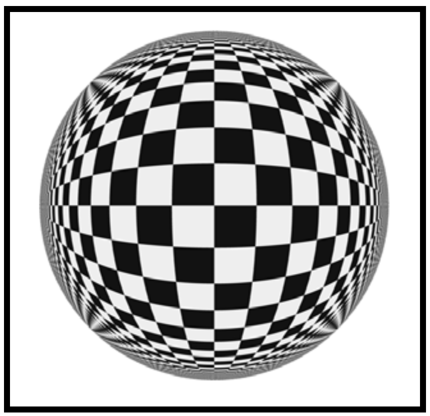
Parameterization of Torus



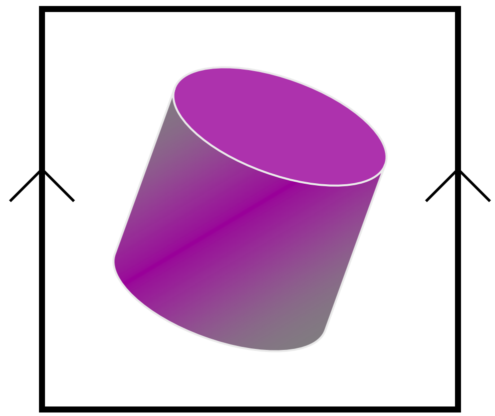
$(\theta_1, \theta_2) \in \mathbb{R}^2$,

problems at $\theta_i = \{0, 2\pi\}$

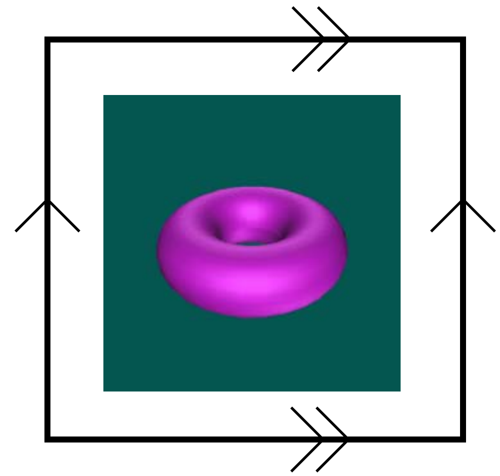
2d Manifolds



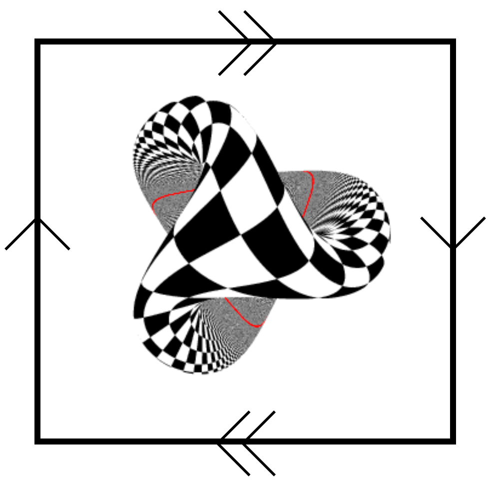
real plane



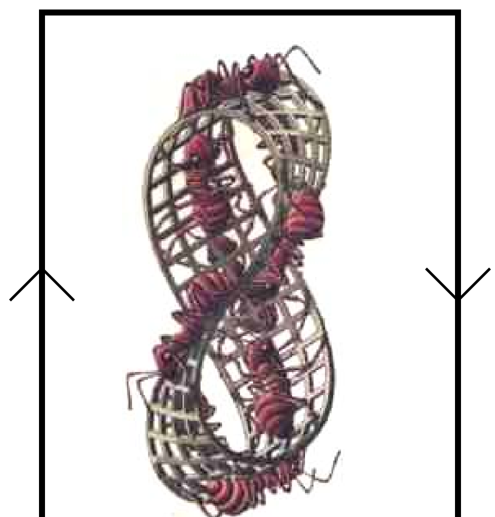
cylinder



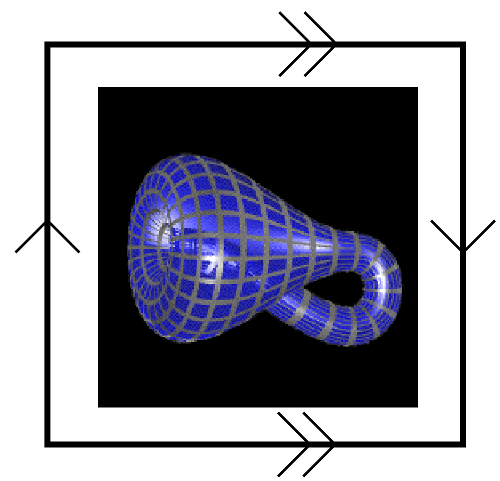
torus



projective plane



mobius strip



klein bottle

Representing Rotations

- Consider S^1 --- rotation in the plane
- The action of a rotation is to, well, rotate --> $R_\theta: \Re^2 \rightarrow \Re^2$
- We can represent this action by a matrix R that is applied (through matrix multiplication) to points in $\Re^2$

$$\begin{matrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{matrix}$$

- Note, we can either think of rotating a point through an angle, or rotate the **coordinate system (or frame)** of the point.

Geometric Transforms

Now, using the idea of homogeneous transforms, we can write:

$$p' = \begin{pmatrix} & R & & T \\ 0 & 0 & & 1 \end{pmatrix} p$$

The group of rigid body rotations SO(2) $\times$ $\Re$(2) is denoted SE(2) (for special Euclidean group)

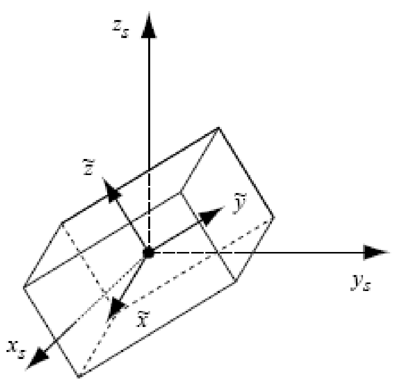
$$R = \begin{bmatrix} \tilde{x}_1 & \tilde{y}_1 \\ \tilde{x}_2 & \tilde{y}_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in SO(2).$$

This space is a type of torus

From 2D to 3D Rotation

- I can think of a 3D rotation as a rotation about different axes:
  - rot(x,θ) rot(y,θ) rot(z,θ)
  - there are many conventions for these (see Appendix E)
    - Euler angles (ZYZ) --- where is the singularity (see eqn 3.8)
    - Roll Pitch Yaw (ZYX)
    - Angle axis
    - Quaternion

- The space of rotation matrices has its own special name: SO(n) (for special orthogonal group of dimension n). It is a manifold of dimension n



$$R = \begin{bmatrix} \tilde{x}_1 & \tilde{y}_1 & \tilde{z}_1 \\ \tilde{x}_2 & \tilde{y}_2 & \tilde{z}_2 \\ \tilde{x}_3 & \tilde{y}_3 & \tilde{z}_3 \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix} \in SO(3)$$

- What is the derivative of a rotation matrix?
  - A tricky question -- what is the topology of that space :)

Geometric Transforms

Now, using the idea of homogeneous transforms, we can write:

$$p' = \begin{pmatrix} & R & & T \\ 0 & 0 & 0 & 1 \end{pmatrix} p$$

The group of rigid body rotations SO(3) $\times$ $\Re$(3) is denoted SE(3) (for special Euclidean group)

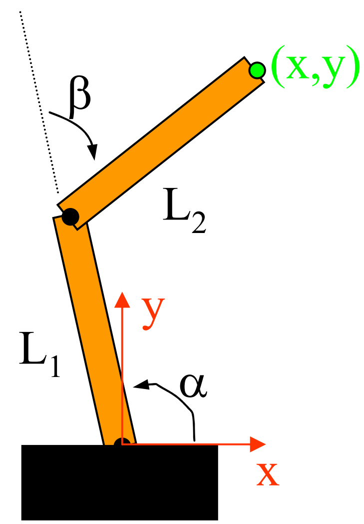
$$SE(n) \equiv \begin{bmatrix} SO(n) & \mathbb{R}^n \\ 0 & 1 \end{bmatrix}$$

What does the inverse transformation look like?

Transforming Velocity

- Recall forward kinematics K: Q $\rightarrow$ *W*
- The *Jacobian* of K is the n $\times$ m matrix with entries
  - $J_{i,j} = d\,K_i \,/\, d\,q_j$
- The Jacobian transforms velocities:
  - dw/dt = J dq/dt
- If square and invertible, then
  - $dq/dt = J^{-1}\, dw/dt$
- Example: our favorite two-link arm...

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_1 c_\alpha \\ L_1 s_\alpha \end{pmatrix} + \begin{pmatrix} L_2 c_{\alpha+\beta} \\ L_2 s_{\alpha+\beta} \end{pmatrix}$$



A Useful Observation

- The Jacobian maps configuration velocities to workspace velocities
- Suppose we wish to move from a point A to a point B in the workspace along a path p(t) (a mapping from some time index to a location in the workspace)
  - dp/dt gives us a velocity profile --- how do we get the configuration profile?
  - Are the paths the same if choose the shortest paths in workspace and configuration space?

Summary

- Configuration spaces, workspaces, and some basic ideas about topology
- Types of robots: holonomic/nonholonomic, serial, parallel
- Kinematics and inverse kinematics
- Coordinate frames and coordinate transformations
- Jacobians and velocity relationships

T. Lozano-Pérez.
Spatial planning: A configuration space approach.
IEEE Transactions on Computing, C-32(2):108-120, 1983.

A Few Final Definitions

- A manifold is *path-connected* if there is a path between any two points.
- A space is *compact* if it is closed and bounded
  - configuration space might be either depending on how we model things
  - compact and non-compact spaces cannot be diffeomorphic!
- With this, we see that for manifolds, we can
  - live with “global” parameterizations that introduce odd singularities (e.g. angle/elevation on a sphere)
  - use atlases
  - embed in a higher-dimensional space using constraints
- Some prefer the later as it often avoids the complexities associated with singularities and/or multiple overlapping maps