CSE276C - Optimization

Henrik I. Christensen

Computer Science and Engineering
University of California, San Diego
http://cri.ucsd.edu

October 2024

Outline

Introduction

- We have discussed approximation and root finding. We can leverage these methods to study optimization.
- Most of robotics is about optimization
- Best trajectory between two points
- Best fit of a model to a swarm of data
- Optimal coverage of an area for fire monitoring
- Energy efficient travel from San Diego to Hawaii by water

Literature

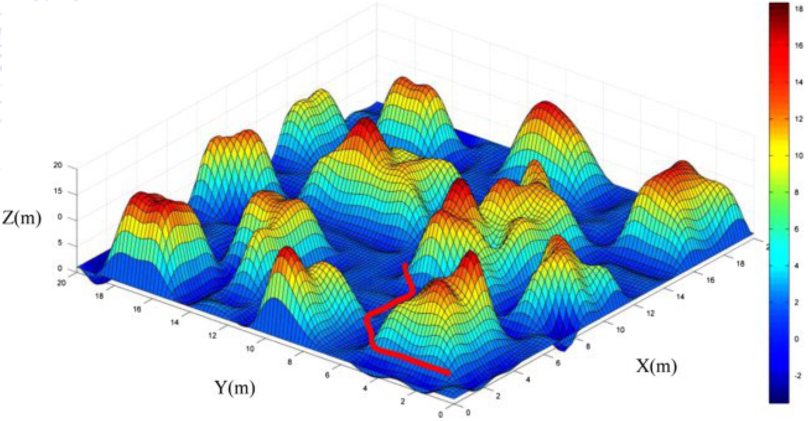
- Numerical Recipes: Chapter 10
- Numerical Renaissance: Chap 14-16. (Part III)

Example 1

- Optimization of trajectories at high speed

Path Planning

- Example potential field



Optimization

- So what is optimization?

Optimization

- So what is optimization?
- Finding extrema for a function over a domain
- Minimum or maximum is immaterial as we can use f or -f
- In many cases we will have local and global extrema
- Consider both deterministic and stochastic approaches

Outline

Golden section

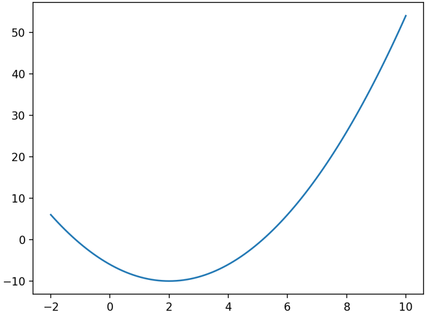
- For bracketing of roots we use bi-section as a basis.
- We can use a similar technique to find an extremum
- We need two points to bracket a root!
- How many points do we need to bracket an extremum?

Golden section

- For bracketing of roots we use bi-section as a basis.
- We can use a similar technique to find an extremum
- We need two points to bracket a root!
- How many points do we need to bracket an extremum?
- We need three points to bracket.
- If we have a triplet $a < b < c$. Iff f(b) is smaller than f(a) and f(c), then we have a minimum within $[a, c]$

Golden Section

- Pick a point between (a,b) or (b,c) and evaluate
- Suppose $x \in (b, c)$ and $f(x) < f(b)$ then our new triple is (b, x, c)
- Consider the function



- How would you choose a new value of x?

Golden Section (cont.)

- Consider (a, b, c)

$$\frac{b-a}{c-a} = w \qquad \frac{c-b}{c-a} = 1 - w$$

- Lets assume $x \in (b, c)$ and

$$\frac{x-b}{c-a} = z$$

- The next bracket is then w+z or 1-w

Golden Section (cont.)

- Consider (a, b, c)

$$\frac{b-a}{c-a} = w \qquad \frac{c-b}{c-a} = 1 - w$$

- Lets assume $x \in (b, c)$ and

$$\frac{x-b}{c-a} = z$$

- The next bracket is then w+z or 1-w
- If we want to make the intervals equal

$$z = 1 - 2w \text{ when } w < \frac{1}{2}$$

- z should be the same distance from b and c and b is from a and c

$$\frac{z}{1-w} = w$$

- we can rewrite to replace z and get the equation

$$w^2 - 3w + 1 = 0 \Rightarrow w = \frac{3-\sqrt{5}}{2} \approx 0.38197$$

- Widely used to select iteration strategies

Parabolic Interpolation

- We covered Brent's method in root finding and in interpolation
- If we have a triple (a, b, c) and the values f(a), f(b), f(c) we can generate a 2nd order interpolation

$$x = b - \frac{1}{2}\frac{(b-a)^2[f(b)-f(c)] - (b-c)^2[f(b)-f(a)]}{(b-a)[f(b)-f(c)] - (b-c)[f(b)-f(a)]}$$

- When would this fail?

Parabolic Interpolation

- We covered Brent's method in root finding and in interpolation
- If we have a triple (a, b, c) and the values f(a), f(b), f(c) we can generate a 2nd order interpolation

$$x = b - \frac{1}{2}\frac{(b-a)^2[f(b)-f(c)]-(b-c)^2[f(b)-f(a)]}{(b-a)[f(b)-f(c)]-(b-c)[f(b)-f(a)]}$$

- When would this fail?
- When the triple pair is co-linear!
- The remedy is to use golden section for the co-linear case

1-D search w. derivative information

- If we have the triple (a, b, c) and f(a), f(b), f(c)
- In addition we have f'(b)
- You can use the sign of f'(b) to choose the next bracket

Outline

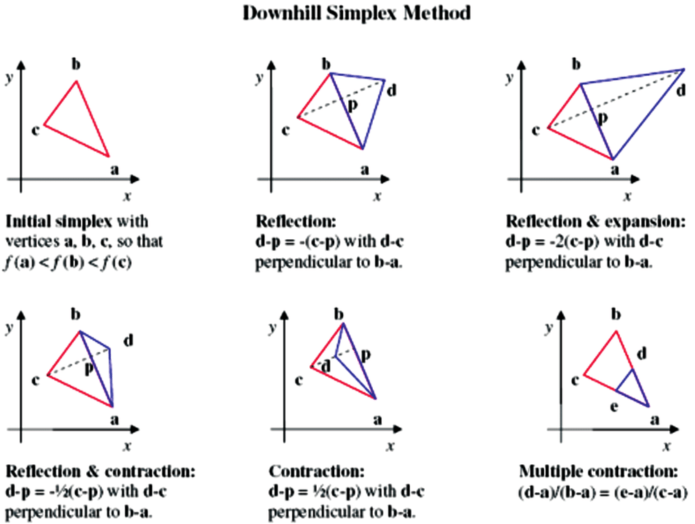
Simplex Method

- Assume we have no gradient information or access to formal model.
- A simplex in N dimensions is composed of N+1 points. Connected by straight lines
  - A 2D simplex is a triangle
  - A 3D simplex is a tetrahedron.
- We have N+1 points $x_1, \ldots, x_{N+1}$

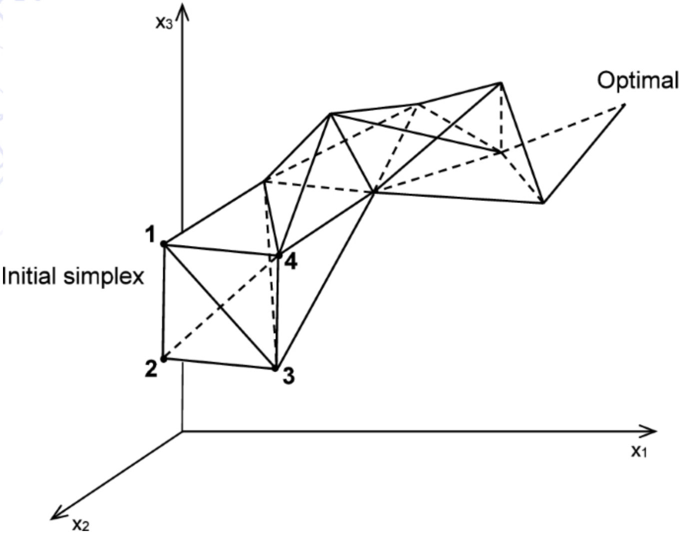
Downhill Simplex Algorithm

- Initial simple
  - Order the values of the vertices: $f(x_1) \leq f(x_2) \leq \ldots \leq f(x_{N+1})$
- Compute x_0, the centroid of all points except x_{N+1}
- **Reflection** compute $x_r = x_0 + \alpha(x_0 - x_{N+1})$, with $\alpha > 0$ if the reflection is better than $f(x_{N-1})$ replace. Restart
- **Expansion** if $f(x_r) < f(x_1)$ compute $x_e = x_0 + \gamma(x_r - x_0)$ if $f(x_e) < f(x_r)$ replace x_{N+1} else replace x_{N+1} with x_r. Restart
- **Contraction** If $f(x_r) > f(x_N)$ compute $x_c = x_0 + \rho(x_{N+1} - x_0)$ with $\rho < .5$. If $f(x_c) < f(x_{N+1})$ replace and restart
- **Shrink** Replaces all points except x_1 with $x_i = x_1 + \sigma(x_i - x_1)$ and restart
- Terminate when update is below a threshold.

Simplex illustration



Simplex example



Outline

Powell's Method

- Assume you have an n-dimensional function $f(\vec{x})$ and a starting point P_0.
- We can use the local gradient to search for an extremum
- We can generate a new estimate

$$P_{new} = P_{old} + \lambda \vec{n}$$

- Locally we can generate a Taylor expansion

$$f(x) = f(P) + \sum_i \frac{\partial f}{\partial x_i} x_i + \frac{1}{2} \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} x_i x_j + \ldots$$

or

$$f(x) \approx \vec{c} - b\vec{x} + \frac{1}{2} \vec{x}^T A \vec{x}$$

where

$$\begin{array}{rcl} \vec{c} & = & f(P) \\ b & = & -\nabla f_P \\ A_{ij} & = & \frac{\partial^2 f}{\partial x_i \partial x_j} \text{ Hessian Matrix} \end{array}$$

- Also remember

$$\nabla f = Ax - b$$

at an extremum

Powell's Method

- Initialize N unit vectors

$$u_i = e_i \; i \in 1 \ldots N$$

1. Start at point P_0
2. For i=1 to N
3. Move along P_i from P_{i-1} along u_i
4. New $u_i = u_{i+1}$
5. Set $u_N = P_n - P_0$
6. Move P_n to minimum value
7. Make $P_0 = P_n$

- Might generate linear degenerate solutions

Outline

Conjugate gradient descent

- If we have the gradient from

$$f(x) \approx \vec{c} - b\vec{x} + \frac{1}{2}\vec{x}^T A\vec{x}$$

- We can do a steepest descent
  1. Start at P_0
  2. Compute $\nabla f(P_i)$
  3. move in the direction of gradient to point P_i
  4. repeat
- We can construct a set of conjugate vectors

$$\begin{aligned}
g_{i+1} &= g_i - \lambda A h_i \\
h_{i+1} &= g_{i+1} + \gamma_i h_i \\
\lambda_i &= \frac{g_i g_j}{h_i A h_i} \\
\gamma_i &= \frac{g_{i+1} g_{i+1}}{g_i g_i}
\end{aligned}$$

Outline

Stochastic Search

- So far we have used direct functional values for optimization.
- The search has been deterministic
- Sometimes the search space is too large
- What if we use a sampling based approach?
- Some possible examples
  - Traveling salesman
  - Layout of silicon for chips
- Loosely based on Boltzmann distribution

$$P(E) = exp(-E/kT)$$

- where E is energy/entropy, T is temperature, and k is the Boltzmann constant.

Metropolis Algorithm

- Transformed into an algorithm by 1953 by Metropolis
- **Algorithm**
- Let $s = s_0$
- For $k = 0$ to k_{max}
  - $T = temperature(k/k_{max}$
  - Pick random neighbor $s_{new} = neighbor(T)$
  - If $(P(S, T) \leq random(0, 1)$
    - $s = s_{new}$
- Return S

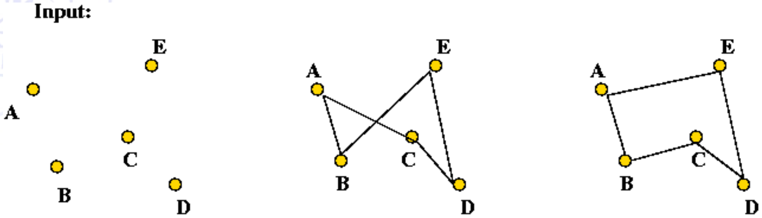
Simulated Annealing

1. Description of possible configurations
2. A way to generate random perturbation of a configuration
3. An objective function whose minimization is the objective
4. A control variable that is lowered over times.

Example - traveling salesman

- A salesman has to visit N cities at locations (x_i, y_i) returning to the original city
- Each city to be visited only once
- Minimize the travel route
- Problem in the optimal sense is known to be NP-hard.

Simple Example - Traveling Salesman



A non–optimal tour:
A B E D C

The optimal tour:
A B C D E

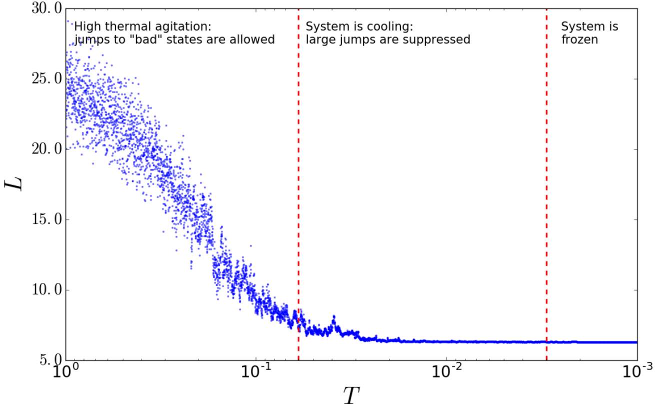
Vacation Planning

- Going to Italy for vacations

SRC https://mirkomiorelli.github.io/SA_TSP/

Vacation Planning - Temperature

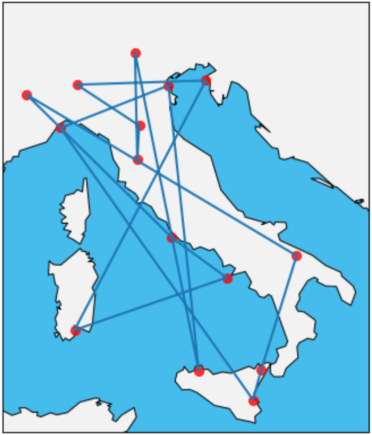
- Going to Italy for vacations



SRC https://mirkomiorelli.github.io/SA_TSP/

Vacation Planning - Random initialization

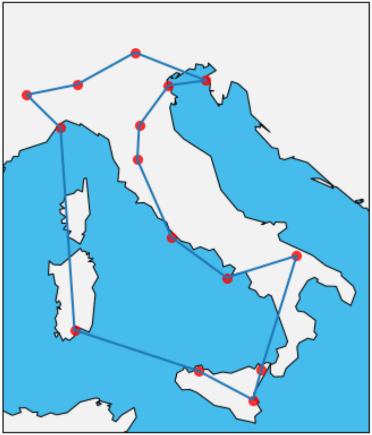
- Going to Italy for vacations



SRC https://mirkomiorelli.github.io/SA_TSP/

Vacation Planning - Random initialization

- Going to Italy for vacations



SRC https://mirkomiorelli.github.io/SA_TSP/

Outline

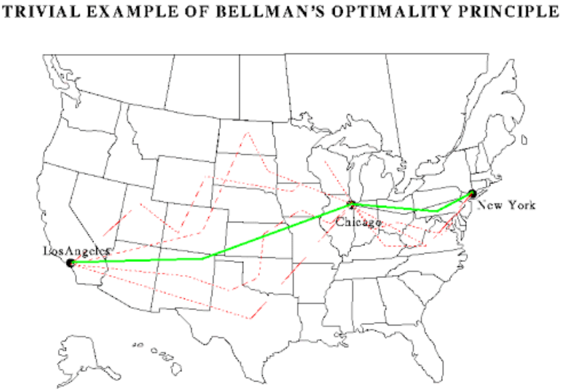
Dynamic Programming

- So far we have considered functional optimization and stochastic optimization
- What if we have a limited set of action to optimize across?
- Say optimizing a set of actions to traverse a graph?
- A strategy to could be
  - Generate a cost-map across the state space
  - Backtrack to find the optimal set of actions

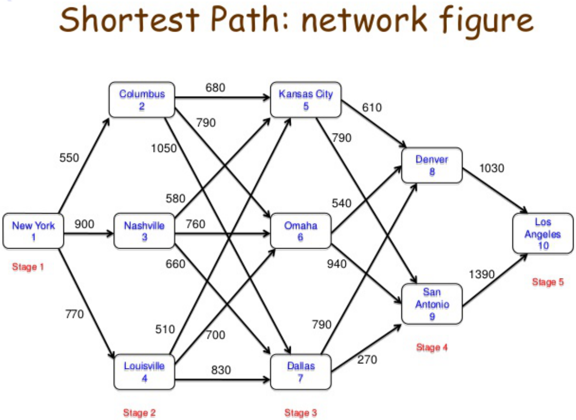
Dynamic programming

- A number of different names / approaches has been used
  - Bellman, Dijkstra, Viterbi, ...
- Selection a state space for optimization
- Identifying a set of possible actions
- Formulation of an objective function

Example navigation



Example navigation



Outline

Summary

- Optimization is a key objective in robotics
  - Robotics is many cases is about formulation of a graph
  - Optimization of an objective function across the graph
- Considered deterministic and stochastic approaches to optimization
- Covered the basics and gave an impression of the fundamentals

Questions