

CSE276C - Differential Geometry

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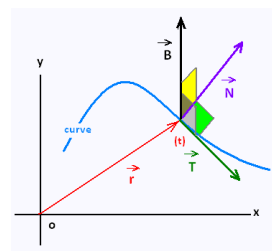
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Introduction

- We can only touch on the basics, but valuable to have basic knowledge
- Differential Geometry is all about moving on a curve / manifold
- Robotics is all about moving considering not only kinematics, but also dynamics
- What motion is possible in a particular space

Basic Concepts

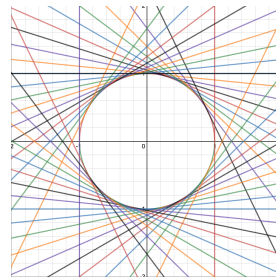
- Tangent vector
 - A vector anchored at a point p
 - Set of possible vectors for p is termed tangent space T_p



Basic Concepts

- Tangent Bundle

- A space along with its tangent vectors
- If \mathbb{R}^n the underlying space and we have a tangent space of \mathbb{R}^n anchored at each of the relevant points
- Space is then $\mathbb{R}^n \times \mathbb{R}^n$
- So a tangent bundle for a circle would be $S^1 \times \mathbb{R}^1$



Basic Concepts

- Vector Field

- A function that maps a manifold to a tangent space
- $M \rightarrow T(M)$ and within it $p \rightarrow v_p \in T_p$
- Frequently denoted $V(p)$ or V_p
- A classic question: does a manifold has a continuously changing vector field that is non-zero?

Basic Concepts

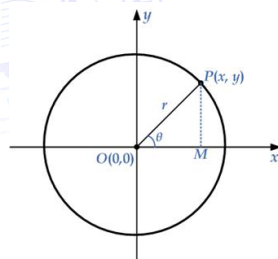
- Vector Field

- A function that maps a manifold to a tangent space
- $M \rightarrow T(M)$ and within it $p \rightarrow v_p \in T_p$
- Frequently denoted $V(p)$ or V_p
- A classic question: does a manifold has a continuously changing vector field that is non-zero?
- The circle example with $M = S^1$ is one such vector field

Geometry of curves in \mathbb{R}^3

- Consider parameterized curves $\alpha(t) = (x(t), y(t), z(t))$
- In general a curve α is a mapping $\alpha : I \rightarrow \mathbb{R}^3$
- I is an interval in \mathbb{R} sometimes we will write it as $(\alpha_1(t), \alpha_2(t), \alpha_3(t))$
- In general $(x(t), y(t), z(t))$ are differentiable
- I.e., has derivatives of all orders throughout I

A simple 2D example

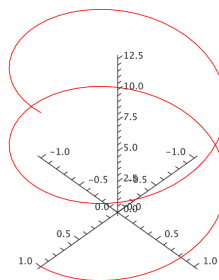


- $\alpha_1(\theta) = (r \cos(\theta), r \sin(\theta))$
- $\theta \in [0, 2\pi] = I$ OR
- $\alpha_2(\theta) = (r \cos(2\theta), r \sin(2\theta))$
- $\theta \in [0, \pi] = I$

Different curves / parameterizations can have the same trace

Simple 3D curve

- $\alpha(t) = (a \cos(t), a \sin(t), bt)$, with $t \in \mathbb{R}$



Velocity vector & Arclength

- The velocity vector of α at time t is the tangent vector of \mathbb{R}^3 given by

$$\alpha'(t) = (\alpha'_1(t), \alpha'_2(t), \alpha'_3(t))$$

- This vector is obviously also the tangent
- The speed of α is $v(t) = \|\alpha'(t)\|$
- The arclength traversed between t_0 and t_1 is

$$\int_{t_0}^{t_1} v(t) dt$$

- You can re-parameterize $\alpha(t)$ as $\beta(s)$ where s is the arc-length, which is the same as representing α at unit speed

Simple Example – Helix

- Consider the helix: $\alpha(t) = (r \cos(t), r \sin(t), qt)$ then
 - Velocity: $\alpha'(t) = (-r \sin(t), r \cos(t), q)$
 - Speed: $v(t) = \sqrt{r^2 + q^2} = c$ a constant
 - Arc-length: $s(t) = \int_0^t c dt = ct$. Thus $t(s) = \frac{s}{c}$
 - Re-parameterized: $\beta(s) = \alpha(\frac{s}{c}) = (r \cos(\frac{s}{c}), r \sin(\frac{s}{c}), q\frac{s}{c})$

Arclength?

- So does the integral

$$s(t) = \int_{t_0}^{t_1} \|\alpha'(t)\| dt$$

always converge?

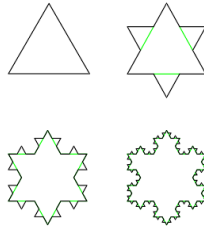
Arclength?

- So does the integral

$$s(t) = \int_{t_0}^{t_1} \|\alpha'(t)\| dt$$

always converge?

- Some curves have infinite arclength (ex fractals - Koch Snowflake)



Vector fields of β

- We can define a set of vector fields for β
 - $T = \beta'$ the unit tangent field
 - $N = \frac{T'}{\|T'\|}$ the principal normal vector field
 - $B = T \times N$ called the bi-normal vector field of β
- The quantity $\|T'\|$ is also named the curvature function $K(s) = \|T'(s)\|$
- The triple (T, N, B) is called the Frenet Frame field of β

Curvature

- Let $\alpha : I \rightarrow \mathbb{R}^3$ be a curve parameterized by arclength
- Curvature is then defined as $\|\alpha''(s)\| = K(s)$
- $\alpha'(s)$ – the tangent vector of s
- $\alpha''(s)$ – the change in the tangent vector
- $R(s) = 1/K(s)$ – is called the radius of curvature

Simple examples

- Straight line

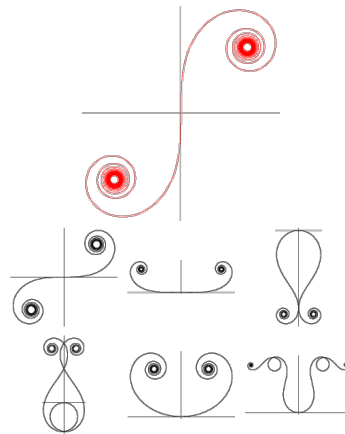
$$\begin{aligned}\alpha(s) &= us + v, \quad u, v \in \mathbb{R}^2 \\ \alpha'(s) &= u \\ \alpha''(s) &= 0 \Rightarrow \|\alpha''(s)\| = 0\end{aligned}$$

- Circle

$$\begin{aligned}\alpha(s) &= (a \cos(s/a), a \sin(s/a)), \quad s \in [0, 2\pi a] \\ \alpha'(s) &= (-\sin(s/a), \cos(s/a)) \\ \alpha''(s) &= (-\cos(s/a)/a, -\sin(s/a)/a) \Rightarrow \|\alpha''(s)\| = 1/a\end{aligned}$$

Curvature examples

- Cornu Spiral - $K(s) = s$
- Generalized Cornu Spirals - $K(s)$ - Polynomial of s



Normals

- When α is parameterized by arc length

$$\alpha'(s) \cdot \alpha'(s) = 1$$

- From Vector Calculus

- If $f, g: I \rightarrow \mathbb{R}^3$ and $f(t) \cdot g(t) = \text{const}$ for all t
- then

$$f'(t) \cdot g(t) = -f(t) \cdot g'(t)$$

for $f \cdot f$ this is only true for $f'(t) \cdot f(t) = 0$

- This implies that

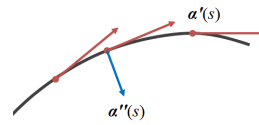
$$\alpha''(s) \cdot \alpha'(s) = 0$$

or $\alpha''(s)$ is orthogonal to $\alpha'(s)$

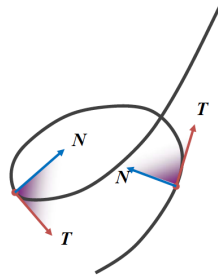
- Its proportional to the normal of the curve

Normals

- $\alpha'(s) = T(s)$ – Tangent Vector
- $||\alpha'(s)||$ – arc length
- $\alpha''(s) = T'(s)$ – normal direction
- $||\alpha''(s)||$ – curvature
- If $||\alpha''(s)|| \neq 0$ then
 $\alpha''(s) = T'(s) = K(s)N(s)$



Osculating Plane



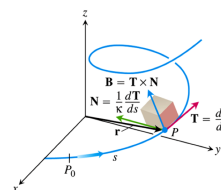
- The local plane determined by the unit tangent and the normal vectors - $T(s)$ and $N(s)$ is called the osculating plane at s

Source: M. Ben-Chen,
Stanford

The Bi-normal Vector

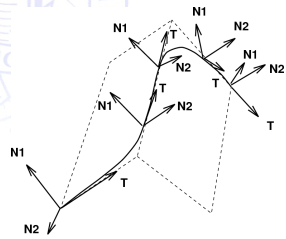
- The binormal is defined for $K(s) \neq 0$ by

$$B(s) = T(s) \times N(s)$$
- The bi-normal defines the osculating plane



Source: R. Gardner, ETSU

The Frenet Frame



Source: A. J. Hanson, LBL

- The system $\{T(s), N(s), B(s)\}$ for an ortho-normal basis for \mathbb{R}^3 called the Frenet Frame
- The obvious question - How does it change along a curve? I.e., what are $T'(s)$, $N'(s)$, and $B'(s)$?

$T'(s)$

- We have already covered $T'(s)$

$$T'(s) = K(s)N(s)$$

- As it is in the direction of $N(s)$ it is orthogonal to $B(s)$ and $T(s)$.

$N'(s)$

- We know that $N(s) \cdot N(s) = 1$
- From our earlier lemma (vector calculus) $N'(s) \cdot N(s) = 0$
- We know $N(s) \cdot T(s) = 0$ from the lemma $N'(s) \cdot T(s) = -N(s) \cdot T'(s)$
- Given $K(s) = N(s) \cdot T'(s)$
- It must be true that $N'(s) \cdot T(s) = -K(s)$

Torsion

- For the parameterized curve $\alpha : I \rightarrow \mathbb{R}^3$ the torsion of α is defined by

$$\tau(s) = N'(s) \cdot B(s)$$

- We can then express

$$N'(s) = K(s)T(s) + \tau(s)B(s)$$

Curvature vs Torsion

- **Curvature** indicates how much the normal changes in the direction of the tangent
- **Torsion** indicates how much the normal change in the direction orthogonal to the osculating plane
- Curvature is always positive, the torsion can be negative
- Neither depend on the choice of parameterization

$B'(s)$

- We know that $B(s) \cdot B(s) = 1$
- From the lemma we know $B'(s) \cdot B(s) = 0$
- We further know: $B(s) \cdot T(s) = 0$ and $B(s) \cdot N(s) = 0$
- From the lemma:

$$B'(s) \cdot T(s) = -B(s) \cdot T'(s) = B(s) \cdot K(s)N(s) = 0$$

- We get

$$B'(s) \cdot N(s) = -B(s) \cdot N'(s) = -\tau(s)$$

and from this we have

$$B'(s) = -\tau(s)N(s)$$

The Frenet Formulas

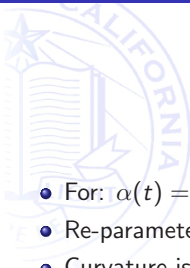


$$\begin{aligned} T'(s) &= K(s)N(s) \\ N'(s) &= -K(s)T(s) + \tau(s)B(s) \\ B'(s) &= -\tau(s)N(s) \end{aligned}$$

In Matrix Form

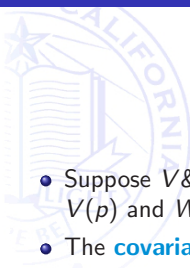
$$\begin{pmatrix} T'(s) & N'(s) & B'(s) \end{pmatrix} = \begin{pmatrix} T(s) & N(s) & B(s) \end{pmatrix} \begin{pmatrix} 0 & K(s) & 0 \\ K(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix}$$

Example - Back to the helix



- For: $\alpha(t) = (a \cos(t), a \sin(t), bt)$
- Re-parameterized: $\alpha(s) = (a \cos(s/c), a \sin(s/c), bs/c)$ where $c = \sqrt{a^2 + b^2}$
- Curvature is then: $K(s) = \frac{a}{a^2 + b^2}$
- Torsion is then $\tau(s) = \frac{b}{a^2 + b^2}$
- Note for this example both curvature and torsion are constants

Covariant Derivatives and Lie Brackets



- Suppose V & W are two vector fields in \mathbb{R}^n so that for each point $p \in \mathbb{R}^n$ $V(p)$ and $W(p)$ are vectors in \mathbb{R}^n
- The **covariant derivative** of W wrt V is

$$(\nabla_V W)(p) = \frac{d}{dt} W(p + tV_p)|_{t=0}$$

- $\nabla_V W$ measures the change in W as one moves along V

Examples - covariant derivatives

- In \mathbb{R}^2 $W(p) = (1,0)$ and $V(p) = (0,1)$ for all p
- The $\nabla_v W = \nabla_w V = 0$
- For a circle in 2D, $p = (x, y) \in \mathbb{R}^2$

$$W = \frac{(x, y)}{\sqrt{x^2 + y^2}} \text{ and } V = \frac{(-y, x)}{\sqrt{x^2 + y^2}}$$

- Then $\nabla_v W = \frac{v}{\sqrt{x^2 + y^2}}$ and of course $\nabla_w V = 0$

A few things about covariant derivatives

- $\nabla_v W$ is an n-dimensional vector
- $\nabla_v(aW + bU) = a\nabla_v W + b\nabla_v U$
- $\nabla_{fV+gU} W = f\nabla_v W + g\nabla_u W$

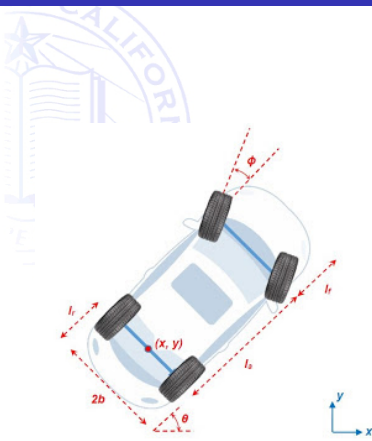
Lie Bracket

- The **Lie Bracket** $[V, W]$ of the two vector fields is defined to be

$$[V, W] = \nabla_v W - \nabla_w V$$

- Basically measure flow in the directions of $V, -V, W, -W$
- Lets illustrate this with a real robot example

Parallel Parking



- The configuration - (x, y, θ)
- The controls are (v, ϕ)
- The controls are

$$\begin{aligned}\dot{x} &= v \cos \phi \cos \theta \\ \dot{y} &= v \cos \phi \sin \theta \\ \dot{\theta} &= \frac{v}{l} \sin \phi\end{aligned}$$

- We can consider nominal motion $(1, \phi_1)$ and $(1, \phi_2)$ as wheel directions

Parallel Parking - Cont

- Two vector fields

$$V_i = V_i(x, y, \theta) = (\cos \phi_i \cos \theta, \cos \phi_i \sin \theta, \frac{\sin \phi_i}{l})$$

- Then

$$\nabla_{V_1} V_2 = (\nabla(\cos \phi_1 \cos \theta) V_2, \nabla(\cos \phi_1 \sin \theta) V_2, \nabla(\frac{\sin \phi_1}{l}) V_2)$$

skipping calculations

$$\nabla_{V_1} V_2 = \frac{\sin \phi_1 \cos \phi_2}{l} (-\sin \theta, \cos \theta, 0)$$

and similarly for the

$$[V_1, V_2] = \frac{\sin(\phi_1 - \phi_2)}{l} (-\sin \theta, \cos \theta, 0)$$

So we can move perpendicular to the axis as long as $(\phi_1 - \phi_2) \neq 0$

Moving to manifolds

- Smooth Manifolds

- A manifold is a set M with an associated one-to-one map $\phi : U \rightarrow M$ from an open subset $U \subset \mathbb{R}^m$ called a global chart or coordinate system of M



Smooth Manifolds

- A smooth manifold is a pair (M, \mathcal{A}) where:
 - M is a set
 - \mathcal{A} is a family of 1-1 charts: $\phi : U \rightarrow M$ from some open subset $U = U_\phi \subset \mathbb{R}^m$ for M

Differentiable and smooth functions

- $f : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^q$
 $(y_1, \dots, y_q) = f(x_1, \dots, x_n)$
- f is of a class C^r if f has continuous partial derivatives

$$\frac{\partial^{r_1 + \dots + r_n} y_k}{\partial x_1^{r_1} \dots \partial x_n^{r_n}}$$

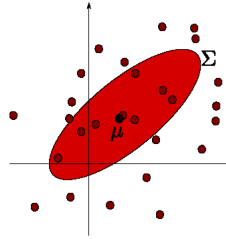
- If $r = \infty$, then f is **smooth**, the main focus in robotics

Diffeomorphism

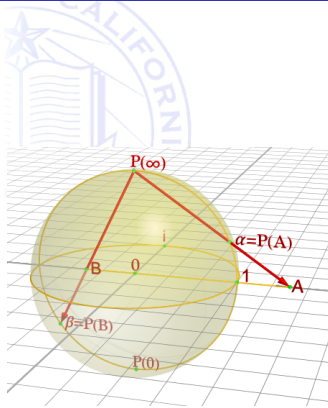
- When $n = q$
 - if f is 1-1, f and f^{-1} are both C^r
 - $\Rightarrow f$ is a **C^r -diffeomorphism**
 - Smooth diffeomorphisms are simply referred as diffeomorphisms
- Inverse Function Theorem:
 - f diffeomorphism $\Rightarrow \det(J_x f) \neq 0$
 - $\det(J_x f) \neq 0 \Rightarrow f$ is local diffeomorphism in a neighborhood of x

Example - Gaussian Distribution

- The space of n-dimensional Gaussian distributions is a smooth manifold
- Global chart: $(\mu, \Sigma) \in \mathbb{R}^n \times \mathcal{P}(n)$



Manifolds can generate multiple charts



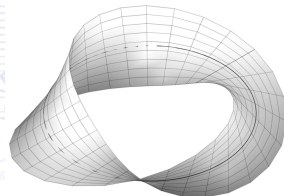
- The sphere $S^2 = \{(x, y, z), x^2 + y^2 + z^2 = 1\}$ has multiple projections/charts
- We can project from the North Pole, of a point $P = (x, y, z)$ given by

$$\phi(P) = \left(\frac{x}{1-z}, \frac{y}{1-z} \right)$$

- is a large coordinate system around the south pole

Manifolds requiring multiple charts

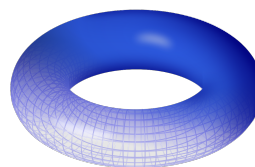
The Moebius Strip



$$u \in [0, 2\pi], v \in [-1/2, 1/2]$$

$$\begin{pmatrix} \cos(u) \left(1 + \frac{1}{2} v \cos\left(\frac{u}{2}\right) \right) \\ \sin(u) \left(1 + \frac{1}{2} v \cos\left(\frac{u}{2}\right) \right) \\ \frac{1}{2} v \sin\left(\frac{u}{2}\right) \end{pmatrix}$$

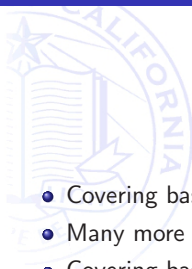
2D Torus



$$(u, v) \in [0, 2\pi]^2, R \gg r > 0$$

$$\begin{pmatrix} \cos(u) (R + r \cos(v)) \\ \sin(u) (R + r \cos(v)) \\ r \sin(v) \end{pmatrix}$$

Summary



- Covering basics of movement along curves
- Many more derivations can be provided for movement on manifolds
- Covering basic characteristics of curves and manifolds
- Definition of the Frenet frame and associated characteristics
- Brief coverage of covariant derivatives and Lie bracket