1\. (a) $P(W_2<0 \mid W_1>0)$

$= P(W_2<W_1,\ |W_2-W_1|>|W_1-W_0|)$

$= P(W_2<W_1) \times P(|W_2-W_1|>|W_1-W_0|)$

$= 50\% \times 50\% = 25\%$

(b) $P(W_1 \times W_2 < 0)$

$= P(W_1<0, W_2>0) + P(W_1>0, W_2<0)$

$= P(W_2>0 \mid W_1<0) \times P(W_1<0) + P(W_2<0 \mid W_1>0) P(W_1>0)$

$= \frac{1}{4}$

(c). $P(W_1<0 \cap W_2<0)$

$= P(W_1<0) \times P(|W_2-W_1|<|W_1-W_0|) + P(W_1<0) \times P(W_2 \text{ downside} \mid W_1<0)$

$= \frac{3}{8}$

2.? $E(|W_{t+\Delta t}-W_t|)$

based on Brownian Motion, $W_{t+\Delta t}-W_t \sim N(0, \Delta t)$

$E(|x|) = \sqrt{\frac{2}{\pi}}\sigma$ for $x \sim N(\mu, \sigma^2)$ I show the process of prove at last

So $E(|W_{t+\Delta t}-W_t|) = \sqrt{\frac{2}{\pi}}\sqrt{\Delta t}$

3\. W_t be standard Brownian Motion, $s<t$.

$V[(W_t-W_s)^2] = E(W_t-W_s)^4 - [E(W_t-W_s)^2]^4$

$= 2(t-s)^2$

4\. W_t be a Brownian Motion, derive STD for dx_t

$dx_t = \mu dt + \sigma dW_t$ $\quad f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt$

(a). $x_t = W_t^2$ $\quad dx_t = dW_t^2 = 2W_t dW_t + dt$.

b. $x_t = t + e^{W_t}$

$dx_t = dt + e^{W_t} = e^{W_t}dW_t + dt + \frac{1}{2}e^{W_t}dt$

c. $X_t = W_t^3 - 3tW_t$

$$dX_t = dW_t^3 - 3tW_t = (3W_t^2 - 3t)\,dW_t - 3W_t\,dt + 3W_t\,dt$$
$$= (3W_t^2 - 3t)\,dW_t$$

d. $X_t = e^{t+W_t}$

$$dX_t = de^{t+W_t} = de^t \cdot e^{W_t} = (e^t e^{W_t})\,dW_t + \tfrac{3}{2}(e^t e^{W_t})\,dt$$

e. $X_t = e^{\frac{t}{2}} \sin(W_t)$

$$dX_t = d\,e^{\frac{t}{2}} \sin(W_t) = e^{\frac{t}{2}} \cos(W_t)\,dW_t + \tfrac{1}{2} e^{\frac{t}{2}} \sin(W_t)\,dt$$
$$- \tfrac{1}{2} e^{\frac{t}{2}} \sin(W_t)\,dt$$
$$= e^{\frac{t}{2}} \cos(W_t)\,dW_t$$

f. $X_t = e^{W_t - \frac{t}{2}}$

$$dX_t = de^{W_t - \frac{t}{2}} = de^{W_t} \cdot e^{-\frac{t}{2}} = e^{W_t} \cdot e^{-\frac{t}{2}}\,dW_t$$

5. $$\begin{cases} dY_t = b(t)\, Y_t\, dW_t \\ dZ_t = A(t)\,dt + B(t)\,dW_t \end{cases}$$

$X_t = Y_t Z_t$

$$dX_t = dY_t Z_t = Y_t \cdot dZ_t + Z_t \cdot dY_t + dY_t \cdot dZ_t$$
$$= Y_t (A(t)\,dt + B(t)\,dW_t) + Z_t (b(t)\, Y_t\, dW_t)$$
$$+ (b(t)\, Y_t\, dW_t) \cdot (A(t)\,dt + B(t)\,dW_t)$$
$$= Y_t (A_t\,dt + B_t\,dW_t + Z_t b_t\,dW_t + B_t b_t\,dt)$$

6. W_t and $\widetilde{W}_t$ denote two independent Brownian motions.

$$Y_t = \frac{W_t}{\widetilde{W}_t} = W_t \cdot \widetilde{W}_t^{-1}$$

$$dY_t = dW_t \tilde{W}_t^{-1} = \frac{\partial Y_t}{\partial W_t} dW_t + \frac{\partial Y_t}{\partial \tilde{W}_t} d\tilde{W}_t + \frac{1}{2} \frac{\partial Y}{\partial W_t^2} dW_t^2$$

$$= \tilde{W}_t^{-1} dW_t - W_t \hat{W}_t^{-2} d\tilde{W}_t$$

7. $dr_t = \theta dt + \sigma dW_t$

θ and σ are both constants

a. $\int_0^T dr_t = \int_0^T \theta dt + \sigma dW_t$

$$r_T - r_0 = \int_0^T \theta \overset{\to t0}{dt} + \int_0^T \sigma dW_t$$

$$r_T - r_0 = T\theta + W_T \sigma$$

$$r_T = r_0 + T\theta + W_T \sigma$$

$$E[\int_0^T r_t dt] = E\{\int_0^T r_0 + t\theta + W_t \sigma \, dt.\}$$ 求积分

$$= E(r_0 T + \frac{1}{2} T^2 \theta + T W_t \sigma)$$

$E(W_t) = 0$ $\quad E[\int_0^T r_t dt] = r_0 T + \frac{1}{2} T^2 \theta$

b. $V[\int_0^T r_t dt] = V[\int_0^T r_0 + t\theta + W_t \sigma dt]$

$$= V[\int_0^T \theta t dt + \int_0^T \sigma W_t dt] + [\int_0^T r_0 dt]$$

$$= V[\frac{1}{2}\theta T^2 + \int_0^T \sigma W_t dt]$$

$$= V[\int_0^T \sigma W_t dt] = E[\int_0^T \sigma W_t dt]^2$$

$$E[\int_0^T \sigma W_t dt]^2 = \sigma^2 E[\int_0^T W_t dt]^2 = \sigma^2 E[\int_0^T W_t dt \cdot \int_0^T W_s ds]$$

$$= \sigma^2 E[\int_0^T \int_0^T W_t W_s dt ds] = \sigma^2 \int_0^T \int_0^T \min(t,s) dt ds$$

$$= \sigma^2 \int_0^T [\int_0^t s dt ds + \int_t^T t dt ds]$$

$$= \sigma^2 [\int_0^T \frac{1}{2} t^2 dt + \int_0^T t(T-t) dt]$$

$$= \sigma^2 [\frac{T^3}{6} + \frac{T^3}{2} - \frac{T^3}{3}] = \sigma^2 \frac{T^3}{3}$$

2. Provement

$E(|W_{t+\Delta t} - W_t|) \sim N(0, \Delta t) = \sqrt{\Delta t} \cdot N(0,1)$

$E(|x|) = \int_{-\infty}^{\infty} |x| f(x) dx \qquad f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad , \mu=0, \sigma=1$

$\therefore f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$
\begin{aligned}
E(|x|) &= \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx. \\
&= \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} x\, e^{-\frac{x^2}{2}} dx + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \left(-x\, e^{-\frac{x^2}{2}}\right) dx \\
&= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Big|_{-\infty}^{0} - \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Big|_{0}^{+\infty} \\
&= \frac{1}{\sqrt{2\pi}} e^0 - 0 - 0 + \frac{1}{\sqrt{2\pi}} e^0 \\
&= \frac{2}{\sqrt{2\pi}} = \sqrt{\frac{2}{\pi}}
\end{aligned}
$$