

QF620 Stochastic Modelling in Finance  
Assignment 2/4  
Due Date: 11-Oct-2023

1. Assume that a stock price  $S_t$  follows the Black-Scholes lognormal process:

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

- (a) Use Itô's formula to derive the stochastic differential equation for the process  $S_t^2$ .
- (b) Solve the stochastic differential equations of  $S_t$  and  $S_t^2$ .
- (c) Use your solutions to verify the following expectations:

$$\begin{aligned}\mathbb{E}[S_t] &= Se^{rt} \\ \mathbb{E}[S_t^2] &= S^2 e^{(2r+\sigma^2)t},\end{aligned}$$

2. Digital (or binary) options are characterized by the binary state of their payoffs. A cash-or-nothing digital call option pays \$1 if  $S_T > K$  on maturity, and 0 otherwise. We can write the payoff as

$$V_{\text{Cash Digital}}(T) = \mathbb{1}_{S_T > K}.$$

Using the Black-Scholes model, the stock price follows a lognormal process

$$dS_t = rS_t dt + \sigma S_t dW_t^*.$$

Derive a valuation formula for this call option:

$$V_{\text{Cash Digital}}(0) = e^{-rT} \mathbb{E}[\mathbb{1}_{S_T > K}].$$

3. An asset-or-nothing digital call option pays  $S_T$  if  $S_T > K$  on maturity, and 0 otherwise. We can write the payoff as

$$V_{\text{Asset Digital}}(T) = S_T \mathbb{1}_{S_T > K}.$$

Again using the Black-Scholes model, derive a valuation formula for the asset-or-nothing digital call option:

$$V_{\text{Asset Digital}}(0) = e^{-rT} \mathbb{E}[S_T \mathbb{1}_{S_T > K}].$$