QF620 Stochastic Modelling in Finance
Assignment 1/4
Due Date: 25-Sep-2023

1. Let W_t denote a standard Brownian motion. Calculate the following probabilities:
   (a) $\mathbb{P}(W_2 < 0 | W_1 > 0)$
   (b) $\mathbb{P}(W_1 \times W_2 < 0)$
   (c) $\mathbb{P}(W_1 < 0 \cap W_2 < 0)$

2. Let W_t denote a standard Brownian motion. Evaluate the following expectation
$$\mathbb{E}[|W_{t+\Delta t} - W_t|],$$
where $|\cdot|$ denote absolute value.

3. Let W_t denote a standard Brownian motion. Let $s < t$, determine the variance
$$V[(W_t - W_s)^2].$$

4. Let W_t denote a Brownian motion. Derive the stochastic differential equation for dX_t and group the drift and diffusion coefficients together for the following stochastic processes:
   (a) $X_t = W_t^2$
   (b) $X_t = t + e^{W_t}$
   (c) $X_t = W_t^3 - 3tW_t$
   (d) $X_t = e^{t+W_t}$
   (e) $X_t = e^{\frac{t}{2}} \sin(W_t)$
   (f) $X_t = e^{W_t - \frac{t}{2}}$

5. Consider 2 stochastic processes Y_t and Z_t, following the dynamics
$$\begin{cases} dY_t = b(t) Y_t dW_t \\ dZ_t = A(t)dt + B(t)dW_t. \end{cases}$$
Define a new stochastic process X_t as $X_t = Y_t Z_t$, write down the stochastic differential equation for dX_t.

6. Let W_t and $\tilde{W}_t$ denote two independent Brownian motions, derive the SDE for the stochastic variable $Y_t = \frac{W_t}{\tilde{W}_t}$.

7. Consider an interest rate model following the stochastic differential equation:

$$dr_t = \theta dt + \sigma dW_t,$$

where θ and σ are both constants. Determine

(a)

$$\mathbb{E}\left[\int_0^T r_t \, dt\right]$$

(b)

$$V\left[\int_0^T r_t \, dt\right]$$