Introduction

1 Finite element method

For the simple, one dimensional case, the exact solution to the Helmholtz problem is know. Finding the exact solution for more complex and higher dimensional problems turns out to be hard and impractical. Often, an approximation to the exact solution suffices for engineering purposes. A widly used method to find such approximations is know as the *Galerkin finite element method*.

Using this method, an approximation to the real solution is found by transforming the problem in a system linear equations. This results in a sparse linear system for which many solving techniques have been studied []. Although the linear systems resulting from this method are sparse, for an accurate aproximation of solution to higher dimensional or heavily oscillating problems a very large sytem can be needed.

1.1 One dimensional Helmholtz problem

As an example, we will consider a simple one dimensional wave problem. Suppose we have the following conditions for $u(x)$ on $[0,1]$:

$$-\frac{d^2u}{dx^2} - k^2u = f, \quad \text{on } \Omega = (0,1), \tag{1}$$

$$u(0) = 0, \tag{2}$$

$$\frac{du}{dx}(1) - iku(1) = 0. \tag{3}$$

It can be shown that the exact solution to is given by

$$u(x) = \frac{e^{ikx}}{k}\int_0^x \sin(ks)f(s)ds + \frac{\sin(kx)}{k}\int_x^1 e^{iks}f(s)ds, \tag{4}$$

which is periodic with wavelenth $\lambda = \frac{2\pi}{k}$.

While this problem has a simple solution, exact solutions to more complex and higher order problems are difficult, nigh impossible to find in a simmilar matter. To find a good approximation, we have to restrict the problem to one that can be solved with a computer.

1.2 Galerking finite element method

Since the original problem is infinite dimensional, we simplify the problem by restricting the approximation of the real solution to a finite dimensional search space. Suppose we have a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 1,2,3$. We define the function space $L_2(\Omega)$ of square integrable function on Ω by saying that $f \in L_2(\Omega)$ if

$$\|f\| := \left(\int_\Omega |f(x)|^2 d\Omega\right)^{\frac{1}{2}} < \infty. \tag{5}$$

Furthermore, we say $f \in H^1_{(0}(\Omega)$ if $f(0) = 0$ and

$$\|\nabla f\|^2 + \|f\|^2 < \infty. \tag{6}$$

For solving the above system with a finite element method, we will first rewrite the problem in its weak form. We multiply both sides with a test function $v \in H^1_{(0}(\Omega)$ and integrate afterwards to obtain

$$-\int_0^1 u''(x)v(x) - k^2u(x)v(x)dx = \int_0^1 f(x)v(x)dx. \tag{7}$$

By substituting boundary condition (3) and taking the fact that $v(0) = (0)$ into consideration, we obtain

$$\int_0^1 u'(x)v'(x)dx - k^2\int_0^1 u(x)v(x)dx - iku(1)v(1) = \int_0^1 f(x)v(x)dx. \tag{8}$$

Instead of solving the above equation for the exact solution $u(x)$, we want to approximate it by finding $U(x) \approx u(x)$. To do this, we define a finite element mesh X_h on Ω, by

$$X_h := \{x_i; x_i = ih, i = 0, 1, \ldots, N\}, \tag{9}$$

where $h = 1/N$. We limit our search space to $S_h(0,1) \subset H^1_{(0}(\Omega)$, the space of piecewise continuous linear functions with nodal values at the points in X_h, satisfying (2). This function space is spanned by the set of hat functions defined as

$$\chi_j(x) = \begin{cases} \frac{1}{h}(x - x_{j-1}), & x \in [x_{j-1}, x_j], \\ \frac{1}{h}(x_{j+1} - x), & x \in [x_j, x_{j+1}], \\ 0 & \text{elsewhere}, \end{cases} \tag{10}$$

for $j = 1, 2, \ldots, N-1$ and for $j = N$ by

$$\chi_N(x) = \begin{cases} \frac{1}{h}(x - x_{j-1}), & x \in [x_{j-1}, 1], \\ 0 & \text{elsewhere}. \end{cases} \tag{11}$$

Now, if we require that both U and v are in $S_h(0,1)$, then we can write

$$U(x) = \sum_{j=1}^{N} u_j \chi_j(x), \tag{12}$$

and (8) transforms to

$$\sum_{j=1}^{N} \left[\int_0^1 \chi_j'(x)\chi_m'(x)dx - k^2 \int_0^1 \chi_j(x)\chi_m(x)dx \right] u_j - iku_N\chi_m(1) = \int_0^1 f(x)\chi_m(x)dx, \tag{13}$$

for $m = 1, 2, \ldots, N$.

1.3 Linear system

1.4 Error estimates

2 Higher dimensional problem

2.1 Linear system

2.2 Mesh generation

3 Unbounded domains

3.1 Absorbing boundary conditions

4 Large wavenumbers and error estimates