1 Higher dimensional problem

For higher dimensional problems, again we consider the general notations for the Helmholtz problem. For a domain $\Omega \in \mathbb{R}^2$ with boundary Γ, the Helmholtz problem is stated as follows

$$\Delta u + k^2 u = 0, \quad \text{in } \Omega, \tag{1}$$

$$\frac{\partial u}{\partial n} + \beta u = g, \quad \text{on } \Gamma, \tag{2}$$

where $k \in \mathbb{R}$, $\beta \in \mathbb{C}$ and $\partial/\partial n$ is the outward normal derivative. Note that

$$\frac{\partial u}{\partial n} = \nabla u \cdot n, \quad \Delta u = \nabla^2 u.$$

Again, we transform the problem to a weak formulation by first multiplying (1) by a test function $v \in H^1(\Omega)$ and integrating it to get

$$\int_\Omega v \Delta u \, \mathrm{d}\Omega + \int_\Omega k^2 uv \, \mathrm{d}\Omega = 0. \tag{3}$$

By applying Green's identity

$$\int_\Omega v \Delta u \, \mathrm{d}\Omega + \int_\Omega \nabla v \cdot \nabla u \, \mathrm{d}\Omega = \int_\Gamma v(\nabla u \cdot n) \, \mathrm{d}\Gamma,$$

we find that

$$\int_\Gamma v \frac{\partial u}{\partial n} \, \mathrm{d}\Gamma - \int_\Omega \nabla v \cdot \nabla u - k^2 uv \, \mathrm{d}\Omega = 0.$$

From multiplying boundary equation (2) with v and integrating, we obtain

$$\int_\Gamma v \frac{\partial u}{\partial n} \, \mathrm{d}\Gamma = -\beta \int_\Gamma uv \, \mathrm{d}\Gamma + \int_\Gamma vg \, \mathrm{d}\Gamma.$$

By substituting these equations into (3) we get the weak formulation of our problem; find $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla v \cdot \nabla u - k^2 uv \, \mathrm{d}\Omega + \beta \int_\Gamma uv \, \mathrm{d}\Gamma = \int_\Gamma vg \, \mathrm{d}\Gamma \tag{4}$$

for all $v \in H^1(\Omega)$.

As for the one dimensional problem, we will restrict our search space to a finite dimensional function space $V \subset H^1(\Omega)$. We define V to be the span of basis functions χ_j, $j = 1, 2, \ldots, N$. When we require that $U, v \in V$, we are able to write

$$U(x) = \sum_{j=1}^{N} u_j \chi_j(x), \tag{5}$$

where we have to determine the values of u_j. The problem can now be reformulated as

$$\sum_{j=1}^{N} \left[\int_\Omega \nabla \chi_j \cdot \nabla \chi_m - k^2 \chi_j \chi_m \, \mathrm{d}\Omega + \beta \int_\Gamma \chi_j \chi_m \, \mathrm{d}\Gamma \right] u_j = \int_\Gamma \chi_m g \, \mathrm{d}\Gamma,$$

for $m = 1, 2, \ldots N$. Expressed as a linear system, this becomes

$$A\mathbf{u} = \mathbf{f},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1N} \\ a_{21} & a_{22} & \ldots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \ldots & a_{NN} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} \int_\Gamma \chi_1 g \, \mathrm{d}\Gamma \\ \int_\Gamma \chi_2 g \, \mathrm{d}\Gamma \\ \vdots \\ \int_\Gamma \chi_N g \, \mathrm{d}\Gamma \end{pmatrix}.$$

Here, the matrix entries a_{jm} of A are defined as

$$a_{jm} = \int_\Omega \nabla \chi_j \cdot \nabla \chi_m - k^2 \chi_j \chi_m \, \mathrm{d}\Omega + \beta \int_\Gamma \chi_j \chi_m \, \mathrm{d}\Gamma.$$

In the same manner as the one dimensional case, solving the linear system and substituting $\mathbf{u}$ into (5) yields our approximation $U(x)$ to $u(x)$.

1.1 Linear system

An important problem with the application of the finite element method is the shape and size of the linear system that needs to be solved. When we need to keep our approximation at a certain level of accuracy and we increase the number of dimensions, the size of the linear system can grow rapidly.

Suppose we want to achieve an accuracy of 0.01 (i.e. 1%) and our error τ is bounded by h^2, with $h = 1/N$ defined as our step size in each dimension. In the one dimensional case, we would have a total of 10 elements. Matrix A would be of size 10×10. In two dimensions, the resulting system would have 100 elements and A would have size 100×100. In three dimensions, we would need a 1000×1000 matrix for the same accuracy. We see that the number of elements increases extremely fast when the number of dimensions in our problem grow.

In real applications, the step size needed to achieve an accuracy of 0.01 could turn out to be many times smaller than in our example above. In these situations, the resulting linear system could grow beyond the size at which solving and storing the linear systems with current technology becomes impractical. For very large systems, we are not able to use direct methods to solve the system. Instead, we have to use iterative methods.

The efficiency of these iterative methods depends for a great deal on the *bandwidth* of the matrix A. In the one dimensional case, we saw that the bandwidth is 3. Although by a suitable choice of basis functions χ_m the matrix A can remain sparse in higher dimensions, the bandwidth typically increases with the number of dimensions. For a two dimensional problem, the bandwidth would be of order N. For three dimensions, the furthest non-zero element would be at a distance of order N^2 from the diagonal. Again, this adds to the cost of solving large systems using a finite element approach.

1.2 Mesh generation

Another important aspect of the finite element method is the generation of the mesh (finite elements) on which we define our basis functions. In the one dimensional case this is very straightforward as described in section **??**. For two dimensions, there exist different commercial and free packages, many of which are based on Delaunay triangulation. These algorithms are very effective and can produce meshes for a wide range of geometries.

Mesh creation packages for three dimensional problems also exist. Again, the difficulty in creating meshes for these problems increases. In contrast to the packages for two dimensional meshes, these algorithms are not judged on their performance, but on the percentage of 'problematic' elements they produce.

2 Unbounded domains

The finite element method is a good fit for bounded domains, but to extend its application to unbounded domains, we have to put in some extra constraints. The way this is done, is by adding an additional boundary condition to model the behavior of waves at infinity. This condition is called the *Sommerfeld radiation condition*. It ensures that no waves are reflected from ∞. We add the following constraints to (1);

$$\frac{1}{R^{(d-1)/2}} \left(\frac{\partial u}{\partial R} - iku \right) \to 0, \quad \text{as } R \to \infty, \quad \Omega = \mathbb{R}^d, \ d = 1, 2, 3. \tag{6}$$

Solutions to Helmholtz problems on infinite domains that satisfy this constraint are also called *radiating* solutions.

However, this condition alone does not solve the problem that we have to discretize an unbounded domain. Since computers only have a finite amount of memory, we need to find a way to discretize an infinite amount of space into a finite set of elements. A common approach is to add an artificial boundary to the domain, replacing the unbounded domain with a bounded one. In order for the solution to the bounded problem to be sufficiently close to the solution for the unbounded domain, an *absorbing boundary condition* is applied. This ensures that there is no reflection or scattering of waves at the boundary.

As an example we consider the one dimensional Helmholtz problem on $\Omega = \mathbb{R}$. The exact solution to the static problem is given by

$$u(x) = Ae^{ikx} + Be^{-ikx},$$

where A and B are determined by the initial conditions. The dynamic solution is given by

$$P(x,t) = Ae^{i(kx-\omega t)} + Be^{-i(kx+\omega t)}$$

where A is the amplitude of the wave travelling from 0 to ∞, and B is the amplitude of the wave travelling from ∞ to 0. Applying the boundary condition

$$\frac{\omega}{k}\frac{\partial P}{\partial x}(x_0) + \frac{\partial P}{\partial t}(x_0) = 0$$

at a point $x_0 \in \Omega$ eliminates the incoming wave. This is called a *non-reflecting boundary condition* at x_0.

In higher dimensions, construction such a NRBC can be difficult, because the direction in which waves travel in the exact solution have to be known. In this case a ABC can be used to approximate the NRBC. Other methods for solving the Helholtz problem on an unbounded domain are the *Perfectly Matched Layers* and the *infinite element* method [**?**, p. 20].

3 Large wave numbers and error estimates

For higher dimensional problems, a large wavelength k can cause additional problems that can lead to a less accurate approximation to the exact solution. One of the approaches used to counter this problem is using higher order piecewise polynomial functions instead of piecewise linear function. This method is called the *hp* approach. When the search space exists of piecewise polynomial functions of order p, the error in our approximation is bounded by

$$\frac{\|u-U\|}{\|u\|} \leq C_1\left(\frac{hk}{2p}\right)^p + C_2 k\left(\frac{hk}{2p}\right)^{2p}.$$

Here, the first term represents the *approximation error*, and the second term is the *pollution error*, which is caused by errors in modeling the exact wavelength propagating through the computation of the approximation. We see that an increase in p leads to a great reduction of the pollution error, without needing to decrease h.

Another way to reduce the pollution error is by using specific basic functions that provide a good fit for high frequency scattering problems. Two methods that use this approach are the *generalized finite element methods* and the *partition of unity method* [**?**, p. 21]. One last problem that occurs when k becomes large is the highly oscillatory behavior of the integrals in (4). The numerical approximations of these integrals becomes more expensive as k increases, which increases the pollution error.