1 Helmholtz equation

An important parial differential equation in engineering is the *Laplace equation*. In two dimensions, it has the following form:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1}$$

When there is a non-zero forcing term, the equation is called the *Poisson equation*:

$$\nabla^2 u = g(x, y). \tag{2}$$

They are both examples of *elliptic* equations. Finite Difference Methods are often used to solve this kind of problems. We will derive some of the specific applications of FDM in one and two dimensions.

The Helmholtz problem considers an unknow, but uniquely determined function. Suppose we have a region R in the xy-plane, on which the function $u = u(x, y)$ takes its values. Here, u satisfies

$$\begin{cases} \nabla^2 u + fu = g \\ u(x, y) = q(x, y) \quad \text{on boundary of } R, \end{cases} \tag{3}$$

with $f = f(x, y)$ and $g = g(x, y)$ smooth functions defined on R.

2 FDM 1D Case

All Finite Difference Methods are created by discritizing u on a grid on R. The second derivative of u at a certain point in R is approximated by taking a linear combination of the values of u at the neighbouring points.

- Notation
- Grid size
- Reference image

$$u_{i+1} = u_i + h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} + \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 u}{\partial x^4} + \mathcal{O}(h^5) \tag{4}$$

$$u_{i-1} = u_i - h\frac{\partial u}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u}{\partial x^2} - \frac{h^3}{6}\frac{\partial^3 u}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 u}{\partial x^4} + \mathcal{O}(h^5) \tag{5}$$

By adding these two Tailor series, we obtain an estimate for the second order derivative of u at x_i:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{h^2}[u_{i+1} - 2u_i + u_{i-1}] + \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}. \tag{6}$$

We see that it is second order accurate.

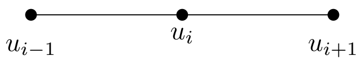

Figure 1: Image comes here