1 Finite difference method

In the finite difference method, we discetize the exact solution of the Helmholtz problem $u(x)$ on a grid R defined on a domain $\Omega \subseteq \mathbb{R}^d$, $d = 1, 2, 3$. Suppose we have a one dimensional problem with $\Omega = (0, 1)$. We want to approximate u in a fixed number of points, say N. We create a uniform grid

$$X_h = \{x_i \mid x_i = hi,\ i = 0, 1, \ldots, N\} \tag{1}$$

where we take our gridsize $h = 1/N$. We denote $u_i = u(x_i)$. Now, we want to discretize our equation for u on Ω

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + fu = g, \quad \text{on } \Omega. \tag{2}$$

We can approximate the second derivate of u in a point x_i by using the Taylor expansion of u at x_i. The value of u at points x_{i+1} and x_{i-1} are given by

$$u_{i+1} = u_i + h\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{h^2}{2}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{h^3}{6}\frac{\mathrm{d}^3 u}{\mathrm{d}x^3} + \frac{h^4}{24}\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} + \mathcal{O}(h^5) \tag{3}$$

$$u_{i-1} = u_i - h\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{h^2}{2}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - \frac{h^3}{6}\frac{\mathrm{d}^3 u}{\mathrm{d}x^3} + \frac{h^4}{24}\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} + \mathcal{O}(h^5). \tag{4}$$

By adding these equations, we obtain an estimate for $\mathrm{d}^2 u/\mathrm{d}x^2$ in the point x_i,

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}(x_i) \approx \frac{1}{h^2}\left[u_{i+1} - 2u_i + u_{i-1}\right]. \tag{5}$$

This method provides an approximation of order $\mathcal{O}(h^2)$, since the local error is

$$-\frac{h^2}{12}\frac{\mathrm{d}^4 u}{\mathrm{d}x^4}(x_i).$$

When we refine our grid, and let $h \to 0$, the error in our approximation to the exact solution u goes to 0 as rapidly as h^2. Of course, the computational effort necessary will increase as we make our grid smaller.

1.1 Five-point formula

We can easily expand the finite difference approach to two dimensions, by approximating $\nabla^2 u = \mathrm{d}^2 u/\mathrm{d}x^2 + \mathrm{d}^2 u/\mathrm{d}y^2$ as

$$\nabla^2 u(x, y) \approx \frac{1}{h^2}\left[u(x + h, y) + u(x - h, y) + u(x, y + h) + u(x, y - h) - 4u(x, y)\right]. \tag{6}$$