1 Finite difference method

In the finite difference method, we discetize the exact solution of the Helmholtz problem $u(x)$ on a grid R defined on a domain $\Omega \subseteq \mathbb{R}^d$, $d = 1, 2, 3$. Suppose we have a one dimensional problem with $\Omega = (0, 1)$. We want to approximate u in a fixed number of points, say N. We create a uniform grid

$$X_h = \{x_i \mid x_i = hi,\ i = 0, 1, \ldots, N\} \tag{1}$$

where we take our gridsize $h = 1/N$. We denote $u_i = u(x_i)$. Now, we want to discretize our equation for u on Ω

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + fu = g, \quad \text{on } \Omega. \tag{2}$$

We can approximate the second derivate of u in a point x_i by using the Taylor expansion of u at x_i. We can approximate the value of u at points x_{i+1} and x_{i-1} by

$$u_{i+1} \approx u_i + h\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{h^2}{2}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{h^3}{6}\frac{\mathrm{d}^3 u}{\mathrm{d}x^3} + \frac{h^4}{24}\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} \tag{3}$$

$$u_{i-1} \approx u_i - h\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{h^2}{2}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - \frac{h^3}{6}\frac{\mathrm{d}^3 u}{\mathrm{d}x^3} + \frac{h^4}{24}\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} \tag{4}$$