Introduction

1 Finite Element Method in one dimension

When searching for an approximate solution to the Helmholtz problem using a Finite Element approach, we aim to approximate a function u by constructing a finite linear system and solving the linear equations. For this approximation to be possible, we want to restrict our search space for $U(x) \approx u(x)$ to a finite dimensional function space.

In this section, we will show how an approximation for u can be found using the *Galerkin finite element method* in both one and higher dimensions. We will give error estimates in both cases and we will address the problems that arise when dealing with heavily oscillating functions.

1.1 Search space

Suppose we have a bounded domain in $\mathbb{R}^n$, $n = 1, 2, 3$. We define the function space $L_2(\Omega)$ of square integrable function on Ω by saying that $f \in L_2(\Omega)$ if

$$\|f\| := \left(\int_\Omega |f(x)|^2 d\Omega \right)^{\frac{1}{2}} < \infty. \tag{1}$$

We define the *Sobolev space* $H^1(\Omega)$ by saying $f \in H^1(\Omega)$ if

$$\|\nabla f\|^2 + \|f\|^2 < \infty. \tag{2}$$

Finaly, we say $f \in H^1_{(0}(\Omega)$ if $f \in H^1(\Omega)$ and $f(0) = 0$.

1.2 Exact solution

As an example, we will consider a simple one dimensional wave problem. Suppose we have the following conditions for a time-harmonic function $u(x)$ on $[0, 1]$:

$$\begin{cases} -\dfrac{d^2u}{dx^2} - k^2 u = f & \text{on } \Omega = (0,1) \\ u(0) = 0 \\ \dfrac{du}{dx}(1) - iku(1) = 0 \end{cases} \tag{3}$$

where we have a Dirichlet boundary condition at $x = 0$ and an impedance or Robin boundary equation in $x = 1$. Here, the forcing term $f \in L_2(\Omega)$.

It can be shown that the exact solution to is given by

$$u(x) = \frac{e^{ikx}}{k} \int_0^x \sin(ks) f(s) ds + \frac{\sin(kx)}{k} \int_x^1 e^{iks} f(s) ds, \tag{4}$$

which is periodic with wavelenth $\lambda = \frac{2\pi}{k}$. While this problem has a simple solution, exact solutions to more complex and higher order problems are difficult, nigh impossible to find in a simmilar matter.

For solving the above system with a finite element method, we will first rewrite the problem in its weak form. We multiply both sides with a test function $v \in H^1_{(0}(\Omega)$ and integrate afterwards to obtain

$$-\int_0^1 u''(x)v(x) - k^2 u(x)v(x)dx = \int_0^1 f(x)v(x)dx. \tag{5}$$

We integrate the first term by part to get

$$[-u'(x)v(x)]_0^1 + \int_0^1 u'(x)v'(x)dx - k^2 \int_0^1 u(x)v(x)dx = \int_0^1 f(x)v(x)dx. \tag{6}$$

By subsituting the impedance boundary condition (3) and taking the fact that $v(0) = (0)$ into consideration, we obtain

$$\int_0^1 u'(x)v'(x)dx - k^2 \int_0^1 u(x)v(x)dx - iku(1)v(1) = \int_0^1 f(x)v(x)dx. \tag{7}$$

1.3 Galerking finite element method

To solve (7), we will start by defining a finite element mesh

$$X_h := \{x_i; 0 = x_0 < x_1 < \ldots < x_N = 1\}, \tag{8}$$

on $\Omega = (0, 1)$. For simplicity, we will use a *uniform* mesh, where all elements x_i have size $h = 1/N$. We limit our search space to $S_h(0, 1) \subset H^1_{(0}(\Omega)$ as the space of piecewise continuous linear functions with nodal values at points in X_h, satisfying (**??**) at $x = 0$. This function space is spanned by the set of hat functions defined as

$$\chi_j(x) = \begin{cases} \frac{1}{h}(x - x_{j-1}), & x \in [x_{j-1}, x_j], \\ \frac{1}{h}(x_{j+1} - x), & x \in [x_j, x_{j+1}], \\ 0 & \text{elsewhere}, \end{cases} \tag{9}$$

for $j = 1, 2, \ldots, N-1$ and for $j = N$ by

$$\chi_N(x) = \begin{cases} \frac{1}{h}(x - x_{j-1}), & x \in [x_{j-1}, 1], \\ 0 & \text{elsewhere}, \end{cases} \tag{10}$$

Now we search for an approximate solution by requiring $U, v \in S_h(0, 1)$.

1.4 Linear system

1.5 Error estimates

2 Higher dimensional problem

2.1 Linear system

2.2 Mesh generation

3 Unbounded domains

3.1 Absorbing boundary conditions

4 Large wavenumbers and error estimates