1 Higher dimensional problem

For higher dimensional problems, again we consider the general notations for the Helmholtz problem. For a domain $\Omega \in \mathbb{R}^2$ with boundary Γ, the Helmholtz problem is stated as follows

$$\Delta u + k^2 u = 0, \quad \text{in } \Omega, \tag{1}$$

$$\frac{\partial u}{\partial n} + \beta u = g, \quad \text{on } \Gamma, \tag{2}$$

where $k \in \mathbb{R}$, $\beta \in \mathbb{C}$ and $\partial/\partial n$ is the outward normal derivative. Note that

$$\frac{\partial u}{\partial n} = \nabla u \cdot n, \quad \Delta u = \nabla^2 u.$$

Again, we transform the problem to a weak formulation by first multiplying (1) by a test function $v \in H^1(\Omega)$ and integrating it to get

$$\int_\Omega v\Delta u \;\mathrm{d}\Omega + \int_\Omega k^2 uv \;\mathrm{d}\Omega = 0. \tag{3}$$

By applying Green's identity

$$\int_\Omega v\Delta u \;\mathrm{d}\Omega + \int_\Omega \nabla v \cdot \nabla u \;\mathrm{d}\Omega = \int_\Gamma v(\nabla u \cdot n) \;\mathrm{d}\Gamma,$$

we find that

$$\int_\Gamma v\frac{\partial u}{\partial n} \;\mathrm{d}\Gamma - \int_\Omega \nabla v \cdot \nabla u - k^2 uv \;\mathrm{d}\Omega = 0.$$

From multiplying boundary equation (2) with v and integrating, we obtain

$$\int_\Gamma v\frac{\partial u}{\partial n} \;\mathrm{d}\Gamma = -\beta \int_\Gamma uv \;\mathrm{d}\Gamma + \int_\Gamma vg \;\mathrm{d}\Gamma.$$

By substituting these equations into (3) we get the weak formulation of our problem; find $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla v \cdot \nabla u - k^2 uv \;\mathrm{d}\Omega + \beta \int_\Gamma uv \;\mathrm{d}\Gamma = \int_\Gamma vg \;\mathrm{d}\Gamma \tag{4}$$

for all $v \in H^1(\Omega)$.

As for the one dimensional problem, we will restrict our search space to a finite dimensional function space $V \subset H^1(\Omega)$. We define V to be the span of basis functions χ_j, $j = 1, 2, \ldots, N$. When we require that $U, v \in V$, we are able to write

$$U(x) = \sum_{j=1}^N u_j \chi_j(x), \tag{5}$$

where we have to determine the values of u_j. The problem can now be reformulated as

$$\sum_{j=1}^N \left[\int_\Omega \nabla\chi_j \cdot \nabla\chi_m - k^2 \chi_j\chi_m \;\mathrm{d}\Omega + \beta \int_\Gamma \chi_j\chi_m \;\mathrm{d}\Gamma\right] u_j = \int_\Gamma \chi_m g \;\mathrm{d}\Gamma, \tag{6}$$

for $m = 1, 2, \ldots N$. Expressed as a linear system, this becomes

$$A\mathbf{u} = \mathbf{f},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1N} \\ a_{21} & a_{22} & \ldots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \ldots & a_{NN} \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} \int_\Gamma \chi_1 g \;\mathrm{d}\Gamma \\ \int_\Gamma \chi_2 g \;\mathrm{d}\Gamma \\ \vdots \\ \int_\Gamma \chi_N g \;\mathrm{d}\Gamma \end{pmatrix}.$$

Here, the matrix entries a_{jm} of A are defined as

$$a_{jm} = \int_\Omega \nabla\chi_j \cdot \nabla\chi_m - k^2 \chi_j\chi_m \;\mathrm{d}\Omega + \beta \int_\Gamma \chi_j\chi_m \;\mathrm{d}\Gamma.$$

1.1 Linear system

1.2 Mesh generation

2 Unbounded domains

2.1 Absorbing boundary conditions

3 Large wavenumbers and error estimates