Finite Difference and Finite Element Methods for Helmholtz scattering problems

Hidde-Jan Jongsma

RuG

April 15, 2011

Outline

What is FDM/FEM?

Frequently used for acoustic electromagnetic scattering problems.
Approach:

- Discretize domain Ω.
- Create new constraints that mimic 'real' ones.
- Find solution to these constrains that *approximate* the exact solution.

Unbounded domains or complex geometries $\rightarrow$ use FEM.

Helmholtz equations

Definition

The Helmholtz equations are given by

$$\Delta u + k^2 u = 0, \quad \text{in } \Omega, \tag{1}$$

$$\frac{\partial u}{\partial n} + \beta u = g, \quad \text{on } \Gamma, \tag{2}$$

for $u : \Omega \to \mathbb{R}$.

Finite difference method

- Grid on domain $\Omega \subseteq \mathbb{R}^d$, $d = 1, 2, 3$.
- Approximate $\nabla^2 u$ by finite difference.

$$\nabla^2 u + fu = g, \text{ on } \Omega.$$

Grid on 1D $\Omega = (0, 1)$:

$$X_h = \{x_i \mid x_i = hi, \ i = 0, 1, \ldots, N\}$$

Finite difference

We define $u_i = u(x_i)$. Use Taylor series to approximate $\mathrm{d}^2 u/\mathrm{d}x^2$:

$$u_{i+1} = u_i + h\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{h^2}{2}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{h^3}{6}\frac{\mathrm{d}^3 u}{\mathrm{d}x^3} + \frac{h^4}{24}\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} + \mathcal{O}(h^5) \qquad (3)$$

$$u_{i-1} = u_i - h\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{h^2}{2}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - \frac{h^3}{6}\frac{\mathrm{d}^3 u}{\mathrm{d}x^3} + \frac{h^4}{24}\frac{\mathrm{d}^4 u}{\mathrm{d}x^4} + \mathcal{O}(h^5). \qquad (4)$$

This results in

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2}(x_i) \approx \frac{1}{h^2}\left[u_{i+1} - 2u_i + u_{i-1}\right], \qquad (5)$$

which has a truncation error of the form $-\frac{h^2}{12}\frac{\mathrm{d}^4 u}{\mathrm{d}x^4}(x_i)$.

Five-point formula

For two dimensions, the commonly used form is the *five-point formula*.

$$\nabla^2 u(x,y) \approx \frac{1}{h^2}\big(u(x+h,y) + u(x-h,y) + u(x,y+h) + u(x,y-h) - 4u(x,y)\big).$$

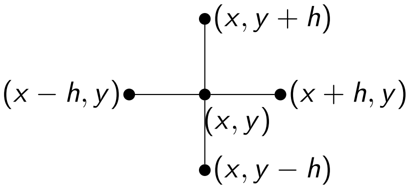

Figure: Five-point stencil

Finite element method

- Approximate solution to weak form of problem.
- Uses search space of functions defined on set of *finite elements*.

Definition

$f \in L_2(\Omega)$:

$$\|f\| := \left(\int_\Omega |f(x)|^2 \, \mathrm{d}\Omega \right)^{\frac{1}{2}} < \infty.$$

$f \in H^1(\Omega)$:

$$\|\nabla f\|^2 + \|f\|^2 < \infty.$$

$f \in H^1_{(0}(\Omega)$ if $f \in H^1(\Omega)$ and $f(0) = 0$.

Weak formulation

The weak form of the problem is given by

$$-\int_0^1 u''(x)v(x) - k^2 u(x)v(x)\,\mathrm{d}x = \int_0^1 f(x)v(x)\,\mathrm{d}x,$$

where $v \in H^1_{(0}(\Omega)$. We integrate the first term by parts to get

$$\int_0^1 u'(x)v'(x)\,\mathrm{d}x - k^2 \int_0^1 u(x)v(x)\,\mathrm{d}x - iku(1)v(1) = \int_0^1 f(x)v(x)\,\mathrm{d}x.$$

Basis functions

We refine ours search space to the piecewise linear function $S_h(0,1)$.
This is spanned by the *basis functions*, $j = 1, \dots, N-1$

$$\chi_j(x) = \begin{cases} \frac{1}{h}(x - x_{j-1}), & x \in [x_{j-1}, x_j], \\ \frac{1}{h}(x_{j+1} - x), & x \in [x_j, x_{j+1}], \\ 0 & \text{elsewhere}, \end{cases},$$

$$\chi_N(x) = \begin{cases} \frac{1}{h}(x - x_{j-1}), & x \in [x_{j-1}, 1], \\ 0 & \text{elsewhere}. \end{cases}$$

Approximation

We require $U(x) \in S_h(0,1)$, we can write

$$U(x) = \sum_{j=1}^{N} u_j \chi_j(x).$$

Our problem reduces to:

$$\sum_{j=1}^{N} \left[\int_0^1 \chi_j'(x)\chi_m'(x)\,\mathrm{d}x - k^2 \int_0^1 \chi_j(x)\chi_m(x)\,\mathrm{d}x \right] u_j - iku_N\chi_m(1) = \int_0^1 f(x)\chi_m(x)\,\mathrm{d}x,$$

for $m = 1, 2, \ldots, N$.

Linear system

This gives rise to a linear system of the form

$$(A - k^2 B - ikC)u = f,$$

where

$$A_{ij} = \int_0^1 \chi_i'(x)\chi_j'(x)\,\mathrm{d}x, \quad B_{ij} = \int_0^1 \chi_i(x)\chi_j(x)\,\mathrm{d}x.$$

$$A = \begin{pmatrix} \frac{2}{h} & -\frac{1}{h} & 0 & \dots & 0 \\ -\frac{1}{h} & \frac{2}{h} & -\frac{1}{h} & \ddots & \vdots \\ 0 & -\frac{1}{h} & \frac{2}{h} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\frac{1}{h} \\ 0 & \dots & 0 & -\frac{1}{h} & \frac{1}{h} \end{pmatrix}, \quad B = \begin{pmatrix} \frac{2h}{3} & \frac{h}{6} & 0 & \dots & 0 \\ \frac{h}{6} & \frac{2h}{3} & \frac{h}{6} & \ddots & \vdots \\ 0 & \frac{h}{6} & \frac{2h}{3} & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{h}{6} \\ 0 & \dots & 0 & \frac{h}{6} & \frac{h}{3} \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}, \quad f = \begin{pmatrix} \int_0^1 f(x)\chi_1(x)\,\mathrm{d}x \\ \int_0^1 f(x)\chi_2(x)\,\mathrm{d}x \\ \vdots \\ \int_0^1 f(x)\chi_N(x)\,\mathrm{d}x \end{pmatrix}.$$

Matrix $(A - k^2B - ikC)$ is *sparse*, *tridiagonal*.

Difficulties and pitfalls

Error estimate

$$\frac{\|u - U\|}{\|u\|} \leq C_1 kh + C_2 k^3 h^2,$$

where C_1, C_2 independent of k, h.

Higher wavenumber $\rightarrow$ *smaller mesh size.*

Size of linear system grows very rapidly, as does the *bandwidth* of the system matrix.

Higher order problem

- Mesh generation non-trivial (Delaunay triangulation).
- Rapidly growing linear system.
- Higher wavenumber means more computational effort.

Remedy: order of piece wise polynomial search space functions.

Special boundary conditions for unbounded domains: *absorbent boundary condition*, *non-reflecting boundary condition*.

Conclusion

- FDM is still used, but FEM is more flexible.
- If the whole domain is important $\rightarrow$ use FEM.
- Unbounded domains can be tackled using special BC's.
- Sparse matrices with low bandwidth.

If you have an unbounded domain and are only interested in surface of an object: use BEM.

Thank you for listening