1 Finite difference method

In the finite difference method, we discetize the exact solution of the Helmholtz problem $u(x)$ on a grid R defined on a domain $\Omega \subseteq \mathbb{R}^d$, $d = 1, 2, 3$. Suppose we have a one dimensional problem with $\Omega = (0, 1)$. We want to approximate u in a fixed number of points, say N. We create a uniform grid

$$X_h = \{x_i \mid x_i = hi,\ i = 0, 1, \dots, N\} \tag{1}$$

where we take our gridsize $h = 1/N$. We denote $u_i = u(x_i)$. Now, we want to discretize our equation for u on Ω

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + fu = g, \quad \text{on } \Omega. \tag{2}$$

We can approximate the second derivate of u in a point x_i by using the Taylor expansion of u at x_i. The value of u at points x_{i+1} and x_{i-1} are given by

$$u_{i+1} = u_i + h\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{h^2}{2}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \frac{h^3}{6}\frac{\mathrm{d}^3u}{\mathrm{d}x^3} + \frac{h^4}{24}\frac{\mathrm{d}^4u}{\mathrm{d}x^4} + \mathcal{O}(h^5) \tag{3}$$

$$u_{i-1} = u_i - h\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{h^2}{2}\frac{\mathrm{d}^2u}{\mathrm{d}x^2} - \frac{h^3}{6}\frac{\mathrm{d}^3u}{\mathrm{d}x^3} + \frac{h^4}{24}\frac{\mathrm{d}^4u}{\mathrm{d}x^4} + \mathcal{O}(h^5). \tag{4}$$

By adding these equations, we obtain an estimate for $\mathrm{d}^2u/\mathrm{d}x^2$ in the point x_i,

$$\frac{\mathrm{d}^2u}{\mathrm{d}x^2}(x_i) \approx \frac{1}{h^2}\left[u_{i+1} - 2u_i + u_{i-1}\right]. \tag{5}$$

This method provides an approximation of order $\mathcal{O}(h^2)$, since the local error is

$$-\frac{h^2}{12}\frac{\mathrm{d}^4u}{\mathrm{d}x^4}(x_i).$$

When we refine our grid, and let $h \to 0$, the error in our approximation to the exact solution u goes to 0 as rapidly as h^2. Of course, the computational effort necessary will increase as we make our grid smaller.

1.1 Five-point formula

We can easily expand the finite difference approach to two dimensions, by approximating $\nabla^2 u = \mathrm{d}^2u/\mathrm{d}x^2 + \mathrm{d}^2u/\mathrm{d}y^2$ as

$$\nabla^2 u(x, y) \approx \frac{1}{h^2}\left[u(x+h, y) + u(x-h, y) + u(x, y+h) + u(x, y-h) - 4u(x, y)\right]. \tag{6}$$

This equation is known as the five-point formula, since it incorporates the value of u at five different points in the plane. An arrangement of points used for the approximation of a differential equation is called a *stencil*. The stencil used in the five-point method is shown in Figure 1 and is named the five-point stencil.

The five-point formula can be modified to account for an irregular grid. However, this method is often only order $\mathcal{O}(h)$ accurate, where h is a bound on the stepsize. This method is commonly used at the boundary of the domain. For small h, one can shift the boundary points such that the regular five-point method can be used. The errors introduced by this transformation have been shown to be no greater than those introduced by the use of an irregular grid.

1.2 Linear System

In the two dimensional case, we transform the Helmholtz problem to

$$\frac{1}{h^2}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}\right] + f_{ij}u_{ij} = g_{ij}, \tag{7}$$

alternatively written as

$$-u_{i+1,j} - u_{i-1,j} - u_{i,j+1} - u_{i,j-1} + (4 - h^2 f_{ij})u_{ij} = -h^2 g_{ij}, \tag{8}$$

where

$$u_{ij} = u(ih, jh), \quad f_{ij} = f(ih, jh) \quad \text{and } g_{ij} = g(ih, jh).$$

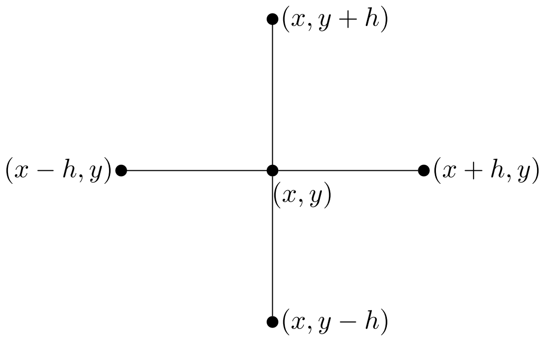

Figure 1: Five-point stencil.