Application of Malliavin Calculus in Finance

Xinda Wu

May 5, 2020

Abstract

The first part will be a brief explanation of what the Greeks are. The main goal in this talk is to derive some central results from [1], namely the Malliavin weights for the Greeks. These results rely on Malliavin calculus, so the second part will give a summary of some relevant theorems that will be needed. Part 3 is the discussion on [1].

1 Introduction

Investments in financial market usually come with risk and the reduction of risk becomes vital in asset management. One simple form risk reduction method, called *hedging*, is to diversify the investment into different assets that are negatively correlated, such as bonds versus stocks.

In 1970s, new types of financial assets called derivatives, such as options, were designed and introduced. Basically, it is a contract that derives its value from another underlying asset, which could be stocks, commodities. These derivatives provide a efficient way to reduce portfolio risk.

Given the large amount of capital involved, hedging away risk associated with a financial derivative become important and it turns out that the necessary strategy required to hedge away the risk can be found through a set of quantities known as the *sensitivity parameters*, commonly referred as the *Greeks*.

1.1 Greeks

The Greeks are quantities representing the sensitivity of the price of the derivatives to a change in underlying parameters on which the value of an instrument or portfolio of financial instruments is dependent. They are also called risk sensitivities, risk measures, or hedge parameters.

They are vital tools in risk management. Each Greek measures the sensitivity of the value of a portfolio to a small change in a given underlying parameter, so that component risks may be treated in isolation, and the portfolio rebalanced accordingly to achieve a desired exposure; see for example *Delta Hedging*.

Delta

Under the assumptions of the Black-Scholes market(Non-arbitrage, frictionless, continuous trading, possibility to short selling, etc.), Stoke prices are modeled by the geometric Brownian motion, given by the SDE:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, S_0 = x$$

Given a financial derivative in the form of a call option on some underlying stock S_t, the option price is given by $u(t, S_t)$. The delta, denoted by the Greek letter Δ, is referred to as the sensitivity of the option with respect to the stock price x, and is a measure of how movements in the stock price affect the option value. It is formally defined as the derivative of the option value with respect to the stock price: $\Delta := \frac{\partial u}{\partial x}(t, S_t)$.

Gamma

The sensitivity of the Δ with respect to the stock price x, known as the gamma, denoted by Γ, defined as $\Gamma := \frac{\partial}{\partial x}\Delta = \frac{\partial^2 u}{\partial x^2}(t, S_t)$. It is a measure of how often a position must be re-hedged in order to maintain a delta neutral position, so to minimize the amount of necessary re-hedging and the corresponding cost.

Vega

The sensitivity to the volatility of the underlying asset, denoted by ν, defined as $\nu := \frac{\partial u}{\partial \sigma}$. Vega hedging means including additional options to the portfolio with the goal of achieving $\nu = 0$.

Rho

The sensitivity to the interest rate, denoted by ρ, commonly defined as $\rho := \frac{\partial u}{\partial r}$, where r is the risk free interest rate from the Black-Scholes market. More generally, it is defined as the derivative with respect to the model drift, i.e $\rho := \frac{\partial u}{\partial \mu}$.

Other Greeks

There many more Greeks of higher orders such as Speed, Vanna, Ultima, etc.

1.2 Numerical Calculation

The underlying asset is assumed to be given by $\{X_t, t \geq 0\}$ which is a markov process with real value and whose dynamics are described by the SDE

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t \tag{1}$$

where $\{W_t, 0 \leq t \leq T\}$ is a Brownian motion. The coefficient b and σ are assumed to satisfy usual conditions in order to ensure the existence and uniqueness of a continuous adapted solution of equation (1). Given $0 < t_1 \leq ... \leq t_m = T$, we consider the function

$$u(x) = \mathbb{E}[\phi(X(t_1), ..., X(t_m)|X(0) = x], \tag{2}$$

$u(x)$ describes the price of a contingent claim defined by a payoff function ϕ involving the times $(t_1, ..., t_m)$, see [2, 3]. Examples of such contingent claims include both usual and path dependent options and more sophiticated derivatives. The function $u(x)$ can then be computed by Monte Carlo methods. Besides, financial applications also require us to compute the Greeks, the derivatives of $u(x)$. A natural approach to this numerical problem is to compute by Monte Carlo simulation the finite difference approximation of the differentials. People tried different finite difference methods, such as forward finite difference, central difference, but they either perform poorly when the payoff function is not smooth enough or have low convergence rate. After that, alternative methods were suggested, however they also have similar drawbacks. In 1999 (see [1]), a new approach was introduced to calculate the Greeks by using Malliavin calculus. It is shown that all the differentials of interest can be expressed as

$$\mathbb{E}[\,\pi\,\phi(X(t_1), ..., X(t_m)|X(0) = x\,], \tag{3}$$

where π is a random variable to be determined later on, known as the Malliavin weight. Therefore the required differential can be computed numerically by Monte Carlo simulation and the estimator achieves the $n^{-1/2}$ usual convergence rate. An important advantage of this differential formula is that the weight π does not depend on the payoff function ϕ. This makes it possible to apply even when the payoff function is discontinuous, eliminating the weakness of past methods. The main goal of this talk is to derive the Malliavin weights for some of the most important Greeks.

2 Results from Malliavin Calculus

Denote the standard Brownian motion by W_t for $t \in [0, T]$ on complete probability space $(\Omega, \mathcal{F}, P)$ such that $W_0 = 0,\ a.s.$. The σ-algebra generated by the Brownian Motion W_t is denoted by $\mathcal{F}_t$.

Property 1 (The Chain Rule) *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous differentiable function with bounded partial derivatives and $F \in \mathbb{D}^{1,2}$ a random variable then $\phi(F) \in \mathbb{D}^{1,2}$ and:*

$$D_t\phi(F) = \phi'(F)D_tF, \quad t \geq 0 \ a.s.$$

Property 2 (Malliavin Derivative and First Variation) *Let $\{X_t, t \geq 0\}$ be an $\mathbb{R}$ value Itô process whose dynamics are diven by the SDE:*

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t,$$

where b and σ are continuously differentiable with bounded derivatives. Let $\{Y_t, t \geq 0\}$ be the associated first variation process defined as $Y_t := \frac{\partial}{\partial x} X_t$ whose dynamics are driven by the SDE:

$$dY_t = b'(X_t)Y_t dt + \sigma'(X_t)Y_t dW_t, \ Y_0 = 1, \tag{4}$$

Then the process $\{X_t, t \geq 0\}$ belongs to $\mathbb{D}^{1,2}$ and its Malliavin derivative is given by:

$$D_s X_t = Y_t Y_s^{-1} \sigma(X_s) 1_{\{s \leq t\}}, \ s \geq 0 \ a.s.$$

Property 3 (Divergence operator δ) *Let u be a stochastic process. Then $u \in Dom(\delta)$ if for any $\phi \in \mathbb{D}^{1,2}$, we have*

$$\mathbb{E}[< D\phi, u >_H] := \mathbb{E}[\int_0^\infty D_t \phi u(t) dt] \leq C(u) \|\phi\|_{1,2}.$$

If $u \in Dom(\delta)$, we define $\delta(u)$ by:

$$\mathbb{E}[\, \phi \, \delta(u) \,] = \mathbb{E}[< D\phi, u >_H] \ for \ any \ \phi \in \mathbb{D}^{1,2}. \tag{5}$$

Property 4 (Itô integral and δ) *The domain of the divergence operator δ contains that of the Itô integral, i.e. $L^2(\Omega \times \mathbb{R}_+) \subset Dom(\delta)$. Let u be an adapted stochatic process in $L^2(\Omega \times \mathbb{R}_+)$. Then the divergence operator δ coinsides with the Itô integral:*

$$\delta(u) = \int_0^\infty u_t dW_t.$$

3 Malliavin Weights for the Greeks

We will now derive the Malliavin weights for the Δ, ρ, ν. Without loss of generality, in order to avoid the multidimensional technicalities, restrict our discussion on one dimensional case. Also, in [1] the payoff function depends on m states of the underlying financial asset, but we make a slight simplification and only consider payoff functions depending on the terminal point X_T. Following the assumptions and notations in 1.2, denote the value of the option as

$$u(x) = \mathbb{E}[\, \phi(X_T^x) \,] \tag{6}$$

Assume from now on, ϕ has a bounded derivative and

$$\mathbb{E}[\, \phi(X_T^x)^2 \,] < \infty \tag{7}$$

in order to use the chain rule(property 1).
Note that similar arguments as in the proof of the following propositions for first-order Greeks can be applied for that of higher-order Greeks.

3.1 Delta - Variations in the initial condition

Define the set of square integrable functions a whose integral over $[0,T]$ equals 1 as:

$$\mathcal{A} := \{a \in L^2([0,T]) \mid \int_0^T a(t)dt = 1\} \tag{8}$$

In addition, we need following lemmas.

Lemma 1 (Exchange the expectation and the derivative) *Suppose $F^\theta \in \mathbb{R}$ is a random variable that depends on some parameters $\theta \in \mathbb{R}$ and suppose for almost every $\omega \in \Omega$ that the mapping $\theta \mapsto F^\theta(\omega)$ is continuously differentiable in $[a,b]$ and that*

$$\mathbb{E}[\sup_{\theta \in [a,b]} \left| \frac{\partial F^\theta}{\partial \theta} \right|] < \infty.$$

Then the mapping $\theta \mapsto \mathbb{E}[F^\theta]$ is differentiable in (a,b), and for $\theta \in (a,b)$ we can change the order of the derivative and the expectation:

$$\frac{\partial}{\partial \theta}\mathbb{E}[F^\theta] = \mathbb{E}[\frac{\partial}{\partial \theta}F^\theta].$$

Remark: It is not difficult to show that $F^\theta = \phi(X_T^x)$ with $\theta = x$, satisfies the above lemma.

The following lemma allows us to assume a smoothness condition for the payoff function ϕ. Denote the asset price X_t by X_t^θ to signify the dependence on some parameter θ.

Lemma 2 *Let $\theta \mapsto \pi^\theta$ be a process such that $\theta \mapsto \psi(\theta) := \|\pi^\theta\|_{L^2(P)}$ is locally bounded. Assume that:*

$$\frac{\partial}{\partial \theta}\mathbb{E}[\phi(X_T^\theta)] = \mathbb{E}[\phi(X_T^\theta)\pi^\theta]$$

is valid for all $\theta \in C_c^\infty(\mathbb{R})$(infinitely differentiable with compact support). Then we can extend this equality to all $\phi \in L^2(\mathbb{R})$.

Lemma 3 *Let $a \in \mathcal{A}$ as in (8). Then*

$$Y_T = \int_0^T (D_s X_T) Y_s \sigma(X_s)^{-1} a(s) ds.$$

Proposition 1 (Malliavin weight for Δ) *for $x \in \mathbb{R}$ and $a \in \mathcal{A}$, we have:*

$$\frac{\partial}{\partial x}u(x) = \mathbb{E}^x[\,\phi(X_T)\int_0^T a(t)[Y_t\sigma(X_t)^{-1}]dW_t\,],$$

so the Malliavin weight for Δ is $\pi^\Delta = \int_0^T a(t)[Y_t\sigma(X_t)^{-1}]dW_t$.

3.2 Rho - Variations in the drift coefficient

The previous Malliavin weight was derived using Malliavin calculus, but π^ρ requires a different approach. The weight is found by calculating the Gateaux derivative, a generalization of the partial derivative to Banach spaces, which is done in the drift direction through a perturbed process. The perturbed stochastic differential equation is the original equation (1) with a small length added in the drift direction. By using Girsanov's theorem the perturbed process is reduced to the original stochastic differential equation where we can derive the weight.

For some variable $\epsilon > 0$ and some bounded function $\gamma : [0,T] \times \mathbb{R} \mapsto \mathbb{R}$, the *perturbed process* X_t^ϵ is defined by its dynamics:

$$dX_t^\epsilon = [b(X_t^\epsilon) + \epsilon\gamma(X_t^\epsilon)]dt + \sigma(X_t^\epsilon)dW_t,\ X_0^\epsilon = x \tag{9}$$

note that $\epsilon = 0$ returns us to X_t as in (1). Associated with this perturbed process (9) is the perturbed option value:

$$u^\epsilon(x) = \mathbb{E}^x[\phi(X_T^\epsilon)]. \tag{10}$$

Proposition 2 (Malliavin weight for ρ) *The function $\epsilon \mapsto u^\epsilon(x)$ is differentiable in $\epsilon = 0$ for any $x \in \mathbb{R}$, and we have*

$$\frac{\partial}{\partial\epsilon} u^\epsilon(x)\bigg|_{\epsilon=0} = \mathbb{E}^x[\phi(X_T)\int_0^T \gamma(X_t)\sigma(X_t)^{-1}dW_t].$$

so we have $\pi^\rho = \int_0^T \gamma(X_t)\sigma(X_t)^{-1}dW_t$.

3.3 Vega - Variations in the diffusion coefficient

In a similar way as for Rho, we define the perturbed process:

$$dX_t^\epsilon = b(X_t^\epsilon) + [\sigma(X_t^\epsilon) + \epsilon\tilde{\sigma}(X_t^\epsilon)]dW_t,\ X_0^\epsilon = x. \tag{11}$$

where we assume $\tilde{\sigma}(X_t^\epsilon) > 0$ and $\sigma(X_t^\epsilon) + \epsilon\tilde{\sigma}(X_t^\epsilon) > 0$ for all $t \in [0,T]$. When $\epsilon = 0$ the equation (11) becomes the same as (1).
Define $Y_t^\epsilon := \frac{\partial}{\partial x} X_t^\epsilon$, the first variation process w.r.t. x, driven by:

$$dY_t^\epsilon = b'(X_t^\epsilon)Y_t^\epsilon + [\sigma'(X_t^\epsilon) + \epsilon\tilde{\sigma}'(X_t^\epsilon)]Y_t^\epsilon dW_t,\ Y_0^\epsilon = 1. \tag{12}$$

And define $Z_t^\epsilon = \frac{\partial}{\partial\epsilon} X_t^\epsilon$, the first variation process w.r.t. ϵ, driven by:

$$dZ_t^\epsilon = b'(X_t^\epsilon)Z_t^\epsilon + \tilde{\sigma}(X_t^\epsilon)dW_t + [\sigma'(X_t^\epsilon) + \epsilon\tilde{\sigma}'(X_t^\epsilon)]Z_t^\epsilon dW_t,\ Z_0^\epsilon = 0. \tag{13}$$

When $\epsilon = 0$ these three process will be denoted as X_t, Y_t and Z_t, respectively. Based on these process, define $\beta(t) := \frac{Z_t}{Y_t}$ for $t \in [0,T]$. Note that $\beta(0) = 0$.

Proposition 3 (Malliavin weight for ν) *For any $a \in \mathcal{A}$:*

$$\frac{\partial}{\partial\epsilon} u^\epsilon(x)\bigg|_{\epsilon=0} = \mathbb{E}^x[\,\phi(X_T)\delta(Y_t\sigma(X_t)^{-1}\tilde{\beta}_a(T))\,],$$

so $\pi^\nu = \delta(Y_t\sigma(X_t)^{-1}\tilde{\beta}_a(T))$, where $\tilde{\beta}_a(T) := (\beta(T) - \beta(0))a(t) = \beta(T)a(t)$.

References

[1] E. Fournié, J.-M- Lasry, J. Lebuchoux, P.-L. Lions, N. Touzi. *Applications of Malliavin calculus to Monte Carlo methods in finance I.* Finance and Stochastics, 1999.

[2] A. Eberle. *Intoduction to Stochastic Analysis.* Lecture note, 2019.

[3] D. Nualart. *The Malliavin calculus and related topics.* Springer-Verlag, Berlin, 2006.

[4] Di Nunno, G., Øksendal, B., Proske, F.. *Malliavin Calculus for Lévy Processes with Applications to Finance.* Springer, 2008.