PyRossGeo: The mean field approach of the geographical compartmental model for infective diseases

Hideki Kobayashi and Rajesh Singh

May 16, 2020

1. Model: Weighted interaction

We now consider a structured metapopulation $\boldsymbol{x} = \{x_{in} \,|\, i = 1, \dots M \times L,\ n = 1, \dots N_d\}$ consisting of M age-compartments and L classes of epidemiological states. We also add another $n = 1, \dots, N_d$, which specifies a geographical node, while the dynamics has a time scale of 1 hour. The interaction between the nodes is captured by the rate of infection of a susceptible individual. Although we apply this approach to SIR model, It can be applied to arbitrary models. This method uses 3 models depending on the time period called HOME, TRANSPORT and WORK.

In the HOME period means everyone is in the home node, the time progress of the system can be represented as the simple SIR model:

$$\begin{aligned}
\frac{\mathrm{d}S_{in}}{\mathrm{d}t} &= -\lambda_{in} S_{in} \\
\frac{\mathrm{d}I_{in}}{\mathrm{d}t} &= \lambda_{in} S_{in} - \gamma I_{in} \\
\frac{\mathrm{d}R_{in}}{\mathrm{d}t} &= \gamma I_{in} \\
\lambda_{in} &= \beta \sum_{j=1}^{M} C_{ij} \frac{I_{jn}}{N^H_{jn}}
\end{aligned}$$

whereS_{in}, I_{in} and R_{in} are defined as Susceptible, Infectious and Recovered of the residence in the node n and the age group$i$, respectively.N^H_{in} is the number of the people who live in n in age group i.

In the WORK period, someone moves to another node, and someone stays in the home node. Here we focus on the behavior of $\hat{\boldsymbol{S}}_{in}$ and $\hat{\boldsymbol{I}}_{in}$, which are the total value of the Susceptible and Infectious in the node n and age group i during WORK period. The definition of these are in following. The time evolution of the model is represented as:

$$\begin{aligned}
\frac{\mathrm{d}\hat{\boldsymbol{S}}_{in}}{\mathrm{d}t} &= -\lambda^W_{in} \hat{\boldsymbol{S}}_{in} \\
\frac{\mathrm{d}\hat{\boldsymbol{I}}_{in}}{\mathrm{d}t} &= \lambda^W_{in} \hat{\boldsymbol{S}}_{in} - \gamma \hat{\boldsymbol{I}}_{in} \\
\lambda^W_{in} &= r^W \beta \sum_{j=1}^{M} C_{ij} \frac{\hat{\boldsymbol{I}}_{jn}}{N^w_{jn}} \\
\hat{\boldsymbol{S}}_{in} &= \sum_{m=1}^{N_d} \frac{D_{i,nm}}{N^H_{im}} S_{im} = \sum_{m=1}^{N_d} P^H_{i,nm} S_{im} \\
\hat{\boldsymbol{I}}_{in} &= \sum_{m=1}^{N_d} \frac{D_{i,nm}}{N^H_{in}} I_{im} = \sum_{m=1}^{N_d} P^H_{i,nm} I_{im}
\end{aligned}$$

where $D_{i,nm}$, which can be obtained from the census dataset, is the number of people working at index n and living in m in the age group i, $N^H_{in} = \sum_m D_{i,mn}$ is the total number of residence people, $N^w_{in} = \sum_m D_{i,nm}$ is the total number of the people working at the node n in the age group i and $P^R_{i,nm} = D_{i,nm}/N^H_{in}$. Under the assumption that the term of S_{in} in both sides of the ODE of $\hat{\boldsymbol{S}}_{im}$ are equal to each other, S_{in} satisfies the equation as:

$$P^H_{i,mn}\frac{\mathrm{d}S_{in}}{\mathrm{d}t} = -\lambda^W_{im}P^H_{i,mn}S_{in}$$

The ODE of S_{in} can be obtained by summation for m in both sides. Thus, we can represent the ODE of S_{in} and I_{in} as:

$$\begin{aligned}
\frac{\mathrm{d}S_{in}}{\mathrm{d}t} &= -\sum_{m=1}^{N_d}\lambda^W_{im}P^H_{i,mn}S_{in} \\
\frac{\mathrm{d}I_{in}}{\mathrm{d}t} &= \sum_{m=1}^{N_d}\lambda^W_{im}P^H_{i,mn}S_{in} - \gamma I_{in}
\end{aligned}$$

Although we DO NOT need to expand the ODE of S_{in} and I_{in} in simulations, the expanded formula is written down for a reference:

$$\begin{aligned}
\frac{\mathrm{d}S_{in}}{\mathrm{d}t} &= -\sum_{m=1}^{N_d}\lambda^W_{im}P^H_{i,mn}S_{in} \\
&= -r^w\beta\sum_{j=1}^{M}\sum_{m=1}^{N_d}C_{ij}\frac{\hat{\boldsymbol{I}}_{jm}}{N^W_{jm}}P^H_{i,mn}S_{in} \\
&= -r^w\beta\sum_{j=1}^{M}\sum_{m,k=1}^{N_d}C_{ij}\frac{P^H_{j,mk}P^H_{i,mn}}{N^W_{jm}}I_{jk}S_{in}
\end{aligned}$$

The expanded formula insist that the incremantaion value of S_{in} includes the contact between I_{jk} and S_{in} in the node m. We call this effect as "second order effect".

In the TRANSPORT period, we focus on the behaviour of $S_{i,nm} = P^H_{i,nm}S_{im}$ and $\hat{\boldsymbol{I}}_{i,nm}$ that is the total value of the Infectious who will go from node m to node n. The time evolution of the model is represented as:

$$
\begin{aligned}
\frac{\mathrm{d}S_{i,nm}}{\mathrm{d}t} &= -\lambda^T_{i,nm} S_{i,nm} \\
\frac{\mathrm{d}\hat{\boldsymbol{I}}_{i,nm}}{\mathrm{d}t} &= \lambda^T_{i,nm} - \gamma \hat{\boldsymbol{I}}_{i,nm} \\
\lambda^T_{i,nm} &= r^T \beta \sum_{j=1}^{M} C_{ij} \frac{\hat{I}_{j,nm}}{N^T_{j,nm}} \\
S_{i,nm} &= P^H_{i,nm} S_{im} \\
\hat{\boldsymbol{I}}_{i,nm} &= \sum_{<kl>} I_{i,kl} \frac{d_{kl}}{d_{nm}} \\
N^T_{i,nm} &= \sum_{<kl>} R_{i,kl} \frac{d_{kl}}{d_{nm}}
\end{aligned}
$$

where $N^T_{i,nm}$ is the effective number of the commuter in the age group i using the route between n and m, $< kl >$ indicate summation of all pair of neighbour node k and l included in the shortest path between node n and m and d_{nm} is the distance between node n and m along the shortest path. $R_{i,nm} = \sum_{<kl>} D_{i,kl}$ is the number of the commuter in the age group i using the route between the neighbor node n and m. The node n and m are included in the shortest path between node pair (k, l) took up all combinations satisfying the conditions. Similarly $I_{i,nm}$ is defined as $I_{i,nm} = \sum_{<kl>} D_{i,kl} I_{il}$.

Furthermore, we define a more coarse-grained $I_{i,nm}$, which we call $I^{CG}_{i,nm}$ here, represented as:

$$
I^{CG}_{i,nm} = P^H_{i,nm} I_{im} - \left(N^T_{i,nm} - D_{i,nm}\right) \frac{\sum_k I_{ik}}{\sum_k N_{ik}}.
$$

This definition implies that we capture the contribution from people moving m to n into $I^{CG}_{i,nm}$ accurately, but we replace the other inputs with a mean value. When we use $I^{CG}_{i,nm}$ instead of $I_{i,nm}$, the computational rate can be improved.

The ODE of S_{in} is obtained by the summation for m in the equation of $S_{i,mn}$. Thus, the equations are represented as:

$$
\begin{aligned}
\frac{\mathrm{d}S_{in}}{\mathrm{d}t} &= -\sum_{m=1}^{N_d} \lambda^T_{i,mn} P^H_{i,mn} S_{in} \\
\frac{\mathrm{d}I_{in}}{\mathrm{d}t} &= \sum_{m=1}^{N_d} \lambda^T_{i,mn} P^R_{i,mn} S_{in} - \gamma I_{in}
\end{aligned}
$$

The expanded formula of S_{in} in TRANSPORT period is also written down for a reference:

$$\begin{aligned}\frac{\mathrm{d}S_{in}}{\mathrm{d}t} &= -\sum_{m=1}^{N_d} \lambda_{i,mn}^{T} P_{i,mn}^{H} S_{in} \\ &= -r^T \beta \sum_{j=1}^{M} \sum_{m=1}^{N_d} \sum_{<k,l>} C_{ij} \frac{P_{j,kl}^{H} I_{lj}}{N_{j,mn}^{T}} P_{i,mn}^{H} S_{in}\end{aligned}$$

The computation rate of this approach is swift. Additionally, this model can include the "second-order" effect. If there is frequent movement between nodes, the network model is identical with the standard model. The advantage of network models is that the model can estimate the effect of travel restrictions directory.