COMPUTATIONS WITH A FAMILY OF THREE-DIMENSIONAL KLOOSTERMAN SUMS

HIDEKI HILL
SUPERVISED BY DR. CAMERON FRANC

ABSTRACT. We introduce background on a special family of three-dimensional representations of the modular group and discuss the classification of these representations into cases of finite image and congruence. We define a Kloosterman sum associated to products of these representations. In congruence cases, we determine exact formulations of finitely many Kloosterman sums using elementary results from number theory and complex analysis. We dissect computations of a Kloosterman sum in the case of a non-congruence finite image representation in an attempt to better understand how the sums are determined.

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1. Background

This section provides an overview of the basic group theoretic definitions and group representation theory relevant to our study of matrix-valued Kloosterman sums.

1.1. **The modular group.**

Begin by considering the following important group of matrices:

$$\mathrm{SL}_2(\mathbf{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a, b, c, d \in \mathbf{Z},\ ad - bc = 1 \right\}.$$

This group is known as the special linear group and arises in a number of places in mathematics, especially in number theory. For example, this group gives the possible basis changes for working inside a rank 2 integral lattice and is thus important to studying their symmetries. This explains the group's importance in the study of elliptic curves and modular forms, among others. The special linear group has infinite order and is known to be generated by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Notice that T has infinite order, since one can prove by mathematical induction that

$$T^n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

Definition 1.1. Modular Group. Define the *modular group* to be the quotient group

$$\Gamma := \mathrm{SL}_2(\mathbf{Z})/\{\pm I\}$$

where, as usual, $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Now define $R = ST$, and it is known that the modular group is generated as a free (non-abelian) product of cyclic groups $\langle S \rangle$ of order 2 and $\langle R \rangle$ of order 3. That is,

$$\Gamma = \langle S \rangle * \langle R \rangle$$

Alperin [1] gives an elementary proof of this fact. It is now clear that, like the special linear group, the modular group is generated by S and T. Thus any element of Γ can be written as a product of the generators S and T.

Example 1.2. We have claimed it is possible to express any element of Γ in terms of the matrices S and T. As a basic example, let $g = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \in \Gamma$. We will illustrate an analog of the Euclidean algorithm to see how this decomposition works.

The goal is to right-multiply g by powers of S and T until the product is one of S or S^{-1}. By the group structure of Γ, we may then invert the operations to isolate g in terms of S and T.

In order to make g look like S, the first step is to annihilate the upper left entry. The right action of S swaps the columns then negates the new first column. So we apply S on the right of g resulting in

$$gS = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}.$$

One can see that this matrix looks like S, except the bottom right entry is nonzero. Luckily, the right action of T adds the first column to the second column in the intuitive way. So we apply T on the right of the product gS resulting in

$$gST = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = S.$$

By right-multiplying both sides by $(ST)^{-1} = T^{-1}S^{-1}$, we see that our word for g is

$$g = ST^{-1}S^{-1}.$$

We will need to perform computations like this extensively because we will study representations of Γ defined entirely by their action on generators. Hence to evaluate a given representation on an arbitrary element of this group of infinite order, it is critical to express it as a word in S and T.

Below we will be interested in various subgroups of finite index in the modular group. These will arise as the kernels of certain particularly nice representations of the modular group. Before turning to this, let us introduce some important finite index subgroups of the modular group.

Definition 1.3. Principal congruence subgroup. The *principal congruence subgroup* of Γ of level $N \geq 1$ is the group

$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv I \pmod{N} \right\} / \{\pm I\}.$$

A subgroup $H \subseteq \Gamma$ is said to be a *congruence subgroup* if it contains a subgroup $\Gamma(N) \subseteq H$ for some N.

It is not completely obvious, but one can show that the map reducing an element of Γ entry-wise modulo N gives a surjection:

$$\Gamma \to \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})/\{\pm I\}$$

whose kernel is exactly $\Gamma(N)$. It follows from the first isomorphism theorem and the fact that $\Gamma(N)$ is the kernel of the reduction mod N map that $\Gamma(N)$ has finite index with

$$[\Gamma : \Gamma(N)] = |\mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z})/\{\pm I\}|.$$

An exact formula for the value of this index can be extracted using this fact, but we will not need to do so. An interested reader can find such formulas in [3].

Notice then, by the subgroup correspondence theorem, that if $H \subseteq \Gamma$ is a congruence subgroup, it too must be of finite index. It turns out that the modular group has many subgroups of finite index that are not congruence subgroups; these are much harder to understand.

One example is a number of subgroups $H \subseteq \Gamma$ of index 7 – see [4]. The existence of these non-congruence yet finite index subgroups is somewhat surprising: one can make analogous definitions of congruence subgroups of $\mathrm{SL}_n(\mathbf{Z})$ for $n \geq 3$ and it is a nontrivial theorem that *every* subgroup of finite index of these groups is congruence. Thus, the modular group is somewhat special in that, in addition to congruence subgroups, it contains many interesting finite index non-congruence subgroups that cannot be defined by simple congruence conditions. Below we will see that in terms of difficulty of understanding, our computations in such non-congruence cases sit somewhere between the relatively easy congruence case and the incomprehensible general case.

1.2. **Group representation basics.** This section gives an overview of the group representation theory needed to define our family of representations of the modular group. The following basic definitions found in [7] on representations of finite groups apply to this thesis without change. As we shall discuss below, the first major difference is that Maschke's theorem (theorem 3.2.8, [7]) does not apply to the modular group: there exist non-decomposable but not irreducible representations of finite dimension. See Example 1.14 below.

Definition 1.4. Group representation. For a vector space V over a field K of characteristic zero and a group G, a *representation* of G acting on V is a homomorphism $\rho : G \to GL(V)$. If V has finite dimension n, we say ρ has degree or dimension n.

Example 1.5. An important example is to let the symmetric group S_n act on $\mathbf{C}^n$ by permuting coordinates. These are linear transformations, so this defines a representation

$$\rho\colon S_n \to \mathrm{GL}(\mathbf{C}^n) = \mathrm{GL}_n(\mathbf{C}).$$

If we fix a basis for V, then the images $\rho(g)$ for $g \in G$ can be expressed as $n \times n$ invertible matrices. We shall work explicitly with bases in our computations below. Changing basis corresponds to conjugating the representation, as one learns in studying linear transformations in linear algebra.

We will often evaluate representations of an element of G at an element of V. The notation used henceforth will describe a representation of a certain element of a group G, and then evaluated at a certain element of the vector space V.

$$\rho_g(v) := \rho(g)(v).$$

Here $g \in G$ and $v \in V$. We often even abuse notation by writing simply gv for $\rho_g(v)$.

To motivate the following definitions, we will make analogies to group theory. First, consider a subgroup: it retains the structure of its parent group. A similar notion in representation theory is a subspace $W \subseteq V$ who, when restricting the vector space V to W, retains the properties of the representation on its own.

Definition 1.6. G-invariant subspace. Given a representation $\rho : G \to GL(V)$, a subspace $W \subseteq V$ is called *G-invariant* if for every $g \in G$ and for every $w \in W$ we have $\rho_g(w) \in W$. Equivalently, we say G is *stable* under the action of ρ.

Example 1.7. Continuing from Example 1.5, let $U \subseteq \mathbf{C}^n$ be the subspace of elements whose coordinates sum to zero, and let $V \subseteq \mathbf{C}^n$ be the subspace of elements whose coordinates are all equal. Then U and V are both stable under the permutation action of S_n. Note that U has dimension $n-1$ and V is one-dimensional.

Following the study of the subgroup and the normal subgroup is the simple group, whose only normal subgroups are the trivial ones. An analogous notion in representation theory is a representation whose only G-invariant subspaces are trivial.

Definition 1.8. Irreducible representation. A nonzero representation $\rho : G \to GL(V)$ is called *irreducible* if the only G-invariant subspaces of V are the trivial subspaces V and $\{0\}$.

Example 1.9. Continuing from Example 1.7, both the subspaces U and V are irreducible representations of S_n. While it is an unobvious fact that U is irreducible, it is easier to see that V is irreducible. We saw that V is one-dimensional, so its only subspaces are $\{0\}$ and V itself. It follows that there cannot be any G-invariant subspaces other than the trivial ones.

As explained in [7], irreducible representations underlie all finite-dimensional group representations, it is thus important to understand and identify them. Irreducible representations of infinite discrete groups like the modular group are not the only basic building blocks of the representation theory of such groups, but they still form an important subset of such representations.

We now have the background necessary to discuss degree n representations of the modular group. Since the degree of a finite representation is determined by the dimension of its associated vector space, we can consider a simplest non-trivial case of the modular group acting on a vector space of dimension 1.

Example 1.10. Degree 1 representations of the modular group. Degree 1 representations of a group are always irreducible; consider the representations

$$\rho : \Gamma \to \mathrm{GL}(\mathbf{C}).$$

First, notice that $\mathrm{GL}(\mathbf{C}) = \mathbf{C}^{\times}$, the multiplicative group of nonzero complex numbers. We have seen that Γ is freely generated by S and R of orders 2 and 3, respectively. To specify the action of ρ on Γ, it is sufficient to specify the images $\rho(S)$ and $\rho(R)$ using the relation $\rho(S)^2 = \rho(R)^3 = 1$. Hence it must be the case that $\rho(S) = \pm 1$ and $\rho(R) = \zeta^a$, where $\zeta = e^{2\pi i/3}$. We have described the orders of the images of the generators $\rho(R)$ and $\rho(S)$, so there are exactly $2 \cdot 3 = 6$ representations of Γ, and we have described them concretely.

It is typical to write $\chi \colon \Gamma \to \mathbf{C}^{\times}$ for the one-dimensional representation uniquely determined by the condition $\chi(T) = e^{2\pi i/6}$. It turns out that this generates the one dimensional characters in the sense that the powers χ^a for $a = 0, 1, 2, 3, 4, 5$ are all the one-dimensional representations of Γ.

Definition 1.11. Direct sum of subspaces. A sum of subspaces V_1, V_2, $\ldots$, V_k of a vector space V given by

$$\{v_1 + v_2 + \cdots + v_k \mid v_i \in V_i\}$$

is called a *direct sum* if each element of the set is uniquely expressed as a sum of elements from each subspace. We denote a direct sum by

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k.$$

Example 1.12. A clear example is the complex numbers expressed as a direct sum of purely imaginary numbers with the real numbers:

$$\mathbf{C} = \mathbf{R} \oplus \{bi \mid b \in \mathbf{R}\}$$

This sum is expressed as the set

$$\{a + bi \mid a, b \in \mathbf{R}\},$$

which is the familiar description of $\mathbf{C}$. Note that the intersection of the summands in this case is $\{0\}$; it is an elementary theorem in linear algebra that a sum of spaces is a direct sum if and only if the intersection of the subspaces is $\{0\}$ – a proof is found in [2]. Also note that these subspaces are the ones generated by the basis vectors of **C**, $(1,0)$ and $(0,i)$.

Since representations are homomorphisms between groups, we can consider combinations of representations. The following gives an intuitive definition of direct sums of representations.

Given $\rho : G \to GL(V_1)$ and $\phi : G \to GL(V_2)$, the direct sum of ρ and ϕ is defined to be the representation

$$\rho \oplus \phi : G \to GL(V_1 \oplus V_2).$$

If we choose bases for V_1 and V_2 so that ρ and ϕ are matrix valued maps, then $(\rho \oplus \phi)(g)$ is the block matrix obtained by putting $\rho(g)$ and $\phi(g)$ diagonally into a block matrix form with the appropriate dimensions. The off-diagonal block entries are then all zero.

A natural question is how far one can reduce a representation into a direct sum of irreducibles, like the decomposition of a vector space into a direct sum of subspaces. Irreducible representations will act as the prime building blocks of representations.

Definition 1.13. Completely reducible representation. A representation $\rho : G \to GL(V)$ is called *completely reducible* if V is a direct sum of G-invariant subspaces. That is, ρ is completely reducible if there exist a finite number of G-invariant subspaces $V_i \subseteq V$ such that G acts irreducibly on each V_i where V is the direct sum of the V_i.

As previously mentioned, Maschske's theorem states that representations of finite groups over a field of characteristic zero are completely reducible, a claim generally false for infinite groups. Since we will be taking the domain of our representations to be the modular group of infinite order, this theorem is where this thesis deviates from the introductory material of the book. To illustrate this fact, consider the following example with $G = (\mathbf{Z}, +)$.

Example 1.14. Define a homomorphism $\rho\colon \mathbf{Z} \to \mathrm{GL}_2(\mathbf{C})$ by

$$\rho(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Since 1 generates **Z** additively, this action determines ρ uniquely. Notice that if V is the span of the column vector $v = (1,0)^t$, then V is stable under $\rho(1)$ and thus under the entire representation since v is an eigenvector of eigenvalue 1. However, one finds in the study of linear algebra that $\rho(1)$ is an easy example of a matrix that is not diagonalizable. Therefore, we cannot split this representation up into a direct sum of two one-dimensional irreducibles, as this equates to diagonalize $\rho(1)$. Thus, this ρ gives an example of a representation of **Z** that is neither irreducible nor completely reducible.

We will now discuss how to describe and compute representations of the modular group in more detail. Let $\rho : \Gamma \to GL(V)$ and let $M \in \Gamma$. We have seen that M can be expressed as a product of S, S^{-1}, and powers of T via a modification of the Euclidean algorithm.

$$M = ST^{e_1}S^{-1}T^{e_2} \cdots ST^{e_r}S^{-1}$$

As we have seen, the action of ρ on S and T uniquely determines the action on the whole group. Suppose we know where ρ maps S and T. Then we can apply the ρ to the product expansion of M element-wise, since ρ is a homomorphism that respects the structure and operation of matrix multiplication:

$$\rho(M) = \rho(S)\rho(T)^{e_1}\rho(S)^{-1}\rho(T)^{e_2}\cdots\rho(S)\rho(T)^{e_r}\rho(S)^{-1}$$

An important case for representations ρ of Γ is when $\operatorname{im}\rho$ is a finite group. There exist many such representations: for example, basically every finite simple group is a quotient of the modular group. These give rise to representations on the cosets that are finite-dimensional whose image is the given finite simple group. We will focus mainly below on representations with finite image who may be further categorized by the following definition:

Definition 1.15. Congruence representation. Let ρ be a representation of the modular group. If $\ker\rho$ is a congruence subgroup of ρ then we say that ρ is a congruence representation. Otherwise we say that ρ is non-congruence.

1.3. **Partial totient functions.** We have seen that the kernels of certain representations will be important congruence subgroups. In working with these finite image subgroups, expressing certain congruence relations will be vital to defining their representations. We introduce the familiar totient function due to Euler:

Definition 1.16. Euler's totient function. For $n \geq 1$, the *totient function* at n, denoted $\varphi(n)$, is the number of integers relatively prime to n. Equivalently, the size of the multiplicative group of units modulo n.

$$\varphi(n) := |\{1 \leq x < n \mid gcd(x,n) = 1\}| = \left|(\mathbf{Z}/n\mathbf{Z})^\times\right|$$

Example 1.17. Though we will not necessarily need to refer to special cases of the Totient function, a fundamental and clear example is that of prime numbers. By definition, the only divisors of p are 1 and p. Notice that it is still true that $gcd(p,1) = 1$ (it is trivial that $gcd(p,p) = p$), implying that every integer $d < p$ relatively prime to p. So it becomes clear that when p is prime,

$$\varphi(p) = p - 1.$$

More generally if p^r is a prime power then $\varphi(p^r) = p^r - p^{r-1}$ and this can be used to give a general formula for $\varphi(n)$ depending on the factorization of n.

The following modification of the totient function is found in [5] and provides a concise description relating relative primality and congruence.

Definition 1.18. Partial totient function. For $c \geq 1$ with $0 \leq a < c$ and a positive integer N the *partial totient function* at (c,a,N), denoted $\varphi(c,a,N)$ or $\varphi_{c,N}(a)$, is the number of integers relatively prime to c and congruent to $a \mod N$.

$$\varphi(c,a,N) = |\{1 \leq x < c \mid gcd(x,c) = 1, x \equiv a \mod N\}|$$

Similarly to the totient function, we may equivalently state this definition as the number of units modulo c who are congruent to $a \mod N$.

Example 1.19. We will illustrate on $c = 5$. Let $a = 1$ and $N = 3$. Then the set of integers between 1 and 5 who are congruent to $1 \mod 3$ is $\{1,4\}$, and each of these two numbers are relatively prime to 5 since 5 is prime. So we find that $\varphi(5,1,3) = 2$.

We will need to consider linear combinations of the functions $\varphi(c,a,N)$ as a varies over the congruence classes mod N. Therefore, let us introduce the following notation that will simplify such expressions:

$$\varphi_{c,N}(a_0, a_1, \ldots, a_{N-1}) := a_0\varphi(c, 0, N) + a_1\varphi(c, 1, N) + \ldots + a_{N-1}\varphi(c, N-1, N).$$

In most cases below we will have $\sum_{j=0}^{N-1} a_j \equiv 0 \pmod N$ but we do not insist on this at the moment. For example, when $a_i = 1$ for every $0 \le i \le N-1$, it is clear that:

$$\sum_{k=0}^{N-1} 1 \cdot \varphi(c, k, N) = \varphi_{c,N}(1, 1, \ldots, 1) = \varphi(c).$$

One can frequently use this relation to reduce to considering the case where $\sum_{j=0}^{N-1} a_j = 0$.

Example 1.20. We saw how to find a partial totient function in Example 1.19. Let us continue the computation by verifying the above formula for $c = 5$ and $N = 3$.

$$\varphi(5, 0, 3) + \varphi(5, 1, 3) + \varphi(5, 2, 3) = 1 + 2 + 1 = 4$$

We know from example 1.17 that $\varphi(5) = 4$ since 5 is prime. Indeed, we see that the formulas agree.

We will shortly study certain formulations of Kloosterman sums where these partial totient functions will be useful; we will explicitly consider relative primality and congruence at the same time.

2. The family of three-dimensional representations

Tuba-Wenzl [8] computed all irreducible representations of the modular group of dimension at most 5. These break up into families with a number of components. Our aim below is to study the Kloosterman sums associated to a particular two-parameter family of three-dimensional representations. Up to twisting by one-dimensional characters, this family is the universal three-dimensional irreducible representation of the modular group.

We are primarily interested in representations ρ with finite image. It turns out that there are only finitely many such representations of dimension one or two. Therefore, this three-dimensional family that we will examine below is the natural first place to look for an infinite set of non-abelian Kloosterman sums that we may be able to deduce theorems from: we shall introduce this family of representations now.

Let x, y, z be complex numbers such that

(1) $xyz = 1$, and
(2) $x^2 + yz \neq 0$, $y^2 + xz \neq 0$ and $z^2 + xy \neq 0$.

It is immediately clear that it is suitable to make the substitution $z = (xy)^{-1}$. The following matrices define the aforementioned two-parameter family of irreducible representations of the modular group in $GL_3(\mathbf{C})$:

$$\rho(T) = \begin{pmatrix} x & y^{-2} + y & y \\ 0 & y & y \\ 0 & 0 & x^{-1}y^{-1} \end{pmatrix}, \qquad \rho(S) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Note that Tuba-Wenzl use this basis for ρ so that they can handle all irreducible representations, even if $\rho(T)$ is not diagonalizable. However, notice that x and y

are the eigenvalues of $\rho(T)$. Only finitely many choices of x and y lead to non-diagonalizable $\rho(T)$, thus there is no issue in changing basis to diagonalize $\rho(T)$. This will more simply isolate the parameters x and y in $\rho(T)$ and the resulting Kloosterman sums will be simplified.

To perform this diagonalization, standard techniques are applied to our representation of T by conjugating a diagonal representation ϕ by an upper triangular representation P in the usual form:

$$\rho(T) = P\phi P^{-1}$$

where

$$P = \begin{pmatrix} 1 & \frac{y^3+1}{y^3-xy^2} & \frac{x^2y+xy^2}{(x^2y-1)(xy^2-1)} \\ 0 & 1 & \frac{xy^2}{1-xy^2} \\ 0 & 0 & 1 \end{pmatrix}.$$

Notice that P is only well-defined if $x, y \neq 0$, $x \neq y$, $xy^2 \neq 1$ and $x^2y \neq 1$. In these avoidable cases, $\rho(T)$ is not diagonalizable. In all other cases, we can isolate this conjugate representation $\phi = \phi_{x,y}$ through reverse conjugation by P on $\rho(T)$:

$$\phi_{x,y} = P^{-1}\rho_{x,y}P$$

resulting in

$$\phi_{x,y}(T) = \begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & x^{-1}y^{-1} \end{pmatrix}.$$

Applying the same reverse conjugation to our representation of S gives the matrix:

$$\phi_{x,y}(S) = \begin{pmatrix} \frac{-x(xy^2+1)}{(y-x)(x^2y-1)} & \frac{-(x+1)(y+1)(x^2-x+1)(y^2-y+1)}{y(y-x)^2(x^2y-1)} & \frac{y(x+1)(xy+1)(x^2-x+1)(x^2y^2-xy+1)}{(y-x)(xy^2-1)(x^2y-1)^2} \\ \frac{xy^2}{xy^2-1} & \frac{y(x^2y+1)}{(y-x)(xy^2-1)} & \frac{xy^2(xy+1)(x^2y^2-xy+1)}{(x^2y-1)(xy^2-1)^2} \\ 1 & \frac{-(y+1)(y^2-y+1)}{(x-y)y^2} & \frac{xy(x+y)}{(xy^2-1)(x^2y-1)} \end{pmatrix}$$

While complicated to write down, one can work with this matrix easily using a computer algebra system. We will want to consider the special cases when our representation is of finite image and, further, congruence. The following results allow us to control these conditions by choosing specific values of x, y. We will soon see that evaluation by these nice values of x, y greatly simplify the above representation $\phi_{x,y}(S)$.

From Franc-Mason [6] it is known that

(1) $\phi_{x,y}$ is of finite image if and only if there exists an nth root of unity ζ with $x = -\zeta$ and $y^2 = \zeta^{-1}$;
(2) $\phi_{x,y}$ is moreover congruence if and only if $n \mid 24$.

The above results imply that we see infinitely many irreducible representations of the modular group of dimension three with finite image, though only finitely many are congruence. These define the necessary guidelines of which inputs x, y we might consider when computing and formulating Kloosterman sums, who are wholly determined by these representations and at which x, y they are evaluated. The following section focuses on formulating sums composed of congruence representations.

3. The exponential sums

Definition 3.1. Using the diagonalized representation $\phi_{x,y}$, define for integers $c \geq 1$ the family of Kloosterman sums:

$$K(x,y,c) = 4 \cdot \sum_{\substack{d=1 \\ \gcd(c,d)=1}}^{c} \phi_{x,y} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^t \phi_{x^{-1},y^{-1}} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$$

We multiply the sum by 4 to simplify computations in following sections.

Proposition 3.2. *The transpose of the the Kloosterman sum is given by*

$$K(x,y,c)^t = K(x^{-1}, y^{-1}, c).$$

Proof. This follows immediately from the definition of $K(x,y,c)$. □

In this notation, for each pair (c,d) we use Bezout's Identity to solve the unit determinant equation $ad - bc = 1$ and use the resulting pair (a,b) to compute the term in the sum. We will show that this is well-defined and independent of (a,b).

Proposition 3.3. *Given a positive integer c with relatively prime integer d, the Kloosterman sum does not depend on the choice of pair (a,b) obtained in accordance with Bezout.*

Proof. We may consider any alteration to the pair (a,b) as corresponding to left-multiplying the matrix by a power of T; without loss of generality we may consider multiplying only once by T. By the homomorphism properties of the representation ϕ, it follows that

$$\begin{aligned}
&\phi_{x,y}(T \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))^t \phi_{x^{-1},y^{-1}}(T \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)) \\
&= \phi_{x,y}(T) \phi_{x,y} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^t \phi_{x^{-1},y^{-1}} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \phi_{x^{-1},y^{-1}}(T) \\
&= \phi_{x,y} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^t \phi_{x,y}(T) \phi_{x^{-1},y^{-1}}(T) \phi_{x^{-1},y^{-1}} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)
\end{aligned}$$

Notice that $\phi_{x,y}(T)^{-1} = \phi_{x^{-1},y^{-1}}(T)$, so the dependence on T (thus the dependence on the alteration to the pair (a,b)) vanishes:

$$\begin{aligned}
&\phi_{x,y} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^t \phi_{x,y}(T) \phi_{x^{-1},y^{-1}}(T) \phi_{x^{-1},y^{-1}} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \\
&= \phi_{x,y} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^t \phi_{x,y}(T) \phi_{x,y}(T)^{-1} \phi_{x^{-1},y^{-1}} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \\
&= \phi_{x,y} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^t \phi_{x^{-1},y^{-1}} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)
\end{aligned}$$

Therefore this expression for $K(x,y,c)$ is independent of the choice of (a,b). □

This thesis focuses on formulating the Kloosterman sums for the representation $\phi_{x,y}$ in its finite image case (suggesting an nth root of unity), further, in its congruence case (suggesting a 24th root of unity).

In accordance with Franc-Mason [6], our inputs x, y, z should take the forms

$$\begin{aligned} x &= -e^{2\pi i\cdot m/24}, \\ y &= e^{-2\pi i\cdot m/(2\cdot 24)}, \\ z &= -e^{-2\pi i\cdot m/(2\cdot 24)}. \end{aligned}$$

Notice above that we have $y^2 = -x^{-1}$. Using this substitution $x = -y^{-2}$, we have the following nice result for our representation of S:

$$\phi_{x,y}(S) = \begin{pmatrix} 0 & 1 & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{4} \\ 1 & 1 & -\frac{1}{2} \end{pmatrix}$$

We will focus on the case where $n \mid 24$ so that we have congruence. It follows that for roots of unity $e^{2\pi i\cdot m/n}$, we may take $1 \leq m < n$ with $\gcd(m, n) = 1$. The only case that we shall omit is the 2nd roots of unity (± 1) since this will give the relation $x = y$ which we have seen leads to an undiagonalizable $\rho(T)$.

Example 3.4. To compute $K(x, y, 1)$, we find that $c = 1$ corresponds to the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. While any matrix with $c = 1, gcd(c, d) = 1$, and determinant 1 would work, we have shown that the sum does not depend on the pair (a, b) so we pick values who are easier to work with. In Example 1.2, it was shown that the matrix may be decomposed into a product as $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) = ST^{-1}S^{-1}$. By the homomorphism properties of the representation $\phi_{x,y}$,

$$\phi_{x,y}(ST^{-1}S^{-1}) = \phi_{x,y}(S)\phi_{x,y}(T)^{-1}\phi_{x,y}(S)^{-1},$$

resulting in

$$\phi_{x,y}\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) = \begin{pmatrix} \frac{-x^2(xy^2+1)}{(y-x)(x^2y-1)} & \frac{-(y+1)(x+1)(y^2-y+1)(x^2-x+1)}{(y-x)^2(x^2y-1)} & \frac{-(x+1)(xy+1)(x^2-x+1)(x^2y^2-xy+1)}{x(y-x)(x^2y-1)^2(xy^2-1)} \\ \frac{x^2y^2}{xy^2-1} & \frac{y^2(x^2y+1)}{(y-x)(xy^2-1)^2} & \frac{y(xy+1)(x^2y^2-xy+1)}{(x^2y-1)(xy^2-1)^2} \\ x & \frac{-(y+1)(y^2-y+1)}{y(x-y)} & \frac{(x+y)}{(xy^2-1)(x^2y-1)} \end{pmatrix}.$$

Substituting $x = -y^{-2}$ as discussed above, the Kloosterman sum becomes

$$K(-y^{-2}, y, 1) = 4 \cdot \begin{pmatrix} \frac{5}{4} & \frac{3}{-4y^3} & \frac{-3}{8y^3} \\ \frac{-3y^3}{4} & \frac{9}{4} & \frac{1}{8} \\ \frac{-3y^3}{8} & \frac{1}{8} & \frac{9}{16} \end{pmatrix} = \begin{pmatrix} 5 & -3y^{-3} & -\frac{3}{2}y^{-3} \\ -3y^3 & 9 & \frac{1}{2} \\ -\frac{3}{2}y^3 & \frac{1}{2} & \frac{9}{4} \end{pmatrix}$$

We now see how multiplication by 4 simplifies our sum, and this remains useful in following results.

4. Formulas for congruence representations

This section is dedicated to presenting formulations of the Kloosterman sums in the case where the representation $\phi_{x.y}$ is congruence. Formulations with inputs of $e^{2\pi i\cdot m/24}$ for $1 \leq m < 24.$ are given, along with their proofs. These cases will be presented in lowest terms; for each root of unity $e^{2\pi i\cdot n/m}$ with $m \mid 24$, we will only consider the numerators n who are relatively prime to the denominator m. If the fraction can be reduced, then we have already seen (or will soon see) the same result in another computation.

The following proofs follow the structure of those given by Franc-Locke [5] in the (2×2) representations of Γ. While the context of that paper is not identical to this thesis, the techniques carry over with straightforward modifications.

4.1. The 3rd and 6th roots of unity and equivalences between them.

Proposition 4.1. ζ_6, ζ_6^{-1}
Let y be a primitive 6th root of unity and set $x = -y^{-2}$. Then for all $c \geq 1$,

$$K(y,c) = \begin{cases} \begin{pmatrix} 5 & 3 & \frac{3}{2} \\ 3 & 9 & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} & c = 1 \\ \varphi(c) \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} & 2 \mid c \\ \varphi(c) \begin{pmatrix} 5 & 3 & 0 \\ 3 & 9 & 0 \\ 0 & 0 & \frac{9}{4} \end{pmatrix} & 2 \nmid c \end{cases}$$

Proof. By computing the terms in the sum, we see a finite number of representations, with $\phi_{x,y}$ congruence of level 2:

$$(\phi_{x,y}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))^t \phi_{x^{-1},y^{-1}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \begin{cases} \frac{1}{4}\begin{pmatrix} 5 & 3 & \frac{3}{2} \\ 3 & 9 & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} & 2 \nmid c \text{ and } 2 \nmid d \\ \frac{1}{4}\begin{pmatrix} 5 & 3 & -\frac{3}{2} \\ 3 & 9 & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} & 2 \nmid c \text{ and } 2 \mid d \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & 2 \mid c \end{cases}$$

By definition, the Kloosterman sum is given by

$$K(y,c) = 4 \sum_{\substack{d=1 \\ gcd(c,d)=1}}^{c} (\phi_{x,y}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))^t \phi_{x^{-1},y^{-1}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).$$

In the special case $c = 1$, we see that the sum is exactly equal to four times one iteration of the first matrix listed since we multiply the sum by 4 to cancel the factor of $\frac{1}{4}$.

In the case $2 \mid c$, we find that

$$K(y,c) = 4 \sum_{\substack{d=1 \\ 2 \nmid d}}^{c} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 4 \cdot \varphi(c) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \varphi(c) \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

since the sum over all d coprime to c can be equivalently formulated as $\varphi(c)$ iterations of the matrix. In the case $2 \nmid c$, we can split the sum into two sums over separate matrices:

$$K(y,c) = 4 \sum_{\substack{d=1 \\ gcd(c,d)=1}}^{c} (\phi_{x,y}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))^t \phi_{x^{-1},y^{-1}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$$

$$= \sum_{\substack{d=1 \\ 2\nmid d \\ \gcd(c,d)=1}}^{c} \begin{pmatrix} 5 & 3 & \frac{3}{2} \\ 3 & 9 & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} + \sum_{\substack{d=1 \\ 2\mid d \\ \gcd(c,d)=1}}^{c} \begin{pmatrix} 5 & 3 & -\frac{3}{2} \\ 3 & 9 & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix}$$

$$= \varphi(c,1,2) \begin{pmatrix} 5 & 3 & \frac{3}{2} \\ 3 & 9 & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} + \varphi(c,0,2) \begin{pmatrix} 5 & 3 & -\frac{3}{2} \\ 3 & 9 & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix}$$

Notice that if $2 \nmid c$, $\varphi(c,1,2) = \varphi(c,0,2) = \frac{\varphi(c)}{2}$, so

$$K(y,c) = \frac{\varphi(c)}{2} \left(\begin{pmatrix} 5 & 3 & \frac{3}{2} \\ 3 & 9 & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} + \begin{pmatrix} 5 & 3 & -\frac{3}{2} \\ 3 & 9 & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} \right) = \varphi(c) \begin{pmatrix} 5 & 3 & 0 \\ 3 & 9 & 0 \\ 0 & 0 & \frac{9}{4} \end{pmatrix}$$

This concludes the case $2 \nmid c$ and concludes the proof. □

Proposition 4.2. ζ_3, ζ_3^{-1}
Let y be a 3rd root of unity (in this case each root is primitive) and set $x = -y^{-2}$. Then for all $c \geq 1$,

$$K(y,c) = \begin{cases} \begin{pmatrix} 5 & -3 & -\frac{3}{2} \\ -3 & 9 & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} & c = 1 \\ \varphi(c) \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} & 2 \mid c \\ \varphi(c) \begin{pmatrix} 5 & 0 & -\frac{3}{2} \\ 0 & 9 & 0 \\ -\frac{3}{2} & 0 & \frac{9}{4} \end{pmatrix} & 2 \nmid c \end{cases}$$

Proof. Similarly to the previous proposition, we have a finite number of representations depending on the parity of c since $\phi_{x,y}$ is again congruence of level 2:

$$(\phi_{x,y}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))^t \phi_{x^{-1},y^{-1}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \begin{cases} \frac{1}{4} \begin{pmatrix} 5 & -3 & -\frac{3}{2} \\ -3 & 9 & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} & 2 \nmid c \text{ and } 2 \nmid d \\ \frac{1}{4} \begin{pmatrix} 5 & 3 & -\frac{3}{2} \\ 3 & 9 & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} & 2 \nmid c \text{ and } 2 \mid d \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & 2 \mid c \end{cases}$$

In the special case $c = 1$, we see that the sum is exactly equal to four times the first representation. Next, $2 \mid c$ implies $K(y,c) = 4\varphi(c)I_3$ and when $2 \nmid c$,

$$\begin{aligned}
K(y,c) &= 4 \sum_{\substack{d=1 \\ gcd(c,d)=1}}^{c} (\phi_{x,y}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))^t \phi_{x^{-1},y^{-1}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \\
&= \sum_{\substack{d=1 \\ 2\nmid d \\ \gcd(c,d)=1}}^{c} \begin{pmatrix} 5 & -3 & -\frac{3}{2} \\ -3 & 9 & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} + \sum_{\substack{d=1 \\ 2\mid d \\ \gcd(c,d)=1}}^{c} \begin{pmatrix} 5 & 3 & -\frac{3}{2} \\ 3 & 9 & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} \\
&= \varphi(c,1,2) \begin{pmatrix} 5 & -3 & -\frac{3}{2} \\ -3 & 9 & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} + \varphi(c,0,2) \begin{pmatrix} 5 & 3 & -\frac{3}{2} \\ 3 & 9 & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} \\
&= \frac{\varphi(c)}{2} \left(\begin{pmatrix} 5 & -3 & -\frac{3}{2} \\ -3 & 9 & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} + \begin{pmatrix} 5 & 3 & -\frac{3}{2} \\ 3 & 9 & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} \right) = \varphi(c) \begin{pmatrix} 5 & 0 & -\frac{3}{2} \\ 0 & 9 & 0 \\ -\frac{3}{2} & 0 & \frac{9}{4} \end{pmatrix}
\end{aligned}$$

Which concludes the case $2 \nmid c$ and concludes the proof. □

4.2. **The 4th and 12th roots of unity.**

Proposition 4.3. $e^{2\pi i\cdot 1/4}, e^{2\pi i\cdot 7/12}, e^{2\pi i\cdot 11/12}$
Let y be one of the values listed and set $x=-y^{-2}$. Then for all $c\geq 1$,

$$K(y,c)=\begin{cases}\begin{pmatrix}5 & 3 & \frac{3}{2}\\ -3 & 9 & \frac{1}{2}\\ -\frac{3}{2} & \frac{1}{2} & \frac{9}{4}\end{pmatrix} & c=1\\ \varphi(c)\begin{pmatrix}4&0&0\\0&4&0\\0&0&4\end{pmatrix} & 4\mid c\\ \begin{pmatrix}5\varphi(c) & 3\varphi_{c,4}(1,1,-1,-1) & 0\\ 0 & 9\varphi(c) & 0\\ \frac{3}{2}\varphi_{c,4}(-1,-1,1,1) & 0 & \frac{9}{4}\varphi(c)\end{pmatrix} & c\equiv 1 \mod 4\\ \varphi(c)\begin{pmatrix}4&0&0\\0&16&0\\0&0&1\end{pmatrix} & c\equiv 2\mod 4\\ \begin{pmatrix}5\varphi(c) & 3\varphi_{c,4}(1,-1,-1,1) & 0\\ 0 & 9\varphi(c) & 0\\ \frac{3}{2}\varphi_{c,4}(-1,1,1,-1) & 0 & \frac{9}{4}\varphi(c)\end{pmatrix} & c\equiv 3\mod 4\end{cases}$$

Proof. We begin with a finite number of matrices with conditions resulting from the congruence level of 4 of $\phi_{x,y}$:

$$(\phi_{x,y}\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right))^t\phi_{x^{-1},y^{-1}}\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)=\begin{cases}\frac{1}{4}\begin{pmatrix}5 & 3 & \frac{3}{2}\\ -3 & 9 & \frac{1}{2}\\ -\frac{3}{2} & \frac{1}{2} & \frac{9}{4}\end{pmatrix} & \begin{matrix}c\equiv 1, d\equiv 1 \mod 4\\ c\equiv 3, d\equiv 3\mod 4\end{matrix}\\ \frac{1}{4}\begin{pmatrix}5 & -3 & -\frac{3}{2}\\ 3 & 9 & \frac{1}{2}\\ \frac{3}{2} & \frac{1}{2} & \frac{9}{4}\end{pmatrix} & \begin{matrix}c\equiv 1, d\equiv 3 \mod 4\\ c\equiv 3, d\equiv 1\mod 4\end{matrix}\\ \frac{1}{4}\begin{pmatrix}5 & -3 & \frac{3}{2}\\ -3 & 9 & -\frac{1}{2}\\ \frac{3}{2} & -\frac{1}{2} & \frac{9}{4}\end{pmatrix} & c\equiv \pm 1, d\equiv 2\mod 4\\ \frac{1}{4}\begin{pmatrix}5 & 3 & -\frac{3}{2}\\ 3 & 9 & -\frac{1}{2}\\ -\frac{3}{2} & -\frac{1}{2} & \frac{9}{4}\end{pmatrix} & c\equiv \pm 1, d\equiv 0\mod 4\\ \begin{pmatrix}1&0&0\\0&4&0\\0&0&\frac{1}{4}\end{pmatrix} & c\equiv 2, d\equiv \pm 1\mod 4\\ \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} & c\equiv 0, d\equiv\pm 1\mod 4\end{cases}$$

In the special case $c=1$, the sum is exactly equal to four times the first representation. The cases $c\equiv 2 \mod 4$ and $4\mid c$ are clear for the reason that we are summing over d

coprime to c. Next, the case $c \equiv 1 \mod 4$:

$$
\begin{aligned}
K(y,c) &= 4 \sum_{\substack{d=1 \\ gcd(c,d)=1}}^{c} (\phi_{x,y}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))^t \phi_{x^{-1},y^{-1}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \\
&= \sum_{\substack{d=1 \\ d\equiv 0 \\ \gcd(c,d)=1}}^{c} \begin{pmatrix} 5 & 3 & -\frac{3}{2} \\ 3 & 9 & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} + \sum_{\substack{d=1 \\ d\equiv 1 \\ \gcd(c,d)=1}}^{c} \begin{pmatrix} 5 & 3 & \frac{3}{2} \\ -3 & 9 & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} \\
&+ \sum_{\substack{d=1 \\ d\equiv 2 \\ \gcd(c,d)=1}}^{c} \begin{pmatrix} 5 & -3 & \frac{3}{2} \\ -3 & 9 & -\frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} + \sum_{\substack{d=1 \\ d\equiv 3 \\ \gcd(c,d)=1}}^{c} \begin{pmatrix} 5 & -3 & \frac{3}{2} \\ 3 & 9 & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} \\
&= \varphi(c,0,4) \begin{pmatrix} 5 & 3 & -\frac{3}{2} \\ 3 & 9 & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} + \varphi(c,1,4) \begin{pmatrix} 5 & 3 & \frac{3}{2} \\ -3 & 9 & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} \\
&+ \varphi(c,2,4) \begin{pmatrix} 5 & -3 & \frac{3}{2} \\ -3 & 9 & -\frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} + \varphi(c,3,4) \begin{pmatrix} 5 & -3 & \frac{3}{2} \\ 3 & 9 & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{9}{4} \end{pmatrix}
\end{aligned}
$$

And, after distributing each partial totient function through to each entry of its respective matrix, we find that this is equal to

$$
\begin{pmatrix} 5\varphi(c) & 3\varphi_{c,4}(1,1,-1,-1) & \frac{3}{2}\varphi_{c,4}(-1,1,1,-1) \\ 3\varphi_{c,4}(1,-1,-1,1) & 9\varphi(c) & \frac{1}{2}\varphi_{c,4}(-1,1,-1,1) \\ \frac{3}{2}\varphi_{c,4}(-1,-1,1,1) & \frac{1}{2}\varphi_{c,4}(-1,1,-1,1) & \frac{9}{4}\varphi(c) \end{pmatrix}.
$$

When $c \equiv 1 \mod 4$, it is a fact that $\varphi(c,0,4) = \varphi(c,1,4)$ and similarly for 2 and 3. So we find that the linear combinations vanish in the cases where these values cancel. As an illustration, consider the $(2,1)$-entry of the above matrix:

$$
3\varphi_{c,4}(1,-1,-1,1) = 3(\varphi(c,0,4) - \varphi(c,1,4)) + (-\varphi(c,2,4) + \varphi(c,3,4)) = 0
$$

The same cancellation occurs in other entries, explaining why we see zero entries. Acknowledging the fact that $\sum_{k=0}^{3} \varphi(c,k,4) = \varphi(c)$, this completes the proof in the case $c \equiv 1 \mod 4$. The case $c \equiv 3 \mod 4$ is identical except for swapping the summation of the respective matrix representations over $d \equiv 1$ and $d \equiv 3$. $\square$

Proposition 4.4. $e^{2\pi i\cdot 3/4}, e^{2\pi i\cdot 1/12}, e^{2\pi i\cdot 5/12}$
Let y be one of the listed values and set $x = -y^{-2}$. Then for all $c \geq 1$,

$$K(y,c) = \begin{cases} \begin{pmatrix} 5 & 3 & \frac{3}{2} \\ -3 & 9 & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} & c = 1 \\ \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} & 4 \mid c \\ \begin{pmatrix} 5\varphi(c) & 3\varphi_{c,4}(1,1,-1,-1) & \frac{3}{2}\varphi_{c,4}(-1,-1,1,1) \\ 3\varphi_{c,4}(1,1,-1,-1) & 9\varphi(c) & 0 \\ 0 & 0 & \frac{9}{4}\varphi(c) \end{pmatrix} & c \equiv 1 \mod 4 \\ \begin{pmatrix} 4 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 1 \end{pmatrix} & c \equiv 2 \mod 4 \\ \begin{pmatrix} 5\varphi(c) & 0 & \frac{3}{2}\varphi_{c,4}(-1,1,1,-1) \\ 3\varphi_{c,4}(1,-1,-1,1) & 9\varphi(c) & 0 \\ 0 & 0 & \frac{9}{4}\varphi(c) \end{pmatrix} & c \equiv 3 \mod 4 \end{cases}$$

Proof. We begin again with finitely many representations of congruence level of 4. Notice here that compared to the preceding proposition, the only difference is the first two matrix representations listed have swapped with respect to their congruence conditions.

$$(\phi_{x,y}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))^t \phi_{x^{-1},y^{-1}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \begin{cases} \frac{1}{4}\begin{pmatrix} 5 & -3 & -\frac{3}{2} \\ 3 & 9 & \frac{1}{2} \\ \frac{3}{2} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} & \begin{array}{c} c \equiv 1, d \equiv 1 \mod 4, \\ c \equiv 3, d \equiv 3 \mod 4 \end{array} \\ \frac{1}{4}\begin{pmatrix} 5 & 3 & \frac{3}{2} \\ -3 & 9 & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} & \begin{array}{c} c \equiv 1, d \equiv 3 \mod 4, \\ c \equiv 3, d \equiv 1 \mod 4 \end{array} \\ \frac{1}{4}\begin{pmatrix} 5 & -3 & \frac{3}{2} \\ -3 & 9 & -\frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} & c \equiv \pm 1, d \equiv 2 \mod 4 \\ \frac{1}{4}\begin{pmatrix} 5 & 3 & -\frac{3}{2} \\ 3 & 9 & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} & c \equiv \pm 1, d \equiv 0 \mod 4 \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} & c \equiv 2, d \equiv \pm 1 \mod 4 \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & c \equiv 0, d \equiv \pm 1 \mod 4 \end{cases}$$

The proof uses the same techniques and cancellations as the preceding case. In particular, notice that the representations are the same except swapping the congruence conditions of the first two matrices. □

4.3. **The 8th and 24th roots of unity.** This section explores the final congruence cases to be characterized. One case will be proved in detail in same manner as the preceding propositions, the other cases are analogous and follow the same structure.

We will require the following partial totient function linear combinations (see Definition 1.18) to cleanly display the results. Begin by letting $\zeta = \frac{i+1}{\sqrt{2}}$ with $\overline{\zeta}$ denoting the usual complex conjugation. Then define

$$\begin{aligned}
A_1 &= \varphi_{c,8}(1, \zeta, i, -\overline{\zeta}, -1, -\zeta, -i, \overline{\zeta}) & B_1 &= \varphi_{c,8}(1, -\overline{\zeta}, -i, \zeta, -1, \overline{\zeta}, i, -\zeta)\\
A_2 &= \varphi_{c,8}(-1, \zeta, -i, -\overline{\zeta}, 1, -\zeta, i, \overline{\zeta}) & B_2 &= \varphi_{c,8}(-1, -\overline{\zeta}, i, \zeta, 1, -\overline{\zeta}, -i, -\zeta)\\
A_3 &= \varphi_{c,8}(1, \overline{\zeta}, -i, -\zeta, -1, -\overline{\zeta}, i, \zeta) & B_3 &= \varphi_{c,8}(1, -\zeta, i, \overline{\zeta}, -1, \zeta, -i, -\overline{\zeta})\\
A_4 &= \varphi_{c,8}(-1, \overline{\zeta}, i, -\zeta, 1, -\overline{\zeta}, -i, \zeta) & B_4 &= \varphi_{c,8}(-1, -\zeta, -i, \overline{\zeta}, 1, \zeta, i, -\overline{\zeta})
\end{aligned}$$

$$\begin{aligned}
C_1 &= \varphi_{c,8}(1, -\zeta, i, \overline{\zeta}, -1, \zeta, -i, -\overline{\zeta}) & D_1 &= \varphi_{c,8}(1, \overline{\zeta}, -i, -\zeta, -1, -\overline{\zeta}, i, \zeta)\\
C_2 &= \varphi_{c,8}(-1, -\zeta, -i, \overline{\zeta}, 1, \zeta, i, -\overline{\zeta}) & D_2 &= \varphi_{c,8}(-1, \overline{\zeta}, i, -\zeta, 1, -\overline{\zeta}, -i, \zeta)\\
C_3 &= \varphi_{c,8}(1, -\overline{\zeta}, -i, \zeta, -1, \overline{\zeta}, i, -\zeta) & D_3 &= \varphi_{c,8}(1, \zeta, i, -\overline{\zeta}, -1, -\zeta, -i, \overline{\zeta})\\
C_4 &= \varphi_{c,8}(-1, -\overline{\zeta}, i, \zeta, 1, \overline{\zeta}, -i, -\zeta) & D_4 &= \varphi_{c,8}(-1, \zeta, -i, -\overline{\zeta}, 1, -\zeta, i, \overline{\zeta})
\end{aligned}$$

One may observe patterns and similarities between each of the defined variables above, though we will not make use of these patterns in particular.

Proposition 4.5. $e^{2\pi i\cdot 1/8}, e^{2\pi i\cdot 11/24}, e^{2\pi i\cdot 19/24}$
Let y be one of the values listed and set $x = -y^{-2}$. Then for all $c \geq 1$,

$$K(y,c) = \begin{cases} \begin{pmatrix} 5 & 3\zeta & \frac{3}{2}\zeta \\ 3\overline{\zeta} & 9 & \frac{1}{2} \\ -\frac{3}{2}\overline{\zeta} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} & c = 1 \\ \varphi(c)\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} & 8 \mid c \\ \begin{pmatrix} 5\varphi(c) & 3A_1 & \frac{3}{2}A_2 \\ 3A_3 & 9\varphi(c) & 0 \\ \frac{3}{2}A_4 & 0 & \frac{9}{4}\varphi(c) \end{pmatrix} & c \equiv 1 \mod 8 \\ \begin{pmatrix} 4\varphi(c) & 0 & 0 \\ 0 & 10\varphi(c) & 3i\varphi_{c,8}(-1,1,-1,1) \\ 0 & 3i\varphi_{c,8}(1,-1,1,-1) & \frac{9}{4}\varphi(c) \end{pmatrix} & c \equiv 2 \mod 8 \\ \begin{pmatrix} 5\varphi(c) & 3B_1 & \frac{3}{2}B_2 \\ 3B_3 & 9\varphi(c) & 0 \\ \frac{3}{2}B_4 & 0 & \frac{9}{4}\varphi(c) \end{pmatrix} & c \equiv 3 \mod 8 \\ \varphi(c)\begin{pmatrix} 4 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 1 \end{pmatrix} & c \equiv 4 \mod 8 \\ \begin{pmatrix} 5\varphi(c) & 3C_1 & \frac{3}{2}C_2 \\ 3C_3 & 9\varphi(c) & 0 \\ \frac{3}{2}C_4 & 0 & \frac{9}{4}\varphi(c) \end{pmatrix} & c \equiv 5 \mod 8 \\ \begin{pmatrix} 4\varphi(c) & 0 & 0 \\ 0 & 10\varphi(c) & -3i\varphi_{c,8}(-1,1,-1,1) \\ 0 & -3i\varphi_{c,8}(1,-1,1,-1) & \frac{9}{4}\varphi(c) \end{pmatrix} & c \equiv 6 \mod 8 \\ \begin{pmatrix} 5\varphi(c) & 3D_1 & \frac{3}{2}D_2 \\ 3D_3 & 9\varphi(c) & 0 \\ \frac{3}{2}D_4 & 0 & \frac{9}{4}\varphi(c) \end{pmatrix} & c \equiv 7 \mod 8 \end{cases}$$

Proof. We begin with a finite number of representations where congruence conditions are given modulo 8, since $\phi_{x,y}$ is congruence of level 8. Let $\zeta = \frac{i+1}{\sqrt{2}}$, with $\overline{\zeta}$ denoting the usual complex conjugation. Then

$$
(\phi_{x,y}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))^t \phi_{x^{-1},y^{-1}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \begin{cases}
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & c \equiv 0 \\
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} & c \equiv 4 \\
\dfrac{1}{4}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 10 & 3i \\ 0 & -3i & \frac{5}{2} \end{pmatrix} & \begin{array}{l} c \equiv 2, d \equiv 3 \\ c \equiv 2, d \equiv 7 \\ c \equiv 6, d \equiv 1 \\ c \equiv 6, d \equiv 5 \end{array} \\
\dfrac{1}{4}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 10 & -3i \\ 0 & 3i & \frac{5}{2} \end{pmatrix} & \begin{array}{l} c \equiv 2, d \equiv 1 \\ c \equiv 2, d \equiv 5 \\ c \equiv 6, d \equiv 3 \\ c \equiv 6, d \equiv 7 \end{array} \\
\dfrac{1}{4}\begin{pmatrix} 5 & 3 & -\frac{3}{2} \\ 3 & 9 & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} & \begin{array}{l} c \equiv 1, d \equiv 0 \\ c \equiv 3, d \equiv 0 \\ c \equiv 5, d \equiv 0 \\ c \equiv 7, d \equiv 0 \end{array} \\
\dfrac{1}{4}\begin{pmatrix} 5 & -3 & \frac{3}{2} \\ -3 & 9 & -\frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} & \begin{array}{l} c \equiv 1, d \equiv 4 \\ c \equiv 3, d \equiv 4 \\ c \equiv 5, d \equiv 4 \\ c \equiv 7, d \equiv 4 \end{array} \\
\dfrac{1}{4}\begin{pmatrix} 5 & -3i & \frac{3}{2}i \\ 3i & 9 & -\frac{1}{2} \\ -\frac{3}{2}i & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} & \begin{array}{l} c \equiv 1, d \equiv 6 \\ c \equiv 3, d \equiv 2 \\ c \equiv 5, d \equiv 6 \\ c \equiv 7, d \equiv 2 \end{array} \\
\dfrac{1}{4}\begin{pmatrix} 5 & 3i & -\frac{3}{2}i \\ -3i & 9 & -\frac{1}{2} \\ \frac{3}{2}i & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} & \begin{array}{l} c \equiv 1, d \equiv 2 \\ c \equiv 3, d \equiv 6 \\ c \equiv 5, d \equiv 2 \\ c \equiv 7, d \equiv 6 \end{array} \\
\dfrac{1}{4}\begin{pmatrix} 5 & 3\overline{\zeta} & \frac{3}{2}\overline{\zeta} \\ 3\zeta & 9 & \frac{1}{2} \\ \frac{3}{2}\zeta & \frac{1}{2} & \frac{9}{4} \end{pmatrix} & \begin{array}{l} c \equiv 1, d \equiv 7 \\ c \equiv 3, d \equiv 5 \\ c \equiv 5, d \equiv 3 \\ c \equiv 7, d \equiv 1 \end{array} \\
\dfrac{1}{4}\begin{pmatrix} 5 & -3\overline{\zeta} & -\frac{3}{2}\overline{\zeta} \\ -3\zeta & 9 & \frac{1}{2} \\ -\frac{3}{2}\zeta & \frac{1}{2} & \frac{9}{4} \end{pmatrix} & \begin{array}{l} c \equiv 1, d \equiv 3 \\ c \equiv 3, d \equiv 1 \\ c \equiv 5, d \equiv 7 \\ c \equiv 7, d \equiv 5 \end{array} \\
\dfrac{1}{4}\begin{pmatrix} 5 & 3\zeta & \frac{3}{2}\zeta \\ 3\overline{\zeta} & 9 & \frac{1}{2} \\ \frac{3}{2}\overline{\zeta} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} & \begin{array}{l} c \equiv 1, d \equiv 1 \\ c \equiv 3, d \equiv 3 \\ c \equiv 5, d \equiv 5 \\ c \equiv 7, d \equiv 7 \end{array} \\
\dfrac{1}{4}\begin{pmatrix} 5 & -3\zeta & -\frac{3}{2}\zeta \\ -3\overline{\zeta} & 9 & \frac{1}{2} \\ -\frac{3}{2}\overline{\zeta} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} & \begin{array}{l} c \equiv 1, d \equiv 5 \\ c \equiv 3, d \equiv 7 \\ c \equiv 5, d \equiv 1 \\ c \equiv 7, d \equiv 3 \end{array}
\end{cases}
$$

As usual, the special case $c = 1$ is clearly the representation corresponding to $c \equiv 1, d \equiv 1 \mod 8$; the proof is also standard in the cases when $8 \mid c$ and $c \equiv 4 \mod 8$.

Next, the case $c \equiv 2$:

$$\begin{aligned}
K(y,c) &= 4 \sum_{\substack{d=1 \\ gcd(c,d)=1}}^{c} (\phi_{x,y}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))^t \phi_{x^{-1},y^{-1}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \\
&= \sum_{\substack{d=1 \\ d\equiv 1 \\ \gcd(c,d)=1}}^{c} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 10 & -3i \\ 0 & 3i & \frac{5}{2} \end{pmatrix} + \sum_{\substack{d=1 \\ d\equiv 3 \\ \gcd(c,d)=1}}^{c} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 10 & 3i \\ 0 & -3i & \frac{5}{2} \end{pmatrix} \\
&+ \sum_{\substack{d=1 \\ d\equiv 5 \\ \gcd(c,d)=1}}^{c} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 10 & -3i \\ 0 & 3i & \frac{5}{2} \end{pmatrix} + \sum_{\substack{d=1 \\ d\equiv 7 \\ \gcd(c,d)=1}}^{c} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 10 & 3i \\ 0 & -3i & \frac{5}{2} \end{pmatrix} \\
&= \varphi(c,1,8) \begin{pmatrix} 4 & 0 & 0 \\ 0 & 10 & -3i \\ 0 & 3i & \frac{5}{2} \end{pmatrix} + \varphi(c,3,8) \begin{pmatrix} 4 & 0 & 0 \\ 0 & 10 & 3i \\ 0 & -3i & \frac{5}{2} \end{pmatrix} \\
&+ \varphi(c,5,8) \begin{pmatrix} 4 & 0 & 0 \\ 0 & 10 & -3i \\ 0 & 3i & \frac{5}{2} \end{pmatrix} + \varphi(c,7,8) \begin{pmatrix} 4 & 0 & 0 \\ 0 & 10 & 3i \\ 0 & -3i & \frac{5}{2} \end{pmatrix}
\end{aligned}$$

Distributing the appropriate products and sums, we find that

$$K(y,c) = \begin{pmatrix} 4\varphi(c) & 0 & 0 \\ 0 & 10\varphi(c) & 3i\varphi_{c,8}(-1,1,-1,1) \\ 0 & 3i\varphi_{c,8}(1,-1,1,-1) & \frac{9}{4}\varphi(c) \end{pmatrix}$$

Which concludes the case $c \equiv 2 \mod 8$. The proof in the case $c \equiv 6 \mod 8$ is analogous.

As one might expect based on previous results, the proofs for the cases $c \equiv 1, 3, 5, 7 \mod 8$ follow the same structure while permuting representations and congruence conditions. We will next illustrate the proof in the case $c \equiv 1 \mod 8$, where the remaining cases are again analogous.

$$
\begin{aligned}
K(y,c) &= 4 \cdot \sum_{\substack{d=1 \\ gcd(c,d)=1}}^{c} \left(\phi_{x,y}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right)^t \phi_{x^{-1},y^{-1}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \\
&= \sum_{\substack{d=1 \\ d\equiv 0 \\ \gcd(c,d)=1}}^{c} \begin{pmatrix} 5 & 3 & -\frac{3}{2} \\ 3 & 9 & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} + \sum_{\substack{d=1 \\ d\equiv 1 \\ \gcd(c,d)=1}}^{c} \begin{pmatrix} 5 & 3\zeta & \frac{3}{2}\zeta \\ 3\overline{\zeta} & 9 & \frac{1}{2} \\ \frac{3}{2}\overline{\zeta} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} \\
&+ \sum_{\substack{d=1 \\ d\equiv 2 \\ \gcd(c,d)=1}}^{c} \begin{pmatrix} 5 & 3i & -\frac{3}{2}i \\ -3i & 9 & -\frac{1}{2} \\ \frac{3}{2}i & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} + \sum_{\substack{d=1 \\ d\equiv 3 \\ \gcd(c,d)=1}}^{c} \begin{pmatrix} 5 & -3\overline{\zeta} & -\frac{3}{2}\overline{\zeta} \\ -3\zeta & 9 & \frac{1}{2} \\ -\frac{3}{2}\zeta & \frac{1}{2} & \frac{9}{4} \end{pmatrix} \\
&+ \sum_{\substack{d=1 \\ d\equiv 4 \\ \gcd(c,d)=1}}^{c} \begin{pmatrix} 5 & -3 & \frac{3}{2} \\ -3 & 9 & -\frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} + \sum_{\substack{d=1 \\ d\equiv 5 \\ \gcd(c,d)=1}}^{c} \begin{pmatrix} 5 & -3\zeta & -\frac{3}{2}\zeta \\ -3\overline{\zeta} & 9 & \frac{1}{2} \\ -\frac{3}{2}\overline{\zeta} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} \\
&+ \sum_{\substack{d=1 \\ d\equiv 6 \\ \gcd(c,d)=1}}^{c} \begin{pmatrix} 5 & -3i & \frac{3}{2}i \\ 3i & 9 & -\frac{1}{2} \\ -\frac{3}{2}i & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} + \sum_{\substack{d=1 \\ d\equiv 7 \\ \gcd(c,d)=1}}^{c} \begin{pmatrix} 5 & 3\overline{\zeta} & \frac{3}{2}\overline{\zeta} \\ 3\zeta & 9 & \frac{1}{2} \\ \frac{3}{2}\zeta & \frac{1}{2} & \frac{9}{4} \end{pmatrix} \\
&= \varphi(c,0,8) \begin{pmatrix} 5 & 3 & -\frac{3}{2} \\ 3 & 9 & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} + \varphi(c,1,8) \begin{pmatrix} 5 & 3\zeta & \frac{3}{2}\zeta \\ 3\overline{\zeta} & 9 & \frac{1}{2} \\ \frac{3}{2}\overline{\zeta} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} \\
&+ \varphi(c,2,8) \begin{pmatrix} 5 & 3i & -\frac{3}{2}i \\ -3i & 9 & -\frac{1}{2} \\ \frac{3}{2}i & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} + \varphi(c,3,8) \begin{pmatrix} 5 & -3\overline{\zeta} & -\frac{3}{2}\overline{\zeta} \\ -3\zeta & 9 & \frac{1}{2} \\ -\frac{3}{2}\zeta & \frac{1}{2} & \frac{9}{4} \end{pmatrix} \\
&+ \varphi(c,4,8) \begin{pmatrix} 5 & -3 & \frac{3}{2} \\ -3 & 9 & -\frac{1}{2} \\ \frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} + \varphi(c,5,8) \begin{pmatrix} 5 & -3\zeta & -\frac{3}{2}\zeta \\ -3\overline{\zeta} & 9 & \frac{1}{2} \\ -\frac{3}{2}\overline{\zeta} & \frac{1}{2} & \frac{9}{4} \end{pmatrix} \\
&+ \varphi(c,6,8) \begin{pmatrix} 5 & -3i & \frac{3}{2}i \\ 3i & 9 & -\frac{1}{2} \\ -\frac{3}{2}i & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} + \varphi(c,7,8) \begin{pmatrix} 5 & 3\overline{\zeta} & \frac{3}{2}\overline{\zeta} \\ 3\zeta & 9 & \frac{1}{2} \\ \frac{3}{2}\zeta & \frac{1}{2} & \frac{9}{4} \end{pmatrix}
\end{aligned}
$$

By summing the product of each element with its partial totient function, we find that the diagonal entries are as we expected. We see that entries $(3,2),(2,3)$ alternate between $\pm\frac{1}{2}$ for every matrix, so the sum vanishes in both cases (note that this happens in the proofs for $c \equiv 3,5,7$ as well). This happens because

$$
\begin{aligned}
\varphi(c,0,8) &= \varphi(c,1,8), \\
\varphi(c,2,8) &= \varphi(c,4,8), \\
\varphi(c,3,8) &= \varphi(c,6,8), \\
\varphi(c,4,8) &= \varphi(c,7,8),
\end{aligned}
$$

so we find due to the alternating coefficients on each of these partial totients that

$$\tfrac{1}{2}(\varphi_{c,8}(-1,1,-1,1,-1,1,-1,1) = 0.$$

Recall that this defines a linear combination from Definition 1.18. While we cannot say that the remaining partial totients always vanish, it is certainly the case that they sometimes will. It is now clear why A_1, A_2, A_3, A_4 have been defined as such. When $c \equiv 1 \mod 8$,

$$K(y,c) = \begin{pmatrix} 5\varphi(c) & 3A_1 & \frac{3}{2}A_2 \\ 3A_3 & 9\varphi(c) & 0 \\ \frac{3}{2}A_4 & 0 & \frac{9}{4}\varphi(c) \end{pmatrix}$$

The proofs in the remaining cases $c \equiv 3, 5, 7$ are analogous, and it is clear that adding them will be needlessly repetitive, thus we shall give the remaining results without proof. □

Proposition 4.6. $e^{2\pi i \cdot 3/8}, e^{2\pi i \cdot 1/24}, e^{2\pi i \cdot 17/24}$
Let y be one of the values listed and set $x = -y^{-2}$. Then for all $c \geq 1$,

$$K(y,c) = \begin{cases} \begin{pmatrix} 5 & -3\overline{\zeta} & -\frac{3}{2}\overline{\zeta} \\ -3\zeta & 9 & \frac{1}{2} \\ -\frac{3}{2}\zeta & \frac{1}{2} & \frac{9}{4} \end{pmatrix} & c = 1 \\ \varphi(c)\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} & 8 \mid c \\ \begin{pmatrix} 5\varphi(c) & -3A_4 & -\frac{3}{2}A_3 \\ -3A_2 & 9\varphi(c) & 0 \\ -\frac{3}{2}A_1 & 0 & \frac{9}{4}\varphi(c) \end{pmatrix} & c \equiv 1 \mod 8 \\ \begin{pmatrix} 4\varphi(c) & 0 & 0 \\ 0 & 10\varphi(c) & -3i\varphi_{c,8}(-1,1,-1,1) \\ 0 & -3i\varphi_{c,8}(1,-1,1,-1) & \frac{9}{4}\varphi(c) \end{pmatrix} & c \equiv 2 \mod 8 \\ \begin{pmatrix} 5\varphi(c) & -3B_4 & -\frac{3}{2}B_3 \\ -3B_2 & 9\varphi(c) & 0 \\ -\frac{3}{2}B_1 & 0 & \frac{9}{4}\varphi(c) \end{pmatrix} & c \equiv 3 \mod 8 \\ \varphi(c)\begin{pmatrix} 4 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 1 \end{pmatrix} & c \equiv 4 \mod 8 \\ \begin{pmatrix} 5\varphi(c) & 3D_1 & \frac{3}{2}D_2 \\ 3D_3 & 9\varphi(c) & 0 \\ \frac{3}{2}D_4 & 0 & \frac{9}{4}\varphi(c) \end{pmatrix} & c \equiv 5 \mod 8 \\ \begin{pmatrix} 4\varphi(c) & 0 & 0 \\ 0 & 10\varphi(c) & 3i\varphi_{c,8}(-1,1,-1,1) \\ 0 & 3i\varphi_{c,8}(1,-1,1,-1) & \frac{9}{4}\varphi(c) \end{pmatrix} & c \equiv 6 \mod 8 \\ \begin{pmatrix} 5\varphi(c) & 3C_1 & \frac{3}{2}C_2 \\ 3C_3 & 9\varphi(c) & 0 \\ \frac{3}{2}C_4 & 0 & \frac{9}{4}\varphi(c) \end{pmatrix} & c \equiv 7 \mod 8 \end{cases}$$

Proposition 4.7. $e^{2\pi i\cdot 5/8}, e^{2\pi i\cdot 7/24}, e^{2\pi i\cdot 23/24}$
Let y be one of the values listed and set $x=-y^{-2}$. Then for all $c\geq 1$,

$$K(y,c)=\begin{cases}\begin{pmatrix}5 & -3\zeta & -\frac{3}{2}\zeta\\ -3\overline{\zeta} & 9 & \frac{1}{2}\\ -\frac{3}{2}\overline{\zeta} & & \end{pmatrix} & c=1\\ \varphi(c)\begin{pmatrix}4&0&0\\0&4&0\\0&0&4\end{pmatrix} & 8\mid c\\ \begin{pmatrix}5\varphi(c) & -3A_2 & -\frac{3}{2}A_1\\ -3A_4 & 9\varphi(c) & 0\\ -\frac{3}{2}A_3 & 0 & \frac{9}{4}\varphi(c)\end{pmatrix} & c\equiv 1 \mod 8\\ \begin{pmatrix}4\varphi(c) & 0 & 0\\ 0 & 10\varphi(c) & 3i\varphi_{c,8}(-1,1,-1,1)\\ 0 & 3i\varphi_{c,8}(1,-1,1,-1) & \frac{9}{4}\varphi(c)\end{pmatrix} & c\equiv 2 \mod 8\\ \begin{pmatrix}5\varphi(c) & -3B_2 & -\frac{3}{2}B_1\\ -3B_4 & 9\varphi(c) & 0\\ -\frac{3}{2}B_3 & 0 & \frac{9}{4}\varphi(c)\end{pmatrix} & c\equiv 3 \mod 8\\ \varphi(c)\begin{pmatrix}4&0&0\\0&16&0\\0&0&1\end{pmatrix} & c\equiv 4 \mod 8\\ \begin{pmatrix}5\varphi(c) & -3C_2 & -\frac{3}{2}C_1\\ -3C_4 & 9\varphi(c) & 0\\ -\frac{3}{2}C_3 & 0 & \frac{9}{4}\varphi(c)\end{pmatrix} & c\equiv 5 \mod 8\\ \begin{pmatrix}4\varphi(c) & 0 & 0\\ 0 & 10\varphi(c) & -3i\varphi_{c,8}(-1,1,-1,1)\\ 0 & -3i\varphi_{c,8}(1,-1,1,-1) & \frac{9}{4}\varphi(c)\end{pmatrix} & c\equiv 6 \mod 8\\ \begin{pmatrix}5\varphi(c) & -3D_2 & -\frac{3}{2}D_1\\ -3D_4 & 9\varphi(c) & 0\\ -\frac{3}{2}D_3 & 0 & \frac{9}{4}\varphi(c)\end{pmatrix} & c\equiv 7 \mod 8\end{cases}$$

Proposition 4.8. $e^{2\pi i\cdot 7/8}, e^{2\pi i\cdot 5/24}, e^{2\pi i\cdot 13/24}$
Let y be one of the values listed and set $x = -y^{-2}$. Then for all $c \geq 1$,

$$K(y,c) = \begin{cases} \begin{pmatrix} 5 & 3\overline{\zeta} & \frac{3}{2}\overline{\zeta} \\ 3\zeta & 9 & \frac{1}{2} \\ \frac{3}{2}\zeta & \frac{1}{2} & \frac{9}{4} \end{pmatrix} & c = 1 \\ \varphi(c)\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} & 8 \mid c \\ \begin{pmatrix} 5\varphi(c) & 3A_3 & \frac{3}{2}A_4 \\ 3A_1 & 9\varphi(c) & 0 \\ \frac{3}{2}A_2 & 0 & \frac{9}{4}\varphi(c) \end{pmatrix} & c \equiv 1 \mod 8 \\ \begin{pmatrix} 4\varphi(c) & 0 & 0 \\ 0 & 10\varphi(c) & -3i\varphi_{c,8}(-1,1,-1,1) \\ 0 & -3i\varphi_{c,8}(1,-1,1,-1) & \frac{9}{4}\varphi(c) \end{pmatrix} & c \equiv 2 \mod 8 \\ \begin{pmatrix} 5\varphi(c) & 3B_3 & \frac{3}{2}B_4 \\ 3B_1 & 9\varphi(c) & 0 \\ \frac{3}{2}B_2 & 0 & \frac{9}{4}\varphi(c) \end{pmatrix} & c \equiv 3 \mod 8 \\ \varphi(c)\begin{pmatrix} 4 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 1 \end{pmatrix} & c \equiv 4 \mod 8 \\ \begin{pmatrix} 5\varphi(c) & 3C_3 & \frac{3}{2}C_4 \\ 3C_1 & 9\varphi(c) & 0 \\ \frac{3}{2}C_2 & 0 & \frac{9}{4}\varphi(c) \end{pmatrix} & c \equiv 5 \mod 8 \\ \begin{pmatrix} 4\varphi(c) & 0 & 0 \\ 0 & 10\varphi(c) & 3i\varphi_{c,8}(-1,1,-1,1) \\ 0 & 3i\varphi_{c,8}(1,-1,1,-1) & \frac{9}{4}\varphi(c) \end{pmatrix} & c \equiv 6 \mod 8 \\ \begin{pmatrix} 5\varphi(c) & 3D_3 & \frac{3}{2}D_4 \\ 3D_1 & 9\varphi(c) & 0 \\ \frac{3}{2}D_2 & 0 & \frac{9}{4}\varphi(c) \end{pmatrix} & c \equiv 7 \mod 8 \end{cases}$$

We have now seen the structure and formulation of each sum in the congruence case and may present plots (Figures 1 and 2) relating growth rates of diagonal entries as c grows. Inspecting these plots, it is apparent that the magnitude of the diagonal entries of $K(y,c)$ grow linearly with c. We have also seen in the congruence case computations that the diagonal entries of the Kloosterman sums bound the off-diagonal entries, especially as c grows.

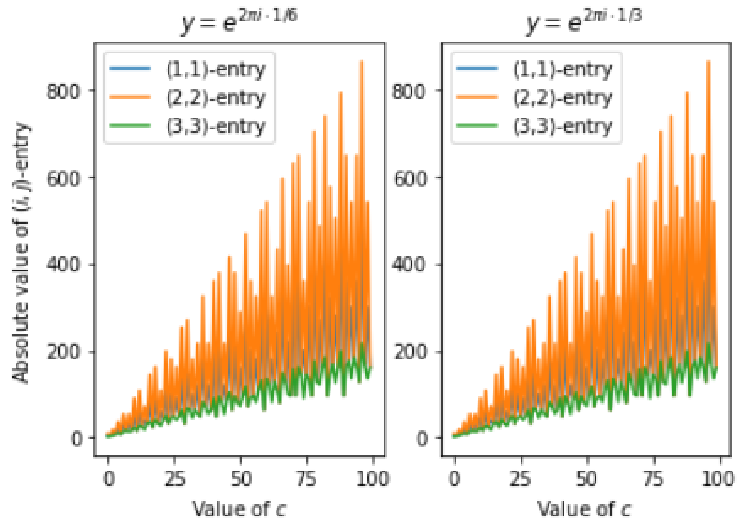

FIGURE 1. Absolute value of diagonal entries of $K(y, c)$ with c for primitive 6th and 3rd roots of unity.

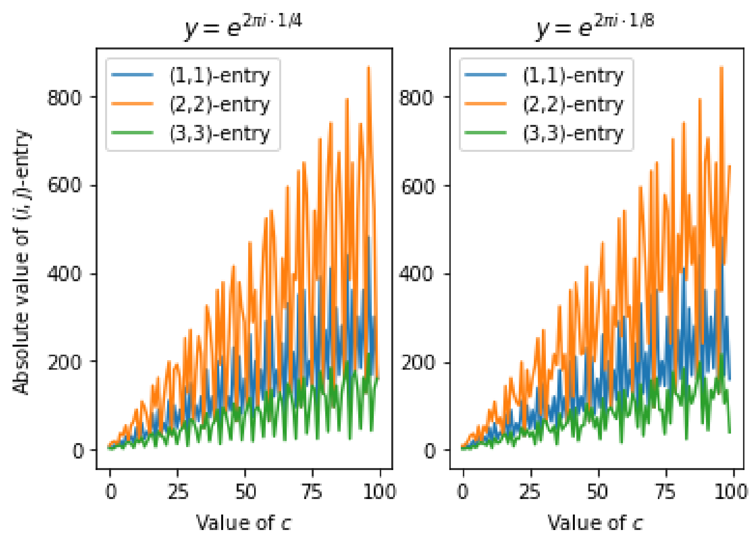

FIGURE 2. Absolute value of diagonal entries of $K(y, c)$ with c for primitive 4th and 8th roots of unity.

5. SUMS IN THE CASE OF FINITE IMAGE WITHOUT CONGRUENCE

Each of the preceding propositions have been in the congruence case which necessarily places them in the finite image case. Up to this point, we have discussed the existence of cases of finite image without congruence; we have remarked how these cases are particularly interesting. This section aims to extract information from these perplexing yet familiar sums.

Recall that we made use of the finite image condition described by Franc-Mason [6] for the representation $\phi_{x,y}$ of the modular group:

$$y = e^{2\pi i m/n}$$
$$x = -y^{-2}$$

The preceding computations required the additional condition that $n \mid 24$ in order to formulate the sums based on the congruence of c, d. Franc and Mason [6] determined that there are infinitely many such cases of finite image without congruence, but we shall explore one case of a 9th root of unity, beginning with computations using the first primitive root, which we shall denote ζ_9 for the remainder of this section:

$$y = \zeta_9 \implies x = -\zeta_9^{-2}$$

Remark 5.1. While the fact that $9 \nmid 24$ tells us that we do not have congruence, it is worth noting that 9 is not relatively prime to 24, in fact $gcd(9, 24) = 3$. This may contribute to the fewer number of matrix terms appearing the sums (compared to other finite image cases) that we will see soon. One might predict a class of "worst cases" for understanding these sums, namely computing the sums for an input value of y an nth root of unity for a prime n. This intuition appears to be correct, since computations in these cases suggest that there are a far greater number of distinct matrix terms that appear in the sums which makes it difficult to classify them and find patterns. Exploring a sum in this prime case is beyond the scope of this thesis, but is certainly worth investigating.

As claimed above, the case we will explore contains a finite number of distinct matrices appearing in its associated Kloosterman sum, and we proceed by listing each of these terms as usual. We are not able to determine beforehand when each term appears in the sum as we could in the congruence case. The most we can currently say depends on the parity of c, d, which allows for three distinct cases to be explored:

(1) $2 \mid c, 2 \nmid d$
(2) $2 \nmid c, 2 \nmid d$
(3) $2 \nmid c, 2 \mid d$

Shortly, we will see that nine total matrices can be classified evenly into each of the above cases. We will also attempt to gain insight in classifying when each matrix term might appear in the sum.

Remark 5.2. In the preceding congruence cases we were able to associate to three distinct roots of unity a single Kloosterman sum. Despite the fact that we are exploring a case without congruence, computations suggest that this property carries over. That is, each unique formulation of a Kloosterman sum corresponds to a family of three inputs. In this 9th root of unity case, we find two distinct Kloosterman sums:

$K(\zeta_9^{\alpha}, c) = K(\zeta_9^{\beta}, c)$ whenever

(1) $\alpha, \beta \in \{1, 4, 7\}$, or
(2) $\alpha, \beta \in \{2, 5, 8\}$.

We find that these equalities are likely due to the relations:

$$y_1 = \zeta_9 = e^{2\pi i \cdot 1/9} \implies x = -e^{-2\pi i \cdot 2/9} = -e^{2\pi i \cdot 7/9} = -y_3$$
$$y_2 = \zeta_9^4 = e^{2\pi i \cdot 4/9} \implies x = -e^{-2\pi i \cdot 8/9} = -e^{2\pi i \cdot 1/9} = -y_1$$
$$y_3 = \zeta_9^7 = e^{2\pi i \cdot 7/9} \implies x = -e^{-2\pi i \cdot 14/9} = -e^{2\pi i \cdot 4/9} = -y_2$$

Applying the the same analysis on case (2) above yields similar relations. Recall from Definition 1.1 that the modular group is obtained as a quotient of $SL_2(\mathbf{Z})$ by $\pm I$, contributing to cancellations of sign in the corresponding Kloosterman sums to yield the same results in each case.

Begin by letting $y = \zeta_9^k$ where $k \in \{1, 4, 7\}$ as described above in case (1), and let $\omega := \zeta_9^3$ be the first primitive third root of unity. The matrices are labeled for notational simplicity in following results. For $c \geq 1$, each term in the Kloosterman sum is given by:

$$(\phi_{x,y}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))^t \phi_{x^{-1},y^{-1}}\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \begin{cases} I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & 2 \mid c \\ M_2 = \dfrac{1}{4}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 13 & 3(\omega + \frac{1}{2}) \\ 0 & -3(\omega + \frac{1}{2}) & \frac{7}{4} \end{pmatrix} & 2 \mid c \\ M_3 = \dfrac{1}{4}\begin{pmatrix} 4 & 0 & 0 \\ 0 & 13 & -3(\omega + \frac{1}{2}) \\ 0 & 3(\omega + \frac{1}{2}) & \frac{7}{4} \end{pmatrix} & 2 \mid c \\ M_4 = \dfrac{1}{4}\begin{pmatrix} 5 & -3 & -\frac{3}{2} \\ -3 & 9 & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} & 2 \nmid c, 2 \nmid d \\ M_5 = \dfrac{1}{4}\begin{pmatrix} 5 & 3(\omega + 1) & \frac{3}{2}(\omega + 1) \\ -3\omega & 9 & \frac{1}{2} \\ -\frac{3}{2}\omega & \frac{1}{2} & \frac{9}{4} \end{pmatrix} & 2 \nmid c, 2 \nmid d \\ M_6 = \dfrac{1}{4}\begin{pmatrix} 5 & 3\omega & -\frac{3}{2}\omega \\ 3(\omega + 1) & 9 & \frac{1}{2} \\ \frac{3}{2}(\omega + 1) & \frac{1}{2} & \frac{9}{4} \end{pmatrix} & 2 \nmid c, 2 \nmid d \\ M_7 = \dfrac{1}{4}\begin{pmatrix} 5 & -3(\omega + 1) & \frac{3}{2}(\omega + 1) \\ 3\omega & 9 & -\frac{1}{2} \\ -\frac{3}{2}\omega & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} & 2 \nmid c, 2 \mid d \\ M_8 = \dfrac{1}{4}\begin{pmatrix} 5 & 3 & -\frac{3}{2} \\ 3 & 9 & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} & \frac{9}{4} \end{pmatrix} & 2 \nmid c, 2 \mid d \\ M_9 = \dfrac{1}{4}\begin{pmatrix} 5 & 3\omega & -\frac{3}{2}\omega \\ -3(\omega + 1) & 9 & \frac{1}{2} \\ \frac{3}{2}(\omega + 1) & \frac{1}{2} & \frac{9}{4} \end{pmatrix} & 2 \nmid c, 2 \mid d \end{cases}$$

One can see that these representations look similar to each of the previous cases we have looked at: we find that most six of the matrices have diagonal entries of $(5, 9, 4)$, and that there exist related matrices that may have nice cancellations when adding adjacent matrices in the list. It is also clear that the three distinct cases with respect to the parity of c, d correspond to three distinct matrices each.

In an attempt to gain insight of when each term may appear in the sum, we present two tables of select values relating how many times each of the above matrices appears for which c in the summation of $K(y,c)$. The matrix M_1 is simply the three-dimensional identity matrix, so we label it as I_3 in the tables. Omitted table entries correspond to none of that matrix appearing in its sum. The tables are divided into three sections for each permutation on the parity of c, d.

TABLE 1. Number of times M_i appears as a term in $K(y,c)$ for $c \leq 100$

	$2 \mid c$			$2 \nmid c, 2 \mid d$			$2 \nmid c, 2 \nmid d$		
c	I_3	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9
1					1				
2			1						
3				1			1		
4		2							
5						2			2
6	2								
7				1	1	1	1	1	1
8	2		2						
9				2		1	2		1
10	2	2							
11				2	2	1	2	2	1
12	2	2							
13				4		2	4		2
14		2	4						
15				3		1	3		1
16	2	2	4						
17				2	2	4	2	2	4
19				3	3	3	3	3	3
20		4	4						
21				3	1	2	3	1	2
23				2	4	5	2	4	5
25				3	4	3	3	4	3
27				6	3		6	3	
29				3	4	7	3	4	7
31				5	4	6	5	4	6
32	6	2	8						
33				5	2	3	5	2	3
35				3	7	2	3	7	2
64	14	8	10						
75				9	6	5	9	6	5
81				11	7	9	11	7	9
99				14	7	9	14	7	9
100	16	12	12						

Notice that, as expected, the sum of each row is exactly Euler's $\varphi(c)$ since the sum is over all d coprime to c (see Example 1.16). It is also evident that when $2 \nmid c$, a matrix from the $2 \mid d$ family appearing a number of times corresponds to another matrix in the $2 \nmid d$ family that same number of times. That is, we see a symmetry with respect to the number of matrices appearing the case $2 \mid d$ versus the case $2 \nmid d$. This observation is formalized in the following conjecture.

TABLE 2. Number of times M_i appears as a term in $K(y, c)$ for $c > 100$

	$2 \mid c$			$2 \nmid c$			$2 \nmid c$		
c	I_3	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9
101				15	18	17	15	18	17
128	22	18	24						
200	32	18	30						
256	42	36	50						
300	44	16	20						
400	52	52	56						
451				65	70	65	65	70	65
512	94	60	102	63	51	52	63	51	52
600	66	46	48						
625				71	89	90	71	89	90
700	88	80	72						
800	114	98	108						
900	74	94	72						
1000	176	98	126						
1001				126	121	113	126	121	113
1024	150	174	188						
$10,000$	1460	1262	1278						

Conjecture 5.3. *Let $y = \zeta_9^k$, where $k \in \{1, 4, 7\}$. Then for all odd $c \geq 2$, the Kloosterman sum is given by*

$$K(y, c) = \alpha(M_4 + M_7) + \beta(M_5 + M_8) + \gamma(M_6 + M_9)$$

where $\alpha, \beta, \gamma \in \mathbf{Z}$ such that $\alpha+\beta+\gamma = \frac{\varphi(c)}{2}$. Similarly for all even $c \geq 2$, the Kloosterman sum is given as an integer linear combination of the matrices I_3, M_2, M_3.

Remark 5.4. It is a reasonable assumption that a construction very similar to that in Conjecture 5.3 can be written down in the case

$$y = \zeta_9^k, k \in \{2, 5, 8\}.$$

Computations in the congruence case lead us to predict that the only difference between these cases would be permuting a subset of the matrices and congruence conditions on c, d.

Conjecture 5.3 implies that in the case $2 \nmid c$, there is a one-to-one correspondence between matrices appearing in the case $2 \mid d$ and matrices appearing in the case $2 \nmid d$; these matrices will appear the same number of times in the Kloosterman sum. Moreover, computations allow us to predict exactly which of the matrices are grouped together. This is a remarkable fact, since we may determine the conditions in which one matrix appears in $K(y, c)$ and immediately know how many times its corresponding matrix appears.

As c grows, Table 1 suggests that that every matrix (up to the obvious restrictions arising from the parity of c) makes a significant contribution the sum. This is expected given that in the previous computations in Section 4, we were able to assign a matrix to distinct congruence classes of c and d. Less intuitively, however, Tables 1 and 2 do not suggest a convergence of these integers to a uniform distribution across the matrices constituting the sum; this is unexpected since this convergence is exactly what happened in the congruence case sums.

Example 5.5. Let $y = \zeta_9^k$ for $k \in \{1, 4, 7\}$ and consider the relatively large value $c = 10,000 = 10^4$ with $\varphi(10^4) = 4000$. A priori, one might expect to see three integers who sum to 4000 spread approximately uniformly across the matrices I_3, M_2, M_3: perhaps distributions of

$$K(y, 10^4) = 1333 \cdot I_3 + 1333 \cdot M_2 + 1334 \cdot M_3,$$

or

$$K(y, 10^4) = 1331 \cdot I_3 + 1334 \cdot M_2 + 1335 \cdot M_3.$$

According to Table 2, the values are actually

$$K(y, 10^4) = 1460 \cdot I_3 + 1262 \cdot M_2 + 1278 \cdot M_3.$$

Notice also that we cannot make any general claims comparing the ordering of these numbers, since none are suggested by Tables 1 and 2. This is *not* proof that these numbers do not eventually become uniformly distributed, only an observation that up to $c = 10^4$, it is not the case. Greater values of c, both odd and even, should be checked to gain more insight. Both tables suggest that there exist values for which these integers are nearby. For example, $c = 7$ has exactly 1 of each matrix in its sum, $c = 451$ sees a difference of only 5 between its numbers, and $c = 700$ sees a relative difference of 8 between each of its numbers.

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