INTRODUCTION TO MATROID AND ITS REPRESENTABILITY

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Abstract. This paper introduces a mathematical structure called Matroid which abstracts the concept of linear independence. The goal of this paper is to discuss the representability of matroid, which will be defined later in this paper, and introduce some examples, sketches of proofs about some of the important theorems and conjectures about representability and others.

Contents

1. Introduction to Matroid

Introduction the similarity of linearly independence and cycles

Definition 1.1. A matroid $M = (E, \mathcal{I})$ is a pair such that E is a finite set of elements, and $\mathcal{I}$ is a family of subsets of E with the following properties:

- $\emptyset \in \mathcal{I}$
- For any $A \in \mathcal{I}$, any subset of A is in $\mathcal{I}$.
- For any $A, B \in \mathcal{I}$ such that $|A| < |B|$, there always exists $x \in B - A$ such that $A \cup \{x\} \in \mathcal{I}$.

E is called the ground set, $\mathcal{I}$ is called the independent sets. A subset of E is called independent if and only if it is in $\mathcal{I}$.

2. Some basic definitions/results

This chapter will introduce some of the basic results, definition which will be necessary to start discussing the representability. (But I started to think that there may not be too many results I need...)

Theorem 2.1. *All maximal independent sets have the same size.*

3. Introduction to representability

This chapter will introduce definitions and results that are necessary to discuss the representability.

More specifically, this chapter will introduce:

- Why we care about representability
  - If all matroids are representable, we are just giving a vector space another name.
  - Therefore, it is important that not all matroids are representable.
  - Show some examples of unrepresentable matroids

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  - Now that we know the existence, the question is, which one?
- The mathematical definition of "representability"

Definition 3.1. Let $M = (E, \mathcal{I})$, $X \subseteq E$ be given. $M \backslash X$ denotes a deletion of X in M and is defined to be $(E - X, \{I \in \mathcal{I} \mid X \cap I = \emptyset\})$

Note that $M \backslash e$ for some element $e \in E$ is equivalent to $M \backslash \{e\}$.

Theorem 3.2. *Let $M = (E, \mathcal{I})$, $X \subseteq E$ be given. $M \backslash X$ is indeed a matroid*

Proof. Let $M' = (E', \mathcal{I}') = M \backslash X$. Since $\emptyset \in \mathcal{I}$ and $X \cap \emptyset = \emptyset$, $\emptyset \in \mathcal{I}$. Let $I \in \mathcal{I}', J \subseteq I$. Since $I \in \mathcal{I}$, $J \in \mathcal{I}$. Since $J \subseteq I$ and $I \cap X = \emptyset$, $J \cap X = \emptyset$. Therefore, $J \in \mathcal{I}'$. Let $A, B \in \mathcal{I}'$ such that $|A| < |B|$. Since $A, B \in \mathcal{I}'$, we know that $A, B \in \mathcal{I}$. Therefore, we can find $x \in B - A$ such that $(A \cup \{x\}) \in \mathcal{I}$. Since $(A \cup \{x\}) \subseteq (A \cup B)$ and $X \cap A = X \cap B = \emptyset$, $X \cap (A \cup \{x\}) = \emptyset$. Therefore, $A \cup \{x\} \in \mathcal{I}'$. Thus we have found such $x \in B - A$ that $A \cup \{x\} \in \mathcal{I}'$. Since $M' = M \backslash X$ follows the three properties, it is indeed a matroid. $\square$

Definition 3.3. Let $M = (E, \mathcal{I})$, $e \in E$ be given. M/e denotes contraction of M by e and $M/e = \begin{cases} M \backslash e, \text{if } e \text{ is a loop,} \\ (E - \{e\}, \{I \in \mathcal{I} \mid e \notin I, (I \cup \{e\}) \in \mathcal{I}\}), \text{ otherwise.} \end{cases}$

Theorem 3.4. *Contraction by an element indeed generates a matroid.*

Proof. If e is a loop, M/e is obviously a matrod since we know that deletion always generates a matroid. Suppose otherwise. Let $\mathcal{I}'$ denote the independent sets of M/e. First, $\emptyset \in \mathcal{I}, e \notin \emptyset$. Since e is not a loop, $(\emptyset \cup \{e\}) \in \mathcal{I}$. Therefore, $\emptyset \in \mathcal{I}'$. Let $I \in \mathcal{I}', J \subseteq I$. Since $I \in \mathcal{I}$, $J \in \mathcal{I}$. Since $e \notin I$, $e \notin J$. Since $(I \cup \{e\}) \in \mathcal{I}$ and $J \subseteq I$, $(J \cup \{e\}) \in \mathcal{I}$. Therefore, $J \in \mathcal{I}'$. Let $A, B \in \mathcal{I}'$ such that $|A| < |B|$. Let $A' = A \cup \{e\}, B' = B \cup \{e\}$. Since $A, B \in \mathcal{I}'$, $A', B' \in \mathcal{I}$. Since $e \notin A, e \notin B$, $|A'| < |B'|$. Let $x \in B' - A'$ such that $A' \cup \{x\} \in \mathcal{I}$. Since $B' - A' = B - A$, $x \in B - A$. For such x, we just showed that $A \cup \{e\} \cup \{x\} \in \mathcal{I}$. Also, $x \neq e$ since $e \in A'$. Therefore, $A \cup \{x\} \in \mathcal{I}'$. Hence, we have found $x \in B - A$ such that $A \cup \{x\} \in \mathcal{I}'$. Since this follows three properties given in the definition, this is indeed a matroid. Therefore, contraction by an element indeed generates a matroid. $\square$

Here are a few simple yet useful results about deletion.

Theorem 3.5. *Let a matroid $M = (E, \mathcal{I}), X \subseteq E$ be given. Let A be an independent set in $M \backslash X$. Then A is independent in M.*

Proof. The independent sets of $M \backslash X$ is $\{I \in \mathcal{I} \mid (I \cap X) = \emptyset\}$. It is easy to see that it is a subset of $\mathcal{I}$. Since A is in the subset of $\mathcal{I}$, A must be in $\mathcal{I}$. $\square$

In other words, this means that deletion never "adds" a new element to the independent sets.

Theorem 3.6. *The deletion of a loop does not change the independent sets.*

Proof. Let $M = (E, \mathcal{I}), e \in E, \{e\} \notin \mathcal{I}$. The independent sets of $M \backslash e$ is $\{I \in \mathcal{I} \mid (I \cap \{e\}) = \emptyset\}$. Since $\{e\}$ is a loop, no independent set can contain e. Therefore, the independent sets of $M \backslash e$ is identical to $\mathcal{I}$. $\square$

Now that we have defined contraction of a matroid by an element, we can define contraction by a subset of a ground set.

Definition 3.7. Let $M = (E, \mathcal{I}), X = \{x_1, \cdots, x_k\} \subseteq E$. M/X is defined to be $(((M/x_1)/x_2) \cdots)/x_k)$.

It is not obvious that this is well-defined. In other words, it is not obvious that the order of contraction does not matter. The following theorem shows that the order does not matter.

Theorem 3.8. *For any given matroid $M = (E, \mathcal{I})$, $(M/e)/f = (M/f)/e$ for any $e \neq f \in E$.*

Proof. There are a few cases.

(1) e, f are both loops.
$(M/e)/f = (M\backslash e)/f$ Since deletion of a loop does not change the independent sets, f is a loop in $(M\backslash e)$. Therefore, $(M/e)/f = (M\backslash e)\backslash f$. Again, deletion of f does not change the independent sets since f is a loop in (M/e). Therefore, we have $(M/e)/f = (E - \{e, f\}, \mathcal{I})$. By symmetry, $(M/e)/f = (M/f)/e$.

(2) One of e, f is a loop, and the other one is not.
Without loss of generality, assume e is a loop. $(M/e)/f = (M\backslash e)/f$. Since deletion of a loop does not change the independent set, the independent set of (M/e) is $\mathcal{I}$. Therefore, the independent set of $(M/e)/f$ is $\mathcal{I}' = \{I \in \mathcal{I} \mid f \notin I, (I \cup \{f\}) \in \mathcal{I}\}$. On the other hand, it is easy to see that $\mathcal{I}'$ is identical to the independent sets of (M/f). Since contraction by an element does not add new elements to the independent sets, e is a loop in (M/f). Since deletion by a loop does not change the independent sets, the independent sets of $(M/f)/e$ is $\mathcal{I}'$. Now we confirmed that $(M/e)/f$ and $(M/f)/e$ have the same independent sets. Therefore, $(M/e)/f = (M/f)/e$.

(3) Neither of them is a loop, and $\{e, f\} \in \mathcal{I}$.
The independent set of M/e is $\mathcal{I}' = \{I \in \mathcal{I} \mid e \notin I, (I \cup \{e\}) \in \mathcal{I}\}$. Since $\{e, f\} \in \mathcal{I}$, $\{f\} \in \mathcal{I}'$. Therefore, f is not a loop in M/e. Hence, the independent sets of $(M/e)/f$ is $\mathcal{I}'' = \{I \in \mathcal{I}' \mid f \notin I, (I \cup \{f\}) \in \mathcal{I}'\}$. $\mathcal{I}''$ is actually equivalent to $S = \{I \in \mathcal{I} \mid e \notin I, f \notin I, (I \cup \{e, f\}) \in \mathcal{I}\}$. We can prove $\mathcal{I}'' = S$ by starting to show that $\mathcal{I}'' \subseteq S$. Let $I \in \mathcal{I}''$. Since I is an independent set of $(M/e)/f$, we know that $e, f \notin I$. Since $(I \cup \{f\}) \in \mathcal{I}'$, we also know that $((I \cup \{f\}) \cup \{e\}) \in \mathcal{I}$. Therefore, $(I \cup \{e, f\}) \in \mathcal{I}$. Thus $I \in S$, and $\mathcal{I}'' \subseteq S$. Now, we want to show that $S \subseteq \mathcal{I}''$. Let $I \in S$. By the definition of S, we know that $e, f \notin I$. Since $(I \cup \{e, f\}) \in \mathcal{I}$, we know that $(I \cup \{e\}) \in \mathcal{I}$. Since $((I \cup \{f\}) \cup \{e\}) \in \mathcal{I}$ and $e \notin (I \cup \{f\})$, we know that $(I \cup \{f\}) \in \mathcal{I}'$. Since $f \notin I$ and $(I \cup \{f\}) \in \mathcal{I}'$, $I \in \mathcal{I}''$. Hence, $S \subseteq \mathcal{I}''$.

Combining these two results, we know that $S = \mathcal{I}''$. By the symmetry, $(M/e)/f$ and $(M/f)/e$ have the same independent sets. Therefore $(M/e)/f = (M/f)/e$.

(4) Neither of e, f is a loop, but $\{e, f\} \notin \mathcal{I}$.
The independent set of M/e is $\mathcal{I}' = \{I \in \mathcal{I} \mid e \notin I, (I \cup \{e\}) \in \mathcal{I}\}$. Since $\{e, f\} \notin \mathcal{I}$, f is a loop in M/e. Therefore, $(M/e)/f = (M/e)\backslash f$. Since the deletion of a loop does not change the independent sets, the independent sets of $(M/e)/f$ is $\mathcal{I}'$. By applying the same argument, the independent set of $(M/f)/e$ is $\mathcal{I}'' = \{I \in \mathcal{I} \mid f \notin I, (I \cup \{f\}) \in \mathcal{I}\}$. We want to show that $\mathcal{I}' = \mathcal{I}''$. By the symmetry, it suffices to show that $\mathcal{I}' \subseteq \mathcal{I}''$. Let $I \in \mathcal{I}'$. Since $\{e, f\}$ is dependent, $f \notin I$. (Otherwise, $I \cup \{e\}$ would be

dependent.) Since both $(I \cup \{e\})$ and $\{f\}$ are independent, we can grow $\{f\}$ by adding elements from $(I \cup \{e\})$ until they have the same size. Since $\{e, f\}$ is dependent, we never add e. In other words, we add every element from $\mathcal{I}$. It means that $I \cup \{f\}$ is independent. Therefore, $I \in \mathcal{I}''$, and thus $\mathcal{I}' \subseteq \mathcal{I}''$. By symmetry, $\mathcal{I}'' \subseteq \mathcal{I}'$. Therefore, $\mathcal{I}' = \mathcal{I}''$.

Therefore, in any case, $(M/e)/f = (M/f)/e$. □

List of topics I could discuss further:

- Maybe talk about dual matroid
  - redefine contraction/deletion in terms of dual
- More properties about contraction deletion
  - $(M/e)\backslash f = (M\backslash f)/e$?
  - Therefore, any minor can be represented as $(M\backslash A)/B$

3.1. What do contraction and deletion mean in graphs and vector spaces?

3.2. Why do these matter?

Because if a matroid is representable over some field $\mathbb{F}$, its minor is always representable over $\mathbb{F}$.

3.3. The introduction of Rota's conjecture.

4. More discussion on matroid representability

This chapter will be directly about matroid representability.

Theorem 4.1. *If a matroid $M = (E, \mathcal{I})$ only contains at most 3 non-loop elements, it is regular.*

Proof. meh □

Definition 4.2. Fano matroid. A set of vertices is independent if it has at most three points and is not a line.

Theorem 4.3. *Fano plane is representable over F_2, but not anything else.*

Proof. meh □

Possible topics I am thinking of discussing

- (Sketch of) proofs of some small cases of Rota's conjecture
- List of interesting matroids that are not representable

References

[1] "WHAT IS A MATROID?" Oxley
[2] "On Matroid Representability and Minor Problems" Hlineny
[3] "Solving Rotas Conjecture" Geelen, Gerards, Whittle