INTRODUCTION TO A MATROID AND ITS REPRESENTABILITY

HIDENORI SHINOHARA

Abstract. This paper introduces a matroid. A matroid is a mathematical structure that abstracts the concept of linear independence. The goal of this paper is to discuss the representability of a matroid with several examples and to introduce an important conjecture related to the representability of a matroid.

1. Introduction to Matroid

A matroid is a mathematical structure that abstracts the concept of linear independence. A linearly independent set of vectors has a lot of good properties. Interestingly, a lot of sets of other mathematical objects often share those properties. For example, any subset of a linearly independent set of vectors is always linearly independent. Consider a subset of edges of a graph that does not contain any cycle. Then any subset of it does not contain any cycle. The matroid theory is an attempt to mathematically formalize those properties and investigate in them.

Definition 1.1. A matroid $M = (E, \mathcal{I})$ is a pair such that E is a finite set of elements, and $\mathcal{I}$ is a family of subsets of E with the following properties:

- $\emptyset \in \mathcal{I}$
- For any $A \in \mathcal{I}$, any subset of A is in $\mathcal{I}$.
- For any $A, B \in \mathcal{I}$ such that $|A| < |B|$, there always exists $x \in B - A$ such that $A \cup \{x\} \in \mathcal{I}$.

E is called the ground set, $\mathcal{I}$ is called the independent sets. A subset of E is called independent if and only if it is in $\mathcal{I}$. In this paper, we will put our focus on a matroid with a finite ground set.

There are some basic matroids that are important in the following discussions. We will start by introducing a column matroid. A column matroid is constructed from a matrix over a field $\mathbb{F}$.

Definition 1.2. Let a matrix A with m rows over $\mathbb{F}$ be given. A column matroid M of A is a matroid with a ground set $\{1, 2, \cdots m\}$. A subset of E is independent in M if and only if the set of the corresponding column vectors is linearly independent.

Theorem 1.3. *A column matroid is indeed a matroid.*

Proof. First, an empty set of vectors is linearly independent by definition. Suppose there exists a linearly independent set of vectors that has a linearly dependent subset. Let $A = \{a_1, \cdots, a_k\} \subseteq B = \{b_1, \cdots, b_n\}$ be such sets. Since A is linearly dependent, there exist constants $c_1, \cdots, c_k$ such that $c_1a_1 + \cdots + c_ka_k = 0$ and

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not all c_i's are 0. However, that implies that we can find constants $d_1, \cdots, d_n$ such that $d_1b_1 + \cdots + d_kb_k = 0$ and not all d_i's are 0. This is a contradiction since B is supposed to be independent. Therefore, such subset must not exist, and thus all subsets of linearly independent sets are linearly independent. Now we want to prove the third property. Let $A = \{a_1, \cdots, a_k\}, B = \{b_1, \cdots, b_n\}$ be linearly independent sets, and assume that $k < n$. Suppose for each $i = 1, \cdots, n$, $b_i \in span\{a_1, \cdots, a_k\}$. It means that the span of B is a subspace of the span of A, which has a smaller dimension than n. That is a contradiction. Therefore, there exists i such that $b_i \notin span\{a_1, \cdots, a_k\}$. For such i, $\{a_1, \cdots, a_k, b_i\}$ should be linearly independent. Since a column matroid satisfies the three properties, it is indeed a matroid. □

Definition 1.4. A uniform matroid $U_{r,k}$ is a matroid such that $E = \{1, \cdots, k\}$ and $\mathcal{I} = \{X \mid X \subseteq E, |X| \leq r\}$.

It is easy to see that a uniform matroid is indeed a matroid.

Here are some important results that will show up later in this paper.

Theorem 1.5. *All maximal independent sets have the same size.*

Proof. Let X, Y be maximal independent sets of some matroid. Suppose $|X| \neq |Y|$. Without loss of generality, $|X| < |Y|$. By the third property of a matroid, there exists $e \in Y - X$ such that $X \cup \{e\}$ is independent. It is a contradiction since X is a maximal independent set. Therefore $|X| = |Y|$. □

This is indeed true in linear algebra. Given a matrix, any maximal independent subset of column vectors always has the same size. In linear algebra, this number is often referred to as the *dimension* of a vector space or the *rank* of a matrix. In the matroid theory, we use the term *rank* as well.

Definition 1.6. The rank of a matroid is the size of a maximal independent set.

The rank of a column matroid is equal to the rank of the matrix since the subset of a ground set is independent if and only if the subset of column vectors is linearly independent.

Definition 1.7. Let $M = (E, \mathcal{I})$ be given. $e \in E$ is called a loop if $\{e\} \notin \mathcal{I}$.

We will conclude this chapter by introducing the notion of isomorphic matroids.

Definition 1.8. Let $M_1 = (E_1, \mathcal{I}_1), M_2 = (E_2, \mathcal{I}_2)$ be given. M_1, M_2 are isomorphic to each other if there exists a bijective mapping $\phi : E_1 \to E_2$ such that $\forall X \subseteq E_1, X \in \mathcal{I}_1 \iff \{\phi(e) : e \in X\} \in \mathcal{I}_2$.

Two isomorphic matroids have the same structure.

2. Introduction to representability

We will start this chapter by defining the representability of a matroid.

Definition 2.1. A matroid $M = (E, \mathcal{I})$ is representable over a field $\mathbb{F}$ if there exists a matrix A over $\mathbb{F}$ such that the column matroid of A is isomorphic to M.

Therefore, if the matroid indeed succeeded in abstracting the concept of linear independence, there should be some matroids that are *not* representable over some fields. If all matroids are representable over every field, it means that we are simply discussing linear algebra using different terms.

Before introducing some unrepresentable matroids, we will start by introducing a nice property of representable matroids.

Theorem 2.2. *Let $M = (E, \mathcal{I})$ be a matroid that is representable over $\mathbb{F}$. Let r be a rank of M and $k = |E|$. Then there exists a matrix $A \in \mathbb{F}^{r\times k}$ such that M is isomorphic to the column matroid of A.*

Proof. Let B a matrix over $\mathbb{F}$ such that the column matroid of B is isomorphic to M. It is easy to see that the number of columns of B is k. From linear algebra, we know that elementary row operations preserve the linear independence of column vectors. Let R be the reduced row echelon form of B. Since R must have a rank of r, it only has r leading zeros. In other words, R has exactly r non-zero row vectors. Removing zero rows clearly does not affect the linear independence. Therefore, we found a matrix in $\mathbb{F}^{r\times k}$ whose column matroid is isomorphic to M. □

This property is useful when proving that a matroid is unrepresentable over some field.

Here are some matroids that are not representable over some fields to show that not all matroids are representable over every field.

Theorem 2.3. *$U_{2,4}$ is not representable over $GF(2)$.*

A matroid is called a *binary matroid* if it is representable over $GF(2)$.

Proof. We know that $U_{2,4}$ has a rank of 2. If $U_{2,4}$ is representable over $GF(2)$, there exists a matrix $A \in GF(2)^{2\times 4}$ such that A's column matroid is isomorphic to $U_{2,4}$. Since $GF(2)$ only has two elements, there are only four possible column vectors of size 2. $\left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ A must not contain $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ since it is a loop. Since A has 4 columns and there are only 3 different column vectors, we know that there are two columns in A that have the exact same column vectors. This is a contradiction since a subset of such two elements will not be independent. Therefore, $U_{2,4}$ is not representable over $GF(2)$. □

$U_{2,4}$ is representable over some field such as $\mathbb{R}$. For example, the column matroid of $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ is isomorphic to $U_{2,4}$.

Now, we will introduce Fano matroid. Fano matroid can be constructed from Fano plane. Fano matroid is one of the examples of matroids that are representable over $GF(2)$, but not over $\mathbb{R}$. We will start by defining Fano matroid mathematically.

Definition 2.4. Fano matroid is a matroid with a ground set $\{1, 2, \cdots, 7\}$. A set of vertexes is independent if it satisfies one of the followings:

(1) it contains less than or equal to 2 elements,
(2) it contains exactly 3 points and they are not on the same line.

Any set of vertexes that have more than 3 elements is dependent.

For example, $\{4, 5\}$ and $\{1, 2, 3\}$ are independent, but $\{1, 4, 2\}$, $\{1, 5, 7\}$, and $\{4, 5, 6\}$ are not independent as each of them is on one line.

Theorem 2.5. *Fano matroid is indeed a matroid.*

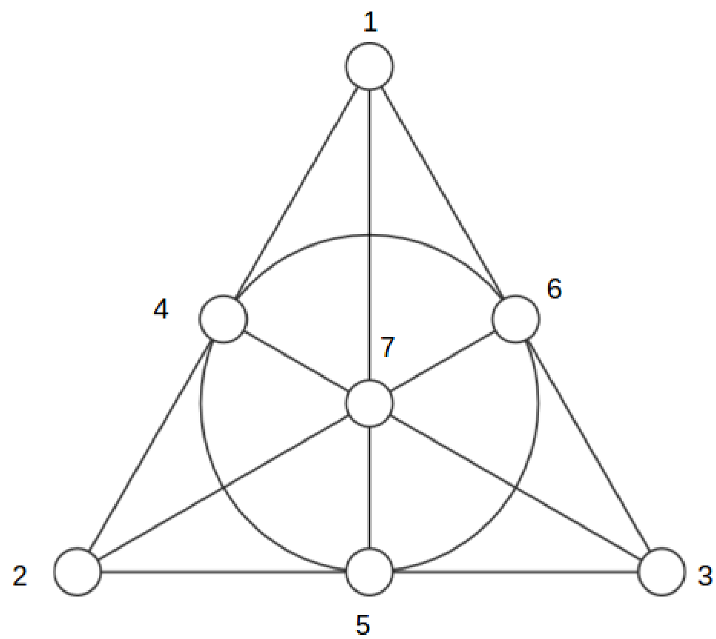

FIGURE 1. Fano plane

Proof. An empty set is independent since it contains less than 2 elements. Any independent set has at most three elements, so any proper subset of it has at most two elements. Therefore, any subset of independent sets is always independent. The third property can be proved by checking each case. Since any set with 2 or fewer elements is independent, we only need to care about the case when we have a set with 2 elements and a set of three elements. Let $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2\}$ be independent sets. It is easy to see from the figure that there must be exactly one line that goes through both b_1 and b_2. Let x denote the third point on such line. Then, adding any element other than b_1, b_2, x to B will generate an independent set of three elements. Therefore, we just need to make sure that $A \neq \{b_1, b_2, x\}$. That cannot be the case since $\{b_1, b_2, x\}$ is dependent and A is independent. Therefore, there must be an element $a \in A - B$ such that $B \cup \{a\}$ is independent. Since Fano matroid satisfies all three properties, it is indeed a matroid. □

Here is an interesting property of Fano matroid

Theorem 2.6. *If Fano matroid is representable over a field* $\mathbb{F}$, $1 + 1 = 0$ *in that field.*

Proof. Suppose Fano matroid is representable over a given field $\mathbb{F}$. Let $A \in \mathbb{F}^{3\times 7}$ such that A's column matroid is isomorphic to Fano matroid. Let R be a row reduced echelon form of A. Then the first three columns should be identical to I_3 since R has a rank of 3. Since $\{1, 2, 4\}$ is dependent, $R_{3,4}$ is 0. Applying the same argument to $\{2, 3, 5\}, \{1, 3, 6\}$, we get the following: $\begin{pmatrix} 1 & 0 & 0 & ? & 0 & ? & ? \\ 0 & 1 & 0 & ? & ? & 0 & ? \\ 0 & 0 & 1 & 0 & ? & ? & ? \end{pmatrix}$

Since multiplying a non-zero constant to some column does not affect on the linearly independency, assume that the first non-zero elements of 4, 5, 6th columns are all 1. $\begin{pmatrix}1 & 0 & 0 & 1 & 0 & 1 & ?\\ 0 & 1 & 0 & ? & 1 & 0 & ?\\ 0 & 0 & 1 & 0 & ? & ? & ?\end{pmatrix}$

Let $a = R_{2,4}, b = R_{3,5}$. $\begin{pmatrix}1 & 0 & 0 & 1 & 0 & 1 & ?\\ 0 & 1 & 0 & a & 1 & 0 & ?\\ 0 & 0 & 1 & 0 & b & ? & ?\end{pmatrix}$ Since $\{4,5,6\}$ is dependent, $R_{3,6}$ must be $-ab$. $\begin{pmatrix}1 & 0 & 0 & 1 & 0 & 1 & ?\\ 0 & 1 & 0 & a & 1 & 0 & ?\\ 0 & 0 & 1 & 0 & b & -ab & ?\end{pmatrix}$ We need to look into the seventh column. The seventh column actually cannot contain any zero. For example, suppose $R_{1,7} = 0$. Then, $\{2,3,7\}$ would be dependent. That's a contradiction. Similar arguments apply to the case of $R_{2,7} = 0, R_{3,7} = 0$. By multiplying a non-zero constant, we get: $\begin{pmatrix}1 & 0 & 0 & 1 & 0 & 1 & 1\\ 0 & 1 & 0 & a & 1 & 0 & ?\\ 0 & 0 & 1 & 0 & b & -ab & ?\end{pmatrix}$ Since $\{3,4,7\}$ is dependent, $R_{2,7} = a$.

And since $\{1,5,7\}$ is dependent, $R_{3,7} = bR_{2,7} = ab$. $\begin{pmatrix}1 & 0 & 0 & 1 & 0 & 1 & 1\\ 0 & 1 & 0 & a & 1 & 0 & a\\ 0 & 0 & 1 & 0 & b & -ab & ab\end{pmatrix}$ $\{2,6,7\}$ is dependent as well. By observing those three columns, it is easy to see that $-(-ab)+ab$ must be 0. Since neither of a, b is 0, $ab(1+1) = 0$ implies $1+1 = 0$. (If a is 0, 1st column and 4th column would be identical. The similar argument applies to b.) □

This theorem is powerful, since this implies that Fano matroid is not representable over $\mathbb{R}$.

Corollary 2.7. *Fano matroid is not representable over* $\mathbb{R}$.

Proof. In $\mathbb{R}$, $1 + 1 \neq 0$. Therefore, Fano matroid is not representable over $\mathbb{R}$. □

Theorem 2.8. *If a matroid* $M = (E, \mathcal{I})$ *only contains at most 3 non-loop elements, it is representable over any field* $\mathbb{F}$.

A matroid is called *regular* if it can be represented over any field.

Proof. I'll use rank here. □

The number of matroids increases exponentially as the size of the ground set increases.

3. More discussion on matroid representability

This chapter will introduce a new concept, a matroid minor, which is crucial when discussing the matroid representability. Rota's conjecture will be also introduced at the end of the chapter. (I am thinking of adding a sketch of the proof by the end of the semester if I can understand the outline)

In order to introduce a matroid minor, we first need to introduce two operations on matroids. Deletion and contraction.

Definition 3.1. Let $M = (E, \mathcal{I})$, $X \subseteq E$ be given. $M \backslash X$ denotes a deletion of X in M and is defined to be $(E - X, \{I \in \mathcal{I} \mid X \cap I = \emptyset\})$

Note that $M\backslash e$ for some element $e \in E$ is equivalent to $M\backslash\{e\}$.

Theorem 3.2. *Let $M = (E, \mathcal{I})$, $X \subseteq E$ be given. $M\backslash X$ is indeed a matroid*

Proof. Let $M' = (E', \mathcal{I}') = M\backslash X$. Since $\emptyset \in \mathcal{I}$ and $X \cap \emptyset = \emptyset$, $\emptyset \in \mathcal{I}$. Let $I \in \mathcal{I}', J \subseteq I$. Since $I \in \mathcal{I}$, $J \in \mathcal{I}$. Since $J \subseteq I$ and $I \cap X = \emptyset$, $J \cap X = \emptyset$. Therefore, $J \in \mathcal{I}'$. Let $A, B \in \mathcal{I}'$ such that $|A| < |B|$. Since $A, B \in \mathcal{I}'$, we know that $A, B \in \mathcal{I}$. Therefore, we can find $x \in B - A$ such that $(A \cup \{x\}) \in \mathcal{I}$. Since $(A \cup \{x\}) \subseteq (A \cup B)$ and $X \cap A = X \cap B = \emptyset$, $X \cap (A \cup \{x\}) = \emptyset$. Therefore, $A \cup \{x\} \in \mathcal{I}'$. Thus, we have found such $x \in B - A$ that $A \cup \{x\} \in \mathcal{I}'$. Since $M' = M\backslash X$ follows the three properties, it is indeed a matroid. $\square$

Definition 3.3. Let $M = (E, \mathcal{I})$, $e \in E$ be given. M/e denotes contraction of M by e and $M/e = \begin{cases} M\backslash e, \text{if } e \text{ is a loop,} \\ (E - \{e\}, \{I \in \mathcal{I} \mid e \notin I, (I \cup \{e\}) \in \mathcal{I}\}), \text{ otherwise.} \end{cases}$

Theorem 3.4. *Contraction by an element indeed generates a matroid.*

Proof. If e is a loop, M/e is obviously a matroid since we know that deletion always generates a matroid. Suppose otherwise. Let $\mathcal{I}'$ denote the independent sets of M/e. First, $\emptyset \in \mathcal{I}, e \notin \emptyset$. Since e is not a loop, $(\emptyset \cup \{e\}) \in \mathcal{I}$. Therefore, $\emptyset \in \mathcal{I}'$. Let $I \in \mathcal{I}', J \subseteq I$. Since $I \in \mathcal{I}$, $J \in \mathcal{I}$. Since $e \notin I$, $e \notin J$. Since $(I \cup \{e\}) \in \mathcal{I}$ and $J \subseteq I$, $(J \cup \{e\}) \in \mathcal{I}$. Therefore, $J \in \mathcal{I}'$. Let $A, B \in \mathcal{I}'$ such that $|A| < |B|$. Let $A' = A \cup \{e\}, B' = B \cup \{e\}$. Since $A, B \in \mathcal{I}'$, $A', B' \in \mathcal{I}$. Since $e \notin A, e \notin B$, $|A'| < |B'|$. Let $x \in B' - A'$ such that $A' \cup \{x\} \in \mathcal{I}$. Since $B' - A' = B - A$, $x \in B - A$. For such x, we just showed that $A \cup \{e\} \cup \{x\} \in \mathcal{I}$. Also, $x \neq e$ since $e \in A'$. Therefore, $A \cup \{x\} \in \mathcal{I}'$. Hence, we have found $x \in B - A$ such that $A \cup \{x\} \in \mathcal{I}'$. Since this follows three properties given in the definition, this is indeed a matroid. Therefore, contraction by an element indeed generates a matroid. $\square$

Here are a few simple yet useful results about deletion.

Theorem 3.5. *Let a matroid $M = (E, \mathcal{I}), X \subseteq E$ be given. Let A be an independent set in $M\backslash X$. Then A is independent in M.*

Proof. The independent sets of $M\backslash X$ is $\{I \in \mathcal{I} \mid (I \cap X) = \emptyset\}$. It is easy to see that it is a subset of $\mathcal{I}$. Since A is in the subset of $\mathcal{I}$, A must be in $\mathcal{I}$. $\square$

In other words, this means that deletion never "adds" a new element to the independent sets.

Theorem 3.6. *The deletion of a loop does not change the independent sets.*

Proof. Let $M = (E, \mathcal{I}), e \in E, \{e\} \notin \mathcal{I}$. The independent sets of $M\backslash e$ is $\{I \in \mathcal{I} \mid (I \cap \{e\}) = \emptyset\}$. Since $\{e\}$ is a loop, no independent set can contain e. Therefore, the independent sets of $M\backslash e$ is identical to $\mathcal{I}$. $\square$

Now that we have defined contraction of a matroid by an element, we can define contraction by a subset of a ground set.

Definition 3.7. Let $M = (E, \mathcal{I}), X = \{x_1, \cdots, x_k\} \subseteq E$. M/X is defined to be $(((M/x_1)/x_2)\cdots)/x_k)$.

It is not obvious that this is well-defined. In other words, it is not obvious that the order of contraction does not matter. The following theorem shows that the order does not matter.

Theorem 3.8. *For any given matroid $M = (E, \mathcal{I})$, $(M/e)/f = (M/f)/e$ for any $e \neq f \in E$.*

Proof. There are a few cases.

(1) e, f are both loops.
$(M/e)/f = (M\backslash e)/f$ Since deletion of a loop does not change the independent sets, f is a loop in $(M\backslash e)$. Therefore, $(M/e)/f = (M\backslash e)\backslash f$. Again, deletion of f does not change the independent sets since f is a loop in (M/e). Therefore, we have $(M/e)/f = (E - \{e, f\}, \mathcal{I})$. By symmetry, $(M/e)/f = (M/f)/e$.

(2) One of e, f is a loop, and the other one is not.
Without loss of generality, assume e is a loop. $(M/e)/f = (M\backslash e)/f$. Since deletion of a loop does not change the independent set, the independent set of (M/e) is $\mathcal{I}$. Therefore, the independent sets of $(M/e)/f$ is $\mathcal{I}' = \{I \in \mathcal{I} \mid f \notin I, (I \cup \{f\}) \in \mathcal{I}\}$. On the other hand, it is easy to see that $\mathcal{I}'$ is identical to the independent sets of (M/f). Since contraction by an element does not add new elements to the independent sets, e is a loop in (M/f). Since deletion by a loop does not change the independent sets, the independent sets of $(M/f)/e$ is $\mathcal{I}'$. Now we confirmed that $(M/e)/f$ and $(M/f)/e$ have the same independent sets. Therefore, $(M/e)/f = (M/f)/e$.

(3) Neither of them is a loop, and $\{e, f\} \in \mathcal{I}$.
The independent set of M/e is $\mathcal{I}' = \{I \in \mathcal{I} \mid e \notin I, (I \cup \{e\}) \in \mathcal{I}\}$. Since $\{e, f\} \in \mathcal{I}$, $\{f\} \in \mathcal{I}'$. Therefore, f is not a loop in M/e. Hence, the independent sets of $(M/e)/f$ is $\mathcal{I}'' = \{I \in \mathcal{I}' \mid f \notin I, (I \cup \{f\}) \in \mathcal{I}'\}$. $\mathcal{I}''$ is actually equivalent to $S = \{I \in \mathcal{I} \mid e \notin I, f \notin I, (I \cup \{e, f\}) \in \mathcal{I}\}$. We can prove $\mathcal{I}'' = S$ by starting to show that $\mathcal{I}'' \subseteq S$. Let $I \in \mathcal{I}''$. Since I is an independent set of $(M/e)/f$, we know that $e, f \notin I$. Since $(I \cup \{f\}) \in \mathcal{I}'$, we also know that $((I \cup \{f\}) \cup \{e\}) \in \mathcal{I}$. Therefore, $(I \cup \{e, f\}) \in \mathcal{I}$. Thus $I \in S$, and $\mathcal{I}'' \subseteq S$. Now, we want to show that $S \subseteq \mathcal{I}''$. Let $I \in S$. By the definition of S, we know that $e, f \notin I$. Since $(I \cup \{e, f\}) \in \mathcal{I}$, we know that $(I \cup \{e\}) \in \mathcal{I}$. Since $((I \cup \{f\}) \cup \{e\}) \in \mathcal{I}$ and $e \notin (I \cup \{f\})$, we know that $(I \cup \{f\}) \in \mathcal{I}'$. Since $f \notin I$ and $(I \cup \{f\}) \in \mathcal{I}'$, $I \in \mathcal{I}''$. Hence, $S \subseteq \mathcal{I}''$.
Combining these two results, we know that $S = \mathcal{I}''$. By the symmetry, $(M/e)/f$ and $(M/f)/e$ have the same independent sets. Therefore $(M/e)/f = (M/f)/e$.

(4) Neither of e, f is a loop, but $\{e, f\} \notin \mathcal{I}$.
The independent set of M/e is $\mathcal{I}' = \{I \in \mathcal{I} \mid e \notin I, (I \cup \{e\}) \in \mathcal{I}\}$. Since $\{e, f\} \notin \mathcal{I}$, f is a loop in M/e. Therefore, $(M/e)/f = (M/e)\backslash f$. Since the deletion of a loop does not change the independent sets, the independent sets of $(M/e)/f$ is $\mathcal{I}'$. By applying the same argument, the independent set of $(M/f)/e$ is $\mathcal{I}'' = \{I \in \mathcal{I} \mid f \notin I, (I \cup \{f\}) \in \mathcal{I}\}$. We want to show that $\mathcal{I}' = \mathcal{I}''$. By the symmetry, it suffices to show that $\mathcal{I}' \subseteq \mathcal{I}''$. Let $I \in \mathcal{I}'$. Since $\{e, f\}$ is dependent, $f \notin I$. (Otherwise, $I \cup \{e\}$ would be dependent.) Since both $(I \cup \{e\})$ and $\{f\}$ are independent, we can grow $\{f\}$ by adding elements from $(I \cup \{e\})$ until they have the same size. Since $\{e, f\}$ is dependent, we never add e. In other words, we add every element

from $\mathcal{I}$. It means that $I \cup \{f\}$ is independent. Therefore, $I \in \mathcal{I}''$, and thus $\mathcal{I}' \subseteq \mathcal{I}''$. By symmetry, $\mathcal{I}'' \subseteq \mathcal{I}'$. Therefore, $\mathcal{I}' = \mathcal{I}''$.

Therefore, in any case, $(M/e)/f = (M/f)/e$. □

Now that we have defined contraction and deletion, we can define a *minor* of a matroid.

Definition 3.9. A minor of a matroid is a matroid that can be obtained by some (possibly zero) a number of contraction and deletion.

Therefore, most matroids have more than one minors. To define the matroid minor more concisely, we will prove the following theorem.

Theorem 3.10. *Let a matroid $M = (E, \mathcal{I})$ and $e \neq f \in E$ be given. Then $(M/e)\backslash f = (M\backslash f)/e$.*

Proof. There are a few cases.

(1) Both e and f are loops. This case is easy to prove since neither contraction nor deletion add any new elements to the independent sets. In other words, f is a loop in (M/e) and e is a loop in $(M\backslash f)$. $(M/e)\backslash f = (M\backslash e)\backslash f = (M\backslash f)\backslash e = (M\backslash f)/e$
(2) e is a loop, but f is not a loop. $(M/e)\backslash f = (M\backslash e)\backslash f$ since e is a loop. $(M\backslash f)/e = (M\backslash f)\backslash e$ since e is a loop in $(M\backslash f)$. We know that the order of deletion does not matter, so they are equivalent.
(3) e is not a loop, but f is a loop. Since the deletion of f does not change the independent sets, M and $(M\backslash f)$ both have the same independent sets, although their ground sets are not identical. Therefore, (M/e) and $(M\backslash f)/e$ have the same independent sets from the definition of contraction.
(4) Neither e nor f is a loop. Let $X \subseteq E - \{e, f\}$. X is independent in $(M/e)\backslash f$ if and only if X is independent in (M/e). X is independent in (M/e) if and only if $(X \cup \{e\})$ is independent in M. On the other hand, X is independent in $(M\backslash f)/e$ if and only if $X \cup \{e\}$ is independent in $(M\backslash f)$. $X \cup \{e\}$ is independent in $(M\backslash f)$ if and only if $X \cup \{e\}$ is independent in M. Therefore, X is independent in $(M/e)\backslash f$ if and only if X is independent in $(M\backslash f)/e$.

Therefore, in every case, $(M/e)\backslash f$ is identical to $(M\backslash f)/e$. □

By this theorem, we know that any series of operations can be expressed as $(M/A)\backslash B$ where A, B are disjoint subsets of E. Therefore, the following definition is equivalent to the previous definition.

Definition 3.11. Let $M = (E, \mathcal{I})$ be given. Let A, B be disjoint subsets of E. Then a matroid $(M/A)\backslash B$ is called a minor of M.

Moreover, if $A \cup B \neq \emptyset$, we call $(M/A)\backslash B$ a *proper minor*.

Of course, we could have defined a minor as $(M\backslash A)/B$ instead of $(M\ A)\backslash B$.

3.1. **What do contraction and deletion mean in graphs and vector spaces?** In vector spaces, contraction can be considered as projection to its orthogonal vector. Maybe insert a figure.

3.2. **Why do these matter?** The discussion of contraction and deletion is very important when discussing the representability of matroids since if a matroid is representable over some field $\mathbb{F}$, its minor is always representable over $\mathbb{F}$.

Theorem 3.12. *Let a matroid $M = (E, \mathcal{I})$ such that it is representable over $\mathbb{F}$. Any minor of M is representable over $\mathbb{F}$.*

Proof. It suffices to show that M/e and $M\backslash e$ are both representable over $\mathbb{F}$ for any $e \in E$. Let r be a rank of M, $n = |E|$. Let $e \in E$ be given. Let $A = \big(u_1 u_2 \cdots u_n\big) \in \mathbb{F}^{r\times n}$ be a matrix such that the column matroid of M is isomorphic to A. Without loss of generality, we can assume $E = \{1, 2, \cdots, n\}$ and $e = n$.

(1) First, we prove the case of deletion. We claim that $M\backslash e$ is isomorphic to the column matroid of $\big(u_1 u_2 \cdots u_{n-1}\big)$. We prove so by comparing the independent sets of each matroid. By definition, $M\backslash e = (E - \{e\}, \{I \in \mathcal{I} \mid e \notin I\})$. Let $U = \{u_{i_1}, u_{i_2}, \cdots, u_{i_k}\}$ be a subset of $\{u_1, u_2, \cdots, u_{n-1}\}$. We want to show that U is linearly independent if and only if $\{i_1, i_2, \cdots i_k\}$ is linearly independent in $M\backslash e$. Suppose U is linearly independent. Then $\{i_1, i_2, \cdots i_k\}$ is independent in the column matroid of A. Since $\{i_1, i_2, \cdots i_k\}$ is in $\mathcal{I}$ and does not contain $e = n$, it is independent in $M\backslash e$ as well. Suppose U is linearly dependent. Then $\{i_1, i_2, \cdots i_k\}$ is dependent in the column matroid of A. Since $\{i_1, i_2, \cdots i_k\}$ is not in $\mathcal{I}$ it is dependent in $M\backslash e$ as well. Therefore, $M\backslash e$ is representable over $\mathbb{F}$.

(2) Next, we prove the case of contraction. First, assume e is a loop. Then the corresponding column of A must be a zero vector. It is easy to see that the removal of the corresponding column will yield a matrix whose column vector is isomorphic to $M/e = M\backslash e$. Now, assume that e is not a loop. Then the corresponding column of A must not be a zero vector. We prove the proposition by comparing the independent sets of each matroid. By definition, $M/e = (E - \{e\}, \{I \in \mathcal{I} \mid e \notin I, (e \cup I) \in \mathcal{I}\})$. We claim that M/e is isomorphic to the column matroid of $B \in \mathbb{F}^{r\times n-1}$, where ith column of B, b_i, is $u_i - \dfrac{u_i \cdot u_n}{u_n \cdot u_n} u_n$. (This makes sense since we are assuming that u_n is not a zero vector.) Let $X = \{i_1, i_2, \cdots, i_k\} \subseteq E - \{n\}$ be given.
X is independent in M/e
$\iff$ $X \cup \{n\}$ is independent in M
$\iff$ $\{i_1, i_2, \cdots, i_k, n\}$ is independent in M
$\iff$ $\{u_{i_1}, u_{i_2}, \cdots, u_{i_k}, u_n\}$ is linearly independent
We are going to take a close look at this set of vectors. Let $c_1, c_2, \cdots, c_k, c \in \mathbb{F}$ be given such that $c_1 u_{i_1} + \cdots + c_k u_{i_k} + c u_n = 0$.

$$c_1 b_{i_1} + \cdots + c_k b_{i_k} = \sum_{j=1,\cdots,k} c_j b_{i_j} = c_1 u_{i_1} + c_2 u_{i_2} + \cdots + c_k u_{i_k} - \sum_{j=1,\cdots,k} c_j \frac{u_i \cdot u_n}{u_n \cdot u_n} u_n =$$

$$-c u_n - \sum_{j=1,\cdots,k} c_j \frac{u_i \cdot u_n}{u_n \cdot u_n} u_n = -\Big(c + \sum_{j=1,\cdots,k} c_j \frac{u_i \cdot u_n}{u_n \cdot u_n}\Big) u_n = -\frac{\Big(c u_n \cdot u_n + \sum_{j=1,\cdots,k} c_j (u_i \cdot u_n)\Big)}{u_n \cdot u_n} u_n =$$

$$-\frac{(c_1 u_1 + c_2 u_2 + \cdots + c_k u_k + c u_n) \cdot u_n}{u_n \cdot u_n} u_n = 0.$$

Therefore, $\{u_{i_1}, u_{i_2}, \cdots, u_{i_k}, u_n\}$ is linearly independent
$\iff$ $\{b_{i_1}, b_{i_2}, \cdots, b_{i_k}\}$ is linearly independent
$\iff$ $\{i_1, i_2, \cdots, i_k\}$ is independent in the column matroid of B.
Therefore, the column matroid of B is isomorphic to M/e.

Hence, we have proved that any minor of M is always representable. □

However, neither the converse nor the inverse of this theorem is always true. Any matroid has a representable minor since $U_{0,k}$ is a minor of any matroid. Also, $U_{2,4}$ is not a binary matroid, but any minor of it only contains at most 3 elements, so we know that any minor of $U_{2,4}$ is regular by the theorem. Rota's conjecture is about unrepresentable matroids any of whose minor is representable. It will be discussed in the next chapter.

4. The introduction of Rota's conjecture

Conjecture 4.1. *For each finite field $\mathbb{F}$, there are, up to isomorphism, only finitely many excluded minors for the class of $\mathbb{F}$-representable matroids.*

(From *Solving Rota's conjecture*)

In other words, given a finite field $\mathbb{F}$, let F be a family of all matroids that are representable over $\mathbb{F}$. By the theorem from the previous chapter, we know that F is minor-closed, i.e., any minor of any element in F is in F. Then an excluded minor is a matroid $M \notin F$ such that any minor of M is in F. The conjecture states that for any finite field $\mathbb{F}$, there are only finitely many excluded minors.

Geelen, Gerards, Whittle announced that they have solved Rota's Conjecture. Here is a brief summary of their proof.

As matroid theory is partially an abstraction of graph theory, they have used several theorems from graph theory. Here is one of the theorems from graph theory. Note that the minor of a graph is a graph that can be obtained by contracting some edges, and it does not include deletion of edges.

Theorem 4.2. *Let F be a minor-closed family of graphs, that is, $\forall G \in F$, any minor of G is in F. Then there are only finitely many graphs H such that $H \notin F$ and any minor of H is in F. In other words, each minor-closed class of graphs has only finitely many excluded minors.*

This is called as *Well-Quasi-Ordering Theorem*. One special case of this theorem is a set of planar graphs. Any minor of a planar graph is also planar. Therefore, a set S of planar graphs is minor-closed. Kuratowski's theorem indeed states that there are only two excluded minors, $K_{3,3}, K_5$.

This theorem is very similar to Rota's Conjecture as they are both about a minor-closed family of mathematical objects and claim that there are only finitely many excluded minors.

Here is a generalized version of WQO theorem.

Theorem 4.3. *Let a finite field $\mathbb{F}$ be given and F be a minor-closed family of $\mathbb{F}$-representable matroids. Then there are only finitely many $\mathbb{F}$-representable matroids M such that $M \notin F$ and any minor of M is in F. In other words, for each finite field $\mathbb{F}$ and each minor-closed class of $\mathbb{F}$-representable matroids, there are only finitely many $\mathbb{F}$-representable excluded minors.*

References

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