INTRODUCTION TO MATROID AND ITS REPRESENTABILITY

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ABSTRACT. This paper introduces a mathematical structure called Matroid which abstracts the concept of linear independence. The goal of this paper is to discuss the representability of matroid, which will be defined later in this paper, and introduce some examples, sketches of proofs about some of the important theorems and conjectures about representability and others.

CONTENTS

1. INTRODUCTION TO MATROID

Introduction the similarity of linearly independence and cycles

Definition 1.1. A matroid $M = (E, \mathcal{I})$ is a pair such that E is a finite set of elements, and $\mathcal{I}$ is a family of subsets of E with the following properties:

- $\emptyset \in \mathcal{I}$
- For any $A \in \mathcal{I}$, any subset of A is in $\mathcal{I}$.
- For any $A, B \in \mathcal{I}$ such that $|A| < |B|$, there always exists $x \in B - A$ such that $A \cup \{x\} \in \mathcal{I}$.

E is called the ground set, $\mathcal{I}$ is called the independent sets. A subset of E is called independent if and only if it is in $\mathcal{I}$.

2. SOME BASIC DEFINITIONS/RESULTS

This chapter will introduce some of the basic results, definition which will be necessary to start discussing the representability.

Theorem 2.1. *All maximal independent sets have the same size.*

3. INTRODUCTION TO REPRESENTABILITY

This chapter will introduce definitions and results that are necessary to discuss the representability.

More specifically, this chapter will introduce:

- Why we care about representability
  - If all matroids are representable, we are just giving a vector space another name.
  - Therefore, it is important that not all matroids are representable.
  - Show some examples of unrepresentable matroids
  - Now that we know the existence, the question is, which one?

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- The mathematical definition of "representability"

Definition 3.1. Let $M = (E, \mathcal{I})$, $X \subseteq E$ be given. $M \backslash X$ denotes a deletion of X in M and is defined to be $(E - X, \{I \in \mathcal{I} \mid X \cap I = \emptyset\})$

Note that $M \backslash e$ for some element $e \in E$ is equivalent to $M \backslash \{e\}$.

Theorem 3.2. *Let $M = (E, \mathcal{I})$, $X \subseteq E$ be given. $M \backslash X$ is indeed a matroid*

Proof. Let $M' = (E', \mathcal{I}') = M \backslash X$. Since $\emptyset \in \mathcal{I}$ and $X \cap \emptyset = \emptyset$, $\emptyset \in \mathcal{I}$. Let $I \in \mathcal{I}', J \subseteq I$. Since $I \in \mathcal{I}$, $J \in \mathcal{I}$. Since $J \subseteq I$ and $I \cap X = \emptyset$, $J \cap X = \emptyset$. Therefore, $J \in \mathcal{I}'$. Let $A, B \in \mathcal{I}'$ such that $|A| < |B|$. Since $A, B \in \mathcal{I}'$, we know that $A, B \in \mathcal{I}$. Therefore, we can find $x \in B - A$ such that $(A \cup \{x\}) \in \mathcal{I}$. Since $(A \cup \{x\}) \subseteq (A \cup B)$ and $X \cap A = X \cap B = \emptyset$, $X \cap (A \cup \{x\}) = \emptyset$. Therefore, $A \cup \{x\} \in \mathcal{I}'$. Thus we have found such $x \in B - A$ that $A \cup \{x\} \in \mathcal{I}'$. Since $M' = M \backslash X$ follows the three properties, it is indeed a matroid. □

Definition 3.3. Let $M = (E, \mathcal{I})$, $e \in E$ be given. M/e denotes contraction of M by e and $M/e = \begin{cases} M \backslash e, \text{if } e \text{ is a loop,} \\ (E - \{e\}, \{I \in \mathcal{I} \mid e \notin I, (I \cup \{e\}) \in \mathcal{I}\}), \text{ otherwise.} \end{cases}$

Theorem 3.4. *Contraction by an element indeed generates a matroid.*

Proof. If e is a loop, M/e is obviously a matrod since we know that deletion always generates a matroid. Suppose otherwise. Let $\mathcal{I}'$ denote the independent sets of M/e. First, $\emptyset \in \mathcal{I}, e \notin \emptyset$. Since e is not a loop, $(\emptyset \cup \{e\}) \in \mathcal{I}$. Therefore, $\emptyset \in \mathcal{I}'$. Let $I \in \mathcal{I}', J \subseteq I$. Since $I \in \mathcal{I}$, $J \in \mathcal{I}$. Since $e \notin I$, $e \notin J$. Since $(I \cup \{e\}) \in \mathcal{I}$ and $J \subseteq I$, $(J \cup \{e\}) \in \mathcal{I}$. Therefore, $J \in \mathcal{I}'$. ** Read this part more carefully.** Let $A, B \in \mathcal{I}'$ such that $|A| < |B|$. Let $A' = A \cup \{e\}, B' = B \cup \{e\}$. Since $A, B \in \mathcal{I}'$, $A', B' \in \mathcal{I}$. Since $e \notin A, e \notin B$, $|A'| < |B'|$. Let $x \in B' - A'$ such that $A' \cup \{x\} \in \mathcal{I}$. For such x, we just showed that $A \cup \{e\} \cup \{x\} \in \mathcal{I}$. Also, $x \neq e$ since $e \in A'$. Therefore, $A \cup \{x\} \in \mathcal{I}'$. Hence, we have found $x \in B - A$ such that $A \cup \{x\} \in \mathcal{I}'$. Since this follows three properties given in the definition, this is indeed a matroid. □

Now that we have defined contraction of a matroid by an element, we can define contraction by a subset of a ground set.

Definition 3.5. Let $M = (E, \mathcal{I}), X = \{x_1, \cdots, x_k\} \subseteq E$. M/X is defined to be $(((M/x_1)/x_2) \cdots)/x_k)$.

It is not obvious that this is well-defined. In other words, it is not obvious that the order of contraction does not matter. The following theorem shows that the order does not matter.

Theorem 3.6. *For any given matroid $M = (E, \mathcal{I})$, $(M/e_1)/e_2 = (M/e_2)/e_1$ for any $e_1 \neq e_2 \in E$.*

Proof. Prove it! □

3.1. **What do contraction and deletion mean in graphs and vector spaces?**

3.2. **Why do these matter?** Because if a matroid is representable over some field

$$\mathbb{F}$$

, its minor is always representable over

$$\mathbb{F}$$

.

3.3. **The introduction of Rota's conjecture.**

4. More discussion on matroid representability

I am planning to write several chapters about matroid representability. Some ideas I have at this point include:

- (Sketch of) proofs of some small cases of Rota's conjecture
- List of interesting matroids that are not representable